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THE SCATTERING PROBLEM FOR A DISCRETE STURM-LIOUVILLE OPERATOR

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ABSTRACT. In this paper properties of the discrete Sturm-Liouville operator are considered, and the scattering problem for this operator is studied using asymptotic formulas for orthogonal polynomials with matrix coefficients.

Bibliography: 22 titles.

Interest in the discrete Sturm-Liouville problem arises in connection with a number of problems in mathematical physics (scattering theory, the arrays of Toda and Langmuir, approximation of a Sturm-Liouville differential operator by finite-difference operators, the study of spectra, and other problems). A discrete Sturm-Liouville operator is undoubtedly simpler than a differential one and thus serves as an easier example for seeing and understanding various subtle properties of this kind of operator. Nevertheless, the process of studying Sturm-Liouville operators went in the opposite direction historically. Ideas which arose in the study of differential operators were carried over to discrete operators. Such is the case for scattering theory. Developed by Gel′fand and Levitan [1], Agranovich and Marchenko [2], and Faddeev [3], it was carried over to discrete operators by the same methods (see [4]–[6]), and the inverse scattering problem was reduced to a Gel′fand-Levitan-Marchenko integral equation.

Another scheme for solving the inverse scattering problem, based on asymptotic properties of orthogonal polynomials and continued fraction expansions of analytic functions, was proposed in [7]. In the present article this approach is extended to the investigation of discrete Sturm-Liouville operators in $l_2(\mathbf{C}^N)$.

§1. Discrete Sturm-Liouville operators. Orthogonal polynomials with matrix coefficients. Definitions. Main properties

Let $\mathbf{C}^N$ be N-dimensional (complex) Euclidean space, and $l_2(\mathbf{C}^N)$ the Hilbert space of sequences $y = \{y_k\}_0^\infty$ such that $y_k \in \mathbf{C}^N$ for all $k = 0, 1, \dots$ and

$$\|y\|_{l_2} = \left\{ \sum_{k=0}^{\infty} (y_k, y_k) \right\}^{1/2} < \infty.$$

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Let $L^2(\mathbf{C}^N)$ be the Hilbert space of $\mathbf{C}^N$-valued functions $v(t)$, $t \in \Delta \subseteq \mathbf{R}^1$, that are measurable (with respect to the norm $\|\cdot\|_{\mathbf{C}^N}$) and such that

$$\|v\|^2_{L^2(\mathbf{C}^N)} = \int_\Delta \|v(t)\|^2_{\mathbf{C}^N}\,dt < \infty.$$

The notation $\mathscr{H}^\delta(\mathbf{C}^N)$ stands for the class of all $\mathbf{C}^N$-valued analytic functions $u(\lambda)$ in the unit disk $K_1 \subset \mathbf{C}^1$ such that the integral

$$\int_0^{2\pi} \|u(re^{it})\|^\delta_{\mathbf{C}^N}\,dt \qquad (0 \leqslant r \leqslant 1)$$

is uniformly bounded with respect to r. It is known (see [8]) that the class $\mathscr{H}^2(\mathbf{C}^N)$ can be identified with a certain subspace of $L^2(\mathbf{C}^N)$.

Let $L^2_\sigma(\mathbf{C}^N)$ be the space of $\mathbf{C}^N$-valued functions that are measurable and bounded in the norm given by the inner product

$$F(t) = (f_1, f_2, \dots, f_N), \qquad G(t) = (g_1, g_2, \dots, g_N),$$
$$(F, G) = \int_{-\infty}^{\infty} (d\sigma F, G),$$

where the inner product under the integral sign is that in $\mathbf{C}^N$, and $\sigma(t)$ is a nondecreasing left-continuous Hermitian matrix-valued function ($\sigma(-\infty) = 0$) (see [9], §86).

1. Let

$$\mathscr{L} = \begin{pmatrix} V_0 & E_0 & 0 & 0 & \cdots \\ E_0^* & V_1 & E_1 & 0 & \cdots \\ 0 & E_1^* & V_2 & E_2 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

be the Jacobi matrix with matrix parameters $\{V_j, E_j\}_0^\infty$. Assume that

$$V_j^* = V_j, \qquad \det E_j \neq 0,$$

where $(\cdot)^*$ is the Hermitian conjugate. The matrices V_j and E_j are interpreted as operators on $\mathbf{C}^N$ with the corresponding norm $\|\cdot\|$.

The matrix $\mathscr{L}$ determines a selfadjoint operator in $l_2(\mathbf{C}^N)$, and it is bounded if

$$\sup_j \left(\|V_j\|, \|E_j\|\right) < \infty.$$

Namely, if $x = \{x_k\}_0^\infty \in l_2(\mathbf{C}^N)$, then $\mathscr{L}x = \{y_0, y_0, \dots\}$, where

$$\begin{aligned} y_0 &= V_0x_0 + E_0x_1, \\ y_1 &= E_0^*x_0 + V_1x_1 + E_1x_2, \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ y_k &= E_{k-1}^*x_{k-1} + V_kx_k + E_kx_{k+1}, \\ &\cdots\cdots\cdots\cdots\cdots\cdots \end{aligned} \tag{1.1}$$

The operator $\mathscr{L}$ is called a discrete Sturm-Liouville operator defined in $l_2(\mathbf{C}^N)$.

2. Let I be the identity operator on $l_2(\mathbf{C}^N)$ and $\mathscr{I}$ the identity operator on $\mathbf{C}^N$. We define the polynomials $Q_n(\lambda)$, $n = 0, 1, \dots$, by the following relations. Let

$$\begin{gathered} Q_0(\lambda) \equiv \mathscr{I}, \qquad E_0Q_1(\lambda) = (\mathscr{I}\lambda - V_0), \\ E_kQ_{k+1}(\lambda) = (\mathscr{I}\lambda - V_k)Q_k(\lambda) - E_{k-1}^*Q_{k-1}(\lambda). \end{gathered} \tag{1.2}$$

The system $\{Q_n(\lambda)\}_0^\infty$ of matrix polynomials plays a fundamental role in the investigation of discrete Sturm-Liouville operators. The polynomial $Q_n(\lambda)$ can be written in the form

$$Q_n(\lambda) = \beta_n\lambda^n + \beta_{n-1}\lambda^n + \cdots + \beta_0,$$

where the matrices $\beta_0, \beta_1, \dots, \beta_n$ depend also on n.

Let $u_1,\dots,u_N$ be the standard basis in $\mathbf{C}^N$:

$$u_j = (0,0,\dots,1,\dots,0), \qquad j = 1,2,\dots,N.$$

The symbol $\hat{u}_j$ denotes the vector in $l_2(\mathbf{C}^N)$ given by

$$\hat{u}_j = (u_j, 0, 0, \dots).$$

Let E_λ be the spectral resolution of the identity for the operator $\mathscr{L}$. Here $\lambda \in [-\infty, \infty]$, although only $\lambda \in \mathscr{S}(\mathscr{L})$ are essential for us, where $\mathscr{S}(\mathscr{L})$ denotes the spectrum of $\mathscr{L}$. Let

$$\sigma(\lambda) = \left((E_\lambda \hat{u}_i, \hat{u}_j)\right)_{i,j=1}^{N}.$$

The Hermitian matrix-valued function $\sigma(\lambda)$ is nondecreasing on $(-\infty,\infty)$: $d\sigma(\lambda) \geqslant 0$. We show that the closed linear span of the vectors

$$\left\{\mathscr{L}^{i_1}\hat{u}_1,\dots,\mathscr{L}^{i_N}\hat{u}_N\right\}_{i_1,\dots,i_N=0}^{\infty}$$

coincides with $l_2(\mathbf{C}^N)$. The subspaces e_k are defined by

$$e_k = \{0,\dots,0,\mathbf{C}^N,0,\dots\}, \qquad k = 0,1,\dots.$$

Let us prove that the linear span of the elements

$$\left\{\mathscr{L}^{i_1}\hat{u}_1,\dots,\mathscr{L}^{i_N}\hat{u}_N\right\}_{0\leqslant i_k\leqslant s}, \qquad k = 1,2,\dots,N, \tag{1.3}$$

coincides with $e_0 \oplus e_1 \oplus \cdots \oplus e_s$. The assertion is obvious for $s = 0$. Suppose that it is true for $s = 0,1,\dots,\nu$. Then the vector

$$A_{\nu,j} = (0,0,\dots,0,u_j,0,\dots)$$

is a linear combination of (1.3) with $s = \nu$. We have

$$\mathscr{L}A_{\nu,j} = (0,0,\dots,0, E_\nu^* u_j, V_\nu u_j, E_{\nu+1}u_j, 0,\dots). \tag{1.4}$$

Setting $j = 1,\dots,N$, we get the vectors $E_{\nu+1}u_1, E_{\nu+1}u_2,\dots,E_{\nu+1}u_N$, in the $(\nu+1)$st place, and these form a basis in $\mathbf{C}^N$ by virtue of the condition $\det E_{\nu+1} \neq 0$. The linear span of the vectors (1.4) thus contains certain vectors of the form

$$B_j = (0,\dots,0,a_j,b_j,u_j,0,\dots)$$

and, consequently, the linear span of the vectors (1.3) with $s = \nu + 1$ contains $e_{\nu+1}$. The assertion is thereby proved.

By known facts in operator theory (see [9]), the operator $\mathscr{L}$ has multiplicity N and is unitarily equivalent to the operator of multiplication by λ in the space $L_\sigma^2(\mathbf{C}^N)$.

3. We now prove that the polynomials $Q_n(\lambda)$, $n = 0,1,\dots,$ are orthogonal with respect to the measure σ, i.e.,

$$\int_{-\infty}^{\infty} Q_n(\lambda)\, d\sigma(\lambda) Q_m^*(\lambda) = \delta_{n,m}\mathscr{I}. \tag{1.5}$$

To prove (1.5) we must develop a certain apparatus. In the space

$$H = \underbrace{l_2(\mathbf{C}^N) \oplus l_2(\mathbf{C}^N) \oplus \cdots \oplus l_2(\mathbf{C}^N)}_{N}$$

let us introduce the operator

$$\Lambda = \begin{pmatrix} \mathscr{L} & 0 & 0 & \cdots & 0 \\ 0 & \mathscr{L} & 0 & \cdots & 0 \\ 0 & 0 & \mathscr{L} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \mathscr{L} \end{pmatrix}.$$

Any scalar matrix

$$A = \|a_{ij}\|_{i,j=1}^{N}, \qquad a_{ij} \in \mathbf{C},$$

can be interpreted as an operator on H. Obviously, Λ commutes with any scalar matrix. Thus, the operators $Q_n(\Lambda)$, $n = 0, 1, \ldots$, are defined in H. Let $\hat{U}_k$ denote the element of H given by

$$\hat{U}_k(A_{k,1}, A_{k,2}, \ldots, A_{k,N}), \qquad k = 0, 1, \ldots,$$

where

$$A_{k,j} = (0, 0, \ldots, 0, u_j, 0, \ldots) \in l_2(\mathbf{C}^N)$$

and

$$u_j(0, 0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbf{C}^N.$$

We prove that

$$Q_n(\Lambda)\hat{U}_0 = \hat{U}_n. \tag{1.6}$$

This relation is true for $n = 0$. Suppose that it holds for $n = 0, 1, \ldots, \nu$. Then

$$\begin{aligned} Q_{\nu+1}(\Lambda)\hat{U}_0 &= E_\nu^{-1}(\mathscr{I}\Lambda - V_\nu)Q_\nu(\Lambda)\hat{U}_0 - E_\nu^{-1}E_{\nu-1}^{*}Q_{\nu-1}(\Lambda)\hat{U}_0 \\ &= E_\nu^{-1}(\mathscr{I}\Lambda - V_0)\hat{U}_\nu - E_\nu^{-1}E_{\nu-1}^{*}\hat{U}_{\nu-1}. \end{aligned}$$

Note that

$$\Lambda\hat{U}_\nu = (\mathscr{L}A_{\nu,1}, \mathscr{L}A_{\nu,2}, \ldots, \mathscr{L}A_{\nu,N}), \qquad \nu = 0, 1, \ldots.$$

Further,

$$\mathscr{L}A_{\nu,j} = (0, \ldots, 0, E_{\nu-1}u_j, V_\nu u_j, E_\nu^{*}u_j, 0, \ldots).$$

Let

$$a_j = \Big(\underbrace{0, \ldots, 0, E_{\nu-1}u_j}_{\nu}, 0, \ldots\Big), \qquad b_j = \Big(\underbrace{0, \ldots, 0, 0, V_\nu u_j}_{\nu+1}, 0, \ldots\Big),$$

$$c_j = \Big(\underbrace{0, \ldots, 0, 0, 0, E_\nu^{*}u_j}_{\nu+2}, 0, \ldots\Big).$$

It is clear that $a_j, b_j, c_j \in l_2(\mathbf{C}^N)$, $\mathscr{L}A_{\nu,j} = a_j + b_j + c_j$, and

$$\Lambda\hat{U}_\nu = (a_1, a_2, \ldots, a_N) + (b_1, b_2, \ldots, b_N) + (c_1, c_2, \ldots, c_N).$$

We show that

$$\begin{gathered} (a_1, a_2, \ldots, a_N) = E_{\nu-1}^{*}\hat{U}_{\nu-1}, \qquad (b_1, b_2, \ldots, b_N) = V_\nu\hat{U}_\nu, \\ (c_1, c_2, \ldots, c_N) = E_\nu\hat{U}_{\nu+1}. \end{gathered} \tag{1.7}$$

These relations are verified in the same way, so we consider only the first one. We have

$$E_{\nu-1}^{*}\hat{U}_{\nu-1} = E_{\nu-1}^{*}(A_{\nu-1,1}, A_{\nu-1,2},\dots,A_{\nu-1,N}) = (t_1, t_2,\dots,t_N).$$

Since the vectors $\{A_{\nu-1,j}\}_{j=1}^{N}$ have zeros at all the l_2-coordinates except the $(\nu-1)$st, the same applies to the vectors $\{t_j\}_1^N$. Let us look at the $(\nu-1)$st coordinate of the vectors $\{t_j\}$. Denoting it by $\{t_j\}_{\nu-1}$, we get

$$([t_1]_{\nu-1}, [t_2]_{\nu-1},\dots,[t_N]_{\nu-1}) = E_{\nu-1}^{*}(u_1, u_2,\dots,u_N).$$

Thus, $[t_j]_{\nu-1}$ is the jth row of the matrix $E_{\nu-1}^{*}$. On the other hand, the $(\nu-1)$st l_2-coordinate of the vector a_j is equal to $E_{\nu-1}u_j$, which is the jth column of $E_{\nu-1}$. Since the jth column of $E_{\nu-1}$ coincides with the jth row of $E_{\nu-1}^{*}$, the first relation in (1.7) is proved. The fact that V_j is a Hermitian matrix should be used in proving the second relation in (1.7).

