On the Eigen-values of Certain Hermitian Forms

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Chapter I

Introduction

1.1. Definition of *L*-forms. In the years 1907–1911 O. Toeplitz [**21, 22, 23, 24**]* studied a class of quadratic forms whose matrix is of the following type:

$$\begin{bmatrix} & & & & \cdots & & & & \\ \cdots & c_{-n+1} & \cdots & c_0 & c_1 & c_2 & \cdots & c_{n+1} & \cdots \\ \cdots & c_{-n} & \cdots & c_{-1} & c_0 & c_1 & \cdots & c_n & \cdots \\ \cdots & c_{-n-1} & \cdots & c_{-2} & c_{-1} & c_0 & \cdots & c_{n-1} & \cdots \\ & & & & \cdots & & & & \end{bmatrix}. \tag{1.1}$$

The elements c_n are given complex constants. Toeplitz designated these forms as L-forms and investigated in detail their relation to the analytic function defined in a neighborhood of the unit circle by the Laurent series $\Sigma\, c_n z^n$, $n = -\infty, \cdots, \infty$; this series is assumed to be convergent in a certain circular ring $r_1 < |z| < r_2$, $r_1 < 1 < r_2$. It is obvious that these matrices are connected with the *infinite* cyclic group, just as the finite cyclic matrix

$$\begin{bmatrix} c_0 & c_1 & c_2 & \cdots & c_n \\ c_n & c_0 & c_1 & \cdots & c_{n-1} \\ c_{n-1} & c_n & c_0 & \cdots & c_{n-2} \\ & & & \cdots & \\ c_1 & c_2 & c_3 & \cdots & c_0 \end{bmatrix} \tag{1.2}$$

is associated with the *finite* cyclic group. The main result of Toeplitz is that the spectrum of the L-form is identical with the complex values the Laurent series assumes on the unit circle $|z| = 1$.

1.2. Hermitian forms. The case $c_{-n} = \bar{c}_n$ is of particular importance; the matrix (1.1) is in this case a Hermitian one and the associated Laurent series represents a *real* function $f(\theta)$ on the unit circle $z = e^{i\theta}$, $-\pi \leqq \theta < \pi$. The coefficients c_n are of course the Fourier coefficients of the function $f(\theta)$:

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)e^{-in\theta}\,d\theta, \qquad n = 0, \pm 1, \pm 2, \cdots. \tag{1.3}$$

* The numbers refer to the Bibliography at the end of the present paper.

1.3. Toeplitz forms. This topic is closely related to the fundamental investigations of C. Carathéodory and L. Fejér [**1, 2**]. This was the first instance in which the sections of (1.1), called Toeplitz matrices, were considered. These finite matrices have the form

$$(c_{\nu-\mu}), \qquad \nu, \mu = 0, 1, \cdots, n, \tag{1.4}$$

where $c_{-m} = \bar{c}_m$. The principal problem dealt with by Carathéodory & Fejér is the following: Let $c_0, c_1, \cdots, c_n$ be given complex numbers. For all power series $\varphi(z)$ whose $n + 1$ first coefficients are these numbers, and which are convergent for $|z| < 1$, we ask for the maximum of the minimum (greatest lower bound) of Re $[\varphi(z)]$, $|z| < 1$. The answer is that this "maximum minimorum" exists and coincides with the smallest eigen-value $\lambda_1^{(n)}$ of the matrix (1.4). This result is connected with another objective of the theory, namely with the characterization of the class of harmonic functions regular and positive in the interior of the unit circle, by their Fourier coefficients.

We consider now the same matrices (1.4) from the following somewhat different view-point.

Let $f(\theta)$ be a given real function of period 2π, integrable in the interval $-\pi, \pi$ in the Lebesgue sense. We form the Fourier coefficients (1.3) of $f(\theta)$. Obviously, $c_{-n} = \bar{c}_n$ so that the finite matrix (1.4) is Hermitian. We call (1.4) the Toeplitz matrices and the corresponding Hermitian forms

$$\begin{aligned} T_n(f) &= \sum_{\mu=0}^{n} \sum_{\nu=0}^{n} c_{\nu-\mu}\, u_\mu \bar{u}_\nu \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta)\, | u_0 + u_1 z + u_2 z^2 + \cdots + u_n z^n |^2 \, d\theta, \end{aligned} \tag{1.5}$$

$$z = e^{i\theta},\ n \leq 0, 1, 2, \cdots$$

the Toeplitz forms associated with the function $f(\theta)$. We point out that (1.4) is *not* a cyclic matrix in general.

1.4. Relation to orthogonal polynomials. The relation of these Hermitian forms to the theory of orthogonal polynomials is rather obvious. Let $f(\theta)$ be a function which is non-negative but not a null-function. We denote by $\{\varphi_n(z)\}$ the system of polynomials which are orthonormal on the unit circle $z = e^{i\theta}$ with the weight $f(\theta)$; that is [**19,** Chapter XI],

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) \varphi_n(z) \overline{\varphi_m(z)}\, d\theta = \delta_{nm}, \qquad z = e^{i\theta};\ n, m = 0, 1, 2, \cdots. \tag{1.6}$$

The polynomial $\varphi_n(z) = k_n z^n + \cdots$, $k_n > 0$, can be characterized by the property that $\sum u_\nu z^\nu = k_n^{-1} \varphi_n(z)$ yields the minimum of the form (1.5) under the condition $u_n = 1$. Denoting by D_n the determinant of (1.5), $D_n > 0$, we have

$$k_n = (D_{n-1}/D_n)^{\frac{1}{2}}, \tag{1.7}$$

and the minimum in question is $k_n^{-2} = D_n/D_{n-1}$. From this fact it can be concluded that [**19,** p. 293, (12.3.3)]

$$\lim_{n \to \infty} \frac{D_n}{D_{n-1}} = \exp\left\{\frac{1}{2\pi} \int_{-\pi}^{\pi} \log f(\theta)\, d\theta\right\}. \tag{1.8}$$

This equation holds for an arbitrary function $f(\theta) \geqq 0$ which is integrable in the Lebesgue sense; the right hand-side (the "geometric mean" of $f(\theta)$) must be replaced by 0 if $\log f(\theta)$ is not Lebesgue-integrable. From (1.8) it follows, as is well-known, that

$$\lim_{n \to \infty} \frac{\log D_n}{n+1} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log f(\theta)\, d\theta. \tag{1.9}$$

Concerning a refinement of (1.9), *cf.* [**22**].

The polynomials which are orthogonal on the unit circle, have a close connection with those orthogonal on a finite real interval [**19,** p. 287, Theorem 11.5]. These polynomials in turn are associated with what are called Hankel forms, that is, forms whose matrix is of the type $(g_{\mu+\nu})$.

1.5. Eigen-values of Toeplitz forms. We denote the eigen-values of the finite Hermitian form (1.5) (that is, the eigen-values of $T_n(f) - \lambda U$, where U is the unit form) by

$$\lambda_1^{(n)}, \lambda_2^{(n)}, \cdots, \lambda_{n+1}^{(n)}, \tag{1.10}$$

ordering these values in *increasing* (non-decreasing) order. Let m and M denote the minimum (greatest lower bound) and maximum (least upper bound) of $f(\theta)$, respectively, where the cases $m = -\infty$, $M = +\infty$ are not excluded. We have then

$$m \leqq \lambda_1^{(n)} \leqq \lambda_2^{(n)} \leqq \cdots \leqq \lambda_{n+1}^{(n)} \leqq M. \tag{1.11}$$

In 1917 Szegö proved [**11,** see also **19,** p. 194] that these values $\lambda_\nu^{(n)}$ behave asymptotically as the ordinates taken on by the function $f(\theta)$ at $n + 1$ equally spaced points θ over the period $-\pi, \pi$. The precise meaning of this assertion is the following: *let $F(\lambda)$ be an arbitrary Riemann-integrable function defined in the interval $m \leqq \lambda \leqq M$, where m and M are now finite. We have then*

$$\lim_{n \to \infty} \frac{F(\lambda_1^{(n)}) + F(\lambda_2^{(n)}) + \cdots + F(\lambda_{n+1}^{(n)})}{n+1} = \frac{1}{2\pi} \int_{-\pi}^{\pi} F[f(\theta)]\, d\theta. \tag{1.12}$$

A special case of this theorem, equivalent to the general assertion, is the following: Let α and β be two real numbers, $\alpha < \beta$; denoting by $N = N(n; \alpha, \beta)$

the number of the eigen-values (1.10) which satisfy the condition $\alpha \leqq \lambda_\nu^{(n)} \leqq \beta$, we have

$$\lim_{n\to\infty} \frac{N(n;\alpha,\beta)}{n+1} = \frac{l}{2\pi} \tag{1.13}$$

where l is the Lebesgue measure of the set of θ-values for which $\alpha \leqq f(\theta) \leqq \beta$.

Another interesting special case (also equivalent to the general assertion) is the following: $m > 0$, $F(\lambda) = \log \lambda$. We regain then the relation (1.9).

A particular consequence of (1.12) is that for a fixed value of ν:

$$\lim_{n\to\infty} \lambda_\nu^{(n)} = m, \tag{1.14}$$

provided m denotes the "essential" lower bound of $f(\theta)$. Similarly for a fixed ν:

$$\lim_{n\to\infty} \lambda_{n+2-\nu}^{(n)} = M \tag{1.15}$$

where M is the "essential" upper bound of $f(\theta)$.

1.6. Content of the present paper. The goal of this investigation lies in three different directions.

First, we subject the equations (1.14) and (1.15) to a further analysis. Assuming a certain regularity of the function $f(\theta)$ in the nieghborhood of the point where the minimum m of $f(\theta)$ is attained (*cf.* 3.2), we prove the following limit relation. *Let ν be fixed while $n \to \infty$; we have then*

$$\lim_{n\to\infty} n^2(\lambda_\nu^{(n)} - m) = c\pi^2\nu^2, \qquad \nu = 1, 2, \cdots, \tag{1.16}$$

where c is a positive constant depending on $f(\theta)$.

