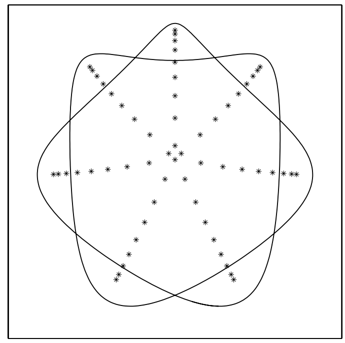

Chapter 9
Singular Values

The asymptotic behavior of the singular values of the matrices $T_n(b)$ is, in a sense, a mirror image of the topological properties of the symbol b: Roch and Silbermann's splitting phenomenon says that if b has no zeros on $\mathbf{T}$ and winding number k about the origin, then the $|k|$ smallest singular values of $T_n(b)$ go exponentially fast to zero while the remaining $n - |k|$ singular values stay away from zero. We also determine the limiting set of the singular values of $T_n(b)$ as $n \to \infty$, and we prove the Avram-Parter theorem, which identifies the corresponding limiting measure. As an application of the Avram-Parter theorem, we show that if x is a randomly chosen vector of length n and n is large, then $\|T_n(b)x\|_2^2$ clusters sharply around a certain value which, moreover, is much smaller than one would predict.

9.1 Approximation Numbers

For $j \in \{0, 1, \ldots, n\}$, let $\mathcal{F}_j^{(n)}$ denote the collection of all $n \times n$ matrices of rank at most j. The jth *approximation number with respect to the ℓ^p norm* of an $n \times n$ matrix A_n is defined by

$$a_j^{(p)}(A_n) = \mathrm{dist}_p(A_n, \mathcal{F}_j^{(n)}) := \min\left\{\|A_n - F_n\|_p : F_n \in \mathcal{F}_j^{(n)}\right\}.$$

Clearly, $0 = a_n^{(p)}(A_n) \le a_{n-1}^{(p)}(A_n) \le \cdots \le a_1^{(p)}(A_n) \le a_0^{(p)}(A_n) = \|A_n\|_p$. Put

$$\sigma_j^{(p)}(A_n) = a_{n-j}^{(p)}(A_n).$$

Thus, $0 = \sigma_0^{(p)}(A_n) \le \sigma_1^{(p)}(A_n) \le \cdots \le \sigma_{n-1}^{(p)}(A_n) \le \sigma_n^{(p)}(A_n) = \|A_n\|_p$. It is well known that in the case $p = 2$ the numbers $\sigma_j^{(p)}(A_n)$ are the *singular values* of a_n, that is, the nonnegative square roots of the eigenvalues $\lambda_j(A_n^* A_n)$ ($j = 1, \ldots, n$) of the matrix $A_n^* A_n$:

$$\sigma_j(A_n) := \sigma_j^{(2)}(A_n) = \sqrt{\lambda_j(A_n^* A_n)}. \tag{9.1}$$

The following results are standard.

Theorem 9.1 (singular value decomposition). *If A_n is an $n \times n$ matrix, then there exist unitary matrices U_n and V_n such that*

$$A_n = U_n \operatorname{diag}\,(\sigma_1(A_n), \dots, \sigma_n(A_n))\; V_n.$$

Theorem 9.2. *If $1 \le p \le \infty$ and A_n is an $n \times n$ matrix, then*

$$\sigma_1^{(p)}(A_n) = \begin{cases} 1/\|A_n^{-1}\|_p & \text{if } A_n \text{ is invertible}, \\ 0 & \text{if } A_n \text{ is not invertible}. \end{cases}$$

Theorem 9.3. *If A_n, B_n, C_n are $n \times n$ matrices and $1 \le p \le \infty$, then*

$$\sigma_j^{(p)}(A_n B_n C_n) \le \|A_n\|_p\, \sigma_j^{(p)}(B_n)\, \|C_n\|_p$$

for every $j \in \{0, 1, \dots, n\}$.

