Linear State Space Models
Additional Material and Examples

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Complements material in `https://lectures.quantecon.org/jl/linear_models.html`. Also see (Ljungqvist and Sargent, 2012, Chapter 2)

1 New Linear Model with Stochastics

1.1 Stochastic Model

Following `https://lectures.quantecon.org/jl/linear_models.html`,

$$x_{t+1} = Ax_t + Cw_{t+1} \tag{1}$$
$$y_t = Gx_t \tag{2}$$

where $x_t \in \mathbb{R}^n$, and $w_t \in \mathbb{R}^m$ with $w_t \sim N(0, I)$

2 Impulse Response

Given a one-time shock to w_{t+1}, how does this evolve over j (if all $w_{t+j} = 0$ for all $j > 1$)? Intuitively, start with $x_t = 0$, and then look at the evolution of x_{t+j}

$$A^{j-1}Cw_{t+1} \to \text{ IRF of } x_{t+j} \tag{3}$$
$$GA^{j-1}Cw_{t+1} \to \text{ IRF of } y_{t+j} \tag{4}$$

Also, note that if we recursively sum up the discounted future:

- $G(I - \beta A)^{-1}Cw_{t+1} \;\to$ IRF for the present value of y_t (5)

Alternatively could use $x_t > 0$ and the IRF is simply the change from the deterministic evolution (due to linearity of the process).

3 Asset Pricing Stochastic Linear Model

From equations (1) and (2) respectively, we have:

$$\begin{aligned} x_{t+1} &= A \cdot x_t + C \cdot w_{t+1} \text{ (evolution)} \\ y_t &= G \cdot x_t \text{ (observation)} \end{aligned}$$

The risk-neutral pricing equation is given by:

$$P_t = y_t + \beta \mathbb{E}_t \left[P_{t+1} \right] \tag{6}$$

Solution: Using Guess-and-Verify method:

$$P_t = H \cdot x_t \text{ (our guess, for some undetermined } H \text{ to be decided)} \tag{7}$$

Verify: Plug in the pricing function at t and $t+1$ from equation (6):

$$H \cdot x_t = y_t + \beta \mathbb{E}_t \left[H \cdot x_{t+1} \right] \tag{8}$$

Using equation (1):

$$\begin{aligned} H \cdot x_t &= y_t + \beta \mathbb{E}_t \left[H \cdot (A \cdot x_t + C \cdot w_{t+1}) \right] && (9) \\ &= y_t + (\beta H \cdot A \cdot x_t) + (\beta H \cdot C \cdot \mathbb{E}_t \left[w_{t+1} \right]) && (10) \end{aligned}$$

By linearity of expectations, and using $\mathbb{E}_t \left[w_{t+1} \right] = 0$

$$H \cdot x_t = G \cdot x_t + \beta H \cdot A \cdot x_t \tag{11}$$

Using the method of undetermined coefficients, the following must hold for the above equation:

$$\begin{aligned} H &= G + \beta H \cdot A && (12) \\ \Rightarrow H(I - \beta A) &= G && (13) \\ \Rightarrow H &= G(I - \beta A)^{-1} && (14) \end{aligned}$$

Plugging this expression of H into our guess in equation (7)

$$P_t = G(I - \beta A)^{-1} x_t \text{ (identical to non-stochastic version)} \tag{15}$$

3.1 Stochastic Example

Using a second-order autoregressive process for y_t:

$$y_{t+1} = \gamma + \rho_1 y_t + \rho_2 y_{t-1} + \sigma \underbrace{w_{t+1}}_{\text{gaussian noise}} \tag{16}$$

$$\mathbb{E}_t \left[w_{t+1} \right] = 0 \tag{17}$$

$$\mathbb{E}_t \left[w_{t+1} w_{t+1} \right] = 1 \tag{18}$$

We need to convert this into a state space:

A Guess: We guess a state : $x_t = \begin{bmatrix} 1 \\ y_t \\ y_{t-1} \end{bmatrix}$.

