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Orthogonal Matrix Polynomials Satisfying Second-Order Differential Equations

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1 Introduction

The aim of this paper is to develop a general method that leads us to introduce and study classes of examples of orthonormal matrix polynomials $(P_n)_n$ satisfying a right-hand side second-order differential equations of the form

$$P_n''(t)A_2(t) + P_n'(t)A_1(t) + P_n(t)A_0(t) = \Gamma_n P_n(t), \quad n \geq 0, \tag{1.1}$$

where A_2, A_1, and A_0 are matrix polynomials (which do not depend on n) of degrees not bigger than 2, 1, and 0, respectively, and Γ_n are Hermitian matrices (i.e., each orthonormal matrix polynomial P_n is an eigenvector of the right-hand side second-order differential operator $\ell_{2,R} = D^2A_2(t) + D^1A_1(t) + D^0A_0(t)$).

All matrices have a common size $N \times N$ and D stands for the usual differentiation operator.

Some readers will be intrigued or annoyed by the fact that the coefficients in (1.1) appear on the *right* side of the argument. This concern will be addressed in Section 2.

We recall some basic definitions, see for instance [1, 4, 5, 6, 7, 10, 11]. A matrix weight W is an $N \times N$ matrix of measures supported in the real line such that $W(A)$ is positive semidefinite for any Borel set $A \subset \mathbb{R}$. We also assume that W has finite moments and that $\int P(t)dW(t)P^*(t)$ is nonsingular if the leading coefficient of the matrix polynomial P is nonsingular.

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This last condition is necessary and sufficient to guarantee the existence of a sequence $(P_n)_n$ of matrix polynomials orthogonal with respect to W and P_n of degree n and with nonsingular leading coefficient. Such condition is fulfilled, in particular, when $dW = W(t)d\mu$ with μ a scalar positive measure and W a Hermitian matrix of integrable functions with respect to μ satisfying that $W(t)$ is positive definite at infinitely many points in the support of μ.

Just as in the scalar case, a sequence of orthonormal matrix polynomials $(P_n)_n$ satisfies a three-term recurrence relation

$$tP_n(t) = A_{n+1}P_{n+1}(t) + B_nP_n(t) + A_n^*P_{n-1}(t), \quad n \geq 0, \tag{1.2}$$

where $P_{-1}(t) = \theta$, A_n are nonsingular matrices, and B_n are Hermitian (here and in the rest of this paper, we write θ for the null matrix, the dimension of which can be determined from the context). This three-term recurrence relation characterizes the orthonormality of a sequence of matrix polynomials with respect to a positive definite matrix of measures.

We remark that the polynomials $R_n(t) = U_nP_n(t)$ with $U_nU_n^* = I$ are also orthonormal with respect to the same positive definite matrix of measures with respect to which the $(P_n)_n$ are orthonormal and satisfy a three-term recurrence relation as (1.2) with coefficients $U_{n-1}A_nU_n^*$ instead of A_n and $U_nB_nU_n^*$ instead of B_n. Here and in the rest of the paper, I denotes the identity matrix, whose dimension will be determined from the context.

By choosing an appropriate sequence of unitary matrices U_n, it is then clear that (if desirable) one can assume that the sequence of eigenvalue matrices in (1.1) is *diagonal*.

Many basic results of the theory of scalar orthogonal polynomials, like Favard's theorem, quadrature formulae, and asymptotic properties (such as Markov's theorem, ratio, weak, and zero asymptotics), have been extended to orthogonal matrix polynomials by Durán (see [5, 6] and the references therein). A different source for the problem considered in this paper is the study of the *bispectral problem* pursued by Grünbaum in a series of papers which started with [3] and continued in the context of orthogonal polynomials. (See [8, 9] and the references therein.)

In this paper, we add to the list of properties mentioned above some fundamental examples of families of orthogonal matrix polynomials. These families satisfy a right-hand side second-order differential equations as in (1.1). The existence of these orthogonal families strongly depends on the noncommutativity of the matrix product, the existence of nonzero singular matrices, and the size N.

Many other authors, starting from Kreĭn [10, 11], have contributed to the theory of matrix-valued orthogonal polynomials. For some of these references, see [1, 4, 5, 6, 7, 9].

We say that two weight matrices $W_1(t)$ and $W_2(t)$ are similar if there exists a nonsingular matrix T (independent of t) such that $W_1 = TW_2T^*$. Given this notion of similarity, it is important to single out two special cases.

We say that a weight matrix W *reduces to a lower size* if there exists a nonsingular matrix T for which

$$W(t) = T \begin{pmatrix} Z_1(t) & \theta \\ \theta & Z_2(t) \end{pmatrix} T^*, \tag{1.3}$$

where Z_1 and Z_2 are weight matrices of lower size. Notice that the orthonormal matrix polynomials with respect to W are then

$$P_n(t) = \begin{pmatrix} P_{n,1} & \theta \\ \theta & P_{n,2} \end{pmatrix} T^{-1}, \quad n \geq 0, \tag{1.4}$$

where $(P_{n,i})_n$ are the orthonormal matrix polynomials with respect to Z_i, $i = 1, 2$. Analogously, we say that W *reduces to scalar weights* if there exists a nonsingular matrix T for which

$$W(t) = TD(t)T^*, \tag{1.5}$$

with $D(t)$ diagonal. This is clearly an extreme case of the situation considered earlier.

If we assume that for some real number a, $W(a) = I$, then W reduces to scalar weights if and only if $W(t)W(s) = W(s)W(t)$ for all t and s. The commutativity condition on the weight matrix $W(t)$ gives a *convenient* way of checking if one is dealing with a case that reduces to scalar weights.

Notice that if a sequence of orthonormal polynomials with respect to W satisfies a second-order differential equation as (1.1), so does a sequence of orthonormal polynomials with respect to the weight matrix TWT^* for any nonsingular matrix T: just take the new differential coefficients equal to $TA_i(t)T^{-1}$. Then, when looking for weight matrices W having a sequence of orthonormal polynomials satisfying a second-order differential equation as in (1.1), it is not a restriction to assume that $W(a) = I$ for certain real number a as long as $W(t)$ is not singular for some t.

The practical consequence of the above considerations is that when looking for examples of size N satisfying (1.1), we assume that our weight matrix does not reduce to a lower size or to scalar weights either, and that for certain real a, we have $W(a) = I$. We observe that in [9] one finds a notion of similarity for the pair $\{W, \ell_{2,R}\}$. This notion allows one to distinguish certain cases that are considered equivalent under the present definition.

