# Equilibrium problem for the eigenvalues of banded block Toeplitz matrices

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#### Abstract

We consider banded block Toeplitz matrices  $T_n$  with n block rows and columns. We show that under certain technical assumptions, the normalized eigenvalue counting measure of  $T_n$  for  $n \to \infty$  weakly converges to one component of the unique vector of measures that minimizes a certain energy functional. In this way we generalize a recent result of Duits and Kuijlaars for the scalar case. Along the way we also obtain an equilibrium problem associated to an arbitrary algebraic curve, not necessarily related to a block Toeplitz matrix.

For banded block Toeplitz matrices, there are several new phenomena that do not occur in the scalar case: (i) The total masses of the equilibrium measures do not necessarily form a simple arithmetic series but in general are obtained through a combinatorial rule; (ii) The limiting eigenvalue distribution may contain point masses, and there may be attracting point sources in the equilibrium problem; (iii) More seriously, there are examples where the connection between the limiting eigenvalue distribution of  $T_n$  and the solution to the equilibrium problem breaks down. We provide sufficient conditions guaranteeing that no such breakdown occurs; in particular we show this if  $T_n$  is a Hessenberg matrix.

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# 1 Introduction

Let  $r \in \mathbb{N} := \{1, 2, 3, \ldots\}$  and let there be given a set of  $r \times r$  matrices

$$A_k \in \mathbb{C}^{r \times r}, \qquad k = -\alpha, \dots, \beta,$$

for some  $\alpha, \beta \in \mathbb{N}$ . These matrices are encoded by the matrix-valued Laurent polynomial (also called symbol)

$$A(z) = A_{-\alpha}z^{-\alpha} + \ldots + A_{\beta}z^{\beta}. \tag{1.1}$$

For  $n \in \mathbb{N}$  define the block Toeplitz matrix  $T_n(A)$  associated to the symbol A(z) by

$$T_n(A) = \left(A_{i-j}\right)_{i,j=1}^n \in \mathbb{C}^{rn \times rn},\tag{1.2}$$

where we put  $A_k \equiv 0$  if  $k > \beta$  or  $k < -\alpha$ . Explicitly,

$$T_{n}(A) = \begin{pmatrix} A_{0} & \dots & A_{-\alpha} & & & 0 \\ \vdots & \ddots & & \ddots & & \\ A_{\beta} & & \ddots & & \ddots & \\ & \ddots & & \ddots & & A_{-\alpha} \\ & & \ddots & & \ddots & \vdots \\ 0 & & & A_{\beta} & \dots & A_{0} \end{pmatrix}_{m \times m}$$
(1.3)

In this paper we are interested in the limiting behavior of the eigenvalues of  $T_n(A)$  for  $n \to \infty$ . It is known that under certain technical assumptions [21], the eigenvalue counting measure has a weak limit supported on a certain curve  $\Gamma_0$  in the complex plane. An example of this phenomenon is shown in Figure 1 for the symbol

$$A(z) = \begin{pmatrix} z^2 & 1\\ z^{-1} + z & 0 \end{pmatrix}; {1.4}$$

see Böttcher-Grudsky [3] for many more illustrations of this type.

In the case of scalar banded Toeplitz matrices, r = 1, it was recently shown by Duits-Kuijlaars [9] that the limiting eigenvalue distribution of  $T_n(A)$  satisfies a (vector) equilibrium problem that is constructed out of the symbol. The goal of this paper is to extend this result to the block case r > 1.

Let us first review some known results in the literature, following the exposition in [6, 9]. We denote the eigenvalue spectrum of  $T_n(A)$  by

$$\operatorname{sp} T_n(A) = \{ \lambda \in \mathbb{C} \mid \det(T_n(A) - \lambda I_{rn}) = 0 \},$$

where in general we use  $I_k$  to denote the identity matrix of size k by k. Following Schmidt-Spitzer [17], we define two limiting sets of the spectrum: we define

$$\liminf_{n\to\infty} \operatorname{sp} T_n(A)$$

to be the set of all  $\lambda \in \mathbb{C}$  for which there exists a sequence  $(\lambda_n)_{n \in \mathbb{N}}$ , with  $\lambda_n \in \operatorname{sp} T_n(A)$  converging to  $\lambda$ . Similarly we define

$$\limsup_{n \to \infty} \operatorname{sp} T_n(A)$$

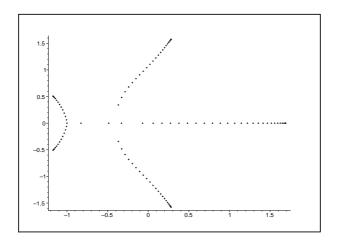


Figure 1: Point plot in the complex plane of the eigenvalues of the banded block Toeplitz matrix  $T_n(A)$ , n=50, with r=2 and symbol (1.4), computed in Maple with 60 digit precision. For  $n\to\infty$  the eigenvalues accumulate on a curve  $\Gamma_0\subset\mathbb{C}$  which consists of six analytic arcs connecting the points -1, -0.42,  $-1.17\pm0.51i$ ,  $0.28\pm1.58i$  and 1.70 (using two digits of precision). Three arcs are emanating with equal angles from -1 and four arcs from -0.42. The limiting eigenvalue distribution of  $T_n(A)$  for  $n\to\infty$  exists as an absolutely continuous measure  $\mu_0$  on  $\Gamma_0$ .

to be the set of all  $\lambda \in \mathbb{C}$  for which there exists a sequence  $(\lambda_n)_{n \in \mathbb{N}}$ , with  $\lambda_n \in \operatorname{sp} T_n(A)$  having a subsequence converging to  $\lambda$ .

Under certain assumptions [21], the above limiting sets can be described in terms of the algebraic equation

$$0 = f(z, \lambda) := \det(A(z) - \lambda I_r). \tag{1.5}$$

Note that each entry of the matrix  $A(z) - \lambda I_r$  is a Laurent polynomial in z, by virtue of (1.1). Hence  $f(z,\lambda) = \det(A(z) - \lambda I_r)$  is a Laurent polynomial in z as well, and we can write it in the form

$$f(z,\lambda) = \sum_{k=-a}^{p} f_k(\lambda) z^k, \tag{1.6}$$

for certain  $p, q \in \mathbb{N} \cup \{0\}$ . The coefficients  $f_k(\lambda)$  are polynomials in  $\lambda$  of degree at most r. More precisely,

$$\deg f_k = \begin{cases} r, & \text{if } k = 0, \\ \le r - 1, & \text{if } k \ne 0, \end{cases}$$

$$\tag{1.7}$$

on account of (1.5)–(1.6). We assume that the numbers p, q in (1.6) are such that the outermost coefficients  $f_{-q}(\lambda)$  and  $f_p(\lambda)$  are not identically zero as a function of  $\lambda$ . To avoid trivial cases we will always assume that

$$\min(p, q) \ge 1. \tag{1.8}$$

This is justified since if  $\min(p,q) = 0$ , then  $\det(T_n(A) - \lambda I_{rn}) = C_0(\lambda) f_0(\lambda)^{n+\alpha}$  for a certain rational function  $C_0(\lambda)$ , by Proposition 5.4 below. In that case the eigenvalues of  $T_n(A)$  are trivially obtained.

For any  $\lambda \in \mathbb{C}$  with  $f_p(\lambda) \neq 0$ , we consider

$$z^{q} f(z, \lambda) = \sum_{k=-q}^{p} f_{k}(\lambda) z^{k+q}$$
(1.9)

as a polynomial in z of degree p + q. We order its roots  $z = z(\lambda)$  (counting multiplicities) by absolute value as

$$0 \le |z_1(\lambda)| \le |z_2(\lambda)| \le \dots \le |z_{p+q}(\lambda)|. \tag{1.10}$$

If  $\lambda$  is such that two or more subsequent roots in (1.10) have the same absolute value, then we may arbitrarily label them so that (1.10) is satisfied. For the special values of  $\lambda$  for which  $f_p(\lambda)=0$ , the polynomial (1.9) has degree less than p+q, say p+q-j, and in that case we order its roots  $z_1(\lambda),\ldots,z_{p+q-j}(\lambda)$  as in (1.10) and set  $z_{p+q-j+1}(\lambda)=\ldots=z_{p+q}(\lambda)=\infty$ , compare with [6]. We also use the latter convention if  $\lambda\in\mathbb{C}$  is such that  $f(z,\lambda)\equiv 0$ . Thus in that case we put  $z_j(\lambda)=\infty$  for all  $j=1,\ldots,p+q$ .

Each of the roots  $z_i(\lambda)$  is finite and non-zero, except when  $\lambda$  belongs to the set

$$\Lambda := \{ \lambda \in \mathbb{C} \mid f_{-q}(\lambda) f_p(\lambda) = 0 \}. \tag{1.11}$$

By virtue of (1.7), the set  $\Lambda$  has cardinality  $|\Lambda| \leq 2r - 2$ . In particular  $\Lambda$  is empty in the scalar case r = 1.

Define the set

$$\Gamma_0 := \{ \lambda \in \mathbb{C} \mid |z_q(\lambda)| = |z_{q+1}(\lambda)| \}. \tag{1.12}$$

For the case of scalar banded Toeplitz matrices r=1 it is known that  $\Gamma_0$  is a curve consisting of a finite number of analytic arcs and having no isolated points, and moreover the eigenvalues of  $T_n(A)$  accumulate on  $\Gamma_0$  in the sense that

$$\liminf_{n \to \infty} \operatorname{sp} T_n(A) = \limsup_{n \to \infty} \operatorname{sp} T_n(A) = \Gamma_0.$$
(1.13)

These results were shown by Schmidt and Spitzer [17]. The same authors also showed that the limiting eigenvalue distribution  $\mu_0$  of  $T_n(A)$  exists as an absolutely continuous measure on  $\Gamma_0$ . An explicit expression for the measure  $\mu_0$  was obtained by Hirschman [12]. An alternative expression for  $\mu_0$  is given by (1.18) below with k = 0, cf. [9]. Further results about  $\mu_0$  in the scalar case r = 1 can be found in [3, 4, 9, 12, 19].

For the case of banded block Toeplitz matrices, r > 1, Widom [21] showed that the above results remain essentially valid, provided that the following hypotheses H2 and H3 hold true. The hypothesis H1 is stated for further reference.

- H1. The set  $\Lambda$  in (1.11) is empty.
- H2. The set  $\Gamma_0$  in (1.12) is a subset of  $\mathbb{C}$  of 2-dimensional Lebesgue measure zero.
- H3. The set  $G_0$  in (1.14) below has finite cardinality.

In the hypothesis H3 we define the set

$$G_0 := \{ \lambda \in \mathbb{C} \setminus \Gamma_0 \mid C_0(\lambda) = 0 \}, \tag{1.14}$$

with

$$C_0(\lambda) := \det\left(\frac{1}{2\pi i} \int_{\sigma_0} z^{\mu-\nu} (A(z) - \lambda I_r)^{-1} \frac{dz}{z}\right)_{\mu,\nu=1,\dots,\alpha}, \quad \text{for } \lambda \in \mathbb{C} \setminus \Gamma_0,$$
 (1.15)

with  $\alpha$  in (1.1), and where  $\sigma_0$  is a counterclockwise oriented closed Jordan curve enclosing z=0 and the points  $z_j(\lambda)$ ,  $j=1,\ldots,q$ , but no other roots of  $f(z,\lambda)=0$ . In (1.15) the determinant is taken of a matrix of size  $r\alpha$  by  $r\alpha$  and the integral is defined entry-wise. For background,

generalizations and alternative representations for the function  $C_0(\lambda)$  we refer to [4, 5, 21, 22] ( $C_0$  corresponds to the function  $E[\varphi]$  in [21, 22]), see also Prop. 5.4 below.

Widom shows that under the above hypotheses H2 and H3, one has that

$$\liminf_{n \to \infty} \operatorname{sp} T_n(A) = \limsup_{n \to \infty} \operatorname{sp} T_n(A) = \Gamma_0 \cup G_0.$$
(1.16)

It can be shown that  $G_0$  is empty in the scalar case r = 1, and then (1.16) reduces to (1.13). Under H2 and H3, Widom also observes that Hirschman's expression [12] for the limiting eigenvalue distribution  $\mu_0$  remains valid.

If hypothesis H1 fails then the limiting eigenvalue distribution of  $T_n(A)$  may contain point masses. This is implicit in Widom [21] and will be described in detail in this paper. On the other hand, if H2 fails then Widom's results are not true in the stated form. Usually they remain valid in a modified form however, see Sections 2.3 and 6.2 below.

The failure of hypothesis H3 is more serious, and it may cause the results to break down (see e.g. Section 6.3). Therefore it is important to provide sufficient conditions guaranteeing that H3 holds true. We will provide two such conditions; in both cases H2 will hold true as well.

#### **Proposition 1.1.** (Sufficient conditions for H2 and H3).

- (a) Suppose that the set  $\mathbb{C} \setminus \Gamma_0$  is connected and moreover  $\Gamma_0$  does not have any interior points. Then H2 and H3 hold true.
- (b) Suppose that A(z) is the symbol of a lower Hessenberg matrix, in the sense that in the entry-wise expansion  $T_n(A) = (t_{i,j})_{i,j=0}^{r_n-1}$  we have  $t_{i,j} = 0$  whenever j > i+1, i.e., all the entries above the first scalar superdiagonal of  $T_n(A)$  vanish. Then H2 (or more generally H2k below) and H3 hold true.

Proposition 1.1 will be proved in Section 5.2. Incidentally, the assumption (1.8) implies that all the entries on the first scalar superdiagonal of the Hessenberg matrix  $T_n(A)$  in Part (b) are non-zero.

