Global and Local Solutions to Difference Equations

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January 22, 2018

1 Backward vs. Forward Difference Equations

1.1 Backwards

Example: AR(1) process

$$y_{t+1} = a + \rho y_t + \varepsilon_{t+1} \tag{1}$$

- Requires initial condition.
- Think of backwards variables as predetermined, i.e. in agents' information set.

1.2 Forwards

$$p_t = y_t + \beta p_{t+1} \tag{2}$$

- p_t : risk neutral asset price
- y_t : dividend
- p_{t+1} : next period

In expectations:

$$p_t = y_t + \beta \mathbb{E}_t \left[p_{t+1}\right] \tag{3}$$

- Requires boundary condition (e.g. at ∞ where it is (hopefully) stationary)
- Can find stationary value and then use that as the terminal value, and work backwards.

1.3 System of Equations

(1) $y_{t+1} = a + \rho y_t + \varepsilon_{t+1},\ \varepsilon_{t+1} \sim \mathrm{N}(0,1)$

(2) $p_t = y_t + \beta \mathbb{E}_t [p_{t+1}]$. The expectation makes this tricky!

(3) y_0 given

(4) Boundary condition for p_t

- We are interested in finding a decision rule $p_t(y_t)$
$p_t(y_t) \leftrightarrow P(y)$ since Markov 1st order.
- We are also interested in simulation from y_0.

1.3.1 Deterministic Solution

What if $\epsilon_{t+1} = 0$ for all t

$$\bar{y} = a + \rho \bar{y} \ \Rightarrow \ \boxed{\bar{y} = \frac{a}{1-\rho}} \tag{4}$$

Use equation (4) to find stationary p_t:

$$\bar{p} = \bar{y} + \beta \bar{p} \ \Rightarrow \ \boxed{\bar{p} = \frac{\bar{y}}{1-\beta}} \tag{5}$$

From any y_0 initial condition, this $\bar{y}, \bar{p}$ is the <u>deterministic steady state</u>.

Solving Deterministic Version For $\varepsilon_{t+1} = 0$ forever:

1) $y_{t+1} = a + \rho y_t$

2) $p_t = y_t + \beta p_{t+1}$, i.e., can drop expectation.

3) y_0 given

4) $\lim_{t\to\infty} p_t = \frac{a}{1-\rho} \cdot \frac{1}{1-\beta}$

This gives a sequence of unique solution $\{p_t\}_{t=0}^{\infty}$

Example Algorithm: (Shooting Method) Solve for the boundary value problems.

1) Guess p_0

2) Use equation (2) in the deterministic equations list above to get p_{t+1} with y_0 given

3) Use (1) to increment y

4) Use equation (2) to get p_{t+2}, etc., and repeat from step 2 until some large T

5) Check equation (4) if $p_T \approx \frac{a}{1-\rho} \cdot \frac{1}{1-\beta}$, then we are done.
Otherwise, change the initial guess for p_0 and repeat.

1.3.2 Stochastic Solution

- There is no steady-state which is constant, only stationary.
- In this case, due to linearity, we can solve in exact form using techniques on linear stochastic difference equation.
- In general: you cannot find an exact solution to a system of nonlinear stochastic difference equations

Global Solutions

- Since this is a Markov order, we can use dynamic programming to find $p(y)$ for all y in domain.
- Use dynamic programming arguments to find a fixed point.

Example: Asset Pricing We can find $P(y)$ such that:

$$P(y) = y + \beta \int P(a + \rho y + \varepsilon) \underbrace{\mathrm{d}\Phi(\varepsilon; 0, 1)}_{\substack{\text{integrate} \\ \text{over shocks}}} \tag{6}$$

- Use value function iteration, collocation methods, etc.
- Relatively expensive to compute in general ("curse of dimensionality" if the state space increases), but can calculate for any y after fixed point is found.
- Overkill if you are only interested around a point.

2 Deterministic Nonlinear Difference Equations

Neoclassical Growth Models: Planner's problem.

Let $u(c_t)$ be the PDV of utility from consumption.

$$\max_{\{c_t,k_{t+1}\}} \sum_{t=0}^{T} \beta^t u(c_t) \tag{7}$$

$$\text{s.t. } c_t + k_{t+1} \leq f(k_t) + (1-\delta)k_t \tag{8}$$

$$k_{T+1} \geq 0, \text{ or transversality if } T = \infty \tag{9}$$

$$k_0 \text{ given} \tag{10}$$

- β : discount factor
- δ : capital depreciation rate
- k_{t+1} : next period's capital stock
- T: finite horizon for now.