The paper is organized as follows. Section 2 discusses differential operators acting on matrix-valued functions and recalls the relation between symmetric operators ℓ_2 and (1.1) above. Section 3 expresses the symmetry of ℓ_2 as a set of differential equations involving the weight matrix $W(t)$ and the coefficients in (1.1). Section 4 gives a method for solving these equations. Sections 5, 6, and 7 deal with 3 special instances where the equations of Section 4 can be explicitly solved to yield interesting families of matrix-valued orthogonal polynomials satisfying (1.1). In Section 8, we give a small sample of structural properties satisfied by our new families of orthogonal polynomials.

We are confident that the method presented here will continue to bear fruits in the form of many families of useful examples. While in the scalar case the only possible examples are the familiar Hermite, Laguerre, and Jacobi polynomials (see [2]), the complexity of the matrix-valued situation opens the door to an *embarrassment of riches*. The matricial weights that appear in the examples given in Sections 5, 6, and 7 have the form $\rho(t)P(t)$, where $\rho(t)$ is one of the so-called scalar classical weights. The second factor is a matrix-valued polynomial built along with $\rho(t)$ out of some properties that are *only present in the matrix case*. In particular, we use the existence of singular (and yet nonzero) matrices. These and other points will be clear in the next few sections. In particular, most of the examples are inspired by the structure of $\mathfrak{sl}(2)$ and its irreducible representations. The consideration of a two-dimensional solvable subalgebra suffices.

2 Right-hand side second-order differential operators

In considering differential operators, it is customary to write them as linear combinations of products of functions of t multiplied on the right by powers of the differentiation operator.

We could therefore consider right-hand side operators

$$\ell_{2,R} = D^2 A_2 + D^1 A_1 + D^0 A_0, \tag{2.1}$$

as well as left-hand side operators

$$\ell_{2,L} = A_2 D^2 + A_1 D^1 + A_0 D^0. \tag{2.2}$$

We briefly discuss the reason that makes right-hand side operators more natural and interesting in relation with matrix inner products defined by a weight matrix W in the usual form

$$\langle P, Q\rangle_1 = \int P(t)dW(t)Q^*(t), \tag{2.3}$$

while left-hand side differential operators are more convenient when the inner product is defined in the *less* natural way

$$\langle P, Q\rangle_2 = \int Q^*(t)dW(t)P(t). \tag{2.4}$$

It turns out that right-hand side operators are left linear but not right linear; that is, $\ell_{2,R}(CP) = C\ell_{2,R}(P)$, with P a matrix function and C a constant matrix, but, in general, $\ell_{2,R}(PC) \neq \ell_{2,R}(P)C$. Analogously, left-hand side operators are right linear but not left linear: $\ell_{2,L}(PC) = \ell_{2,L}(P)C$, but, in general, $\ell_{2,L}(CP) \neq C\ell_{2,L}(P)$.

The lack of left linearity of the left-hand side operators has certain undesirable consequences.

Lemma 2.1 [4, Lemma 2.1]**.** Let W be a weight matrix and $(P_n)_n$ a sequence of orthonormal polynomials with respect to it. Then, for a right-hand side second-order differential operator $\ell_{2,R}$, the following conditions are equivalent.

(a) The operator $\ell_{2,R}$ is symmetric with respect to the inner product of the form (2.3), that is, $\langle \ell_2(P), Q\rangle = \langle P, \ell_2(Q)\rangle$, for any matrix polynomials P and Q.

(b) The orthonormal polynomial P_n is an eigenvector of $\ell_{2,R}$ with left eigenvalue Γ_n Hermitian: $\ell_{2,R}(P_n) = \Gamma_n P_n$, $n = 0, 1, \ldots$.

For a left-hand side operator, (a) also implies (b) but (b) does not, in general, imply (a). □

We observe that, *from the beginning*, we are assuming that the coefficients of our second-order differential operator are matrix polynomials satisfying a degree condition that insures that the space of matrix polynomials of a given degree is invariant under the action of the differential operator.

A lemma analogous to Lemma 2.1 can be given for left-hand side operators and inner products of the form (2.4). Such an approach is used in [9], a paper that grew out of a progression starting with [8]. In [8], the search for matrix-valued spherical functions for the *complex projective plane* yields a family of matrix-valued polynomials $Q_n(t)$ that satisfy a three-term recursion relation as in (1.2) and a differential equation of the form

$$EQ_n(t)^* = Q_n(t)^*\Lambda_n, \tag{2.5}$$

where the operator E is given by

$$E = A_2(t)\frac{d^2}{dt^2} + A_1(t)\frac{d}{dt} + A_0(t). \tag{2.6}$$

It is also clear that an equation like (1.1) is equivalent to one involving the above operator if one exchanges every coefficient $A_i(t)$ by its adjoint and makes the same replacement for $P_n(t)$. A corresponding change has to be made on the right-hand side of (1.1). We will use the form in expression (1.1) but one should recall that both formulations are entirely equivalent.

There is another reason to consider left-hand side operators as less interesting (than right-hand ones) when the inner product (2.3) is used: as it was proved by Durán (see [4, Theorem 3.2]), in the matrix case all the examples of weight matrices having a symmetric left-hand side second-order operator reduce to the scalar classical examples.

For all the reasons given above, in the rest of the paper, we always consider right-hand side operators. We stress that we will make *no commutativity* assumptions on the coefficients of this differential operator. This brings in certain difficulties, but it opens up the field to interesting examples. The undesirable effect of making some simplifying assumptions, such as $A_0W = WA_0^*$, can be seen in [4].

3 The differential equations for the weight matrix

In this section, we convert the condition of symmetry for the pair made up of a weight matrix W and a right-hand side second-order differential operator $\ell_{2,R}$, namely, $\langle \ell_2(P), Q\rangle_1 = \langle P, \ell_2(Q)\rangle_1$, for any matrix polynomials P and Q, into a set of differential equations relating W and the coefficients of $\ell_2 = \ell_{2,R}$.

To establish Theorem 3.1, we write the coefficients of the differential operator

$$\ell_2 = D^2A_2(t) + D^1A_1(t) + D^0A_0(t), \tag{3.1}$$

as $A_2(t) = t^2A_{2,2} + tA_{2,1} + A_{2,0}$ and $A_1(t) = tA_{1,1} + A_{1,0}$, where $A_{i,j}$ are arbitrary constant matrices and we denote with μ_n, $n \geq 0$, the moments of the matrix weight W, that is, $\mu_n = \int t^n dW(t)$. Recall that, by assumption, $A_0(t) = A_0(0)$ is independent of t. We assume that $dW(t) = W(t)dt$ with a smooth $W(t)$.

Theorem 3.1. The following three sets of conditions are equivalent.

(1) The operator ℓ_2 is symmetric with respect to W.