We now proceed to prove that the polynomials $\{Q_n(\lambda)\}$ are orthonormal. Let

$$Q_n(\lambda) = (p_{ij}(\lambda))_{i,j=1}^{N}, \qquad Q_m(\lambda) = (q_{ij}(\lambda))_{i,j=1}^{N}.$$

Further,

$$d\sigma(\lambda)Q_m^{*}(\lambda) = \left(\sum_{j=1}^{N} q_{kj}(\lambda)\, d(E_\lambda \hat{u}_i, \hat{u}_j)\right)_{i,k=1}^{N}$$

and

$$Q_n(\lambda)\,d\sigma(\lambda)Q_m^{*}(\lambda) = \left(\sum_{i=1}^{N} p_{si}(\lambda)\sum_{j=1}^{N} q_{kj}(\lambda)\, d(E_\lambda \hat{u}_i, \hat{u}_j)\right)_{s,k=1}^{N}.$$

Hence,

$$\int_{-\infty}^{\infty} Q_n(\lambda)\,d\sigma(\lambda)Q_m^{*}(\lambda) = \left(\left(\sum_{i=1}^{N} p_{si}(\mathscr{L})\hat{u}_i, \sum_{i=1}^{N} q_{kj}(\mathscr{L})\hat{u}_j\right)\right)_{s,k=1}^{N}.$$

Note that

$$\left(\sum_{i=1}^{N} p_{si}(\mathscr{L})\hat{u}_i\right)_{s=1}^{N} = (p_{ij}(\mathscr{L}))_{i,j=1}^{N}\cdot\begin{pmatrix}\hat{u}_1\\ \hat{u}_2\\ \vdots\\ \hat{u}_N\end{pmatrix} = Q_n(\Lambda)\hat{U}_0 = \hat{U}_n.$$

Similarly,

$$\left(\sum_{i=1}^{N} q_{kj}(\mathscr{L})\hat{u}_j\right)_{k=1}^{N} = \hat{U}_m.$$

Thus,

$$\int_{-\infty}^{\infty} Q_n(\lambda)\, d\sigma(\lambda)Q_m^{*}(\lambda) = 0, \qquad n \neq m,$$

and

$$\int_{-\infty}^{\infty} Q_n(\lambda)\, d\sigma(\lambda)Q_n^{*}(\lambda) = ((u_i, u_j))_{i,j=1}^{N} = \mathscr{I}.$$

This proves (1.5).

4. We consider the question of determining the operator $\mathscr{L}$, i.e., the matrices $\{V_j, E_j\}$, from the spectral measure σ. For this we form the Markov function of the measure σ:

$$\hat{\sigma}(z) = \int_{-\infty}^{\infty} \frac{d\sigma(\lambda)}{z - \lambda}.$$

The function $\hat{\sigma}(z)$ is analytic outside the spectrum $\mathscr{S}$ of $\mathscr{L}$. In a neighborhood of the point at infinity

$$\hat{\sigma}(z) = \sum_{k=0}^{\infty} \frac{s_k}{z^{k+1}}, \qquad s_k = \int_{-\infty}^{\infty} \lambda^k \, d\sigma(\lambda).$$

We consider the product

$$Q(z)\hat{\sigma}(z) = \int_{-\infty}^{\infty} \frac{Q_k(z) - Q_k(\lambda)}{z - \lambda} d\sigma(\lambda) + \int_{-\infty}^{\infty} \frac{Q_k(\lambda)}{z - \lambda} d\sigma(\lambda).$$

The first term is the polynomial (with matrix coefficients)

$$P_k(z) = \int_{-\infty}^{\infty} \frac{Q_k(z) - Q_k(\lambda)}{z - \lambda} d\sigma(\lambda). \tag{1.8}$$

The second term has a series expansion at infinity of the form

$$\int_{-\infty}^{\infty} \frac{Q_k(\lambda)}{z - \lambda} d\sigma(\lambda) = \frac{A_{k,1}}{z^{k+1}} + \frac{A_{k,2}}{z^{k+2}} + \cdots,$$

because

$$\int_{-\infty}^{\infty} Q_k(\lambda)\lambda^{\nu} \, d\sigma(\lambda) = 0, \qquad \nu = 0, 1, k - 1.$$

Thus,

$$Q_k(z)\hat{\sigma}(z) - P_k(z) = \frac{A_{k,1}}{z^{k+1}} + \cdots. \tag{1.9}$$

The polynomials $\{P_k(z)\}_{k=0}^{\infty}$ defined by (1.8) are called *polynomials of the second kind* with respect to the measure σ.

We say that a matrix-valued rational function is *proper* if it can be represented in the form

$$R(z) = Q^{-1}(z)P(z),$$

where Q and P are matrix polynomials. (Obviously, the fraction $(z - a)b(z - a)^{-1}$ is not a proper rational function if a and b do not commute.)

It follows from (1.9) that the regular rational fraction

$$\pi_k(z) = Q_k^{-1}(z)P_k(z)$$

has the property that

$$\hat{\sigma}(z) - \pi_k(z) = \frac{c}{z^{2k+1}} + \cdots.$$

The fraction $\pi_k(z)$ is called the *kth Padé fraction* for $\hat{\sigma}(z)$.

The relation (1.9) determines the proper fraction $\pi_k(z)$ uniquely to within a factor. Indeed, (1.9) is equivalent to the orthogonality relations

$$\int_{-\infty}^{\infty} Q_k(\lambda)\lambda^{\nu} \, d\sigma(\lambda) = 0, \qquad \nu = 0, 1, \ldots, k - 1,$$

which determine the polynomial $Q_k(\lambda)$ to within a normalizing factor.

Since Q_k satisfies the recursion relation

$$E_k Q_{k+1}(z) = (z\mathscr{I} - V_k)Q_k(z) - E_{k-1}^* Q_{k-1}(z), \qquad Q_0 = \mathscr{I}, \quad Q_{-1} = 0,$$

the polynomials of the second kind $P_k(z)$ in (1.8) satisfy the same recursion relation

$$E_k P_{k+1} = (z\mathscr{I} - V_k)P_k - E_{k-1}^* P_{k-1}, \qquad P_0 = 0, \quad P_1 = E_0^{-1}.$$

It can be shown that the fractions $\pi_k(z) = Q_k^{-1}P_k$ are convergents for a continued fraction of the form

$$\hat{\sigma}(z) = \cfrac{1}{z\mathscr{I} - V_0 - E_0 \cfrac{1}{z\mathscr{I} - V_1 - E_1 \cfrac{1}{z\mathscr{I} - V_2 - \ddots - E_n \cfrac{1}{z\mathscr{I} - V_{n+1} - \ddots} E_n^*} E_1^*} E_0^*}.$$

Thus, all the matrices $\{V_j, E_j\}$ can be found from the measure σ by means of various algorithms connected with continued fractions only in the case $E_k = E_k^*$.

5. Let $\mathscr{L}_0$ be the operator for which all the V_j are 0 and $E_j = \mathscr{I}/2$. It is not hard to see that this operator decomposes into a sum of operators

$$\mathscr{L}_0 = \underbrace{l_0 \oplus l_0 \oplus \cdots \oplus l_0}_{N},$$

where l_0 is the operator acting in $l_2(\mathbf{C})$ by the formula

$$l_0\{x_k\} = \left\{\tfrac{1}{2}x_{k-1} + \tfrac{1}{2}x_{k+1}\right\}, \qquad x_{-1} = 0.$$

The spectrum of $\mathscr{L}_0$ thus consists of the interval $[-1,1]$ with the absolutely continuous matrix-valued measure σ. We shall consider operators $\mathscr{L}$ close to $\mathscr{L}_0$. It was shown in [6] that if the Marchenko condition

$$\sum_{n=0}^{\infty} n\left(\|E_n - \tfrac{1}{2}\mathscr{I}\| + \|V_n\|\right) < \infty \tag{1.10}$$

holds, then the spectrum of $\mathscr{L}$ consists of $[-1,1]$ and a finite number of points outside $[-1,1]$. We should note that the connection between the smoothness of the measure σ and the rate of convergence in $E_n \to \mathscr{I}/2$ and $V_n \to 0$ has not yet been sufficiently studied, in our view. We leave this question aside and, when necessary, formulate the results in terms of σ. In particular, we do not investigate here the question of conditions on the parameters $\{E_j, V_j\}$ necessary and sufficient for the Szegő condition to hold:

$$\int_{-1}^{1} \frac{\ln\det\sigma'(\lambda)}{\sqrt{1-\lambda^2}}\,d\lambda > -\infty. \tag{1.11}$$

If $N = 1$, then, as proved in [7], condition (1.11) is equivalent to the convergence

$$0 < \prod_{n=0}^{\infty} 4E_n^2 < \infty, \qquad E_n \in \mathbf{R}^1. \tag{1.12}$$

(Here the operators involved have only finitely many eigenvalues outside $[-1,1]$, and the spectrum on $[-1,1]$ is infinite.) It is possible that an analogous modification of condition (1.12) is equivalent to (1.11) even when $N > 1$.

In what follows we always assume that the spectrum of $\mathscr{L}$ consists of $[-1,1]$ and a finite number of points outside $[-1,1]$, and that the measure σ on $[-1,1]$ satisfies the Szegő condition (1.11).

6. The usual way of obtaining the asymptotic behavior of orthogonal polynomials involves passing from the measure given on $[-1,1]$ to a measure given on the circle and studying the asymptotic properties of orthogonal polynomials on the circle.

Let σ be the measure on $[-1,1]$ given by the function $\sigma(\lambda)$. It will be assumed that $\sigma(-1)=0$.

We set

$$\eta(\theta)=\begin{cases}-\sigma(\cos\theta), & 0\leqslant\theta\leqslant\pi,\\ \sigma(\cos\theta), & \pi\leqslant\theta\leqslant 2\pi.\end{cases} \tag{1.13}$$

The Hermitian matrix-valued function $\eta(\theta)$ is nondecreasing on $[0,2\pi]$ and determines a measure η on $[0,2\pi]$. We construct polynomials $\Phi_n(z)$ which are orthogonal on the circle:

$$\frac{1}{2\pi}\int_0^{2\pi}\Phi_n(e^{i\theta})\,d\eta(\theta)\Phi_n(e^{i\theta})^*=\delta_{n,m}\mathscr{I}; \tag{1.14}$$

here $\Phi_n(z)=\mathscr{A}_n z^n+\cdots$, and $\Phi_n(z)^*=\mathscr{A}_n^*\bar{z}^n+\cdots$. Let $\mathscr{A}_n>0$.

We also construct another orthonormal system $\{\Psi_n(z)\}$ of polynomials:

$$\frac{1}{2\pi}\int_0^{2\pi}\Psi_n(e^{i\theta})^*\,d\eta(\theta)\Psi_m(e^{i\theta})=\delta_{n,m}\mathscr{I};$$

here $\Psi_n(z)=\mathscr{B}_n z^n+\cdots$. The system $\{\Psi_n(z)\}$ will be said to be *associated* with the system $\{\Phi_n(z)\}$.

The polynomials $Q_n(\lambda)$ can be expressed as follows in terms of the polynomials $\{\Phi_n(z)\}$ and $\{\Psi_n(z)\}$, which are orthogonal with respect to the measure $\eta(\theta)$ in (1.13):

$$Q_n(\lambda)=\alpha_n\left[\Phi_{2n}(z)z^{-n}+z^n\Psi_{2n}^*(1/z)\right], \tag{1.15}$$

where $\lambda=\frac{1}{2}(z+1/z)$, and α_n is a constant matrix for normalization. To prove this fact we should first argue that the expression on the right-hand side of (1.15) is a polynomial. To do this let us write (1.14) in the form

$$\int_0^{2\pi}\Phi_n(e^{i\theta})\,d\eta(\theta)e^{-ik\theta}=0,\qquad k=0,1,\ldots,n-1.$$

Let $\Phi_n(e^{i\theta})=\sum_0^n\lambda_k e^{ik\theta}$. We make the change of variables $\theta\to 2\pi-\theta$. Then

$$\int_0^{2\pi}\sum_{k=0}^n\lambda_k e^{-ik\theta}\,d\eta(2\pi-\theta)e^{ik\theta}=0,\qquad k=0,1,\ldots,n-1,$$

or

$$\int_0^{2\pi}\left(\sum_{k=0}^n\lambda_k^* e^{ik\theta}\right)^*d\eta(\theta)e^{ik\theta}=0,\qquad k=0,1,\ldots,n-1.$$

This implies that

$$\Psi_n(e^{i\theta})=\sum_{k=0}^n\lambda_k^* e^{ik\theta}.$$

Therefore, in (1.15)

$$Q_n(\lambda) = \alpha_n \left[\sum_{k=0}^{2n} \lambda_k z^{k-n} + \sum_{k=0}^{2n} \lambda_k z^{n-k} \right],$$

i.e., $Q_n(\lambda)$ is a polynomial.