Finally we discuss the results of Duits-Kuijlaars [9]. These authors noticed that in addition to the set  $\Gamma_0$  in (1.12), an important role is played by the sets

$$\Gamma_k := \{ \lambda \in \mathbb{C} \mid |z_{q+k}(\lambda)| = |z_{q+k+1}(\lambda)| \}, \tag{1.17}$$

for  $k = -q + 1, \ldots, p - 1$ . In the scalar case r = 1, each set  $\Gamma_k$  is a curve consisting of finitely many analytic arcs. We equip every analytic arc of  $\Gamma_k$  with an orientation and we define the +-side (or --side) as the side on the left (or right) of the arc when traversing it according to its orientation.

Duits and Kuijlaars then define the measure

$$d\mu_k(\lambda) = \frac{1}{2\pi i} \sum_{i=1}^{q+k} \left( \frac{z'_{j+}(\lambda)}{z_{j+}(\lambda)} - \frac{z'_{j-}(\lambda)}{z_{j-}(\lambda)} \right) d\lambda$$
 (1.18)

on the curve  $\Gamma_k$ . Here  $d\lambda$  denotes the complex line element on each analytic arc of  $\Gamma_k$ , according to the chosen orientation of  $\Gamma_k$ . In addition,  $z_{j+}(\lambda)$  and  $z_{j-}(\lambda)$  denote the boundary values of  $z_j(\lambda)$  from the +-side and --side of  $\Gamma_k$ , respectively. These boundary values exist for all but finitely many points. The definition (1.18) is independent of the choice of the orientation of  $\Gamma_k$ .

In the scalar case r = 1, it is shown in [9] that the measures  $\mu_k$  are the minimizers of a certain (vector) equilibrium problem from potential theory. Moreover,  $\mu_k$  is the weak limit for  $n \to \infty$ 

of the normalized counting measures of the kth generalized eigenvalues of the Toeplitz matrix  $T_n(A)$ . The usual eigenvalues correspond to k=0.

In this paper we wish to extend these results to the block case  $r \geq 2$ . Instead of hypothesis H2 we are then led to the following generalization H2k:

H2k. Each set  $\Gamma_k$  in (1.17),  $k = -q + 1, \ldots, p - 1$ , is a subset of  $\mathbb C$  of 2-dimensional Lebesgue measure zero.

To the algebraic curve  $f(z, \lambda) = 0$  we will associate an equilibrium problem, even when hypotheses H1 and/or H2k fail. Then the equilibrium problem may contain point sources (if H1 fails) or the definition of  $\Gamma_k$  in (1.17) needs to be modified (if H2k fails).

The measure  $\mu_0$  will be one of the measures involved in the equilibrium problem. This measure will be the absolutely continuous part of the limiting eigenvalue distribution of  $T_n(A)$ , provided that hypothesis H3, or a suitable analogue thereof if H2 fails, holds true. In particular this will be the case for the two situations in Prop. 1.1. There is also an interpretation of the measure  $\mu_k$ ,  $k \neq 0$ , as the absolutely continuous part of the limiting distribution of the kth generalized eigenvalues of  $T_n(A)$ , in the spirit of [6, 9]; this will be briefly discussed in Section 3.

In the next section, we associate an equilibrium problem to an arbitrary algebraic curve  $f(z,\lambda)=0$  as in (1.6), which is not necessarily defined from a block Toeplitz matrix. In Section 3 we apply this to banded block Toeplitz matrices  $T_n(A)$ . In Section 4 we specialize our results to the case where  $T_n(A)$  has a scalar banded structure. Section 5 contains the proofs of our main results. Section 6 illustrates our results for some examples. Finally, Section 7 contains some concluding remarks.

# 2 Equilibrium problem associated to an arbitrary algebraic curve

#### 2.1 Definitions

In this section we show how an equilibrium problem can be associated to an arbitrary algebraic curve. We consider an algebraic curve which is written in the form

$$f(z,\lambda) = \sum_{k=-q}^{p} f_k(\lambda) z^k = 0,$$
(2.1)

where  $f_k(\lambda)$ , k = -q, ..., p are polynomials, and where  $p, q \ge 1$  are such that the outermost polynomials  $f_{-q}(\lambda)$  and  $f_p(\lambda)$  are not identically zero. Note that the numbers p and q in (2.1) do not have an absolute meaning; indeed by multiplying f with  $z^j$ ,  $j \in \mathbb{Z}$ , the indices p and q are shifted to p + j and q + j respectively. The reason why we write (2.1) in its present form is because of the applications to banded block Toeplitz matrices.

Denote

$$r := \max_{k \in \{-q, \dots, p\}} \deg f_k. \tag{2.2}$$

This definition of r is compatible with the one used before, by virtue of (1.7).

Define the roots  $z_j = z_j(\lambda)$ , j = 1, ..., p + q as in (1.10), and define the sets  $\Gamma_k$ , k = -q + 1, ..., p - 1 as in (1.17). The structure of the set  $\Gamma_k$  is given by the following result.

**Lemma 2.1.** (Structure of  $\Gamma_k$ ). Let  $k \in \{-q+1, \ldots, p-1\}$ . Then any point  $\lambda_0 \in \mathbb{C}$  has an open neighborhood  $U \subset \mathbb{C}$  whose intersection with  $\Gamma_k$  is either empty, the singleton  $\{\lambda_0\}$ , the entire

neighborhood U, or a finite union of analytic arcs moving from  $\lambda_0$  to the boundary  $\partial U$  of the neighborhood U, with the arcs intersecting only at the point  $\lambda_0$ . A similar statement holds true for  $\lambda_0 = \infty$  provided that we consider  $\Gamma_k$  on the Riemann sphere  $\overline{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ . The isolated points of  $\Gamma_k$  all belong to  $\Lambda$  in (1.11).

Lemma 2.1 was observed for k=0 by Widom [21, Page 312], based on the similar result for the scalar case by Schmidt and Spitzer [17]. The proof for  $k \neq 0$  is exactly the same. See also Prop. 2.10 below for further information on  $\Gamma_k$ .

In addition to the set  $\Gamma_k$  we also introduce

$$\widetilde{\Gamma}_k := \Gamma_k \setminus \{ \text{isolated points of } \Gamma_k \} 
= \operatorname{cls} \{ \lambda \in \mathbb{C} \setminus \Lambda \mid |z_{a+k}(\lambda)| = |z_{a+k+1}(\lambda)| \},$$
(2.3)

for  $k = -q + 1, \dots, p - 1$ , where cls denotes the closure of a subset of  $\mathbb{C}$ .

Our next goal is to provide an expression for the total mass of the measure  $\mu_k$  in (1.18). To this end we need some auxiliary definitions. The next definition is a variant of the so-called *Newton polygon*, see e.g. [11].

**Definition 2.2.** (The numbers  $m_k$ ). We denote by  $k \mapsto m_k$  the smallest concave function on  $\{-q, \ldots, p\}$  for which  $m_k \ge \deg f_k$  for all k. Formally,

$$m_k = \max_{i \le k \le j} \left( \frac{j-k}{j-i} \deg f_i + \frac{k-i}{j-i} \deg f_j \right),$$

where the maximum is taken over all integers i, j with  $-q \le i \le k$  and  $k \le j \le p$  and with the equalities i = k and j = k not holding simultaneously.

A graphical interpretation of Definition 2.2 is as follows: consider the grid points  $(k, m_k) \in \mathbb{Z}^2$ ,  $k = -q, \ldots, p$ , and draw a line segment between  $(k, m_k)$  and  $(k+1, m_{k+1})$ , for  $k = -q, \ldots, p-1$ . This then results in a curve which lies above the grid points  $(k, \deg f_k) \in \mathbb{Z}^2$ , and which is the 'lowest' concave, piecewise linear curve with this property.

Let us illustrate Definition 2.2 for two examples.

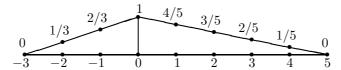


Figure 2: Illustration for Example 2.3.

**Example 2.3.** For the situation in Duits-Kuijlaars [9] we have deg  $f_k = 1$  if k = 0 and deg  $f_k = 0$  if  $k \neq 0$ . Then we easily find that

$$(m_k)_{k=-q}^p = \left(0, \frac{1}{q}, \frac{2}{q}, \dots, 1, \dots, \frac{2}{p}, \frac{1}{p}, 0\right).$$
 (2.4)

Let us illustrate this if q=3 and p=5. In that case  $(\deg f_k)_{k=-3}^5=(0,0,0,1,0,0,0,0,0)$ , and Figure 2 shows how to construct the concave, piecewise linear curve lying above the grid points  $(k, \deg f_k)$ . From the figure we can then read off that  $(m_k)_{k=-3}^5=(0,\frac{1}{3},\frac{2}{3},1,\frac{4}{5},\frac{3}{5},\frac{2}{5},\frac{1}{5},0)$ . Finally, we observe that the number  $m_k$  in (2.4),  $k=-q+1,\ldots,p-1$ , is precisely the total mass of the measure  $\mu_k$  in [9].

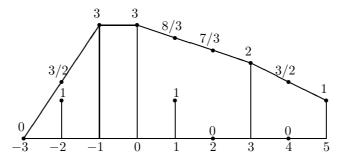


Figure 3: Illustration for Example 2.4.

**Example 2.4.** Assume that q = 3, p = 5 and  $(\deg f_k)_{k=-3}^5 = (0, 1, 3, 3, 1, 0, 2, 0, 1)$ . Proceeding in a similar way as before, we find that

$$(m_k)_{k=-3}^5 = \left(0, \frac{3}{2}, 3, 3, \frac{8}{3}, \frac{7}{3}, 2, \frac{3}{2}, 1\right),$$

as illustrated in Figure 3.

Recall the definition of the set  $\Lambda$  in (1.11), and choose an arbitrary but fixed labeling of the elements of this set, i.e.,  $\Lambda =: \{\lambda_l\}_{l=1}^L$ . Note that under hypothesis H1 we have that  $L := |\Lambda| = 0$ , so in that case we can ignore all the arguments involving the numbers  $(\lambda_l)_{l=1}^L$  in what follows.

We need the following analogue of Definition 2.2.

**Definition 2.5.** (The numbers  $m_k^{(l)}$ ). Fix  $l \in \{1, \ldots, L\}$ . We denote by  $k \mapsto m_k^{(l)}$  the largest convex function on  $\{-q, \ldots, p\}$  for which  $m_k^{(l)} \leq \text{mult}_{\lambda - \lambda_l} f_k$  for all k, where  $\text{mult}_{\lambda - \lambda_l} f_k$  denotes the multiplicity of  $\lambda - \lambda_l$  as a factor of  $f_k(\lambda)$ . Formally,

$$m_k^{(l)} = \min_{i \le k \le j} \left( \frac{j-k}{j-i} \text{mult}_{\lambda - \lambda_l} f_i + \frac{k-i}{j-i} \text{mult}_{\lambda - \lambda_l} f_j \right),$$

where the minimum is taken over all integers i, j with  $-q \le i \le k$  and  $k \le j \le p$  and with the equalities i = k and j = k not holding simultaneously.

A graphical interpretation of Definition 2.5 is as follows: consider the grid points  $(k, m_k^{(l)}) \in \mathbb{Z}^2$ ,  $k = -q, \ldots, p$ , and draw a line segment between  $(k, m_k^{(l)})$  and  $(k+1, m_{k+1}^{(l)})$ , for  $k = -q, \ldots, p-1$ . This then results in a curve which lies below the grid points  $(k, \operatorname{mult}_{\lambda - \lambda_l} f_k) \in \mathbb{Z}^2$ , and which is the 'highest' convex, piecewise linear curve with this property.

**Example 2.6.** Assume that q = 3, p = 5 and suppose that  $\lambda_l \in \Lambda$  is such that  $(\text{mult}_{\lambda - \lambda_l} f_k)_{k=-3}^5 = (1, 1, 0, 2, 1, 0, 1, 1, 2)$ . Figure 4 then shows how to construct the convex, piecewise linear curve lying below the grid points  $(k, \text{mult}_{\lambda - \lambda_l} f_k)$ . We can then read off that

$$(m_k^{(l)})_{k=-3}^5 = \left(1, \frac{1}{2}, 0, 0, 0, 0, \frac{1}{2}, 1, 2\right).$$

**Definition 2.7.** (The numbers  $\tilde{m}_k$ ). We define

$$\tilde{m}_k := m_k - \left( m_k^{(1)} + \ldots + m_k^{(L)} \right),$$
(2.5)

for k = -q + 1, ..., p - 1, where  $m_k$  and  $m_k^{(l)}$  are as in Definitions 2.2 and 2.5.

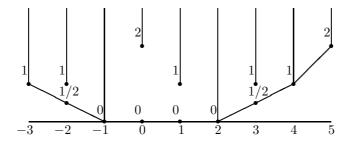


Figure 4: Illustration for Example 2.6.

The number  $\tilde{m}_k$  in (2.5) will be the total mass of the measure  $\mu_k$  in (1.18). Note that we defined  $\tilde{m}_k$  for  $k = -q + 1, \dots, p - 1$ . We could also define  $\tilde{m}_k$  for k = -q or k = p, by using the same definition (2.5). But in that case it is easy to see that  $\tilde{m}_{-q} = \tilde{m}_p = 0$ .

**Lemma 2.8.** The numbers  $\tilde{m}_k$  in (2.5) satisfy

$$\tilde{m}_k \ge 0, \tag{2.6}$$

for any  $k = -q + 1, \dots, p - 1$ . These inequalities are strict if (1.7) holds.