(2) For $n \geq 2$,

$$A_{2,2}\mu_n + A_{2,1}\mu_{n-1} + A_{2,0}\mu_{n-2} = \mu_n A_{2,2}^* + \mu_{n-1}A_{2,1}^* + \mu_{n-2}A_{2,0}^*; \tag{3.2}$$

for $n \geq 1$,

$$\begin{aligned} 2(1-n)\big(A_{2,2}\mu_n + A_{2,1}\mu_{n-1} + A_{2,0}\mu_{n-2}\big) - \big(A_{1,1}\mu_n + A_{1,0}\mu_{n-1}\big) \\ = \mu_n A_{1,1}^* + \mu_{n-1}A_{1,0}^*; \end{aligned} \tag{3.3}$$

and for $n \geq 0$

$$\begin{aligned} &n(n-1)\big(A_{2,2}\mu_n + A_{2,1}\mu_{n-1} + A_{2,0}\mu_{n-2}\big) \\ &\quad + n\big(A_{1,1}\mu_n + A_{1,0}\mu_{n-1}\big) + A_0\mu_n = \mu_n A_0^*. \end{aligned} \tag{3.4}$$

(3) One has the boundary conditions that

$$A_2(t)W(t), \qquad \big(A_2(t)W(t)\big)' - A_1(t)W(t) \tag{3.5}$$

should have vanishing limits at each of the endpoints of the support of $W(t)$, and the weight matrix W satisfies

$$A_2 W = W A_2^*, \tag{3.6}$$

$$2\big(A_2 W\big)' = W A_1^* + A_1 W, \tag{3.7}$$

$$\big(A_2 W\big)'' - \big(A_1 W\big)' + A_0 W = W A_0^*. \tag{3.8}$$

□

Proof. The first and third set of conditions are shown to be equivalent by modifying the proof of Lemma 2.2 in [4], refraining from the use of the relation $A_0 W = W A_0^*$ which was assumed in [4]. The equivalence with the third set of conditions is considered in [4] and is best handled by using integration by parts as in [9] (taking into account the boundary conditions (3.5)). ■

Not all the conditions given above are of equal importance. For instance, condition (3.7) is, under mild conditions, a consequence of (3.6) and (3.8). In spite of the *redundant character* of these conditions, we will see in Section 4 that (3.7) plays an important role in finding the general solution of the set of three equations (3.6), (3.7), and (3.8). Since the solution of these three equations constitutes the main tool of the paper, we feel it useful to think of these equations as being *on the same footing*.

4 A general method for solving the differential equations for the weight matrix

Equations (3.6), (3.7), and (3.8), briefly rewritten as

$$\begin{aligned} &A_2W = WA_2^*,\\ &2\big(A_2W\big)' - A_1W = WA_1^*,\\ &\big(A_2W\big)'' - \big(A_1W\big)' + A_0W = WA_0^*, \end{aligned} \tag{4.1}$$

constitute the basis of all the analysis that follows.

In this section, we obtain a *description* of all solutions for equations (4.1). We refrain from saying that we have solved the equations since the description in question involves the requirement that the matrix function (4.15) satisfies a certain Hermitian condition for all t.

We will strive to pull out of $W(t)$ an appropriate "scalar weight" denoted by $\rho(t)$ resulting in an expression for $W(t)$ that reduces to

$$W(t) = \rho(t)I \tag{4.2}$$

in the scalar case. This real-valued $\rho(t)$ will be arbitrary for the time being.

We now return to the task of solving the three equations in (4.1).

We are interested in the case when $A_2(t)$ is a *real-valued scalar matrix*: $A_2(t) = a_2(t)I$. We assume that $a_2(t)$ does not vanish inside the support of $W(t)$.

The first equation in (4.1) is trivially satisfied.

The second one

$$2\big(A_2W\big)' = WA_1^* + A_1W \tag{4.3}$$

is best handled after we introduce some auxiliary functions. The final product of this process will be a particularly appealing form for the weight matrix $W(t)$.

With a so far unspecified $\rho(t)$, define $c(t)$ by

$$c = \frac{\big(\rho a_2\big)'}{\rho}. \tag{4.4}$$

Now, define a matrix-valued function $F(t)$ by the relation

$$A_1(t) = 2a_2(t)F(t) + c(t)I. \tag{4.5}$$

We note that in the scalar case, $F(t)$ is identically zero. Herein lies the *main difference* from the scalar case. We will see that $F(t)$ gives rise to some nontrivial family of matrices $T(t)$ that give a *factorization* of $W(t)$.

Finally, introduce $Z(t)$ by the recipe $W = Z\rho$.

In terms of these functions, it is straightforward to see (A_2 is scalar) that the second equation in (4.1) becomes

$$Z'(t) = F(t)Z(t) + Z(t)F^*(t). \tag{4.6}$$

If we define the matrix-valued function $T(t)$ by $T'(t) = F(t)T(t)$, $T(a) = I$, we have

$$Z(t) = T(t)Z(a)T^*(t), \tag{4.7}$$

or finally, we get the *factorized* form of our weight matrix

$$W(t) = \frac{\rho(t)}{\rho(a)}T(t)W(a)T^*(t). \tag{4.8}$$

The choice of the value a is a matter of convenience.

This is the appealing form of $W(t)$ advertised above. In the scalar case, when, as we noticed earlier, $F(t)$ vanishes, it allows us to identify $W(t)$ with $\rho(t)$.

Factorizations as in (4.8) play a very important role in many areas of mathematics. Famous instances of them are connected with the names Riemann-Hilbert, Birkhoff, Wiener-Hopf, and Gohberg-Kreĭn.

Now, we consider the third equation in (4.1). By multiplying it by two and subtracting it from the derivative of the second one, we get

$$\left(WA_1^* - A_1W\right)' = 2\left(WA_0^* - A_0W\right) \tag{4.9}$$

or, using (4.5) and its adjoint, we get

$$\left(a_2WF^* - Fa_2W\right)' = \left(WA_0^* - A_0W\right). \tag{4.10}$$

Using the second equation in (4.1) a couple of times to replace $(a_2W)'$ and then using (4.5) and its adjoint once again, we get

$$W(t)\Big(a_2\left(F^*\right)' + a_2\left(F^*\right)^2 + cF^* - A_0^*\Big) = \left(a_2F' + a_2F^2 + cF - A_0\right)W(t). \tag{4.11}$$

If we finally put $\chi(t) = T^{-1}(t)(a_2(t)F'(t) + a_2(t)F^2(t) + c(t)F(t) - A_0)T(t)$ and use the above-obtained expression for $W(t)$, this is equivalent to the requirement that the product

$$\chi(t)W(a) \tag{4.12}$$

is Hermitian for all t.

