Matrix valued orthogonal polynomials

Luisa Miranian

UC Berkeley, Department of Mathematics

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Scalar valued orthogonal polynomials: definition

Orthogonality of monic polynomials $\{p_n(x)\}_{n=0}^{\infty}$ with respect to a positive measure $\mu(x)$ defined on $\mathbb{R}$:

$$\langle p_n, p_m \rangle_\mu = \int_{\mathbb{R}} p_n(x) p_m(x) \mathrm{d}\mu(x) = S_n \delta_{nm}, \quad n, m \geq 0,$$

where S_n is the norm $\langle p_n, p_n \rangle_\mu$, is equivalent to a three term recursion relation (Favard's theorem):

$$x p_n(x) = p_{n+1}(x) + b_n p_n(x) + a_n p_{n-1}(x), \quad a_n > 0, \quad b_n \in \mathbb{R}, \quad n \geq 0$$

In matrix form the above can be rewritten as $x\mathbf{P} = L\mathbf{P}$, i.e.

$$x \begin{pmatrix} p_0(x) \\ p_1(x) \\ p_2(x) \\ \vdots \end{pmatrix} = \begin{pmatrix} b_0 & 1 & & & \\ a_1 & b_1 & 1 & & \\ & a_2 & b_2 & 1 & \\ & & \ddots & \ddots & \ddots \end{pmatrix} \begin{pmatrix} p_0(x) \\ p_1(x) \\ p_2(x) \\ \vdots \end{pmatrix}$$

Classical orthogonal polynomials: Bochner property

Bochner (1929): classified all orthogonal polynomials $\{p_n\}_{n=0}^{\infty}$ satisfying

$$(c_2x^2 + c_1x + c_0)p_n''(x) + (d_1x + d_0)p_n'(x) = \lambda_n p_n(x)$$

Hermite: $\mu(x) = \mathrm{e}^{-x^2}$, $x \in (-\infty, \infty)$:

$$H_n(x)'' - 2xH_n(x)' = -2nH_n(x)$$

Laguerre: $\mu(x) = x^\alpha \mathrm{e}^{-x}$, $\alpha > -1$, $x \in (0, \infty)$:

$$xL_n^\alpha(x)'' + (\alpha + 1 - x)L_n^\alpha(x)' = -nL_n^\alpha(x)$$

Jacobi: $\mu(x) = x^\alpha(1-x)^\beta$, $\alpha, \beta > -1$, $x \in (0, 1)$:

$$x(1-x)P_n^{(\alpha,\beta)}(x)'' + (\alpha + 1 - (\alpha + \beta + 2)x)P_n^{(\alpha,\beta)}(x)' =$$

$$-n(n + \alpha + \beta + 1)P_n^{(\alpha,\beta)}(x)$$

Bessel: the support of the orthogonality measure is the unit circle.

Matrix orthogonal polynomials

Matrix valued polynomials on the real line:

$$C_n x^n + C_{n-1}x^{n-1} + \cdots + C_0, \quad C_i \in \mathbb{C}^{k\times k}, \quad x \in \mathbb{R}.$$

Krein (1949): introduced matrix valued orthogonal polynomials

- Measure: $\mu(\mathrm{d}x) = W(x)\mathrm{d}x$ with Hermitian weight function $W(x) \in \mathbb{C}^{k\times k}$ supported on the real line, $k \geq 1$
- n-th moment of the measure $\mu(\mathrm{d}x)$:

$$\mu_n = \int x^n \mu(\mathrm{d}x) = \int x^n W(x)\mathrm{d}x; \quad \mu_n \in \mathbb{C}^{k\times k}$$

Define matrix valued inner product as:

$$\langle P, Q\rangle_\mu = \int_{\mathbb{R}} P^*(x)\mathrm{d}\mu(x)Q(x); \quad P, Q \in \mathbb{C}^{k\times k}[x].$$