The orthogonality relation can be proved in a way analogous to that for the case $N = 1$. Indeed, let

$$T_k(\lambda) = \frac{1}{2}\left(z^k + \frac{1}{z^k}\right), \qquad \lambda = \frac{1}{2}\left(z + \frac{1}{z}\right),$$

be the Tchebycheff polynomial, and let $k \leqslant n$. Then

$$\begin{aligned} I_{k,n} &= \int_{-1}^{1} Q_n(\lambda) T_k(\lambda)\, d\sigma(\lambda) = \tfrac{1}{2}\int_0^{2\pi} Q_n(\cos\theta) \cos k\theta \, d\eta(\theta) \\ &= \tfrac{1}{4}\alpha_n \int_0^{2\pi} \left[\Phi_{2n}(e^{i\theta}) + e^{2ni\theta}\Psi_{2n}^*(e^{-i\theta})\right] e^{-in\theta}(e^{ik\theta} + e^{-ik\theta})\, d\eta(\theta) \\ &= \tfrac{1}{4}\alpha_n \int_0^{2\pi} \Phi_{2n}(e^{i\theta})[e^{-i(k+n)\theta} + e^{-i(n-k)\theta}]\, d\eta(\theta) \\ &\quad + \tfrac{1}{4}\alpha_n \int_0^{2\pi} \Psi_{2n}(e^{i\theta})^*[e^{i(n-k)\theta} + e^{i(n+k)\theta}]\, d\eta(\theta). \end{aligned}$$

If $k < n$, then clearly $I_{k,n} = 0$. But if $k = n$, then

$$\begin{aligned} I_{n,n} &= \tfrac{1}{4}\alpha_n \int_0^{2\pi} \Phi_{2n}(e^{i\theta})\, d\eta(\theta) e^{-2in\theta} + \tfrac{1}{4}\alpha_n \int_0^{2\pi} \Psi_{2n}(e^{i\theta})^*\, d\eta(\theta) e^{2ni\theta} \\ &= \tfrac{1}{4}\alpha_n\left(\mathscr{A}_n^{-1} + \mathscr{B}_n^{-1}\right). \end{aligned}$$

From the symmetry of the measure in (1.13) it follows that $\mathscr{A}_n = \mathscr{B}_n^*$, so $I_{n,n} \neq 0$, which proves (1.15).

REMARK. Generally speaking, orthonormal polynomials on the circle and on a segment are determined by the orthonormality relations only to within a factor: an orthogonal transformation in $\mathbf{C}^N$. If we require the relations

$$\mathscr{A}_n > 0, \qquad \mathscr{A}_n = \mathscr{A}_n^*, \tag{1.16}$$

where $\mathscr{A}_n$ is the leading coefficient, then the orthonormal polynomials are uniquely determined. Indeed, let $\varphi_n(z)$ be the nth orthogonal polynomial on the circle with leading coefficient $\mathscr{I}$, and let

$$C = \int_0^{2\pi} \varphi_n(e^{i\theta})\, d\eta(\theta) \varphi_n(e^{i\theta})^* > 0.$$

If $\Phi_n = \mathscr{A}_n \cdot \varphi_n$, then the normalization condition amounts to the equality

$$\mathscr{A} C \mathscr{A}^* = \mathscr{I}. \tag{1.17}$$

It is clear that (1.17) has a unique solution in the class (1.16), by the theorem on a Hermitian positive square root.

We assume that the polynomials $\Phi_n(z)$ and $\Psi_n(z)$ are normalized in just this way. The polynomials $Q_n(\lambda)$, which are orthonormal not necessarily in the canonical way proved above, coincide with

$$\alpha_n\left[\Phi_{2n}(z) z^{-n} + z^n \Psi_{2n}^*(1/z)\right]$$

to within the factor α_n.

§2. Asymptotic properties of polynomials that have matrix coefficients and are orthogonal on the circle or on an interval

As mentioned at the end of the previous section (subsection 6), it is common to obtain the asymptotic properties of polynomials orthogonal on a closed interval as a corollary to the corresponding properties of polynomials orthogonal on the circle. The asymptotic behavior of polynomials with matrix coefficients and orthogonal on the circle was the subject of [22] (see also [17] and [18]). In subsection 1 of the present section we formulate Theorem 1, which characterizes this asymptotic behavior. Most of the assertions in Theorem 1 are known and are proved in [22]. But we have not encountered the rest (in particular, the asymptotic behavior of the polynomials on the support of the measure, needed for the scattering problem) in the literature on this question known to us; therefore, we give proofs carried out by the standard devices for these assertions.

1. THEOREM 1. *Let $d\eta(\theta)$ be a Hermitian matrix-valued nonnegative measure, $\theta \in [0, 2\pi]$, and let $P(\theta) = \eta'(\theta)$ be its absolutely continuous component, which exists almost everywhere in $[0, 2\pi]$. Let $\{\Phi_n(z) = \mathscr{A}_n z^n + \cdots\}$ be a system of polynomials orthonormal with respect to the measure η on the circle, and let $\{\Psi_n(z) = \mathscr{B}_n z^n + \dots\}$ be the system associated with it. If the Szegő condition*

$$\ln\det P(\theta) \in L_1[0, 2\pi] \tag{2.1}$$

holds, then the following assertions are true:

1) *The limits*

$$\lim_{n\to\infty} \mathscr{A}_n = \mathscr{A}, \qquad \lim_{n\to\infty} \mathscr{B}_n = \mathscr{B} \tag{2.2}$$

exist and are finite.

2) *Outside the unit disk*

$$\begin{aligned} \Phi_n(z) &= z^n\{D^*(z^{-1})\}^{-1} + o(1), \\ \Psi_n(z) &= z^n\{\tilde{D}^*(z^{-1})\}^{-1} + o(1), \qquad |z| > 1, \end{aligned} \tag{2.3}$$

where $o(1)$ is understood in the sense of uniform convergence for $|z| \geqslant R > 1$, and $D(z)$ and $\tilde{D}(z)$ are certain matrix-valued functions that are analytic and regular in $|z| < 1$ and do not have zeros there.

3) $D(z), \tilde{D}(z) \in \mathscr{H}^2(\mathbf{C}^N)$.

4)

$$D(e^{i\theta})D(e^{i\theta})^* = P(\theta), \qquad \tilde{D}(e^{i\theta})^*\tilde{D}(e^{i\theta}) = P(\theta) \tag{2.4}$$

almost everywhere in $[0, 2\pi]$.

5)

$$\Phi_n(e^{i\theta}) = e^{in\theta}D(e^{i\theta})^{*-1} + o(1), \qquad \Psi_n(e^{i\theta}) = e^{in\theta}\tilde{D}(e^{i\theta})^{*-1} + o(1),$$

where $o(1)$ is understood in the sense of convergence in $L_{2\sigma}$ or in Lebesgue measure.

6) *There exists a Hermitian matrix-valued integrable function $M(\theta)$ uniquely determined by $P(\theta)$ such that* $\operatorname{tr} M(\theta) = \ln\det P(\theta)$ *and*

$$D(z) = \mathop{\overset{\curvearrowright}{\int}}\nolimits_0^{2\pi} \exp\left\{\frac{e^{i\theta} + z}{e^{i\theta} - z} M(\theta)\, d\theta\right\}, \tag{2.5}$$

where $\curvearrowright\!\!\!\int_0^{2\pi}$ *is the multiplicative integral*

$$\curvearrowright\!\!\!\int_0^{2\pi} \exp\{F(\theta)\,d\theta\} = \lim_{\Delta\theta_i \to \infty} \prod_{i=0}^{n-1} e^{F(\varphi_i)\Delta\theta_i},$$

$$0 = \theta_0 \leqslant \varphi_0 \leqslant \theta_1 \leqslant \varphi_1 \leqslant \theta_2 \leqslant \cdots \leqslant \theta_{n-1} \leqslant \varphi_{n-1} \leqslant \theta_n = 2\pi.$$

PROOF. Parts 1), 2), and 4) were proved in [22] (see Theorems 17, 18, and 20). The proof of 3) repeats word-for-word (with Theorem 9 in [22] taken into account) that of the corresponding assertion in the scalar case (see [10], Russian p. 26, English p. 21).

Since $D(z)$, $\tilde{D}(z) \in \mathscr{H}^2(\mathbf{C}^n)$, the limit boundary values exist almost everywhere in $[0, 2\pi]$:

$$\lim_{r\to 1-0} D(re^{i\theta}) = \dot{D}(e^{i\theta}), \qquad \lim_{r\to 1-0} \tilde{D}(re^{i\theta}) = \dot{\tilde{D}}(e^{i\theta}).$$

Everywhere in what follows, the limit boundary values of the function $D(z)$ will be understood to be the function

$$D(e^{i\theta}) = \chi_{E_1}\chi_{E_2}\dot{D}(e^{i\theta}), \qquad \theta \in [0, 2\pi],$$

where χ_{E_1} is the characteristic function of the set $E_1 \subseteq [0, 2\pi]$ on which $\eta(\theta)$ has a finite and positive derivative, and χ_{E_2} is the characteristic function of the set $E_2 \subseteq [0, 2\pi]$ on which $\dot{D}(e^{i\theta})$ exists;

$$\chi_E(\theta) = \begin{cases} 1, & \theta \in E, \\ 0, & \theta \notin E. \end{cases}$$

Obviously, $D(e^{i\theta})$ is equivalent to $\chi_{E_2}(\theta)\dot{D}(e^{i\theta})$.

5) Let us first prove the relations in 5) in the sense of convergence in the L^2_η-norm. According to [22] (formula (82)),

$$\Phi_n^{\#}(e^{i\theta})\mathscr{A}_n = \sum_{\nu=0}^{n} \Psi\nu(e^{i\theta})\Psi_\nu^*(0),$$

where

$$\Phi_n^{\#}(z) = z^n\Phi_n^*(1/z).$$

The right-hand side of the given expression is an expansion in an orthonormal system in L^2_η. Since the Fourier coefficients of this expression (see [22], formula (81)) are in $l_2(\mathbf{C}^N)$, the sequence $\{\Phi_n^{\#}(e^{i\theta})\mathscr{A}_n\}$ in L^2_η has a limit:

$$\lim_{n\to\infty}\{\Phi_n^{\#}(e^{i\theta})\mathscr{A}_n\} = \sum_{\nu=0}^{\infty} \Psi_\nu(e^{i\theta})\Psi_\nu^*(0),$$

where the series converges in the L^2_η-norm. According to part 1) of Theorem 1,

$$\lim_{n\to\infty}\{\Phi_n^{\#}(e^{i\theta})\mathscr{A}_n\} = \lim_{n\to\infty}\{\Phi_n^{\#}(e^{i\theta})\}D^{-1}(0),$$

and, consequently,

$$\lim_{n\to\infty}\{\Phi_n(e^{i\theta})\} = \sum_{\nu=0}^{\infty} \Psi_\nu(e^{i\theta})\Psi_\nu^*(0)D(0). \tag{2.6}$$

To see that the series being considered converges precisely to $D^{-1}(e^{i\theta})$ in L^2_η, it suffices to show that the Fourier coefficients of $D^{-1}(e^{i\theta})$ (call them a_n) with respect to $\{\Psi_\nu(e^{i\theta})\}$ in L^2_η coincide with the coefficients in the expansion (2.6). According to the stipulation in the proof of part a) of the given theorem,

$$a_n^* = \frac{1}{2\pi}\int_0^{2\pi} D^{-1}(e^{i\theta})\,d\eta(\theta)\Psi_n(e^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} D^{-1}(e^{i\theta})P(\theta)\Psi_n(e^{i\theta})\,d\theta,$$

and by (2.4) (and the mean value theorem),

$$a_n^* = \frac{1}{2\pi}\int_0^{2\pi} D(e^{i\theta})^*\Psi_n(e^{i\theta})\,d\theta = D^*(0)\Psi_n(0).$$

Thus, we have shown that

$$\Phi_n^{\#}(e^{i\theta}) = e^{in\theta}\Phi_n(e^{i\theta})^* = D^{-1}(e^{i\theta}) + o(1), \tag{2.7}$$

where $o(1)$ should be understood in the sense of convergence in $L^2_\eta(\mathbf{C}^N)$. We now show that the Szegő condition (2.1) (and even the weaker condition that $\eta'(\theta) > 0$ for almost all $\theta \in [0, 2\pi]$) allows us to understand the $o(1)$ in (2.7) in the sense of convergence in Lebesgue measure. If $\eta'(\theta) > 0$ almost everywhere, then Lebesgue measure $\mu(\theta)$ is absolutely continuous with respect to the measure $\eta(\theta)$. Indeed,

$$\eta(E) = \int_E d\eta(\theta) \geqslant \int_E \eta'(\theta)\,d\theta.$$

The right-hand side is strictly greater than zero if $\mu(E) > 0$. Therefore, $\mu(E) = 0$ if $\eta(E) = 0$, i.e., Lebesgue measure μ is absolutely continuous with respect to the matrix-valued measure η. But η, in turn, is absolutely continuous with respect to the measure ν given by the scalar function $\nu(\theta) = \sum_{i=1}^N \eta_{ii}(\theta)$ (see [13], §1). Therefore, writing

$$E_n = \left\{\theta\colon \left\|\Phi_n^{\#}(e^{i\theta}) - D(e^{i\theta})\right\|_{\mathbf{C}^N} > \varepsilon\right\},$$

we have, by the Radon-Nikodým theorem, an absolutely ν-integrable function $f(\theta)$ such that

$$\mu(E_n) = \int_{E_n} f(\theta)\,d\nu(\theta).$$

The right-hand side of the given expression tends to zero as $n \to \infty$ for any $\varepsilon > 0$, since $(\Phi_n^*(e^{i\theta}) - D(e^{i\theta})) \to 0$ as $n \to \infty$ in the L^2_η-norm, and hence [13] also in the measure ν. This completes the proof of part 5) of Theorem 1.