We have thus proved the following theorem.

Theorem 4.1. Let ρ, a_2, A_1, and A_0 be a (real) scalar function, a (real) scalar polynomial with degree at most 2, and matrix polynomials with degrees less than or equal to 1 and 0, respectively. Define $A_2(t) = a_2(t)I$, the scalar function c as

$$c(t) = \frac{\big(\rho(t)a_2(t)\big)'}{\rho(t)}, \tag{4.13}$$

and the matrix function F as in (4.5). Write T for the solution of the differential equation

$$T'(t) = F(t)T(t), \qquad T(a) = I, \tag{4.14}$$

and define the matrix function χ as

$$\chi(t) = T^{-1}(t)\big(a_2(t)F'(t) + a_2(t)F^2(t) + c(t)F(t) - A_0\big)T(t). \tag{4.15}$$

If the matrix function $\chi(t)W(a)$ is Hermitian for all t, then the matrix weight

$$W(t) = \frac{\rho(t)}{\rho(a)}T(t)W(a)T^*(t) \tag{4.16}$$

satisfies the differential equations (4.1).

The converse is also true. □

The next three sections display a variety of interesting examples. All our examples have one thing in common: of the matrices A and B that will make up the matrix-valued function $F(t)$ introduced above, only one of them is allowed to be nonzero. We leave the consideration of *mixed cases* for a future publication. A few more words on this issue appear in Section 5.

We aim at obtaining only examples when $W(t)$ does not reduce to lower sizes. This will have the effect that either A or B will be assumed to be *unitarily irreducible*. This property, coupled with others, that either A or B will inherit from $F(t)$ as a consequence of Theorem 4.1, appears to restrict our choices for A or B rather severely. For instance, in Section 5, we will be considering only cases when they are essentially *nilpotent*. We have strong reasons to believe that this is not a real restriction and that our examples are very representatives of the general situation.

5 Examples with $A_2 = I$

In the scalar case, the only weight function having a symmetric second-order differential operator with $A_2 = 1$ is the Hermite weight function $w(t) = e^{-t^2}$ up to a linear change of variable.

We then take $\rho(t) = e^{-t^2}$, $A_2(t) = I$, and $a = 0$ (so that we look for weight matrices W satisfying $W(0) = I$). This gives for the function c (see (4.13)) the expression

$$c(t) = -2t, \tag{5.1}$$

and for the matrix function F in (4.5),

$$F(t) = tI + \frac{A_1(t)}{2}. \tag{5.2}$$

Taking into account that A_1 is a polynomial of degree at most 1, we can write

$$F(t) = 2Bt + A. \tag{5.3}$$

Here, as well as in Sections 6 and 7, the matrices A and B that determine the corresponding $F(t)$ are independent of t.

Hence, we are looking for the solutions T of the differential equation

$$T'(t) = (2Bt + A)T(t), \qquad T(0) = I. \tag{5.4}$$

If A and B do not commute, then in general one can only give a formal series solution for the differential equation (5.4). Such expressions in terms of *time ordering* are familiar in many fields, including quantum field theory. These expressions make it rather laborious to check whether the function χ in (4.15) is Hermitian. We study here two special, but rather interesting, cases: when either A or B vanishes. This may seem too modest a goal. See, however, the last paragraph of the next section.

5.1 Weight matrices of the form $e^{-t^2}e^{At}e^{A^*t}$

By solving equation (5.4) for the case $B = \theta$, we find $T(t) = e^{At}$, and then the weight matrix we are looking for has the form

$$W(t) = e^{-t^2}e^{At}e^{A^*t}, \tag{5.5}$$

where A is an $N \times N$ matrix.

We assume that A is not unitarily equivalent to block-diagonal (in particular, A is not a normal matrix), otherwise the weight matrix $e^{-t^2}e^{At}e^{A^*t}$ reduces to lower sizes (see (1.3)).

From Theorem 4.1, we have to check whether there is a convenient choice of A_0 so that the matrix function

$$\chi(t) = T^{-1}(t)\big(A_2(t)F'(t) + A_2(t)F^2(t) + c(t)F(t) - A_0\big)T(t) \tag{5.6}$$

is Hermitian, that is, we have to find those matrices A, not unitarily equivalent to block-diagonal, for which there exists A_0 such that the matrix function

$$\chi(t) = A^2 - 2tA - e^{-At}A_0e^{At}. \tag{5.7}$$

is Hermitian.

Using the standard notation

$$\mathrm{ad}_X^0 Y = Y, \qquad \mathrm{ad}_X^1 Y = [X, Y], \qquad \mathrm{ad}_X^2 Y = \big[X, [X, Y]\big], \tag{5.8}$$

and, in general, $\mathrm{ad}_X^{n+1} Y = [X, \mathrm{ad}_X^n Y]$, where $[X, Y] = XY - YX$, we can write (5.7) as

$$\begin{aligned}\chi(t) &= A^2 - 2At - e^{-t\,\mathrm{ad}_A}(A_0)\\ &= \big(A^2 - A_0\big) - t\big(2A - \mathrm{ad}_A A_0\big) - \sum_{n\geq 2}\frac{(-1)^n t^n}{n!}\mathrm{ad}_A^n A_0.\end{aligned} \tag{5.9}$$

The existence of a matrix A_0 such that (5.7) holds is equivalent to the existence of a matrix A_0 such that the following matrices

$$A^2 - A_0, \qquad 2A - \big[A, A_0\big], \qquad \mathrm{ad}_A^n A_0, \quad n \geq 2. \tag{5.10}$$

are Hermitian.

We now exhibit a maximal example for nilpotent matrices A of order N.

Theorem 5.1. For arbitrary given numbers $v_1, \ldots, v_{N-1} \in \mathbb{C}$, with $v_1 \cdots v_{N-1} \neq 0$, consider the nilpotent matrix of order N defined by

$$A_{N,v_1,\ldots,v_{N-1}} = \begin{pmatrix} 0 & v_1 & 0 & 0 & \cdots & 0\\ 0 & 0 & v_2 & 0 & \cdots & 0\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & 0 & \cdots & 0\\ 0 & 0 & 0 & 0 & \cdots & v_{N-1}\\ 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}. \tag{5.11}$$

Then, the weight matrix $W(t) = e^{-t^2} e^{A_{N,\nu_1,\dots,\nu_{N-1}} t} e^{A^*_{N,\nu_1,\dots,\nu_{N-1}} t}$ satisfies the differential equation

$$W''(t) + \left[\left(2tI - 2A_{N,\nu_1,\dots,\nu_{N-1}}\right)W(t)\right]' + A_0 W(t) = W(t)A_0^*, \tag{5.12}$$

where the matrix A_0 is given by

$$A_0 = A^2_{N,\nu_1,\dots,\nu_{N-1}} - \operatorname{diag}\left(2(N-1), 2(N-2), 2(N-3), \dots, 2, 0\right). \tag{5.13}$$

Moreover, this example is maximal in the following sense: if the weight matrix $e^{-t^2} e^{At} e^{A^* t}$, with A nilpotent of order N, satisfies a second-order differential equation as given by (3.8) with $A_2 = I$, then A is unitarily equivalent to a matrix $A_{N,\nu_1,\dots,\nu_{N-1}}$ for certain numbers $\nu_i \in \mathbb{C}$, $i = 1, \dots, N-1$. □

Proof. It is enough to check the Hermitian conditions given in (5.10) for $A = A_{N,\nu_1,\dots,\nu_{N-1}}$ and A_0 as in (5.13).

