Money in Utility Model and Monetary Neutrality

Jesse Perla
University of British Columbia

February 5, 2018

1 Role of Money

- In an RBC model, only the price level matters (on homogeneous good or differentiated price index with monopolistic competition).
- Money serves only as a numeraire: “cashless".
- With price frictions, there can be interesting price dispersions, etc., but there isn't a role for monetary policy. **Why?**
  - 1 period bond and holding money have identical risk profiles, but bonds dominate in returns.
  - No reason to hold cash.

Approaches:

(1) Perhaps money is just a unit of account? Numeraire

(2) Cash in advance: hold money to pay for things you may want later

(3) Money in utility: some unmodelled service value of having real money stocks (kind of cheating, but useful).

(4) Monetary search: smooths difficulties in barter

2 Money In Utility

References: See Walsh Chapter 2, Gali Section 2.5.

2.1 Environment

- Labour supply is inelastic for simplicity
- Normalize population: $N_t = 1$.
- Consumer has k_{t-1} units of capital per-capita, and owns and operates the firm with production function $f(k_{t-1})$
- Homogeneous good
- Quantity of \$, M_t nominal money supply
- P_t is the price of the homogeneous good
- Deterministic for simplicity

2.2 Consumer's Problem

- c_t per-capita consumption
- M_t/P_t is real holdings of money (in goods).
- Let $m_t = \frac{M_t}{P_t}$, per capital real money holdings.
- $U(c_t, \frac{M_t}{P_t})$: utility of per capita consumption and from money services (M_t enters through m_t), discounted by β.
- Assume: $\boldsymbol{\partial}_m U(c, 0) = \infty$, i.e. Inada conditions on money (always want to hold money).

The consumer's problem is given by:

$$\max_{\{c_t, k_t, B_t, M_t\}} \sum_{t=0}^{\infty} \beta^t U(c_t, M_t/P_t) \tag{1}$$

subject to the nominal budget constraint of household (for owning firm / capital):

$$\underbrace{f(k_{t-1})}_{\text{production}} + \underbrace{(1-\delta)k_{t-1}}_{\substack{\text{depreciated} \\ \text{capital}}} + \overbrace{(1+i_{t-1})}^{\substack{\text{nominal} \\ \text{interest}}} \underbrace{\frac{B_{t-1}}{P_t}}_{\substack{\text{bond holdings} \\ \text{before period}}} + \underbrace{\frac{M_{t-1}}{P_t}}_{\text{money holdings}} = \underbrace{c_t}_{\text{consumption}} + \underbrace{k_t}_{\substack{\text{capital for} \\ \text{next period}}} + \underbrace{\frac{M_t}{P_t} + \frac{B_t}{P_t}}_{\substack{\text{real bond /} \\ \text{money holdings}}} \tag{2}$$

A few things:

- This is using a different timing than before, k_{t-1} is the capital at the start of the period, and they choose k_t. Same for bonds, money.

Real bond / money holdings:: Define inflation as the ratio of prices of bonds or money, then the real value of bonds:

$$1+\pi_t \equiv \frac{P_t}{P_{t-1}} \tag{3}$$
$$b_t \equiv \frac{B_t}{P_t} \tag{4}$$

At time t (divide by P_t):

$$\frac{B_{t-1}}{P_t} = b_{t-1} \cdot \frac{P_{t-1}}{P_t} = \frac{b_{t-1}}{1+\pi_t} \tag{5}$$

Therefore, the consumer's problem sequentially is rewritten by:

$$\max_{c_t,k_t,b_t,m_t} \sum_{t=0}^{\infty} \beta^t u(c_t, m_t) \tag{6}$$
$$\text{s.t. } c_t + k_t + m_t + b_t \leq f(k_{t-1}) + (1-\delta)k_{t-1} + \underbrace{\frac{(1+i_{t-1})b_{t-1} + m_{t-1}}{1+\pi_t}}_{\text{nominal value inflated away}} \tag{7}$$

This is identical to the neoclassic growth model. Taking the first-order conditions and rearranging:

$$\overbrace{\underbrace{\boldsymbol{\partial}_m u(c_t, m_t)}_{\substack{\text{direct}\\ \text{utility}}} + \underbrace{\beta \cdot \frac{\boldsymbol{\partial}_c u(c_{t+1}, m_{t+1})}{1+\pi_{t+1}}}_{\substack{\text{adds to direct wealth}\\ \text{for consumption tomorrow}}}}^{\text{marginal benefit of holding money}} = \overbrace{\boldsymbol{\partial}_c u(c_t, m_t)}^{\substack{\text{marginal benefit}\\ \text{of consumption}}} \tag{8}$$

This will determine a money demand equation.

