Stochastic Difference Equations

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January 25, 2016, Draft Version: 260

1 New Linear Model with Stochastics

1.1 Stochastic Model

- Add stochastic evolution of the state variable
- References: (Ljungqvist and Sargent, 2012, Chapter 2), and Hansen and Sargent (2013) for online MATLAB codes
- Keep Markov, m i.i.d. shocks: $\varepsilon_{t+1} \in \mathbb{R}^m$, $C \in \mathbb{R}^{n \times m}$
- Simplicity: Assume $\varepsilon \sim \mathrm{N}(0_m, I_m)$, i.e. ε is multivariate Gaussian; ε_{t+1} i.i.d.; $\mathbb{E}_t[\varepsilon_{t+1}] = 0$ (uncorrelated)
- Use C to correlate and adjust variance

Therefore, under the i.i.d shocks, the linear model with stochastics are represented by:

$$\boxed{\begin{aligned} \underbrace{x_{t+1}}_{n\times 1} &= \underbrace{A}_{n\times n} \cdot \underbrace{x_t}_{n\times 1} + \underbrace{C}_{n\times m} \cdot \underbrace{\varepsilon_{t+1}}_{m\times 1} \quad (1) \\ \underbrace{y_t}_{1\times 1} &= \underbrace{G}_{1\times n} \cdot \underbrace{x_t}_{n\times 1} \quad (2) \end{aligned}}$$

Recall from equation (1) that if $C = 0$, then:

$$x_{t+1} = A \cdot x_t \tag{3}$$

Taking the expectations of equation (1) at time t:

$$\underbrace{\mathbb{E}_t\left[x_{t+1}\right]}_{\text{time } t \text{ info of } t+1} = \mathbb{E}_t\Bigg[\underbrace{Ax_t + C\varepsilon_{t+1}}_{\text{plugged in evolution}}\Bigg] \tag{4}$$

$$= A \cdot x_t + C \cdot \mathbb{E}_t\left[\varepsilon_{t+1}\right] \tag{5}$$

$\mathbb{E}_t\left[\varepsilon_{t+1}\right] = 0$ by definition, without any assumption of normality

$$\boxed{\mathbb{E}_t\left[x_{t+1}\right] = A \cdot x_t} \tag{6}$$

1.2 Law of Iterated Expectations

Notation:

$$\mathbb{E}_t\left[x_{t+1}\right] = \mathbb{E}\left[x_{t+1}|x_t\right] \text{ (where } x_t \text{ is the only known state)} \tag{7}$$

Given $t+1$ time's information, we can make a forecast for $t+2$:

$$\mathbb{E}_t\left[x_{t+2}\right] = \mathbb{E}_t\left[\mathbb{E}_{t+1}\left[x_{t+2}\right]\right] \tag{8}$$

$$= \mathbb{E}_t\left[Ax_{t+1}\right] \tag{9}$$

$$= A\ \mathbb{E}_t\left[x_{t+1}\right] \tag{10}$$

using equation (6):

$$\mathbb{E}_t\left[x_{t+2}\right] = A^2 \cdot x_t \tag{11}$$

More generally, for a random variable X we have a special case of the the **law of iterated expectations**:

$$\mathbb{E}_t\left[X\right] = \mathbb{E}_t\left[\mathbb{E}_{t+1}\left[X\right]\right] \tag{12}$$

2 Forecasts

Using equation (1) at time $t+1$ and then plugging in for $t+1$:

$$x_{t+2} = Ax_{t+1} + C\varepsilon_{t+2} \tag{13}$$

$$= A(Ax_t + C\varepsilon_{t+1}) + C\varepsilon_{t+2} \tag{14}$$

$$= A^2x_t + AC\varepsilon_{t+1} + C\varepsilon_{t+2} \tag{15}$$

In general,

$$x_{t+j} = A^j x_t + A^{j-1} C\varepsilon_{t+1} + A^{j-2} C\varepsilon_{t+2} + \ldots + C\varepsilon_{t+j} \tag{16}$$

where $A^j x_t$ is known at time t, and the rest unknown

$$\Rightarrow x_{t+j} = A^j x_t + \sum_{i=1}^{j} A^{i-1}\ C\ \varepsilon_{t+j-1+i} \tag{17}$$

Take the expectations at time t and using the linearity of expectations : $(\mathbb{E}\left[\varepsilon_{t+j}\right] = 0\ \forall j)$:

$$\boxed{\mathbb{E}_t\left[x_{t+j}\right] = A^j \cdot x_t} \text{ (forecasting formula)} \tag{18}$$