**Lemma 2.9.** (Criteria for  $\tilde{m}_k = 0$ ). In Lemma 2.8, the following statements give equivalent conditions for having equality in (2.6):

- (i)  $\tilde{m}_k = 0$  for some  $k \in \{-q+1, \dots, p-1\}$ ,
- (ii)  $\tilde{m}_k = 0$  for all  $k = -q, \ldots, p$ ,
- (iii)  $f(z,\lambda) = g(\lambda)\tilde{f}(h(\lambda)z)$ , where g is a polynomial and h a rational function of  $\lambda$ , and where  $\tilde{f}(z) = c_{-q}z^{-q} + \dots + c_{p}z^{p}$  is a Laurent polynomial with coefficients  $c_{k}$  not depending on  $\lambda$ .

Lemmas 2.8 and 2.9 are proved in Section 5.3. The nonnegativity of  $\tilde{m}_k$  can also be deduced from the fact that it is the total mass of the positive measure  $\mu_k$ . To avoid trivial statements, we will often tacitly assume that none of the equivalent conditions in Lemma 2.9 is satisfied.

Here is an addendum to Lemma 2.1:

**Proposition 2.10.** (Connected components of  $\Gamma_k$ ). Suppose that hypothesis H2k holds true. Then the number of compact, connected components of  $\Gamma_k$  is  $\leq m_k$  (and hence  $\leq r$ ). Moreover, for each compact, connected component C of  $\Gamma_k$ , denote by  $\mu_k(C)$  the total mass of the restriction of the measure  $\mu_k$  in (1.18) to C, with  $\mu_k(C) = 0$  if C is an isolated point of  $\Gamma_k$ . Then we have that

$$\mu_k(C) \in \mathbb{N}, \quad \text{if hypothesis H1 holds,}$$
 (2.7)

and in general,

$$\mu_k(C) + \sum_{\lambda_l \in \Lambda \cap C} m_k^{(l)} \in \mathbb{N}, \tag{2.8}$$

where the sum runs over all l = 1, ..., L with  $\lambda_l \in \Lambda \cap C$ .

Prop. 2.10 will be proved in Section 5.5. The proposition was observed before by Widom [21, Page 315] in the special case where k = 0 and  $m_0^{(l)} = 0$  for all l; note that (2.8) then reduces to (2.7). In the scalar case r = 1 it generalizes Ullman's result [19] that  $\Gamma_0$  is connected.

#### 2.2 The equilibrium problem

Now we associate an equilibrium problem to (2.1). First we do this under the hypothesis H2k. We will closely follow [6, 9]. For any measure  $\mu$  on  $\mathbb{C}$  define its logarithmic energy as

$$I(\mu) = \iint \log \frac{1}{|x - y|} d\mu(x) d\mu(y).$$

Similarly, for any measures  $\mu, \nu$  on  $\mathbb{C}$  define their mutual energy as

$$I(\mu, \nu) = \iint \log \frac{1}{|x - y|} \ d\mu(x) \ d\nu(y).$$

**Definition 2.11.** We call a vector of positive measures  $\vec{\nu} = (\nu_{-q+1}, \dots, \nu_{p-1})$  admissible if  $\nu_k$  has finite logarithmic energy,  $\nu_k$  is supported on  $\widetilde{\Gamma}_k$  in (2.3), and it has total mass  $\nu_k(\widetilde{\Gamma}_k) = \tilde{m}_k$  for every  $k = -q+1, \dots, p-1$ , recall (2.5)–(2.6).

**Definition 2.12.** The energy functional J is defined by

$$J(\vec{\nu}) = \sum_{k=-q+1}^{p-1} I(\nu_k) - \sum_{k=-q+1}^{p-2} I(\nu_k, \nu_{k+1}) + \sum_{l=1}^{L} \sum_{k=-q+1}^{p-1} (-m_{k-1}^{(l)} + 2m_k^{(l)} - m_{k+1}^{(l)}) \int \log \frac{1}{|\lambda - \lambda_l|} d\nu_k(\lambda). \quad (2.9)$$

The (vector) equilibrium problem is to minimize the energy functional (2.9) over all admissible vectors of positive measures  $\vec{\nu}$ .

Note that the numbers  $-m_{k-1}^{(l)} + 2m_k^{(l)} - m_{k+1}^{(l)}$  in (2.9) are all nonpositive because of the convexity of  $k \mapsto m_k^{(l)}$ .

The equilibrium problem can be understood intuitively as follows, compare with [6, 9]. On each of the curves  $\widetilde{\Gamma}_k$  (recall the assumption  $\mathrm{H}2k$ ) we put charged particles with total charge  $\widetilde{m}_k$ . The particles on each curve repel each other. The particles on two consecutive curves attract each other, with a strength that is half as strong as the repulsion on each individual curve. Particles on different curves that are non-consecutive do not interact directly. Moreover, if  $2m_k^{(l)} \neq m_{k-1}^{(l)} + m_{k+1}^{(l)}$  then we have an attracting external field acting on the particles on the curve  $\widetilde{\Gamma}_k$ . The external field is induced by an attracting point charge (also called sink) at  $\lambda = \lambda_l$ . We refer to [15, 16] for background on equilibrium problems with external fields, and to [14] for vector equilibrium problems.

Note that if hypothesis H1 holds true then (2.9) reduces to

$$J(\vec{\nu}) = \sum_{k=-q+1}^{p-1} I(\nu_k) - \sum_{k=-q+1}^{p-2} I(\nu_k, \nu_{k+1}).$$
 (2.10)

This is the energy functional in [9]; it also appears in the theory of Nikishin systems [14]. The following theorem generalizes a result in [6] and [9].

**Theorem 2.13.** (Equilibrium problem associated to an algebraic curve). Consider an algebraic curve as in (2.1) and define the sets  $\Gamma_k$ ,  $\widetilde{\Gamma}_k$ ,  $k = -q+1, \ldots, p-1$  as in (1.17) and (2.3). Assume that hypothesis H2k holds true. Then

- (a) The vector of measures  $\vec{\mu} = (\mu_k)_{k=-q+1}^{p-1}$  defined in (1.18) is admissible.
- (b) There exist constants  $l_k \in \mathbb{R}$  such that

$$2\int \log \frac{1}{|\lambda - x|} d\mu_k(x) - \int \log \frac{1}{|\lambda - x|} d\mu_{k+1}(x) - \int \log \frac{1}{|\lambda - x|} d\mu_{k-1}(x) + \sum_{l=1}^{L} (-m_{k-1}^{(l)} + 2m_k^{(l)} - m_{k+1}^{(l)}) \log \frac{1}{|\lambda - \lambda_l|} = l_k, \quad (2.11)$$

for  $\lambda \in \widetilde{\Gamma}_k$ ,  $k \in \{-q+1, \ldots, p-1\}$ . Here we let  $\mu_{-q}$  and  $\mu_p$  be the zero measures.

(c)  $\vec{\mu} = (\mu_k)_{k=-q+1}^{p-1}$  is the unique solution to the equilibrium problem in Def. 2.12.

Theorem 2.13 will be proved in Section 5.5. Note that the equalities in Part (b) are nothing but the Euler-Lagrange variational conditions of the equilibrium problem, see also [6, 9].

#### 2.3 Roots with identically equal modulus

Now we extend Theorem 2.13 to the case where hypothesis  $\mathrm{H}2k$  fails, i.e., the case where one or more sets  $\Gamma_k$  have non-zero 2-dimensional Lebesgue measure in  $\mathbb{C}$ . By Lemma 2.1 this implies that  $\Gamma_k$  contains an open disk U. Inside this disk, two or more roots  $z(\lambda)$  of  $f(z,\lambda)$  have identically equal modulus as functions of  $\lambda$ . If the disk U is disjoint from  $\Lambda$ , then we can label these roots so that they are analytic functions in U. The maximum modulus principle then implies that they are identically equal as functions of  $\lambda$ , up to a constant factor of modulus 1.

In this case we adapt the definition of the sets  $\Gamma_k$  as follows:

 $\Gamma_k = \{\lambda \in \mathbb{C} \setminus \Lambda \mid |z_{q+k}(\lambda)| = |z_{q+k+1}(\lambda)| \text{ and, possibly after relabeling the roots, the function } z_{q+k}/z_{q+k+1} \text{ takes infinitely many values on each open neighborhood } U \text{ of } \lambda\}$ 

$$\cup \{\lambda_l \in \Lambda \mid m_k^{(l)} > 0\}.$$
 (2.12)

This new definition guarantees that  $\Gamma_k$  is a curve:

**Lemma 2.14.** Fix  $k \in \{-q+1, \dots, p-1\}$ .

- (a) The set  $\Gamma_k$  in (2.12) is a finite union of analytic arcs and points, with all of its isolated points belonging to  $\Lambda$ , and it satisfies Lemma 2.1.
- (b) For any simply connected domain  $U \subset \mathbb{C} \setminus (\Gamma_k \cup \Lambda)$ , we can choose an ordering of the roots  $z_j(\lambda)$  as in (1.10) such that  $\prod_{j=1}^{q+k} z_j(\lambda)$  is analytic for  $\lambda \in U$ . Moreover, we can uniquely define the logarithmic derivative  $\left(\prod_{j=1}^{q+k} z_j(\lambda)\right)' / \prod_{j=1}^{q+k} z_j(\lambda)$  as a meromorphic function in  $\mathbb{C} \setminus \Gamma_k$  with poles at the points in  $\Lambda$ .

Lemma 2.14 is proved in Section 5.4.

Due to Lemma 2.14, we can uniquely define the measure  $\mu_k$  on  $\Gamma_k$  (more precisely on  $\Gamma_k$ ) by means of (1.18). We have the following generalization of Theorem 2.13.

**Theorem 2.15.** (Equilibrium problem with roots of identically equal modulus). Consider the setting of Theorem 2.13 but assume that hypothesis H2k fails. Define the curves  $\Gamma_k$ ,  $\widetilde{\Gamma}_k$  as in (2.12) and (2.3) and the measures  $\mu_k$  as in (1.18), taking into account Lemma 2.14. Then Theorem 2.13 remains valid.

This theorem is proved in Section 5.5.

# 3 The measure $\mu_0$ as the limiting eigenvalue distribution of the banded block Toeplitz matrix $T_n(A)$

Using the results of the previous section, we can associate a vector equilibrium problem to the algebraic equation  $f(z,\lambda)=0$  in (1.5) that is defined from the banded block Toeplitz matrix  $T_n(A)$ . We want to show that the measure  $\mu_0$  in the equilibrium problem is the absolutely continuous part of the limiting distribution of the eigenvalues of  $T_n(A)$ . As discussed before, this will require the hypothesis H3 (or a suitable analogue thereof if H2 fails) to hold true. The next theorem should be compared with Widom's result [21, Theorem 6.1]. We define the normalized eigenvalue counting measure  $\mu_{0,n}$  of  $T_n(A)$  as

$$\mu_{0,n} := \frac{1}{n} \sum_{\lambda \in \text{sp } T_n(A)} \delta_{\lambda}, \tag{3.1}$$

where  $\delta_{\lambda}$  is the Dirac measure at  $\lambda$  and each eigenvalue is counted according to its multiplicity.

**Theorem 3.1.** (Limiting eigenvalue distribution of  $T_n(A)$ ). Let A(z) be such that the assumptions in parts (a) and/or (b) of Prop. 1.1 are satisfied and define  $\Gamma_0$ ,  $G_0$  as in (1.12) and (1.14). Then

$$\liminf_{n \to \infty} \operatorname{sp} T_n(A) = \limsup_{n \to \infty} \operatorname{sp} T_n(A) = \Gamma_0 \cup G_0, \tag{3.2}$$

and

$$\lim_{n \to \infty} \int \phi(z) \ d\mu_{0,n}(z) = \int \phi(z) \ d\mu_0(z) + \sum_{l=1}^{L} m_0^{(l)} \phi(\lambda_l)$$
 (3.3)

for every bounded continuous function  $\phi$  on  $\mathbb{C}$ .

Moreover, for each  $\lambda \in G_0$  there is a positive integer  $j \in \mathbb{N}$  (more precisely, the multiplicity of  $\lambda$  as a zero of  $G_0$ ) such that for every sufficiently small open disk U around  $\lambda$ , one has

$$|U \cap \operatorname{sp} T_n(A)| = j, \tag{3.4}$$

for all n sufficiently large, where we take into account eigenvalue multiplicities.

Theorem 3.1 shows that the limiting eigenvalue distribution of  $T_n(A)$  for  $n \to \infty$  consists of the absolutely continuous part  $\mu_0$  together with a point mass of mass  $m_0^{(l)}$  at each  $\lambda_l \in \Lambda$ ,  $l = 1, \ldots, L$ . The theorem also shows that  $G_0$  attracts isolated eigenvalues in the spectrum of  $T_n(A)$ . The theorem will be proved in Section 5.6.

It can be checked that  $m_0^{(l)} > 0$  implies  $\lambda_l \in \Gamma_0$ . The point  $\lambda_l$  can then either be an isolated point of  $\Gamma_0$  or it can lie on one or more analytic arcs of  $\Gamma_0$ .

Incidentally, the occurrence of point masses at the points of  $\{\lambda_l \in \Lambda \mid m_0^{(l)} > 0\}$  can already be seen at the level of the finite n matrices  $T_n(A)$ :

**Proposition 3.2.** Let A(z) in (1.1) be the symbol of an arbitrary banded block Toeplitz matrix. Then there exists a constant  $c \in \mathbb{R}$  such that

(a) For each  $\lambda_l \in \Lambda$ , l = 1, ..., L, we have that

$$\operatorname{mult}_{\lambda-\lambda_l} \det T_n(A(z) - \lambda I_r) \ge m_0^{(l)} n - c, \quad \text{for all } n \in \mathbb{N}.$$
 (3.5)

(b) We have that

$$\deg_{\lambda} \det T_n(A(z) - \lambda I_r) \le m_0 n + c, \quad \text{for all } n \in \mathbb{N}.$$
 (3.6)

Prop. 3.2 is established in Section 5.6.