Set up the Lagrangian:

$$\mathcal{L} = \sum_{t=0}^{T} \beta^t \left(u(c_t) + \lambda_t \left[f(k_t) + (1-\delta)k_t - c_t - k_{t+1} \right] \right) + \lambda_{T+1}\beta^{T+1}k_{T+1} \tag{11}$$

2.1 Optimality Conditions and Solution

The first-order necessary conditions are:

$$[c_t] : u'(c_t) - \lambda_t = 0 \tag{12}$$

$$[k_{t+1}] : -\lambda_t + \beta\lambda_{t+1} \left[f'(k_{t+1}) + (1-\delta) \right] = 0 \qquad \text{(for } t = 0.\dots, T-1) \tag{13}$$

$$[k_{T+1}] : \beta^{T+1}\lambda_{T+1}k_{T+1} = 0 \qquad \text{(complementarity)} \tag{14}$$

If infinite horizon, then this ends up as a transversality condition:

$$\lim_{T\to\infty} \beta^T \lambda_T k_{T+1} = 0 \tag{15}$$

Rewrite to form nonlinear 1st order difference equations:

$$u'(c_t) = \lambda_t \tag{16}$$

$$\lambda_t = \beta\lambda_{t+1} \left[f'(k_{t+1}) + (1-\delta) \right] \tag{17}$$

$$k_{t+1} + c_t = f(k_t) + (1-\delta)k_t \tag{18}$$

$$\beta^T \lambda_T k_{T+1} = 0 \tag{19}$$

Boundary / initial values:

$$k_0 \text{ given} \tag{20}$$
$$k_{T+1} = 0 \text{ from complementarity} \tag{21}$$

Let: $f(k) = k^{\alpha}, \Rightarrow f'(k) = \alpha k^{\alpha-1}$;
$u(c) = \log(c), \Rightarrow u'(c) = \frac{1}{c}$

Rewriting the system:

$$c_{t+1} = c_t \beta \left[f'(k_{t+1}) + (1-\delta) \right] \tag{22}$$
$$k_{t+1} = f(k_t) + (1-\delta)k_t - c_t \tag{23}$$

Nonlinear terms: $c_t \cdot f'(k_{t+1}),\ f(k_t)$
We can't write it as $x_{t+1} = Ax_t$.
- Could guess c_0, iterate, ... to k_∞ with shooting method;
(can get closer to a form with consistent lag).

2.2 Exact Solution Method: Shooting

1. Guess λ_0, given k_0
2. For each t:
   (a) Use equation (16) to get c_t
   (b) Use c_t, k_t in equation (18) to get k_{t+1}
   (c) Use λ_t, k_{t+1} in (17) to get λ_{t+1}
   (d) Repeat step 2 with λ_{t+1} and k_{t+1}
3. See if $k_{T+1} = 0$ whether the boundary condition is fulfilled.
   If not, update λ_0, try again.

Example:

$$c_{t+1} = c_t \beta \left[f'(f(k_t) + (1-\delta)k_t - c_t) + (1-\delta) \right] \quad \text{(forward)} \tag{24}$$
$$k_{t+1} = f(k_t) + (1-\delta)k_t - c_t \quad \text{(backwards)} \tag{25}$$
$$k_0 \text{ given} \quad \text{(initial)} \tag{26}$$
$$c_\infty = f(k_\infty) - \delta k_\infty, \quad \text{(boundary)} \tag{27}$$

where:

$$1 = \beta \left[f'(k_\infty) - (1-\delta) \right] \tag{28}$$

Infinite Horizon:
- Steady state capital solves equation $\boxed{1 = \beta \left[f'(k_\infty) - (1-\delta) \right]}$
- Pick large T, check $k_T \approx k_\infty$ as boundary condition
- Steady state c_∞ given k_∞ with $\boxed{k_\infty + c_\infty = f(k_\infty + (1-\delta)k_\infty}$
- "Exact" since can solve for any initial condition and get entire sequence to arbitrary precision by adjusting T, λ_0

3 Roadmap to Solving DSGE Models

Model:

a) Set of optimization / choices for agents

b) Stochastic process of exogenous variables

c) Equilibrium conditions and resource constraints

d) Parameters for all of these

The typical DSGE setup uses a local solution, using the general approach:

Step 1:

- Find FONC for models choices, including Lagrange multipliers
- Add these FONC and budget constraints to list of equations
- This may make assumptions on local convexity for the equations. And take note of 2nd order conditions for convexity, generally tested after.

Step 2:

- Convert stochastic processes into a first-order Markov form for linear gaussian state space model
- Ensure all shocks in the state space are i.i.d. and normal

Step 3:

- Same for resource and equilibrium conditions

Step 4:

- Could write the entire set of equations as a system of stochastic nonlinear difference equations for some operator $F(\cdot)$
- Keep track of inequality constraints! If they could bind, change approaches. Otherwise, the system is:

$$\mathbb{E}_t\left[F\left(\{x_t\},\{x_{t+1}\},\{x_{t-1}\},u_t,\varepsilon_t\right)\right]=0 \tag{29}$$

For example, $\mathbb{E}_t\left[\begin{bmatrix}\beta p_{t+1}-p_t-y_t\\ y_{t+1}-\rho y_t-\epsilon_{t+1}\end{bmatrix}\right]=0$

Given: initial conditions for some variables

The tricky part: boundary values for others (e.g., some endogenous prices)

Step 5:

- Linearize this complicated system so that we have a system of stochastic linear first order equations. Some forwards, some backwards.