A similar result holds for the eigen-values $\lambda_{n+2-\nu}^{(n)}$ where ν is fixed, $n \to \infty$. If $m \neq 0$ we can write (1.16) in the form

$$\lim_{n\to\infty} (\lambda_\nu^{(n)}/m)^{-n^2} = e^{-c'\pi^2\nu^2}, \qquad c' = c/m. \tag{1.17}$$

Second, we generalize the Toeplitz matrices in the following direction. Let $f(x, \theta)$ be a given real function of two variables, periodic in θ with period 2π, and defined for $0 \leqq x \leqq 1$. Assuming that $f(x, \theta)$ is Lebesgue-integrable in θ, we form the Fourier coefficients

$$\psi_n(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x,\theta)e^{-in\theta}\,d\theta, \qquad n = 0, \pm 1, \pm 2, \cdots, \tag{1.18}$$

of $f(x, \theta)$ as function of θ; $\psi_{-n}(x) = \bar{\psi}_n(x)$. We consider the Hermitian matrix

$$\left(\psi_{\nu-\mu}\left(\frac{\mu+\nu}{2n}\right)\right), \qquad \mu, \nu = 0, 1, 2, \cdots, n. \tag{1.19}$$

In the case of a function f independent of x, these matrices are of the Toeplitz type. We investigate again the eigen-values of (1.19) generalizing the limit relation (1.12), at least for a certain class of functions f. The following theorem might be expected as a generalization of (1.9). *Let the matrix* (1.19) *be positive definite and* Δ_n *its determinant. We have then* $f(x, \theta) \geqq 0$, $\Delta_n > 0$, *and the following limit relation holds:*

$$\lim_{n\to\infty} \frac{\log \Delta_n}{n+1} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \int_0^1 \log f(x, \theta)\, dx\, d\theta. \tag{1.20}$$

We prove this under the condition that f is a trigonometric polynomial in θ and $\psi_n(x)$ is continuous (even under a slightly more general condition (2.1)). Again from (1.20) the distribution theorem on eigen-values follows.

Thirdly, we consider the analogue of these matrices and forms in the "continuous" case; that is, we discuss the relation of these problems to the theory of integral equations. We consider such equations with a kernel which has a similar structure as the Toeplitz matrices have.

Chapter 2 deals with the eigen-values of matrices of the type (1.18).

Chapter 3 deals with relations of the form (1.16) for the "extreme" eigen-values.

Chapter 4 is devoted to the discussion of the eigen-values of integral equations of a certain class.

1.7. Probabilistic background and conjectures. Toeplitz matrices and their continuous analogues are intimately connected with various problems in Probability Theory and Statistics. They occur, in particular, in the theory of discrete and continuous stationary time series (see *e.g.* Grenander [**6**]). They also occur quite naturally in problems related to random walk with absorbing barriers. In fact, the principal result of Chapter 3 of the present paper was suggested by the latter problems. Since this background is of independent interest and since a great variety of problems arising in this connection are still unsolved, it seems desirable to give here a detailed account.

Let $X_1, X_2, \cdots$ be a sequence of independent identically distributed random variables each having mean 0 and variance σ^2. Let $-1 < \xi < 1$ and consider the probability

$$P_n(\xi) = \text{Prob. } \{|s_1 + \xi n^{\frac{1}{2}}| < n^{\frac{1}{2}}, |s_2 + \xi n^{\frac{1}{2}}| < n^{\frac{1}{2}}, \cdots, |sn + \xi n^{\frac{1}{2}}| < n^{\frac{1}{2}}\}, \tag{1.21}$$

where

$$s_k = X_1 + X_2 + \cdots + X_k.$$

It was proved by Erdös & Kac [**3**] that

$$\lim_{n\to\infty} P_n(\xi) = \sum_{j=1}^{\infty} e^{-\pi^2 j^2/(8\sigma^2)} \varphi_j(\xi) \int_{-1}^{+1} \varphi_j(x)\, dx, \tag{1.22}$$

where

$$\varphi_{2l}(\xi) = \sin l\pi\xi, \qquad \varphi_{2l+1}(\xi) = \cos (l + \tfrac{1}{2})\pi\xi.$$

Now, let us assume that each X has the same density function $\rho(x)$, which is assumed to satisfy

1. $$\rho(-x) = \rho(x),$$

2. $$\int_{-\infty}^{\infty} x^2\rho(x)\,dx = \sigma^2.$$

Consider the integral equation

$$\int_{-n^{\frac{1}{2}}}^{n^{\frac{1}{2}}} \rho(x - y)\psi(y)\,dy = \lambda\psi(x). \tag{1.23}$$

Denote by $\lambda_1(n^{\frac{1}{2}})$, $\lambda_2(n^{\frac{1}{2}})$, $\cdots$ the eigen-values and by $\psi_1(x; n^{\frac{1}{2}})$, $\psi_2(x; n^{\frac{1}{2}})$, $\cdots$ the corresponding normalized eigen-functions. The eigen-values are assumed to be ordered in decreasing order of their absolute values.

We have

$$\begin{aligned} P_n(\xi) &= \int_{-n^{\frac{1}{2}}}^{n^{\frac{1}{2}}} \cdots \int_{-n^{\frac{1}{2}}}^{n^{\frac{1}{2}}} \rho(x_1 - \xi n^{\frac{1}{2}})\rho(x_2 - x_1) \cdots \rho(x_n - x_{n-1})\,dx_1 \cdots dx_n \\ &= \sum_{j=1}^{\infty} \lambda_j^n(n^{\frac{1}{2}})\psi_j(\xi n^{\frac{1}{2}}; n^{\frac{1}{2}}) \int_{-n^{\frac{1}{2}}}^{n^{\frac{1}{2}}} \psi_j(x; n^{\frac{1}{2}})\,dx. \end{aligned} \tag{1.24}$$

Comparing (1.24) with (1.22) suggests the conjecture that

$$\lim_{n\to\infty} \lambda_j^n(n^{\frac{1}{2}}) = e^{-\pi^2 j^2/(8\sigma^2)}. \tag{1.25}$$

This, unfortunately, we are unable to prove. However, the discrete analogue of (1.25) can be proved, as will be shown in Chapter 3. This analogue can be formulated as follows. Instead of assuming that the X's have a density function $\rho(x)$ we assume that they are discrete, *i.e.*,

$$\text{Prob. } \{X_j = k\} = c_k, \qquad k = 0, \pm 1, \pm 2, \cdots; \qquad \sum_{R=-\infty}^{\infty} c_k = 1,$$

and furthermore

1. $$c_{-k} = c_k,$$

2. $$\sum_{k=-\infty}^{\infty} k^2 c_k = \sigma^2.$$

Denoting by $\lambda_j(n)$ the jth eigen-value (in order of decreasing absolute values) of the matrix

$$\begin{pmatrix} c_0 & c_1 & c_2 & \cdots & c_{n-1} \\ c_1 & c_0 & c_1 & \cdots & c_{n-2} \\ c_2 & c_1 & c_0 & \cdots & c_{n-3} \\ & \cdot & & \cdots & \cdot \\ c_{n-1} & c_{n-2} & c_{n-3} & \cdots & c_0 \end{pmatrix},$$

we have

$$\lim_{n\to\infty} (\lambda_j(n))^{n^2} = e^{-\pi^2 j^2/(8\sigma^2)}. \tag{1.26}$$

[The fact that $n^{\frac{1}{2}}$ in (1.25) is replaced by n in (1.26) is, of course, of no consequence.] Actually, the fact that the c_k's are probabilities and hence non-negative is of no importance, as the proof in Chapter 3 will show. However, the condition that

$$\sum_{k=-\infty}^{\infty} k^2 \mid c_k \mid < \infty$$

is of crucial importance. If this condition is violated, (1.26) will certainly fail. A probabilistic argument again throws some light on the situation.

If the c_k's are again probabilities ($c_k = c_{-k}$) and if

$$\lim_{n\to\infty} \left(\sum_{k-\infty}^{\infty} c_k \cos \frac{k\xi}{n} \right)^n = e^{-|\xi|}, \tag{1.27}$$

which will be the case if $c_k \sim k^{-2}$, we find ourselves in the so-called domain of attraction of the Cauchy law. A probabilistic argument analogous to the one given above combined with a result of Kac & Pollard [8] leads to the conjecture that in this case (1.26) should be replaced by

$$\lim_{n\to\infty} (\lambda_j(n))^n = e^{-\mu_j} \tag{1.28}$$

where μ_j is the jth eigen-value of the integral equation

$$\tfrac{1}{4} \int_{-1}^{1} \log \frac{1 - xy + [(1-x^2)(1-y^2)]^{\frac{1}{2}}}{1 - xy - [(1-x^2)(1-y^2)]^{\frac{1}{2}}} \psi(y)\, dy = \mu\psi(x). \tag{1.29}$$

This we are unable to prove, but we feel that the conjecture is of sufficient interest to be stated explicitly. In the more general case

$$c_k \sim k^{-1-\alpha}, \qquad 0 < \alpha < 2,$$

one is in the domain of attraction of the stable law with exponent α, and the probabilistic argument can again be repeated. Unfortunately we do not know the precise analogue of the μ_j's.

What happens in case the c_k's do not belong to any of the above domains of attraction is at the moment beyond even conjecture.

Chapter II

On Certain Generalizations of Toeplitz Matrices

2.1. Theorem. In this chapter we consider functions $f(x, \theta)$ satisfying the following *Condition A:*

(1) *$f(x, \theta)$ is real-valued and periodic in θ with period 2π; the variable x ranges in the interval $0 \leqq x \leqq 1$;*

(2) *the coefficients $\psi_n(x)$ of the Fourier expansion*

$$f(x, \theta) \sim \sum_{n=-\infty}^{\infty} \psi_n(x) e^{in\theta} \tag{2.1}$$

are continuous;

(3) *there exists a constant M such that*

$$\sum_{n=-\infty}^{\infty} \max | \psi_n(x) | \leqq M, \tag{2.2}$$

where the maxima are taken in the interval $0 \leqq x \leqq 1$.

Obviously, $\psi_{-n}(x) = \overline{\psi_n(x)}$. The function $f(x, \theta)$ is continuous.

We form now the following three generalizations of the Toeplitz matrices:

$$A_n = \left(\psi_{\nu-\mu}\left(\frac{\mu+\nu}{2n+2}\right)\right), \qquad B_n = \left(\psi_{\nu-\mu}\left(\frac{\min(\mu,\nu)}{n+1}\right)\right), \qquad C_n = \left(\psi_{\nu-\mu}\left(\frac{\max(\mu,\nu)}{n+1}\right)\right); \tag{2.3}$$

in all these cases $\mu, \nu = 0, 1, \cdots, n$. These matrices are of the Hermitian type. If $f(x, \theta)$ is independent of x, all three types reduce to Toeplitz matrices.

Let us consider the matrix A_n. Denoting its eigen-values by $\lambda_1^{(n)}, \lambda_2^{(n)}, \cdots, \lambda_{n+1}^{(n)}$ we have

$$| \lambda_\nu^{(n)} | \leqq M, \qquad \nu = 1, 2, \cdots, n+1. \tag{2.4}$$

This estimate is a consequence of the following useful

Lemma: *Let $A = (a_{\mu\nu})$, $\mu, \nu = 0, 1, \cdots, n$, be a Hermitian matrix. We assume that the sum of the moduli of the elements in each row is under a common bound M:*

$$|a_{\mu 0}| + |a_{\mu 1}| + \cdots + |a_{\mu n}| \leqq M.$$

Then we have for the eigen-values λ_ν of A:

$$| \lambda_\nu | \leqq M.$$

The assertion is equivalent to the fact that the Hermitian form $\sum a_{\mu\nu} u_\mu \bar{u}_\nu$ associated with A is in the absolute value not greater than M provided $\sum | u_k |^2 = 1$. Now for each eigen-value λ we have with proper u_k, not all zero,

$$a_{\mu 0} u_0 + a_{\mu 1} u_1 + \cdots + a_{\mu n} u_n = \lambda u_\mu, \qquad \mu = 0, 1, \cdots, n.$$

Let $\max | u_k | = | u_\mu |$; then

$$| \lambda | \, | u_\mu | \leqq (| a_{\mu 0} | + | a_{\mu 1} | + \cdots + | a_{\mu n} |) \, | u_\mu | \leqq M \, | u_\mu |,$$

and this yields the assertion.

