Question 1: Simulating a Time Series

Take the stochastic process that follows

$$y_{t+1} = \gamma + \theta y_t + \sigma w_{t+1} \tag{1}$$

where $w_{t+1} \sim \mathrm{N}(0, 1)$.

1.1 Simulation

Let $\gamma = 1, \sigma = 1$, and $y_0 = 0$. Also let $\Theta \equiv \{0.1, 0.5, 0.98\}$

(a) Simulate the data generating process $\{y_t\}_{t=0}^{T}$ for $T = 1000$ for each $\theta \in \Theta$.

(b) Plot a rolling mean of the process, i.e. for each $1 \leq \tau \leq T$, plot $\frac{1}{\tau}\sum_{t=1}^{\tau} y_t$ for each $\theta \in \Theta$

1.2 Stationary Distribution

For each $\theta \in \Theta$

(a) For some large T (maybe 30 is high enough? You can play around with it) simulate y_T for a large I (maybe 200, but feel free to play around with it). Plot a histogram of the stationary distribution for y_T (which we can call y_∞ if we assume that T is large enough).

(b) Numerically find the mean and variance of this as an ensemble average, i.e. $\mathbb{E}[y_\infty] \approx \sum_{i=1}^{I} \frac{y_T^i}{I}$ and $\mathbb{V}[y_\infty] \approx \sum_{i=1}^{I} \frac{(y_T^i)^2}{I} - \mathbb{E}[y_\infty]^2$

(c) Find the stationary distribution for y_∞ from theory (hint: when it is in our linear state space model, we can use formulas for stationarity), and plot an overlay of the normal distribution with the stationary distribution vs. the histogram.

1.3 (Optional) Show the Evolution of the Stationary Distribution

(a) Instead of just showing the histogram of the stationary distribution for y_T, show an animation of the histogram for $1 \leq \tau \leq T$.[1] This gives you a sense of how large T needs to be before the stationary distribution converges.

(b) Play around with adding an interactive way to edit the σ and γ coefficients in a Jupyter notebook.[2]

[1] See `http://docs.juliaplots.org/latest/animations/` for some notes on animations, although there are many ways to do it.

[2] One way to do this is with `https://github.com/JuliaGizmos/Interact.jl`, but I would love to see different ways to accomplish this.

Question 2: Linear Asset Pricing

Assume that the aggregate component of payoffs follows

$$Z_{t+1} = \alpha + \rho_2 Z_t + \rho_3 Z_{t-1} + \sigma_1 w_{1,t+1} \tag{2}$$

And the idiosyncratic component of payoffs follows

$$z_{t+1} = \rho_1 z_t + \sigma_2 w_{2,t+1} + \theta w_{2,t} + c w_{1,t+1} \tag{3}$$

While the payoff for a given $\{Z_t, z_t\}$ state is

$$y_t = \lambda Z_t + (1-\lambda) z_t \tag{4}$$

where $\begin{bmatrix} w_{1,t+1} \\ w_{2,t+1} \end{bmatrix} \sim N(0, I)$.

Define the expected PDV of payoffs as

$$p_t \equiv \mathbb{E}_t \left[\sum_{j=0}^{\infty} \beta^j y_{t+j} \right] \tag{5}$$

2.1 Setting up the Model

For parameter values, use $\alpha = 0.1, \rho_1 = 0.8, \rho_2 = 0.75, \rho_3 = 0.1, \theta = 0.1, c = 0.05, \lambda = 0.5, \beta = 0.95, \sigma_1 = 0.1, \sigma_2 = 0.05$

(a) Setup this problem in our canonical linear state space model

(b) Verify that these parameter values would have a steady-state (hint, use the eigenvalues from the LSS model)

(c) From theory, find an expression for p_t from the LSS model in terms of the underlying state

(d) Write down a formula for the Impulse Response Function to a 1 standard deviation shock to $w_{1,t+1}$. Do the same for $w_{2,t+1}$.[3]

(e) Calculate—first using theory, and then plugging in the parameter values—the stationary distribution of the $\{Z_t\}$ process (note that you will not need to deal with a joint distribution).

Note: you do not need to type this in LaTeX in the jupyter notebook (though you could if you wish). Instead, you can attach any algebra to the notebook as an image or just append to the pdf you submit.

2.2 Simulating the Model

Let Z_0 be the mean of the stationary distribution of Z_t you calculated previously and $z_0 = z_1 = w_{2,0} = 0$ (or, if you do the next optional part, you could draw from the stationary distribution of the joint $\{Z_t, z_t, w_{2,t}\}$ process). Let $T = 20$

[3]Hint: using the formulas form our notes, if these shocks have a distribution of $N(0,1)$, then the shock w_{t+1} to use is either $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ or $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

(a) Simulate $I = 5$ paths of $\{Z_t, z_t, y_t, p_t\}_{t=1}^{T}$

(b) Plot the I paths for the y_t and p_t series in a way you think is easy to interpret

(c) Plot an IRF using your formula above for shocks to $w_{1,t+1}$ and $w_{2,t+1}$

2.3 (Optional) Stationary Distributions

(a) Calculate—first using theory, and then plugging in the parameter values—the stationary distribution of the joint $\{Z_t, z_t\}$ process[4]

(b) Calculate—first using theory, and then plugging in the parameter values—the stationary distribution of y_t

(c) Calculate—first using theory, and then plugging in the parameter values—the stationary distribution of p_t

2.4 (Optional) Interactive Plot

(a) Recreate the first plot in `https://lectures.quantecon.org/jl/linear_models.html#ensemble-interpretations` for y_t with the simulation. For this, you should choose something like $T = 30$ and increase the number of sample paths $I = 30$ or something like that.[5]

(b) Play around with adding an interactive way to edit the parameters of the simulation in a Jupyter notebook.[6]

[4] Hint: You will end up calculating the joint stationary distribution of all of the state (being careful to consider if there is a constant term or not when choosing the stationary distribution formula). At that point, you may need to take the marginal distribution of the joint multivariate normal distribution. You can look at `https://en.wikipedia.org/wiki/Multivariate_normal_distribution` if you want.

[5] Hint: the code for this figure is in `https://github.com/QuantEcon/QuantEcon.lectures.code/blob/master/linear_models/paths_and_hist.jl`

[6] One way to do this is with `https://github.com/JuliaGizmos/Interact.jl`, but I would love to see different ways to accomplish this.