Using equation (1), we set up the evolution equations:

$$x_{t+1} = A \cdot x_t + C \cdot w_{t+1} \tag{19}$$

$$\begin{bmatrix} 1 \\ y_t \\ y_{t-1} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \gamma & \rho_1 & \rho_2 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ y_t \\ y_{t-1} \end{bmatrix} + \begin{bmatrix} 0 \\ \sigma \\ 0 \end{bmatrix} w_{t+1} \tag{20}$$

Using equation (2), the observation equation is:

$$\begin{aligned} y_t &= G \cdot x_t \\ &= \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ y_t \\ y_{t-1} \end{bmatrix} \end{aligned} \tag{21}$$

A Different Formulation (1): This time, let the evolution equation be the following:

$$x_{t+1} = B + A \cdot x_t + C \cdot w_{t+1} \tag{22}$$

Using the guess $x_t = \begin{bmatrix} y_t \\ y_{t-1} \end{bmatrix}$, the linear state space model of the AR(2) process from equation (16) is:

$$\begin{bmatrix} y_{t+1} \\ y_t \end{bmatrix} = \begin{bmatrix} \gamma \\ 0 \end{bmatrix} + \begin{bmatrix} \rho_1 & \rho_2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} y_t \\ y_{t-1} \end{bmatrix} + \begin{bmatrix} \sigma \\ 0 \end{bmatrix} w_{t+1} \tag{23}$$

A Different Formulation (2): Let the evolution equation be:

$$x_{t+1} = A \cdot x_t + w_{t+1},\ \ w_{t+1} \sim \mathrm{N}\left(0, \Sigma\right) \tag{24}$$

Using the guess $x_t = \begin{bmatrix} 1 \\ y_t \\ y_{t-1} \end{bmatrix}$,

$$\Sigma = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \sigma^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{25}$$

where

$$\Sigma = CC' = \begin{bmatrix} 0 & \sigma & 0 \end{bmatrix}' \begin{bmatrix} 0 \\ \sigma \\ 0 \end{bmatrix}' \tag{26}$$

Principle:

- We can always convert to a 1st order difference equation.
- Choose the state carefully (augmenting the state).
- Equation (19) is a **Vector Auto-Regression (VAR)**.

4 Playing with State Spaces/Orthogonalizing Shocks

Example: State Space Example Using a mixed moving average autoregressive model:

$$\underbrace{y_{t+1}}_{\text{scalar}} = \alpha + \rho y_t + \underbrace{w_{t+1} + \gamma w_t}_{\text{moving average}} \tag{27}$$

Using the guess $x_t = \begin{bmatrix} 1 \\ y_t \\ w_t \end{bmatrix}$, the system is given by:

$$\begin{bmatrix} 1 \\ y_{t+1} \\ w_{t+1} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \alpha & \rho & \gamma \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ y_t \\ w_t \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} w_{t+1} \tag{28}$$

Note that the canonical state space form

$$x_{t+1} = A \cdot x_t + C \cdot w_{t+1} \tag{29}$$

has w_{t+1} as orthogonal. But shocks are often correlated. How can we put them in canonical form?

4.1 Correlated Shocks and Cholesky Decomposition

Example: Take the following process (not in our canonical state space form!)

$$x_{t+1} = A \cdot x_t + \eta_{t+1} \tag{30}$$

$$\text{where } \eta_{t+1} = \begin{bmatrix} \eta_{1t+1} \\ \eta_{2t+1} \end{bmatrix} \tag{31}$$

$$\text{and } \eta_{t+1} \sim \mathrm{N}\left(\underbrace{\begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}}_{\equiv \alpha}, \underbrace{\begin{bmatrix} \sigma_1^2 & c \\ c & \sigma_2^2 \end{bmatrix}}_{\equiv \Sigma} \right) \tag{32}$$

We want a matrix C such that:

$$\alpha + C \cdot w_{t+1} \sim \mathrm{N}(\alpha, \Sigma) \text{ for } w_{t+1} \sim \mathrm{N}(0, I_2) \tag{33}$$

Then we have another version of (30), in a slightly different canonical form than our usual setup (i.e, with a constant term) :

$$x_{t+1} = A \cdot x_t + \alpha + C \cdot w_{t+1} \tag{34}$$

which can be further simplified to the canonical form. The α constant is easy to remove by adding a 1 to the state, so let's focus on the case $\alpha = 0$.

What if $\eta_{t+1} \sim \mathrm{N}\left(0, \begin{bmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{bmatrix}\right)$?