Given a Hilbert space H and $A \in \mathcal{B}(H)$, we define

$$\Sigma(A) = \left\{\sigma \in [0, \infty) : \sigma^2 \in \operatorname{sp} A^*A\right\}.$$

In particular, for an $n \times n$ matrix A_n we have

$$\Sigma(A_n) = \{\sigma_1(A_n), \dots, \sigma_n(A_n)\}.$$

Finally, we set

$$\Sigma_p(A_n) = \left\{\sigma_1^{(p)}(A_n), \dots, \sigma_n^{(p)}(A_n)\right\}.$$

9.2 The Splitting Phenomenon

The splitting phenomenon is the most striking property of the singular values (approximation numbers) of Toeplitz matrices. It is described by the following theorem. An illustration is in Figure 9.1.

Theorem 9.4 (Roch and Silbermann). *Let b be a Laurent polynomial and suppose $T(b)$ is Fredholm of index $k \in \mathbf{Z}$. Let α be any number satisfying (1.23) and $1 \le p \le \infty$. Then the $|k|$ first approximation numbers $\sigma_1^{(p)}(T_n(b)), \dots, \sigma_{|k|}^{(p)}(T_n(b))$ of $T_n(b)$ go to zero with exponential speed,*

$$\sigma_{|k|}^{(p)}(T_n(b)) = O(e^{-\alpha n}), \tag{9.2}$$

while the remaining $n - |k|$ approximation numbers $\sigma_{|k|+1}^{(p)}(T_n(b)), \dots, \sigma_n^{(p)}(T_n(b))$ stay away from zero,

$$\sigma_{|k|+1}^{(p)}(T_n(b)) \ge d > 0 \tag{9.3}$$

for all sufficiently large n, where d is a constant depending only on b.

