Introduction to Linear Difference Equations

Jesse Perla
University of British Columbia

January 2, 2018

1 Linear DSGE Models

1.1 Goal for First Part of Course

DSGE Framework:

- (L)inear : system of first-order linear Gaussian difference equations
- (D)ynamic : difference equations
- (S)tochastic : (linear) Gaussian shocks
- (G)eneral (E)quilibrium : [# of equations = # of unknowns] (prices are variables)
- Rational Expectations equations
- Forecasting the future based on knowledge of stochastic processes

1.2 Simple Motivation:

- Risk-neutral (deterministic) asset pricing (linear preferences)
- Deterministic dividend stream: $\{y_t\},\ t = 0, \ldots, \infty$
- Discounted factor $\beta,\ 0 < \beta < 1$

1.3 Sequential and Recursive Form

With this setup, Price P_t = PDV (present discounted value).

$$P_t = \sum_{j=0}^{\infty} \beta^j \ y_{t+j} \tag{1}$$

Factor and take out β,

$$= y_t + \beta \left[\sum_{j=0}^{\infty} \beta^j \ y_{t+j+1} \right] \tag{2}$$

From (1), iterate forward one time period

$$P_{t+1} = \sum_{j=0}^{\infty} \beta^j \ y_{t+j+1} \tag{3}$$

Therefore, combining (2) and (3) the equation can be written recursively

$$P_t = y_t + \beta P_{t+1} \tag{4}$$

1.4 Linear First Order Difference Equation

$$P_t = y_t + \beta P_{t+1} \tag{5}$$

is the linear first-order difference equation in P_t, where:

- <u>Difference equation:</u> $P_t \ P_{t+1}$ terms
- <u>First-order:</u> maximum lag is 1 period ("$t+1$" and "t")
- <u>Linear:</u> No $P_t^2, \ y_t P_t, \ P_{t+1} P_t, \ldots$ terms

1.5 Simple Example with Constant Dividends

$$P_t = \bar{y} \sum_{t=0}^{\infty} \beta^t = \frac{\bar{y}}{1-\beta} \quad \text{(unique sequential solution).} \tag{6}$$

In recursive form: $P_t = \bar{y} + \beta P_{t+1}$.

A Guess Using Guess-and-Verify method:

$$P_t = \bar{P} \ \forall t \quad \text{(our guess)} \tag{7}$$

$$\Rightarrow \bar{P} = \bar{y} + \beta \bar{P} \tag{8}$$

Therefore,

$$\bar{P} = \frac{\bar{y}}{1-\beta} \quad \forall \ |\beta| < 1. \tag{9}$$

This is consistent with the sequential form, but the solution is not necessarily unique!

Another Guess To show this, we guess that the following guess holds **for at least one** $c \in \mathbb{R}$ and for all $t \geq 0$,

$$P_t = \frac{\bar{y}}{1-\beta} + c\beta^{-t} \tag{10}$$

Reorganizing

$$\frac{\bar{y}}{1-\beta} + c\beta^{-t} = \bar{y} + \beta\left[\frac{\bar{y}}{1-\beta} + c\beta^{-(t+1)}\right] \tag{11}$$

$$= \bar{y} + \frac{\beta}{1-\beta}\bar{y} + c\beta^{-t} \tag{12}$$

Cancelling, we see that this actually holds **for any** $c \in \mathbb{R}$

$$c\beta^{-t} = c\beta^{-t} \tag{13}$$

Interpretations:

- $c\beta^{-t}$ is a bubble term: prices rise because they are expected to rise
- Transversality conditions often eliminate the bubble term
- Sequential problem will be a solution to the recursive problem, but the recursive problem will not necessarily be unique
- Mechanically, difference equations are like differential equations (contains "general and particular solution")

Role of $|\beta| < 1$: to ensure non-explosiveness (eigenvalues for more complicated systems).