The first condition in (5.10) is straightforward from the definition of A_0 (5.13) since $A^2_{N,\nu_1,\dots,\nu_{N-1}} - A_0$ is diagonal. A simple calculation shows also that

$$2A_{N,\nu_1,\dots,\nu_{N-1}} + A_0 A_{N,\nu_1,\dots,\nu_{N-1}} - A_{N,\nu_1,\dots,\nu_{N-1}} A_0 = \theta, \tag{5.14}$$

and the second condition in (5.10) follows from here. Since $A_{N,\nu_1,\dots,\nu_{N-1}} = [A_{N,\nu_1,\dots,\nu_{N-1}}, A_0]/2$, we have that $A_{N,\nu_1,\dots,\nu_{N-1}}$ commutes with $[A_{N,\nu_1,\dots,\nu_{N-1}}, A_0]$, and it follows that $\operatorname{ad}^n_{A_{N,\nu_1,\dots,\nu_{N-1}}} A_0 = \theta$, $n \geq 2$, and hence, the rest of the Hermitian conditions in (5.10) are satisfied too.

We do not give the details needed to prove that these families of examples are maximal. ■

One can obtain examples from the ones above by replacing the numbers ν_i by blocks V_i. These examples are maximal for nilpotent matrices A of any order.

Since there are nilpotent matrices of order N which are not unitarily equivalent to $A_{N,\nu_1,\dots,\nu_{N-1}}$, $\nu_i \in \mathbb{C}$, $i = 1, \dots, N-1$, for instance $\left(\begin{smallmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{smallmatrix}\right)$, our maximal example shows that not all weight matrices of the form (5.5) arising from nilpotent matrices A of order N satisfy a differential equation as (3.8) with $A_2 = I$.

It is straightforward to see that Theorem 5.1 is also true for the matrices A of the form $A = \alpha I - A_{N,\nu_1,\dots,\nu_{N-1}}$, $\alpha \in \mathbb{C}$; the coefficients A_1 and A_0 for the differential equation are then $A_1(t) = -2tI + 2A$ and $A_0 = A^2 - \operatorname{diag}(2(N-1), 2(N-2), \dots, 2, 0)$.

Recall that in solving (5.4) we have been rather modest and considered only the case when either A or B vanishes, instead of just asking that they should commute. Using the techniques developed above, it can be proved that, for instance, for $N = 2$, the weight matrices $e^{-t^2}T(t)W(0)T^*(t)$, with $T'(t) = (2Bt + A)T(t)$, $T(0) = I$, A and B nilpotent, and $AB = BA$, do not satisfy a second-order differential equation as (3.8), with $A_2 = I$, unless $A = \theta$ or $B = \theta$.

5.2 Weight matrices of the form $e^{-t^2}e^{Bt^2}e^{B^*t^2}$

By solving the equation (5.4) for the case $A = \theta$, we find $T(t) = e^{Bt^2}$, and then the weight matrix we are looking for has the form

$$W(t) = e^{-t^2}e^{Bt^2}e^{B^*t^2}, \tag{5.15}$$

where B is an $N \times N$ matrix.

We again assume that B is not unitarily equivalent to block-diagonal, otherwise the weight matrix $e^{-t^2}e^{Bt^2}e^{B^*t^2}$ reduces to lower sizes.

From Theorem 4.1, we have to check whether there is a convenient choice of A_0 so that the matrix function (5.6) is Hermitian. This means that we have to find those matrices B not unitarily equivalent to block-diagonal for which there exists A_0 such that the matrix function

$$\chi(t) = 2B + 4\left(B^2 - B\right)t^2 - e^{-Bt^2}A_0e^{Bt^2} \tag{5.16}$$

is Hermitian, or equivalently,

$$\begin{aligned}\chi(t) &= 2B + 4\left(B^2 - B\right)t^2 - e^{-t^2\,\mathrm{ad}_B}\left(A_0\right) \\ &= \left(2B - A_0\right) + \left(4B^2 - 4B + \mathrm{ad}_B\,A_0\right)t^2 - \sum_{n\ge 2}\frac{(-1)t^{2n}}{n!}\,\mathrm{ad}_B^n\,A_0\end{aligned} \tag{5.17}$$

is Hermitian.

The existence of a matrix A_0 such that (5.16) holds is then equivalent to the existence of a matrix A_0 such that the following matrices

$$2B - A_0, \qquad 4B^2 - 4B + \left[B, A_0\right], \qquad \mathrm{ad}_B^n\,A_0, \quad n \ge 2. \tag{5.18}$$

are Hermitian.

We first concentrate our attention on nilpotent matrices B of order N (and thus with rank $N - 1$), for which we display the following maximal example.

Theorem 5.2. For given numbers $\nu_1, \ldots, \nu_{N-1} \in \mathbb{C}$, $\nu_1 \cdots \nu_{N-1} \neq 0$, consider the nilpotent matrix of order N defined by

$$B_{N,\nu_1,\ldots,\nu_{N-1}} = \begin{pmatrix} 0 & \nu_1 & -\nu_1\nu_2 & \nu_1\nu_2\nu_3 & \cdots & (-1)^N \nu_1\nu_2\cdots\nu_{N-1} \\ 0 & 0 & \nu_2 & -\nu_2\nu_3 & \cdots & (-1)^{N-1}\nu_2\cdots\nu_{N-1} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -\nu_{N-2}\nu_{N-1} \\ 0 & 0 & 0 & 0 & \cdots & \nu_{N-1} \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}. \tag{5.19}$$