Our principal result is the following

Theorem: *Let $f(x, \theta)$ satisfy Condition A. We denote by $\lambda_\nu^{(n)}$ the eigen-values of the matrix A_n, $| \lambda_\nu^{(n)} | \leqq M$ [cf. (2.1)–(2.4)]. If $F(\lambda)$ is any Riemann-integrable function defined for $-M \leqq \lambda \leqq M$, we have*

$$\lim_{n\to\infty} \frac{F(\lambda_1^{(n)}) + F(\lambda_2^{(n)}) + \cdots + F(\lambda_{n+1}^{(n)})}{n+1} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \int_0^1 F[f(x, \theta)] \, dx \, d\theta. \tag{2.5}$$

2.2. Special cases. If $f(x, \theta)$ is independent of x this theorem yields that of G. Szegö **[15]**.

The following special cases are of interest:

(*1*) Let s be any positive integer; then

$$\lim_{n\to\infty} \frac{(\lambda_1^{(n)})^s + (\lambda_2^{(n)})^s + \cdots + (\lambda_{n+1}^{(n)})^s}{n+1} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \int_0^1 [f(x, \theta)]^s \, dx \, d\theta. \tag{2.6}$$

(*2*) Let $\alpha < \beta$, and let us denote by $N(n; \alpha, \beta)$ the number of the eigen-values $\lambda_\nu^{(n)}$ for which $\alpha \leqq \lambda_\nu^{(n)} \leqq \beta$ holds. We have then

$$\lim_{n\to\infty} \frac{N(n; \alpha, \beta)}{n+1} = \frac{\sigma}{2\pi} \tag{2.7}$$

where σ is the area of the sub-domain in the rectangle $-\pi \leqq \theta \leqq \pi$, $0 \leqq x \leqq 1$, in which $\alpha \leqq f(x, \theta) \leqq \beta$.

(*3*) Let $\lambda_\nu^{(n)} \geqq m > 0$ for all ν and n. We have then (see (2.3))

$$\lim_{n\to\infty} (\det A_n)^{1/(n+1)} = \exp\left\{\frac{1}{2\pi} \int_{-\pi}^{\pi} \int_0^1 \log f(x, \theta) \, dx \, d\theta\right\}. \tag{2.8}$$

From any of these special cases the general theorem (2.5) follows.

2.3. Remarks. (*a*) A sufficient condition for $\lambda_\nu^{(n)} \geqq m > 0$ is the following:

$$\min \psi_0(x) \geqq \sum_{\substack{n=-\infty,\cdots,\infty;\\ n\neq 0}} \max \, | \psi_n(x) | + m.$$

This condition implies that $\psi_0(x) \geqq m > 0$.

Indeed,

$$A_n - mU = \prod_{\mu=0}^{n} \left\{ \psi_0 \left(\frac{\mu}{n+1} \right) - m \right\} (U + K),$$

where U is the unit matrix and the elements of the matrix K are of the form $\psi_{\nu-\mu}(x)\{\psi_0(x') - m\}^{-\frac{1}{2}}\{\psi_0(x'') - m\}^{-\frac{1}{2}}$; here $\mu \neq \nu$ and x, x', x'' are certain values in 0, 1. In view of the lemma in (2.1), the eigen-values of K are in the absolute value not greater than

$$\sum_{\substack{n=-\infty,\cdots\infty;\\ n\neq 0}} \max|\psi_n(x)|\,(\min[\psi_0(x)-m])^{-1} \leqq 1$$

so that the form $A_n - mU$ is non-negative.

(*b*) We observe that $\lambda_\nu^{(n)} \geqq m$ (for each ν and n) implies $f(x, \theta) \geqq m$. It suffices to show this for $m = 0$. Now from $\lambda_\nu^{(n)} \geqq 0$ we infer that

$$\sum_{\mu,\nu=0,1,\cdots,n} \psi_{\nu-\mu}\left(\frac{\mu+\nu}{2n+2}\right) u_\mu \bar{u}_\nu \geqq 0.$$

Let $0 < x - \epsilon < x + \epsilon < 1$. We choose $u_\nu = e^{-i\nu\theta}$, θ real, provided ν satisfies the conditions

$$x - \epsilon \leqq \nu/(n+1) \leqq x + \epsilon; \tag{2.9}$$

otherwise let $u_\nu = 0$. Hence, if μ and ν satisfy the inequalities (2.9),

$$\sum \psi_{\nu-\mu}\left(\frac{\mu+\nu}{2n+2}\right) e^{i(\nu-\mu)\theta} \geqq 0. \tag{2.10}$$

Let q be a fixed positive number. We form the qth Fejér mean of the trigonometric polynomial (2.10). According to the fundamental property of these means we have

$$\sum_{|\nu-\mu|\leqq q} \left(1 - \frac{|\nu-\mu|}{q}\right) \psi_{\nu-\mu}\left(\frac{\mu+\nu}{2n+2}\right) e^{i(\nu-\mu)\theta} \geqq 0. \tag{2.11}$$

Let $M = M(x, \epsilon, n)$ be the number of the terms $\mu = \nu$ in (2.10) or (2.11), so that $M(x, \epsilon, n) \to \infty$ as $n \to \infty$ (x, ϵ fixed). If k is fixed, $|k| \leqq q$, the number of the terms $\nu - \mu = k$ is $M(x, \epsilon, n) + O(1)$ as $n \to \infty$. Dividing (2.11) by $M(x, \epsilon, n)$ and passing to the limit $n \to \infty$ and then to $\epsilon \to 0$, we obtain

$$\sum_{|k|\leqq q} (1 - |k|/q)\psi_k(x)e^{ik\theta} \geqq 0. \tag{2.12}$$

Since this is true for each q, we have $f(x, \theta) \geqq 0$.

2.4. Proof of the theorem. In order to prove the theorem formulated in 2.1, it suffices to prove the special case (2.6). Let ϵ be arbitrary and let p be chosen so that

$$\sum_{|n|>p} \max|\psi_n(x)| < \epsilon. \tag{2.13}$$

We consider the trigonometric polynomial of the pth degree

$$f_p(x) = \sum_{n=-p}^{p} \psi_n(x) e^{in\theta}. \tag{2.14}$$

We denote the matrix of the first type in (2.3) associated with $f_p(x, \theta)$ by $A_n^{(p)}$ and write

$$A_n = A_n^{(p)} + D. \tag{2.15}$$

The sum of the moduli of the elements in any row of D is less than ϵ so that the same holds for the eigen-values of D, hence also for the values of $A_n - A_n^{(p)}$ as $\Sigma \mid u_k \mid^2 = 1$. By the Lemma of Weyl-Courant* we have for the eigen-values $\lambda_\nu^{(n)}$ and $\lambda_\nu^{(n,p)}$ of A_n and $A_n^{(p)}$ (ordered in non-decreasing order)

$$\mid \lambda_\nu^{(n)} - \lambda_\nu^{(n,p)} \mid < \epsilon. \tag{2.16}$$

If s is a positive integer, we have, since $\mid \lambda_\nu^{(n)} \mid \leqq M$, $\mid \lambda_\nu^{(n,p)} \mid \leqq M$,

$$\mid (\lambda_\nu^{(n)})^s - (\lambda_\nu^{(n,p)})^s \mid < sM^{s-1}\epsilon.$$

A similar estimate holds for $[f(x, \theta)]^s - [f_p(x, \theta)]^s$. Consequently it is sufficient to prove (2.6) for a trigonometric polynomial $f_p(x, \theta)$.

2.5. Continuation of the proof. To simplify the notation we assume that $f(x, \theta)$ itself is a trigonometric polynomial of degree p. Hence the matrix A_n will have elements different from zero only in $2p + 1$ "lines" parallel to the main diagonal. Replacing all these elements except those in the "line r" by 0 $(-p \leqq r \leqq p)$, a matrix B_{nr} will result. Clearly,

$$A_n = \sum_{r=-p}^{p} B_{nr}. \tag{2.17}$$

We denote by $\mathrm{Tr}(A)$ the *trace* of an arbitrary matrix A. Hence the sum $(\lambda_1^{(n)})^s + (\lambda_2^{(n)})^s + \cdots + (\lambda_{n+1}^{(n)})^s$ occurring in (2.6) is $\mathrm{Tr}(A_n^s)$ where A_n^s is the sth power of the matrix A_n. Now

$$A_n^s = \sum B_{nr_1} B_{nr_2} \cdots B_{nr_s}, \tag{2.18}$$

where $r_1, r_2, \cdots, r_s$ run independently from $-p$ to p. Thus it suffices to show that

$$\lim_{n\to\infty} \frac{T_r(B_{nr_1} B_{nr_2} \cdots B_{nr_s})}{n+1} = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(r_1+r_2+\cdots+r_s)\theta} d\theta \int_0^1 \psi_{r_1}(x)\psi_{r_2}(x) \cdots \psi_{r_s}(x)\, dx. \tag{2.19}$$

* See below, 3.3(a).

The right-hand side is, of course, zero unless $\Sigma r_1 = 0$ in which case it reduces to the second integral (integral in x).

For the sake of convenience we use for the matrices $B_{nr_1}, B_{nr_2}, \cdots, B_{nr_s}$ the following alternate notations:

$$(g_{\mu\nu}^{(1)}),\ (g_{\mu\nu}^{(2)}),\ \cdots,\ (g_{\mu\nu}^{(s)}), \tag{2.20}$$

where $\mu, \nu = 0, 1, \cdots, n$. In these matrices all terms are zero except those for which $v - \mu = r_1, r_2, \cdots, r_s$, respectively. The general element $g_{\mu\nu}$ of their product is

$$\sum g_{\mu\nu_1}^{(1)} g_{\nu_1\nu_2}^{(2)} \cdots g_{\nu_{s-2}\nu_{s-1}}^{(s-1)} g_{\nu_{s-1}\nu}^{(s)} \tag{2.21}$$

where $\nu_1, \nu_2, \cdots, \nu_{s-1}$ vary independently from 0 to n. Now all terms of $g_{\mu\nu}$ will vanish except if

$$\nu_1 - \mu = r_1, \qquad \nu_2 - \nu_1 = r_2, \cdots, \qquad \nu - \nu_{s-1} = r_s. \tag{2.22}$$

A consequence of these conditions is that $\nu - \mu = \Sigma r_i$. But the elements $g_{\mu\nu}$ contribute to the trace only if $\mu = \nu$ so that (2.19) will be trivial if $\Sigma r_i \neq 0$.

