Question 1: CES with Idiosyncratic Quality

(The following is a variation of the monopolistic competition algebra in the notes.) Assume that a consumer has CES preferences over measure N of goods, labeled $i \in [0, N] \equiv \Omega$ (i.e., i is the 'name' of a good on a continuum, rather than a discrete number). Each good is produced by a single CRS firm making a static, frictionless pricing decision. The consumer provides 1 unit of labor inelastically at a competitive real market wage of w

The following summarize quality, productivity, and demand

- The idiosyncratic productivity of firm i is $z(i)$.
- The idiosyncratic quality of firm i is $a(i)$.
- The aggregate productivity for all firms is Z
- The aggregate quality for all products is A
- The nominal marginal cost for all firms is Pw, where P will be the price index, and w is the real wage determined in equilibrium
- Define idiosyncratic real profits of the firm as $\pi(i)$ where aggregate real profits are $\int_\Omega \pi(i)\mathrm{d}i$. The consumer owns a perfectly diversified portfolio of the firms.
- Let $p(i)$ be the price set by firm i, $y(i)$ be the market clearing demand for firm i, and Y be the real income of the consumer which includes the real wages and the aggregate profits.

1.1 Calculate the Demand Function

The consumer's problem is, given equilibrium $p(i), w$, and Π

$$U = \max_{y(i)} \left[\int_\Omega (Aa(i)y(i))^\rho \,\mathrm{d}i\right]^{1/\rho} \tag{1}$$

$$\text{s.t.} \int_\Omega p(i)y(i)\mathrm{d}i \leq Pw + P\Pi \equiv PY \tag{2}$$

where P is the price index. Define $\kappa \equiv 1/(1-\rho)$,[1]

$$P \equiv \left[\int_\Omega \left(\frac{p(i)}{a(i)}\right)^{1-\kappa} \mathrm{d}i\right]^{1/(1-\kappa)} \tag{3}$$

(a) Calculate the demand function $y(i)$ in terms of $a(i), p(i)/P, Y$

(b) What is U as a function of aggregates? Interpret

[1] Alternatively,you can define P to include the A term, completely encapsulating all of the productivity and quality terms.

1.2 Calculate the Supply, Revenue, Profits

Firm i has an marginal productivity as the product of an aggregate and idiosyncratic term, $Zz(i)$. The firm's problem is to choose the price and labor demand to maximize *real* profits:

$$\pi(i) = \max_{\ell(i),y(i),p(i)} \left\{\tfrac{p(i)}{P}y(i) - w\ell(i)\right\} \tag{4}$$

$$\text{s.t. } y(i) = Zz(i)\ell(i) \tag{5}$$

$$y(p) \text{ from the consumer's demand} \tag{6}$$

With aggregate profits,

$$\Pi \equiv \int_\Omega \pi(i)\mathrm{d}i \tag{7}$$

(a) What is the firm's optimal choice of $p(i)/P$ as a function of $a(i), z(i), A, Z, Y, w$?[2]

(b) What is the firm's real revenue $\frac{p(i)}{P}y(i)$ in terms of these variables? Real profits $\pi(i)$? Labor demand $\ell(i)$?

(c) Substitute the optimal price $p(i)/P$ into the definition of the price index to get an expression for the real wages in terms of $z(i), a(i), Z, A$)?

(d) Define a suitable aggregate 'productivity', $\tilde{Z}$, which encapsulates all of the idiosyncratic quality and productivity variables $a(i), A, z(i), Z$.

1.3 Calculate the Equilibrium

(a) Define a suitable aggregate 'productivity', $\tilde{Z}$, which encapsulates all of the idiosyncratic quality and productivity variables $a(i), A, z(i), Z$.

(b) Express w in terms of $\tilde{Z}$

(c) Simplify Π as much as you can.

(d) Simplify Y and then U as much as you can.

1.4 Calculating for a Given Distribution

Assume that N is fixed. For each firm i in the economy, the distribution of $z(i)$ and $a(i)$ are *independent* and distributed $z(i) \sim \text{Pareto}(z_m, \alpha_z)$ and $a(i) \sim \text{Pareto}(a_m, \alpha_a)$. Call the pdfs of these distributions $f_a(a)$ and $f_z(z)$. The pdf of a $\text{Pareto}(x_m, \alpha)$ distribution is:

$$f(x) = \alpha\frac{x_m^\alpha}{x^{\alpha+1}}$$

for $x \geq x_m$, and 0 otherwise.

[2] For this questions, and all similarly phrased, this is a superset of all allowed variables. Some may drop out of these expressions.

(a) For any function $h(\cdot)$, generically express $\int_\Omega h(z(i))\mathrm{d}i$ in terms of the pdf $f_z(z)$ and $h(\cdot)$.[3]

(b) General this to functions $h(z, a)$ of two variables where z and a are independently distributed. Hint: can use two integrals due to independence.

(c) Calculate $\tilde{Z}$ in terms of Z, A and the Pareto model parameters (where Z and A are parameters since exogenous).

