MEAN AND APPARENT PLACE COMPUTATIONS IN THE NEW IAU SYSTEM. III. APPARENT, TOPOCENTRIC, AND ASTROMETRIC PLACES OF PLANETS AND STARS

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ABSTRACT

A set of algorithms is presented for computing the apparent directions of planets and stars on any date to milliarcsecond precision. The expressions are consistent with the new IAU astronomical reference system for epoch J2000.0. The algorithms define the transformation between fundamental reference data, such as star and radio-source catalogs and planetary ephemerides, and astrometric observables.

I. INTRODUCTION

This paper presents a set of algorithms for computing the apparent, topocentric, and astrometric places of planets and stars within the new IAU fundamental astronomical reference system. The new IAU reference system is that defined by the IAU (1976) System of Astronomical Constants and related resolutions passed by the IAU in 1979 and 1981. The new system has been in use since 1984 in the international ephemerides and by the Earth rotation and precise-time community. The new FK5 fundamental star catalog (in preparation; available in machine-readable form) embodies this system. It is generally understood that in astronomical publications, data referred to the epoch designated "J2000.0," which is Julian Date 2451545.0, are consistent with this system; proper use of such data requires adherence to the various IAU resolutions mentioned above in any calculations or procedures. The system has been described by Kaplan (1981), Melbourne *et al.* (1983), and in the Supplement to the *Astronomical Almanac* of 1984.

This paper is part of a four-paper series describing the implications of the new system for astrometric computations and the data derived therefrom. Paper I (Smith *et al.* 1989) describes the computations relevant to the construction of fundamental astrometric catalogs, and, in particular, the conversion of data from old star catalogs to the new system. Paper II (Yallop *et al.* 1989) presents the results of Paper I in a simple-to-apply 6×6 matrix form for transforming data from the FK4 (B1950.0) system to the FK5 (J2000.0) system. This paper, Paper III, describes the algorithms for obtaining angular "observables" from reference data. In particular, this paper describes the procedures for obtaining apparent, topocentric, and astrometric places of planets and stars. Paper IV (Hughes *et al.* 1989) explores the computational considerations at the milliarcsecond level of precision.

The developments presented here draw heavily upon previous work by Scott and Hughes (1964), Emerson (1973), Mueller (1969), and Murray (1981,1983). Sections III and IV follow (with some modifications) the procedures outlined in the *Astronomical Almanac* by Yallop, where numerical examples may be found. The individual algorithms for precession, aberration, etc., are not new, but the application of the recent IAU resolutions to these computations has received little attention. More importantly, this paper presents a unified development of a number of frequently required astrometric computations that heretofore have been scattered throughout the literature and considered only within a narrow context. The presentation format is intended to facilitate the practical application of the algorithms even by nonspecialists.

II. BASIC CONCEPTS

a) Reference Points

A compiled astrometric star catalog typically consists of tabulations of "mean places," associated proper motions, and other relevant information for some appropriately chosen stars, applicable to a particular reference epoch such as J2000.0. The mean place of a star can best be described as representing the direction of the star as it would be seen from the solar system barycenter at the reference epoch, in the coordinate system defined by the Earth's mean equator and equinox at the reference epoch, if the masses of the Sun and other solar system bodies were negligible. The proper motion is essentially the time derivative of the mean place. The mean place is thus a fundamental reference point for star catalogs but it bears little relation to the position at which a star will be observed from Earth on a particular date.

The transformation described in Papers I and II is a "mean place to mean place" conversion. That is, it transforms catalog mean places and proper motions from one reference epoch (B1950.0) to another (J2000.0). The complexity of the transformation results from the changes in constants, timescales, and procedures mandated by the IAU for epoch J2000.0 catalog data. A specific complication arises from the fact that mean places in star catalogs of standard epoch B1950.0 invariably were adjusted for the so-called "E terms" of aberration and therefore were not true mean places according to the above definition. For star catalogs of standard epoch J2000.0, the listed star positions are not to be contaminated by the E terms. (See Paper I for a more complete discussion.) Therefore, in the developments in this paper, we assume that a star's catalog mean place is its true mean place without any E term adjustment.

Another reference point, the "apparent place" of a star, has been defined which is more pertinent to observations on a particular date. The apparent place of a star represents the position of the star as it would be seen from the center of mass of the Earth at some date, in the coordinate system defined by the Earth's true equator and equinox of date, if the Earth and its atmosphere were transparent and nonrefracting. The transformation between the catalog mean place of a star and its apparent place on some particular date is an astrometric computation of fundamental importance.

Of course, the word "star," as used in this paper, is a generic term referring to any observable body outside of the solar system; included are extragalactic radio sources, pulsars, infrared objects, etc. However, except for a few pulsars, proper-motion information is not available (at the present time) for these objects.

For a planet or other solar system body there is, of course, no catalog mean place; the corresponding data are ephemerides, with respect to the solar system barycenter, in some well-defined nonrotating rectangular coordinate system. The ephemeris, whatever its actual representation, can be thought of as a time series of position vectors, each vector representing the position of the body at some date within the barycentric reference system. The concept of apparent place, as stated above for stars, is equally applicable to solar system bodies. Again, the transformation between the barycentric ephemeris of a solar system body and its apparent place on some particular date is of great practical importance.

Since we do not actually observe from the center of the Earth, we must also consider "topocentric place," which is the apparent direction of an object as it would be seen by a real observer on the surface of the Earth, neglecting atmospheric refraction. More precisely, the topocentric place represents the position of a star or planet as it would be seen from a specific location on Earth at some date and time, in the coordinate system defined by the Earth's true equator and equinox of date, if the atmosphere were nonrefracting.

Two other reference points, which we term the "virtual place" and "local place" of a star or planet, are useful when measurements of an object's position are to be made differentially with respect to other objects in the same general direction. The virtual and local place correspond to the apparent and topocentric place but are simpler to compute, since calculations relating to the orientation of the final coordinate system are omitted. The virtual place is thus simply an apparent place expressed in the coordinate system of the original reference data, that is, the coordinate system defined by the Earth's mean equator and equinox at the reference epoch. The local place is simply a topocentric place expressed in the coordinate system of the reference data. For differential measurements the coordinate system in which the object positions are expressed is relatively unimportant; in practice, its orientation is considered arbitrary and subject to determination from the observations themselves. The virtual and local place can also be useful in the computation of the circumstances of coordinate-system independent phenomena.

Traditionally, in differential work, if the field of view is sufficiently small, some physical effects that alter an object's apparent direction have been neglected. They are assumed to apply equally to all objects within a small field so that differential measurements are insensitive to them. These latter assumptions effectively define another reference point, called the "astrometric place," which is even simpler to compute than the virtual or local place. Although the astrometric place is well rooted in astrometry because of its convenience and simplicity, given current computing power it probably should be avoided in the reduction of high-precision observations.

b) Fundamental Coordinate System

In the following developments, it is assumed that fundamental reference data such as star positions and proper motions or the barycentric ephemeris of a planet are expressed in what we call the "space-fixed" coordinate system (or frame). The spatial origin of the space-fixed system is the solar system barycenter, and the orientation of the coordinate axes is defined by the Earth's mean equator and equinox at the standard epoch t_0, which for this paper is assumed to be J2000.0, JD 2451545.0 in the TDB timescale (see below). Specifically, our fundamental space-fixed system is a right-handed system oriented such that the xy plane is parallel to the Earth's mean equator at epoch t_0, the x axis points toward the mean equinox of t_0, and the z axis points towards the mean north celestial pole of t_0. Either spherical or rectangular coordinates may be used within this system. The positions listed in recently published star catalogs (such as the FK5) or solar system ephemerides (such as the DE 200 ephemeris from the Jet Propulsion Laboratory) are referred to such a coordinate system.

This space-fixed coordinate system is considered to be inertial in the classical (or special relativistic) sense; the solar-system barycenter is assumed to be unaccelerated. Obviously, we can introduce other, similar inertial systems: those with coordinate axes identically oriented and moving with a constant velocity with respect to the fundamental space-fixed system. Specific systems of the latter variety that are useful in the following developments are those with origins that coincide, at a specific epoch, in both position and velocity with that of either the center of the Earth or the observer. These coordinate systems suffice for the developments presented here, which are basically classical (Euclidian/Newtonian); relativity appears as a post-Newtonian approximation in only a few places.

c) Timescales

Throughout the following sections, reference is made to two fundamental timescales defined by the IAU: TDB, Barycentric Dynamical Time, and TDT, Terrestrial Dynamical Time. See Kaplan (1981) for a brief description of these timescales as provided by the relevant IAU commissions, or Winkler and Van Flandern (1977) or Moyer (1981a,b) for more complete discussions. TDB is a theoretical timescale which is a measure of "coordinate time" (in the terminology of general relativity) for a coordinate system whose spatial origin is the solar system barycenter, such as our space-fixed frame. As such, TDB cannot be maintained by any real clock; it is mathematically derived from TDT (expressions are given below). TDT, on the other hand, is for practical purposes readily available to high precision. Practically, TDT is simply a constant offset from TAI, International Atomic Time: TDT = TAI + $32^{\mathrm{s}}.184$. TAI, in turn, is an integral number of seconds offset from UTC, Coordinated Universal Time. UTC is widely distributed since it is the basis for the worldwide system of civil time. The TAI–UTC offset is 24 s in 1989; this offset advances whenever a "leap second" is introduced into UTC. Thus, during 1989 TDT is simply UTC + $56^{\mathrm{s}}.184$.

d) Algorithm Overview

We develop the transformations between the reference points defined above beginning with stellar or planetary reference data assumed to be expressed in the fundamental space-fixed system. We use position vectors in rectangular coordinates throughout; all three dimensions are considered relevant. It is clear that for a solar system body, the barycentric ephemeris represents its motion through space with respect to the space-fixed system. It may not be quite so obvious that the catalog mean place and proper motion of a star, together with its parallax and radial velocity (if known), can also be transformed into a representation of its motion through space with respect to the three-dimensional space-fixed reference frame. Deferring for now the discussion of how this might be done, it is clear that if the space motion of a star and a solar system body can be similarly represented, then the algorithms used for the transformations between the reference points must be virtually identical in the two cases.