#### The measure $\mu_k$ and the kth generalized eigenvalues of $T_n(A)$ : discussion

Fix  $k \in \{-q+1, \dots, p-1\}$  and define the cyclic shift matrix

$$S := \begin{pmatrix} 0 & z \\ I_{r-1} & 0 \end{pmatrix}. \tag{3.7}$$

Let  $\lambda$  be a parameter and consider the 'shifted' symbol

$$S^{-k}(A(z) - \lambda I_r) =: A_{-\alpha_k}(\lambda) z^{-\alpha_k} + \ldots + A_{\beta_k}(\lambda) z^{\beta_k}, \tag{3.8}$$

for suitable  $\alpha_k, \beta_k \in \mathbb{N}$ . We may assume that  $\alpha_k, \beta_k$  are such that the coefficients  $A_{-\alpha_k}$  and  $A_{\beta_k}$  in (3.8) are not identically zero, although this will not be essential. Note that for k = 0 we can take  $\alpha_0 = \alpha$  and  $\beta_0 = \beta$  as in (1.1).

We consider the 'shifted' block Toeplitz matrix  $T_n(S^{-k}(A(z) - \lambda I_r))$ . Note that for  $k \geq 0$ , this block Toeplitz matrix is obtained from  $T_n(A) - \lambda I_{rn}$  by skipping its first k rows and adding k new rows at the bottom of the matrix, subject to the block Toeplitz structure. A similar description holds for k < 0, see also [6, 9].

We define the kth generalized spectrum of  $T_n(A)$  as

$$\operatorname{sp}_{k} T_{n}(A) = \{ \lambda \in \mathbb{C} \mid \det(T_{n}(S^{-k}(A(z) - \lambda I_{r}))) = 0 \}. \tag{3.9}$$

Inspired by Duits-Kuijlaars [9], one may hope to interpret the measure  $\mu_k$ ,  $k \neq 0$ , as the absolutely continuous part of the weak limit of the normalized counting measures of the kth generalized eigenvalues of  $T_n(A)$ . This limiting distribution should then also have a point mass of mass  $m_k^{(l)}$  at  $\lambda = \lambda_l$ ,  $l = 1, \ldots, L$ .

of mass  $m_k^{(l)}$  at  $\lambda = \lambda_l$ ,  $l = 1, \ldots, L$ . It turns out that these ideas can indeed be established, provided that a suitable analogue H3k of hypothesis H3 holds true. Let us define the following analogues of the objects  $G_0$  and  $C_0(\lambda)$  in (1.14)–(1.15):

$$G_k := \{ \lambda \in \mathbb{C} \setminus \Gamma_k \mid C_k(\lambda) = 0 \}, \tag{3.10}$$

and

$$C_k(\lambda) := \det\left(\frac{1}{2\pi i} \int_{\sigma_k} z^{\mu-\nu} (A(z) - \lambda I_r)^{-1} S^k \frac{dz}{z}\right)_{\mu,\nu=1,\dots,\alpha_k}, \quad \text{for } \lambda \in \mathbb{C} \setminus \Gamma_k, \quad (3.11)$$

with  $\alpha_k$  in (3.8), and where  $\sigma_k$  is a counterclockwise oriented closed Jordan curve enclosing z=0 and the points  $z_j(\lambda)$ ,  $j=1,\ldots,q+k$ , but no other roots of  $f(z,\lambda)=0$ . In (3.11) the determinant is taken of a matrix of size  $r\alpha_k$  by  $r\alpha_k$  and the integral is again defined entry-wise.

The hypothesis H3k now reads as follows:

H3k. The set  $G_k$  in (3.10) has finite cardinality.

Define the normalized counting measure

$$\mu_{k,n} := \frac{1}{n} \sum_{\lambda \in \operatorname{sp}_k T_n(A)} \delta_{\lambda}, \tag{3.12}$$

where again each root is counted according to its multiplicity.

**Proposition 3.3.** Let  $k \in \{-q+1,\ldots,p-1\}$  be such that the hypotheses H2k and H3k hold true. Then the statements (3.2)–(3.4) in Theorem 3.1 remain true, provided that we replace everywhere  $\Gamma_0$ ,  $G_0$ ,  $\mu_{0,n}$ ,  $\mu_0$ ,  $m_0^{(l)}$  and sp by  $\Gamma_k$ ,  $G_k$ ,  $\mu_{k,n}$ ,  $\mu_k$ ,  $m_k^{(l)}$  and sp<sub>k</sub> respectively.

Unfortunately the hypothesis H3k is very delicate to handle, and we have been unable to obtain sufficient conditions in the style of Prop. 1.1 for a reasonably large class of symbols A(z). For this reason, we will not discuss generalized eigenvalues any further in this paper.

# 4 A case study: scalar banded matrices

In this section we specialize our results to the case where  $T_n(A)$  is a scalar banded matrix with non-vanishing outer diagonals. More precisely, we assume that  $T_n(A)$  is a banded block Toeplitz matrix as in (1.3), that can be written in the scalar form

$$T_n(A) = \begin{pmatrix} a_0^{(0)} & \dots & a_0^{(-q)} & & & & 0 \\ \vdots & \ddots & & \ddots & & & \\ a_p^{(p)} & & \ddots & & \ddots & & \\ & \ddots & & \ddots & & & a_{rn-q-1}^{(-q)} & & & & \\ & & \ddots & & \ddots & & \vdots & & \\ 0 & & & a_{rn-1}^{(p)} & \dots & a_{rn-1}^{(0)} \end{pmatrix} , \tag{4.1}$$

where the numbers  $a_i^{(k)} \in \mathbb{C}$  are such that

$$a_i^{(k)} = a_{i \bmod r}^{(k)}, \tag{4.2}$$

for all  $i \in \mathbb{N} \cup \{0\}$  and  $k = -q, \dots, p$ , and with

$$a_i^{(p)} \neq 0, \qquad a_i^{(-q)} \neq 0,$$
 (4.3)

for all i = 0, ..., r - 1. To avoid trivial cases we again assume that  $\min(p, q) \ge 1$ . We will see in a moment that the notations p and q in (4.1) are consistent with those used before in (1.6).

The representations (1.3) and (4.1) are related as follows:

$$\alpha := \lceil q/r \rceil, \qquad \beta := \lceil p/r \rceil, \tag{4.4}$$

and the matrices  $A_k$ ,  $k = -\alpha, ..., 0$  in (1.3) are obtained by taking the submatrix formed by the first r rows of (4.1) and partitioning it in blocks of size  $r \times r$  as follows:

$$\begin{pmatrix} a_0^{(0)} & \dots & a_0^{(-q)} & 0 \\ \vdots & & \ddots & \\ a_{r-1}^{(r-1)} & \dots & \dots & a_{r-1}^{(-q)} \end{pmatrix} = (A_0 \quad A_1 \quad \dots \quad A_{-\alpha}). \tag{4.5}$$

Here we add  $r\alpha - q = r\lceil q/r \rceil - q$  zero columns at the right of the matrix in the left hand side of (4.5) in order to have compatible matrix dimensions. Similarly the matrices  $A_k$ ,  $k = 0, \ldots, \beta$  are obtained by taking the submatrix formed by the first r columns of (4.1) and partitioning it in blocks of size  $r \times r$ .

One checks that the symbol A(z) can be written as

$$A(z) = \sum_{k=-q}^{p} \operatorname{diag}(a_0^{(k)}, \dots, a_{r-1}^{(k)}) S^k,$$
(4.6)

where S is the cyclic shift matrix in (3.7). There is also the alternative representation

$$DA(z^r)D^{-1} = \sum_{k=-a}^{p} z^k \operatorname{diag}(a_0^{(k)}, \dots, a_{r-1}^{(k)})\widetilde{S}^k, \tag{4.7}$$

where  $D := \operatorname{diag}(1, z, \dots, z^{r-1})$  and

$$\widetilde{S} := \begin{pmatrix} 0 & 1 \\ I_{r-1} & 0 \end{pmatrix}. \tag{4.8}$$

One may argue that (4.7) is more natural than (4.6), in the sense that it gives the same weight  $z^k$  to all the entries on the kth scalar diagonal of the matrix (4.1). From this representation we also obtain that

$$f(z^r, \lambda) = \det\left(-\lambda I_r + \sum_{k=-q}^p z^k \operatorname{diag}(a_0^{(k)}, \dots, a_{r-1}^{(k)})\widetilde{S}^k\right),$$
 (4.9)

recall (1.5).

**Proposition 4.1.** (Structure of f). Let  $T_n(A)$  be as in (4.1)–(4.3). Then  $f(z, \lambda)$  in (4.9) can be written in the form

$$f(z,\lambda) = f_{-q}(\lambda)z^{-q} + \dots + f_0(\lambda) + \dots + f_p(\lambda)z^p, \tag{4.10}$$

where all the coefficients  $f_k(\lambda)$ ,  $k = -q, \ldots, p$  are polynomials in  $\lambda$ . The outermost coefficients take the values

$$f_{-q}(\lambda) \equiv f_{-q} = (-1)^{q(r-1)} \prod_{k=0}^{r-1} a_k^{(-q)}, \qquad f_p(\lambda) \equiv f_p = (-1)^{p(r-1)} \prod_{k=0}^{r-1} a_k^{(p)}, \tag{4.11}$$

so hypothesis H1 holds true. For general k, the degree of the polynomial  $f_k(\lambda)$  is bounded by

$$\deg f_k \le \begin{cases} \frac{q+k}{q}r, & \text{for } k = -q, \dots, 0, \\ \frac{p-k}{p}r, & \text{for } k = 0, \dots, p. \end{cases}$$
(4.12)

*Proof.* Equations (4.10)–(4.11) follow immediately from (4.9). To prove (4.12) one can use a combinatorial argument in the style of [9, Proof of Prop. 2.5]; a simpler proof will be obtained in Example 5.3 below.

Corollary 4.2. Under the conditions of Prop. 4.1, we have that

$$(m_k)_{k=-q}^p = (\tilde{m}_k)_{k=-q}^p = \left(0, \frac{r}{q}, \frac{2r}{q}, \dots, r, \dots, \frac{2r}{p}, \frac{r}{p}, 0\right).$$
 (4.13)

So the total masses  $\tilde{m}_k$  of the measures  $\mu_k$  form a simple arithmetic series in the same way as in the scalar Toeplitz case, see Example 2.3. The energy functional of the equilibrium problem reduces to (2.10).

#### 5 Proofs

In this section we prove our main results.

#### 5.1 Some preliminaries

First we single out some preliminaries which will be repeatedly used in the proofs.

#### Asymptotics of the roots $z_i(\lambda)$

Consider an algebraic curve  $f(z,\lambda)=0$  as in (2.1) and define the roots  $z_j(\lambda)$  as in (1.10) and curves  $\Gamma_k$  as in (2.12). Let  $j\in\{1,\ldots,p+q\}$  be fixed. It is well-known that there exist constants  $s_j\in\mathbb{R},\,c_j\in\mathbb{C}\setminus\{0\}$  and  $\kappa_j\in\mathbb{N}$  such that

$$z_j(\lambda) = c_j \lambda^{s_j} \left( 1 + O\left(\lambda^{-1/\kappa_j}\right) \right), \tag{5.1}$$

as  $\lambda \to \infty$  with  $\lambda \in \mathbb{C} \setminus \bigcup_k \Gamma_k$ , with possibly a different constant  $c_j$  for each connected component of  $\mathbb{C} \setminus \bigcup_k \Gamma_k$  in which we let  $\lambda \to \infty$ . Obviously,

$$s_1 \le \dots \le s_{p+q},\tag{5.2}$$

because of the ordering (1.10) of the roots  $z_i(\lambda)$ .

The expansion (5.1) is an instance of a *Puiseux series* and the next lemma is a well-known result for the Newton polygon. We include the proof for completeness.

**Lemma 5.1.** The numbers  $s_i$  in (5.1) are such that

$$\sum_{j=1}^{q+k} s_j = \deg(f_{-q}) - m_k, \tag{5.3}$$

for any k = -q, ..., p, with  $m_k$  as in Definition 2.2.

*Proof.* We start from the factorization

$$f(z,\lambda) = \frac{f_p(\lambda)}{z^q} \prod_{j=1}^{p+q} (z - z_j(\lambda)).$$

By expanding this product in powers of z, we see that the coefficient  $f_k(\lambda)$  in (2.1) is given by

$$f_k(\lambda) = (-1)^{p-k} f_p(\lambda) \sum_{S} \prod_{j \in S} z_j(\lambda), \tag{5.4}$$

where the summation runs over all subsets  $S \subset \{1, \ldots, p+q\}$  with |S| = p-k, for any  $k \in \{-q, \ldots, p\}$ . Then we obtain

$$f_k(\lambda)/f_p(\lambda) = O\left(\sum_{S} |\lambda|^{\sum_{j \in S} s_j}\right) = O\left(|\lambda|^{\sum_{j \in S_k} s_j}\right), \quad \lambda \to \infty,$$
 (5.5)

for any  $k \in \{-q, \ldots, p\}$ , where in the second step we define  $S_k := \{q + k + 1, \ldots, p + q\}$ . Hence

$$\sum_{j=q+k+1}^{p+q} s_j \ge \deg f_k - \deg f_p. \tag{5.6}$$

Moreover if  $s_{q+k} < s_{q+k+1}$  then equality must hold in (5.6), since in that case  $S = S_k$  yields the unique dominant summand in the middle term of (5.5). In particular this holds for k = -q:

$$\sum_{j=1}^{p+q} s_j = \deg f_{-q} - \deg f_p. \tag{5.7}$$

By subtracting (5.7) from (5.6) we then get

$$\deg f_{-q} - \sum_{j=1}^{q+k} s_j \ge \deg f_k,\tag{5.8}$$

with equality if  $s_{q+k} < s_{q+k+1}$ .