Money Supply Let the nominal money stock evolve <u>exogenously</u> with growth rate g_{mt}:

$$\frac{M_t}{M_{t-1}} = 1 + g_{mt} \tag{9}$$
$$\Rightarrow \frac{M_t}{P_t} \cdot \frac{P_{t-1}}{M_{t-1}} = \frac{P_{t-1}}{P_t}(1+g_{mt}) \tag{10}$$

Using the definition of inflation: $\frac{P_t}{P_{t-1}} \equiv 1 + \pi_t$, and substituting in the expressions for the real money stock: $\frac{M_t}{P_t} \equiv m_t$:

$$\Rightarrow m_t = \left(\frac{1 + g_{mt}}{1 + \pi_t}\right) m_{t-1} \tag{11}$$

This expression is the real money supply, which evolves according to the exogenous growth rate, g_{mt} and the endogenous inflation rate π_t

Three Assets: Money, Bonds and Capital.
The indifference comes out of the FONC in equation (8):

$$\frac{\boldsymbol{\partial}_m u(c_t, m_t)}{\boldsymbol{\partial}_c u(c_t, m_t)} = 1 - \frac{1}{1 + \pi_{t+1}} \cdot \beta \frac{\boldsymbol{\partial}_c u(c_{t+1}, m_{t+1})}{\boldsymbol{\partial}_c u(c_t, m_t)} \qquad \text{(money)} \tag{12}$$

$$= 1 - \frac{1}{(1 + r_t)(1 + \pi_{t+1})} \qquad \text{(capital)} \tag{13}$$

$$= \frac{i_t}{1 + i_t} \qquad \text{(bonds)} \tag{14}$$

where the real return on capital is given by:

$$1 + r_t \equiv \boldsymbol{\partial}_k f(k_t) + 1 - \delta \tag{15}$$

Rearrange the equations (13) and (14) to get:

$$1 + i_t = [\boldsymbol{\partial}_k f(k_t) + 1 - \delta]\,(1 + \pi_{t+1}) = (1 + r_t)(1 + \pi_{t+1}) \tag{16}$$

Note that for r_t and π_{t+1}, since $(1 + r_t)(1 + \pi_{t+1}) \approx 1 + r_t + \pi_{t+1}$

$$\boxed{i_t = r_t + \pi_{t+1}} \tag{17}$$

This gives the Fisher equation (a relationship on real versus nominal rates), where:
nominal interest rates on bonds = real returns + expected inflation

2.3 Steady State Equations

Variables: c^*, m^*, k^*, π^*.
Prices: i^*, r^*.

Note: $\pi^* = \dfrac{P_t}{P_{t-1}}$ constant inflation

$$1 = \beta \left[\boldsymbol{\partial}_k f(k^*) + 1 - \delta \right] \qquad \text{(neoclassical steady state capital)} \tag{18}$$

$$\frac{1+i^*}{1+\pi^*} = \left[\boldsymbol{\partial}_k f(k^*) - \delta \right] \equiv r^* \qquad \text{(Fisher equation on real vs. nominal rates)} \tag{19}$$

$$\boldsymbol{\partial}_m u(c^*, m^*) - \beta \left[\boldsymbol{\partial}_k f(k^*) + (1-\delta) \right] \boldsymbol{\partial}_c u(c^*, m^*) + \beta \frac{\boldsymbol{\partial}_c u(c^*, m^*)}{1+\pi^*} = 0 \tag{20}$$

and: $i^* \approx \pi^* + r^*$.

A few points to notice:

(1) Money in utility has no effect on steady state capital, output, etc. Neither does inflation rate.

(2) (Without proof) Lump sum taxes/transfers finance all changes in nominal money supply, so: $c^* = f(k^*) - \delta k^*$,
i.e. output-replacement of capital, makes sense as a resource constraint.

It seems like little has changed from the neoclassical growth model:

(1) Production function and discount rate determine steady state capital and real returns to capital.