The matrix C is easy to compute here since the shocks are independent:

$$C = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix}, \text{(i.e., take square root of the variances)} \tag{35}$$

Note that,

$$C \cdot C' = \Sigma \text{ , so } C \text{ is kind of a "square root" of } \Sigma \tag{36}$$

What about if:

$$\Sigma = \begin{bmatrix} \sigma_1^2 & \rho \\ \rho & \sigma_2^2 \end{bmatrix} \tag{37}$$

Note that all variance-covariance matrices are symmetric and positive definite. How can we find the matrix C like equation (36) in the case of the above variance-covariance matrix?

The answer is a Cholesky decomposition which find a lower-triangular (or upper-triangular) C, such that (36) is true and:

$$C \cdot w_{t+1} \sim \mathrm{N}\left(0, \Sigma\right) \tag{38}$$

In linear algebra, we are doing a change of basis on the η_{t+1} to an orthogonal basis (i.e., orthogonal and unit w_{it+1}).
Cholesky decompositions can be done on a computer, and come up all over VARs and empirical macroeconomics.

Example: Using a 2×2 example:

$$\text{If } \Sigma = \begin{bmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{bmatrix} \Rightarrow C = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix} \tag{39}$$

which is consistent with our independent shocks answer.

$$\text{For } \Sigma = \begin{bmatrix} \sigma_1^2 & \rho \\ \rho & \sigma_2^2 \end{bmatrix} \Rightarrow C = \begin{bmatrix} \sigma_1 & 0 \\ \frac{\rho}{\sigma_1} & \sqrt{\sigma_2^2 - \frac{\rho^2}{\sigma_1^2}} \end{bmatrix} \tag{40}$$

Alternatively, if upper-triangular, then transpose:

$$C = \begin{bmatrix} \sigma_1 & \frac{\rho}{\sigma_1} \\ 0 & \sqrt{\sigma_2^2 - \frac{\rho^2}{\sigma_1^2}} \end{bmatrix} \Rightarrow C' \cdot C = \Sigma \tag{41}$$

If you redefine the constant ρ, get a little cleaner expression:

$$\Sigma = \begin{bmatrix} \sigma_1^2 & \rho\sigma_1 \\ \rho\sigma_1 & \sigma_2^2 \end{bmatrix} \Rightarrow C = \begin{bmatrix} \sigma_1 & 0 \\ \rho & \sqrt{\sigma_2^2 - \rho^2} \end{bmatrix} \tag{42}$$

So if:

$$x_{1t+1} = a_1 x_{1t} + \eta_{1t+1} \tag{43}$$

$$x_{2t+1} = a_1 x_{2t} + \eta_{2t+1} \tag{44}$$

where

$$\eta_{it} \sim \mathrm{N}\left(0, \begin{bmatrix} \sigma_1^2 & \rho\sigma_1 \\ \rho\sigma_1 & \sigma_2^2 \end{bmatrix}\right) \tag{45}$$

$$\Rightarrow x_t = \begin{bmatrix} x_{1t} \\ x_{2t} \end{bmatrix}, \; w_{t+1} \sim \mathrm{N}\left(0, I_2\right)$$

Combining the above identity with equations (43) and (44):

$$x_{t+1} = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix} \cdot x_t + \begin{bmatrix} \sigma_1 & 0 \\ \rho & \sqrt{\sigma_2^2 - \rho^2} \end{bmatrix} \cdot w_{t+1} \tag{46}$$

which is in canonical state space form.

Impulses to Orthogonalized Shocks Note that a unit shock $w_{t+1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is transformed to:

$$\eta_{t+1} = \begin{bmatrix} \sigma_1 & 0 \\ \rho & \sqrt{\sigma_2^2 - \rho^2} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{47}$$

$$= \begin{bmatrix} \sigma_1 \\ \rho \end{bmatrix} \tag{48}$$

Hence, this is a 1 standard deviation shock to η_{1t+1}, and has the correlated effect of η_{2t+1}. It would be possible to do something similar to η_{2t+1} as well with a little linear algebra.