Let $\Sigma r_i = 0, \mu = \nu$; we have then

$$\nu_1 = \mu + r_1, \qquad \nu_2 = \mu + r_1 + r_2, \cdots, \qquad \nu_s = \nu = \mu + r_1 + r_2 + \cdots + r_s = \mu, \tag{2.23}$$

and the trace in question will be

$$\begin{aligned} \sum_{\mu=0}^{n} g_{\mu\mu} = {\sum_{\mu=0}^{n}}' \psi_{r_1}\left(\frac{2\mu + r_1}{2n+2}\right) \psi_{r_2}\left(\frac{2\mu + 2r_1 + r_2}{2n+2}\right) \\ \cdot \psi_{r_3}\left(\frac{2\mu + 2r_1 + 2r_2 + r_3}{2n+2}\right) \cdots \psi_{r_3}\left(\frac{2\mu + 2_1 + \cdots + 2r_{s-1} + r_s}{2n+2}\right). \end{aligned} \tag{2.24}$$

In the sum Σ' we are to suppress all terms in which at least one of the quantities $\mu + r_1, \mu + r_1 + r_2, \cdots, \mu + r_1 + \cdots + r_s$ fails to be in the range $0, n$. The number of these exceptional values of μ is obviously bounded [not greater than $2(sp + 1)$] as $n \to \infty$.

Multiplying (2.24) by $(n + 1)^{-1}$ and passing to the limit $n \to \infty$ we obtain

$$\int_0^1 \psi_{r_1}(x)\, \psi_{r_2}(x) \cdots \psi_{r_s}(x)\, dx,$$

and this proves (2.19).

2.6. A special case. We deal with the special case $f(x, \theta) = f(\theta) + g(x)$ under somewhat different conditions than in 2.1 and 2.3. Using another method we prove (2.8)*.

* A further treatment of this case based on an argument similar to that in 2.4 and 2.5 is due to ULLMAN [25].

Let $f(\theta)$ be periodic of period 2π and let $g(x)$ be defined in the interval $0 \leqq x \leqq 1$. We assume that both functions $f(\theta)$ and $g(x)$ are Riemann-integrable and $f(\theta) + g(x)$ is bounded away from zero, that is, $m \leqq f(\theta) + g(x) \leqq M$ where $m > 0$. Using the notation (1.5) we consider the form

$$\begin{aligned} H_n &= T_n(f) + \sum_{v=1}^{n} g(v/n) \mid u_v \mid^2 \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) \mid u_0 + u_1 z + \cdots + u_n z^n \mid^2 d\theta \\ &\qquad + \sum_{v=1}^{n} g(v/n) \mid u_v \mid^2, \qquad z = e^{i\theta}, \end{aligned} \tag{2.25}$$

and denote its determinant by Δ_n. We prove then that

$$\lim_{n \to \infty} \frac{\log \Delta_n}{n+1} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \int_0^1 \log \left[f(\theta) + g(x) \right] dx\, d\theta. \tag{2.26}$$

We note that $T_n(f) = \bar{f} \, \Sigma \mid u_k \mid^2$ where $\bar{f}$ is a certain mean value of $f(\theta)$ depending on u_k. Hence $H_n \geqq mU$, so that $\Delta_n > 0$.

We can find trigonometric polynomials $\underline{f}(\theta)$, $\bar{f}(\theta)$, and functions $\underline{g}(x)$, $\bar{g}(x)$, constant in stretches such that for all θ and x

$$\underline{f}(\theta) \leqq f(\theta) \leqq \bar{f}(\theta), \qquad \underline{g}(x) \leqq g(x) \leqq \bar{g}(x); \tag{2.27}$$

moreover we can determine these functions so that $\underline{f}(\theta) + \underline{g}(x)$ is positive and the integrals

$$\int_{-\pi}^{\pi} \int_0^1 \log \left[\underline{f}(\theta) + \underline{g}(x) \right] dx\, d\theta, \qquad \int_{-\pi}^{\pi} \int_0^1 \log \left[\bar{f}(\theta) + \bar{g}(x) \right] dx\, d\theta \tag{2.28}$$

differ from the integral in (2.26) by an arbitrarily small amount. Hence for the proof of (2.26) we may assume that *$f(\theta)$ is a trigonometric polynomial and $g(x)$ is constant in stretches, $f(\theta) + g(x) > 0$.*

(*a*) An upper estimate of Δ_n can be obtained by using E. Fischer's determinant theorem.* Let us denote the constant values of $g(x)$ by $g_1, g_2, \cdots, g_m$, assumed in intervals of the length $l_1, l_2, \cdots, l_m$, respectively, where $\sum_{i=1}^{m} l_i = 1$. We consider those values of μ, ν for which μ/n and ν/n belong to l_i. The corresponding minor of Δ_n is a certain Toeplitz determinant $\Delta_n^{(i)}$ associated with the function $f(\theta) + g_i$; denoting its order by $n_i + 1$ we have $n_i/n \cong l_i$ as $n \to \infty$. Now by Fischer's theorem

$$\Delta_n \leqq \Delta_n^{(1)} \Delta_n^{(2)} \cdots \Delta_n^{(m)} \tag{2.29}$$

* E. Fischer, "Über den Hadamardschen Determinantensatz," *Archiv der Mathematik und Physik*, (3) **13** (1908), 32–40.

and

$$\frac{\log \Delta_n}{n+1} \leqq \sum_{i=1}^{m} \frac{\log \Delta_n^{(i)}}{n_i+1} \cdot \frac{n_i+1}{n+1}. \tag{2.30}$$

Hence, by (1.9),

$$\begin{aligned} \limsup_{n\to\infty} \frac{\log \Delta_n}{n+1} &\leqq \sum_{i=1}^{m} \frac{1}{2\pi} \int_{-\pi}^{\pi} \log [f(\theta) + g_i]\, d\theta \cdot l_i \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \int_0^1 \log [f(\theta) + g(x)]\, dx\, d\theta. \end{aligned} \tag{2.31}$$

(b) We divide the variables $u_0, u_1, \cdots, u_n$ into groups

$$u_1^{(i)}, u_2^{(i)}, \cdots, u_{n_i}^{(i)}, \qquad i = 1, 2, \cdots, m, \tag{2.32}$$

where the variables of the ith group are those u_ν for which ν/n belongs to l_i. The Hermitian form

$$K_n = \sum_{i=1}^{m} \frac{1}{2\pi} \int_{-\pi}^{\pi} [f(\theta) + g_i] \mid u_1^{(i)} + u_2^{(i)} z + \cdots + u_{n_i}^{(i)} x^{n_i - 1} \mid^2 d\theta, \tag{2.33}$$

$$z = e^{i\theta},$$

will have the determinant $\Delta_n^{(2)} \Delta_n^{(2)} \cdots \Delta_n^{(m)}$. We compare the forms H_n and K_n. We have

$$H_n - K_n = \sum \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) u_i(z) u_j(z)\, d\theta, \qquad i \neq j;\ i, j = 1, 2, \cdots, m, \tag{2.34}$$

where $z = e^{i\theta}$ and

$$u_i(z) = (u_1^{(i)} + u_2^{(i)} z + \cdots + u_{n_i}^{(i)} z^{n_i - 1}) z^{n_1 + n_2 + \cdots + n_{i-1}}. \tag{2.35}$$

(We write $z^{n_1 + \cdots + n_{i-1}} = 1$ for $i = 1$.)

Since $f(\theta)$ is a trigonometric polynomial, the majority of the integrals in (2.34) will vanish; more precisely, all those integrals in which $|i - j| \geqq 2$, provided n is sufficiently large. Even in the integrals with $j = i + 1$ only a bounded number of terms will remain, namely the terms of the form

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) z^{N-\alpha} \bar{z}^{N+\beta}\, d\theta, \quad N = n_1 + n_2 + \cdots + n_i,$$

where $\alpha + \beta$ does not exceed the degree of $f(\theta)$. (Similarly for $j = i - l$). Consequently,

$$K_n \leqq H_n + A \sum | u_k |^2 = H_n^*, \tag{2.36}$$

where A is fixed positive constant and the summation involves only a fixed number, say ρ, terms.

From (2.36) we conclude by the Lemma of 3.3 (a) that

$$\Delta_n^{(1)}\Delta_n^{(2)} \cdots \Delta_n^{(n)} \leqq \Delta_{n\rho}. \tag{2.37}$$

Here $\Delta_{n\rho}$ denotes the determinant of H_n^*, *i.e.*, a determinant arising from Δ_n by increasing certain ρ elements in the main diagonal by the fixed quantity A. The final part of our argument is to show that $\Delta_{n\rho}/\Delta_n$ is bounded as $n \to \infty$ so that

$$\Delta_n^{(1)}\Delta_n^{(2)} \cdots \Delta_n^{(m)} \leqq \Delta_n O(1). \tag{2.38}$$

By comparison of this with (2.29) [or with (2.31)] the assertion will follow.

The proof proceeds by induction. Indeed, for $\rho = 0$, $\Delta_{n\rho} = \Delta_n$. Now $\Delta_{n\rho}$ can be decomposed into the sum of two determinants, the one being of the form $\Delta_{n,\,\rho-1}$ and the other can be written as $A \cdot \Delta'$, where Δ' is a minor of $\Delta_{n\rho}$ corresponding to a certain diagonal element $\mu = \nu$. On the other hand, $\Delta_{n\rho}/\Delta'$ is the minimum of H_n^* under the condition $u_\nu = 1$. As observed after (2.26) we have $H_n \geqq mU$ so that by (2.36) we find for $u_\nu = 1$

$$H_n^* \geqq (A + m)u_\nu^2 = A + m,$$

where m is the minimum of $f(\theta) + g(x)$. Thus $\Delta_{n\rho} \geqq (A + m)\Delta'$ so that

$$\Delta_{n\rho} \leqq \Delta_{n,\,\rho-1} + \frac{A}{A+m}\,\Delta_{n\rho}, \qquad \Delta_{n\rho} \leqq (1 + A/m)\Delta_{n,\,\rho-1}.$$

Repeating this argument we find for $\Delta_{n\rho}/\Delta_n$ the upper bound $(1 + A/m)^\rho$.

Chapter III.

Extreme Eigen-values of Toeplitz Forms

Let $f(\theta)$ be a given Lebesgue integrable function. We define the associated Toeplitz forms $T_n(f)$ by (1.5) and denote their eigen-values by $\lambda_\nu^{(n)}$ [(1.10)]. The purpose of this chapter is to study the asymptotic behavior of the extreme eigen-values $\lambda_\nu^{(n)}$ and $\lambda_{n+2-\nu}^{(n)}$ where ν is fixed and $n \to \infty$. Certain conditions on $f(\theta)$ will be imposed [*cf.* 3.2].

3.1. Special cases. We consider two special cases in which either $f(\theta)$ or $[f(\theta)]^{-1}$ is a trigonometric polynomial of the first order. In the first case the eigen-values can be computed explicitly. The second case is somewhat less simple; of course the trigonometric polynomial in question must be then of constant sign. In both cases the relation (1.16) can be verified easily.