2.1 Prediction Error of Forecasts

The error at time t of $t+j$ is:

$$x_{t+j} - \mathbb{E}_t\left[x_{t+j}\right] \tag{19}$$

So, the expected error with time t information is:

$$\mathbb{E}_t\left[x_{t+j} - \mathbb{E}_t\left[x_{t+j}\right]\right] = 0 \tag{20}$$

The variance-covariance of prediction errors is:

$$\begin{aligned}
\Sigma_j &\equiv \mathbb{E}_t\left[\left(x_{t+j} - \mathbb{E}_t\left[x_{t+j}\right]\right)\left(x_{t+j} - \mathbb{E}_t\left[x_{t+j}\right]\right)'\right]^{1} && (21)\\
&= \mathbb{E}_t\left[\left(A^{j-1}C\varepsilon_{t+1} + A^{j-2}C\varepsilon_{t+2} + \ldots + C\varepsilon_{t+j}\right)\left(A^{j-1}C\varepsilon_{t+1} + A^{j-2}C\varepsilon_{t+2} + \ldots + C\varepsilon_{t+j}\right)'\right] \\
&= \mathbb{E}_t\left[A^{j-1}C\varepsilon_{t+1}\varepsilon'_{t+1}C'(A^{(j-1)})' + \ldots + C\varepsilon_{t+j}\varepsilon'_{t+j}C' + (\text{cross } \varepsilon \text{ terms})\right] && (22)
\end{aligned}$$

Using the linearity of expectations:

$$\begin{aligned}
\mathbb{E}_t\left[\varepsilon_{t+i}\varepsilon'_{t+j}\right] &= 0\ \forall i \neq j \text{ (by assumption)} && (23)\\
\mathbb{E}_t\left[\varepsilon_{t+j}\varepsilon'_{t+j}\right] &= I \text{ (by assumption)} && (24)\\
\Sigma_j &= A^{j-1}CC'(A^{(j-1)})' + A^{j-2}CC'(A^{(j-2)})' + \ldots + CC' && (25)\\
&\boxed{\Sigma_j = \sum_{i=1}^{j} A^{i-1}CC'(A^{(i-1)})'} \text{ (j-step ahead forecast error)} && (26)
\end{aligned}$$

[1] The transpose is required on the second term to give us a covariance matrix

2.2 Possible Questions

How can we forecast the 'infinite future', i.e. $\lim_{j\to\infty} \Sigma_j$?

- If an eigenvalue of $A > 1$ in absolute values, then the right hand side of equation (26) for the limit $j \to \infty$ might diverge
- There will be convergence if $\max\{|\text{eig}(A)|\} < 1$

3 Impulse Response

Given a one-time shock to ε_{t+1}, how does this evolve over j (if all $\varepsilon_{t+j} = [0] \ \forall j > 1$)? Using equations (1) and (2), the impulse response functions (IRFs) of a ε_{t+1} shock is:

- $A^{j-1}C\varepsilon_{t+1} \quad \to$ IRF of x_{t+j} (27)
- $GA^{j-1}C\varepsilon_{t+1} \ \to$ IRF of y_{t+j} (28)

Also, note that if we recursively sum up the discounted future:

- $G(I-\beta A)^{-1}C\varepsilon_{t+1} \ \to$ IRF for the present value of y_t (29)

4 Asset Pricing Stochastic Linear Model

From equations (1) and (2) respectively, we have:

$$
\begin{aligned}
x_{t+1} &= A \cdot x_t + C \cdot \varepsilon_{t+1} \text{ (evolution)} \\
y_t &= G \cdot x_t \text{ (observation)}
\end{aligned}
$$

The risk-neutral pricing equation is given by:

$$P_t = y_t + \beta \mathbb{E}_t \left[P_{t+1}\right] \tag{30}$$

Solution: Using Guess-and-Verify method:

$$P_t = H \cdot x_t \text{ (our guess, for some undetermined } H \text{ to be decided)} \tag{31}$$