(2) The nominal interest rates are simply the real interest plus the expected (exogenous) inflation, in particular the <u>long run</u>:

   (a) **Monetary neutrality:**
   The real equilibrium is independent of the money stock.

   (b) **Monetary super-neutrality:**
   The rate of growth of nominal money supply have no real effects.

However, in transaction dynamics or in the <u>short run</u>, these do not need to hold:

- Welfare contains m_t, so policy can affect welfare if you believe the utility.

- Long-run monetary neutrality is uncontroversial.

2.4 Recursive Problem

An alternative setup is to rewrite the MIU problem recursively (timing is different), where π' is next period:

$$V(w) = \max_{c,k,b,m} [u(c,m) + \beta V(w')] \tag{21}$$

$$\text{s.t. } w = c + k + m + b \tag{22}$$

$$w' = f(k) + (1-\delta)k + \frac{(1+i)b + m}{1+\pi'} \tag{23}$$

Trick: Combine all wealth into a single state variable, and use decisions of today into wealth tomorrow:

$$\Rightarrow \; k = w - c - m - b$$

$$V(w) = \max_{c,k,b,m} \Bigg[u(c,m) + \beta V \Bigg(f(w-c-m-b) + (1-\delta)(w-c-m-b) + \frac{(1+i)b+m}{1+\pi'} \Bigg) \Bigg] \tag{24}$$

- This is a 1-dimensional state space.
- Unconstrained dynamic programming problem.
  We can take FOCs from this and use the envelope condition (standard trick).

3 Dynamics of Money in Utility

3.1 Linearized Model

References: See Walsh 2.5.
We add stochastic productivity (i.e. RBC) with **endogenous** labor supply with n_t hours.

Production

$$y_t = e^{z_t} k_{t-1}^{\alpha} n_t^{1-\alpha} \tag{25}$$

Utility

$$u(c_t, m_t, 1-n_t) = \frac{\left[ac_t^{1-b} + (1-a)m_t^{1-b}\right]^{\frac{1-\Phi}{1-b}}}{1-\Phi} + \Psi\frac{(1-n_t)^{1-\eta}}{1-\eta} \tag{26}$$

The utility function is additively separable in leisure, to make labor supply/consumption easier, but <u>not</u> additively separable in m_t, c_t unless $\Phi = b$; a C.E.S. structure otherwise.

Parameters: a, b, Φ, Ψ, η.

Investment: Define x_t as the investment in period t:

$$x_t \equiv k_t - (1-\delta)k_{t-1} \tag{27}$$

Marginal utility of consumption:

$$\lambda_t \equiv \boldsymbol{\partial}_c u \tag{28}$$

Note: MUC independent of m_t if $\Phi = b$.

The **physical variables** of the model are: $m_t, c_t, n_t, y_t, k_t, x_t, z_t$.

The **price related variables** are: $r_t, \lambda_t, i_t, \pi_t$

3.2 Percent Deviation from Steady State

Define the $\hat{q}_t$ for variable q_t as:

$$q_t \equiv q^{SS}(1+\hat{q}_t) \tag{29}$$

where q^{SS} is the steady state.

Then $\hat{q}_t$ is the <u>percent deviation</u> from the steady state[1].
This defines the percent deviation variables: $\hat{m}_t, \hat{c}_t, \hat{n}_t, \hat{y}_t, \hat{k}_t, \hat{x}_t, \hat{z}_t, \hat{\lambda}_t$.

For the others: $\hat{r}_t \equiv r_t - r^{SS}, \hat{\pi}_t \equiv \pi_t - \pi^{SS}, \hat{i}_t \equiv i_t - i^{SS}$.

Note: for notational convenience, drop all hats so everything is in terms of **percent deviations**.

[1] See older notes on log linearization

3.3 Shocks

Shocks to Productivity: The productivity variable follows an AR(1) process, with the productivity shock mean as 0 (taken to exponent in Cobb-Douglas):

$$z_t = \rho_z z_{t-1} + e_t \tag{30}$$

where $|\rho_z| < 1,\ \sigma_e > 0,\ e_t \sim \mathrm{N}\left(0, \sigma_e^2\right)$.