Denote by  $\widehat{m}_k$  the left hand side of (5.8). Then  $k \mapsto \widehat{m}_k$  is a concave function on  $\{-q, \ldots, p\}$  by virtue of (5.2). From (5.8) we see that  $\widehat{m}_k \geq \deg f_k$ , with equality for each k for which  $2\widehat{m}_k > \widehat{m}_{k-1} + \widehat{m}_{k+1}$ , i.e., for each k for which the concave, piecewise linear function that interpolates between the grid points  $(k, \widehat{m}_k)$  changes slope. Then Definition 2.2 implies that  $\widehat{m}_k = m_k$ , which is (5.3).

Corollary 5.2. Under the assumption (1.7) we have that

$$\begin{cases}
z_j(\lambda) \to 0, & j = 1, \dots, q, \\
z_j(\lambda) \to \infty, & j = q + 1, \dots, p + q,
\end{cases}$$
(5.9)

as  $\lambda \to \infty$  with  $\lambda \in \mathbb{C} \setminus \bigcup_k \Gamma_k$ . In particular, the set  $\Gamma_0$  in (2.12) (or (1.17)) is compact.

*Proof.* The assumption (1.7) implies that  $k \mapsto m_k$  is a strictly increasing function on  $\{-q, \ldots, 0\}$  and strictly decreasing on  $\{0, \ldots, p\}$ . Thus (5.3) implies that  $s_j < 0$  for  $j = 1, \ldots, q$  and  $s_j > 0$  for  $j = q + 1, \ldots, p + q$ . The result then follows from (5.1).

**Example 5.3.** Let  $f(z,\lambda) = 0$  be an algebraic curve as in (4.9) and (4.3). Then

$$s_j = \begin{cases} -r/q, & j = 1, \dots, q, \\ r/p, & j = q + 1, \dots, p + q. \end{cases}$$
 (5.10)

Indeed, by virtue of (5.1) and (4.9) we find that

$$0 = f(z_i(\lambda), \lambda) = \left( (-1)^r \lambda^r + c_1 \lambda^{-qs_j} + c_2 \lambda^{ps_j} \right) (1 + o(1)), \qquad \lambda \to \infty.$$

for certain non-zero constants  $c_1, c_2$ . For this expression to be zero for large  $\lambda$  we must have that two out of the three exponents  $\{r, -qs_j, ps_j\}$  are equal and the third is smaller; this implies that either  $s_j = -r/q$  or  $s_j = r/p$ , for all  $j = 1, \ldots, p+q$ . The fact that  $s_j = -r/q$  occurs with multiplicity q and  $s_j = r/p$  occurs with multiplicity p, is then a consequence of the relation  $\sum_{j=1}^{p+q} s_j = 0$ , recall (5.7) and (4.11). Finally, we note that (5.10) and (5.3) imply (4.13), which in turn leads to (4.12).

Similarly to the above discussion, for any  $l=1,\ldots,L$  there exist constants  $s_j^{(l)} \in \mathbb{R}, \ c_j^{(l)} \in \mathbb{C} \setminus \{0\}$  and  $\kappa_j^{(l)} \in \mathbb{N}$  such that

$$z_{j}(\lambda) = c_{j}^{(l)}(\lambda - \lambda_{l})^{s_{j}^{(l)}} + O\left((\lambda - \lambda_{l})^{s_{j}^{(l)} + 1/\kappa_{j}^{(l)}}\right), \tag{5.11}$$

as  $\lambda \to \lambda_l$  with  $\lambda \in \mathbb{C} \setminus \bigcup_k \Gamma_k$ , with possibly a different value of  $c_j^{(l)}$  for each connected component of  $\mathbb{C} \setminus \bigcup_k \Gamma_k$  in which we let  $\lambda \to \lambda_l$ . The numbers  $s_j^{(l)}$  are such that

$$\sum_{j=1}^{q+k} s_j^{(l)} = \text{mult}_{\lambda - \lambda_l}(f_{-q}) - m_k^{(l)}, \tag{5.12}$$

for any  $k = -q, \ldots, p$  and  $l = 1, \ldots, L$ .

#### Widom's determinant identity

**Proposition 5.4.** (Widom's determinant identity). Let  $\lambda \in \mathbb{C}$  be such that the solutions  $z_j(\lambda)$  of the algebraic equation  $f(z,\lambda) = 0$  in (1.5) are pairwise distinct. Then for all n sufficiently large we have

$$\det T_n(A(z) - \lambda I_r) = \sum_S C_S(\lambda) (w_S(\lambda))^{n+\alpha}, \tag{5.13}$$

where the sum is over all subsets  $S \subset \{1, 2, \dots, p+q\}$  of cardinality |S| = q and for each such S we have

$$w_S(\lambda) = (-1)^q f_{-q}(\lambda) \prod_{j \in S} z_j(\lambda)^{-1}$$
 (5.14)

and

$$C_S(\lambda) = \det\left(\frac{1}{2\pi i} \int_{\sigma_S} z^{\mu-\nu} (A(z) - \lambda I_r)^{-1} \frac{dz}{z}\right)_{\mu,\nu=1,\dots,\alpha}$$
 (5.15)

where  $\sigma_S$  is a counterclockwise oriented closed Jordan curve enclosing z = 0 and the points  $z_j(\lambda)$ ,  $j \in S$ , but no other roots of  $f(z, \lambda) = 0$ .

Note that (5.14) can be written alternatively as

$$w_S(\lambda) = (-1)^p f_p(\lambda) \prod_{j \in \overline{S}} z_j(\lambda),$$

with  $\overline{S} := \{1, 2, \dots, p+q\} \setminus S$ . This expression has maximal modulus among the subsets S of cardinality |S| = q if  $S = S_0$  with

$$S_0 = \{1, \dots, q\}. \tag{5.16}$$

For  $S = S_0$  the definition of  $C_{S_0}$  in (5.15) reduces to the one of  $C_0$  in (1.15).

Prop. 5.4 was obtained in [21, Section 6] by means of the Baxter-Schmidt formula [1]. Note that Prop. 5.4 assumes that n is sufficiently large, say  $n \ge n_0$ , but this is no problem since [21, Section 6, Remark 1] guarantees that the same value of  $n_0$  works for all  $\lambda$ .

Prop. 5.4 assumes that the solutions of the algebraic equation  $f(z, \lambda) = 0$  are pairwise distinct. If this assumption fails then similar determinant formulas can be obtained, by taking a suitable limit of (5.13) and using continuity. This will be hinted at in Section 5.6.

For the scalar case r = 1 it is known that

$$C_S(\lambda) = \prod_{j \in \overline{S}} z_j(\lambda)^q \prod_{j \in \overline{S}, l \in S} (z_j(\lambda) - z_l(\lambda))^{-1},$$

and then Prop. 5.4 reduces to a result in [20].

#### 5.2 Proof of Proposition 1.1

Proof of Proposition 1.1(a). Suppose that  $\mathbb{C}\setminus\Gamma_0$  is connected and moreover  $\Gamma_0$  does not have any interior points, recalling (1.17). From Lemma 2.1 we immediately obtain H2. Next we establish H3. Eq. (5.9) implies that for  $|\lambda|$  large enough we can take the contour  $\sigma_0$  in (1.15) to be the unit circle. Then we easily find that

$$\left(\frac{1}{2\pi i} \int_{\sigma_0} z^{\mu-\nu} (A(z) - \lambda I_r)^{-1} \frac{dz}{z}\right)_{\mu,\nu=1,...,\alpha} = -\lambda^{-1} I_{r\alpha} (1 + O(\lambda^{-1})), \qquad \lambda \to \infty,$$

and therefore  $C_0(\lambda) = (-\lambda^{-1})^{r\alpha}(1 + O(\lambda^{-1})) \neq 0$  as  $\lambda \to \infty$ . Hypothesis H3 then follows from the analyticity of  $C_0(\lambda)$  in  $\mathbb{C} \setminus \Gamma_0$ .

Proof of Proposition 1.1(b). Assume that  $T_n(A)$  is a Hessenberg matrix. Hence by definition,  $T_n(A)$  has the form (4.1) with q = 1 and with superdiagonal entries  $a_i^{(-1)} \neq 0$  for all i. We will need some auxiliary lemmas.

By virtue of (4.10) (where now q=1) we see that the equation  $f(z,\lambda)=0$  has p+1 roots

$$z_j = z_j(\lambda), \qquad j = 1, \dots, p+1,$$
 (5.17)

for a certain  $p \in \mathbb{N}$  (which is not necessarily the same p as in (4.1)). Basic algebraic geometry shows that one can choose a finite union of analytic arcs  $\Gamma \subset \mathbb{C}$  so that the roots  $z_j(\lambda)$ ,  $j = 1, \ldots, p+1$  depend analytically on  $\lambda \in \mathbb{C}$ , except when  $\lambda \in \Gamma$ . We will see in a moment that we can define  $\Gamma$  by means of (1.17), and the  $z_j(\lambda)$  as in (1.10); but we are *not* making these assumptions yet.

We consider the Riemann surface  $\mathcal{R}$  associated to the algebraic equation  $f(z,\lambda) = 0$ : it is a branched (p+1)-sheeted covering of  $\overline{\mathbb{C}}$ , with the analytic function  $z_j(\lambda)$  defined for  $\lambda$  on the jth sheet  $\overline{\mathbb{C}}$ ,  $j = 1, \ldots, p+1$ . These functions have a cut along the appropriate arcs of  $\Gamma$ , and the different sheets of  $\mathcal{R}$  are glued together along these arcs.

**Lemma 5.5.** Let the roots  $z_j(\lambda)$  in (5.17) and the Riemann surface  $\mathcal{R}$  be defined as in the previous paragraph. Then  $\mathcal{R}$  is connected.

*Proof.* Take an arbitrary point  $(\lambda_0, z_0) \in \mathcal{R}$ . Define the set

$$\mathcal{Z} := \{z \in \overline{\mathbb{C}} \mid \text{there exists } \lambda \in \overline{\mathbb{C}} \text{ and a continuous path in } \mathcal{R} \text{ from } (\lambda_0, z_0) \text{ to } (\lambda, z)\}.$$

Then  $\mathcal{Z}$  is a subset of  $\overline{\mathbb{C}}$  which is both open and closed. Hence it must be the entire Riemann sphere  $\overline{\mathbb{C}}$ . In particular it contains the value z=0. But to z=0 there corresponds only  $\lambda=\infty$  (use (4.10)–(4.11) with q=1). Moreover, there is a unique such point  $(\lambda,z)=(\infty,0)$  on the Riemann surface (use (5.9) with q=1). Summarizing, we see that there is a continuous path in  $\mathcal{R}$  from  $(\lambda_0, z_0)$  to this unique reference point  $(\infty, 0)$ . Since this holds true for any  $(\lambda_0, z_0) \in \mathcal{R}$ , the connectedness of  $\mathcal{R}$  follows.

From now on we will order the roots  $z_j = z_j(\lambda)$ , j = 1, ..., p + 1, by increasing modulus as in (1.10). We also define the sets  $\Gamma_k$ , k = 0, ..., p - 1, as in (1.17).

**Lemma 5.6.** Each set  $\Gamma_k$ ,  $k = 0, \dots, p-1$  is a finite union of analytic arcs and points in  $\mathbb{C}$ . Hence, the sets  $\Gamma_k$  can be taken as cuts for the Riemann surface  $\mathcal{R}$ .

*Proof.* The proof boils down to showing that  $\Gamma_k$  does not contain a (two-dimensional) open disk  $U \subset \mathbb{C} \setminus \Lambda$ . In that case we would have two roots  $z_i(\lambda)$  and  $z_j(\lambda)$  that have identically equal modulus in U. Their ratio must be a constant of modulus one. We then obtain a contradiction by using the connectedness of the Riemann surface associated to  $f(z,\lambda) = 0$  (Lemma 5.5), and the fact that there is only one root  $z_1(\lambda)$  that goes to zero if  $\lambda$  goes to  $\infty$  (see (5.9) with q = 1).  $\square$ 

Since q = 1, (5.13) now specializes to the form

$$\det(T_n(A) - \lambda I_{rn}) = \sum_{j=1}^{p+1} C_{j-1}(\lambda) (-z_j(\lambda))^{-n-1},$$
(5.18)

with

$$C_{j-1}(\lambda) := \det\left(\frac{1}{2\pi i} \int_{\sigma} (A(z) - \lambda I_r)^{-1} \frac{dz}{z}\right),\tag{5.19}$$

where  $\sigma$  is a counterclockwise oriented closed Jordan curve enclosing z=0 and the point  $z_j(\lambda)$ , but none of the other roots  $z_i(\lambda)$ ,  $i \in \{1, \ldots, p+1\}$ ,  $i \neq j$ . Here we write  $C_{j-1}$  rather than  $C_j$  to be consistent with (1.15). The function  $C_{j-1}(\lambda)$  is defined for  $\lambda$  in the domain  $\mathcal{D}_j := \mathbb{C} \setminus (\Gamma_{j-1} \cup \Gamma_j)$ .

**Lemma 5.7.** The function  $C_{j-1}(\lambda)$  in (5.19),  $j \in \{1, ..., p+1\}$ , has only isolated zeros in  $\mathcal{D}_j$ .

*Proof.* We must show that  $C_{j-1}(\lambda)$  cannot be identically zero in any open disk in  $\mathbb{C} \setminus \bigcup_k \Gamma_k$ . By the fact that the Riemann surface is connected (Lemma 5.5) and analytic continuation, this would imply that each of the  $C_{j-1}(\lambda)$ ,  $j=1,\ldots,p+1$ , is identically zero in  $\mathcal{D}_j$ . But then (5.18) would imply that  $\det(T_n(A) - \lambda I_{rn}) \equiv 0$  which is clearly a contradiction.