Verify: Plug in the pricing function at t and $t+1$ from equation (30):

$$H \cdot x_t = y_t + \beta \mathbb{E}_t \left[H \cdot x_{t+1}\right] \tag{32}$$

Using equation (1):

$$
\begin{aligned}
H \cdot x_t &= y_t + \beta \mathbb{E}_t \left[H \cdot (A \cdot x_t + C \cdot \varepsilon_{t+1}) \right] && (33)\\
&= y_t + (\beta H \cdot A \cdot x_t) + (\beta H \cdot C \cdot \mathbb{E}_t \left[\varepsilon_{t+1} \right]) && (34)
\end{aligned}
$$

By linearity of expectations, and using $\mathbb{E}_t \left[\varepsilon_{t+1} \right] = 0$

$$
H \cdot x_t = G \cdot x_t + \beta H \cdot A \cdot x_t \tag{35}
$$

Using the method of undetermined coefficients, the following must hold for the above equation:

$$
\begin{aligned}
H &= G + \beta H \cdot A && (36)\\
\Rightarrow H(I - \beta A) &= G && (37)\\
\Rightarrow H &= G(I - \beta A)^{-1} && (38)
\end{aligned}
$$

Plugging this expression of H into our guess in equation (31)

$$
P_t = G(I - \beta A)^{-1} x_t \text{ (identical to non-stochastic version)} \tag{39}
$$

4.1 Stochastic Example

Using a second-order autoregressive process for y_t:

$$
\begin{aligned}
y_{t+1} &= \gamma + \rho_1 y_t + \rho_2 y_{t-1} + \sigma \underbrace{\varepsilon_{t+1}}_{\text{gaussian noise}} && (40)\\
\mathbb{E}_t \left[\varepsilon_{t+1} \right] &= 0 && (41)\\
\mathbb{E}_t \left[\varepsilon_{t+1} \varepsilon_{t+1} \right] &= 1 && (42)
\end{aligned}
$$

We need to convert this into a state space:

A Guess: We guess a state : $x_t = \begin{bmatrix} 1 \\ y_t \\ y_{t-1} \end{bmatrix}$.

Using equation (1), we set up the evolution equations:

$$x_{t+1} = A \cdot x_t + C \cdot \varepsilon_{t+1} \tag{43}$$

$$\begin{bmatrix} 1 \\ y_t \\ y_{t-1} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \gamma & \rho_1 & \rho_2 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ y_t \\ y_{t-1} \end{bmatrix} + \begin{bmatrix} 0 \\ \sigma \\ 0 \end{bmatrix} \varepsilon_{t+1} \tag{44}$$

Using equation (2), the observation equation is:

$$\begin{aligned} y_t &= G \cdot x_t \\ &= \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ y_t \\ y_{t-1} \end{bmatrix} \end{aligned} \tag{45}$$

A Different Formulation (1): This time, let the evolution equation be the following:

$$x_{t+1} = B + A \cdot x_t + C \cdot \varepsilon_{t+1} \tag{46}$$

Using the guess $x_t = \begin{bmatrix} y_t \\ y_{t-1} \end{bmatrix}$, the linear state space model of the AR(2) process from equation (40) is:

$$\begin{bmatrix} y_{t+1} \\ y_t \end{bmatrix} = \begin{bmatrix} \gamma \\ 0 \end{bmatrix} + \begin{bmatrix} \rho_1 & \rho_2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} y_t \\ y_{t-1} \end{bmatrix} + \begin{bmatrix} \sigma \\ 0 \end{bmatrix} \varepsilon_{t+1} \tag{47}$$

A Different Formulation (2): Let the evolution equation be:

$$x_{t+1} = A \cdot x_t + \varepsilon_{t+1},\ \ \varepsilon_{t+1} \sim \mathrm{N}\left(0, \Sigma\right) \tag{48}$$

Using the guess $x_t = \begin{bmatrix} 1 \\ y_t \\ y_{t-1} \end{bmatrix}$,

$$\Sigma = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \sigma^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{49}$$

where

$$\Sigma = CC' = \begin{bmatrix} 0 & \sigma & 0 \end{bmatrix}' \begin{bmatrix} 0 \\ \sigma \\ 0 \end{bmatrix}' \tag{50}$$

Principle:

- We can always convert to a 1st order difference equation.
- Choose the state carefully (augmenting the state).
- Equation (43) is a **Vector Auto-Regression (VAR)**.

References

Hansen, L. P., and T. J. Sargent (2013): *Recursive Models of Dynamic Linear Economies*. Princeton University Press.

Ljungqvist, L., and T. J. Sargent (2012): *Recursive Macroeconomic Theory, Third Edition*, vol. 1 of *MIT Press Books*. The MIT Press.