We have implemented transformation algorithms that take advantage of this fact. The algorithms are implemented as a system of FORTRAN subroutines which have been widely distributed and used. Solar system bodies and stars are treated similarly throughout; the distinction between the two cases results only from the different schemes required to obtain the three-dimensional position vector of the object in space at the relevant time. The formulation used is vector- and matrix-based, modular, and does not employ spherical trigonometry or Besselian Day Numbers at any point. Since the same basic subroutines are used for all objects, both within and outside of the solar system, the position vectors formed and operated on by these routines place each relevant object at its actual distance (in AU) from the solar system barycenter. Objects at unknown distance (parallax undetermined or zero to within the accuracy of measurement) are placed on the "celestial sphere," herein defined to have the somewhat arbitrary radius of 10 Mpc (2.06×10^{12} AU).

In Sec. III, immediately following, we describe in detail the algorithms used to compute the apparent place of a planet, given its ephemeris with respect to the solar system barycenter. In Sec. IV is a description of the modifications required in the algorithms to compute the apparent place of a star, given its catalog mean place and proper motion at some reference epoch (e.g., J2000.0). Section V is a discussion of some considerations involved in the computer implementation of these algorithms. Then, in Sec. VI, topocentric place computations are described. In Sec. VII, virtual place, local place, and astrometric place computations for stars and planets are covered. Finally, in Sec. VIII, we discuss a general method of computing geometry-sensitive observables for modern high-precision observing techniques.

All vectors used below are column vectors.

III. APPARENT PLACE ALGORITHM FOR PLANETS

The algorithm used to compute the apparent place of a planet or other solar system body at an epoch of observation t', given its ephemeris with respect to the solar system barycenter, can be succinctly represented as

$$\mathbf{u}(t') = \mathbf{N}(t)\mathbf{P}(t)\, f\{g[\mathbf{u}(t-\tau) - \mathbf{E}(t)]\}, \tag{1}$$

where:

t' = epoch of observation, in the TDT timescale,
t = epoch of observation, in the TDB timescale,
τ = light travel time from the planet to the Earth, in the TDB timescale, for light arriving at the epoch of observation t,
$\mathbf{u}(t-\tau)$ = position of the planet with respect to the solar system barycenter at epoch $t-\tau$,
$\mathbf{E}(t)$ = position of the Earth with respect to the solar system barycenter at the epoch of observation t,
$g(...)$ = function representing the gravitational deflection of light,
$f(...)$ = function representing the aberration of light,
$\mathbf{P}(t)$ = precession rotation matrix, evaluated for the epoch of observation t,
$\mathbf{N}(t)$ = nutation rotation matrix, evaluated for the epoch of observation t, and
$\mathbf{u}(t')$ = apparent place of the planet at the epoch of observation t', represented as a three-dimensional position vector with origin at the center of mass of the Earth.

This expression is schematic; the full functional forms of f and g, the elements of the $\mathbf{P}$ and $\mathbf{N}$ matrices, and other auxiliary calculations are not indicated. In typical computer implementations, the algorithm is actually evaluated in a stepwise fashion by successive calls to a sequence of subroutines, each of which handles one particular aspect of the problem. The formulations used at each step are described in detail below. Since most of the steps are also used in the computation of the apparent places of stars (see Sec. IV) the formulations and discussion are presented in as much generality as possible, with possible simplifications noted where appropriate.

a) Determine Relevant Time Arguments

Step (a): Express the epoch of observation t' as a TDT Julian Date.

Step (b): Compute T', the number of Julian centuries in the TDT timescale from J2000.0 TDT (JD 2451545.0 TDT):

$$T' = (t' - 2451545.0)/36525. \tag{2}$$

Step (c): Compute the mean anomaly m of the Earth in its orbit, in radians, at the epoch of observation:

$$m = (357.5 + 35999.1\ T')2\pi/360. \tag{3}$$

Step (d): Compute s, the difference, in seconds, between the clock readings in the two timescales (in the sense TDB − TDT), and t, the TDB Julian Date corresponding to the epoch of observation:

$$\begin{aligned} s &= 0.001658 \sin(m + 0.01671 \sin m) \\ t &= t' + s/86400, \end{aligned} \tag{4}$$

where the formula for s is an adaptation from Moyer (1981b).

Step (e): Compute T, the number of Julian centuries in the TDB timescale elapsed since J2000.0 TDB (JD 2451545.0 TDB):

$$T = (t - 2451545.0)/36525. \tag{5}$$

In the expression for s, lunar and planetary terms of order 10^{-5} s have been ignored. Furthermore, the expression for m (step c) strictly requires a time argument in the TDB, not the TDT, timescale (see Moyer (1981a,b) for a complete discussion). However, the algorithm given above is much more precise than is required for the computation of apparent places of stars and most solar system bodies. For stellar apparent places, one can set $s = 0$, $t = t'$, and $T = T'$ with negligible error. For solar system bodies, the same approximation can be used for all bodies except the Moon and close-approaching comets and asteroids, where the error in using the $t = t'$ approximation may approach 1 mas in very unfavorable circumstances.

b) Obtain Ephemeris Data for the Earth and Sun

Step (f): Enter the ephemeris of positions of solar system bodies and extract the position and velocity vectors of the Earth for time t, with respect to the solar system barycenter. (The position and velocity of the center of mass of the Earth, not that of the Earth–Moon barycenter, are the relevant quantities.) In the following developments, it is assumed that the vector components are in units of AU and AU/day, respectively, and are in the coordinate system defined by the Earth's mean equator and equinox of t_0, the reference epoch J2000.0. Call these two vectors $\mathbf{E}(t)$ and $\dot{\mathbf{E}}(t)$.

Step (g): Enter the ephemeris of positions of solar system bodies and extract the position vector of the Sun for time t, with respect to the solar system barycenter. The vector components should be in AU and must be in the coordinate system defined by the Earth's mean equator and equinox of the reference epoch. Call this vector $\mathbf{S}(t)$; form the vector $\mathbf{E}'(t) = \mathbf{E}(t) - \mathbf{S}(t)$, which represents the heliocentric position of the Earth at the time t of observation.

Alternatively, enter the solar system ephemeris and directly extract the heliocentric position vector $\mathbf{E}'(t)$ of the Earth for time t of observation.

The new standard ephemeris of the major bodies in the solar system is the Jet Propulsion Laboratory ephemeris designated DE 200 (Standish 1982). The positions of the Sun, Moon, and planets given in the *Astronomical Almanac* and other international almanacs are now obtained from this ephemeris. Values for the vector components of $\mathbf{E}$, $\dot{\mathbf{E}}$, and $-\mathbf{E}'$ at 1-day intervals are tabulated in the *Astronomical Almanac*. A set of analytical planetary theories (closed-form expressions) fitted to DE 200 has been developed by Bretagnon (1982).

The barycentric position of the Earth is used to form the geocentric position vector of the body of interest (see Secs. III *d* and III *e* below); the barycentric velocity of the Earth is used in the aberration computation (Sec. III *g*); and the heliocentric position of the Earth is used in the computation of the relativistic gravitational deflection of light (Sec. III *f*).

c) Obtain Ephemeris Data for the Planet

Step (h): Enter the ephemeris of positions of solar system bodies and extract the position vector of the planet for time t, with respect to the solar system barycenter. The vector components should be in AU and must be in the coordinate system defined by the Earth's mean equator and equinox of t_0, the reference epoch J2000.0. Call this vector $\mathbf{u}(t)$.

d) Compute Geometric Distance Between Earth and Planet

Step (i): Compute d, the geometric distance between the positions of the centers of mass of the planet and the Earth at time t, in AU:

$$d = |\mathbf{u}(t) - \mathbf{E}(t)|. \tag{6}$$

The geometric distance d is the quantity tabulated in the *Astronomical Almanac* as the "true distance" of solar system bodies. Using d, compute τ, a first approximation to the light travel time between the planet and the Earth:

$$\tau = d/c', \tag{7}$$

where c' is the speed of light expressed in AU/day; it is obtained from $86400/\tau_A$, where τ_A is the light-time for unit distance (1 AU) in seconds. In the IAU (1976) System, $\tau_A = 499.004782$ s, so $c' = 173.144633$ AU/day.

e) Compute Geocentric Position of Planet, Accounting for Light Time

Step (j): Enter the ephemeris of positions of solar system bodies and extract the position vector of the planet for time $t-\tau$, with respect to the solar system barycenter. This vector is $\mathbf{u}(t-\tau)$.

Step (k): Form the vector $\mathbf{u}_3$:

$$\mathbf{u}_3 = \mathbf{u}(t-\tau) - \mathbf{E}(t). \tag{8}$$

The vector $\mathbf{u}_3$ represents an approximation to the geocentric position of the planet as it would be seen from Earth at the epoch of observation t.

Step (l): Next, compute τ', a better approximation to the light travel time between the planet and the Earth:

$$\tau' = |\mathbf{u}_3|/c'. \tag{9}$$

Compare τ' with τ; if they are identical within some small tolerance, continue to step (m). If they are not, then replace the value of τ with the value of τ' ($\tau' \to \tau$) and repeat steps (j)–(l) until the light time converges to within the tolerance permitted. Since the speed of bodies in the solar system is small compared to the speed of light, this process converges rapidly.