2 Differential Equations Limit

2.1 Setup and Notation

Heuristic Limit Setup:

- Let Δ be the length of time period
- $P(t)$: price of asset, $t \in [0, \infty)$
- $y(t)$: flow dividends; in discrete time, $y(t) \cdot \Delta$: period dividends
- $\beta(\Delta)$: discount factor parametrized by period length
- $r > 0$: instantaneous interest rate; $\beta(\Delta) \equiv 1 - \Delta r$

Difference equation:

$$P_t = y_t + \beta P_{t+1} \tag{14}$$

Writing with the substitutions

$$= y(t) \cdot \Delta + (1 - \Delta r)P(t + \Delta) \tag{15}$$

2.2 Limiting ODE and Interpretation

Rearranging (15),

$$\Delta r P(t + \Delta) = \Delta y(t) + P(t + \Delta) - P(t) \tag{16}$$

$$\Rightarrow r P(t + \Delta) = y(t) + \frac{P(t + \Delta) - P(t)}{\Delta}, \tag{17}$$

Taking the limit $\Delta \to 0$

$$r P(t) = y(t) + \boldsymbol{\partial}_t P(t) \tag{18}$$

where:

- $rP(t)$: opportunity cost of purchasing a unit of asset,
- $y(t)$: flow dividends
- $\boldsymbol{\partial}_t P(t)$: capital gains in instant t.

Analyzing:

- LHS > RHS in (18) :keep all the money in the bank)
- RHS > LHS in (18): (keep on buying only assets).
- LHS = RHS (no arbitrage condition).

Using guess-and-verify, we can solve the above ODE.

2.3 Example:

Let $y(t) = \bar{y}$.
General solution would be:

$$P(t) = \frac{\bar{y}}{r} + Ce^{rt} \tag{19}$$

Where:
$\frac{\bar{y}}{r}$ is the PDV of dividends and C is the undetermined coefficient.

How to determine C?

- Initial value: e.g., $p(0) = p_0$
  $\Rightarrow$ Solve for C in the bubble term, or
- Boundary value, e.g., tranversality condition, which gives $C = 0$ (no bubble)

$$\lim_{t\to\infty} e^{-rt} P(t) = 0 \tag{20}$$

3 Bubbles and Stochastic Process Example

This ODE assumed everything as deterministic, but if you allow stochastics, then martingales also work. Take the difference equation in (4),

$$P_t = y_t + \beta P_{t+1} \tag{21}$$

Turning it into a stochastic difference equation by replacing with the expected price tomorrow,

$$P_t = y_t + \mathbb{E}_t \left[P_{t+1}\right] \tag{22}$$

Where $\mathbb{E}_t \left[P_{t+1}\right]$ is the forecast of P_{t+1} based on time 't' information.

3.1 Martingales, and Stochastic Process Example

$$C_{t+1} = \begin{cases} \frac{1}{\lambda} C_t & \text{with prob } \lambda \in [0,1) \\ 0 & \text{with prob } 1-\lambda \end{cases} \tag{23}$$

Taking the expected value,

$$\mathbb{E}_t \left[C_{t+1}\right] = \lambda(\frac{1}{\lambda} C_t) + (1-\lambda) \cdot 0 \tag{24}$$

$$= C_t \tag{25}$$

Definition 1. *Martingale*
A stochastic process X_t is a Martingale if $\mathbb{E}_t \left[X_{t+1}\right] = X_t$.

Interpreting: a martingale is a stochastic process where the expectation of the forecast of tomorrow is the (observed) value today. Inductively, they can forecast any number of periods in the value today.

Example with (23) Take (22), and assume that $y_t = \bar{y}$. Guess that

$$P_t = \frac{\bar{y}}{1-\beta} + \beta^{-t} C_t \tag{26}$$

will solve the stochastic difference equation in (22). Note that with the form of (23), $C_t = 0$ is an absorbing “bubble popped" state.

Verifying the Solution: We can check that the following solved the difference equation: $P_t = \bar{y} + \beta \mathbb{E}_t \left[P_{t+1}\right]$, so:

$$P_t = \begin{cases} \frac{\bar{y}}{1-\beta} + (\beta\lambda)^{-t} C_0 & \text{if bubble hasn't popped, for any } C_0 \\ \dfrac{\bar{y}}{1-\beta} & \text{after bubble pops} \end{cases} \tag{27}$$

This stochastic process works for <u>any</u> martingale. Note that $C_0 \geq 0$ is necessary in practice so we don't have negative bubbles with negative values.