(*a*) The first case is well known [*cf.* for instance **9**, vol. 2, p. 320, problem 70; also below, §3.5]. We have

$$\lambda_\nu^{(n)} = f\left(\theta_0 + \frac{\nu\pi}{n+2}\right), \qquad \nu = 1, 2, \cdots, n+1, \tag{3.1}$$

where $f(\theta_0) = m$ is the minimum of the function $f(\theta)$. Hence

$$\lambda_\nu^{(n)} = m + \left(\frac{\nu\pi}{n+2}\right)^2 \frac{f''(\theta_0)}{2} + \cdots,$$

so that for fixed ν, $n \to \infty$,

$$\lambda_\nu^{(n)} - m \cong \left(\frac{\nu\pi}{n+2}\right)^2 \frac{f''(\theta_0)}{2}.$$

Excluding the case $f(\theta) = \text{const.}$, we have $f''(\theta_0) > 0$. This yields (1.16) with

$$c = \tfrac{1}{2}f''(\theta_0). \tag{3.2}$$

(*b*) The second case is less trivial. Without impairing the generality we may assume that

$$f(\theta) = \frac{1-r^2}{1-2r\cos\theta+r^2}, \qquad 0 < r < 1. \tag{3.3}$$

The equation [*cf.* **9**, vol. 2, p. 319, problem 69]

$$(3.4) \qquad D_n(f-\lambda) = \begin{vmatrix} 1-\lambda & r & r^2 & \cdots & r^n \\ r & 1-\lambda & r & \cdots & r^{n-1} \\ r^2 & r & 1-\lambda & \cdots & r^{n-2} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ r^n & r^{n-1} & r^{n-2} & \cdots & 1-\lambda \end{vmatrix} = 0$$

yields the eigen-values $\lambda_\nu^{(n)}$. Multiplying the second row by r, subtracting it from the first, and performing a similar operation with the columns, we find

$$(3.5) \quad D_n(f-\lambda) = \begin{vmatrix} 1-\lambda-r^2(1+\lambda) & r\lambda & 0 & \cdots & 0 \\ r\lambda & 1-\lambda & r & \cdots & r^{n-1} \\ 0 & r & 1-\lambda & \cdots & r^{n-2} \\ \cdot & \cdot & & \cdots & \cdot \\ \cdot & \cdot & & \cdots & \cdot \\ \cdot & \cdot & & \cdots & \cdot \\ 0 & r^{n-1} & r^{n-2} & \cdots & 1-\lambda \end{vmatrix}$$

$$= [1-\lambda-r^2(1+\lambda)]D_{n-1}(f-\lambda) - r^2\lambda^2 D_{n-2}(f-\lambda),$$

$$n = 2, 3, 4, \cdots.$$

This recurrence formula holds also for $n = 1$ provided we put $D_{-1} = 1$.

In order to find D_n explicitly, we write conveniently

$$(3.6) \qquad \lambda = \frac{1-r^2}{1-2r\cos x + r^2} = f(x)$$

and form the characteristic equation

$$(3.7) \qquad \alpha^2 = [1-\lambda-r^2(1+\lambda)]\alpha - r^2\lambda^2.$$

Since $1-\lambda-r^2(1+\lambda) = -2r\lambda\cos x$, the roots of (3.7) are $-r\lambda e^{\pm ix}$. Now we easily verify that the formula

$$(3.8) \qquad \begin{aligned} D_n(f-\lambda) &= \frac{(-\lambda r)^{n+1}}{1-r^2}\left(\frac{\sin(n+2)x}{\sin x} - 2r\frac{\sin(n+1)x}{\sin x} + r^2\frac{\sin nx}{\sin x}\right) \\ &= \frac{(-\lambda r)^{n+1}}{1-r^2}\, p_n(\cos x) \end{aligned}$$

holds for $n = -1$ and $n = 0$ [$D_{-1} = 1$, $D_0 = 1-\lambda$]. Thus it holds generally.

The expression $p_n(\xi)$ occuring in (3.8) is a polynomial of degree $n+1$ in

$\xi = \cos x$. It has $n + 1$ real and distinct zeros $\cos x_\nu^{(n)}$ where $0 < x_1^{(n)} < x_2^{(n)} < \cdots < x_{n+1}^{(n)} < \pi$ and

$$\lambda_\nu^{(n)} = f(x_{n+2-\nu}^{(n)}), \qquad \nu = 1, 2, \cdots, n + 1. \tag{3.9}$$

Indeed, we have

$$p_n\left(\cos \frac{\nu\pi}{n+1}\right) = (-1)^\nu(1 - r^2), \qquad \nu = 1, 2, \cdots, n,$$

and

$$p_n(1) > 0, \qquad (-1)^{n+1} p_n(-1) > 0.$$

In order to obtain the asymptotic expression for the extreme eigen-values we observe that

$$\begin{aligned} &\lim_{n\to\infty} n^{-1} p_n\left(\cos \frac{z}{n}\right) = (1 - r)^2 \frac{\sin z}{z}, \\ &\lim_{n\to\infty} (-1)^{n+1} n^{-1} p_n\left(- \cos \frac{z}{n}\right) = (1 + r)^2 \frac{\sin z}{z}. \end{aligned} \tag{3.10}$$

Here z is an arbitrary complex number and the relations (3.10) hold uniformly for $|z| \leqq R$. Consequently for fixed ν

$$x_\nu^{(n)} \cong \pi - x_{n+2-\nu}^{(n)} \cong \frac{\nu\pi}{n+2}, \tag{3.11}$$

so that

$$\lambda_\nu^{(n)} = f\left(\pi \frac{\nu\pi + \epsilon}{n+2}\right), \tag{3.12}$$

where $\epsilon = \epsilon_\nu^{(n)} \to 0$ as $n \to \infty$. A similar expression holds for the eigen-values $\lambda_{n+2-\nu}^{(n)}$. Compare (3.12) with (3.1).

3.2. A class of functions $f(\theta)$. We consider now the class of functionl $f(\theta)$ satisfying the following

Condition B: Let $f(\theta)$ be periodic with period 2π and continuous. Let $\min f(\theta) = f(\theta_0) = m$ *and let $\theta = \theta_0$ be the only value of θ* (mod 2π) *for which this minimum is reached. Moreover we assume that $f(\theta)$ has a continuous second derivative $f''(\theta)$ in a certain neighborhood of θ_0. Finally let* $f''(\theta_0) \neq 0$.

Obviously $f''(\theta_0) > 0$, and we have

$$\lim (\theta - \theta_0)^{-2}(f(\theta) - f(\theta_0)) = \tfrac{1}{2}f''(\theta_0). \tag{3.13}$$

We prove the following

Theorem: *Let $f(\theta)$ satisfy Condition B. We have then for a fixed ν and $n \to \infty$:*

$$\lambda_\nu^{(n)} - m \cong f\left(\theta_0 + \frac{\nu\pi}{n}\right) - f(\theta_0) \cong c\,\frac{\nu^2\pi^2}{n^2}, \tag{3.14}$$
$$c = \tfrac{1}{2}f''(\theta_0); \qquad \nu = 1, 2, 3, \cdots.$$

A similar result holds for $\lambda_{n+2\nu}^{(n)}$. As an example we mention

$$f(\theta) = b_0 - \sum_{n=1}^{\infty} b_n \cos n\theta, \tag{3.15}$$

where $b_n > 0$ and $\sum n^2 b_n$ is convergent. Then

$$m = b_0 - \sum_{n=1}^{\infty} b_n, \qquad c = \tfrac{1}{2}\sum_{n=1}^{\infty} n^2 b_n. \tag{3.16}$$

3.3. Preparations. The proof of the Theorem is based on the following steps:
(a) Application of a lemma on Hermitian forms;
(b) Approximation of $f(\theta)$ by appropriate special functions.

(a) *Lemma* (of Weyl-Courant)*: *Let $A = \Sigma a_{\mu\nu} u_\mu \bar{u}_\nu$ and $B = \Sigma b_{\mu\nu} u_\mu \bar{u}_\nu$ be two Hermitian forms, the second positive definite; $\mu, \nu = 0, 1, \cdots, n$. We denote the eigen-values of $A - \lambda B$ by $\lambda_1, \lambda_2, \cdots, \lambda_{n+1}$ (ordered in non-decreasing order). Then λ_ν can be characterized by the following extremum property: Let $\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_{\nu-1}$ be arbitrary vectors in the $n + 1$-dimensional (complex) space and let $\mathbf{u}$ be the vector $(u_0, u_1, \cdots, u_n)$. We denote by ρ the minimum of A/B for all $\mathbf{u}$ which are not zero vectors and satisfy the conditions*

$$(\mathbf{a}_1, \mathbf{u}) = (\mathbf{a}_2, \mathbf{u}) = \cdots = (\mathbf{a}_{\nu-1}, \mathbf{u}) = 0. \tag{3.17}$$

Then $\lambda_\nu =$ max ρ *for all possible choices of the vectors $\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_{\nu-1}$.*

For $\nu = 1$ we have no auxiliary conditions (3.17) and $\lambda_1 = \min A/B$ for all $\mathbf{u}$.

This lemma is so important and its proof so simple that it seems desirable to present it. As is known,

$$\frac{A}{B} = \frac{\Sigma\lambda_\nu |\xi_\nu|^2}{\Sigma |\xi_\nu|^2} \qquad \nu = 1, 2, \cdots, n+1,$$

where the ξ_ν are linearly independent linear forms in the u_ν. By definition $\rho \leqq A/B$ where we may choose *any* vector $\mathbf{u}$ satisfying the linear conditions (3.17). We add to these the conditions $\xi_{\nu+1} = \cdots = \xi_{n+1} = 0$, imposing altogether n linear conditions on $\mathbf{u}$. There is always a non-zero vector $\mathbf{u}$ satisfying these conditions; the corresponding ξ cannot be all zero, so that $\xi_1, \xi_2, \cdots, \xi_\nu$ are not all zero. Hence

$$\rho \leqq \frac{\lambda_1 |\xi_1|^2 + \cdots + \lambda_\nu |\xi_\nu|^2}{|\xi_1|^2 + \cdots + |\xi_\nu|^2} \leqq \lambda_\nu.$$

* Cf. R. Courant-D. Hilbert, Methoden der mathematischen Physik, **1**, Berlin 1924; pp. 16–18.

On the other hand let us take for $\mathbf{a}_1, \cdots, \mathbf{a}_{\nu-1}$ the vectors whose projections are the conjugates of the coefficients of $\xi_1, \cdots, \xi_{\nu-1}$ (as linear forms of $u_0, \cdots, u_n$). Then $\xi_1 = \cdots = \xi_{\nu-1} = 0$, and

$$\rho = \min \frac{\lambda_\nu |\xi_\nu|^2 + \cdots + \lambda_{n+1} |\xi_{n+1}|^2}{|\xi_\nu|^2 + \cdots + |\xi_{n+1}|^2}$$

for all possible $\xi_\nu, \cdots, \xi_{n+1}$ which is exactly λ_ν.