The tolerance permitted depends on the precision desired in the final coordinates and the apparent angular speed of the body as viewed from Earth. The most rapidly moving objects in the sky are the Moon (angular rate approximately 0.5 arcsec/s), Mercury (angular rate at inferior conjunction 0.05 arcsec/s), and the Sun (angular rate 0.04 arcsec/s). However, occasionally an Earth-crossing asteroid or comet may exceed these rates for short periods of time. For a computational precision of one milliarcsecond, therefore, the light time convergence tolerance must be $0.002\ \text{s} = 2\times10^{-8}$ days or less; we use 1×10^{-8} days.

Yallop's development in the *Astronomical Almanac* uses a more complex formula in place of Eq. (9) above. Yallop's formula is essentially that of Murray (1981) and contains a term accounting for the extra relativistic delay due to the Sun's gravitational field. However, the relativistic term adds no more than a few tenths of a millisecond to the light time in the worst case (an outer planet at superior conjunction near the Sun's limb) and can affect the angular coordinates of a solar-system body only at the microarcsecond level. Therefore we have neglected this term; without it, the formula in the *Astronomical Almanac* reduces to Eq. (9). The term is obviously important for radar-ranging observations, but these require a more complex algorithm anyway since in this case it is the round-trip light time that is the observable.

Step (m): Set $\mathbf{u}_2 = \mathbf{u}(t-\tau)$, where τ is the final (converged) value of the light travel time. The vector $\mathbf{u}_2$ represents the position of the body at time $t-\tau$ with respect to the barycenter of the solar system.

f) Evaluate and Include the Effect of the Relativistic Deflection of Light in the Sun's Gravitational Field

Step (n): Form the vector $\mathbf{q}$:

$$\mathbf{q} = \mathbf{E}'(t) + \mathbf{u}_3, \tag{10}$$

where $\mathbf{E}'(t)$ is the heliocentric position of the Earth from Sec. III *b* (step (g)). The vector $\mathbf{q}$ represents the heliocentric position of the body. Alternatively, if the vector $\mathbf{S}(t)$ is available (from step (g)), the vector $\mathbf{q}$ can be formed using

$$\mathbf{q} = \mathbf{u}_2 - \mathbf{S}(t). \quad (11)$$

Step (o): Form the following unit vectors:

$$\hat{\mathbf{u}} = \mathbf{u}_3/|\mathbf{u}_3|,$$
$$\hat{\mathbf{e}} = \mathbf{E}'(t)/|\mathbf{E}'(t)|, \quad (12)$$
$$\hat{\mathbf{q}} = \mathbf{q}/|\mathbf{q}|,$$

and the following dimensionless scalar quantities:

$$g_1 = \frac{2G}{c^2|\mathbf{E}'(t)|A} = \frac{2k^2}{c'^2|\mathbf{E}'(t)|},$$
$$g_2 = 1 + \hat{\mathbf{q}}\cdot\hat{\mathbf{e}}, \quad (13)$$

where G is the heliocentric gravitational constant, c and c' are the speed of light (in m/s and AU/day, respectively), A is the number of meters in 1 AU, and k is the Gaussian gravitational constant. The values of these constants, from the IAU (1976) System, are: $G = 1.327\,124\,38 \times 10^{20}$ m^3 s^{-2}, $c = 299\,792\,458$ m/s, $A = 1.495\,978\,70 \times 10^{11}$ m, and $k^2 = (0.017\,202\,098\,95)^2$ AU^3 day^{-2}. From step (i), $c' = 173.144\,633$ AU/day. The value of g_1 is always close to 2×10^{-8}, while g_2 varies between 0 and $+2$.

Step (p): The deflected geocentric direction of the body is then given by the vector $\mathbf{u}_4$:

$$\mathbf{u}_4 = |\mathbf{u}_3| \left\{ \hat{\mathbf{u}} + \frac{g_1}{g_2} [(\hat{\mathbf{u}}\cdot\hat{\mathbf{q}})\hat{\mathbf{e}} - (\hat{\mathbf{e}}\cdot\hat{\mathbf{u}})\hat{\mathbf{q}}] \right\}. \quad (14)$$

The algorithm for the deflection of light given above is that of Yallop as given in the *Astronomical Almanac* (1984, p. B37), which is an adaptation of Murray's (1981) formulas. The isotropic metric has been assumed. Only the Sun's gravitational field has been included; each of the planets causes a similar effect that is smaller by a factor equal to the ratio of planet's mass to that of the Sun (1/1047 for Jupiter). The gravitational field of the Earth, also ignored here, can deflect light by a few tenths of a milliarcsecond for ground-based observers.

It should be mentioned that using the vector $\mathbf{E}'(t)$ (and $\mathbf{S}(t)$) in the relativistic deflection computation introduces a minor approximation resulting from the use of the barycentric position of the Sun at the epoch of observation t. The resulting error cannot exceed 0.1 mas in the worst case (object observed at the limb of the Sun with barycentric motion of Sun orthogonal to line of sight) and is generally much less. Furthermore, the deflection algorithm itself results from a first-order development which assumes small deviations of the photon track from a straight line in Euclidian space; the error in neglecting second-order effects can reach about 0.5 mas for an object observed at the Sun's limb (Kammeyer 1988).

Figure 1 shows the magnitude of the relativistic deflection of light, as viewed from Earth, for planets and stars as a function of the geocentric angular separation of the observed body from the center of the Sun. The figure was produced from Eq. (14) assuming circular orbits for the planets.

If we are interested in only the magnitude of the gravitational deflection (its direction as viewed from Earth is always radially outward from the Sun), then Eq. (14) can be reduced to a very simple form. Under the assumption that the Earth's distance from the Sun is constant, Eq. (14) is equivalent to the following formula for the magnitude of the small angle describing the gravitational deflection:

$$\phi = g_1 \tan(\psi/2), \quad (15)$$

where $g_1 = 0.004\,07$ arcsec, and ψ is the *heliocentric* angular separation of the Earth and the observed body. The variation in the Earth–Sun distance modulates the value of g_1 by less

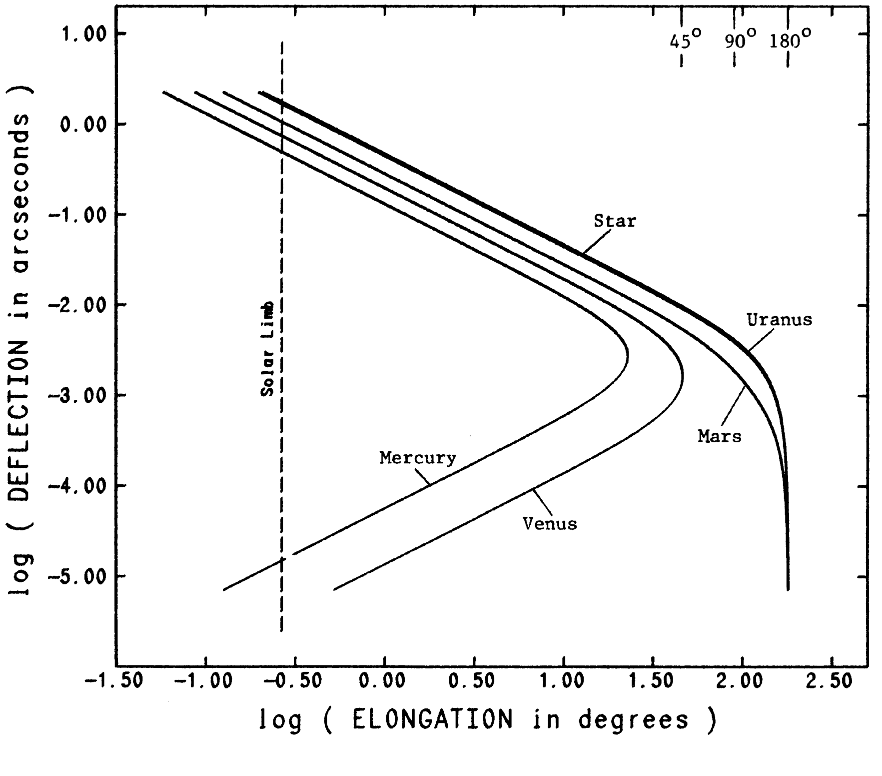

FIG. 1. Relativistic deflection of light in Sun's gravitational field for several bodies. Graph shows the apparent geocentric deflection as a function of the apparent geocentric elongation of the body from the Sun. From left to right, curves are shown for Mercury, Venus, Mars, Uranus, and a star; the curves for Uranus and the star overlap on this scale. For Mercury and Venus, the lower part of the curve represents the deflection when the planet is near inferior conjunction; the upper part, superior conjunction.

than 2%. This simple form of the gravitational deflection formula has been noted previously by Shapiro (1967) and by Fukushima (1982). In this form, the deflection is not explicitly dependent on the distance of the emitting body from the Sun or the Earth. Therefore, to an observer on Earth, the apparent gravitational deflection is the same for all objects that lie anywhere on a given line which extends radially outward from the Sun. This nonintuitive result holds regardless of the orientation of the line with respect to the Earth.

Both Eqs. (14) and (15) have an indeterminacy for light originating beyond the Sun on the extension of the Sun–Earth line; bodies there are hidden by the Sun's disk and unobservable in any event. For these bodies or the Sun itself, the deflection can be considered to be zero so that $\mathbf{u}_4 = \mathbf{u}_3$.

g) Evaluate and Include the Aberration of Light

Step (q): Form the following scalar quantities:

$$\begin{aligned}
\tau &= |\mathbf{u}_4|/c', \\
\beta &= |\dot{\mathbf{E}}(t)|/c', \\
\cos D &= \frac{\mathbf{u}_4 \cdot \dot{\mathbf{E}}(t)}{|\mathbf{u}_4||\dot{\mathbf{E}}(t)|}, \\
\gamma^{-1} &= \sqrt{1-\beta^2}, \\
f_1 &= \beta \cos D, \\
f_2 &= [1 + f_1/(1+\gamma^{-1})]\tau.
\end{aligned} \tag{16}$$

In the above, c' is the speed of light expressed in AU/day, carried from step (i).