6) A matrix-valued function $f(z)$ in $\mathscr{H}^p(\mathbf{C}^N)$ is said to be *outer* (see [8], [14] or [17]) if its determinant admits the representation

$$\det f(z) = \kappa \exp\left[\frac{1}{2\pi}\int_0^{2\pi}\frac{e^{i\theta}+z}{e^{i\theta}-z}\ln k(\theta)\,d\theta\right], \qquad |z| < 1,$$

where $k(\theta) \geqslant 0$, $\ln k(\theta) \in L_1[0, 2\pi]$, and $|\kappa| = 1$.

According to a theorem of Ginzburg ([14], Theorem 2) on parametric representation of outer functions in $\mathscr{H}^p(\mathbf{C}^N)$, the relation (2.5) will be proved if we show that $D(z)$ is an outer function. As follows from Theorem 3 in [14], the function $M(\theta)$ in (2.5) is uniquely determined by the function

$$D(e^{i\theta})\left(D(e^{i\theta})\right)^* = P(\theta).$$

We prove that $D(z)$ is an outer function. Indeed, since $D(z) \in \mathscr{H}^2(\mathbf{C}^N)$, it follows that $\det D(z) \in \mathscr{H}^{2/N}(\mathbf{C}^1)$ (since $|\det D(z)| \leqslant \|D(z)\|^N$), and it follows from (2.4) that the values of $|\det D(z)|^2$ for $z = e^{i\theta}$ coincide with $\det P(\theta)$ almost everywhere. Consequently, the parametric representation for functions in $\mathscr{H}^p(\mathbf{C}^1)$ (see [15]) gives us that $\det D(z)$ admits for $|z| < 1$ the representation

$$\det D(z) = \kappa b(z) \exp\left\{ \frac{1}{4\pi} \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \ln \det P(\theta)\, d\theta \right\}$$
$$\times \exp\left\{ \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} d\Psi(\theta) \right\}; \tag{2.8}$$

here $|\kappa| = 1$, $b(z)$ is the Blaschke product for $\det D(z)$, and $d\Psi(\theta)$ is a measure that is singular with respect to Lebesgue measure (i.e., $\Psi(\theta)$ is a nonincreasing function with derivative equal almost everywhere to zero).

Since $D(z)$ does not have zeros, $b(z) \equiv 1$. Setting $z = 0$ in (2.8), we get that

$$\det D(0) = \exp\left\{ \frac{1}{4\pi} \int_0^{2\pi} \ln \det P(\theta)\, d\theta \right\} \exp\left\{ \frac{1}{2\pi} \int_0^{2\pi} d\Psi(\theta) \right\};$$

$\kappa = 1$ because $D(0) = D^*(0)$. Moreover, since $\int_0^{2\pi} d\Psi(\theta) \leqslant 0$, formula (89) in [22] gives us that $\int_0^{2\pi} d\Psi(\theta) = 0$. Consequently, $\Psi(\theta) \equiv \text{const}$, and so $D(z)$ satisfies the equality

$$\det(D(z)) = \exp\left\{ \frac{1}{4\pi} \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \ln \det P(\theta)\, d\theta \right\},$$

so $D(z)$ is an outer function. This proves Theorem 1.

2. It is clear from part 6) of Theorem 1 that the asymptotic behavior of the orthogonal polynomials on the circle is uniquely determined only by the absolutely continuous component of the measure. This remark is made precise by the following estimate:

$$\left\| \Phi_n(e^{i\theta}) \right\|_{L^2_{\eta_1}} < 2 \left\| \mathscr{I} - \mathscr{A}_n \mathscr{A}^{-1} \right\|_{\mathbf{C}^N}^{1/2}, \tag{2.9}$$

where η_1 is the sum of the jump function and the singular component of the measure η. It follows from Theorem 1 that if the Szegő condition (2.1) holds, then the right-hand side of (2.9) is $o(1)$.

Let us prove (2.9). By (2.4) and the average value theorem,

$$\frac{1}{2\pi} \int_0^{2\pi} \Phi_n(e^{i\theta})\, d\eta_1(\theta) \Phi_n(e^{i\theta})^* = \frac{1}{2\pi} \int_0^{2\pi} \Phi_n^\#(e^{i\theta})^*\, d\eta_1(\theta) \Phi_n^\#(e^{i\theta})$$
$$= \mathscr{I} - \left[\frac{1}{2\pi} \int_0^{2\pi} \Phi_n^\#(e^{i\theta})^* P(\theta) \Phi_n^\#(e^{i\theta})\, d\theta \right]$$
$$= \mathscr{I} - 2\,\mathrm{Re}\left\{ \frac{1}{2\pi i} \int_{|z|=1} (\Phi_n^\#(z))^* D(z) \right\} \frac{dz}{z} + \mathscr{I}$$
$$- \left[\frac{1}{2\pi} \int_0^{2\pi} \Phi_n^\#(e^{i\theta})^* D(e^{i\theta}) D(e^{i\theta})^* \Phi_n^\#(e^{i\theta})\, d\theta \right.$$
$$\left. - 2\,\mathrm{Re}\left\{ \frac{1}{2\pi} \int_0^{2\pi} \Phi_n^\#(e^{i\theta})^* D(e^{i\theta})\, d\theta \right\} + \mathscr{I} \right]$$
$$= 2\mathscr{I} - 2\mathscr{A}_n \mathscr{A}^{-1} - \frac{1}{2\pi} \int_0^{2\pi} \left[\Phi_n^\#(e^{i\theta})^* D(e^{i\theta}) - 1 \right]$$
$$\times \left[\Phi_n^\#(e^{i\theta})^* D(e^{i\theta}) - 1 \right]^* d\theta,$$

i.e.,

$$\frac{1}{2\pi}\int_0^{2\pi} \Phi_n(e^{i\theta})\, d\eta_1(\theta)\Phi_n(e^{i\theta})^* < 2(\mathscr{I} - \mathscr{A}_n\mathscr{A}^{-1}).$$

The estimate (2.9) follows from this. (For $N = 1$, (2.9) was proved in [10].)

3. We now consider $\{Q_n(\lambda)\}$: polynomials orthonormal with respect to a measure σ concentrated on $[-1,1]$. Assume that Szegő's condition holds; that is,

$$\int_{-1}^{1} \frac{\ln \det \sigma'(\lambda)}{\sqrt{1-\lambda^2}}\, d\lambda > -\infty.$$

It can be assumed without loss of generality that $\mu(-1) = 0$.

We construct the polynomials $\Phi_n(z)$ orthonormal on the circle with respect to the measure

$$\eta(\theta) = \begin{cases} -\sigma(\cos\theta), & 0 \leqslant \theta \leqslant \pi, \\ \sigma(\cos\theta), & \pi \leqslant \theta \leqslant 2\pi. \end{cases}$$

As shown in §1.6, the polynomials

$$\hat{Q}_n(\lambda) = \hat{Q}_n\left[\frac{1}{2}\left(z + \frac{1}{z}\right)\right] = z^{-n}\Phi_{2n}(z) + z^n\Phi_{2n}\left(\frac{1}{z}\right), \qquad \lambda = \frac{1}{2}\left(z + \frac{1}{z}\right),$$

are orthogonal on $[-1,1]$. Using the asymptotic formulas in Theorem 2, we get for $z = e^{i\theta}$ that

$$\hat{Q}_n(\cos\theta) = \frac{1}{\sqrt{2\pi}}\left[e^{in\theta}D^{-1}(e^{-i\theta}) + e^{-in\theta}D^{-1}(e^{i\theta})\right] + o(1), \tag{2.10}$$

where $o(1)$ is understood in the L^2_η-sense. The asymptotic formula (2.10) and the estimate (2.9) enable us to find the normalization constant. Let η_1 be the singular component of η. We have that

$$\int_{-1}^{1} \hat{Q}_n(\lambda)\, d\eta(\lambda)\hat{Q}_n(\lambda)^* = \frac{1}{2}\int_0^{2\pi}\left[e^{-in\theta}\Phi_{2n}(e^{i\theta}) + e^{in\theta}\Phi_{2n}(e^{-i\theta})\right]\eta'(\theta)$$
$$\times\left[e^{in\theta}\Phi_{2n}(e^{i\theta})^* + e^{-in\theta}\Phi_{2n}(e^{i\theta})^*\right] d\theta + I_n,$$

where I_n is the analogous integral, except with respect to the measure η_1. By (2.9), $I_n = o(1)$, and by (2.10),

$$\int_{-1}^{1} \hat{Q}_n(\lambda)\, d\eta(\lambda)\hat{Q}_n(\lambda)^* = \frac{1}{4\pi}\int_0^{2\pi}\{D^{-1}(e^{-i\theta})\eta'(\theta)D^{-1}(e^{-i\theta})^*$$
$$+ D^{-1}(e^{i\theta})\,\eta'(\theta)D^{-1}(e^{i\theta})^*\}\, d\theta + o(q).$$

According to (2.4),

$$D(e^{i\theta})D(e^{i\theta})^* = \eta'(\theta) = \eta'(-\theta) = D(e^{-i\theta})D(e^{-i\theta})^*,$$

so

$$\int_{-1}^{1} \hat{Q}_n(\lambda)\, d\eta(\lambda)\hat{Q}_n(\lambda)^* = \mathscr{I} + o(1). \tag{2.10$'$}$$

Thus,

$$Q_n(\cos\theta) = \frac{1}{\sqrt{2\pi}}\left[e^{-in\theta}D^{-1}(e^{i\theta}) + e^{in\theta}D^{-1}(e^{-i\theta})\right] + o(1), \tag{2.11}$$

where $\{Q_n(\lambda)\}$ is the sequence of orthonormal polynomials for the measure σ.

The asymptotic formula for the polynomials $Q(\lambda)$ when $\lambda \notin [-1,1]$ follows from the corresponding asymptotic relations for the polynomials $\Phi_n(z)$ (Theorem 1). For $|z| < 1$

$$Q_n\left[\frac{1}{2}\left(z + \frac{1}{z}\right)\right] = \frac{1}{\sqrt{2\pi}} \frac{1}{z^n} D^{-1}(z)[1 + o(1)] \tag{2.12}$$

uniformly in $|z| \leqslant r < 1$.

§3. The asymptotic behavior of polynomials that have matrix coefficients and are orthogonal on the line

In this section we extend our asymptotic formulas to polynomials orthogonal on the line with respect to a measure concentrated on $[-1,1]$ and at finitely many points $\{\lambda_k\}$ lying outside $[-1,1]$. The analogous problem in the scalar case ($N = 1$) was treated by Gonchar (see [16]).

1. Assume that on $[-1,1]$ the measure σ satisfies the Szegő condition

$$\int_{-1}^{1} \frac{\ln\det\sigma'(\lambda)}{\sqrt{1-\lambda^2}} d\lambda > -\infty, \tag{3.1}$$

while the nonnegative masses $\{\sigma_k\}_{k=1}^{M}$ are concentrated at the points

$$\lambda_k = \frac{1}{2}\left(z_k + \frac{1}{z_k}\right), \qquad -1 < z_k < 1, z_k \neq 0, k = 1,2,\dots,M$$

(here the σ_k, generally speaking, are singular nonnegative-definite matrices).

Let $\{Q_n(\lambda)\}$ be the polynomials orthonormal with respect to the measure σ. The same polynomials with leading coefficient $\mathscr{I}$ are denoted by $\{q_n(\lambda)\}$. We prove by induction on M that the asymptotic formulas (2.11) and (2.12) remain valid also for the polynomials $\{Q_n(\lambda)\}$. Only the function $D(z)$ changes. It will differ from that considered in §2.3 (i.e., the one constructed from $\sigma'(\cos\theta)|\sin\theta|$) by a factor—a matrix-valued Blaschke product.

Accordingly, the induction hypothesis amounts to the validity of formulas (2.11) and (2.12) for $\{Q_n(\lambda)\}$ along with the validity of the following relations:

a)

$$\lim_{n\to\infty} \int_{-1}^{1} \frac{Q_n(\lambda)}{\lambda - \lambda_0} d\sigma(\lambda) Q_n(\lambda)^* = \frac{1}{2\pi}\mathscr{I}\int_0^{2\pi} \frac{d\theta}{\cos\theta - \lambda_0}$$

for any point $\lambda_0 \notin [-1,1] \cup \{\lambda_k\}_{k=1}^{M}$.

b)

$$\lim_{n\to\infty} Q_n(\lambda_k)\sigma_k Q_n(\lambda_k)^* = 0, \qquad k = 1,2,\dots,M.$$

c) If $Q_n(\lambda) = \eta_n\lambda^n + \cdots$, then

$$\lim_{n\to\infty} \frac{\eta_n}{2^n} = \eta = D^{-1}(0).$$

It is also assumed that $D(z)$ is analytic and nonsingular in $\{|z| < 1\} \setminus \{z_k\}_{k=1}^{M}$. The boundary values satisfy $D(e^{i\theta}) \in L^2(\mathbf{C}^N)_{[0,2\pi]}$ and

$$D(e^{i\theta})D(e^{i\theta})^* = \mu'(\cos\theta)|\sin\theta|.$$

For $M = 0$ only a) and c) need to be checked. The former can be established similarly to (2.10′).