Then the weight matrix $W(t) = e^{-t^2} e^{B_{N,\nu_1,\ldots,\nu_{N-1}} t^2} e^{B^*_{N,\nu_1,\ldots,\nu_{N-1}} t^2}$ satisfies the differential equation

$$W''(t) + \left[\left(2I - 4B_{N,\nu_1,\ldots,\nu_{N-1}} \right) tW(t) \right]' + A_0 W(t) = W(t) A_0^*, \tag{5.20}$$

where the matrix A_0 is given by

$$A_0 = 2B_{N,\nu_1,\ldots,\nu_{N-1}} + \operatorname{diag}\left(-4(N-1), -4(N-2), -4(N-3), \ldots, -4, 0 \right). \tag{5.21}$$

Moreover, this example is maximal in the following sense: if the weight matrix $e^{-t^2} e^{Bt^2} e^{B^* t^2}$, with B nilpotent of order N, satisfies a second-order differential equation as given by (3.8) with $A_2 = I$, then B is unitarily equivalent to a matrix $B_{N,\nu_1,\ldots,\nu_{N-1}}$ for certain numbers $\nu_i \in \mathbb{C}$, $i = 1, \ldots, N-1$. □

The first part of the proof can be done by following the steps of Theorem 5.1. We omit the details needed to prove the second part of the theorem.

Since there are nilpotent matrices of order N which are not unitarily equivalent to $B_{N,\nu_1,\ldots,\nu_{N-1}}$, $\nu_i \in \mathbb{C}$, $i = 1, \ldots, N-1$, for instance $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$, Theorem 5.1 shows that not all weight matrices of the form (5.15) provided by nilpotent matrices B of order N satisfy a differential equation as (3.8).

Again, we work with blocks to get more examples involving nilpotent matrices.

Notice that the matrix $B_{N,\nu_1,\ldots,\nu_{N-1}}$, which plays the role of A here, is a simple rational function of the matrix introduced in (5.11).

6 Examples with $A_2 = tI$

In the scalar case, the weight functions admitting a second-order differential operator with $A_2 = t$ are the Laguerre weight functions $w(t) = t^\alpha e^{-t}$, $\alpha > -1$ (up to a linear change of variable, as usual).

We then take $\rho(t) = t^\alpha e^{-t}$, $\alpha > -1$, $A_2(t) = tI$, and $a = 1$ (i.e., we look for weight matrices W satisfying $W(1) = I$). This gives for the function c (see (4.13)) the expression

$$c(t) = (\alpha + 1) - t, \tag{6.1}$$

and for the matrix function F in (4.5),

$$F(t) = \frac{1}{2t}\Big(A_1(t) - (\alpha + 1 - t)I\Big). \tag{6.2}$$

Taking into account that A_1 is a polynomial of degree at most 1, we can write

$$F(t) = A + \frac{B}{t}. \tag{6.3}$$

Hence, we look for the solutions T of the differential equation

$$T'(t) = \left(A + \frac{B}{t}\right)T(t), \quad T(1) = I. \tag{6.4}$$

As in Section 5, we study here two special, although rather interesting, cases: when either A or B vanishes. We notice that a form of this *mixed* equation, for a special choice of A and B, features in the treatment of the hydrogen atom by means of Dirac's equation.

6.1 Weight matrices of the form $t^\alpha e^{-t} e^{At} e^{A^* t}$; $t \geq 0$

By solving the equation (6.4) for the case $B = \theta$, we find $T(t) = e^{At}e^{-A}$ and then the weight matrix we are looking for has the form

$$W(t) = t^\alpha e^{-t} e^{At} e^{A^* t}, \tag{6.5}$$

where A is an $N \times N$ matrix, and $W(1) = e^{-1}e^{A}e^{A^*}$ (it is enough to consider $\tilde{W}(t) = ee^{-A}W(t)e^{-A^*}$ to get a weight matrix for which $\tilde{W}(1) = I$).

As before, we again assume that A is not unitarily equivalent to block-diagonal.

From Theorem 4.1, we have to check whether there is a convenient choice of A_0 so that the matrix function

$$\chi(t)W(1) = T^{-1}(t)\big(A_2(t)F'(t) + A_2(t)F^2(t) + c(t)F(t) - A_0\big)T(t)W(1) \tag{6.6}$$

is Hermitian. This means that we have to find those matrices A, not unitarily equivalent to block-diagonal, for which there exists A_0 such that the matrix function

$$\chi(t)W(1) = e^{A}\left[(\alpha+1)A + \left(A^2 - A\right)t - e^{-At}A_0e^{At}\right]e^{A^*} \tag{6.7}$$

is Hermitian, or equivalently,

$$\begin{aligned}\chi(t)W(1) &= e^{A}\left[(\alpha+1)A + \left(A^2 - A\right)t - e^{-t\,\mathrm{ad}_A}(A_0)\right]e^{A^*} \\ &= e^{A}\left[\left((\alpha+1)A - A_0\right) + \left(A^2 - A + \mathrm{ad}_A A_0\right)t - \sum_{n\geq 2}\frac{(-t)^n}{n!}\,\mathrm{ad}_A^n A_0\right]e^{A^*}\end{aligned} \tag{6.8}$$

is Hermitian.

The existence of a matrix A_0 such that (6.7) holds is then equivalent to the existence of a matrix A_0 such that the following matrices

$$(\alpha+1)A - A_0, \qquad A^2 - A + [A, A_0], \qquad \mathrm{ad}_A^n A_0, \quad n \geq 2. \tag{6.9}$$

are Hermitian.

These equations are similar to those studied in Section 5 (see (5.16)).

We can then deduce the following theorem.

Theorem 6.1. For given numbers $\nu_1, \dots, \nu_{N-1} \in \mathbb{C}$, $\nu_1 \cdots \nu_{N-1} \neq 0$, consider the nilpotent matrix $B_{N,\nu_1,\dots,\nu_{N-1}}$ of order N given by (5.19). Then the weight matrix

$$W(t) = t^{\alpha}e^{-t}e^{B_{N,\nu_1,\dots,\nu_{N-1}}t}e^{B^*_{N,\nu_1,\dots,\nu_{N-1}}t}, \quad t \geq 0, \tag{6.10}$$

satisfies the differential equation

$$(tW)''(t) + \left[\left(t\left(I - 2B_{N,\nu_1,\dots,\nu_{N-1}}\right) - (\alpha+1)I\right)W(t)\right]' + A_0W(t) = W(t)A_0^*, \tag{6.11}$$

where the matrix A_0 is given by

$$A_0 = (\alpha+1)B_{N,\nu_1,\dots,\nu_{N-1}} + \mathrm{diag}\left(-(N-1), -(N-2), -(N-3), \dots, -1, 0\right). \tag{6.12}$$

Furthermore, this example is maximal in the following sense: if the weight matrix $t^\alpha e^{-t}e^{At}e^{A^*t}$, with A nilpotent of order N, satisfies a second-order differential equation as in (3.8), with $A_2 = tI$, then A is unitarily equivalent to a matrix $B_{N,\nu_1,\dots,\nu_{N-1}}$ for certain numbers $\nu_i \in \mathbb{C}$, $i = 1, \dots, N-1$. □

The example of Theorem 6.1 can be generalized working by blocks as in Section 5.