Step (r): The aberrated geocentric direction of the body is then given by the vector $\mathbf{u}_5$:

$$\mathbf{u}_5 = [\gamma^{-1}\mathbf{u}_4 + f_2\dot{\mathbf{E}}(t)]/(1+f_1). \tag{17}$$

Again, we base our algorithms on Murray's (1981) development.

The above algorithm includes relativistic terms, which are of order 1 mas. Therefore, for many applications the much simpler classical formula may be used:

$$\mathbf{u}_5 = \mathbf{u}_4 + \dot{\mathbf{E}}(t)\tau. \tag{18}$$

h) Apply Precession to the Coordinate System

Step (s): Evaluate the three fundamental precession angles:

$$\begin{aligned}
\zeta &= 2306.2182\ T + 0.30188\ T^2 + 0.017998\ T^3, \\
z &= 2306.2181\ T + 1.09468\ T^2 + 0.018203\ T^3, \\
\theta &= 2004.3109\ T - 0.42665\ T^2 - 0.041833\ T^3.
\end{aligned} \tag{19}$$

The above angles are expressed in arcseconds. T (from step (e)) is the number of Julian centuries in the TDB timescale between J2000.0 and the epoch of observation. The above formulas apply only to precession of coordinates from reference epoch J2000.0; they are adaptations of more general formulas given by Lieske *et al.* (1977) and Lieske (1979).

Step (t): Transform the coordinate system to that defined by the mean Earth equator and equinox at the epoch of observation, by forming the precession rotation matrix $\mathbf{P}$ and applying it to the vector $\mathbf{u}_5$:

$$\mathbf{P} = \begin{bmatrix}
\cos\zeta\cos\theta\cos z - \sin\zeta\sin z & -\sin\zeta\cos\theta\cos z - \cos\zeta\sin z & -\sin\theta\cos z \\
\cos\zeta\cos\theta\sin z + \sin\zeta\cos z & -\sin\zeta\cos\theta\sin z + \cos\zeta\cos z & -\sin\theta\sin z \\
\cos\zeta\sin\theta & -\sin\zeta\sin\theta & \cos\theta
\end{bmatrix}, \tag{20}$$

$$\mathbf{u}_6 = \mathbf{P}\mathbf{u}_5. \tag{21}$$

The above rotation matrix is taken from the *Explanatory Supplement* to the Ephemeris (1961), pp. 31 and 34.

i) Apply Nutation to the Coordinate System

Step (u): Obtain the two fundamental nutation angles $\Delta\psi$ and $\Delta\epsilon$. These may be obtained by interpolating the daily values given in the *Astronomical Almanac* to the epoch of observation t. Alternatively, the two 106 term series for $\Delta\psi$ and $\Delta\epsilon$ may be evaluated for the epoch of observation. The complete series and related formulas are given in Seidelmann (1982), Kaplan (1981), and the Supplement to the *Astronomical Almanac* for 1984. The theory of nutation is that of Wahr (1981), which has been adopted by both the IAU and the IUGG.

Step (v): Compute the values for the mean obliquity of the ecliptic ϵ and the true obliquity of the ecliptic ϵ' for the epoch of observation:

$$\epsilon = 84381.448 - 46.8150\ T - 0.00059\ T^2 + 0.001813\ T^3, \tag{22}$$

$$\epsilon' = \epsilon + \Delta\epsilon. \tag{23}$$

The above angles are expressed in arcseconds. The expressions are from Lieske *et al.* (1977); the value of the obliquity at J2000.0, 23°26′21.″448, is from the IAU (1976) System.

Step (w): Transform the coordinate system to that defined by the true Earth equator and equinox at the epoch of observation, by forming the nutation rotation matrix $\mathbf{N}$ and applying it to the vector $\mathbf{u}_6$:

$$\mathbf{N} = \begin{bmatrix}
\cos\Delta\psi & -\sin\Delta\psi\cos\epsilon & -\sin\Delta\psi\sin\epsilon \\
\sin\Delta\psi\cos\epsilon' & \cos\Delta\psi\cos\epsilon\cos\epsilon' + \sin\epsilon\sin\epsilon' & \cos\Delta\psi\sin\epsilon\cos\epsilon' - \cos\epsilon\sin\epsilon' \\
\sin\Delta\psi\sin\epsilon' & \cos\Delta\psi\cos\epsilon\sin\epsilon' - \sin\epsilon\cos\epsilon' & \cos\Delta\psi\sin\epsilon\sin\epsilon' + \cos\epsilon\cos\epsilon'
\end{bmatrix}, \tag{24}$$

$$\mathbf{u}_7 = \mathbf{N}\mathbf{u}_6. \tag{25}$$

The above rotation matrix is taken from Emerson (1973). Note that the above is the complete rotation matrix for nutation without any approximations; the approximate matrix given on p. 43 of the *Explanatory Supplement* to the Ephemeris cannot be used where precisions of better than a half milliarcsecond are required.

One can use the combined precession-nutation matrices given in the *Astronomical Almanac* in place of steps (s)–(w) above. The precession–nutation matrix given in the *Astronomical Almanac* is $\mathbf{R} = \mathbf{NP}$. The values of the elements of this matrix must be interpolated to the epoch of observation. Then, the following formula can be used in place of steps (s)–(w) above:

$$\mathbf{u}_7 = \mathbf{R}\mathbf{u}_5. \tag{26}$$

j) Express Position Vector of the Body in Spherical Coordinates

Step (x): Compute the object's apparent right ascension α' and declination δ', using the three components of vector $\mathbf{u}_7$, $u_7(x)$, $u_7(y)$, and $u_7(z)$:

$$\begin{aligned} w &= \sqrt{u_7(x)^2 + u_7(y)^2}, \\ \alpha' &= \arctan\left(\frac{u_7(y)}{u_7(x)}\right), \\ \delta' &= \arctan\left(\frac{u_7(z)}{w}\right). \end{aligned} \tag{27}$$

Most computers have double-argument arctangent functions which will provide the correct quadrant if the numerators and denominators given above are entered separately. The resulting values for α' and δ' represent the geocentric apparent place of the object at the epoch of observation.

IV. APPARENT PLACE ALGORITHM FOR STARS

The algorithm used to compute the apparent place of a star at an epoch of observation t', given its mean place, proper motion, and other data (as available) at reference epoch t_0, can be succinctly represented as

$$\mathbf{u}(t') = \mathbf{N}(t)\mathbf{P}(t) f\{g[\mathbf{u}(t_0) + \dot{\mathbf{u}}(t_0)\cdot(t - t_0) - \mathbf{E}(t)]\} \tag{28}$$

where:

t' = epoch of observation, in the TDT timescale,
t = epoch of observation, in the TDB timescale,
t_0 = reference epoch (e.g., J2000.0) of the star catalog, in the TDB timescale,
$\mathbf{u}(t_0)$ = catalog mean place of the star at the reference epoch t_0, represented as a three-dimensional position vector with origin at the solar system barycenter,
$\dot{\mathbf{u}}(t_0)$ = space motion of the star at the reference epoch t_0, obtained from the catalog proper motion, parallax, and radial-velocity values, represented as a three-dimensional velocity vector with origin at the solar system barycenter,
$\mathbf{E}(t)$ = position of the Earth with respect to the solar system barycenter at the epoch of observation t,
$g(...)$ = function representing the gravitational deflection of light,
$f(...)$ = function representing the aberration of light,
$\mathbf{P}(t)$ = precession rotation matrix, evaluated for the epoch of observation t,
$\mathbf{N}(t)$ = nutation rotation matrix, evaluated for the epoch of observation t, and
$\mathbf{u}(t')$ = apparent place of the star at the epoch of observation t', represented as a three-dimensional position vector with origin at the center of mass of the Earth.

This expression is quite similar to the one given at the beginning of Sec. III for planets. Most of the algorithms are identical; the difference is due only to the more complex motion of a planet compared to that of a star. In Sec. III we had to obtain the position of the planet from an ephemeris and deal with the light-time problem explicitly; in this section we will assume uniform rectilinear motion for the star and neglect variations in light-time as the star moves. Details of the steps follow.

The developments for a star can also be used for an extragalactic object, where the catalog proper motion and parallax are zero.

a) Determine Relevant Time Arguments

Steps (a)–(e): Follow steps (a)–(e) in Sec. III *a*. This results in the determination of the value of t, the TDB Julian Date corresponding to the epoch of observation, and T, the number of Julian centuries in the TDB timescale elapsed from J2000.0 TDB. For stars, one can skip the computation of m and assume $s = 0$; that is, skip step (c) and for step (d) simply set $t = t'$.

b) Obtain Ephemeris Data for the Earth and Sun

Step (f): Follow step (f) in Sec. III *b* to obtain $\mathbf{E}(t)$ and $\dot{\mathbf{E}}(t)$, the position and velocity vectors of the Earth for time t, with respect to the solar system barycenter.

Step (g): Follow step (g) in Sec. III *b* to obtain $\mathbf{E}'(t)$, the heliocentric position vector of the Earth for time t, or $\mathbf{S}(t)$, the position vector of the Sun for time t with respect to the solar system barycenter.

For apparent places of stars, the necessary ephemeris data can be obtained in a number of ways, since relatively low-precision ephemeris data are required. For stellar apparent place accuracies of a few milliarcseconds, it is necessary only to obtain the vector components to 3 significant digits in position and 5 significant digits in velocity. It is feasible to construct relatively compact closed-form algorithms which provide this accuracy. For example, we have coded a computer subroutine that provides the positions and velocities of the Earth by evaluating a truncated, modified form of Newcomb's theory; no reference to an external file is required. However, for the highest precision, or when the apparent places of planets are being computed, either an external file of positions of solar system bodies or significantly more complex closed-form algorithms must be employed.

c) Express Catalog Data for the Star as a Position and Velocity Vector

Step (h): Let α and δ represent the catalog mean right ascension and declination of the star at t_0, the reference epoch J2000.0. Let μ_α and μ_δ represent the corresponding proper-motion components in seconds of time and arc, respectively, per Julian century (of TDB). Let p represent the parallax of the star in arcseconds, and $\dot{r}$ its radial velocity in km/s.