This proves the assertion.

We note the following simple implication of this lemma. Let A, B; A', B' be two pairs of Hermitian forms, B and B' positive definite, and let $A/B \leqq A'/B'$ for all $\mathbf{u}$. Denoting the eigen-values of these pairs by λ_ν and λ_ν' (ordered in non-decreasing order), we have

$$\lambda_\nu \leqq \lambda_\nu', \qquad \nu = 1, 2, \cdots, n + 1.$$

A consequence of these inequalities is the following. Let us denote by $D(A)$, $D(B)$, $D(A')$, $D(B')$ the determinants of the given forms which we assume now to be all positive definite. Then

$$\frac{D(A)}{D(B)} \leqq \frac{D(A')}{D(B')}.$$

(*b*) *Approximation.* For the proof of the theorem we may assume that $\theta_0 = 0, f(0) = m = 0$. To a given $\epsilon > 0$ we can find two functions $\underline{f}(\theta)$ and $\bar{f}(\theta)$ satisfying the following conditions:

1.
$$\underline{f}(\theta) = (1 - \cos\theta)[a + b(1 - \cos\theta)]^{-1},$$
$$\bar{f}(\theta) = (1 - \cos\theta)[c + d(1 - \cos\theta)]$$

where a, b, c, d are positive constants;

2. for all θ we have

$$\underline{f}(\theta) \leqq f(\theta) \leqq \bar{f}(\theta);$$

3. we have

$$|f''(0) - \underline{f}''(0)| < \epsilon, \; |f''(0) - \bar{f}''(0)| < \epsilon.$$

We form the function $F(\theta) = (1 - \cos\theta)^{-1} f(\theta)$, which is positive and continuous; $F(0) = f''(0)$. First we determine a neighborhood $(-\theta', \theta')$ of $\theta = 0$ in which $F(\theta) < F(0) + \epsilon/2 = c$ holds. Then we choose d so that $d(1 - \cos\theta') > \max F(\theta)$, and the second inequality in *2* will hold. By applying a similar reasoning to $[F(\theta)]^{-1}$ the function $\underline{f}(\theta)$ can be obtained.

If two functions $f_1(\theta)$ and $f_2(\theta)$ satisfy the inequality $f_1(\theta) \leqq f_2(\theta)$, we have for the corresponding Toeplitz forms $T_n(f_i)$:

$$T_n(f_1) \leqq T_n(f_2),$$

so that in view of the lemma we conclude the same inequality for the corresponding eigen-values. Using the approximation in (*b*), it suffices to prove the theorem for the special functions of the form $\underline{f}(\theta)$ and $\bar{f}(\theta)$.

3.4. Lower approximation. The case $\underline{f}(\theta)$ is settled essentially by §3.1 (b) since

$$\frac{1 - \cos\theta}{a + b(1 - \cos\theta)} = \alpha - \beta \frac{1 - r^2}{1 - 2r\cos\theta + r^2}$$
$$= \alpha - \frac{\beta(1 - r^2)}{(1 - r)^2 + 2 + (1 - \cos\theta)}.$$

Here a, b are given positive numbers and α, β, r can be determined in such a manner that α, $\beta > 0$, $0 < r < 1$. Indeed

$$\alpha = \frac{1}{b}; \qquad \beta = \frac{(1 - r)}{1 + r}\frac{1}{b}; \qquad \frac{(1 - r)^2}{2r} = \frac{a}{b}.$$

Thus for a function of the form $\underline{f}(\theta)$ the assertion follows from (3.12).

3.5. Upper approximation. We assume that $c = 1$ so that the function $\bar{f}(\theta)$ has the form

$$\bar{f}(\theta) = (1 - \cos\theta) + d(1 - \cos\theta)^2, \qquad d > 0. \tag{3.18}$$

(a) First we consider the function $f(\theta) = 1 - \cos\theta$. The eigen-values are in this case

$$\lambda_\nu = 1 - \cos\theta_\nu, \qquad \theta_\nu = \frac{\nu\pi}{n + 2}, \qquad \nu = 1, 2, \cdots, n + 1. \tag{3.19}$$

This can be concluded from the identities

$$\begin{aligned} &\frac{1}{2\pi}\int_{-\pi}^{\pi} | u_0 + u_1 z + \cdots + u_n z^n |^2 \, d\theta \\ &\qquad = \sum_{\nu=1}^{n+1} | l_{\nu 0} u_0 + l_{\nu 1} u_1 + \cdots + l_{\nu n} u_n |^2 \\ &\frac{1}{2\pi}\int_{-\pi}^{\pi} \cos\theta \, | u_0 + u_1 z + \cdots + u_n z^n |^2 \, d\theta \\ &\qquad = \sum_{\nu=1}^{n+1} \cos\theta_\nu \, | l_{\nu 0} u_0 + l_{\nu 1} u_1 + \cdots + l_{\nu n} u_n |^2, \end{aligned} \tag{3.20}$$

where $z = e^{i\theta}$, and

$$l_{\nu\mu} = \left(\frac{2}{n + 2}\right)^{\frac{1}{2}} \sin(\mu + 1)\theta_\nu, \qquad \mu = 0, 1, \cdots, n. \tag{3.21}$$

The meaning of the first identity in (3.20) is that $(l_{\nu\mu})$ is an *orthogonal* matrix.

These identities are easy to verify by using the familiar formula

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \varphi(\theta) \, d\theta = \frac{1}{2N} \sum_{\nu=-N+1}^{N} \varphi\left(\frac{\lambda\pi}{N}\right) = \frac{1}{2N} \sum_{\nu=-N+1}^{N} \varphi\left(\frac{(\nu + \frac{1}{2})\pi}{N}\right) \tag{3.22}$$

valid for an arbitrary trigonometric polynomial $\varphi(\theta)$ of degree $2N - 1$. We choose $N = n + 2$ and

$$\varphi(\theta) = \sin (p + 1)\theta \sin (q + 1)\theta, \tag{3.23}$$
$$\varphi(\theta) = \cos \theta \sin (p + 1)\theta \sin (q + 1)\theta$$

where $p, q = 0, 1, \cdots, n$; both polynomials are of degree $2n + 3$ and they vanish for $\theta = 0$ and $\theta = \pi$. Defining θ_ν by (3.19), we have

$$\begin{aligned}\sum_{\nu=1}^{n+1} l_{\nu p} l_{\nu q} &= \frac{2}{n+2} \sum_{\nu=1}^{n+1} \sin (p + 1)\theta_\nu \sin (q + 1)\theta_\nu \\ &= \frac{1}{n+2} \sum_{\nu=-n-1}^{n+2} \sin (p + 1)\theta_\nu \sin (q + 1)\theta_\nu \\ &= \frac{1}{\pi} \int_{-\pi}^{\pi} \sin (p + 1)\theta \sin (q + 1)\theta \, d\theta = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(p-q)\theta} \, d\theta,\end{aligned}$$

$$\begin{aligned}\sum_{\nu=1}^{n+1} \cos \theta_\nu \cdot l_{\nu p} l_{\nu q} &= \frac{2}{n+2} \sum_{\nu=1}^{n+1} \cos \theta_\nu \sin (p + 1)\theta_\nu \sin (q + 1)\theta_\nu \\ &= \frac{1}{n+2} \sum_{\nu=-n-1}^{n+2} \cos \theta_\nu \sin (p + 1)\theta_\nu \sin (q + 1)\theta_\nu \\ &= \frac{1}{\pi} \int_{-\pi}^{\pi} \cos \theta \sin (p + 1)\theta \sin (q + 1)\theta \, d\theta \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos \theta \cdot e^{i(p-q)\theta} \, d\theta,\end{aligned}$$

and this establishes the assertion.

(*b*) We turn now to the general case (3.18). Let us denote the eigen-values of (θ) by $\lambda_\nu^{(n)}$ and those of $1 - \cos \theta$ by λ_ν [see (3.19)]. We introduce the real vectors

$$\mathbf{l}_\nu \colon l_{\nu 0}, l_{\nu 1}, \cdots, l_{\nu n},$$
$$\mathbf{u} \colon u_0, u_1, \cdots, u_n.$$

By the lemma in §3.3 we have $\lambda_\nu^{(n)} \leq T_n(\bar{f})/T_n(1)$ with any non-zero vector $\mathbf{u}$ satisfying certain suitable $\nu - 1$ linear conditions. We choose the constants $A_1, A_2, \cdots, A_\nu$ not all zero such that

$$\mathbf{u} = A_1 \mathbf{l}_1 + A_2 \mathbf{l}_2 + \cdots + A_\nu \mathbf{l}_\nu \tag{3.24}$$

satisfies these conditions. With a proper normalization

$$T_n(1) = A_1^2 + A_2^2 + \cdots + A_\nu^2 = 1. \tag{3.25}$$

On the other hand, $z = e^{i\theta}$,

$$
(3.26)\qquad \begin{aligned} T_n(\bar{f}) &= \frac{1}{2\pi}\int_{-\pi}^{\pi}(1-\cos\theta)\,|\,u_0+u_1z+\cdots+u_nz^n\,|^2\,d\theta \\ &\quad+\frac{d}{2\pi}\int_{-\pi}^{\pi}(1-\cos\theta)^2\,|\,u_0+u_1z+\cdots+u_nz^n\,|^2\,d\theta. \end{aligned}
$$

The first part, in view of (3.20) and (3.25), is

$$
\sum_{h=1}^{n+1}(1-\cos\theta_h)(\mathbf{l}_h,\mathbf{u})^2=\sum_{h=1}^{n+1}\lambda_h(\mathbf{l}_h,\mathbf{u})^2=\sum_{h=1}^{\nu}\lambda_hA_h^2\leqq\lambda_\nu\sum_{h=1}^{\nu}A_h^2=\lambda_\nu\cong\bar{f}\left(\frac{\nu\pi}{n}\right),
$$

provided ν is fixed and $n\to\infty$. We show that the second part is $O(n^{-3})$. Indeed, $(1-\cos\theta)^2=\frac{1}{4}\,|\,1-z\,|^4$, and

$$
(3.27)\qquad \begin{aligned} &\frac{1}{2\pi}\int_{-\pi}^{\pi}|\,(1-z)^2(u_0+u_1z+\cdots+u_nz^n)\,|^2\,d\theta \\ &=\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\sum_{h=0}^{n+2}(u_h+u_{h-2}-2u_{h-1})z^h\right|^2d\theta \\ &=\sum_{h=0}^{n+2}(u_h+u_{h-2}-2u_{h-1})^2;\qquad u_{-2}=u_{-1}=u_{n+1}=u_{n+2}=0. \end{aligned}
$$

Now for fixed ν and h

$$
\begin{aligned} u_h&=A_1l_{1h}+A_2l_{2h}+\cdots+A_\nu l_{\nu h}, \\ u_h^2&\leqq l_{1h}^2+l_{2h}^2+\cdots+l_{\nu h}^2=O(n^{-3}) \end{aligned}
$$

since $l_{\nu h}=O(n^{-\frac{1}{2}}\theta_\nu)=O(n^{-\frac{3}{2}})$. Similar estimates hold for u_{n-h}. Further we have for any h

$$
\begin{aligned} (u_h+u_{h-2}-2u_{h-1})^2&\leqq\sum_{k=1}^{\nu}(l_{kh}+l_{k,\,h-2}-2l_{k,\,h-1})^2, \\ l_{kh}+l_{k,\,h-2}-2l_{k,\,h-1}&=\left(\frac{2}{n+2}\right)^{\frac{1}{2}}[\sin(h+1)\theta_k+\sin(h-1)\theta_k-2\sin h\theta_k] \\ &=\left(\frac{2}{n+2}\right)^{\frac{1}{2}}\cdot2\sin h\theta_k\,(\cos\theta_k-1)=O(n^{-\frac{1}{2}}\theta_k^2) \\ &=O(n^{-\frac{1}{2}}n^{-2})=O(n^{-\frac{5}{2}}) \end{aligned}
$$

and

$$
\sum_{h=0}^{n+2}n^{-5}=O(n^{-4}).
$$

This establishes the assertion.