Matrix orthogonal polynomials

Orthogonality of monic matrix polynomials $\{P_n(x)\}_{n=0}^{\infty}$ with respect to a weight matrix W

$$\langle P_n, P_m \rangle_W = \int_{\mathbb{R}} P_n^*(x) \mathrm{d}W(x) P_m(x) = \delta_{nm} S_n, \quad n, m \geq 0$$

is equivalent to a three term recurrence relation (matrix analog of Favard's theorem, proven by A. Duran)

$$\begin{aligned} & xP_n(x) = P_{n+1}(x) + P_n(x)B_n^* + P_{n-1}(x)A_n^*, \quad n \geq 0 \\ & \det(A_n) \neq 0. \end{aligned}$$

In matrix form, $x\mathbf{P}^* = L\mathbf{P}^*$,

$$x \begin{pmatrix} P_0^*(x) \\ P_1^*(x) \\ P_2^*(x) \\ \vdots \end{pmatrix} = \begin{pmatrix} B_0 & I & & & \\ A_1 & B_1 & I & & \\ & A_2 & B_2 & I & \\ & & \ddots & \ddots & \ddots \end{pmatrix} \begin{pmatrix} P_0^*(x) \\ P_1^*(x) \\ P_2^*(x) \\ \vdots \end{pmatrix}$$

Representation of orthogonal polynomials

- It is well known that scalar orthogonal polynomials can be represented as determinants

$$p_n(x) = \det\left(T_n\right),$$

where

$$T_n = \begin{pmatrix} \mu_0 & \mu_1 & \cdots & \mu_{n-1} & \mu_n \\ \mu_1 & \mu_2 & \cdots & \mu_n & \mu_{n+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \mu_{n-1} & \mu_n & \cdots & \mu_{2n-2} & \mu_{2n-1} \\ 1 & x & \dots & x^{n-1} & x^n \end{pmatrix}.$$

- How could we define *matrix* orthogonal polynomials in terms of moments of the orthogonality measure?

Schur complement

Given a matrix A with partition:

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

Schur complement of A_{22} is:

$$Schur\ complement(A_{22}) = A_{22} - A_{21}A_{11}^{-1}A_{12}$$

The identity below will become useful at reconciling matrix definitions with the scalar ones:

$$\det(A) = \det\left(A_{11}\right)\det\left(A_{22} - A_{21}A_{11}^{-1}A_{12}\right)$$

Matrix valued polynomials: definition

Starting with matrix T_n where I is $k \times k$ identity matrix, $x \in \mathbb{R}$:

$$T_n = \begin{pmatrix} \mu_0 & \mu_1 & \cdots & \mu_{n-1} & \mu_n \\ \mu_1 & \mu_2 & \cdots & \mu_n & \mu_{n+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \mu_{n-1} & \mu_n & \cdots & \mu_{2n-2} & \mu_{2n-1} \\ I & xI & \dots & x^{n-1}I & x^n I \end{pmatrix} \in \mathbb{C}^{k(n+1)\times k(n+1)},$$

define matrix polynomials by taking Schur complements of $x^n I$:

$$P_n(x) = x^n I - \begin{bmatrix} I & xI & \dots & x^{n-1}I \end{bmatrix} \begin{pmatrix} \mu_0 & \mu_1 & \cdots & \mu_{n-1} \\ \mu_1 & \mu_2 & \cdots & \mu_n \\ \vdots & \vdots & \vdots & \vdots \\ \mu_{n-1} & \mu_n & \cdots & \mu_{2n-2} \end{pmatrix}^{-1} \begin{pmatrix} \mu_n \\ \mu_{n+1} \\ \vdots \\ \mu_{2n-1} \end{pmatrix}$$

with $P_0(x) = I$.