If the star's radial velocity $\dot{r}$ is not known, set $\dot{r} = 0$. If the parallax p of the star is unknown, unavailable, or zero to within the accuracy of measurement, set it to some small but finite positive number. The choice is fairly arbitrary, and is needed only to avoid a mathematical indeterminacy; we use 1×10^{-7} arcsec, which effectively places objects of unknown parallax at a radius of 10 Mpc (the "celestial sphere").

Then compute the distance r to the star in AU:

$$\begin{aligned} r &= 1/\sin p \\ &\approx (1/p)(3600 \cdot 360)/2\pi. \end{aligned} \tag{29}$$

Step (i): Form the position vector $\mathbf{u}(t_0)$ of the star at the catalog epoch t_0, with respect to the solar system barycenter, in the rectangular coordinate system defined by the Earth's mean equator and equinox of the catalog epoch, with components in AU:

$$\mathbf{u}(t_0) = \begin{bmatrix} r\cos\delta\cos\alpha \\ r\cos\delta\sin\alpha \\ r\sin\delta \end{bmatrix}. \tag{30}$$

Step (j): Convert proper-motion and radial-velocity values to units of AU/day:

$$\begin{aligned} \mu'_\alpha &= \mu_\alpha 15\cos\delta/(36525p), \\ \mu'_\delta &= \mu_\delta/(36525p), \\ \dot{r}' &= 86400\dot{r}/A \end{aligned} \tag{31}$$

where A is the number of kilometers in 1 AU; the value from the IAU (1976) System of Astronomical Constants is $1.495\,978\,70 \times 10^8$. These three quantities represent orthogonal components of the star's space velocity, with respect to the solar system barycenter, to the extent known from the data available, in units of AU/day. These three quantities are therefore the components of a velocity vector in curvilinear coordinates.

Our use of the distance r as a multiplying factor in Eqs. (30) and (31) (in the form $1/p$) is somewhat unconventional for apparent place algorithms and is discussed in Sec. V.

It is worth considering several cases in which the available data are incomplete. For extragalactic objects, proper motion and parallax are effectively zero. The radial velocity is required only for relatively nearby stars for which foreshortening effects (second-order changes in the apparent position and motion of the star due to the shifting aspect of its motion) are significant. However, the radial velocity is useless in this regard (and should be set to zero) if the parallax (distance) is unknown. If the star's radial velocity is zero or unknown, or has been set to zero because the parallax is not known, then $\dot{r} = \dot{r}' = 0$ and the above velocity vector is tangent to the celestial sphere at the star's catalog position $\mathbf{u}(t_0)$. Conversely, if the proper-motion components are zero, then the star has no known tangential velocity. Also, it may be the case that the proper-motion components are known, but not the parallax. In such a case, if a "reasonable guess" parallax value is not used, the values for either or both of the computed velocity components μ'_α or μ'_δ could be greater than seems physically plausible. However, this is a computational curiosity with no physical meaning or practical effect on the results of the calculation. The subject of incomplete catalog information is discussed extensively in Paper IV.

Step (k): Transform the above velocity components to form the space velocity vector $\dot{\mathbf{u}}(t_0)$ of the star at the catalog epoch t_0, with respect to the solar system barycenter, with components in AU/day:

$$\dot{\mathbf{u}}(t_0) = \begin{bmatrix} -\sin\alpha & -\cos\alpha\sin\delta & \cos\alpha\cos\delta \\ \cos\alpha & -\sin\alpha\sin\delta & \sin\alpha\cos\delta \\ 0 & \cos\delta & \sin\delta \end{bmatrix} \begin{bmatrix} \mu'_\alpha \\ \mu'_\delta \\ \dot{r}' \end{bmatrix}. \tag{32}$$

The above transformation corresponds to two simple rotations. At this point, both the position vector $\mathbf{u}(t_0)$ and the velocity vector $\dot{\mathbf{u}}(t_0)$ are in the same rectilinear coordinate system.

d) Compute and Apply the Space Motion of the Star Between the Epoch of the Catalog and the Epoch of Observation

Step (l): Form the vector $\mathbf{u}_2$:

$$\mathbf{u}_2 = \mathbf{u}(t_0) + \dot{\mathbf{u}}(t_0)\cdot(t - t_0). \tag{33}$$

The vector $\mathbf{u}_2$ represents the position of the star at the epoch of observation t, with respect to the solar system barycenter. Both proper-motion and foreshortening effects are implicitly included in $\mathbf{u}_2$, since the vector $\dot{\mathbf{u}}(t_0)$ represents the star's three-dimensional space velocity.

There is no explicit correction for light time in stellar apparent place computations; it is assumed that the vectors $\mathbf{u}(t_0)$ and $\dot{\mathbf{u}}(t_0)$ implicitly include the light time and its time derivative.

e) Shift the Origin from the Barycenter of the Solar System to the Center of Mass of the Earth

Step (m): Form the vector $\mathbf{u}_3$:

$$\mathbf{u}_3 = \mathbf{u}_2 - \mathbf{E}(t). \tag{34}$$

The vector $\mathbf{u}_3$ represents the geocentric position of the star at the epoch of observation t. This step therefore introduces annual parallax.

f) Evaluate and Include the Effect of the Relativistic Deflection of Light in the Sun's Gravitational Field

Steps (n)–(p): Follow steps (n)–(p) in Sec. III *f* to obtain the vector $\mathbf{u}_4$, the geocentric direction of the star corrected for the relativistic deflection of light.

g) Evaluate and Include the Aberration of Light

Steps (q)–(r): Follow steps (q)–(r) in Sec. III *g*.

h) Apply Precession to the Coordinate System

Steps (s)–(t): Follow steps (s)–(t) in Sec. III *h*.

i) Apply Nutation to the Coordinate System

Steps (u)–(w): Follow steps (u)–(w) in Sec. III *i*.

j) Express Position Vector of the Star in Spherical Coordinates

Step (x): Follow step (x) in Sec. III *j*.

V. NOTES ON COMPUTER IMPLEMENTATION OF APPARENT PLACE ALGORITHMS

We have found it expedient to implement each of the major steps above as a separate computer subroutine. This modular approach allows most of the same subroutines to be used in the apparent place computation of both stars and planets. It also allows for testing of alternative, or simplified, algorithms for special purposes.

It was the desire for such a unified approach to these computations that resulted in the use of position vectors throughout the development for both stars and planets. In many apparent place developments, unit vectors are used for stars and position vectors are used for planets (for example, see Yallop's algorithms in the *Astronomical Almanac*). In such developments, the distance of a star is assumed to be 1 and its parallax becomes a multiplicative factor for the radial velocity in Sec. IV *c* and the Earth's position vector in Sec. IV *e*. In contrast, in the above development "real" position and velocity vectors with ordinary units (AU and AU/day) are used throughout, for both stars and planets. The penalty for this approach is that the zero-parallax case must be avoided. In any case, the two kinds of developments are equivalent and if correctly implemented will yield identical results. The choice has more to do with esthetics than substance.

It should be noted that many of the quantities that are developed during the course of an apparent place computation may be saved and reused during a subsequent computation for a different body, so long as the epoch of observation remains the same. Quantities in this category include the time arguments, the ephemeris data for the Earth and Sun, and the precession and nutation matrices. Computer subroutines can be coded to save such data from one call to the next. Then, if a large number of apparent places are to be computed, it is most efficient to compute the apparent places of all bodies at a given observing epoch before moving to a new observing epoch.

Our experience is that the major computational burden in the above algorithm involves the retrieval of data from the planetary ephemeris in steps (f)–(l) and the evaluation of the two nutation angles $\Delta\psi$ and $\Delta\epsilon$ in step (u). In both cases, there are several ways of obtaining the necessary data. Closed-form algorithms consisting of lengthy trigonometric series are available in both cases. (The 1980 IAU Theory of Nutation is defined by two 106 term trigonometric series.) The convenience of such self-contained algorithms must, however, be weighed against the large number of calculations that are required.

If computation time becomes critical (such as within telescope control systems or in microcomputer implementations) the method of obtaining the ephemeris and nutation data should be reviewed. Self-contained algorithms can often be considerably simplified by truncating small terms from the series, if high precision in the final apparent places is not required. Consideration should also be given to precomputing the required data at fixed intervals and storing the data in an external file that can be efficiently accessed and interpolated. Planetary ephemeris data are frequently distributed in this form anyway. Precomputing and storing the values of the elements of the combined precession-nutation matrix (referred to as $\mathbf{R}$ at the end of step (v)) is also feasible: see Sec. B of the *Astronomical Almanac*.

VI. TOPOCENTRIC PLACE

The topocentric place of a star or planet refers to its direction as it would actually be observed from some place on Earth, neglecting atmospheric refraction. The apparent place that we have developed in Secs. III and IV can be thought of as the observed direction of an object for a fictitious observer located at the center of a transparent, nonrefracting Earth. The difference between the apparent place and the topocentric place is due to the slightly different position and velocity of an observer on the Earth's surface compared with those of the fictitious observer at the Earth's center. The change in the direction of the observed body due to the difference of position is referred to as geocentric parallax, and is significant only for objects in the solar system. It is typically a few arcseconds for most solar system bodies, but reaches about 1° for the Moon. The change in direction due to the difference in velocity (due to the rotation of the Earth) is referred to as diurnal aberration, and is independent of the distance of the observed body. It is always less than 0.32 arcsec.