Combining the above two lemmas, we have now established that H2k and H3 hold true when  $T_n(A)$  has Hessenberg structure. This ends the proof of Proposition 1.1(b).

#### 5.3 Proofs of Lemmas 2.8 and 2.9

*Proof of Lemma 2.8.* From the definition of  $m_k$  we trivially have that

$$m_k \ge \frac{p-k}{p+q} \deg f_{-q} + \frac{k+q}{p+q} \deg f_p,$$
 (5.20)

while from the definition of  $m_k^{(l)}$  it follows that

$$m_k^{(l)} \le \frac{p-k}{p+q} \operatorname{mult}_{\lambda-\lambda_l} f_{-q} + \frac{k+q}{p+q} \operatorname{mult}_{\lambda-\lambda_l} f_p,$$
 (5.21)

for any l = 1, ..., L. Summing (5.21) for all l = 1, ..., L and subtracting this from (5.20), we then obtain the desired inequality (2.6) upon using that

$$\deg f_{-q} = \sum_{l=1}^{L} \operatorname{mult}_{\lambda - \lambda_l} f_{-q}, \qquad \deg f_p = \sum_{l=1}^{L} \operatorname{mult}_{\lambda - \lambda_l} f_p.$$
 (5.22)

Next we check the statement about the strictness of the inequality (2.6). From the arguments in the above paragraph we see that equality in (2.6) can be achieved only if equality holds in both (5.20) and (5.21). For (5.20), this means graphically that the grid point  $(k, m_k)$  lies on the line segment connecting  $(-q, \deg f_{-q})$  and  $(p, \deg f_p)$ . From the definition of the numbers  $m_k$  it then follows that each of the grid points  $(k, \deg f_k)$ ,  $k = -q, \ldots, p$ , must lie below this line segment, in the sense that

$$\deg f_k \le \frac{p-k}{p+q} \deg f_{-q} + \frac{k+q}{p+q} \deg f_p, \tag{5.23}$$

for all  $k = -q, \ldots, p$ . Similarly, equality in (5.21) implies that

$$\operatorname{mult}_{\lambda-\lambda_l} f_k \ge \frac{p-k}{p+q} \operatorname{mult}_{\lambda-\lambda_l} f_{-q} + \frac{k+q}{p+q} \operatorname{mult}_{\lambda-\lambda_l} f_p,$$
 (5.24)

for all  $k = -q, \ldots, p$  and  $l = 1, \ldots, L$ .

Now if (1.7) holds then we have  $\deg f_0 = r$  while the right hand side of (5.23) is at most r-1. So we cannot have  $\tilde{m}_k = 0$  in that case.

Proof of Lemma 2.9. First we show that (i) implies (iii). So suppose that  $\tilde{m}_k = 0$  for some  $k \in \{-q+1,\ldots,p-1\}$ . As observed before, we then have the inequalities (5.23)–(5.24) for all  $k = -q,\ldots,p$ . Summing (5.24) for all l and subtracting this from (5.23), we get

$$\deg f_k - \sum_{l=1}^L \operatorname{mult}_{\lambda - \lambda_l} f_k \le 0, \tag{5.25}$$

for any k = -q, ..., p, where the right hand side was simplified with the help of (5.22). On the other hand, we trivially have that

$$\deg f_k - \sum_{l=1}^L \operatorname{mult}_{\lambda - \lambda_l} f_k \ge 0.$$

So equality holds in (5.25). Tracing back the argument, we must then have equality in both (5.23) and (5.24). Graphically this means that each of the grid points  $(k, \deg f_k)$ ,  $k = -q, \ldots, p$  must lie on the line segment connecting  $(-q, \deg f_{-q})$  and  $(p, \deg f_p)$ , and similarly each of the grid points  $(k, \operatorname{mult}_{\lambda - \lambda_l} f_k)$ ,  $k = -q, \ldots, p$  must lie on the line segment connecting  $(-q, \operatorname{mult}_{\lambda - \lambda_l} f_{-q})$  and  $(p, \operatorname{mult}_{\lambda - \lambda_l} f_p)$ , for any  $l = 1, \ldots, L$ . It is easily seen that these assertions are equivalent to the statement in part (iii), with the rational function  $h(\lambda)$  given by

$$h(\lambda) := \prod_{l=1}^{L} (\lambda - \lambda_l)^{(\text{mult}_{\lambda - \lambda_l}(f_p/f_{-q}))/(p+q)}.$$

So we showed that (i) implies (iii). The proof that (iii) implies (ii) can be obtained (in a simpler way) by reversing the above arguments.

#### 5.4 Proof of Lemma 2.14

The proof of (a) follows again by mimicking the argument of Schmidt and Spitzer [17]. For Part (b), let  $U \subset \mathbb{C} \setminus (\Gamma_k \cup \Lambda)$  be a simply connected domain. For fixed  $\lambda \in U$  let  $j_1, j_2 \in \mathbb{N} \cup \{0\}$  be such that

$$|z_{q+k-j_1-1}(\lambda)| < |z_{q+k-j_1}(\lambda)| = \dots = |z_{q+k+j_2}(\lambda)| < |z_{q+k+j_2+1}(\lambda)|,$$

where we set  $z_0 \equiv 0$  and  $z_{p+q+1} \equiv \infty$  if necessary. Since  $U \subset \mathbb{C} \setminus \Gamma_k$  we have that either  $j_2 \equiv 0$  on U, or else  $j_1$  and  $j_2$  both take a constant value on U. In the case where  $j_2 \equiv 0$  the analyticity of  $\prod_{j=1}^{q+k} z_j(\lambda)$  on U follows immediately from [9, Proof of Prop. 3.5]. So we can focus now on the case where  $j_1$  and  $j_2$  are constant on U. Since  $U \subset \mathbb{C} \setminus \Lambda$ , none of the roots  $z_j$ ,  $j=q+k-j_1,\ldots,q+k+j_2$  can take the value 0 or  $\infty$ . Hence by the fact that U is simply connected, there exists a labeling so that each of the functions  $z_{q+k-j_1}(\lambda),\ldots,z_{q+k+j_2}(\lambda)$  is analytic in U, with the pairwise ratios being constants of modulus 1. On the other hand, the argument in [9, Proof of Prop. 3.5] shows that  $\prod_{j=1}^{q+k-j_1-1} z_j(\lambda)$  is analytic in U. Combining all these observations, we obtain the required analyticity of  $\prod_{j=1}^{q+k} z_j(\lambda)$  in U.

Finally, the statement about the logarithmic derivative follows since if  $z(\lambda)$  and  $\tilde{z}(\lambda)$  are analytic functions of  $\lambda \in U$  that are identically equal up to a constant factor, then their logarithmic derivatives are the same:  $z'(\lambda)/z(\lambda) = \tilde{z}'(\lambda)/\tilde{z}(\lambda)$ , for all  $\lambda \in U$ .

#### 5.5 Proofs of Proposition 2.10 and Theorems 2.13 and 2.15

In this section we prove Prop. 2.10 and Theorems 2.13 and 2.15. The proof of Theorem 2.13 will closely follow [9] and especially [6].

Define the function  $w_k$  by

$$w_k(\lambda) = \prod_{j=1}^{q+k} z_j(\lambda), \quad \lambda \in \mathbb{C} \setminus \Gamma_k,$$
 (5.26)

for k = -q + 1, ..., p - 1. Occasionally we will also consider  $w_k$  for the indices k = -q or k = p. We rewrite (1.18) as

$$d\mu_k(\lambda) = \frac{1}{2\pi i} \left( \frac{w'_{k+}(\lambda)}{w_{k+}(\lambda)} - \frac{w'_{k-}(\lambda)}{w_{k-}(\lambda)} \right) d\lambda.$$
 (5.27)

From Lemma 2.14 we know that  $w'_k/w_k$  exists as a meromorphic function on  $\mathbb{C} \setminus \Gamma_k$  with poles at the points of  $\Lambda$ . The following proposition gives more detailed information.

**Proposition 5.8.** (The function  $w'_k/w_k$ ). Let  $k \in \{-q+1, \ldots, p-1\}$  and recall Definitions 2.2 and 2.5. Then the following statements hold true:

(a) For any  $\lambda_0 \in \mathbb{C} \setminus \Lambda$ , there exists  $\kappa \in \mathbb{N} = \{1, 2, 3, \ldots\}$  such that

$$\frac{w_k'(\lambda)}{w_k(\lambda)} = O((\lambda - \lambda_0)^{-1 + 1/\kappa}),$$

as  $\lambda \to \lambda_0$  with  $\lambda \in \mathbb{C} \setminus \Gamma_k$ . We have  $\kappa = 1$  for all but finitely many  $\lambda_0$ .

(b) Near  $\infty$  there exists  $\kappa \in \mathbb{N}$  such that

$$\frac{w_k'(\lambda)}{w_k(\lambda)} = \frac{\deg(f_{-q}) - m_k}{\lambda} + O(\lambda^{-1-1/\kappa}),$$

as  $\lambda \to \infty$  with  $\lambda \in \mathbb{C} \setminus \Gamma_k$ .

(c) Near the point  $\lambda_l$ ,  $l \in \{1, ..., L\}$ , there exists  $\kappa \in \mathbb{N}$  such that

$$\frac{w_k'(\lambda)}{w_k(\lambda)} = \frac{\text{mult}_{\lambda - \lambda_l}(f_{-q}) - m_k^{(l)}}{\lambda - \lambda_l} + O((\lambda - \lambda_l)^{-1 + 1/\kappa}),$$

as  $\lambda \to \lambda_l$  with  $\lambda \in \mathbb{C} \setminus \Gamma_k$ .

*Proof.* Part (a) can be shown as in [6], for example. Now we turn to proving Part (b). Recalling the notation  $s_j$  in (5.1), we obtain

$$\frac{w_k'(\lambda)}{w_k(\lambda)} = \sum_{j=1}^{q+k} \frac{z_j'(\lambda)}{z_j(\lambda)} = \frac{\sum_{j=1}^{q+k} s_j}{\lambda} + O\left(\lambda^{-1-1/\kappa}\right),$$

as  $\lambda \to \infty$  with  $\lambda \in \mathbb{C} \setminus \Gamma_k$ . On account of Lemma 5.1 we then obtain part (b). The proof of part (c) follows in a similar way from (5.11)–(5.12).

**Proposition 5.9.** For each  $k = -q+1, \ldots, p-1$  we have that  $\mu_k$  in (5.27) is a positive measure on  $\widetilde{\Gamma}_k$  with total mass  $\mu_k(\widetilde{\Gamma}_k) = \widetilde{m}_k$ .

*Proof.* (Compare with [6, Prop 3.4].) Prop. 5.8 implies that the density (5.27) is locally integrable around all the points in  $(\Lambda \cup \{\infty\}) \cap \widetilde{\Gamma}_k$ , and the arguments in [9] show that  $\mu_k$  is a positive measure. The statement that  $\mu_k(\widetilde{\Gamma}_k) = \widetilde{m}_k$  follows from the contour deformation

$$\mu_{k}(\widetilde{\Gamma}_{k}) := \frac{1}{2\pi i} \int_{\widetilde{\Gamma}_{k}} \left( \frac{w'_{k+}(\lambda)}{w_{k+}(\lambda)} - \frac{w'_{k-}(\lambda)}{w_{k-}(\lambda)} \right) d\lambda$$

$$= \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{w'_{k}(\lambda)}{w_{k}(\lambda)} d\lambda + \sum_{l=1}^{L} \operatorname{Res} \left( \frac{w'_{k}(\lambda)}{w_{k}(\lambda)}, \lambda = \lambda_{l} \right), \tag{5.28}$$

where  $\mathcal{C}$  is a clockwise oriented contour surrounding  $\widetilde{\Gamma}_k \cup \Lambda$ , and where  $\operatorname{Res}(h,\lambda)$  denotes the residue of h at  $\lambda$ . Equation (5.28) is valid even if one or more points  $\lambda_l \in \Lambda$  lie on the curve  $\widetilde{\Gamma}_k$ , thanks to the local integrability of  $\mu_k$  around these points. Applying the residue theorem once again, now for the exterior domain of  $\mathcal{C}$ , we find for the first term in (5.28) that

$$\frac{1}{2\pi i} \int_{\mathcal{C}} \frac{w_k'(\lambda)}{w_k(\lambda)} d\lambda = -\text{Res}\left(\frac{w_k'(\lambda)}{w_k(\lambda)}, \lambda = \infty\right). \tag{5.29}$$

From (5.28)–(5.29) and the residue expressions in Prop. 5.8 we then obtain

$$\mu_k(\widetilde{\Gamma}_k) = \left(m_k - \sum_{l=1}^L m_k^{(l)}\right) - \left(\deg f_{-q} - \sum_{l=1}^L \operatorname{mult}_{\lambda - \lambda_l} f_{-q}\right) = \widetilde{m}_k,$$

by virtue of (2.5) and (5.22).

**Proposition 5.10.** For each k we have that

$$\int \frac{d\mu_k(x)}{\lambda - x} = -\frac{w_k'(\lambda)}{w_k(\lambda)} + \sum_{l=1}^L \frac{\text{mult}_{\lambda - \lambda_l}(f_{-q}) - m_k^{(l)}}{\lambda - \lambda_l}, \quad \text{if } \lambda \in \mathbb{C} \setminus \widetilde{\Gamma}_k$$
 (5.30)

and

$$\int \log|\lambda - x| \ d\mu_k(x) = -\log|w_k(\lambda)| + \sum_{l=1}^{L} (\text{mult}_{\lambda - \lambda_l}(f_{-q}) - m_k^{(l)}) \log|\lambda - \lambda_l| + \alpha_k, \quad (5.31)$$

if  $\lambda \in \mathbb{C}$ , for a suitable constant  $\alpha_k$ .