To prove c), note that

$$Q_n(\lambda) = \left[z^{-n}\Phi_{2n}(z) + z^n\Phi_{2n}\left(\frac{1}{z}\right)\right][\mathscr{I} + o(1)],$$

where $o(1)$ does not depend on z and expresses (2.12). Let $\Phi_k(z) = \mathscr{A}_k z^k + \cdots$. Then

$$\eta_n = 2^n \mathscr{A}_{2n}[\mathscr{I} + o(1)],$$

and the validity of c) for $M = 0$ follows from part 1) of Theorem 1.

2. Suppose now that another matrix-valued mass σ_0, concentrated at a point $\lambda_0 \in [-1, 1] \cup \{\lambda_k\}_{k=1}^M$, is added to the measure σ. Let σ_* denote the measure obtained in this way. The polynomials with leading coefficient $\mathscr{I}$ that are orthogonal with respect to the measure σ_* are denoted by $h_n(\lambda)$. We expand the polynomial $(\lambda - \lambda_0)h_n(\lambda)$ with respect to the polynomials $\{Q_k(\lambda)\}$ orthonormal with respect to the measure σ. By the orthogonality,

$$(\lambda - \lambda_0)h_n(\lambda) = \eta_{n+1}^{-1}Q_{n+1}(\lambda) + \alpha_n Q_n(\lambda) + \beta_n Q_{n-1}(\lambda).$$

Clearly,

$$\beta_n = \int_{-\infty}^{\infty} (\lambda - \lambda_0)h_n(\lambda)\, d\sigma(\lambda) Q_{n-1}(\lambda)^* = \int_{-\infty}^{\infty} h_n(\lambda)\, d\sigma_*(\lambda) h_n(\lambda)^* \eta_{n-1}^*.$$

Note that the matrix $\beta_n(\eta_{n-1}^*)^{-1}$ is Hermitian. We see

$$\beta_n \eta_{n-1}^{*-1} = \delta_n, \qquad \delta_n = \delta_n^*.$$

Then

$$(\lambda - \lambda_0)h_n(\lambda) = \eta_{n+1}^{-1}Q_{n+1}(\lambda) + \alpha_n Q_n(\lambda) + \delta_n \eta_{n-1}^* Q_{n-1}(\lambda). \tag{3.2}$$

Let

$$\Delta_n = \max\{\|\eta_{n+1}, \alpha_n\|, \|\eta_{n+1}\delta_n\eta_{n-1}^*\|, 1\}.$$

By (3.2),

$$\eta_{n+1}^{-1}Q_{n+1}(\lambda_0) + \alpha_n Q_n(\lambda_0) + \delta_n \eta_{n-1}^* Q_{n-1}(\lambda_0) = 0.$$

Multiplying this expression on the left by η_{n+1} and on the right by $Q_n^{-1}(\lambda_0)$ and using (2.12), we get that

$$\mathscr{I}/z_0 + \eta_{n+1}\alpha_n + \eta_{n+1}\delta_n\eta_{n-1}^* z_0 = o(\Delta_n).$$

Here $\lambda_0 = \frac{1}{2}(z_0 + 1/z_0)$, $-1 < z_0 < 1$, $z_0 \neq 0$.

Let us write the condition that $h_n(\lambda)$ be orthogonal to $\mathscr{I}$ with respect to σ_*:

$$\begin{aligned} 0 &= \int_{-\infty}^{\infty} h_n(\lambda)\, d\sigma \\ &= \int_{-\infty}^{\infty} \frac{1}{\lambda - \lambda_0}\left[\eta_{n+1}^{-1}Q_{n+1}(\lambda) + \alpha_n Q_n(\lambda) + \delta_n\eta_{n-1}^* Q_{n-1}(\lambda)\right] d\sigma \\ &\qquad + \left[\eta_{n+1}^{-1}Q_{n+1}'(\lambda_0) + \alpha_n Q_n'(\lambda_0) + \delta_n\eta_{n-1}^* Q_{n-1}'(\lambda)\right] \sigma_0. \end{aligned} \tag{3.3}$$

It is not hard to see that

$$\int_{-\infty}^{\infty} \frac{1}{\lambda - \lambda_0} Q_k(\lambda)\, d\sigma(\lambda) = \int_{-\infty}^{\infty} \frac{1}{\lambda - \lambda_0} Q_k(\lambda)\, d\sigma(\lambda) Q_k(\lambda)^* \left(Q_k(\lambda_0)^{-1}\right)^*,$$

because

$$\int_{-\infty}^{\infty} Q_k(\lambda)\, d\sigma(\lambda) \frac{1}{\lambda - \lambda_0}[Q_k(\lambda)^* - Q_k(\lambda_0)^*] = 0.$$

Hence,

$$\int_{-\infty}^{\infty} \frac{1}{\lambda - \lambda_0}\left[\eta_{n+1}^{-1}Q_{n+1}(\lambda) + \alpha_n Q_n(\lambda) + \delta_n\eta_{n-1}^* Q_{n-1}(\lambda)\right] d\sigma(\lambda) = o(2^n \Delta_n z_0^n).$$

Consequently,

$$\left[Q_{n+1}'(\lambda_0) + \eta_{n+1}\alpha_n Q_n'(\lambda_0) + \eta_{n+1}\delta_n\eta_{n-1}^* Q_{n-1}'(\lambda_0)\right] \sigma_0 = o(\Delta_n z_0^n).$$

Consider the expression

$$R_n = \frac{d}{d\lambda}\left[Q_{n+1}\sigma_0 Q_n^{-1} + \eta_{n+1}\alpha_n Q_n\sigma_0 Q_n^{-1} + \eta_{n+1}\delta_n\eta_{n-1}^* Q_{n-1}\sigma_0 Q_n^{-1}\right]_{\lambda=\lambda_0}.$$

Carrying out the differentiation, we get that

$$\begin{aligned} R_n = &-\left(Q_{n+1} + \eta_{n+1}\alpha_n Q_n + \eta_{n+1}\delta_n\eta_{n-1}^* Q_{n-1}\right)\sigma_0 Q_n^{-1} Q_n' Q_n^{-1}\big|_{\lambda=\lambda_0} \\ &+\left(Q_{n+1}' + \eta_{n+1}\alpha_n Q_n' + \eta_{n+1}\delta_n\eta_{n-1}^* Q_{n-1}'\right)\sigma_0 Q_n^{-1}\big|_{\lambda=\lambda_0}. \end{aligned}$$

Therefore,

$$R_n = o(\Delta_n z_0^n).$$

On the other hand, using the asymptotic expression for $Q_n((z + 1/z)/2)$, we get that

$$\begin{aligned} R_n = \frac{2}{(1 - 1/z^2)} \frac{d}{dz}\Big[&\frac{1}{z} D^{-1}(z)\sigma_0 D(z) + \eta_{n+1}\alpha_n D^{-1}(z)\sigma_0 D(z) \\ &+ z\eta_{n+1}\delta_n\eta_{n-1}^* D^{-1}(z)\sigma_0 D(z)\Big]_{z=z_0} + o(\Delta_n). \end{aligned}$$

Hence,

$$\left(-\frac{1}{z^2}\mathscr{I} + \eta_{n+1}\delta_n\eta_{n-1}^*\right) D^{-1}(z_0)\sigma_0 = o(\Delta_n).$$

We return again to (3.3). Let P be the projection onto Ker σ_0. Then (3.5) gives us that

$$\begin{aligned} \Big[&\int_{-\infty}^{\infty} \frac{1}{\lambda - \lambda_0} Q_{n+1}(\lambda)\, d\sigma(\lambda) Q_{n+1}(\lambda)^* Q_{n+1}^{-1}(\lambda_0)^* \\ &+\eta_{n+1}\alpha_n \int_{-\infty}^{\infty} \frac{1}{\lambda - \lambda_0} Q_n(\lambda)\, d\sigma(\lambda) Q_n(\lambda)^* Q_n^{-1}(\lambda_0)^* \\ &+\eta_{n+1}\delta_n\eta_{n-1}^* \int_{-\infty}^{\infty} \frac{1}{\lambda - \lambda_0} Q_{n-1}(\lambda)\, d\sigma(\lambda) Q_{n-1}(\lambda)^* Q_{n-1}^{-1}(\lambda_0)^*\Big] P = 0, \end{aligned}$$

because $\sigma_0 P = 0$. Multiplying this relation by $1/z_0^n$, we use a) and b) of the induction hypothesis and get that

$$\left[z_0\mathscr{I} + \eta_{n+1}\alpha_n + \eta_{n+1}\delta_n\eta_{n-1}^* \frac{1}{z_0}\right] D^*(z_0) P = o(\Delta_n).$$

Thus, we have obtained the three relations:

$$\begin{cases} \mathscr{I}/z_0 + \eta_{n+1}\alpha_n + z_0\eta_{n+1}\delta_n\eta_{n-1}^* = o(\Delta_n), \\ \left(z_0\mathscr{I} + \eta_{n+1}\alpha_n + z_0^{-1}\eta_{n-1}\delta_n\eta_{n-1}^*\right) D^*(z_0) P = o(\Delta_n), \\ \left(-1/z_0^2 + \eta_{n+1}\delta_n\eta_{n-1}^*\right) D^{-1}(z_0)\sigma_0 = o(\Delta_n). \end{cases} \tag{3.4}$$

3. We show that these relations imply the existence of the limits

$$\lim_{n\to\infty} \eta_{n+1}\alpha_n = A, \qquad \lim_{n\to\infty} \eta_{n+1}\delta_n\eta_{n-1}^* = B, \tag{3.5}$$

and we find these limits. Let us first prove that $\Delta_n = O(1)$. Indeed, if $\Delta_{n_k} \to \infty$, then consider the sequences

$$e_k = \frac{\eta_{n_k+1}\alpha_{n_k}}{\Delta_{n_k}}, \qquad u_k = \Delta_{n_k}^{-1}\eta_{n_k+1}\delta_{n_k}\eta_{n_k-1}^*.$$

From them let us select convergent subsequences:

$$\lim_{j\to\infty} e_{k_j} = \alpha, \qquad \lim_{j\to\infty} u_{k_j} = \delta.$$

By the normalization,

$$\max(\|e_k\|, \|u_k\|) = 1.$$

Therefore, either $\alpha \neq 0$ or $\delta \neq 0$. Dividing (3.4) by $\Delta_{n_{k_j}}$ and passing to the limit, we get the relations

$$\alpha + z_0\delta = 0, \qquad (\alpha + z_0^{-1}\delta)D^*(z_0)P = 0, \qquad \delta D^{-1}(z_0)\sigma_0 = 0. \tag{3.6}$$

We show that $\delta^* = \delta$. Indeed,

$$u_k^* = (\eta_{n_k-1}\delta_{n_k}\eta_{n_k+1}^*)/\Delta_{n_k} = \eta_{n_k-1}\eta_{n_k+1}^{-1}u_k\eta_{n_k-1}^{*-1}\eta_{n_k+1}^*.$$

Replacing k by k_j here and letting j go to ∞, we get that $\delta^* = \delta$.

Elimination of α from (3.6) leads to the equalities

$$\delta D^*(z_0)P = 0, \qquad \delta D^{-1}(z_0)(\mathscr{I} - P) = 0;$$

transition to the conjugate equality in the first of them leaves us with

$$PD(z_0)\delta = 0, \qquad \delta D^{-1}(z_0)(\mathscr{I} - P) = 0.$$

Let $\Lambda = D(z_0)\delta D^{-1}(z_0)$. Then

$$P\Lambda = 0, \qquad \Lambda(\mathscr{I} - P) = 0.$$

From this it follows easily that $\Lambda^2 = 0$. Indeed, the first relation gives us that for any $x \in \mathbf{C}^N$

$$\Lambda x \in \operatorname{im}\sigma_0 = (\operatorname{Ker}\sigma_0)^{\perp}.$$

On the other hand, for any $x \in \operatorname{im}\sigma_0$ the second relation implies that $\Lambda x = 0$. Therefore, $\Lambda^2 x = 0$ for any $x \in \mathbf{C}^N$, and so $\delta^2 x = 0$ for any $x \in \mathbf{C}^N$. Since δ is Hermitian,

$$(\delta x, \delta x) = (\delta^2 x, x) = 0.$$

Therefore, $\delta = 0$. Thus, we have arrived at a contradiction, and $\Delta_n = O(1)$.

The sequences $\{\eta_{n+1}\alpha_n\}$ and $\{\eta_{n+1}\delta_n\eta_{n-1}^*\}$ are uniformly bounded. Let

$$A_0 = \lim_{\substack{n\to\infty \\ n\in\tau}} \eta_{n+1}\alpha_n, \qquad B_0 = \lim_{\substack{n\to\infty \\ n\in\tau}} \eta_{n+1}\delta_n\eta_{n-1}^*$$

be arbitrary limit points of these sequences. As soon as we have shown that only the one possibility $A_0 = A$, $B_0 = B$ can hold for (A_0, B_0), we will thereby have proved the existence of the limits (3.5).

Passing to the limit in (3.4) with respect to the subsequence $n \in \tau$, we get

$$\frac{1}{z_0} + A_0 + z_0 B_0 = 0, \qquad \left(z_0 + A_0 + \frac{1}{z_0} B_0\right) D^* P = 0,$$

$$\left(-\frac{1}{z_0^2} + B_0\right) D^{-1}(\mathscr{I} - P) = 0, \qquad D = D(z_0).$$

As before, $B_0^* = B_0$ and, consequently, $A_0^* = A_0$. Elimination of A_0 from these relations gives us that

$$(\mathscr{I} - B_0) D^* P = 0, \qquad (z_0^{-2} + B_0) D^{-1}(\mathscr{I} - P) = 0.$$

In the first relation we pass to the adjoint operators

$$PD(\mathscr{I} - B_0) = 0, \qquad (z_0^{-2} + B_0) D^{-1}(\mathscr{I} - P) = 0, \qquad D = D(z_0). \tag{3.7}$$

If this equation has a selfadjoint solution, then it is unique. Indeed, if there were two such solutions, then their difference $\tilde{\delta}$ would be a selfadjoint operator satisfying the equations

$$PD\tilde{\delta} = 0, \qquad \tilde{\delta} D^{-1}(\mathscr{I} - P) = 0. \tag{3.8}$$

As we saw earlier, (3.8) has the unique Hermitian solution $\tilde{\delta} = 0$. We thus need only determine a Hermitian solution of (3.7).