Matrix valued polynomials: notation

To ease the notation, denote Hankel matrix H_n and vector v_n:

$$H_n = \begin{pmatrix} \mu_0 & \mu_1 & \cdots & \mu_{n-1} \\ \mu_1 & \mu_2 & \cdots & \mu_n \\ \vdots & \vdots & \vdots & \vdots \\ \mu_{n-1} & \mu_n & \cdots & \mu_{2n-2} \end{pmatrix} \in \mathbb{C}^{kn\times kn}; \quad v_n = \begin{pmatrix} \mu_n \\ \mu_{n+1} \\ \vdots \\ \mu_{2n-1} \end{pmatrix}$$

- A family of **scalar** monic orthogonal polynomials is defined as:

$$P_n(x) = \frac{\det(T_n)}{\det(H_n)},$$

  which is exactly what we obtain using definition above in scalar case.
- It is worth observing that:
  - matrices $\{H_n\}_{n=0}^{\infty}$ defined above are symmetric
  - we additionally assume that the weight function $W(x)$ is such that H_n are all **invertible.**

Matrix valued polynomials: recurrence relation

- Matrix polynomials defined above are orthogonal, hence
- there exists three term recurrence relation

$$xP_n(x) = P_{n+1}(x) + P_n(x)b_n^* + P_{n-1}(x)a_n^*, \quad n \geq 0$$

- Coefficients a_n and b_n can be expressed in terms of the moments, e.g.

$$a_n = S_n S_{n-1}^{-1}, \quad \text{where} \;\; S_n = \mu_{2n} - v_n^* H_n^{-1} v_n.$$

- In the scalar case the expression for a_n is:

$$a_n = \frac{H_{n+1} H_{n-1}}{H_n^2} = \frac{S_n}{S_{n-1}}, \;\; \text{since} \;\; S_n = \frac{H_{n+1}}{H_n}.$$

- b_n can also be expressed in terms of the moments of the measure
- Matrices S_n will be used later in defining matrix analog of τ-functions

Matrix orthogonal polynomials: kernel polynomials

- Matrices S_n can be used to generate *orthonormal* polynomials out of the *monic* ones in the following way:

$$\overline{P}_n = P_n S_n^{-1/2}.$$

- Denote a kernel polynomial of degree n by

$$K_n(x,y) = \sum_{i=0}^{n} \overline{P}_i(y) \overline{P}_i^*(x),$$

then

$$K_n(x,y) = \begin{bmatrix} I & yI & \dots & y^n I \end{bmatrix} \begin{pmatrix} \mu_0 & \mu_1 & \dots & \mu_n \\ \mu_1 & \mu_2 & \dots & \mu_{n+1} \\ \vdots & \vdots & \vdots & \vdots \\ \mu_n & \mu_{n+1} & \dots & \mu_{2n} \end{pmatrix}^{-1} \begin{bmatrix} I \\ xI \\ \vdots \\ x^n I \end{bmatrix}$$

Kernel polynomials, Christoffel-Darboux property

- In scalar theory the kernel polynomial is given by:

$$K_n(x,y) = -\det \begin{pmatrix} \mu_0 & \mu_1 & \dots & \mu_{n-1} & 1 \\ \mu_1 & \mu_2 & \dots & \mu_n & x \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \mu_n & \mu_{n+1} & \dots & \mu_{2n} & x^n \\ 1 & y & \dots & y^n & 0 \end{pmatrix}$$

  which agrees with our matrix formula applied in the scalar case

- Matrix formulation of Christoffel-Darboux formula:

$$\sum_{m=0}^{n} \overline{P}_m(y)\overline{P}_m^*(x) = \frac{\overline{P}_n(y)\overline{a}_{n+1}^*\overline{P}_{n+1}^*(x) - \overline{P}_{n+1}(y)\overline{a}_{n+1}\overline{P}_n^*(x)}{x-y},$$

  where $\overline{a}_n = S_n^{1/2} S_{n-1}^{-1/2}$.

Matrix valued orthogonal polynomials: τ-function

Insert an infinite set of time variables $t_1,\ t_2,\dots$ into the measure:

$$\mu_t(\mathrm{d}x) = e^{\sum_{i=1}^{\infty} t_i x^i I}\mu(\mathrm{d}x), \quad \text{where } I \text{ is } k \times k \text{ identity matrix.}$$

In the classical theory τ-function is defined as:

$$\tau_n(t) = \det\left(H_n(t)\right).$$

In matrix case we define τ-function as:

$$\tau_n(t) \equiv S_n(t) = \mu_{2n}(t) - v_n^*(t)H_n^{-1}(t)v_n(t),$$

where $H_n(t)$ and $v_n(t)$ are defined as before, but with time dependence.