Atmospheric refraction also affects the observed direction of celestial objects. In fact, refraction at all wavelengths is orders of magnitude larger than either geocentric parallax or diurnal aberration. However, because of the difficulties of accurately modeling refraction and the need for meteorological data taken at the time of the observation, refraction is usually considered a correction to observations rather than an effect to be taken into account in computing a topocentric place. It will not be considered further here.

The simplest way of computing a topocentric place is to compute an apparent place using the position and velocity vectors of the observer rather than the center of mass of the Earth. That is, modify the vectors $\mathbf{E}(t)$ and $\dot{\mathbf{E}}(t)$ by the position and velocity of the observer relative to the center of mass of the Earth, immediately after step (f). This procedure is equally applicable to both stars and solar system objects. The necessary computations are described below, which should be considered optional extensions to step (f) in Secs. III*b* and IV*b*. The development below requires quantities related to precession and nutation that, in the computation of geocentric apparent places, are not needed before step (s). These quantities should be computed and used here and saved for later use; they are specifically noted as they arise.

a) Determine the Location and Universal Time of Observation

Step (f.1): Determine the universal time of observation, specifically, the epoch of observation in the UT1 timescale. UT1 is affected by unpredictable irregularities in the Earth's rotation but is always within 0.9 s of UTC, the latter defining civil time and broadcast worldwide according to international convention. The difference $\Delta\mathrm{UT} = \mathrm{UT1} - \mathrm{UTC}$ is determined and distributed by the International Earth Rotation Service; the value of ΔUT to within the nearest 0.1 s (denoted DUT) is also coded into UTC time broadcasts.

Step (f.2): Obtain the observer's geodetic latitude ϕ, longitude λ, and height h, the latter in meters. According to the current IAU convention, east longitudes are positive, west negative. The height specifically refers to the height above the Earth's reference ellipsoid, but for most purposes the height above mean sea level (the regional geoid) can be used.

b) Compute the Sidereal Time at the Epoch of Observation

Step (f.3): Obtain the two fundamental nutation angles, $\Delta\psi$ and $\Delta\epsilon$, and compute the mean and true obliquity of the ecliptic, ϵ and ϵ'. See steps (u) and (v) of Sec. III*i* for details. Save all of these quantities for later use.

Step (f.4): Using the UT1 epoch of observation as the argument, compute the Greenwich mean sidereal time s_m:

$$s_\mathrm{m} = 67310\overset{\mathrm{s}}{.}54841 + (876600^\mathrm{h} + 8640184\overset{\mathrm{s}}{.}812866)\,T_u + 0\overset{\mathrm{s}}{.}093104\,T_u^2 - 6\overset{\mathrm{s}}{.}2\times 10^{-6}\,T_u^3, \tag{35}$$

where T_u is the number of centuries of 36 525 days of universal time from 2000 January 1, 12^h UT1 (JD 2451545.0 UT1); see Aoki *et al.* (1982). The Greenwich apparent sidereal time is then

$$s = s_m + \Delta\psi \cos \epsilon'. \tag{36}$$

The quantity $\Delta\psi \cos \epsilon'$ is the "equation of the equinoxes," which must be expressed in units of time. Both the Greenwich mean sidereal time and the Greenwich apparent sidereal time, along with the equation of the equinoxes, are tabulated in the *Astronomical Almanac* for each day of the year at 0^h UT1.

Step (f.5): Compute the local apparent sidereal time s' at the place and epoch of observation:

$$s' = s + \lambda, \tag{37}$$

where λ is the observer's longitude (east positive), expressed in units of time.

c) Determine the Position and Velocity Vectors of the Observer with Respect to the Center of Mass of the Earth

Step (f.6): Compute the following two scalar quantities:

$$C = 1/\sqrt{\cos^2 \phi + (1-f)^2 \sin^2 \phi},$$
$$S = (1-f)^2 C, \tag{38}$$

where ϕ is the observer's geodetic latitude and f is the adopted flattening of the Earth's reference ellipsoid; $f = 0.003\,352\,81$ in the IAU (1976) System of Astronomical Constants.

Step (f.7): Compute the geocentric position and velocity vectors of the observer with respect to the true equator and equinox of date:

$$\mathbf{g}'(t) = \begin{bmatrix} (aC+h)\cos\phi\cos s' \\ (aC+h)\cos\phi\sin s' \\ (aS+h)\sin\phi \end{bmatrix},$$
$$\dot{\mathbf{g}}'(t) = w\hat{\mathbf{k}} \times \mathbf{g}'(t) \tag{39}$$
$$= \begin{bmatrix} -w(aC+h)\cos\phi\sin s' \\ w(aC+h)\cos\phi\cos s' \\ 0 \end{bmatrix},$$

where a is the equatorial radius of the Earth in meters, w is the rotational angular velocity of the Earth in radians/s, ϕ is the geodetic latitude of the observer, h is the height of the observer in meters, $\hat{\mathbf{k}}$ is a unit vector pointing in the $+z$ direction (toward the north celestial pole of date), and s' is the local apparent sidereal time at the time of observation, expressed in angular units. The quantity a is 6 378 140 in the IAU (1976) System and w is $7.292\,115\,146\,7\times10^{-5}$ (Aoki *et al.* 1982). More information on the computation of an observer's geocentric coordinates may be found in the *Astronomical Almanac*, pp. K11–K13, Mueller (1969), and Taff (1981).

The above expressions do not take into account polar motion, which affects the components of the observer's geocentric position vector at the 10 m (0.3 arcsec) level. Neglecting polar motion affects the computed topocentric place of the Moon by several milliarcseconds, with a much smaller effect, inversely proportional to distance, for other bodies. If effects at this level are important, corrections are also required to refer the regional geoid—the coordinate system for the observer's geodetic latitude, longitude, and height—to the Earth's reference ellipsoid; see the *Astronomical Almanac*, pp. K12–K13. Also ignored are the unpredictable variations in the Earth's angular velocity (length of day), but these remain below a part in 10^7 and affect the topocentric place only at the submicroarcsecond level.

If the geocentric "Earth-fixed" position vector of the observer is known, then steps (f.2), (f.5), (f.6), and (f.7) above can be replaced by the following simple procedure. Define $\mathbf{g}$ to be the position vector of the observer in an Earth-fixed, geocentric, right-handed coordinate system with the xy plane the Earth's equator, the xz plane the Greenwich meridian, and the z axis pointed towards the north terrestrial pole. If the components of $\mathbf{g}$ are expressed in meters, then $\mathbf{g}$ is simply rotated by the angle s, the Greenwich apparent sidereal time in angular units, to obtain $\mathbf{g}'(t)$:

$$\mathbf{g}'(t) = \begin{bmatrix} \cos s & -\sin s & 0 \\ \sin s & \cos s & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{g}. \tag{40}$$

The velocity $\dot{\mathbf{g}}'(t)$ follows immediately from the cross-product relation $\dot{\mathbf{g}}'(t) = w\hat{\mathbf{k}} \times \mathbf{g}'(t)$.

Step (f.8): Convert the geocentric position and velocity vectors to units of AU and AU/day, respectively:

$$\mathbf{g}(t) = \mathbf{g}'(t)/A,$$
$$\dot{\mathbf{g}}(t) = \dot{\mathbf{g}}'(t)\cdot 86400/A, \tag{41}$$

where A, the number of meters in 1 AU, is $1.495\,978\,70\times10^{11}$, from the IAU (1976) System. (Note that $\mathbf{g}(t)$ is distinguished from $\mathbf{g}$.)

d) Transform the Instantaneous Position and Velocity Vectors of the Observer to the Space-Fixed Frame

Step (f.9): Transform the vectors $\mathbf{g}(t)$ and $\dot{\mathbf{g}}(t)$ to the coordinate system defined by the Earth's mean equator and equinox of the reference epoch, which is the space-fixed coordinate system in which the position and velocity of the Earth are expressed:

$$\mathbf{G}(t) = \mathbf{P}^{-1}\mathbf{N}^{-1}\mathbf{g}(t) = \mathbf{P}'\mathbf{N}'\mathbf{g}(t) = \mathbf{R}'\mathbf{g}(t),$$
$$\dot{\mathbf{G}}(t) = \mathbf{P}^{-1}\mathbf{N}^{-1}\dot{\mathbf{g}}(t) = \mathbf{P}'\mathbf{N}'\dot{\mathbf{g}}(t) = \mathbf{R}'\dot{\mathbf{g}}(t), \tag{42}$$

where $\mathbf{P}$ and $\mathbf{N}$ are the precession and nutation matrices, developed in steps (s)–(w) of Secs. III*h* and III*i*. $\mathbf{R}$ is the combined precession-nutation matrix given in the *Astronomical Almanac*. Here, the inverse of each matrix is used, which is simply its transpose. The most efficient procedure would be to evaluate the elements of these matrices at this point and save the elements for later use in steps (t) and (w). Steps (s), (u), and (v) could therefore be skipped.

Strictly, the precession and nutation matrices define a transformation between a space-fixed system and a slowly rotating system, the slow rotation being the changing orientation of the Earth's axis due to external torques which the precession and nutation theories describe. Therefore, the conversion of the observer's velocity given in Eq. (42) is missing a Coriolis term. However, the equivalent linear velocity of this rotation is of order 10^{-5} m/s for an observer on the surface of the Earth, comparable to the tracking velocity of large telescopes and negligible.

e) Adjust the Position and Velocity Vectors of the Earth to Include the Position and Velocity of the Observer

Step (f.10): Add the vectors $\mathbf{G}(t)$ and $\dot{\mathbf{G}}(t)$, obtained above, representing the position and velocity of the observer with respect to the center of mass of the Earth, to $\mathbf{E}(t)$ and $\dot{\mathbf{E}}(t)$, obtained in step (f), representing the position and velocity of the center of mass of the Earth with respect to the solar system barycenter:

$$\mathbf{O}(t) = \mathbf{E}(t) + \mathbf{G}(t),$$
$$\dot{\mathbf{O}}(t) = \dot{\mathbf{E}}(t) + \dot{\mathbf{G}}(t). \tag{43}$$

The resulting vectors, $\mathbf{O}(t)$ and $\dot{\mathbf{O}}(t)$, represent the position and velocity vectors of the observer with respect to the solar system barycenter. Then redefine the vectors $\mathbf{E}(t)$ and $\dot{\mathbf{E}}(t)$ to be identical to $\mathbf{O}(t)$ and $\dot{\mathbf{O}}(t)$: $\mathbf{E}(t) = \mathbf{O}(t)$, $\dot{\mathbf{E}}(t) = \dot{\mathbf{O}}(t)$. At this point, proceed with step (g) using the redefined $\mathbf{E}(t)$ and $\dot{\mathbf{E}}(t)$.