Step 6:

- Calculate the deterministic steady state for our linear methods.
- The linearized model will be local to this equilibrium.

Step 7:

- Use our old tools to simulate and estimate.

3.1 Local Solutions

- Global would always be preferable, but isn't always tractable (and often delivers complicated likelihood).
- For much more complicated setups with large state spaces, the curse of dimensionality makes it impossible in many circumstances.
- Instead, we can solve approximately around a particular point.
- Approximate the result locally with a polynomial function.
- e.g., if the global solution for a complicated $p(y)$ was:

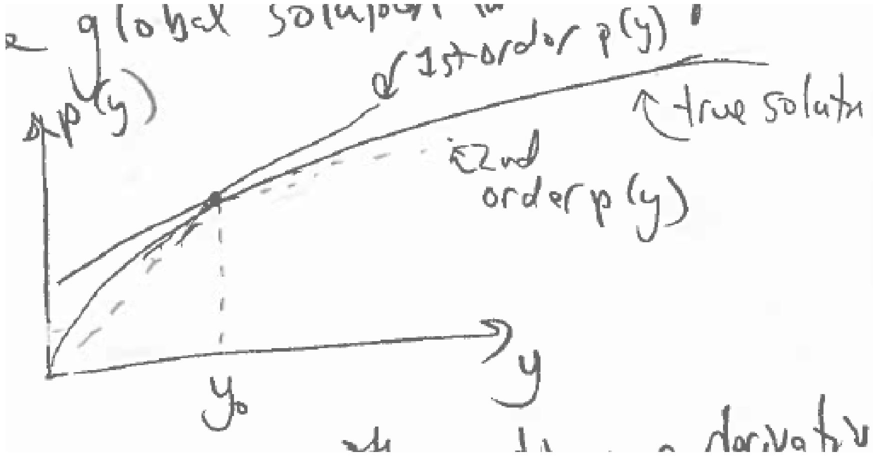

Figure 1: Global Solution of Polynomial

- For <u>smooth</u> functions with continuous derivatives, can achieve arbitrarily close approximation <u>around</u> the point.
- The further away from the point, the worse the approximation.

Taylor Series: (Look up N-dimensional version)
Given $f(x)$ and some $\bar{x}$, and smoothness conditions:

$$f(x) = f(\bar{x}) + f'(\bar{x})(x - \bar{x}) + \frac{1}{2}f''(\bar{x})(x - \bar{x})^2 + \dots \tag{30}$$

1st order approximation around $\bar{x}$:

$$f(x) \approx f(\bar{x}) + f'(\bar{x})(x - \bar{x}) \tag{31}$$

If we could find $f(\bar{x})$ and $f'(\bar{x})$ (only at point $\bar{x}$), we would have an approximation!

<u>Use the steady state!</u>

- In our models, we will use the steady state, e.g. we know $\bar{y} = \frac{a}{1-\rho}$ and $p(\bar{y}) = \bar{p} = \frac{\bar{y}}{1-\beta}$
- In the stochastic setup, we will often use the version of the model with shocks set to 0 as the <u>non-stochastic steady state</u> to take approximations around.
- <u>Key:</u> These simple DSGE techniques (implemented by Dynare, etc.) are only approximations <u>locally</u> around the steady state.

Log linearization

- When the difference equation isn't linear, we will use Taylor series to convert into an approximately linear model to use the linear stochastic difference equations we developed earlier; will linearize

- If you use higher-order polynomials, these are called perturbation methods, we can't use the linear stochastic machinery since nonlinear.

- Dynare, etc. getting better at perturbation methods, but let's stay linear.

- Turns out that things are usually easier if you linearize in logs of the variables, which gives percent deviations from steady state.

- For likelihoods, linearized models can use all of the tools we developed. Higher order approximations requires tools like particle filters.