Observe that the new definition applied to the scalar case would be different from the classical one:

$$\tau_n(t) = \frac{\det(H_{n+1}(t))}{\det(H_n(t))}.$$

Recurrence relations and τ-function

- Given a family of monic matrix orthogonal polynomials satisfying the recursion relation $xP_n(x) = P_{n+1}(x) + P_n(x)b_n^* + P_{n-1}(x)a_n^*$, then

$$b_n^* = \tau_n(t)^{-1}\tau_n'(t)|_{t_1=0} = \ln(\tau_n(t))'_{t_1=0}$$
$$a_n^* = \tau_{n-1}(0)^{-1}\tau_n(0),$$

  where operator "$'$" represents $\frac{\partial}{\partial t_1}$.
- In classical theory expression for b_n is:

$$b_n = \frac{\partial}{\partial t_1} \ln\left(\frac{\det(H_{n+1}(t))}{\det(H_n(t))}\right)$$

  which is consistent with the new definition of $\tau_n(t) = \dfrac{\det(H_{n+1}(t))}{\det(H_n(t))}$.

Orthogonal polynomials and τ-function

- Orthogonal polynomials themselves can be described using τ-function:

$$P_{n+1}(x,t) = xP_n(x,t)\tau_n^{-1}(t)\tau_n\left(t - \left[x^{-1}\right]\right), \text{ where}$$

$$\begin{aligned} \mu_n\left(t - \left[x^{-1}\right]\right) &= \int z^n e^{\sum_{i=1}^{\infty}\left(t_i - \frac{x^{-i}}{i}\right)z^i} W(z)\mathrm{d}z \\ &= \mu_n(t) - \frac{\mu_{n+1}(t)}{x}. \end{aligned}$$

- This expression for $P_n(x,t)$ is equivalent to the one in classical theory:

$$p_n(x,t) = x^n \frac{\tau_n\left(t - [x^{-1}]\right)}{\tau_n(t)},$$

where $\tau_n(t) = \det\left(H_n(t)\right)$.

Orthogonal polynomials of the second type and τ-function

- Similar expression holds for *polynomials of the second type* defined as

$$Q_n(x) = x \int \frac{P_n(z)}{x - z} \mu(\mathrm{d}z)$$

- $Q_n(x)$ satisfy the same recursion relation as $P_n(x)$ but with different initial conditions
- The expression for $Q_n(x, t)$ in terms of τ-function is

$$xQ_{n+1}(x,t) = Q_n(x,t)\tau_n^{-1}(t)\tau_{n+1}\big(t + \big[x^{-1}\big]\big), \text{ where}$$

$$\begin{aligned} \mu_n\big(t + \big[x^{-1}\big]\big) &= \int z^n e^{\sum_{i=1}^{\infty}\left(t_i + \frac{x^{-i}}{i}\right)z^i} W(z)\mathrm{d}z \\ &= \sum_{i=n}^{\infty} \frac{\mu_{n+i}(t)}{x^i}. \end{aligned}$$

Properties of τ-function: useful identities

- let $\{P_n(x,t)\}_{n=0}^{\infty}$ be a family of monic orthogonal matrix polynomials with "time" dependent moments
- let a_n and b_n be the coefficients of the recursion relation with "time" dependence
- let $\dfrac{\partial}{\partial t_1}$ be denoted by "$'$",

then

1. $P'_{n+1}(x,t) = -P_n(x,t)a^*_{n+1}$;
2. $(b^*_n)' = a^*_{n+1} - a^*_n$;
3. $(a^*_n)' = a^*_n b^*_n - b^*_{n-1}a^*_n$.

