Linear Quadratic Control
Additional Material and Examples

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This material is intended to supplement `https://lectures.quantecon.org/jl/lqcontrol.html`

1 Quadratic Objective Functions

(a) 2nd order approximation of more complicated non-linear functions, e.g. cost of inflation on output gap.

(b) Quadratic adjustment costs

(c) Toy models to illustrate concepts.

These were used for much of early macroeconomics: R.E. examples, due to tractability; permanent income model, etc. See Ljungqvist and Sargent (2012, Chapter 5)

2 General Setup: Infinite Horizon Optimal Linear Regulator

The value of the planner with control u_t and state x_t, with the following:

- $x_t \in \mathbb{R}^n$, x_0 given
- $w_{t+1} \in \mathbb{R}^j$: shocks with $w_{t+1} \sim \mathrm{N}(0, \mathbf{I})$
- $u_t \in \mathbb{R}^k$: decision rule
- $R \in \mathbb{R}^{n\times n}, Q \in \mathbb{R}^{k\times k}$: capture objective function
  (can add another $H \in \mathbb{R}^{k\times n}$ for cross terms)

- $A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times k}$: capture state evolution

- $C \in \mathbb{R}^{n \times j}$: Cholesky of covariance matrix

3 Problem

We choose u in the following maximization problem:

$$\max_{\{u_t\}} \mathbb{E}_0 \left[-\sum_{t=0}^{\infty} \beta^t \left[x_t^T R x_t + u_t^T Q u_t \right] \right] \tag{1}$$

$$\text{s.t. } x_{t+1} = A \cdot x_t + B \cdot u_t + C \cdot w_{t+1} \tag{2}$$

$$x_0 \text{ given} \tag{3}$$

$$w_{t+1} \sim \mathrm{N}\left(0, \mathbf{I}\right) \tag{4}$$

This is a quadratic objective function, with linear evolution. Note that the canonical form often has the observation equation built into R.

Recursive Formulation:

$$V(x) = \max_{u(\cdot)} \left[-x^T R x - u^T Q u + \beta \mathbb{E}\left[V(x')\right] \right] \tag{5}$$

$$\text{s.t. } \underbrace{x'}_{\substack{\text{state}\\\text{next}\\\text{period}}} = Ax + Bu + Cw \tag{6}$$

Solution: See Ljungqvist and Sargent (2012), or notes on linear equations.
Given that $V(x)$ is quadratic in the state, we have:

$$V(x) = -x^T P x - d \tag{7}$$

$$u(x) = -Fx \tag{8}$$

for a matrix $P \in \mathbb{R}^{n \times n}$; scalar d; $F \in \mathbb{R}^{k \times n}$: linear control, where P solves:

$$P = R + \beta A^T P A - \beta^2 A^T P B (Q + \beta B^T P B)^{-1} B^T P A \tag{9}$$

Equation (9) is an algebraic matrix Ricatti equation; use numerical methods for $n > 2$:

$$F = \beta (Q + \beta B^T P B)^{-1} B^T P A \tag{10}$$

$$d = \beta (1 - \beta)^{-1} \cdot \mathrm{trace}(P C C^T) \tag{11}$$

Shocks do not affect the choice, but it affects $v(\cdot)$ through d, and $u(\cdot)$ through x_{t+1}.

4 Certainty Equivalence

Key:

- Value depends on shocks through C
- Policy independent of shock distribution
- We can solve the deterministic version with $C = [0]$ to get policy.
- This is the key tractability result of L-Q control.

See (Ljungqvist and Sargent, 2012, Chapter 5, Appendix B) for examples of L-Q approximations of standard models.
Substituting equation (8) into the LQM:

$$\begin{aligned} x_{t+1} &= Ax_t + Bu_t + Cw_{t+1} && (12)\\ &= Ax_t - B \cdot Fx_t + Cw_{t+1} && (13)\\ &= (A - BF) \cdot x_t + Cw_{t+1} && (14) \end{aligned}$$

This is a canonical linear Gaussian state space:

$$\boxed{\mathbb{E}_t\left[x_{t+j}\right] = (A - BF)^j x_t} \quad \text{(forecast)} \tag{15}$$

The forecast of value is different than previous linear asset pricing examples due to Quadratic utility.

5 Example: Exercise 5.12 from Ljungqvist and Sargent (2012)

5.0.1 Problem

Demand:
A firm believes market wide output, Y_t, for a product follows (large number of firms):

$$Y_{t+1} = H_0 + H_1 Y_t + H_2 \nu_t, \;\; Y_0 \text{ given} \tag{16}$$

where the demand shock is also AR(1)

$$\nu_{t+1} = \rho \nu_t + \sigma_\nu w_{t+1} \text{ for } w_{t+1} \sim \mathrm{N}(0,1), \ |\rho| < 1 \tag{17}$$

The price, p_t lies on the demand curve:[1]

$$p_t = A_0 - A_1 Y_t + \nu_t \tag{18}$$

<u>Adjustment Costs:</u>

The firm chooses y_t, the output, to maximize profits. It is subject to a quadratic adjustment cost:

$$\max_{y_{t+1}(\cdot)} \mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t \left(\underbrace{p_t y_t}_{\text{revenue}} - \frac{1}{2}\delta(\underbrace{y_{t+1} - y_t}_{\substack{\text{adjustment}\\ \text{cost}}})^2 \right) \right] \tag{19}$$

where $\delta > 0$ is the cost, with discount rate $\beta \in (0,1)$.
Information: sees Y_t, p_t, y_t at time t. The competitive equilibrium is seen through the law of motion for Y_t, which is independent of y_t.

