Linear State Space Models
Additional Material and Examples

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Complements material in `https://lectures.quantecon.org/jl/linear_models.html`. Also see (Ljungqvist and Sargent, 2012, Chapter 2)

1 New Linear Model with Stochastics

1.1 Stochastic Model

Following `https://lectures.quantecon.org/jl/linear_models.html`,

$$x_{t+1} = Ax_t + Cw_{t+1} \tag{1}$$
$$y_t = Gx_t \tag{2}$$

where $x_t \in \mathbb{R}^n$, and $w_t \in \mathbb{R}^m$ with $w_t \sim N(0, I)$

2 Impulse Response

Given a one-time shock to w_{t+1}, how does this evolve over j (if all $w_{t+j} = 0$ for all $j > 1$)? Intuitively, start with $x_t = 0$, and then look at the evolution of x_{t+j}

$$A^{j-1}Cw_{t+1} \rightarrow \text{ IRF of } x_{t+j} \tag{3}$$
$$GA^{j-1}Cw_{t+1} \rightarrow \text{ IRF of } y_{t+j} \tag{4}$$

Also, note that if we recursively sum up the discounted future:

- $G(I - \beta A)^{-1}Cw_{t+1} \rightarrow$ IRF for the present value of y_t (5)

Alternatively could use $x_t > 0$ and the IRF is simply the change from the deterministic evolution (due to linearity of the process).

3 Asset Pricing Stochastic Linear Model

From equations (1) and (2) respectively, we have:

$$x_{t+1} = A \cdot x_t + C \cdot w_{t+1} \text{ (evolution)}$$
$$y_t = G \cdot x_t \text{ (observation)}$$

The risk-neutral pricing equation is given by:

$$P_t = y_t + \beta \mathbb{E}_t \left[P_{t+1} \right] \tag{6}$$

Solution: Using Guess-and-Verify method:

$$P_t = H \cdot x_t \text{ (our guess, for some undetermined } H \text{ to be decided)} \tag{7}$$

Verify: Plug in the pricing function at t and $t+1$ from equation (6):

$$H \cdot x_t = y_t + \beta \mathbb{E}_t \left[H \cdot x_{t+1} \right] \tag{8}$$

Using equation (1):

$$H \cdot x_t = y_t + \beta \mathbb{E}_t \left[H \cdot (A \cdot x_t + C \cdot w_{t+1}) \right] \tag{9}$$
$$= y_t + (\beta H \cdot A \cdot x_t) + (\beta H \cdot C \cdot \mathbb{E}_t \left[w_{t+1} \right]) \tag{10}$$

By linearity of expectations, and using $\mathbb{E}_t \left[w_{t+1} \right] = 0$

$$H \cdot x_t = G \cdot x_t + \beta H \cdot A \cdot x_t \tag{11}$$

Using the method of undetermined coefficients, the following must hold for the above equation:

$$H = G + \beta H \cdot A \tag{12}$$
$$\Rightarrow H(I - \beta A) = G \tag{13}$$
$$\Rightarrow H = G(I - \beta A)^{-1} \tag{14}$$

Plugging this expression of H into our guess in equation (7)

$$P_t = G(I - \beta A)^{-1} x_t \text{ (identical to non-stochastic version)} \tag{15}$$

3.1 Stochastic Example

Using a second-order autoregressive process for y_t:

$$y_{t+1} = \gamma + \rho_1 y_t + \rho_2 y_{t-1} + \sigma \underbrace{w_{t+1}}_{\text{gaussian noise}} \tag{16}$$

$$\mathbb{E}_t \left[w_{t+1}\right] = 0 \tag{17}$$

$$\mathbb{E}_t \left[w_{t+1} w_{t+1}\right] = 1 \tag{18}$$

We need to convert this into a state space:

A Guess: We guess a state : $x_t = \begin{bmatrix} 1 \\ y_t \\ y_{t-1} \end{bmatrix}$.

Using equation (1), we set up the evolution equations:

$$x_{t+1} = A \cdot x_t + C \cdot w_{t+1} \tag{19}$$

$$\begin{bmatrix} 1 \\ y_t \\ y_{t-1} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \gamma & \rho_1 & \rho_2 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ y_t \\ y_{t-1} \end{bmatrix} + \begin{bmatrix} 0 \\ \sigma \\ 0 \end{bmatrix} w_{t+1} \tag{20}$$

Using equation (2), the observation equation is:

$$\begin{aligned} y_t &= G \cdot x_t \\ &= \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ y_t \\ y_{t-1} \end{bmatrix} \end{aligned} \tag{21}$$

A Different Formulation (1): This time, let the evolution equation be the following:

$$x_{t+1} = B + A \cdot x_t + C \cdot w_{t+1} \tag{22}$$

Using the guess $x_t = \begin{bmatrix} y_t \\ y_{t-1} \end{bmatrix}$, the linear state space model of the AR(2) process from equation (16) is:

$$\begin{bmatrix} y_{t+1} \\ y_t \end{bmatrix} = \begin{bmatrix} \gamma \\ 0 \end{bmatrix} + \begin{bmatrix} \rho_1 & \rho_2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} y_t \\ y_{t-1} \end{bmatrix} + \begin{bmatrix} \sigma \\ 0 \end{bmatrix} w_{t+1} \tag{23}$$

A Different Formulation (2): Let the evolution equation be:

$$x_{t+1} = A \cdot x_t + w_{t+1},\ w_{t+1} \sim \mathrm{N}\left(0, \Sigma\right) \tag{24}$$

Using the guess $x_t = \begin{bmatrix} 1 \\ y_t \\ y_{t-1} \end{bmatrix}$,

$$\Sigma = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \sigma^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{25}$$

where

$$\Sigma = CC' = \begin{bmatrix} 0 & \sigma & 0 \end{bmatrix}' \begin{bmatrix} 0 \\ \sigma \\ 0 \end{bmatrix}' \tag{26}$$

Principle:

- We can always convert to a 1st order difference equation.
- Choose the state carefully (augmenting the state).
- Equation (19) is a **Vector Auto-Regression (VAR)**.

References

Ljungqvist, L., and T. J. Sargent (2012): *Recursive Macroeconomic Theory, Third Edition*, vol. 1 of *MIT Press Books*. The MIT Press.