Let $V = \operatorname{Ker} \sigma_0$. We set $H = (D^{-1} V^{\perp})^{\perp}$. Let R be the orthogonal projection onto H. We set

$$B = R + (\mathscr{I} - R)/z_0^2$$

and prove that B is a selfadjoint operator satisfying the system (3.7). If $x \in H$, then

$$(\mathscr{I} - B)x = 0$$

and, consequently,

$$PD(\mathscr{I} - B)x = 0.$$

But if $x \in H^{\perp}$, then there exists a $y \in V^{\perp}$ such that $x = D^{-1} y$.

We have that

$$PD(\mathscr{I} - B)x = (1 - z_0^{-2}) PDx = (1 - z_0^{-2}) PDD^{-1} y = 0,$$

because $Py = 0$. Accordingly, the equation

$$PD(\mathscr{I} - B) = 0$$

is satisfied. Now if $x \in V$, then $(\mathscr{I} - P)x = 0$ and, consequently,

$$(-z_0^{-2}\mathscr{I} + B) D^{-1}(\mathscr{I} - P)x = 0.$$

But if $x \in V^{\perp}$, then

$$(-z_0^{-2}\mathscr{I} + B) D^{-1}(\mathscr{I} - P)x = (-z_0^{-2}\mathscr{I} + B) D^{-1} x = 0,$$

because $D^{-1}x \in H^{\perp}$ and

$$(-z_0^{-2}\mathscr{I} + B) H^{\perp} = 0.$$

The equation

$$(-z_0^{-2}\mathscr{I} + B) D^{-1}(\mathscr{I} - P) = 0$$

is thus also satisfied. It is also obvious that B is a Hermitian operator on $\mathbf{C}^N$.

Accordingly, we have proved that

$$A = \lim_{n\to\infty} \eta_{n+1}\alpha_n = \left(\frac{1}{z_0} - z_0\right)R - \frac{2}{z_0}\mathscr{I},$$

$$B = \lim_{n\to\infty} \eta_{n+1}\delta_n\eta_{n-1}^* = R + \frac{1}{z_0^2}(\mathscr{I} - R),$$

where R is the orthogonal projection onto $[D^{-1}(\operatorname{Ker}\sigma_0)^\perp]^\perp$, and $D = D(z_0)$.

4. We now pass to the verification of the induction hypothesis. As before, let $h_n(\lambda)$ be the polynomials with leading coefficient $\mathscr{I}$ that are orthogonal with respect to the measure σ_*. Then

$$\eta_{n+1}(\lambda - \lambda_0)h_n(\lambda) = Q_{n+1}(\lambda) + \left[-\frac{2}{z_0}\mathscr{I} + (z_0^{-1} - z_0)R + o(1)\right]Q_n(\lambda) + \left[R + \frac{1}{z_0^2}(\mathscr{I} - R) + o(1)\right]Q_{n-1}(\lambda).$$

Taking $\lambda = \frac{1}{2}(z^{-1} + z)$, $\lambda_0 = \frac{1}{2}(z_0^{-1} + z_0)$, $|z| < 1$, $z = z_k$, and substituting the asymptotic formulas for $Q_n(\lambda)$, we obtain

$$\frac{1}{2}\eta_{n+1}h_n\left[\frac{1}{2}\left(z + \frac{1}{z}\right)\right] = \frac{1}{\sqrt{2\pi}}\frac{1}{z_0}\left\{\frac{z_0 - z}{1 - zz_0}\mathscr{I} + z\frac{z_0^2 - 1}{zz_0 - 1}R\right\}z^{-n}D^{-1}(z)[1 + o(1)].$$

Let

$$\tilde{\mathfrak{B}}(z) = \frac{1}{z_0}\left\{\frac{z_0 - z}{1 - zz_0}(\mathscr{I} - R) + z_0R\right\}.$$

Thus,

$$\frac{1}{2}\eta_{n+1}h_n\left[\frac{1}{2}(z + z^{-1})\right] = \frac{1}{\sqrt{2\pi}}z^{-n}[D(z)\tilde{\mathfrak{B}}^{-1}(z)](1 + o(1)).$$

The factor $\tilde{\mathfrak{B}}(z)$ is a Blaschke factor. Consider the product

$$\tilde{\mathfrak{B}}(e^{i\theta})\tilde{\mathfrak{B}}(e^{i\theta})^* = z_0^{-2}\left\{\left|\frac{z_0 - e^{i\theta}}{1 - z_0e^{i\theta}}\right|(\mathscr{I} - R) + z_0^2R\right\} = \frac{1}{z_0^2}[(\mathscr{I} - R) + z_0^2R].$$

We introduce the factor

$$\beta = \left\{\frac{1}{z_0^2}[(\mathscr{I} - R) + z_0R]\right\}^{-1} = z_0(\mathscr{I} - R) + R$$

and let

$$\mathfrak{B}(z) = \beta\tilde{\mathfrak{B}}(z) = \frac{z_0 - z}{1 - zz_0}(\mathscr{I} - R) + R.$$

The factor β is an invertible matrix that commutes with $\tilde{\mathfrak{B}}(z)$. We have

$$\mathfrak{B}(e^{i\theta})\mathfrak{B}(e^{i\theta})^* = \mathscr{I}, \qquad \mathfrak{B}(0) = \beta. \tag{3.9}$$

Let

$$H_n(\lambda) = \tfrac{1}{2}\beta\eta_{n+1}h_n(\lambda)$$

and

$$D_{\sigma_*}(z) = D(z)\mathfrak{B}^{-1}(z). \tag{3.10}$$

The asymptotic formulas

$$H_n\left[\frac{1}{2}(z+z^{-1})\right]=\frac{1}{\sqrt{2\pi}}z^{-n}D_{\sigma_*}^{-1}(z)[1+o(1)]$$

hold uniformly on compact subsets of $\{|z|<1\}\setminus\{z_k\}_{k=0}^{M}$, and

$$H_n(\cos\theta)=\frac{1}{\sqrt{2\pi}}\left[e^{-in\theta}D_{\sigma_*}^{-1}(e^{i\theta})+e^{in\theta}D_{\sigma_*}(e^{-i\theta})\right]+o(1),$$

where $o(1)$ is understood in the sense of $L_\eta^2(\mathbf{C}^N)_{[0,2\pi]}$, and

$$\eta(\theta)=\begin{cases}-\sigma_*(\cos\theta)|\sin\theta|, & 0\leqslant\theta\leqslant\pi,\\ \sigma_*(\cos\theta)|\sin\theta|, & \pi\leqslant\theta\leqslant 2\pi.\end{cases}$$

The first of these formulas has already been obtained, and the second can be obtained in exactly the same way. We now find the asymptotic behavior of the polynomials $H_n(\lambda)$ at the point $\lambda=\lambda_0$. This asymptotic expression will consist of two parts. We have that

$$\left|\beta^{-1}H_n(\lambda_0)=\frac{1}{2}\frac{d}{d\lambda}\left[Q_{n+1}(\lambda)+\eta_{n+1}\alpha_nQ_n(\lambda)+\eta_{n+1}\beta_nQ_{n+1}(\lambda)\right]\right|_{\lambda=\lambda_0}.$$

An asymptotic expression for the derivatives $Q_n'[\frac{1}{2}(z+z^{-1})]$ can obviously be obtained by differentiating the asymptotic expression for $Q_n[\frac{1}{2}(z+z^{-1})]$. This can be obtained by using the Cauchy formula. We have that

$$\begin{aligned}\beta^{-1}H_n(\lambda_0)&=\frac{1}{\sqrt{2\pi}}\left(1-z_0^{-2}\right)^{-1}\\&\quad\times\Big\{-(n+1)z_0^{-(n+2)}D_{(z_0)}^{-1}-z_0^{-(n+1)}\left[D^{-1}(z)\right]_{z=z_0}'\\&\qquad+\left[-2z_0^{-1}+\left(z_0^{-1}-z_0\right)R\right]\left[-nz_0^{-n-1}D^{-1}(z_0)-z_0^{-n}\left[D^{-1}(z)\right]_{z=z_0}'\right]\\&\qquad+\left[z_0^{-2}(\mathscr{I}-R)+R\right]\left[-(n-1)z_0^{-n}D^{-1}(z_0)-z_0^{-n+1}\left[D^{-1}(z)\right]_{z=z_0}'\right]\Big\}\\&\qquad\times(1+o(1))\\&=\frac{1}{\sqrt{2\pi}}z_0^{-n}RD^{-1}(z_0)(1+o(1)).\end{aligned}$$

Thus, we get that

$$H_n(\lambda_0)=\frac{z_0^{-n}}{\sqrt{2\pi}}\beta RD^{-1}(z_0)[1+o(1)].\tag{3.11}$$

Considering that $\beta R=R$ and

$$D_{\sigma_*}^{-1}(z_0)=RD^{-1}(z_0),$$

we see that the operator $D\sigma_*(z_0)$ does not exist, but the operator $D_{\sigma_*}^{-1}(z_0)=\lim_{z\to z_0}D_{\sigma_*}^{-1}(z)$ exists, although it is singular. The formula for $H_n(\lambda_0)$ takes the previous form

$$H_n(\lambda_0)=\frac{1}{\sqrt{2\pi}}z_0^{-n}D_{\sigma_*}^{-1}(z_0)[1+o(1)];$$

as is easy to see, $\operatorname{Ker}D_{\sigma_*}^{-1}(z_0)=(\operatorname{Ker}\sigma_0)^{\perp}$.

We now find an asymptotic expression for $H_n(\lambda_0)\sigma_0$. To do this we turn to (3.5). It implies that

$$-\beta^{-1}H_n(\lambda_0)\sigma_0 = \frac{1}{2}\int_{-\infty}^{\infty}\frac{1}{\lambda-\lambda_0}Q_{n+1}(\lambda)\,d\sigma(\lambda)Q_{n+1}(\lambda)^*Q_{n+1}^{-1}(\lambda_0)^*$$
$$+\frac{1}{2}\eta_{n+1}\alpha_n\int_{-\infty}^{\infty}\frac{1}{\lambda-\lambda_0}Q_n(\lambda)\,d\sigma(\lambda)Q_n(\lambda)^*Q_n^{-1}(\lambda_0)^*$$
$$+\frac{1}{2}\eta_{n+1}\beta_n\int_{-\infty}^{\infty}\frac{1}{\lambda-\lambda_0}Q_{n-1}(\lambda)\,d\sigma(\lambda)Q_{n-1}(\lambda)^*Q_{n-1}^{-1}(\lambda_0)^*.$$

Using the relation

$$\lim_{n\to\infty}\int_{-\infty}^{\infty}\frac{1}{\lambda-\lambda_0}Q_n(\lambda)\,d\sigma(\lambda)Q_n(\lambda)^*$$
$$=\frac{1}{2\pi}\mathscr{I}\int_0^{2\pi}\frac{d\theta}{\cos\theta-\lambda_0}=-\frac{2}{z_0\left(1-1/z_0^2\right)},$$

where $\lambda_0=\frac{1}{2}(z_0+z_0^{-1})$, we get that

$$-\beta^{-1}H_n(\lambda_0)\sigma_0$$
$$=-\sqrt{2\pi}\left(z_0-z_0^{-1}\right)^{-1}\left\{z_0^{n+1}D(z_0)^*+\left[-2z_0^{-1}+\left(z_0-z_0^{-1}\right)R\right]\right.$$
$$\left.\times z_0^nD(z_0)^*+\left[z_0^{-2}(\mathscr{I}-R)+R\right]z_0^{n-1}D(z_0)^*\right\}(1+o(1))$$
$$=-\sqrt{2\pi}\,z_0^n\left(z-z_0^{-1}\right)\left[z_0\left(1-z_0^{-2}\right)^2\mathscr{I}-z_0\left(1-z_0^{-2}\right)R\right]D(z_0)^*[1+o(1)].$$

Thus,

$$\beta^{-1}H_n(\lambda_0)\sigma_0=\left(\frac{1}{z_0^2}-1\right)\sqrt{2\pi}\,z_0^n[\mathscr{I}-R]D(z_0)^*[1+o(1)].$$

This representation implies at once that

$$H_n(\lambda_0)\sigma_0H_n(\lambda_0)^*=o(1),\qquad n\to\infty.$$

Part b) of the induction hypothesis is thus proved. (We have verified b) only for the point λ_0, but this is completely sufficient in view of equivalence of the points $\{\lambda_k\}_0^M$.)