3.6. Another lower approximation. By using Weierstrass' theorem we can replace the lower approximation $f(\theta)$ in §3.3(b) by a function of the form $g(\theta) = (1 - \cos\theta)\, h(\theta)$ where $h(\theta)$ is a positive trigonometric polynomial. That is, we have for all θ

$$g(\theta) \leqq f(\theta) \tag{3.28}$$

and

$$|f''(0) - g''(0)| < \epsilon. \tag{3.29}$$

For the sake of simplicity we can also assume that $h(\theta)$ is a cosine polynomial. Thus it suffices to prove that the eigen-values $\lambda_\nu^{(n)} \geqq g(\nu\pi/n)(1 + \epsilon)$ where ν is fixed, $n \to \infty$, and $\epsilon = \epsilon(\nu, n) \to 0$.

Let l be the degree of $g(\theta)$ and

$$N = [\tfrac{1}{2}(n + l + 1)], \tag{3.30}$$

so that $N \geqq \frac{1}{2}(n + l + 1)$, $2N - 1 \geqq n + l$; hence (3.22) can be used in the following manner:

$$\frac{T_n(g)}{T_n(1)} = \frac{\sum_{k=-N+1}^{N} g(k\pi/N)u(k\pi/N)}{\sum_{k=-N+1}^{N} u(k\pi/N)} = \frac{\sum_{k=-N+1}^{N} g((k+\frac{1}{2})\pi/N)u((k+\frac{1}{2})\pi/N)}{\sum_{k=-N+1}^{N} u((k+\frac{1}{2})\pi/N)}. \tag{3.31}$$

$$u(\theta) = |u_0 + u_1 z + \cdots + u_n z^n|^2, \qquad z = e_i^\theta.$$

We assume that the u_ν are real. In order to estimate $\lambda_1^{(n)}$ from below, we use the second formula in (3.31), the smallest factor g being $g(\pm\pi/2N) \cong g(\pi/n)$. In order to estimate $\lambda_2^{(n)}$ from below, we can impose on the u_ν one arbitrary (real) condition. We use the first formula in (3.31) with

$$u_0 + u_1 z + \cdots + u_n z^n = 0 \text{ for } z = 1.$$

Since the smallest remaining g is $g(\pm\pi/N) \cong g(2\pi/n)$ we have the assertion. Estimating $\lambda_3^{(n)}$ from below we use again the second formula with

$$u_0 + u_1 z + \cdots + u_n z^n = 0, \qquad z = e^{\pm i\pi/2N}$$

and the smallest remaining factor g is $g(\pm 3\pi/2N) \cong g(3\pi/n)$, *etc.* This establishes the assertion.

3.7. Another class of functions $f(\theta)$. Let $0 < \alpha < 2$. We assume that the function $f(\theta)$ is such that $[f(\theta)]^{2/\alpha}$ satisfies Condition B of §3.2, $\theta_0 = 0$, $m = 0$. This is equivalent to the assumption that $f(\theta)$ has the form

$$f(\theta) = \theta^\alpha g(\theta), \tag{3.32}$$

where $g(\theta)$ is a positive and continuous function and $[f(\theta)]^{2/\alpha}$ has a continuous second derivative in a certain neighborhood of the origin. Under this condition we prove that the eigen-values $\lambda_\nu^{(n)}$ of $f(\theta)$ are still of the order $f(\nu\pi/n)$ in the following sense: Let ν be fixed, $n \to \infty$; then

$$A < \lambda_\nu^{(n)} n^\alpha < B, \tag{3.33}$$

where A and B are two positive constants depending on ν but independent of n [they depend of course on $f(\theta)$].

(a) An essential tool of the proof is the fact that the expression

$$\left\{ \frac{\int_{-\pi}^{\pi} [f(\theta)]^p u(\theta)\, d\theta}{\int_{-\pi}^{\pi} u(\theta)\, \alpha\theta} \right\}^{\frac{1}{p}}, \quad u(\theta) = | u_0 + u_1 z + \cdots + u_n z^n |^2, \qquad z = e^{i\theta}, \tag{3.34}$$

is an increasing function of p.

We compare first the cases $p = 1$ and $p = 2/\alpha$. Using the lemma in §3.3 and the theorem in §3.2, we find

$$\lambda_\nu^{(n)} \leqq f(\nu\pi/n)(1 + \epsilon) \cong (\nu\pi/n)^\alpha g(0), \tag{3.35}$$

where $\epsilon = \epsilon(\nu, n) \to 0$ for fixed ν and $n \to \infty$.

(b) On the other hand we compare the case $p = 1$ with the limiting case $p = 0$ to which the geometric mean

$$\exp \left\{ \frac{\int_{-\pi}^{\pi} \log f(\theta) \cdot u(\theta)\, d\theta}{\int_{-\pi}^{\pi} u(\theta)\, d\theta} \right\} \tag{3.36}$$

corresponds. Now

$$\log f(\theta) > \alpha \log | 1 - e^{i\theta} | + \text{const.}$$

In view of the Fourier expansion

$$\log | 1 - e^{i\theta} | = - \sum_{n=1}^{\infty} \frac{\cos n\theta}{n} \tag{3.37}$$

we find for real u_ν:

$$\begin{aligned} \frac{1}{2\pi} \int_{-\pi}^{\pi} \log | 1 - e^{i\theta} | \cdot u(\theta)\, d\theta &= -\tfrac{1}{2} \sum_{\substack{p, q = 0, 1, \cdots, n; \\ p \neq q}} \frac{u_p u_q}{p - q} \\ &= - \sum_{k=1}^{n} \frac{u_0 u_k + u_1 u_{k+1} + \cdots + u_{n-k} u_n}{k}. \end{aligned}$$

But $u_0 u_k + u_1 u_{k+1} + \cdots + u_{n-k} u_n \leqq u_0^2 + u_1^2 + \cdots + u_n^2$ so that

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \log |1 - e^{i\theta}| \cdot u(\theta)\, d\theta \geqq -\sum_{k=1}^{n} \frac{1}{k} \cdot \frac{1}{2\pi} \int_{-\pi}^{\pi} u(\theta)\, d\theta.$$

This yields for (3.36) the lower bound

$$\exp(-\alpha \log n + \text{const.})$$

which shows indeed that $\lambda_\nu^{(n)}$ is of the order $n^{-\alpha}$.

Chapter IV

On the Eigen-values of Certain Integral Equations

4.1. Statement of the main result. In this chapter we shall discuss some continuous analogues of Szegö's results concerning Toeplitz matrices (see §1.5). It will be more convenient to treat first a somewhat more general case and to postpone the special case until later.

There is a variety of conditions under which our theorem can be proved. To avoid unessential technicalities we have chosen particularly simple although rather restrictive conditions.

Let $\rho(x)$ be an even function in $L(-\infty, \infty)$. Let

$$F(\xi) = \int_{-\infty}^{\infty} \rho(x)e^{i\xi x}\, dx, \tag{4.1}$$

and assume that also $F(\xi) \varepsilon L(-\infty, \infty)$. It follows that the Fourier inversion formula

$$\rho(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\xi)e^{-i\xi x}\, d\xi \tag{4.2}$$

holds and that $\rho(x)$ is continuous.

Furthermore, let $K(x)$ be an even function such that $K(x)$ is continuous almost everywhere,

$$| K(x) | < M,$$

and

$$K(x) \varepsilon L^2(-\infty, \infty).$$

We consider now the integral equation

$$\int_{-\infty}^{\infty} K(\alpha x)\rho(x - y)K(\alpha y)\varphi(y)\, dy = \lambda\varphi(x), \tag{4.3}$$

and denote by $\lambda_1(\alpha), \lambda_2(\alpha), \cdots$, its eigen-values [since $\iint_{-\infty}^{\infty} K^2(\alpha x)\rho^2(x - y)\cdot K^2(\alpha y)\, dx\, dy < \infty$ the spectrum is discrete]. We state the main result of this chapter as follows.

Theorem: *Let $N_\alpha(a, b)$ denote the number of eigen-values $\lambda_j(\alpha)$ which fall within the interval (a, b). If (a, b) does not contain* 0 *and if the sets of points (x, ξ) for which $K^2(x)F(\xi)$ is equal to either a or b are of two-dimensional measure* 0, *then*

$$\lim_{\alpha \to 0} \alpha N_\alpha(a, b) = \frac{1}{2\pi}\Omega(a, b), \tag{4.4}$$

where $\Omega(a, b)$ *is the measure of the plane set* (x, ξ) *defined by the inequalities*

$$a < K^2(x)F(\xi) < b. \tag{4.5}$$

4.2. Proof of the theorem. In order to prove our theorem we need two lemmas, the principal one being the following

Lemma 4.2.1: *Under the conditions of the preceding section we have for* $n \geqq 2$

$$\lim_{\alpha \to 0} \alpha \sum_{j=1}^{\infty} \lambda_j(\alpha)^n = \frac{1}{2\pi} \int_{-\infty}^{\infty} K^{2n}(x)\, dx \int_{-\infty}^{\infty} F^n(\xi)\, d\xi. \tag{4.6}$$

We carry out the proof for $n = 3$, as it contains all the features of the general case.