Properties (2) and (3) could be interpreted as the non-abelian Toda equations

Matrix valued orthogonal polynomials: Bochner's problem

- As mentioned before, in 1929 Bochner characterized all families of scalar orthogonal polynomials satisfying second order differential equations
- In 1997 Durán formulated a problem of characterizing matrix orthonormal polynomials $\{P_n\}_{n=0}^{\infty}$ satisfying $\mathbf{D}P = P\Lambda$, where

$$\mathbf{D} = (\alpha_2 x^2 + \alpha_1 x + \alpha_0)\frac{\mathrm{d}^2}{\mathrm{d}x^2} + (\beta_1 x + \beta_0)\frac{\mathrm{d}}{\mathrm{d}x} + \gamma_0,$$

with $\alpha_2,\ \alpha_1,\ \alpha_0,\ \beta_1,\ \beta_0,\ \gamma_0,\ \Lambda_n \in \mathbb{C}^{k,k}$ and $P = [P_0^*(x),\ P_1^*(x),\ \cdots]$; $\Lambda = [\Lambda_0,\ \Lambda_1,\ \cdots]$ with Λ_n depending on n, but not on x.

- Operator $\mathbf{D}$ being symmetric is equivalent to Λ_n being Hermitian, where the symmetry of $\mathbf{D}$ with respect to matrix weight $W(x)$ is defined as:

$$\langle PD, Q\rangle_W = \langle P, QD\rangle_W.$$

Approaches to Bochner's problem: "ad" condition

- For a family of monic matrix valued polynomials the following conditions are equivalent:

1. $\mathbf{D}P = P\Lambda$, $LP^* = xP^*$,
2. $(\mathrm{ad}L^*)^3(\Lambda) = 0$, where $\mathrm{ad}(A)(B) \equiv AB - BA$ and matrix L is the tri-diagonal matrix containing the coefficients of the recursion relations, i.e.

$$L^{*3}\Lambda - 3L^{*2}\Lambda L^* + 3L^*\Lambda L^{*2} - \Lambda L^{*3} = 0$$

- Grunbaum and Haine first used this condition to revisit the original Bochner's classification and re-derive the classical families.
- This approach is quite general and does not require symmetry of the differential operator, however non-commutativity of matrix multiplication makes it very difficult to attack the Bochner's problem with this tool.

Approaches to Bochner problem: moments equations

Assume the symmetry of the differential operator, then a family of orthogonal polynomials satisfies the second order differential equation if and only if

$$A_2 = A_2^*,$$

$$-2(k+1)A_2 = A_1 + A_1^*$$

$$A_2(k+1)(k+2) + A_1(k+2) + A_0 = A_0^*;$$

where

$$A_2 = \mu_{k+2}\alpha_2 + \mu_{k+1}\alpha_1 + \mu_k\alpha_0,$$

$$A_1 = \mu_{k+2}\beta_1 + \mu_{k+1}\beta_0;$$

$$A_0 = \mu_{k+2}\gamma_0,$$

for $k \geq 0$, where μ_k are the moments of the orthogonality measure.

Approaches to Bochner problem: symmetry equations

From moments equations the following conditions can be derived:

$$W(x)B_2(x) = B_2^*(x)W(x)$$

$$2\Big(W(x)B_2\Big)' = W(x)B_1 + B_1^* W(x),$$

with $W(x)B_2$ vanishing at the boundary of the support of the measure

$$\Big(W(x)B_2\Big)'' - \Big(W(x)B_1\Big)' + W(x)B_0 = B_0^* W(x),$$

with $\left(W(x)B_2\right)' - W(x)B_1$ vanishing at the boundary, where

$$B_2 = x^2\alpha_2 + x\alpha_1 + \alpha_0,$$

$$B_1 = x\beta_1 + \beta_0,$$

$$B_0 = \gamma_0,$$

and $W(x)$ is the orthogonality measure.

Examples generated by solving symmetry equations

- The following examples along with technique used to generate them was developed by Grunbaum and Duran
- In most examples so far the leading coefficient $B_2(x)$ is taken to be scalar, which makes the first symmetry equation trivial.
- After applying some technique to the remaining two symmetry equations several weight matrices can be generated, for example taking $B_2(x) = I$ leads to:

$$W(x) = e^{-x^2} e^{Ax} e^{A^* x},$$

where

$$A = \begin{pmatrix} 0 & \nu_1 & 0 & \cdots & 0 \\ 0 & 0 & \nu_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \nu_{N-1} \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \nu_i \in \mathbb{C} \setminus \{0\}$$

- Coefficients of the differential operator are $B_2 = I$, $B_1 = 2x - 2A$, and $B_0 = A^2 - 2J$, for certain diagonal matrix J.