Money Supply Shocks: From (11). Let the nominal money stock evolve <u>exogenously</u> with growth rate g_{mt}

$$\Rightarrow m_t = \left(\frac{1+g_{mt}}{1+\pi_t}\right) m_{t-1} \tag{31}$$

This gives the expression for real evolution of money supply. Note that if g_m and π are constant, then m is also constant. Let $u_t \equiv g_{mt} - g_m^{SS}$, then this can be log linearized (remembering that $\hat{\pi}_t = \pi_t - \pi^{SS}$ and $\hat{g}_{mt}$ would also be log linearized in differences) as

$$\hat{m}_t = \hat{m}_{t-1} - \hat{\pi}_t + u_t \tag{32}$$

We define a process of u_t directly instead of a specific process on g_{mt}:

$$u_t = \rho_u u_{t-1} + \phi z_{t-1} + \Psi_t,\ \ \Psi_t \sim \mathrm{N}\left(0, \sigma_u^2\right) \tag{33}$$

where ϕ can respond to the real changes.

Then the set of equilibrium conditions, after <u>log linearizing</u> (see Section 2.7 of Walsh) and dropping the hats, are given by the following set of equations:

$$\lambda_t = \Omega_1 c_t + \Omega_2 m_t \qquad \text{(marginal utility of consumption)} \quad (34)$$

where: $\Omega_1 \equiv [(b-\Phi)\gamma - b]$,

$$\Omega_2 \equiv (b-\Phi)(1-\gamma),$$

$$\gamma \equiv \frac{a(c^{SS})^{1-b}}{[a(c^{SS})^{1-b} + (1-a)(m^{SS})^{1-b}]}$$

$$\left(\frac{x^{SS}}{k^{SS}}\right) x_t = k_t - (1-\delta)k_{t-1} \qquad \text{(savings definition in \% deviation)} \quad (35)$$

$$y_t = \alpha k_{t-1} + (1-\alpha)n_t + z_t \qquad \text{(production)} \quad (36)$$

$$\left(\frac{y^{SS}}{k^{SS}}\right) y_t = \left(\frac{c^{SS}}{k^{SS}}\right) c_t + \delta x_t \qquad \text{(resource constraint)} \quad (37)$$

$$r_t = \alpha \left(\frac{y^{SS}}{k^{SS}}\right) \left(\mathbb{E}_t\left[y_{t+1}\right] - k_t\right) \qquad \text{(marginal rate of capital and real return)} \quad (38)$$

$$\lambda_t = \mathbb{E}_t\left[\lambda_{t+1}\right] + r_t \qquad \text{(pricing kernel)} \quad (39)$$

$$y_t - n_t = -\lambda_t + \eta\left(\frac{n_{SS}}{1-n_{SS}}\right) \qquad \text{(labor supply)} \quad (40)$$

$$z_t = \rho_z z_{t-1} + e_t \qquad \text{(productivity shock)} \quad (41)$$

$$i_t = r_t + \mathbb{E}_t\left[\pi_{t+1}\right] \qquad \text{(no-arbitrage for nominal bonds)} \quad (42)$$

$$m_t - c_t = -\left(\frac{1}{b}\right)\left(\frac{1}{i^{SS}}\right) i_t \qquad \text{(money demand)} \quad (43)$$

$$m_t = m_{t-1} - \pi_t + u_t \qquad \text{(evolution of real money stock)} \quad (44)$$

$$u_t = \rho_u u_{t-1} + \phi z_{t-1} + \Psi_t \qquad \text{(money supply shock)} \quad (45)$$

3.4 Separability

- If $\Phi = b$, then λ_t is independent of m_t.
- There are no real effects, i.e. we can solve $k_t, c_t, x_t, y_t, r_t, \lambda_t$ separately. Then the exogenous c_t, r_t can be used to solve for the monetary parts of the equilibrium. Hence, if separable, there will be monetary super-neutrality *even* on transition dynamics.

3.5 Anticipated versus Unanticipated for $\Phi = b$

- Let $\rho_u = \phi = 0$, so u_t is only an i.i.d. shock.
- So, u_t are only *unanticipated*.

- But the shock then has no effect on future money growth, and so $\mathbb{E}_t [\pi_{t+1}]$ doesn't change. Therefore, interest rates don't respond.

- Nothing happens in the real economy: just a 1-period change in inflation, and a level change in money supply.