Therefore, to do a 1 standard deviation shock, the impulse response on x_t at period $t+j$ is:

$$A^{j-1} \cdot Cw_{t+1} \tag{49}$$

and the corresponding PDV of the impulse is the standard

$$(I - \beta A)^{-1} Cw_{t+1} \tag{50}$$

Of course, if the impulse response is on an observable $y_t = Gx_t$, then the IRF and PDV of the IRF for y_t are

$$GA^{j-1} \cdot Cw_{t+1} \tag{51}$$

and

$$G(I - \beta A)^{-1} Cw_{t+1} \tag{52}$$

5 Stationary Distribution

5.1 No constant term

Let $x_{t+1} = Ax_t + Cw_{t+1}$, where $w_{t+1} \sim \mathrm{N}(0, I)$.

Then the stationary distribution of the first-order linear gaussian state space is:

$$x_\infty \sim \mathrm{N}(0, \Sigma_\infty) \tag{53}$$

where x_∞ has unconditional mean 0, and the variance-covariance matrix Σ_∞ solves the discrete Lyapunov equation

$$\Sigma_\infty = A' \Sigma_\infty A + CC' \tag{54}$$

- Can use Matlab command: `dlyap(A,C*C')`. For Julia, see `http://quantecon.github.io/QuantEcon.jl/latest/api/QuantEcon.html#QuantEcon.solve_discrete_lyapunov` Note that all the eigenvalues of A must be less than 1 (or > -1)
- Requires numerical solutions in general
- Solutions may not exist (e.g., non-stationary)

5.2 With Constant Term

If there is a constant, set up an alternative state space form:

$$x_{t+1} = a + A \cdot x_t + C \cdot w_{t+1} \tag{55}$$

i.e., leave the constant out of the state and C.
Then:

$$x_\infty \sim \mathrm{N}\left(\mu_\infty, \Sigma_\infty\right) \tag{56}$$

where Σ_∞ solves the same discrete Lyapunov equation (54) and

$$\mu_\infty \equiv (I - A)^{-1} \cdot a \tag{57}$$

5.2.1 Example 1

Take a scalar example:

$$x_{t+1} = \rho x_t + \sigma w_{t+1} \tag{58}$$

Finding the stationary distribution for $t \to \infty$ we use (54). Substituting (58) and solving for the stationary variance,

$$x_\infty \sim \mathrm{N}\left(0, \frac{\sigma^2}{1-\rho^2}\right) \tag{59}$$

5.2.2 Example 2

$$x_{1t+1} = a_1 + \rho_1 x_{1t} + \rho_2 x_{2t} + \sigma_1 w_{1t+1} + \sigma_2 w_{2t+1} \tag{60}$$

$$x_{2t+1} = \rho_3 x_{2t} + \sigma_3 w_{2t+1} \tag{61}$$

where x_{it} is scalar, and $w_{it+1} \sim \mathrm{N}(0,1)$.

Guess a state: $x_t = \begin{bmatrix} x_{1t} \\ x_{2t} \end{bmatrix}$. Let the corresponding matrices be:

$a = \begin{bmatrix} a_1 \\ 0 \end{bmatrix}$, $A = \begin{bmatrix} \rho_1 & \rho_2 \\ 0 & \rho_3 \end{bmatrix}$, $C = \begin{bmatrix} \sigma_1 & \sigma_2 \\ 0 & \sigma_3 \end{bmatrix}$.

Therefore, the state space is given by:

$$\begin{bmatrix} x_{1t+1} \\ x_{2t+1} \end{bmatrix} = \begin{bmatrix} a_1 \\ 0 \end{bmatrix} + \begin{bmatrix} \rho_1 & \rho_2 \\ 0 & \rho_3 \end{bmatrix} \begin{bmatrix} x_{1t} \\ x_{2t} \end{bmatrix} + \begin{bmatrix} \sigma_1 & \sigma_2 \\ 0 & \sigma_3 \end{bmatrix} \begin{bmatrix} w_{1t+1} \\ w_{2t+1} \end{bmatrix} \tag{62}$$

Then the stationary distribution is given by:

$$x_\infty \sim \mathrm{N}\left(\mu_\infty, \Sigma_\infty\right) \tag{63}$$

References

Ljungqvist, L., and T. J. Sargent (2012): *Recursive Macroeconomic Theory, Third Edition*, vol. 1 of *MIT Press Books*. The MIT Press.