4 Linear State Space Model

4.1 Systems of Linear Difference Equations

General Setup:

- x_t : n-dimensional vector of states
- A : n x n matrix (evolution of state, makes first-order)
- G : 1 x n vector (observation of state components, contemporaneous)
- y_t : a scalar "observable" Furthermore, assume that,

$$x_{t+1} = A \cdot x_t \tag{28}$$
$$y_t = G \cdot x_t \tag{29}$$

We assume that we can see x_t and know A and G.[1]
For dividend y_t and price p_t, the (sequential) pricing equation is:

$$P_t = \sum_{j=o}^{\infty} \beta^j \cdot y_{t+j} \tag{30}$$

Substituting from (29)

$$= \sum_{j=o}^{\infty} \beta^j \cdot G \cdot x_{t+j} \tag{31}$$

Using (28), if $x_{t+1} = A \cdot x_t \;\Rightarrow x_{t+j} = A^j \cdot x_t$. Substituting,

$$= \sum_{j=o}^{\infty} \beta^j \cdot G \cdot A^j \cdot x_t \tag{32}$$

Reorganizing using properties of matrices (remember no commutativity!)

$$= G \cdot \left(\sum_{j=0}^{\infty} (\beta A)^j \right) \cdot x_t \tag{33}$$

Recall: if A is scalar, then

$$\sum_{j=0}^{\infty} (\beta A)^j = \frac{1}{1 - \beta A} \tag{34}$$

[1]If x_t was unobservable, this is a hidden Markov model

if A is a matrix, then:

$$\sum_{j=0}^{\infty}(\beta A)^j = \sum_{j=0}^{\infty}\beta^j A^j = (I-\beta A)^{-1} \quad \text{(matrix inverse)} \tag{35}$$

Using with (33),

$$P_t = G(I-\beta A)^{-1}x_t \tag{36}$$

4.2 Example:

Asset Prices:
$P_t = y_t + \beta P_{t+1},\ P_t = \sum_{j=0}^{\infty}\beta^j \cdot y_{t+j},$

Evolution of Dividends:
$y_{t+1} = \rho_1 y_t + \rho_2 y_{t-1}$ (second-order!),
We guess a state to put in the setup: $x_t = \begin{bmatrix} y_t \\ y_{t-1} \end{bmatrix}$.

The evolution equation for x_t :

$$\underbrace{\begin{bmatrix} y_{t+1} \\ y_t \end{bmatrix}}_{x_{t+1}} = \underbrace{\begin{bmatrix} \rho_1 & \rho_2 \\ 1 & 0 \end{bmatrix}}_{A}\underbrace{\begin{bmatrix} y_t \\ y_{t-1} \end{bmatrix}}_{x_t} \tag{37}$$

$$\Rightarrow x_{t+1} = A \cdot x_t \tag{38}$$

Observation equation:

$$\underbrace{y_t}_{y_t} = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{G}\underbrace{\begin{bmatrix} y_t \\ y_{t-1} \end{bmatrix}}_{x_t} \tag{39}$$

$$\Rightarrow y_t = G \cdot x_t \tag{40}$$

As we now know G, β, A, x_t, we can use (36) to get the asset price P_t.

4.3 Stability

With a scalar example. Let $y_t = \lambda^t$. Calculate the PDV,

$$P_t = \sum_{j=0}^{\infty} \beta^j \cdot y_{t+j} = \sum_{j=0}^{\infty} (\beta\lambda)^j \tag{41}$$

The solution, and condition for convergence of the sum, are,

$$P_t = \frac{1}{1 - \beta\lambda} \quad \text{for} \quad |\lambda| < \frac{1}{\beta} \tag{42}$$

For e.g., if λ is TFP, then income growth shouldn't grow more than P_t, or P_t will explode. Analogously here, we will need to constraint A. As it turns out, the condition is on eigenvalues: $\max \ |\text{eig}(A)| < \dfrac{1}{\beta}$ (multivariate case).