5.0.2 Recursive Setup

We want to maximize the following recursive problem:

$$V(x) = \max_{u(\cdot)} \{-x^T R x - u^T Q u + \beta \mathbb{E}[V(x')]\} \tag{20}$$

$$\text{s.t. } x' = Ax + Bu + Cw \tag{21}$$

Converting to a linear-Gaussian state space: Find a state x_t, u_t and error w_{t+1}:

$$x_t = \begin{bmatrix} y_t \\ Y_t \\ \nu_t \\ 1 \end{bmatrix}, \ u_t = y_{t+1} - y_t \equiv \triangle y_t, \ w_{t+1} = w_{t+1}.$$

[1]The ν_t in (18) isn't strictly necessary for this sort of problem, but may help in aggregation to ensure that a large number of symmetric firms choose the same y_{t+1}, which allows us to let $Y_{t+1} = y_{t+1}$ after finding the optimal policy. This is a general equilibrium approach not fully implemented here.

Evolution:

$$\underbrace{\begin{bmatrix} y_{t+1} \\ Y_{t+1} \\ \nu_{t+1} \\ 1 \end{bmatrix}}_{x'} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & H_1 & H_2 & H_0 \\ 0 & 0 & \rho & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} y_t \\ Y_t \\ \nu_t \\ 1 \end{bmatrix}}_{x} + \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{B} \underbrace{\triangle y_t}_{u} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ \sigma_\nu \\ 0 \end{bmatrix}}_{C} \underbrace{w_{t+1}}_{w} \tag{22}$$

Revenue:

$$P_t y_t = (A_0 - A_1 Y_t + \nu_t) \cdot y_t \tag{23}$$

Now, take the sequential equation and plug in the expression for p_t:

$$\max_{y_{t+1}(\cdot)} \mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t \left(\underbrace{(A_0 - A_1 Y_t + \nu_t)\, y_t}_{x^T R x} - \underbrace{\frac{1}{2} \delta \triangle y_t^2}_{u^T Q u} \right) \right] \tag{24}$$

Note that Q is scalar here, so:

$$-u^T Q u = -\frac{1}{2} \delta \triangle y_t^2 \tag{25}$$

Since $u \equiv \triangle y$:

$$Q \equiv \frac{1}{2} \delta \tag{26}$$

2nd term in the objective function:

$$\triangle y \left(\frac{\delta}{2} \right) \triangle y \leftrightarrow u^T Q u \tag{27}$$

Now, take $x^T R x$:

$$-x^T R x = (A_0 - A_1 Y_t + \nu_t) y_t \tag{28}$$

$$-x^T R x = \begin{bmatrix} y & Y & \nu & 1 \end{bmatrix} \begin{bmatrix} 0 & -\frac{A_1}{2} & \frac{1}{2} & \frac{A_0}{2} \\ -\frac{A_1}{2} & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 \\ \frac{A_0}{2} & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} y \\ Y \\ \nu \\ 1 \end{bmatrix} \tag{29}$$

$$R = -\begin{bmatrix} 0 & -\frac{A_1}{2} & \frac{1}{2} & \frac{A_0}{2} \\ -\frac{A_1}{2} & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 \\ \frac{A_0}{2} & 0 & 0 & 0 \end{bmatrix} \tag{30}$$

How to get used to drawing up states spaces? Practice!

(a) Quadratic forms are symmetric

(b) No squared terms, or all 0's on diagonal

(c) Divide cross terms by 2 since added twice.

Note: We had to be careful on the signs of Q and R to match the signs in the canonical setup.

5.0.3 Summary of Solution

In our model, if A, B, C, Q, R are all defined, then we can use the formulas for certainty equivalence, i.e:

$$V(x) = -x' P x - d \tag{31}$$

$$u(x) = -F \cdot x \tag{32}$$

for $P \in \mathbb{R}^{4\times 4}, F \in \mathbb{R}^4$, where P, F solve the algebraic matrix Ricatti equation (9) defined before.

5.0.4 Simulating

1. Choose $x_0 = \begin{bmatrix} y_0 & Y_0 & \nu_0 & 1 \end{bmatrix}$ for initial conditions you are interested in.
2. Iterate:
   (a) Solve for $u(x) = -F \cdot x$ (can calculate value as well)
   (b) Draw w for shock

(c) Use evolution with $x, u, w \to x'$

(d) Repeat

3. Stationary solution for x?
   $\to$ Could find stationary Y, u_∞ (for large T), then iterate forward for $w = 0$ until $\triangle y = 0$? Or, better yet: use the approach from the VAR notes with the discrete Lyapunov equation to solve for the stationary distribution.

References

Ljungqvist, L., and T. J. Sargent (2012): *Recursive Macroeconomic Theory, Third Edition*, vol. 1 of *MIT Press Books*. The MIT Press.