The formula for $H_n(\lambda_0)\sigma_0$ can be rewritten in the form

$$(H(\lambda_0)\sigma)^*=\sqrt{2\pi}\,z_0^{n-1}\operatorname*{res}_{z=z_0}D_{\sigma_*}(z)[1+o(1)].$$

It is also not hard to see that

$$\operatorname{Ker}\Big(\operatorname*{res}_{z=z_0}D_{\sigma_*}(z)\Big)^*=\operatorname{Ker}(\mathscr{I}-R)D(z_0)=\left[\operatorname{im}D(z_0)-(\mathscr{I}-R)\right]^{\perp}$$
$$=\left\{D(z_0)\left[D^{-1}(z_0)(\operatorname{Ker}\sigma_0)^{\perp}\right]\right\}^{\perp}=\operatorname{Ker}\sigma_0$$

and

$$\operatorname{im}\operatorname*{res}_{z=z_0}D_{\sigma_*}(z)=(\operatorname{Ker}\sigma_0)^{\perp}.$$

We now list some properties of the function $D\sigma_*(z)$. First, note that the function $D\sigma_*(z)$ was only supplemented by a Blaschke factor which added a single first-order pole at z_0. Therefore, its boundary properties remained the same as for the function $D(z)$. From (3.9) and (3.10) it follows that

$$D_{\sigma_*}(e^{i\theta})D_{\sigma_*}(e^{i\theta})^* = \sigma'(\cos\theta)|\sin\theta|, \qquad \theta \in E \subseteq [0,2\pi], \quad \text{meas}\, E = 2\pi.$$

This equality gives us the induction hypothesis a) and the fact that the polynomials $H_n(\lambda)$ are normalized:

$$\int_{-\infty}^{\infty} H_n(\lambda)\, d\sigma(\lambda) H_n(\lambda)^* = \mathscr{I} + o(1).$$

We now turn to part c) of the induction. The leading coefficient in the polynomial $H_n(\lambda)$ is $2^{-1}\beta\eta_{n+1}$. Therefore,

$$\lim_{n\to\infty} 2^{-(n+1)}\beta\eta_{n+1} = \beta\eta = \beta D^{-1}(0) = D_{\sigma_*}^{-1}(0).$$

This completes the induction.

5. We summarize the statements proved in subsections 1–4 and in §2.

THEOREM 2. *Let σ be a matrix-valued measure concentrated on $[-1,1]$ and at finitely many points*

$$\lambda_k = \tfrac{1}{2}(z_k + z_k^{-1}), \qquad -1 < z_k < 1,\, z_k \neq 0,$$

where $k = 1,\dots,M$. Assume that the measure σ satisfies the Szegő condition

$$\int_{-1}^{1} \frac{\ln\det\sigma'(\lambda)}{\sqrt{1-\lambda^2}}\, d\lambda > -\infty.$$

Let σ_k be the matrix-valued mass concentrated at the point λ_k. Fix an arbitrary sequence of orthonormal polynomials

$$H_n(\lambda) = \kappa_n\lambda^n + \cdots, \qquad n = 0,1,2,\dots.$$

Let $\kappa_n = \zeta_n\varepsilon_n$, where $\varepsilon_n > 0$ is a Hermitian matrix and ζ_n is an orthogonal matrix. Assume that the limit

$$\lim_{n\to\infty} \zeta_n \tag{3.12}$$

exists. Then there exists a unique meromorphic matrix-valued function $D(z)$ on the disk $\{|z| < 1\}$ with the following properties:

a) *$D(z)$ is holomorphic in $\{|z| < 1\}\setminus\{z_k\}_{k=1}^M$ and has simple poles at the points $\{z_k\}$.*

b) *$D(z)$ is nonsingular in $\{|z| < 1\}\setminus\{z_k\}_{k=1}^M$ and becomes a member of the class $\mathscr{H}^2(\mathbf{C}^N)$ after the principal parts at $z_1,\dots,z_M$ are subtracted off.*

c) *The boundary values $D(e^{i\theta})$ satisfy the equality*

$$D(e^{i\theta})D(e^{i\theta})^* = \sigma'(\cos\theta)|\sin\theta|$$

for almost all $\theta \in [0,2\pi]$.

d) *There exists a Hermitian matrix-valued integrable function $M(\theta)$, uniquely determined by $\sigma'(\cos\theta)$, such that*

$$\operatorname{tr} M(\theta) = \ln\det\{\sigma'(\cos\theta)|\sin\theta|\}$$

and such that

$$D(z) = \overset{\curvearrowright}{\int_0^{2\pi}} \exp\left\{\frac{e^{i\theta}+z}{e^{i\theta}-z} M(\theta)\, d\theta\right\} \cdot \overset{\frown}{\prod_{k=1}^{M}} \mathfrak{B}_k^{-1}(z)\cdot\zeta,$$

where $\overset{\curvearrowright}{\int_0^{2\pi}}$ *is the multiplicative integral,* $\overset{\curvearrowright}{\prod}_{k=1}^{M}$ *is the matrix-valued Blaschke product with poles at the points* z_k, $k = 1,\dots,M$, *and* ζ *is an orthogonal matrix.*

e) *The relation*

$$H_n[2^{-1}(z + z^{-1})] = (2\pi)^{1/2} z^{-1} D^{-1}(z)[1 + o(1)]$$

holds uniformly on compact subsets of $\{|z| < 1\} \setminus \{z_k\}_{k=1}^{M}$.

f) *For* $\theta \in [0, 2\pi]$

$$H_n(\cos\theta) = (2\pi)^{1/2}[e^{-in\theta} D^{-1}(e^{i\theta}) + e^{in\theta} D^{-1}(e^{-i\theta})] + o(1),$$

where $o(1)$ *is understood in the sense of* $L^2_\eta(\mathbf{C}^N)_{[0,2\pi]}$, *with*

$$\eta(\theta) = \begin{cases} -\sigma(\cos\theta), & 0 \leqslant \theta \leqslant \pi, \\ \sigma(\cos\theta), & \pi \leqslant \theta \leqslant 2\pi, \sigma(-1) = 0 \end{cases}$$

(*the same* $o(1)$ *can be understood in the sense of convergence in Lebesgue measure on* $[0, 2\pi]$).

g) *At the point* z_k

$$H_n[2^{-1}(z_k + z_k^{-1})] = (2\pi)^{-1/2} z_k^{-n} D^{-1}(z_k)[1 + o(1)],$$

where $D^{-1}(z_k)$ *is understood as* $\lim_{z \to z_k} D^{-1}(z)$, *and*

$$\operatorname{Ker} D^{-1}(z_k) = (\operatorname{Ker} \sigma_k)^{\perp}.$$

h) *At* z_k

$$\left(H_n[2^{-1}(z_k + z_k^{-1})]\sigma_k\right)^* = (2\pi)^{1/2} z_k^{n-1} \operatorname*{res}_{z=z_k} D(z)[1 + o(1)],$$

and

$$\operatorname{Ker}\left(\operatorname*{res}_{z=z_k} D(z)\right)^* = \operatorname{Ker}\sigma_k, \qquad \operatorname{im} \operatorname*{res}_{z=z_k} D(z) = (\operatorname{Ker}\sigma_k)^{\perp}.$$

i)

$$\lim_{n\to\infty} \frac{\kappa_n}{2^n} = D^{-1}(0).$$

Only a renaming of quantities is needed to prove Theorem 2. Let $T_n(\lambda) = \eta_n \lambda^n + \cdots$ be polynomials orthonormal with respect to the measure σ and satisfying all the parts a)–i) of Theorem 2, and let $D_1(z)$ be the function corresponding to them. Such orthonormal polynomials exist according to §2 and subsections 1–4 of §3. Let

$$\mathscr{K}_n(\lambda) = \varepsilon_n \lambda^n + \cdots$$

be polynomials orthonormal with respect to the measure σ and satisfying $\varepsilon_n^* = \varepsilon_n$. Clearly,

$$\mathscr{K}_n(\lambda) = \varepsilon_n \eta^{-1} T_n(\lambda)$$

and $\varepsilon_n \eta_n^{-1}$ is an orthogonal matrix on $\mathbf{C}^N$, i.e., $\varepsilon_n \eta_n^{-1} \eta_n^{*-1} \varepsilon_n = \mathscr{I}$. Hence, $\varepsilon_n = \sqrt{\eta_n^* \eta_n}$. Therefore,

$$\lim_{n\to\infty} \varepsilon_n \eta_n^{-1} = \lim_{n\to\infty} \sqrt{\eta_n^* \eta_n}\, \eta_n^{-1} = \sqrt{D_1^{*-1}(0) D_1^{-1}(0)}\, D_1(0).$$

It is now clear that (3.12) gives us the limit relation

$$\lim_{n\to\infty} \zeta_n \varepsilon_n \eta_n^{-1} = \zeta.$$

Thus,

$$H_n(\lambda) = \zeta_n \varepsilon_n \eta_n^{-1} T_n(\lambda) = [\zeta + o(1)] T_n(\lambda).$$

Let

$$D(z) = D_1(z)\zeta^{-1}.$$

It is not hard to see that the function $D(z)$ has all the properties in a)–i).

§4. The scattering problem

1. Let $\mathscr{L}$ be the Sturm-Liouville operator in $l_2(\mathbf{C}^N)$ with parameters $\{E_j, V_j\}_{j=0}^{\infty}$. Assume that the V_j are Hermitian and the E_j are invertible. Let $\{Q_n(\lambda)\}$ be the polynomials determined by the recursion relations

$$Q_0 = \mathscr{I}, \qquad E_0 Q_1(\lambda) = (\lambda\mathscr{I} - V_0),$$
$$E_{n-1}Q_n(\lambda) = (\lambda\mathscr{I} - V_{n-1})Q_{n-1}(\lambda) - E_{n-2}^* Q_{n-2}, \qquad n = 2, 3, \ldots.$$

If $f_0 \in \mathbf{C}^N$, then the vector

$$h = \{Q_0 f_0, Q_1 f_0, Q_2 f_0, \ldots, Q_n f_0, \ldots\}$$

formally satisfies the relation

$$\mathscr{L}h = \lambda h.$$

For λ in the spectrum of $\mathscr{L}$ there arises the problem of determining the asymptotic properties of the "eigenvectors" $h = (f_0, f_1, \ldots)$ as $n \to \infty$. This problem is obviously equivalent to the study of the asymptotic properties of the sequence $\{Q_n(\lambda)\}_{n=0}^{\infty}$ of operators on $\mathbf{C}^N$, and is called the direct scattering problem. The inverse scattering problem consists in recovering the operator $\mathscr{L}$ from the known asymptotic properties of the coordinates of the "eigenvectors" h for λ in the spectrum of $\mathscr{L}$ (see, for example, [6]).

2. In point of fact, Theorem 2 gives a solution of the direct scattering problem. Here we assume that $\mathscr{L}$ is such that its spectral measure σ satisfies the Szegő condition (3.1) on $[-1,1]$ and has only finitely many masses outside $[-1,1]$. It is proved in [7] that in a certain sense the Szegő condition is necessary for the existence of scattering data when $N = 1$. Therefore, the requirements imposed on σ are hardly too restrictive. We have not investigated the question of conditions on the parameters $\{E_j, V_j\}$ such that the spectral measure σ satisfies the requirements of Theorem 2, except for the case $N = 1$ in the second author's paper [7]. It turns out that the Szegő condition is equivalent to the relation

$$0 < \prod_{k=0}^{\infty} 4E_k^2 < \infty, \qquad E_k \in \mathbf{R}^1.$$

As asserted in [6], the number of eigenvalues outside $[-1,1]$ is finite if the Marchenko condition holds:

$$\sum_{k=0}^{\infty} k\left\{\|E_k - \tfrac{1}{2}\mathscr{I}\| + \|V_k\|\right\} < \infty. \tag{4.1}$$

This condition can be weakened somewhat when $N = 1$ (see [7]). We remark that if (4.1) holds, then the limit

$$\lim_{n\to\infty} \frac{1}{2^n} E_{n-1}^{-1} E_{n-2}^{-1} \cdots E_0^{-1}$$

exists. This suffices for us to be able to apply Theorem 2.

Accordingly, the following conditions are required for the solution of the direct scattering problem.

1. The matrix-valued spectral measure σ of $\mathscr{L}$ is concentrated on $[-1,1]$ and at finitely many points $\{\lambda_k\}_{k=1}^{M}$ outside $[-1,1]$.

2. On $[-1,1]$ the spectral measure satisfies the Szegő condition

$$\int_{-1}^{1} \frac{\ln \det \sigma'(\lambda)}{\sqrt{1-\lambda^2}}\,d\lambda > -\infty,$$

where $\sigma'(\lambda)$ is the derivative, which exists almost everywhere on $[-1,1]$.

3. The limit

$$\lim_{n\to\infty} \frac{1}{2^n} E_{n-1}^{-1} E_{n-2}^{-1} \cdots E_0^{-1}$$

exists.

4. The matrices E_j are invertible, and the V_j are Hermitian (selfadjoint).

Under these assumptions the application of Theorem 2 leads to the assertion that the sequence of matrices

$$Y_n(\lambda) = \sqrt{2\pi}\, Q_n\left[\frac{1}{2}(z+z^{-1})\right] D(z), \qquad \lambda = \frac{1}{2}\left(z+\frac{1}{z}\right), z = z_j,$$

is a solution of the system of recursion relations (a solution of the matrix discrete Sturm-Liouville equation)

$$Y_{-1} = 0, \qquad Y_0 = \mathscr{I},$$
$$E_{n-1} Y_n(\lambda) + V_{n-1} Y_{n-1}(\lambda) + E_{n-2}^{*} Y_{n-2}(\lambda) = \lambda Y_{n-1}(\lambda), \qquad n = 1,2,\ldots.$$

For $\lambda \in [-1,1]$, $\lambda = \cos\theta$, the asymptotic behavior of $Y_n(\cos\theta)$ is given by the relation

$$Y_n(\cos\theta) = \mathscr{I} e^{-in\theta} + S(\theta) e^{in\theta} + o(1),$$

where $o(1)$ is understood in the sense of convergence in Lebesgue measure on $[0,2\pi]$.