6.2 Weight matrices of the form $t^\alpha e^{-t}t^B t^{B^*}$

We now solve equation (6.4) for the case $A = \theta$.

The presence of *logarithmic terms* makes this case more subtle than the ones explored so far. This will be clear below.

We have $T(t) = t^B = e^{B\log t}$.

The weight matrix we are looking for has the form

$$W(t) = t^\alpha e^{-t}t^B t^{B^*}, \tag{6.13}$$

where B is an $N \times N$ matrix ($W(t)$ satisfies $W(1) = e^{-1}I$, instead of $W(1) = I$).

From Theorem 4.1, we have to check whether there is a convenient choice of A_0 so that the matrix function

$$\chi(t) = T^{-1}(t)\big(A_2(t)F'(t) + A_2(t)F^2(t) + c(t)F(t) - A_0\big)T(t) \tag{6.14}$$

is Hermitian.

Taking into account our previous choice, we have to find those matrices B for which there exists A_0 such that the matrix function

$$\chi(t) = \left[(B^2 + \alpha B)\frac{1}{t} - B\right] - t^{-B}A_0 t^B \tag{6.15}$$

is Hermitian.

This can be written as

$$\begin{aligned}\chi(t) &= \left[(B^2 + \alpha B)\frac{1}{t} - B\right] - e^{-\log t\,\mathrm{ad}_B}(A_0)\\ &= (B^2 + \alpha B)\frac{1}{t} - (B + A_0) - \sum_{n\ge 1}\frac{(-1)^n \log^n t}{n!}\,\mathrm{ad}_B^n A_0.\end{aligned} \tag{6.16}$$

Here is a strategy to insure that $\chi(t)$ is Hermitian: produce matrices B and B_0 such that

$$\mathrm{ad}_B(B_0) = B_0 \tag{6.17}$$

and set $A_0 = -B + B_0$. This gives

$$\chi(t) = \frac{B^2 + \alpha B - B_0}{t}, \tag{6.18}$$

so that one needs to satisfy (6.17) as well as the requirement that

$$B^2 + \alpha B - B_0 \tag{6.19}$$

is Hermitian. This requirement can also be achieved by expanding the term $1/t$ in (6.16) in powers of $\log t$.

Equation (6.8) suggests introducing the matrices $D, F_{\nu_1,\dots,\nu_{N-1}}$ (where $\nu_i \in \mathbb{C}\backslash\{0\}$, $i = 1, \dots, N-1$) as follows. Denote by D the diagonal matrix of dimension N with entries $d_{ii} = N - i$, $i = 1, \dots, N$, and by $F_{\nu_1,\dots,\nu_{N-1}}$ the matrix with entries $(i, i+1)$ equal to ν_i, $i = 1, \dots, N-1$ and 0 otherwise. This gives $\mathrm{ad}_D(F_{\nu_1,\dots,\nu_{N-1}}) = F_{\nu_1,\dots,\nu_{N-1}}$. The matrix $D^2 + \alpha D - F_{\nu_1,\dots,\nu_{N-1}}$ with eigenvalues given by the entries of $D^2 + \alpha D$ can be put in Jordan form. This means that there exists a nonsingular S such that

$$S\big(D^2 + \alpha D - F_{\nu_1,\dots,\nu_{N-1}}\big)S^{-1} = D^2 + \alpha D \tag{6.20}$$

holds. Such a matrix S can be explicitly given. If we write c_i, $i = 1, \dots, N$, for the diagonal entries of $D^2 + \alpha D$, then

$$S = \big(s_{ij}\big)_{i,j=1,\dots,N}; \quad s_{ij} = \begin{cases} 0, & \text{if } i > j, \\ 1, & \text{if } i = j, \\ \displaystyle\prod_{l=1}^{j-i} \frac{\nu_{i+l-1}}{c_{i+l} - c_i}, & \text{if } i < j. \end{cases} \tag{6.21}$$

This suggests that we can take

$$\begin{aligned} B &= SDS^{-1}, \\ B_0 &= SF_{\nu_1,\dots,\nu_{N-1}}S^{-1}, \end{aligned} \tag{6.22}$$

and then conditions (6.17) and (6.19) are fulfilled.

We have strong reasons to believe that our choice of the strategy above is not too whimsical.

From the definition of B, it follows that t^B is a matrix polynomial of degree $N - 1$ which does not vanish at $t = 0$. Hence, to guarantee the integrability of the weight matrix

W at 0, we have to assume, as in the scalar case, that $\alpha > -1$. The differential equation satisfied by W is then

$$(tW)''(t) + \big[(tI - 2B - (\alpha+1)I)W(t)\big]' + \big(-B + B_0\big)W(t) = W(t)\big(-B^* + B_0^*\big). \tag{6.23}$$

It is worth noting that if $B^2 + \alpha B$ is Hermitian, by taking $A_0 = -B$, we have $\mathrm{ad}_B\, A_0 = \theta$ and the matrix function $\chi(t)$ in (6.16) is then Hermitian. Unfortunately, the orthogonal matrix polynomials with respect to $W(t) = t^\alpha e^{-t} t^B t^{B^*}$, $B^2 + \alpha B$ Hermitian, do not satisfy second-order differential equation as (1.1). The reason is that these weight matrices do not satisfy the boundary conditions (3.5). Indeed, the matrix function $(A_2(t)W(t))' - A_1(t)W(t)$ does not vanish at $t = 0$. In particular, this implies that the first moment of the weight matrix W does not satisfy the condition (3.4) for $n = 0$.

It is a simple calculation to see that the examples given above, as well as those in Theorem 6.1 and in the next section, all satisfy the boundary conditions (3.5).

7 Examples with $A_2 = (1+t)(1-t)I$

In the scalar case, the weight functions going with a symmetric second-order differential operator with $A_2 = (1+t)(1-t)$ are, up to a linear change of variable, the Jacobi weight functions $w(t) = (1+t)^\alpha(1-t)^\beta$, $\alpha, \beta > -1$.

We then take $\rho(t) = (1+t)^\alpha(1-t)^\beta$, $\alpha, \beta > -1$, $A_2(t) = (1+t)(1-t)I$, and $a = 0$ (i.e., we look for weight matrices W satisfying $W(0) = I$).