Continuing at step (g) in Secs. III*b* or IV*b*, the other steps follow as before, except that the elements of the precession and nutation matrices need not be recomputed in steps (s)–(w). The right ascension α' and declination δ' obtained at step (x) represent the topocentric place of the object at the epoch of observation. The topocentric apparent hour angle of the object is given by $h' = s' - \alpha'$, where objects west of the meridian (setting) have positive hour angles.

In many cases the above procedure can be simplified with negligible error to the final topocentric right ascension and declination. The most care is needed for objects in the inner solar system when the highest precision is required. For objects beyond the inner solar system, to milliarcsecond precision, nutation can be ignored throughout. That is, mean sidereal time can be used instead of apparent sidereal time (the equation of the equinoxes can be considered zero) and the nutation rotation in Eq. (42) can be neglected (that is, $\mathbf{N}$ can be considered the unity matrix). Additionally, the difference between the UT1 and UTC timescales can be ignored. However, these simplifications may not result in a real computational saving. The nutation parameters would have to be computed anyway in a later step. Furthermore, these simplifications affect the computed sidereal time at the 1 s level, and will therefore cause errors in the computed topocentric hour angle of this magnitude. Therefore, in many cases, carrying out the full procedure, while saving the values of the relevant nutation and time variables for later use, may be the most prudent course.

VII. DIFFERENTIAL ASTROMETRY: VIRTUAL PLACE, LOCAL PLACE, AND ASTROMETRIC PLACE

For differential astrometric measurements, the algorithms in Secs. III and IV can be simplified. In differential work it is necessary only to consider effects which can alter the angles between the position vectors of observed bodies, i.e., arc lengths on the celestial sphere. The orientation of the coordinate system is not considered of fundamental importance since in most cases the celestial and instrumental coordinate systems are coupled. In any event, in differential observing, the coordinate system is not established until after the fact, during the reduction of the observations.

For the reduction of high-precision differential astrometric observations, therefore, the final precession and nutation rotations need not be performed, and Eqs. (1)*a* and (28)*a* reduce to, respectively,

$$\mathbf{u}(t') = f\{g[\mathbf{u}(t-\tau) - \mathbf{E}(t)]\} \tag{44}$$

and

$$\mathbf{u}(t') = f\{g[\mathbf{u}(t_0) + \dot{\mathbf{u}}(t_0)\cdot(t-t_0) - \mathbf{E}(t)]\}, \tag{45}$$

where all of the symbols have been defined in Secs. III and IV. We term the resulting position the "virtual place" of the planet or star. The virtual place is computed by following steps (a)–(r) in Sec. III or IV then setting $\mathbf{u}_7 = \mathbf{u}_5$ and skipping to step (x). The omission of the final precession and nutation rotations does not introduce any approximations or distortions since only orthogonal transformations are omitted.

The virtual place can be thought of as an apparent place expressed in the coordinate system of the reference epoch. It represents the position of the star or planet as it would be seen from the center of mass of the Earth at some date, in the coordinate system defined by the Earth's mean equator and equinox of the reference epoch, if the Earth and its atmosphere were transparent and nonrefracting.

We can also define a "local place," which is essentially the topocentric place expressed in the coordinate system of the reference epoch. Local place is related to topocentric place in the same way that virtual place is related to apparent place. Specifically, the local place represents the position of a star or planet as it would be seen from a specific location on Earth at some date and time, in the coordinate system defined by the Earth's mean equator and equinox of the reference epoch, if the atmosphere were nonrefracting. To compute it, simply add the procedure given in Sec. VI to the procedure given immediately above for computing virtual place. (Note: the precession and nutation rotations in the Sec. VI procedure should not be omitted.) The local place has utility beyond its use in relative astrometry and will be referred to again in Sec. VIII.

In differential work it has also been customary, if not strictly correct, to neglect both the gravitational deflection of light (function $g(...)$ above) and the aberration of light (function $f(...)$ above). The assumption is that for sufficiently small fields these effects—along with atmospheric refraction—are nearly identical for all observed bodies, so that relative positions are not significantly affected by them. Any residual distortion of the field resulting from the neglect of these effects is assumed to be absorbed into plate constants or similar parameters solved for in the data-reduction process. In this case, Eqs. (1) and (28) reduce to the very simple forms

$$\mathbf{u}(t') = \mathbf{u}(t-\tau) - \mathbf{E}(t) \tag{46}$$

and

$$\mathbf{u}(t') = \mathbf{u}(t_0) + \dot{\mathbf{u}}(t_0)\cdot(t-t_0) - \mathbf{E}(t), \tag{47}$$

and the resulting position is referred to as the "astrometric place" of the planet or star, respectively. The astrometric place is computed by following steps (a)–(m) in Secs. III or IV then setting $\mathbf{u}_7 = \mathbf{u}_3$ and skipping to step (x).

However, it should be recognized that the gravitational deflection of light should not really be ignored in this way since it cannot in principle be absorbed into plate constants or similar reduction parameters: the deflection is a function not only of position but also distance. Although in any part of the sky the direction of the deflection is the same for all bodies, its magnitude is less for solar system bodies than for stars (see Fig. 1). Generally, this detail is of little practical importance since only in a few special cases can it cause errors exceeding 0.01 arcsec.

Because they are simple to compute, astrometric places have been widely used. Another attractive feature is that an astrometric place can be directly plotted on an ordinary star map with negligible error in the resulting field configuration. Astrometric places are therefore used for the ephemerides of faint or fast-moving solar system bodies, such as the ephemerides of minor planets and Pluto in the *Astronomical Almanac*.

For observations of solar system bodies, it is frequently useful to compute a topocentric astrometric place. Only the correction for geocentric parallax is applied. To compute a topocentric astrometric place, simply add the procedure given in Sec. VI to the procedure given immediately above for computing astrometric place. (Again, the precession and nutation rotations in the Sec. VI procedure should not be omitted.) In this case, however, the observer's velocity vector need not be computed since only the vector $\mathbf{E}(t)$, not $\dot{\mathbf{E}}(t)$, is used in the astrometric place computation (aberration is ignored). Topocentric astrometric places of stars are never required since the topocentric correction is vanishingly small.

For the reduction of high-precision differential observations, the local place should be used. With available computing power it is straightforward to compute the local place of all objects within a field and use the ensemble of local places as the starting point for the reduction procedure.

VIII. A GENERAL APPROACH TO ASTROMETRIC OBSERVABLES

The preceding sections have been devoted to the precise definition and computation, within certain specific contexts, of two angular quantities, right ascension and declination. The importance and utility of the traditional equatorial reference system for celestial coordinates is self-evident to any astronomer. Nevertheless, with the exception of equatorially mounted telescopes, modern astronomical instrumentation rarely relates directly to these two angles. Right ascension and declination might be useful to the human observer, but for many types of observations they are insufficient information and most techniques at least require their transformation into quantities that are directly measurable or controllable. Quantities in the latter category might include shaft encoder readouts, interferometric delays, occultation timings, coordinates of star images in a CCD frame, or the arrival times of pulsar pulses. In all cases these quantities express a relationship among two or more time-dependent vectors. Therefore, we seek a general approach to transforming various vectors—which may represent the positions of astronomical objects, the orientation of the Earth, the location of the observer, or the directions of instrumental axes—to a common, space-fixed (inertial) coordinate system in which these relationships can be most simply expressed.

The common space-fixed coordinate system to which all relevant vectors will be transformed is the rectangular, right-handed system with the xy plane parallel to the Earth's mean equator of a specific reference epoch t_0, the x axis pointed toward the mean equinox of t_0, and the z axis pointed towards the mean north celestial pole of t_0. The spatial origin of this system can be considered arbitrary. The barycentric timescale TDB is appropriate for use within this system. (The transformation between various timescales was discussed in Sections II*c* and III*a*.) Generally, t_0 will be the epoch J2000.0, Julian Date 2451545.0 TDB.