The matrix

$$S(\theta) = \left[D(e^{i\theta})\right]^{-1} D(e^{i\theta}) \tag{4.2}$$

is called the scattering matrix, and the relation (4.2) is a consequence of Theorem 2. To determine the scattering data at the point λ_j let

$$f(\lambda) = \begin{cases} \mathscr{I}, & \lambda = \lambda_j, \\ 0, & \lambda \neq \lambda_j. \end{cases}$$

The polynomials $\{Q_n(\lambda)\}$ form a complete orthonormal system in L_σ^2. (The completeness is a consequence of the fact that any continuous matrix-valued function on a finite interval can be approximated with any accuracy by matrix-valued polynomials.) Let

$$A_n = \int_{-\infty}^{\infty} f(\lambda)\, d\sigma(\lambda) Q_n(\lambda)^{*}.$$

Then

$$\sum_{n=0}^{\infty} A_n Q_n(\lambda) \overset{L_\sigma^2}{=} f(\lambda).$$

Furthermore, $A_n = \sigma_j(Q_n(\lambda_j))^*$. Thus,

$$\sum_{n=0}^{\infty} \sigma_j Q_n(\lambda_j)^* Q_n(\lambda_j) = \mathscr{I}. \tag{4.3}$$

Let us now consider all the eigenvectors of $\mathscr{L}$ corresponding to λ_j. Let $F = \{f_0, f_1, f_2, \dots\}$ be an eigenvector of $\mathscr{L}$. Then $f_k = Q_k(\lambda_j) f_0$. The condition $F \in l_2(\mathbf{C}^N)$ and the asymptotic formulas for $Q_k(\lambda_j)$ in Theorem 2 allow us to assert that

$$f_0 \in W_j = (\operatorname{Ker} \sigma_j)^{\perp}.$$

We include the subspaces $\{W_j\}_1^M$ in the scattering data.

Let $\sqrt{\sigma_j}$ denote the selfadjoint positive square root, and $\sigma_j^{-1/2}$ its inverse operator, which acts on W_j. Relation (4.3) implies that for any $x \in W_j$ with $\|x\| = 1$

$$\sum_{n=0}^{\infty} \left\| Q_n(\lambda_j)\sqrt{\sigma_j}\, x \right\|_{\mathbf{C}^N}^2 = 1.$$

Thus, $\{Q_n(\lambda_j)\sqrt{\sigma_j}\, x\}_{n=0}^{\infty} \in l_2$, and it is a normalized eigenvector of $\mathscr{L}$. The asymptotic behavior of the coordinates of the vector is given by Theorem 2:

$$Q_n(\lambda_j)\sqrt{\sigma_j} = \sqrt{2\pi}\, z_j^{n-1}\Big(\operatorname*{res}_{z=z_j} D(z)\Big)^* \sigma_j^{-1/2}[1 + o(1)].$$

Let

$$\Pi_j = \frac{1}{z_j}\sqrt{2\pi}\Big(\operatorname*{res}_{z=z_j} D(z)\Big)^* \sigma_j^{-1/2}. \tag{4.4}$$

Then for $x \in W_j$

$$Q_n(\lambda_j)\sqrt{\sigma_j}\, x = z_j^n \Pi_j x[1 + o(1)].$$

We include the operators Π_j in the scattering data.

3. Thus, by the scattering data of the operator $\mathscr{L}$ we shall understand the collection

$$\{S(\theta), 0 \leqslant \theta \leqslant 2\pi; \lambda_1, \dots, \lambda_M; W_1, W_2, \dots, W_M; \Pi_1, \Pi_2, \dots, \Pi_M\},$$

where $S(\theta)$ is the scattering matrix, $\lambda_1, \dots, \lambda_M$ are the points of the discrete spectrum of $\mathscr{L}$, and the W_j are subspaces of $\mathbf{C}^N$. The subspace W_j is the projection of the eigenspace (λ_j) of $\mathscr{L}$ on the first coordinate (f_0).

The inverse scattering problem consists in determining the operator $\mathscr{L}$ (i.e., its parameters E_k and V_k) from the scattering data. The key to solving this problem is the set of relations

$$D(e^{i\theta})D(e^{i\theta})^* = \sigma'(\cos\theta)|\sin\theta|, \qquad [D(e^{-i\theta})]^{-1}D(e^{i\theta}) = S(\theta),$$

$$W_j = (\operatorname{Ker}\sigma_j)^{\perp}, \quad j = 1, 2, \dots, M, \qquad \operatorname{Ker}\Big(\operatorname*{res}_{z=z_j} D(z)\Big)^* = \operatorname{Ker}\sigma_j,$$

$$\Pi_j = z_j^{-1}\sqrt{2\pi}\Big(\operatorname*{res}_{z=z_j} D(z)\Big)^* \sigma_j^{-1/2},$$

which connect the scattering data and the spectral measure. Therefore, only the spectral measure of $\mathscr{L}$ can be found from the scattering data. The Markov function

$$\hat{\sigma}(z)=\int_{-\infty}^{\infty}\frac{d\sigma(z)}{z-\lambda}=\cfrac{1}{z-V_0-E_0\cfrac{1}{z-V_1-E_1\cfrac{1}{z-V_2-\ddots}E_1^*}E_0^*},\tag{4.5}$$

connecting the spectral measure and the parameters of $\mathscr{L}$ is found in a unique way from the spectral measure. It is immediately clear from (4.5) that $E_0E_0^*$ is the coefficient of $-1/z$ in the power series expansion of the denominator. Just this product is determined from $\hat{\sigma}(z)$. Of course, it is impossible to determine E_0 uniquely from the product $E_0E_0^*$. However, this is possible by the theorem on a unique Hermitian positive-definite square root if we require beforehand that E_0 be a positive Hermitian matrix.

Accordingly, it is assumed that

$$E_k=E_k^*,\qquad E_k>0,\quad k=0,1,2,\ldots.$$

We thereby eliminate one (not very important) reason for the nonuniqueness of the solution of the inverse problem.

Another (essential) reason for nonuniqueness is that the scattering data does not depend on the singular component of the measure σ on $[-1,1]$, while the parameters $\{E_j,V_j\}$ depend on it in an essential way. To remove this reason we should choose a class of operators $\mathscr{L}$ (in terms of the parameters $\{E_j,V_j\}$) in such a way that all its operators have absolutely continuous spectral measures on $[-1,1]$. For $N=1$ the article [5] of Guseĭnov gives us that the class of operators satisfying the Marchenko condition (4.1) is such a class. Another class of such operators was indicated in [7], but is difficult to check the condition for an operator to belong to that class. However, it is not clear (for $N=1$) whether the condition (4.1) is order-sharp. For $N>1$ a theorem on absolute continuity of the measure under the Marchenko condition (4.1) has not been stated anywhere in explicit form, but it is very likely.

We put this question aside and assume that σ is *absolutely continuous* on $[-1,1]$.

4. Suppose that σ satisfies the Szegő condition (3.1) on $[-1,1]$. We consider an algorithm for reconstructing the measure from the scattering data. Taking into account the relations

$$\operatorname{Ker}\Big(\operatorname*{res}_{z=z_j}D(z)\Big)^*=\operatorname{Ker}\sigma_j,\qquad \Pi_j=\frac{1}{z_j}\sqrt{2\pi}\Big(\operatorname*{res}_{z=z_j}D(z)\Big)^*\sigma_j^{-1/2},$$

we find that

$$\operatorname{Ker}\operatorname*{res}_{z=z_j}D(z)=\big(\operatorname{im}\Pi_j\big)^{\perp}=F_j.$$

Using Blaschke products and constructing the factors $\mathfrak{B}_j(z)$ successively, we get the matrix-valued function

$$\mathscr{D}(z)=D(z)\mathfrak{B}_1(z)\mathfrak{B}_2(z)\cdots\mathfrak{B}_M(z),$$

which does not have zeros in $\{|z| < 1\}$ and is analytic there. The factors $\mathfrak{B}_j(z)$ have the form

$$\mathfrak{B}_j(z) = \frac{z - z_j}{1 - zz_j}(\mathscr{I} - R_j) + R_j,$$

where the orthogonal projections R_j are constructed only from the subspaces F_j in an inductive fashion (see [7]). We have that

$$\mathscr{D}(e^{i\theta})[\mathscr{D}(e^{i\theta})]^* = D(e^{i\theta})[D(e^{i\theta})]^* = \sigma'(\cos\theta)|\sin\theta|$$

and

$$\begin{aligned} S_1(\theta) &= [\mathscr{D}(e^{i\theta})]^{-1}\mathscr{D}(e^{i\theta}) \\ &= \mathfrak{B}_M^{-1}(e^{-i\theta})\mathfrak{B}_{M-1}^{-1}(e^{-i\theta}) \cdots \mathfrak{B}_1^{-1}(e^{-i\theta})D^{-1}(e^{-i\theta})D(e^{i\theta})\mathfrak{B}_1(e^{i\theta}) \cdots \mathfrak{B}_M(e^{i\theta}) \\ &= \mathfrak{B}_M^{-1}(e^{-i\theta}) \cdots \mathfrak{B}_1^{-1}(e^{-i\theta})S(\theta)\mathfrak{B}_1(e^{i\theta}) \cdots \mathfrak{B}_M(e^{i\theta}). \end{aligned}$$

Thus, we come to the problem of determining an invertible function $\mathscr{D}(z)$ that is analytic interior to $\{|z| \leqslant 1\}$ and satisfies

$$\mathscr{D}^{-1}(e^{-i\theta})\mathscr{D}(e^{i\theta}) = S_1.$$

Suppose that $\mathscr{D}(z)$ has been determined. Then we find at once that

$$\sigma'(\cos\theta)|\sin\theta| = \mathscr{D}(e^{i\theta})[\mathscr{D}(e^{i\theta})]^*.$$

We find also the main function

$$D(z) = \mathscr{D}(z)\mathfrak{B}_M^{-1}(z)\mathfrak{B}_{M-1}^{-1}(z) \cdots \mathfrak{B}_1^{-1}(z).$$

The relation

$$\Pi_j = z_i\sqrt{2\pi}\Big(\operatorname*{res}_{z=z_j} D(z)\Big)^*\sigma_j^{-1/2}$$

gives us that

$$\sigma_j = 2\pi z_j^{-2}\left[\Pi_j^{-1}\Big(\operatorname*{res}_{z=z_j} D(z)\Big)^*\right]^2.$$

Here σ_j is regarded as an invertible operator on W_j and Π_j as an invertible operator from W_j to $F_j^{\perp} = \operatorname{im}\Pi_j$, and

$$\operatorname{im}\Big(\operatorname*{res}_{z=z_j} D(z)\Big)^* = \Big(\operatorname{Ker}\operatorname*{res}_{z=z_j} D(z)\Big)^{\perp} = \operatorname{im}\Pi_j.$$

We thereby determine σ_j on $W_j = (\operatorname{Ker}\sigma_j)^{\perp}$. The spectral measure is thus completely determined, and the parameters $\{E_j, V_j\}$ of $\mathscr{L}$ are determined by using a continued fraction:

$$E_j = E_j^*, \qquad E_j > 0, \qquad V_j = V_j^*.$$

5. We remark that the relations

$$DD^* = \sigma, \qquad D^{*-1}D = S \tag{4.6}$$

connecting the spectral measure of the operator and the scattering matrix by means of the Szegő functions are well known for differential operators (see [19]). Here the analogue of the Szegő function is the so-called Jost function. However, these relations are not emphasized in investigations concerning the scattering problem for differential operators, a circumstance partially explained by the existence of the Marchenko method (an alternative to the method of Newton, Gel'fand, and Levitan), which is more convenient in physical applications of the inverse scattering problem. However, in our view the relations (4.6) are the key relations in the theoretical setting of the scattering problem.

§5. Conclusion

The authors are quite aware that to solve the scattering problem completely, effective ways are needed to find the solutions of the factorization problems

$$DD^* = \sigma', \qquad \left[D(e^{-i\theta})\right]^{-1} D(e^{i\theta}) = S(\theta). \tag{5.1}$$

It is clear that the methods proposed in other approaches to the scattering problem give various algorithms for constructing solutions of (5.1). We think it is useful to divide the scattering problem into three independent parts.

I. Determination of the spectral measure from the parameters of the Sturm-Liouville operator, and recovery of the operator $\mathscr{L}$ from the spectral measure.

In the case of a discrete Sturm-Liouville operator this problem can be regarded as solved in a theoretical setting. It is perfectly clear that the algorithms for solving the direct one of these problems (determination of the spectral measure from the operator) are closely connected with various computational methods in the theory of orthogonal polynomials. It is common to find the zeros of the orthogonal polynomials and the Christoffel coefficients and to approximate the spectral measure by a discrete measure. There are also other methods leading to smooth approximations of the spectral measure. The situation with the inverse problem (recovery of the operator from the spectral measure) is analogous.

This first part can be solved by methods in [1] in the case of a differential Sturm-Liouville operator, but it would be interesting to analyze other possible approaches involving, for example, approximation of a differential operator by a difference operator.

II. The second part of the scattering problem consists in the relations connecting the spectral measure with the scattering matrix. It is not excluded that the relations (5.1) bear a more general character than the situation in which their validity has already been proved. It would also be interesting to clear up other classes of operators for which (5.1) is true. At the same time, it is possible that the relations (5.1) do not hold in more complicated cases, and are replaced by other relations.

III. The third very essential part involves the solution of the factorization problems (5.1). If $N = 1$, so that $D(z)$ is an analytic scalar function, no problems arise: (5.1) can be solved by standard methods in function theory. Already for $N = 2$ it is impossible to write out an explicit solution of (5.1), and the existing methods (the Riemann problem, the inverse problem method) are not very effective, because they take into account the specific nature of the problem only to a small degree.

Of course, this division of the scattering problem into three parts is conditional, though it has long been present in not quite explicit form in the mathematical literature (see [20], [18] and [21]).

It is possible that the solution of the factorization problem in combination with the inverse spectral problem (III + I) gives simpler computational algorithms, so that the three parts of the scattering problem indicated above are to a large extent of theoretical interest. Nevertheless, it seems to us that such a partition of the problem into parts relating essentially to different areas of function theory is useful.

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