This gives for the function c (see (4.13)) the expression

$$c(t) = (\alpha+1)(1-t) - (\beta+1)(1+t); \tag{7.1}$$

and for the matrix function $F(t)$ in (4.5),

$$F(t) = \frac{1}{2(1+t)(1-t)}\big(A_1(t) - \big[(\alpha+1)(1-t) - (\beta+1)(1+t)\big]I\big). \tag{7.2}$$

Taking into account that A_1 is a polynomial of degree at most 1, we can write

$$F(t) = \frac{A}{1+t} + \frac{B}{1-t}. \tag{7.3}$$

Hence, we look for the solutions T of the differential equation

$$T'(t) = \left(\frac{A}{1+t} + \frac{B}{1-t}\right)T(t), \quad T(0) = I. \tag{7.4}$$

It is interesting to note that this equation, for arbitrary constant matrices A and B, is sometimes labelled the *abstract hypergeometric equation*. It has also surfaced recently in connection with the so-called Knizhnik-Zamolodchikov equation. It is clear that the techniques in [12] should have a bearing in the solution of this equation.

Because of the symmetry between 1 and -1, we study here only the special case when B vanishes. We will see that this gives a fairly interesting example. For a few other examples in the case $N = 2$, the reader can see [9] and its references. Incidentally, [9, Example 5.1] features a one-parameter family of *commuting* differential operators of order two. One can thus exhibit a *first-order differential operator*, a situation that does not arise in the scalar case. The examples in [9] involve *nonzero* constant matrices A and B.

Our choice gives $T(t) = (1+t)^A = e^{A\log(1+t)}$.

The weight matrix we are looking for has the form

$$W(t) = (1+t)^{\alpha}(1-t)^{\beta}(1+t)^{A}(1+t)^{A^*}, \tag{7.5}$$

where A is an $N \times N$ matrix.

From Theorem 4.1, we have to check whether there is a convenient choice of A_0 so that the matrix function

$$\chi(t) = T^{-1}(t)\big(A_2(t)F'(t) + A_2(t)F^2(t) + c(t)F(t) - A_0\big)T(t) \tag{7.6}$$

is Hermitian for all t. Taking into account our previous choice, we have to find those matrices A for which there exists A_0 such that the matrix function

$$\chi(t) = \left[2\big(\alpha A + A^2\big)\frac{1}{1+t} - \big(A^2 + \alpha A + (\beta+1)A\big)\right] - (1+t)^{-A}A_0(1+t)^{A} \tag{7.7}$$

is Hermitian, which can be written as

$$\begin{aligned}\chi(t) &= \left[2\big(\alpha A + A^2\big)\frac{1}{1+t} - \big(A^2 + \alpha A + (\beta+1)A\big)\right] - e^{-\log(1+t)\,\mathrm{ad}_A}\big(A_0\big)\\ &= 2\big(\alpha A + A^2\big)\frac{1}{1+t} - \big(A^2 + \alpha A + (\beta+1)A + A_0\big)\\ &\quad - \sum_{n\geq 1}\frac{(-1)^n\log^n(1+t)}{n!}\,\mathrm{ad}_A^n A_0.\end{aligned} \tag{7.8}$$

Proceeding, as in Section 6.2, it is clear that a possible strategy to insure that $\chi(t)$ is Hermitian goes as follows. Find matrices A and B_0 such that

$$\mathrm{ad}_A\,(B_0) = B_0, \tag{7.9}$$

and set $A_0 = -A(1+\alpha+\beta) - A^2 + B_0$. This gives

$$\chi(t) = \frac{2A^2 + 2\alpha A - B_0}{1+t} \tag{7.10}$$

so that one needs to satisfy (7.9) as well as the requirement that

$$2A^2 + 2\alpha A - B_0 \tag{7.11}$$

is Hermitian.

It is clear how the rest of the construction in Section 6.2 can be adapted to this case too.

8 Some examples of structural formulas

We close this paper by pointing out that the families of orthogonal matrix polynomials satisfying second-order differential equations, introduced above, also satisfy a rich variety of structural formulas as those satisfied by the classical orthogonal families of Jacobi, Laguerre, and Hermite. We will give a full account of these relations in a forthcoming publication. As a sample, we include here what is possibly the simplest example.

In Section 5.1, take $A = \left(\begin{smallmatrix} 0 & a \\ 0 & 0 \end{smallmatrix}\right)$; this gives the weight matrix

$$W(t) = \begin{pmatrix} 1+a^2t^2 & at \\ at & 1 \end{pmatrix} e^{-t^2}. \tag{8.1}$$

The theory developed above can be used to see that the orthogonal *monic* polynomials $(\widehat{P}_n)_n$ with respect to the weight matrix $W(t)$ satisfy the second-order differential equation

$$\widehat{P}_n''(t) + \widehat{P}_n'(t) \begin{pmatrix} -2t & 2a \\ 0 & -2t \end{pmatrix} + \widehat{P}_n(t) \begin{pmatrix} -2 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} -2n-2 & 0 \\ 0 & -2n \end{pmatrix} \widehat{P}_n(t). \tag{8.2}$$

They also satisfy the Rodrigues' formula

$$\begin{aligned} \widehat{P}_n(t) = (-2)^{-n} \begin{pmatrix} 1 & 0 \\ 0 & \dfrac{2}{a^2n+2} \end{pmatrix} \left[e^{-t^2} \left(\begin{pmatrix} 1+a^2t^2 & at \\ at & 1 \end{pmatrix} + \begin{pmatrix} \dfrac{a^2n}{2} & 0 \\ 0 & 0 \end{pmatrix} \right) \right]^{(n)} \\ \times \begin{pmatrix} 1 & -at \\ -at & 1+a^2t^2 \end{pmatrix} e^{t^2}. \end{aligned} \tag{8.3}$$

One gets for the L^2 norm of $\widehat{P}_n$, $n \geq 0$, that

$$\|\widehat{P}_n\|_2^2 = \sqrt{\pi} n! 2^{-n} \operatorname{diag}\left(\frac{2+a^2(n+1)}{2}, \frac{2}{2+a^2 n}\right). \tag{8.4}$$

The coefficients in the three-term recurrence for the orthonormal matrix polynomials $(P_n)_n$ (see (1.2)) are then

$$\begin{aligned} A_{n+1} &= \sqrt{\frac{n+1}{2}} \operatorname{diag}\left(\sqrt{\frac{2+a^2(n+2)}{2+a^2(n+1)}}, \sqrt{\frac{2+a^2 n}{2+a^2(n+1)}}\right) \\ B_n &= \frac{a}{\sqrt{(2+a^2 n)(2+a^2(n+1))}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad n \geq 0. \end{aligned} \tag{8.5}$$

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