Remark 5.11. As in [6], each  $\lambda_l \in \Lambda \setminus \widetilde{\Gamma}_k$  (or  $\lambda_l \in \Lambda$ ) is a removable singularity for the right hand side of (5.30) (or (5.31) respectively) due to the continuity of the corresponding left hand side.

Proof of Proposition 5.10. (Compare with [6, Prop 3.5].) We use the contour deformation

$$\int_{\widetilde{\Gamma}_{k}} \frac{d\mu_{k}(x)}{\lambda - x} := \frac{1}{2\pi i} \int_{\widetilde{\Gamma}_{k}} \frac{1}{\lambda - x} \left( \frac{w'_{k+}(x)}{w_{k+}(x)} - \frac{w'_{k-}(x)}{w_{k-}(x)} \right) dx$$

$$= \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{1}{\lambda - x} \frac{w'_{k}(x)}{w_{k}(x)} dx - \frac{w'_{k}(\lambda)}{w_{k}(\lambda)} + \sum_{l=1}^{L} \frac{1}{\lambda - \lambda_{l}} \operatorname{Res} \left( \frac{w'_{k}(x)}{w_{k}(x)}, x = \lambda_{l} \right),$$

where C is a clockwise oriented contour surrounding  $\widetilde{\Gamma}_k \cup \Lambda \cup \{\lambda\}$ . The first term in the right hand side vanishes since the residue of the integrand at infinity is zero. From the residue expressions in Prop. 5.8 we then get (5.30). Eq. (5.31) follows from this by integration, see also [9].

Finally, we can finish the proofs of Theorems 2.13 and 2.15:

Proof of Theorem 2.13(a)–(b). With Propositions 5.9 and 5.10 in place, Theorem 2.13(a)–(b) now follows in the same way as in [6].  $\Box$ 

*Proof of Theorem 2.13(c)*. First we show that the energy functional  $J(\vec{\nu})$  in (2.9) is bounded from below. To this end we rewrite  $J(\vec{\nu})$  as

$$J(\vec{\nu}) = \sum_{k=-q+1}^{p-2} \frac{\tilde{m}_k \tilde{m}_{k+1}}{2} I\left(\frac{\nu_k}{\tilde{m}_k} - \frac{\nu_{k+1}}{\tilde{m}_{k+1}}\right) + \sum_{k=-q+1}^{p-1} \frac{\tilde{m}_k}{2} (-m_{k-1} + 2m_k - m_{k+1}) I\left(\frac{\nu_k}{\tilde{m}_k}\right) + \sum_{l=1}^{L} \sum_{k=-q+1}^{p-1} \frac{\tilde{m}_k}{2} (m_{k-1}^{(l)} - 2m_k^{(l)} + m_{k+1}^{(l)}) \left(I\left(\frac{\nu_k}{\tilde{m}_k}\right) - 2\int \log \frac{1}{|\lambda - \lambda_l|} \frac{d\nu_k(\lambda)}{\tilde{m}_k}\right).$$
 (5.32)

This formula is easily shown with the help of (2.9) and (2.5).

The terms in the first sum in (5.32) are all nonnegative [18]. For the second sum in (5.32), we observe that  $-m_{k-1} + 2m_k - m_{k+1} = s_{q+k+1} - s_{q+k} > 0$  by virtue of (5.3). So these coefficients are all nonnegative, and they are non-zero precisely when  $s_{q+k+1} > s_{q+k}$ . But for such k the curve  $\widetilde{\Gamma}_k$  is compact and so  $I(\nu_k)$  is bounded from below. Finally, for the double sum in (5.32) we have that  $m_{k-1}^{(l)} - 2m_k^{(l)} + m_{k+1}^{(l)} = s_{q+k}^{(l)} - s_{q+k+1}^{(l)} \geq 0$ , recall (5.11)–(5.12). So these coefficients are all nonnegative, and they are non-zero precisely when  $s_{q+k}^{(l)} > s_{q+k+1}^{(l)}$ . But for such k we have that  $\lambda_l \not\in \widetilde{\Gamma}_k$ , and then standard arguments from potential theory show that the expression between brackets in the double sum in (5.32) is minimized precisely when  $\nu_k$  is (a constant times) the balayage of the Dirac point mass at  $\lambda_l$  onto the curve  $\widetilde{\Gamma}_k$ ; in particular this expression is bounded from below as well [16, Chapter 2].

Summarizing, we have now established that the energy functional  $J(\vec{\nu})$  is bounded from below. Then the proof of Theorem 2.13(c) follows from part (b) in the same way as in [6].

Proof of Proposition 2.10. Let us first prove (2.8) if  $\Lambda \cap C = \emptyset$ . Applying contour deformation, we then find that

$$\mu_k(C) := \frac{1}{2\pi i} \int_C \left( \frac{w'_{k+}(\lambda)}{w_{k+}(\lambda)} - \frac{w'_{k-}(\lambda)}{w_{k-}(\lambda)} \right) d\lambda = \frac{1}{2\pi i} \int_{\gamma} \frac{w'_{k}(\lambda)}{w_{k}(\lambda)} d\lambda,$$

where  $\gamma$  is the disjoint union of one or more closed Jordan curves in  $\mathbb{C} \setminus \Gamma_k$ . More precisely,  $\gamma$  consists of a clockwise oriented loop surrounding the outer boundary of C, and a counterclockwise oriented loop inside each of the 'holes' of C. Now since  $\frac{w_k'(\lambda)}{w_k(\lambda)} = (\log w_k(\lambda))'$ , the integral of this quantity over any closed Jordan curve in  $\mathbb{C} \setminus \Gamma_k$  is obviously an integral multiple of  $2\pi i$ . So we obtain (2.8) if  $\Lambda \cap C = \emptyset$ . The same argument also works if  $\Lambda \cap C \neq \emptyset$  provided that we take into account the residue from the pole of  $w_k'/w_k$  at each  $\lambda_l \in \Lambda \cap C$ , Prop. 5.8(c), thereby noting that  $\mathrm{mult}_{\lambda - \lambda_l} f_{-q} \in \mathbb{N} \cup \{0\} \subset \mathbb{Z}$ .

Finally, denote by K the number of compact, connected components of  $\Gamma_k$ . By summing (2.8) over all such components C we get the following upper bound on K:

$$K \le \sum_{C} \left( \mu_k(C) + \sum_{\lambda_l \in \Lambda \cap C} m_k^{(l)} \right) \le \tilde{m}_k + \sum_{l=1}^{L} m_k^{(l)} \le m_k,$$

by virtue of (2.5).

*Proof of Theorem 2.15.* Thanks to Lemma 2.14, the above proof of Theorem 2.13 yields Theorem 2.15 as well.  $\Box$ 

#### 5.6 Proof of Theorem 3.1 and Proposition 3.2

*Proof of Prop. 3.2.* Fix  $\lambda_l \in \Lambda$  and a subset  $S \subset \{1, 2, \dots, p+q\}$  of cardinality |S| = q. From (5.14) we have that

$$w_S(\lambda) = O\left((\lambda - \lambda_l)^{\text{mult}_{\lambda - \lambda_l}(f_{-q}) - \sum_{j=1}^q s_k^{(l)}}\right) = O\left((\lambda - \lambda_l)^{m_0^{(l)}}\right), \tag{5.33}$$

as  $\lambda \to \lambda_l$  with  $\lambda \in \mathbb{C} \setminus \bigcup_k \Gamma_k$ , where the last step follows from (5.12). Prop. 3.2(a) then follows from (5.33) and (5.13), provided that there is a disk U around  $\lambda_l$  such that for all but finitely many  $\lambda \in U$  the roots to  $f(z,\lambda) = 0$  are pairwise different. But this condition is generic and the case where it fails follows by an easy continuity argument.

The proof of Prop. 3.2(b) is similar.

Theorem 3.1 can be obtained from Widom's determinant identity, Prop. 5.4, in the same way as in [6, 9]. We outline the main steps.

#### Proposition 5.12. We have that

$$\lim_{n \to \infty} \int_{\mathbb{C}} \frac{d\mu_{0,n}(x)}{\lambda - x} = \int_{\mathbb{C}} \frac{d\mu_0(x)}{\lambda - x} + \sum_{l=1}^{L} \frac{m_0^{(l)}}{\lambda - \lambda_l},\tag{5.34}$$

uniformly on compact subsets of  $\mathbb{C} \setminus (\Gamma_0 \cup G_0)$ .

Remark 5.13. As in [6], each  $\lambda_l \in \Lambda \setminus (\Gamma_0 \cup G_0)$  is a removable singularity for the right hand side of (5.34), due to the continuity of the left hand side.

Proof of Proposition 5.12. As mentioned before, the dominant term in Prop. 5.4 for n large is obtained by taking  $S = S_0 := \{1, 2, ..., q\}$ . Then we find in the same way as in [9, Proof of Corollary 5.3] and [6, Proof of Prop. 4.2] that

$$\lim_{n \to \infty} \int_{\mathbb{C}} \frac{d\mu_{0,n}(x)}{\lambda - x} = \lim_{n \to \infty} \frac{1}{n} \sum_{\lambda_i \in \operatorname{sp} T_n(A)} \frac{1}{\lambda - \lambda_i} = \lim_{n \to \infty} \frac{1}{n} \frac{\left(\det T_n(A(z) - \lambda I_r)\right)'}{\det T_n(A(z) - \lambda I_r)}$$

$$= \frac{w'_{S_0}(\lambda)}{w_{S_0}(\lambda)} = -\frac{w'_0(\lambda)}{w_0(\lambda)} + \sum_{l=1}^L \frac{\operatorname{mult}_{\lambda - \lambda_l} f_{-q}}{\lambda - \lambda_l}, \quad (5.35)$$

uniformly on compact subsets of  $\mathbb{C} \setminus (\Gamma_0 \cup G_0)$ , where the last equality in (5.35) follows from (5.26) and (5.14). Finally, Prop. 5.10 shows that the right hand side of (5.35) equals the right hand side of (5.34).

*Proof of Theorem 3.1.* From the convergence of the Cauchy transforms in Prop. 5.12 we obtain

$$\mu_{0,n} \rightarrow \mu_0 + \sum_{l=1}^L m_0^{(l)} \delta_{\lambda_l}$$

in the weak-star sense, i.e., (3.3) holds for every continuous function  $\phi$  that vanishes at infinity. Gerschgorin's circle theorem implies that there is a compact set K such that all the measures  $\{\mu_{0,n}\}_n$  are supported in K. Therefore the assumption that  $\phi$  vanishes at infinity is redundant and we obtain (3.3) for all bounded continuous functions.

Finally, we establish the claim that  $G_0$  attracts isolated eigenvalues. Let  $\lambda_0 \in G_0$  and take a sufficiently small disk U around  $\lambda_0$ . Then from Prop. 5.4 we find that

$$w_{S_0}(\lambda)^{-n-\alpha} \det T_n(A - \lambda I_r) = C_{S_0}(\lambda) + O(c^n), \qquad \lambda \in U, \tag{5.36}$$

for some absolute constant c with |c| < 1. We claim that  $w_{S_0}(\lambda)$  tends to a non-zero constant if  $\lambda \to \lambda_0$ . This is obvious if  $\lambda_0 \notin \Lambda$ ; if  $\lambda_0 = \lambda_l \in \Lambda$  then it follows by mimicking (5.33) and noting that  $m_0^{(l)} = 0$  due to our assumption that  $\lambda_0 = \lambda_l \in G_0 \subset \mathbb{C} \setminus \Gamma_0$ . From (5.36), Hurwitz' theorem then implies that for all n sufficiently large, there are precisely j eigenvalues (counting multiplicities) of  $T_n(A)$  inside U, with j being the multiplicity of  $\lambda_0$  as a zero of  $C_{S_0} = C_0$ .

Finally let us note that, strictly speaking, the above applications of Prop. 5.4 again require that there is a disk U around  $\lambda_0$  such that for all but finitely many  $\lambda \in U$  the roots to  $f(z,\lambda) = 0$  are pairwise different. But this constraint can again be circumvented by an easy continuity argument.