A Chebyshev example

- The following example appears in a book of Berezanski, and then Castro and Grunbaum
- Consider recursion relation coefficients given as:

$$b_0 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad b_n = \frac{1}{2}\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad a_n = \frac{1}{4}I, \quad n \geq 1.$$

- Monic polynomials generated from the recursion relations satisfy the following differential equations:

$$\left[\begin{pmatrix} 1 & x \\ -x & -1 \end{pmatrix}\frac{\mathrm{d}}{\mathrm{d}x} + \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}\right] P_n^*(x) = P_n^*(x)\begin{pmatrix} 0 & n \\ -1-n & 0 \end{pmatrix}$$

as well as

$$\left[\begin{pmatrix} x & 1 \\ -1 & -x \end{pmatrix}\frac{\mathrm{d}}{\mathrm{d}x} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right] P_n^*(x) = P_n^*(x)\begin{pmatrix} n+1 & 0 \\ 0 & -n \end{pmatrix}.$$

A Chebyshev example, continued

- The orthogonality weight matrix is given by:

$$W(x) = \frac{1}{\sqrt{1-x^2}} \begin{pmatrix} 1 & x \\ x & 1 \end{pmatrix}, \quad -1 < x < 1.$$

- Polynomials $P_n^*(x)$ can be expressed in the following way:

$$P_n^*(x) = \frac{1}{2^n} \begin{pmatrix} U_n(x) & -U_{n-1}(x) \\ -U_{n-1}(x) & U_n(x) \end{pmatrix},$$

  where $U_n(x)$ are scalar Chebyshev polynomials.

- Polynomials $P_n^*(x)$ also satisfy the following zero-th order differential equation:

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} P_n^*(x) = P_n^*(x) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

- The example above is one of the simplest ones, but it already illustrates several major differences between the scalar and matrix set up.

New phenomena: algebra of differential operators

For a fixed family $\{P_n(x)\}_{n=0}^{\infty}$ of matrix orthogonal polynomials consider the algebra over $\mathbb{C}$

$$\mathcal{D}(W) = \left\{ D = \sum_{i=0}^{k} \partial^i F_i(x) : P_n D = \Lambda_n(D) P_n,\ n = 0, 1, 2, \ldots \right\}$$

Scalar case [M, 2005]: It was proved that if $\mathcal{F}$ is the second order differential operator (Hermite, Laguerre or Jacobi), then any operator $\mathcal{U}$ such that $\mathcal{U} p_n = \lambda_n p_n$

$$\mathcal{U} = \sum_{i=0}^{k} c_i \mathcal{F}^i, \quad c_i \in \mathbb{C} \Rightarrow \mathcal{D}(\omega) \simeq \mathbb{C}[t].$$

New phenomena

Scalar case:

- Only even order differential operators are possible
- Operators having a fixed family of polynomials as eigenfunctions can be only powers of the original second order operator associated with the given family

Matrix case: The algebra can be noncommutative and generated by several elements

- Existence of several linearly independent second order differential operators having a fixed family of MOP as eigenfunctions
- Existence of families of MOP satisfying odd order differential equations
- Existence of several families of orthogonal polynomials satisfying the same fixed differential operator. This phenomenon is considered below.

Adding a Dirac delta distribution

All examples we consider are of the form

$$\gamma W + \zeta M(x_0)\delta_{x_0}, \quad \gamma > 0, \zeta \geq 0, \quad x_0 \in \mathbb{R},$$

where W is a weight matrix having several linearly independent symmetric second order differential operators and $M(t_0)$ certain positive semidefinite matrix.