3.2 Example: Consumption Euler Equation

$$\max_{C_t, A_{t+1}} \mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t u(c_t) \right] \tag{32}$$

$$\text{s.t. } A_{t+1} = R(A_t + y_t - c_t) \tag{33}$$

where:
c_t : consumption
A_{t+1} : savings
R: gross interest rate
y_t: stochastic income.
$y_t - c_t$: savings from income

Lagrangian (simple deterministic version):

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \left[u(c_t) + \underbrace{\lambda_t}_{\text{L.M.}} \left(R(A_t + y_t - c_t) - A_{t+1} \right) \right] \tag{34}$$

FONC:

$$\partial_{c_t} \mathcal{L} : \beta^t u'(c_t) - \beta^t \lambda_t \cdot R = 0 \tag{35}$$

$$\Rightarrow u'(c_t) = R\lambda_t \tag{36}$$

$$\partial_{A_{t+1}} \mathcal{L} : -\beta^t \lambda_t + \beta^{t+1} \cdot R \cdot \lambda_{t+1} = 0 \tag{37}$$

$$\Rightarrow \frac{\lambda_{t+1}}{\lambda_t} = \frac{1}{\beta R} \tag{38}$$

Combine:

$$\frac{u'(c_{t+1})}{u'(c_t)} = \frac{1}{\beta R} \tag{39}$$

$$\Rightarrow \boxed{u'(c_t) = \beta \cdot R \cdot u'(c_{t+1})} \text{ (Euler equation)} \tag{40}$$

If c_t was stochastic due to stochastic y_t:

$$u'(c_t) = \beta \cdot R \cdot \mathbb{E}_t\left[u'(c_{t+1})\right] \tag{41}$$

i.e. use time t information set:

If $u(c_t) = \frac{c_t^{1-\gamma}}{1-\gamma} \Rightarrow u'(c_t) = c_t^{-\gamma}$

$$\Rightarrow c_t^{-\gamma} = \beta R \mathbb{E}_t\left[c_{t+1}^{-\gamma}\right] \tag{42}$$

Note: $\left(\mathbb{E}_t\left[c_{t+1}^{-\gamma}\right]\right)^{-\frac{1}{\gamma}} \neq \mathbb{E}_t\left[c_{t+1}\right]$ due to Jensen's inequality.

3.3 Log Linearization

Let $\hat{x}_t = \log\left(\frac{x_t}{\bar{x}}\right) = \log(x_t) - \log(\bar{x})$.
This is the % deviation of x_t from steady state $\bar{x}$.

Note: $x_t = \bar{x} \cdot \left(\frac{x_t}{\bar{x}}\right) = \bar{x} \exp\left(\log\left(\frac{x_t}{\bar{x}}\right)\right) = \bar{x} \exp\left(\hat{x}_t\right)$
$\rightarrow$ Substitute, for the series x_t.

Taylor Series: At steady state: $\hat{x}_t = 0$, i.e. no deviation from $\bar{x}$; use as the point for approximation of $\hat{x}_t$.
Note: $e^x \approx 1 + x$, first order Taylor around 0
$\Rightarrow x_t \approx \bar{x}(1 + \hat{x}_t) \Rightarrow \frac{x_t}{\bar{x}} \approx 1 + \hat{x}_t$.

We can also show for $y_t \rightarrow \hat{y}_t$ that:

$$x_t y_t \approx \bar{x}\bar{y}(1 + \hat{x}_t + \hat{y}_t) \tag{43}$$

And for some $f(x_t)$:

$$f(x_t) \approx f(\bar{x})(1 + \eta \hat{x}_t) \tag{44}$$

where $\eta \equiv \boldsymbol{\partial}_x f(\bar{x}) \cdot \frac{\bar{x}}{f(\bar{x})}$.

So, for example:

$$\begin{aligned} f(x_t) &= x_t^{-\gamma} && (45) \\ \Rightarrow f'(\bar{x}) &= -\gamma \bar{x}^{-\gamma-1} && (46) \\ \Rightarrow \eta &= \frac{-\gamma \bar{x}^{-\gamma-1}}{\bar{x}^{-\gamma}} \cdot \bar{x} = -\gamma && (47) \\ \Rightarrow x_t^{-\gamma} &\approx \bar{x}^{-\gamma} (1 - \gamma \hat{x}_t) && (48) \end{aligned}$$

Back to consumption Euler equation:

$$c_t^{-\gamma} = \beta R \mathbb{E}_t \left[c_{t+1}^{-\gamma} \right] \tag{49}$$

Steady state $\bar{c}$, $\hat{c}_t = \log \left(\frac{c_t}{\bar{c}} \right)$

$$\begin{aligned} \bar{c}^{-\gamma}(1 - \gamma \hat{c}_t) &= \beta R \mathbb{E}_t \left[\bar{c}^{-\gamma}(1 - \gamma \hat{c}_{t+1}) \right] && (50) \\ (1 - \gamma \hat{c}_t) &= \beta R (1 - \gamma \mathbb{E}_t \left[\hat{c}_{t+1} \right]) && (51) \end{aligned}$$

(The steady state drops out and is now linear).

Google for: "Intro to log-linearization","log-linearization tricks", "log-linearization cheat sheet".
$\rightarrow$ Practice!