# 6 Examples

#### 6.1 Example 1: a non-degenerate case

We now illustrate our main results for a small-size example where each of the hypotheses H1, H2k and H3 holds true. Consider the symbol

$$A(z) = \begin{pmatrix} b_1 & a_1 + c_1 z \\ c_2 + a_2/z & b_2 \end{pmatrix}, \tag{6.1}$$

where we assume for convenience that each of the numbers  $a_j, c_j, j \in \{1, 2\}$ , is non-zero. Then the block Toeplitz matrix  $T_n(A)$  has the tridiagonal form

$$T_n(A) = \begin{pmatrix} b_1 & a_1 & & & 0 \\ c_2 & b_2 & a_2 & & & \\ \hline & c_1 & b_1 & a_1 & & \\ & & c_2 & b_2 & \ddots & \\ \hline & & & \ddots & \ddots & \end{pmatrix}_{2n \times 2n}.$$

A little calculation shows that

$$f(z,\lambda) = -c_1c_2z + ((b_1 - \lambda)(b_2 - \lambda) - a_1c_2 - a_2c_1) - \frac{a_1a_2}{z}$$
  
=:  $-\frac{c_1c_2}{z}(z - z_1(\lambda))(z - z_2(\lambda)),$ 

where as usual the roots are ordered such that  $|z_1(\lambda)| \leq |z_2(\lambda)|$ . We now have p = q = 1 and hence there is only one relevant set

$$\Gamma_0 = \{ \lambda \in \mathbb{C} \mid |z_1(\lambda)| = |z_2(\lambda)| \}.$$

The coefficients  $C_S(\lambda)$  in Prop. 5.4 are labeled by index sets  $S \subset \{1,2\}$  with |S| = 1; hence  $S = \{1\}$  or  $S = \{2\}$ . It can be shown that

$$C_S(\lambda) = -\frac{1 + \frac{c_1}{a_1} z_i}{c_1 c_2 (z_i - z_i)},$$

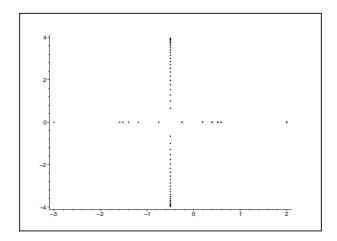


Figure 5: Eigenvalues for the symbol (6.1), with the values (6.3) and n=30, computed in Maple with 60 digit precision. There are 2n=60 eigenvalues in total of which 48 live on the vertical line segment  $[x_4, x_3] \approx [-1/2 - 3.97i, -1/2 + 3.97i]$ , 10 live on the horizontal line segment  $[x_1, x_2] \approx [-1.61, 0.61]$ , and the final 2 are outliers lying extremely close to  $\lambda = -3$  and  $\lambda = 2$ , respectively, cf. (6.4).

where we put  $S = \{i\}$ ,  $i \in \{1, 2\}$ , and where  $j \in \{1, 2\}$  is the index different from i. In particular, we have  $C_S(\lambda) = 0$  if and only if  $z_i(\lambda) = -a_1/c_1$ . From (6.1), this implies in turn that

$$\det \begin{pmatrix} b_1 - \lambda & 0 \\ * & b_2 - \lambda \end{pmatrix} = 0.$$

Hence, we can only have  $C_S(\lambda) = 0$  if  $\lambda = b_1$  or  $\lambda = b_2$ . For these two special  $\lambda$ -values, the second solution to  $f(z,\lambda) = 0$  is  $z_j(\lambda) = -a_2/c_2$ ; therefore we obtain that

$$G_0 = \begin{cases} \{b_1, b_2\}, & \text{if } |a_1/c_1| \le |a_2/c_2|, \\ \emptyset, & \text{otherwise.} \end{cases}$$

$$(6.2)$$

We now turn the above discussion into a numerical example by setting

$$a_1 = 1/2, \ a_2 = -3, \ b_1 = 2, \ b_2 = -3, \ c_1 = 4, \ c_2 = -3.$$
 (6.3)

In this case, the discriminant of  $f(z, \lambda) = 0$  equals

$$\lambda^4 + 2\lambda^3 + 16\lambda^2 + 15\lambda - 63/4$$
.

whose four roots are  $x_1 \approx -1.61$ ,  $x_2 \approx 0.61$ ,  $x_3 \approx -1/2 + 3.97i$  and  $x_4 \approx -1/2 - 3.97i$ . These are the branch points of  $f(z, \lambda) = 0$ . It turns out that the set  $\Gamma_0 \subset \mathbb{C}$  consists of two line segments, one vertically connecting the branch points  $x_3$  and  $x_4$  and the other one horizontally connecting the branch points  $x_1$  and  $x_2$ . The two line segments intersect at  $\lambda = -1/2$ .

For the values (6.3), the first case in (6.2) applies and so we have

$$G_0 = \{b_1, b_2\} = \{-3, 2\}. \tag{6.4}$$

Thus for n large,  $T_n(A)$  has an isolated eigenvalue near  $\lambda = -3$  and near  $\lambda = 2$ , both of multiplicity one. Finally, Theorem 2.13 implies that the limiting eigenvalue distribution  $\mu_0$  of  $T_n(A)$  is precisely the equilibrium measure of the set  $\Gamma_0$ .

These considerations are confirmed in Figure 5.

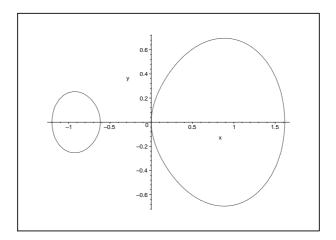


Figure 6: Support  $\Gamma_0$  of the limiting eigenvalue distribution for the symbol (6.5). It consists of two closed Jordan curves in the complex  $\lambda$ -plane and an isolated point at  $\lambda = -1$ .

#### 6.2 Example 2: a degenerate case, I

Next we study an example where both H1 and H2 fail. Consider the symbol [21, page 321]

$$A(z) = \begin{pmatrix} 0 & z^{-1} - z \\ 1 + z & z^{-1} + z^2 \end{pmatrix}.$$
 (6.5)

Then one has that

$$f(z,\lambda) = \det(A(z) - \lambda I_2) = (1 - \lambda)z^2 + z + (\lambda^2 - 1) + (-1 - \lambda)z^{-1}.$$
 (6.6)

Hence hypothesis H1 is violated.

Observe that the following factorization holds,

$$f(z,\lambda) = \frac{1}{z}((1-\lambda)z + 1)(z^2 - \lambda - 1). \tag{6.7}$$

Hence the three roots are given by  $\{z_1(\lambda), z_2(\lambda), z_3(\lambda)\} = \{\frac{1}{\lambda-1}, (\lambda+1)^{1/2}, -(\lambda+1)^{1/2}\}$ , where the labeling should be taken according to increasing absolute value. Since two of the three roots have the same absolute value in the entire complex  $\lambda$ -plane, hypothesis H2k is violated as well.

Let us first check the point sources. The set  $\Lambda = \{\lambda_l\}_{l=1}^L$  in (1.11) is such that L=2 and  $\lambda_1 = -1$ ,  $\lambda_2 = 1$ , and the relevant data are given by

	k = -1	k = 0	k = 1	k=2	_
$m_k$	1	2	3/2	1	
$m_k^{(1)} \\ m_k^{(2)} \\ \tilde{m}_k$	1	1/2	0	0	
$m_{k}^{(2)}$	0	0	0	1	
$\tilde{m}_k$	0	3/2	3/2	0	

From this we see that the measures  $\mu_0$  and  $\mu_1$  both have total mass  $\tilde{m}_0 = \tilde{m}_1 = 3/2$ . Taking into account Section 2.3, cf. (2.12), the curves  $\Gamma_0$  and  $\Gamma_1$  are defined as

$$\Gamma_1 = \{ \lambda \in \mathbb{C} \mid |\lambda + 1|^{1/2} |\lambda - 1| = 1 \}, \qquad \Gamma_0 = \Gamma_1 \cup \{-1\}.$$

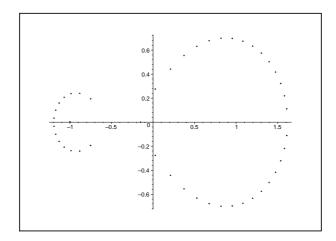


Figure 7: Eigenvalues for n = 30 for the symbol (6.5), computed in Maple with 60 digit precision. There are 2n = 60 eigenvalues in total of which 15 coalesce at  $\lambda = -1$ . Note that the eigenvalues closely approximate the curve in Figure 6.

The curve  $\widetilde{\Gamma}_0 = \widetilde{\Gamma}_1$  is plotted in Figure 6. On this curve, the measures  $\mu_0$  and  $\mu_1$  are defined according to the density (1.18).

The failure of H2 implies, as mentioned before, that the definitions of H3 and  $G_0$  need to be modified. Let us do this with an ad-hoc calculation. The coefficients  $C_S(\lambda)$  in Prop. 5.4 are labeled by index sets  $S \subset \{1, 2, 3\}$  with |S| = 1; it can be shown that

$$C_S(\lambda) = \begin{cases} -\frac{\lambda^2}{(\lambda+1)(1-(\lambda+1)(\lambda-1)^2)}, & \text{if } S \text{ labels the root } z(\lambda) = \frac{1}{\lambda-1}, \\ \frac{1\pm(\lambda+1)^{1/2}}{2(\lambda+1)(1\pm(1-\lambda)(\lambda+1)^{1/2})}, & \text{if } S \text{ labels the root } z(\lambda) = \pm(\lambda+1)^{1/2}. \end{cases}$$

After some simplifications, (5.13) then reduces to

$$\det T_n(A(z) - \lambda I_2) = \frac{\lambda (\lambda^2 - 1)^{n+1}}{(\lambda + 1)(\lambda^2 - \lambda - 1)} - \begin{cases} \frac{(\lambda + 1)^{n/2}}{\lambda^2 - \lambda - 1}, & \text{if } n \text{ is even,} \\ \frac{\lambda (\lambda + 1)^{(n-1)/2}}{\lambda^2 - \lambda - 1}, & \text{if } n \text{ is odd.} \end{cases}$$

From this, it is easy to see that Theorem 3.1 can indeed be applied. Thus the limiting eigenvalue distribution of  $T_n(A)$  consists of the absolutely continuous part  $\mu_0$  on  $\widetilde{\Gamma}_0$ , and a point mass of mass 1/2 at  $\lambda_1 = -1$ . Moreover,  $T_n(A)$  does not have isolated eigenvalues for n large, neither for n even nor for n odd. This reproduces the result in [21, page 321]. The comparison with the eigenvalues of  $T_n(A)$  for n = 30 is shown in Figure 7.

Finally, the energy functional (2.9) reduces to

$$I(\nu_0) + I(\nu_1) - I(\nu_0, \nu_1) - \frac{1}{2} \int \log \frac{1}{|\lambda + 1|} d\nu_1(\lambda) - \int \log \frac{1}{|\lambda - 1|} d\nu_1(\lambda).$$
 (6.8)

Theorem 2.15 implies that  $(\mu_0, \mu_1)$  is the minimizer of this functional over all pairs of measures  $(\nu_0, \nu_1)$  supported on  $\widetilde{\Gamma}_0 = \widetilde{\Gamma}_1$ , with total masses  $\tilde{m}_0 = \tilde{m}_1 = 3/2$ . The last two terms in (6.8) can be interpreted as an attraction of  $\mu_1$  towards the points  $\lambda = -1$  and  $\lambda = 1$ .

#### 6.3 Example 3: a degenerate case, II

We discuss a variant of the previous example. Consider the symbol

$$A(z) = \begin{pmatrix} z^2 - 1 & z^{-1} - z \\ 0 & z^{-1} + 1 \end{pmatrix}.$$
 (6.9)

The algebraic equation  $f(z, \lambda) = 0$  is again given by (6.7). The triangularity of A(z) implies the following factorization for the finite n determinants:

$$\det T_n(A(z) - \lambda I_2) = \det T_n(z^2 - 1 - \lambda) \det T_n(z^{-1} + 1 - \lambda) = (-1 - \lambda)^n (1 - \lambda)^n.$$

So the limiting eigenvalue distribution of  $T_n(A)$  has a pure point spectrum with point masses at  $\lambda = -1$  and  $\lambda = 1$ . In particular, it is *not* related to the measure  $\mu_0$  on the set  $\widetilde{\Gamma}_0$  in Fig. 6. Thus Theorem 3.1 breaks down in this case. The reason is that several coefficients in Widom's formula are identically zero, and so H3 (actually a modification thereof since H2 fails) is not valid.

It is straightforward to generalize the above idea: Whenever the symbol  $A(z) - \lambda I_r$  is block upper triangular, or can be reduced into block upper triangular form by means of suitable row and column transformations, then det  $T_n(A(z) - \lambda I_r)$  factorizes into two smaller-size block Toeplitz determinants. For such symbols A(z), the hypotheses H1 and H2k typically hold true while H3 and Theorem 3.1 both fail.

Finally, one may argue that the above counterexamples to Theorem 3.1 are harmless, in the sense that in each case the eigenvalue problem for  $T_n(A)$  can be reduced into two smaller-size eigenvalue problems. One may wish to construct more interesting examples for which Theorem 3.1 fails. One way to construct such examples is from the symbol

$$A(z) = \begin{pmatrix} a(z) & 0 & a_{1,3}(z) \\ 0 & a(z) & a_{2,3}(z) \\ a_{3,1}(z) & a_{3,2}(z) & a_{3,3}(z) \end{pmatrix}, \tag{6.10}$$

where a(z),  $a_{i,3}(z)$  and  $a_{3,i}(z)$ , i=1,2,3, are given Laurent polynomials in z. By suitably fine-tuning these Laurent polynomials, and especially the exponents of their highest and lowest degree terms in z, one may construct symbols A(z) for which H1 and H2k hold true, H3 and Theorem 3.1 both fail, and for which no reduction to block upper triangular form is possible. We leave the details to the interested reader.

# 7 Concluding remarks

- 1. Generalizations. The main Theorem 3.1 was stated under the following condition: either  $\mathbb{C} \setminus \Gamma_0$  is connected and  $\Gamma_0$  does not have any interior points; or  $T_n(A)$  is a Hessenberg matrix. It is an open problem to generalize this theorem to other classes of banded block Toeplitz matrices.
- 2. Applications. We expect that our main Theorems 2.13/2.15 and 3.1 may be used to obtain some results in the theory of multiple and matrix orthogonal polynomials on the real line. In fact, recently several papers appeared [8, 13, 23] which apply the results of Duits and Kuijlaars [9] on scalar banded Toeplitz matrices, to the context of multiple orthogonal polynomials. The recurrence relations of these polynomials lead to a banded Hessenberg matrix. Typically this matrix is not exactly Toeplitz but only asymptotically. More generally, the orthogonality weights may be varying with n, which leads to so-called locally Toeplitz matrices [8, 13, 23]. We anticipate that more applications of this type may arise in the future, possibly leading to block (rather than

scalar) Toeplitz matrices. A first application of this kind is given in [2]. Finally, we also anticipate that our results could be used in the context of matrix orthogonal polynomials on the real line, see e.g. [7, 10].

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