Scalar case $(\omega + m\delta_{x_0})$

- Second order: there are no symmetric second order differential operators.
- Fourth order: x_0 at the endpoints of the support, which is not symmetric with respect to the original weight (Krall, 1941):

$$\text{Laguerre type } e^{-x} + M\delta_0$$
$$\text{Legendre type } 1 + M(\delta_{-1} + \delta_1)$$
$$\text{Jacobi type } (1-x)^{\alpha} + M\delta_0$$

Adding a Dirac delta distribution: example

Adding Dirac delta to the matrix weight and the example below are due to de la Iglesia and Duran.

$$D = \partial^2 F_2(x) + \partial^1 F_1(x) + \partial^0 F_0(x),$$

$$F_2(x) = \begin{pmatrix} 1 - ax & -1 + a^2x^2 \\ -1 & 1 + ax \end{pmatrix}$$

$$F_1(x) = \begin{pmatrix} -2a - 2x & 2a + 2(2 + a^2)x \\ 0 & -2x \end{pmatrix}$$

$$F_0(x) = \begin{pmatrix} -1 & 2\frac{2+a^2}{a^2} \\ \frac{4}{a^2} & 1 \end{pmatrix}$$

$$M = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

$\Rightarrow$ D is symmetric with respect to the family of weight matrices

$$\Upsilon(D) = \left\{ \gamma e^{-x^2} \begin{pmatrix} 1 + a^2x^2 & ax \\ ax & 1 \end{pmatrix} + \zeta \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \delta_0(x), \quad \gamma > 0, \zeta \geq 0, x \in \mathbb{R} \right\}$$

Applications

Quantum mechanics

[Durán–Grünbaum] *P A M Dirac meets M G Krein: matrix orthogonal polynomials and Dirac´s equation*, J. Phys. A: Math. Gen. (2006).

Time-and-band limiting

[Durán–Grünbaum] *A survey on orthogonal matrix polynomials satisfying second order differential equations*, J. Comput. Appl. Math. (2005).

Quasi-birth-and-death processes

[Grünbaum–MdI] *Matrix valued orthogonal polynomials arising from group representation theory and a family of quasi-birth-and-death processes*, SIMAX (2008).

References

- T.S. Chihara, *An introduction to orthogonal polynomials*, Gordon and Breach Science Publishers, 1978
- S. Bochner, *Über Sturm-Liouvillesche Polynomsysteme,* Math. Z. 29 (1929), 730–736
- A. J. Duran, *A generalization of Favard's theorem for polynomials satisfying a recurrence relation,* J. Approximation theory, 74 (1997), pp. 7818-7829.
- M. Castro, A. Grünbaum, *Orthogonal matrix polynomials satisfying first order differential equations: a collection of instructive examples*, to appear in Journal of Nonlinear Mathematical Physics.
- M.G. Krein, *Infinite J-matrices and a matrix moment problem,* Dokl. Akad. Nauk SSSR, 69 nr. 2, 2 (1949), pp. 125-128.
- A. J. Duran, *Matrix inner product having a matrix symmetric second order differential operator,* Rocky Mount. J. Math. 27 (1997), pp. 585-600.
- F.A. Grünbaum and L. Haine, *Orthogonal Polynomials Satisfying Differential Equations: The Role of the Darboux Transformation*, CRM Proceedings, Lecture Notes (9), AMS Providence, RI 1996, pp. 143-154.

References

- Durán, A. J. y Grünbaum, F. A., *A survey on orthogonal matrix polynomials satisfying second order differential equations*, J. Comput. Appl. Math. **178** (2005), 169–190.
- Durán, A. J. y de la Iglesia, M. D., *Second order differential operators having several families of orthogonal matrix polynomials as eigenfunctions*, Internat. Math. Research Notices, Vol. 2008, Article ID rnn084, 24 pages.
- Miranian, L., *Matrix valued orthogonal polynomials on the real line: some extensions of the classical theory*, J. Phys. A: Math. Gen. **38** (2005), 5731–5749.
- Miranian, L., *On classical orthogonal polynomials and differential operators*, J. Phys. A: Math. Gen. **38** (2005), 6379–6383.