(d) Calculate Y in terms of only model and density parameters.

(e) Interpret the dependence of Y on N and P on N

If you are getting confused on algebra for this part of the question, setup the integrals/etc. properly and then move on.

1.5 (Optional) Efficiency

In this monopolistically competitive environment, the previous section should give you a U related to fundamental, non-price variables (e.g. $\bar{Z}$). Given the inelastic labor supply, L and the allocation to the various intermediaries, $\ell(i)$, make sure you understand the connection to U.

Consider a planner's problem where they choose the $\ell(i)$ and $y(i)$ directly–subject to the same production function–rather than relying on the decentralized prices under monopolistic competition. i.e.,

$$U = \max_{y(i),\ell(i)} \left[\int_\Omega (Aa(i)y(i))^\rho \,\mathrm{d}i\right]^{1/\rho} \tag{8}$$

$$\text{s.t. } y(i) = Zz(i)\ell(i) \tag{9}$$

$$\int_\Omega \ell(i)\mathrm{d}i = 1 \tag{10}$$

(a) Solve this problem for the optimal $\ell(i)$ and compare to the setup in monopolistic competition. Can you conclude anything about Pareto efficiency of decentralization with monopolistic competition?

1.6 (Optional) Identification of Quality and Productivity

Take the equation for firm i's revenue and profits.

(a) Given data on aggregate income Y, aggregate price index P, and aggregates A and Z, can you use the revenue and profits to estimate $z(i)$ from data? What about $a(i)$?

(b) Can you separately identify them, and if not, is there a way you can combine them into a single variable which can be estimated (hint: look for linear combinations in logs, etc.)?

[3]Hint: You want to get rid of the i and convert over to just the densities. If there is a density of $Nf_(z)$ of i firms with productivity z, then when integrating over all of the firms $i \in \Omega$, you can simply integrate over the density.Keep in mind that these densities add up to 1, so you need to make sure you normalize by N correctly.

(c) At the aggregate level. Given data on real GDP, Y and a cost of living index, P, which of the aggregate/idiosyncratic quality and productivity parameters can you separately estimate? Relate this to the interpretation of $\tilde{Z}$

Question 2: Neoclassical Growth

Take the planner's problem for neoclassical growth:

$$\max_{\{c_t, k_{t+1}\}} \sum_{t=0}^{T} \beta^t u(c_t) \tag{11}$$

$$\text{s.t. } c_t + k_{t+1} \leq f(t, k_t) + (1-\delta)k_t \tag{12}$$

$$+ \text{ transversality condition if } T = \infty, \text{ or } k_{T+1} = 0 \text{ if finite} \tag{13}$$

2.1 Stationary Solution

Assume that $u(c) = \frac{c^{1-\gamma}}{1-\gamma}$, $f(t,k) = z_t k^\alpha$ where z_t is an (exogenously given) aggregate productivity.

(a) Form a system of difference equations for T finite and T infinite ensuring you have the correct boundary conditions.

(b) Assume $T = \infty$ and that $\lim_{t\to\infty}\left(\frac{z_{t+1}}{z_t}\right) = 1 + g$ for some constant $g \geq 0$. Derive the stationary growth rate of output, and the proportion of output consumed (i.e., $\lim_{t\to\infty}\left(\frac{f(t+1,k_{t+1})}{f(t,k_t)}\right)$ and $\lim_{t\to\infty} c_t/f(t,k_t)$)?

2.2 Simple Dynamics

For this part, you could use either `Dolo.jl` or directly implement the shooting method yourself.[4] Use the above setup with $z_t = 1$ for all t.

(a) For $T = \infty$, find the stationary level of capital $\bar{k}$ and consumption $\bar{c}$.

(b) Assume that $T = \infty$ and that $k_0 = 2\bar{k}$. Using the shooting method discussed in class, find the sequence of c_t and k_{t+1} for $t = 0, \dots T$ for some large T at which point the system seems to have converged.

(c) Let $T = 10$ and $k_0 = .9\bar{k}$ (i.e., capital stock starts at a fraction of what the infinite horizon stationary level would be.) Using the shooting method or Dolo.jl, find the sequence of c_t and k_{t+1} for $t = 0, \dots T$

(d) Now try for $T = 100$. How does the majority of the dynamics compare to the infinite horizon case? (It should be very close to equilibrium for most t. This is called a "Turnpike".)

2.3 (Optional) Compare Dolo and the Shooting Method

Solve the simple dynamics with both methods and compare.

2.4 (Optional) Shooting with and Without Foresight

[4] Hint: the Dolo example on perfect foresight models should be useful.

Assume at time 0, that the economy has been running at steady state for a long time with $\delta = .1$. Let $k_0 = k_\infty$. Consider 2 scenarios:

1. The agents learn, at time $t = 0$, that $\delta = .05$ for all $t = 10, \ldots$
2. The agents don't learn this until $t = 10$ at which point there is an immediate change to $\delta = .05$ forever.

Using shooting and/or Dolo solve for both scenarios to the new steady state and compare.