From the general theory of integral equations we have

$$\sum_1^{\infty} \lambda_j^3(\alpha)$$

$$= \iiint_{-\infty}^{\infty} K^2(\alpha x_1)K^2(\alpha x_2)K^2(\alpha x_3)\rho(x_1 - x_2)\rho(x_2 - x_3)\rho(x_3 - x_1)\, dx_1\, dx_2\, dx_3,$$

Introducing the new variables

$$\alpha x_3 = t_3, \qquad \alpha x_2 = t_3 + \alpha t_2, \qquad \alpha x_1 = t_3 + \alpha t_2 + \alpha t_1$$

we obtain

$$\sum_1^{\infty} \lambda_j^3(\alpha)$$

$$= \iiint_{-\infty}^{\infty} K^2(t_3 + \alpha t_2 + \alpha t_1)K^2(t_3 + \alpha t_2)K^2(t_3)\rho(t_1)\rho(t_2)\rho(t_1 + t_2)\, dt_1\, dt_2\, dt_3,$$

and since $K^2(x) < M^2$ and $|\rho(x)| < A$ we see that the integrand is dominated by

$$M^4AK^2(t_3)\,|\rho(t_1)|\,|\rho(t_2)|,$$

which is absolutely integrable. As $\alpha \to 0$ the integrand approaches (for almost every t_3)

$$K^6(t_3)\rho(t_1)\rho(t_2)\rho(t_1 + t_2),$$

and hence by dominated convergence we have

$$\lim_{\alpha \to 0} \alpha \sum_1^{\infty} \lambda_j^3(\alpha) = \int_{-\infty}^{\infty} K^6(t_3)\, dt_3 \iint_{-\infty}^{\infty} \rho(t_1)\rho(t_2)\rho(t_1 + t_2)\, dt_1\, dt_2.$$

Our lemma now follows from the fact that

$$\iint_{-\infty}^{\infty} \rho(t_1)\rho(t_2)\rho(t_1 + t_2)\, dt_1\, dt_2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} F^3(\xi)\, d\xi.$$

Once Lemma 4.2.1 has been established, the proof of the main theorem, except for a minor complication, proceeds along standard lines.

We first show by an adaptation of a familiar argument (see the lemma in §2.1) that $|\lambda_j(\alpha)| < C$, where C is independent of α.

Let $\varphi(x)$ be an eigen-function belonging to the eigen-value λ and set

$$\psi(x) = \frac{\varphi(x)}{K(\alpha x)} \quad \text{if} \quad K(\alpha x) \neq 0,$$

$$\psi(x) = 0 \quad \text{if} \quad K(\alpha x) = 0.$$

It is then clear from (4.3) that

$$\int_{-\infty}^{\infty} \rho(x - y)K^2(\alpha y)\psi(y)\, dy = \lambda\psi(x),$$

and it follows that $\psi(x)$ is continuous and bounded in $(-\infty, \infty)$. Let

$$L = \operatorname*{l.u.b.}_{-\infty<x<\infty} |\psi(x)|$$

and let $\{x_k\}$ be a sequence such that

$$|\psi(x_k)| \to L \text{ as } k \to \infty.$$

We have

$$|\lambda|\,|\psi(x_k)| \leqq LM^2 \int_{-\infty}^{\infty} |\rho(x - y)|\, dy = LM^2 \int_{-\infty}^{\infty} |\rho(y)|\, dy.$$

Letting $k \to \infty$ and noting that $L \neq 0$ we obtain

$$|\lambda| \leqq M^2 \int_{-\infty}^{\infty} |\rho(y)|\, dy = C.$$

Now let (a, b) be an interval contained in $(-C, C)$ and not containing 0. Let $\gamma(x)$ be the characteristic function of (a, b) and define as usual the functions $\gamma_\delta^-(x)$ and $\gamma_\delta^+(x)$ as follows

$$\gamma_\delta^-(x) = \begin{cases} 1, & a + \delta \leqq x \leqq b - \delta, \\ \dfrac{x - a}{\delta}, & a \leqq x \leqq a + \delta, \\ \dfrac{b - x}{\delta}, & b - \delta \leqq x < b, \\ 0, & \text{otherwise} \end{cases}$$

$$\gamma_\delta^+(x) = \begin{cases} 1 \quad , & a \leqq x \leqq b, \\ \dfrac{x - a + \delta}{\delta}, & a - \delta \leqq x \leqq a, \\ \dfrac{b + \delta - x}{\delta}, & b \leqq x \leqq b + \delta, \\ 0 \quad , & \text{otherwise.} \end{cases}$$

In imitating the standard procedure we must now approximate $\gamma_\delta^-(x)$ and $\gamma_\delta^+(x)$ in $(-C, C)$ by polynomials. However, since our lemma was proved only for $n \geqq 2$ we can use only polynomials in which no terms of degree 0 or 1 occur. The following trivial lemma takes care of this difficulty.

Lemma 4.2.2: *If $f(x)$, $-C \leqq x \leqq C$, is such that*

$$\lim_{x \to 0} \frac{f(x)}{x^2}$$

exists, then $f(x)$ can be approximated in $(-C, C)$ uniformly by polynomials in which no terms of degree 0 *or* 1 *occur.*

The proof is trivial as it is sufficient to notice that the function $g(x)$ defined by the formulas

$$g(x) = \frac{f(x)}{x^2}, \quad x \neq 0,$$

and

$$g(x) = \lim_{x \to 0} \frac{f(x)}{x^2},$$

is continuous and can thus be approximated uniformly by polynomials. Since $|x| \leqq C$ our lemma follows immediately.

If δ is sufficiently small $\gamma_\delta^+(x)$ satisfies the conditions of our lemma ($\gamma_\delta^-(x)$ satisfies the condition automatically), and hence we can carry out the steps of the usual proof and thus complete the proof of our theorem.

4.3. An application to the zeros of Bessel functions. We set

$$\rho(x) = \tfrac{1}{2} e^{-|x|}, \qquad K(x) = e^{-|x|}$$

so that the integral equation assumes the form

$$\tfrac{1}{2} \int_{-\infty}^{\infty} e^{-\alpha|x|} \, e^{-|x-y|} \, e^{-\alpha|y|} \, \varphi(y) \, dy = \lambda \varphi(x).$$

The eigen-values of this equation can be found by imitating the procedure used by M. L. Juncosa [7] in a case of a similar integral equation, and the result is as follows.

Let $r_1, r_2, r_3, \cdots$, be the positive roots of $J'_{1/\alpha}(x) = 0$ and $t_1, t_2, t_3, \cdots$, the positive roots of $J_{1/\alpha}(x) = 0$. Then the eigen-values are

$$(r_1\alpha)^{-2},\ (t_1\alpha)^{-2},\ (r_2\alpha)^{-2},\ (t_2\alpha)^{-2},\ \cdots .$$

Now $N_\alpha(a, b)$ is simply the number of r's and t's which satisfy the inequalities

$$\alpha^{-1}b^{-\frac{1}{2}} < r < \alpha^{-1}a^{-\frac{1}{2}}, \qquad \alpha^{-1}b^{-\frac{1}{2}} < t < \alpha^{-1}a^{-\frac{1}{2}}$$

and since the r's and t's interlace, $N_\alpha(a, b)$ is asymptotically twice the number of r's in the interval $(\alpha^{-1}b^{-\frac{1}{2}}, \alpha^{-1}a^{-\frac{1}{2}})$.

An elementary computation gives

$$\Omega(a, b) = f(b) - f(a), \text{ where } f(x) = 2\{(x^{-1} - 1)^{\frac{1}{2}} - \text{arc tan } (x^{-1} - 1)^{\frac{1}{2}}\},$$

and thus setting $b = 1$ we obtain

$$\lim_{\alpha\to 0} \alpha M_\alpha(a) = \pi^{-1}\{(a^{-1} - 1)^{\frac{1}{2}} - \text{arc tan } (a^{-1} - 1)^{\frac{1}{2}}\},$$

where $M_\alpha(a)$ is the number of roots of $J_{1/\alpha}(x) = 0$ which are less than $\alpha^{-1}a^{-\frac{1}{2}}$. Putting $\alpha^{-1} = n$, $a^{-\frac{1}{2}} = y$, $y > 1$, we get

$$\lim_{n\to\infty} n^{-1}N_n(y) = \pi^{-1}\{y^2 - 1)^{\frac{1}{2}} - \text{arc tan } (y^2 - 1)^{\frac{1}{2}}\}, \tag{4.7}$$

where $N_n(y)$ is the number of roots of $J_n(x) = 0$ which are less than $ny^{\frac{1}{2}}$.

We claim now that if $r_n(n)$ is the nth positive root of $J_n(x) = 0$ the limit

$$\lim_{n\to\infty} n^{-1}r_n(n)$$

exists. For suppose that

$$\limsup_{n\to\infty} n^{-1}r_n(n) = B, \qquad \liminf_{n\to\infty} n^{-1}r_n(n) = A.$$

For infinitely many n

$$n^{-1}r_n(n) > B - \epsilon,$$

and hence for these n, $N_n(B - \epsilon) \leqq n$. From (4.7) it follows that

$$\pi^{-1}\{((B - \epsilon)^2 - 1)^{\frac{1}{2}} - \text{arc tan } ((B - \epsilon)^2 - 1)^{\frac{1}{2}}\} \leqq 1.$$

Similarly we get

$$\pi^{-1}\{((A + \epsilon)^2 - 1)^{\frac{1}{2}} - \text{arc tan } ((A + \epsilon)^2 - 1)^{\frac{1}{2}}\} \geqq 1.$$

Thus $A = B$ and it follows immediately that

$$\lim_{n\to\infty} n^{-1}r_n(n) = 1 + x_1^2, \tag{4.8}$$

where x_1 is the first positive root of the equation

$$\tan x = x. \tag{4.9}$$

4.4. A special case. If

$$K(x) = 1, \qquad |x| < 1,$$

$$K(x) = 0, \qquad |x| \geqq 1,$$

and $\alpha = A^{-1}$, our integral equation becomes

$$\int_{-A}^{A} \rho(x - y)\varphi(y)\, dy = \lambda\varphi(x), \tag{4.10}$$

and our main theorem assumes the following form:

If $N_A(a, b)$ *is the number of eigen-values of* (4.10) *falling within* (a, b) *and if* (a, b) *does not contain* 0, *then*

$$\lim_{A\to\infty} \frac{1}{2A} N_A(a, b) = \frac{1}{2\pi} \left| \underset{\xi}{E}\{a < F(\xi) < b\} \right|, \tag{4.11}$$

provided the sets where $F(\xi) = a$ *or* $F(\xi) = b$ *are of measure* 0.

Here, as usual, $\underset{\xi}{E}\{\ \}$ denotes the set of ξ's for which the relation in brackets holds, and $|E|$ denotes the measure of the pertinent set.

Formula (4.11) is the analogue of Szegö's result (1.13).

Finally, it can be shown by the application of the Weyl-Courant lemma that $F(\xi)\ \varepsilon\ L$ implies absolute convergence of $\sum \lambda_j(a)$, where $\lambda_j(a)$ are the eigen-values of (4.10).

Thus the Fredholm determinant of (4.10) is

$$D_a(\lambda) = \prod_{j=1}^{\infty} (1 - \lambda\lambda_j),$$

and from (4.11) we obtain for sufficiently small $|\lambda|$

$$\lim_{a\to\infty} (D_a(\lambda))^{\frac{1}{a}} = \exp\left\{\frac{1}{\pi}\int_0^\infty \log(1 - \lambda F(\xi))\, d\xi\right\} \tag{4.12}$$

which is the analogue of Szegö's result (1.9).

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