The general approach that we outline here consists of five broad steps: (1) defining the applicable vector relationships which yield scalar observables; (2) expressing the "terrestrial" vectors, those tied in some way to the Earth's crust or ground-based instrumentation, in a suitable Earth-fixed coordinate system; (3) transforming these vectors to the space-fixed system; (4) expressing the topocentric direction of an astronomical object, as seen by the ground-based instrumentation, in the space-fixed system; and (5) carrying out the vector operations defined in step (1) to yield the needed quantities. We now discuss these steps in some detail.

a) Define the Required Vector Relationships

First, the vector expressions that lead to scalar observables must be defined. This step is very technique- and instrument-specific. In many cases the required quantities are, to a first approximation, the dot product of two vectors. For example, suppose $\hat{\mathbf{s}}$ represents the unit vector in the direction of a celestial object. Then the zenith distance of the object is arccos $(\hat{\mathbf{s}}\cdot\hat{\mathbf{V}})$, where $\hat{\mathbf{V}}$ is a unit vector pointing toward the local vertical. Interferometric delay is basically $\mathbf{B}\cdot\hat{\mathbf{s}}/c$, where $\mathbf{B}$ is the baseline vector and c is the speed of light. Similarly, the arrival time of a pulsar pulse at a terrestrial observer is different by $(\mathbf{E}+\mathbf{G})\cdot\hat{\mathbf{s}}/c$ from its arrival time at the solar system barycenter, where $\mathbf{G}$ is the observer's geocentric position vector and $\mathbf{E}$ is the position vector of the Earth with respect to the barycenter. These relationships are obviously purely geometric. The particular quantity required may also depend on nongeometric effects such as atmospheric propagation or instrumental response, effectively adding scalar terms to the expressions. Here, however, we limit ourselves to the geometry of the observations.

b) Express the Terrestrial Vectors in an Earth-Fixed Coordinate System

We next treat those vectors from the first step that are associated with Earth-based instrumentation or in some other way are directly tied to the Earth's crust. Examples would include the local vertical, the location of the geographic pole, an interferometer baseline, or other instrumental axes. These vectors are to be expressed in the Earth-fixed, rotating, rectangular, right-handed system with origin at the center of mass of the Earth (for the cases where an origin needs to be specified) in which the xy plane is the Earth's equator, the xz plane is the Greenwich meridian, and the z axis points towards the north geographic pole. In some cases it may be expedient to first express these vectors in a local horizon-based system then rotate them, first by geodetic latitude, then by longitude, into the above system. Three simple but illustrative examples of vectors expressed in the above-defined Earth-fixed coordinate system are given below (all symbols used have been defined in Sec. VI).

Geocentric position vector of observer:

$$\mathbf{g}=\begin{bmatrix}(aC+h)\cos\phi\cos\lambda\\(aC+h)\cos\phi\sin\lambda\\(aS+h)\sin\phi\end{bmatrix}; \tag{48a}$$

Unit vector toward local vertical (neglecting geophysical deflection):

$$\hat{\mathbf{v}}=\begin{bmatrix}\cos\phi\cos\lambda\\\cos\phi\sin\lambda\\\sin\phi\end{bmatrix}; \tag{48b}$$

Unit vector toward geographic pole (or, along polar axis of ideal equatorial mount):

$$\hat{\mathbf{z}}=\begin{bmatrix}0\\0\\1\end{bmatrix}. \tag{48c}$$

c) Transform the Terrestrial Vectors into the Space-Fixed System

An arbitrary vector, **x**, expressed in the Earth-fixed system, is transformed into the space-fixed system by a series of rotations, embodied in four matrices as follows:

$$\mathbf{X}(t) = \mathbf{P}^{-1}\mathbf{N}^{-1}\mathbf{S}\mathbf{W}\ \mathbf{x}$$
$$= \mathbf{P}'\mathbf{N}'\mathbf{S}\mathbf{W}\ \mathbf{x}. \qquad (49)$$

This transformation has been previously discussed, in the context of VLBI observations, by Cannon (1978) and Ma (1978). In the above, **P** and **N** are the precession and nutation matrices, developed in steps (s)–(w) of Secs. III*h* and III*i*. Here, as in Sec. VI*d*, the inverse of each of these two matrices is used, which in both cases is simply the transpose. The **S** matrix is the "spin," or Earth-rotation matrix; it is responsible for most of the time dependency of the transformed vector (although all four above matrices are time dependent) and is defined as follows:

$$\mathbf{S} = \begin{bmatrix} \cos s & -\sin s & 0 \\ \sin s & \cos s & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad (50)$$

where s is the Greenwich apparent sidereal time of the observation, developed in Sec. VI*b*. The **W** matrix is the "wobble," or polar-motion matrix, and is defined as follows:

$$\mathbf{W} = \begin{bmatrix} 1 & 0 & -X \\ 0 & 1 & Y \\ X & -Y & 1 \end{bmatrix}, \qquad (51)$$

where X and Y are the conventionally defined coordinates of the pole of rotation with respect to the Conventional International Origin (loosely, the geographic pole), expressed in radians. The coordinates X and Y are published monthly by the International Earth Rotation Service; they amount at most to a few tenths of an arcsecond, change slowly (typically a few milliarcseconds per day), and for many applications can be neglected (**W** becomes the unity matrix).

Transform the geocentric position vector of the observer to the space-fixed frame using the above transformation ($\mathbf{g} \rightarrow \mathbf{G}(t)$). Similarly transform all other relevant terrestrial vectors (e.g., $\hat{\mathbf{v}} \rightarrow \hat{\mathbf{V}}(t)$, $\hat{\mathbf{z}} \rightarrow \hat{\mathbf{Z}}(t)$, etc.). Note that the definition and transformation of the geocentric position vector of the observer described here is equivalent to that in Secs. VI*c* and VI*d*, except that in Sec. VI*d* polar motion was ignored so the **W** matrix never appears.

Additionally, transform to the space-fixed system the orthonormal basis vectors of the coordinate system defined by the Earth's true equator and equinox of date. These three unit vectors are not strictly in the Earth-fixed system but are closely related to it. If $\hat{\mathbf{e}}_z$ is the unit vector in the direction of the true celestial pole of date, $\hat{\mathbf{e}}_x$ is the unit vector in the direction of the equinox of date, and $\hat{\mathbf{e}}_y$ is the unit vector orthogonal to $\hat{\mathbf{e}}_z$ and $\hat{\mathbf{e}}_x$ defining a right-handed system, then these three vectors, expressed in the space-fixed system, are:

$$\hat{\mathbf{e}}_z = P'N' \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$
$$\hat{\mathbf{e}}_x = P'N' \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad (52)$$
$$\hat{\mathbf{e}}_y = \hat{\mathbf{e}}_z \times \hat{\mathbf{e}}_x.$$

For many types of observations only $\hat{\mathbf{e}}_z$ is needed, however.

d) Obtain the Topocentric Direction of the Celestial Body in the Space-Fixed System

The topocentric direction of the star or planet as seen by the terrestrial observer, expressed relative to the axes of the space-fixed system, is simply the local place of the body, as described in Sec. VII. The detailed computational steps are those in Secs. VII and VI and either Sec. IV (for stars) or Sec. III (for planets). The most logical progression is to work backwards from Sec. VII.

In carrying out the computational steps, note that, in essence, the computations described in Sec. VI have already been performed. The purpose of Sec. VI is simply the determination of the geocentric position and velocity of the observer, expressed in the space-fixed system. In the steps immediately above, however, the geocentric position vector of the observer in the space-fixed system has already been computed, and the geocentric velocity vector is readily available. Specifically, if $\mathbf{G}(t)$ is the geocentric position vector of the observer, and $\hat{\mathbf{e}}_z$ is the unit vector in the direction of the celestial pole of date, then the velocity vector of the observer is simply $\dot{\mathbf{G}}(t) = w\hat{\mathbf{e}}_z \times \mathbf{G}(t)$; w is the rotational angular velocity of the Earth and all vectors are expressed in the space-fixed system. (The value of w is $7.292\,115\,146\,7 \times 10^{-5}$ radians/s; conversion of $\mathbf{G}(t)$ and $\dot{\mathbf{G}}(t)$ to units of AU and AU/day, respectively, is required.) All that remains of the Sec. VI procedure is step (f.10) in Sec. VI*e*, where the position and velocity vectors of the observer are added to the corresponding vectors of the center of mass of the Earth.

The remaining steps of the local place computation are unchanged until the last step (step (x)). That step should be replaced by a simple normalization of the vector $\mathbf{u}_7$: $\hat{\mathbf{s}} = \mathbf{u}_7/|\mathbf{u}_7|$. The unit vector $\hat{\mathbf{s}}$ then represents the direction of the star or planet, as seen by the observer, relative to the axes of the space-fixed coordinate system.

e) Carry Out the Necessary Vector Operations to Obtain Scalar Observables

At this point, all of the relevant vectors have been transformed into the space-fixed coordinate system defined by the Earth's mean equator and equinox of the reference epoch t_0. The operations on these vectors defined in the first step above can therefore be carried out. All ordinary astrometric quantities can in fact be obtained from vector relationships. For example, using only previously defined vectors, we have:

$$\text{Zenith distance of object} = \arccos[\hat{\mathbf{s}}\cdot\hat{\mathbf{V}}(t)],$$
$$\text{Topocentric declination of object} = \arcsin(\hat{\mathbf{s}}\cdot\hat{\mathbf{e}}_z), \qquad (53)$$
$$\text{Topocentric right ascension of object} = \arctan\left[\frac{\hat{\mathbf{s}}\cdot\hat{\mathbf{e}}_y}{\hat{\mathbf{s}}\cdot\hat{\mathbf{e}}_x}\right].$$

Many other relationships can be derived as well; see, for example, Murray (1983).

The approach outlined in this section is primarily useful in deriving quantities that essentially are a measure of the angle between a terrestrial vector and a celestial vector. We have established a common coordinate system for relating vectors associated with well-defined directions, locations, or objects on the rotating Earth with those associated with the directions of celestial bodies. With some modifications, however, much of the above can be applied to other types of observations that are geometry sensitive, for example, pulsar timings, VLBI observations, and eclipse and occultation circumstances.

IX. CONCLUSION

The adoption of a new set of astronomical constants by the IAU in 1976, along with the related IAU resolutions concerning dynamical timescales, the equinox, nutation, and Universal Time, are having a profound effect on the computation of the most basic astronomical angles when precisions of an arcsecond or better are required. The changes were intended to eliminate known sources of systematic errors and facilitate the construction of highly precise astronomical reference frames, limited only by the observational uncertainties. With the computing power currently available, very sophisticated models of the observing geometry are possible, effectively eliminating computational approximations as an error source.

In this paper we have presented a series of algorithms for computing the apparent direction of astronomical objects to milliarcsecond precision. Effectively, these algorithms are a practical implementation of the new IAU system, forming the basic link between astrometric observations and the fundamental reference data derived therefrom.

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