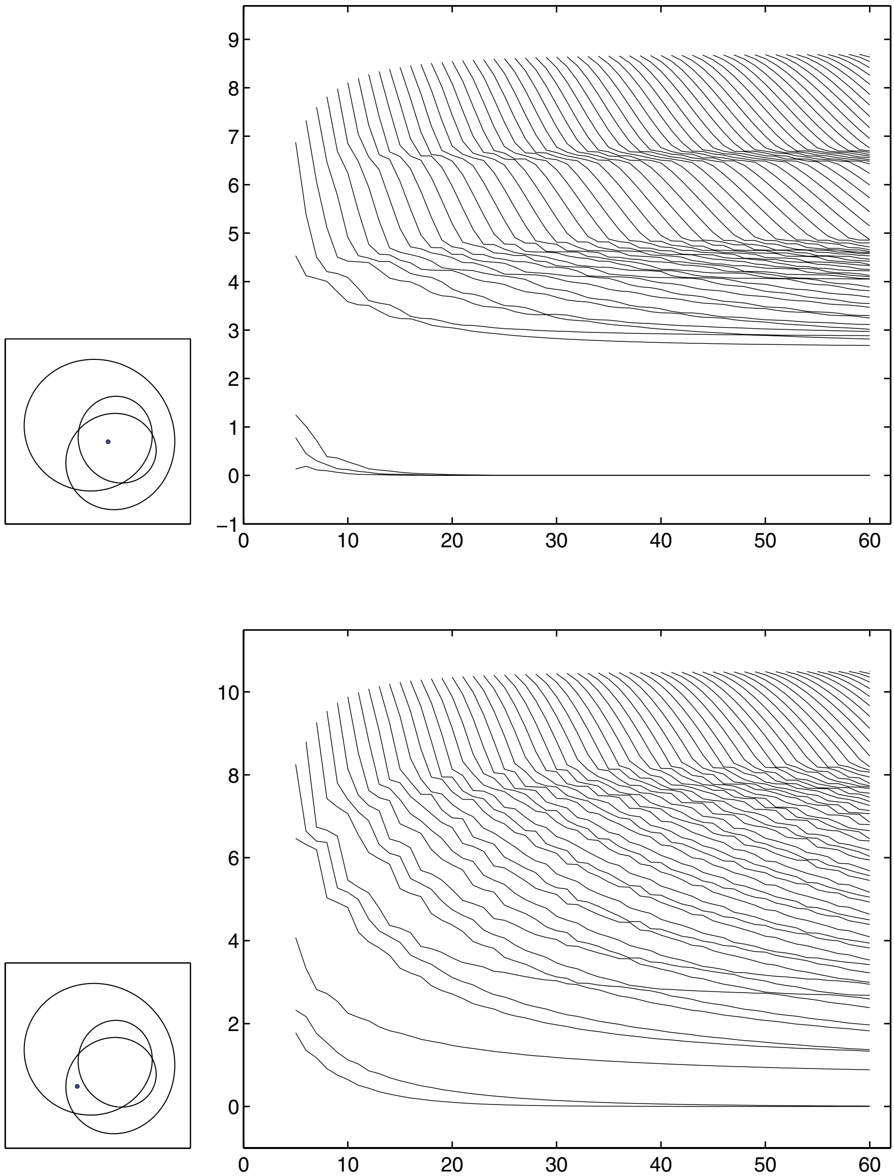

Figure 9.1. *In the small pictures we see the range of the symbol* $a(t) = t^{-1} - it + 2it^2 - 5it^3$*, which has winding numbers* 3 *and* 2 *about the origin and the point* $-3 - 2i$*, respectively. The singular values of* $T_n(a)$ *and* $T_n(a + 3 + 2i)$ *for* $5 \le n \le 60$ *are shown in the top and bottom pictures. As predicted by Theorem* 9.4*, three of them go to zero in the top picture and two of them approach zero in the bottom picture. The rest stays away from zero.*

Proof. We first prove (9.2). For the sake of definiteness, let us assume that $k > 0$; there is nothing to be proved for $k = 0$, and the case $k < 0$ can be reduced to the case $k > 0$ by passage to adjoints. Recall that χ_m is defined by $\chi_m(t) := t^m$ ($t \in \mathbf{T}$) and that $\mathcal{P}_n^+$ is the set of all analytic polynomials of degree at most $n-1$. Let $b = b_- \chi_{-k} b_+$ be a Wiener-Hopf factorization of b. If n is sufficiently large, then

$$c_n(t) := \left(\sum_{\ell=0}^{n-k} (b_+^{-1})_\ell t^\ell \right)^{-1}$$

is a function in W. Let $F_n : \mathcal{P}_n^+ \to \mathcal{P}_n^+$ be the linear operator that sends f to $\Pi_n(c_n \chi_{-k} b_- f)$, where Π_n is the orthogonal projection of $L^2(\mathbf{T})$ onto $\mathcal{P}_n^+$. For $j = 0, 1, \ldots, k-1$, the function $\chi_j c_n^{-1}$ belongs to $\mathcal{P}_n^+$, and we have

$$F_n(\chi_j c_n^{-1}) = \Pi_n(c_n \chi_{-k} b_- \chi_j c_n^{-1}) = \Pi_n(\chi_{j-k} b_-) = 0.$$

Hence, $\dim \operatorname{Im} F_n = n - \dim \operatorname{Ker} F_n \le n-k$. Let G_n be the matrix representation of F_n with respect to the basis $\{\chi_0, \chi_1, \ldots, \chi_{n-1}\}$ of $\mathcal{P}_n^+$. Then $G_n \in \mathcal{F}_{n-k}^{(n)}$ and thus, $\sigma_k^{(p)}(T_n(b)) = a_{n-k}^{(p)}(T_n(b)) \le \|T_n(b) - G_n\|_p$. Since $T_n(b) - G_n$ is the matrix representation of the linear operator $\mathcal{P}_n^+ \to \mathcal{P}_n^+$, $f \mapsto \Pi_n((b_+ - c_n)\chi_{-k} b_- f)$ in the basis $\{\chi_0, \chi_1, \ldots, \chi_{n-1}\}$, it follows that

$$\sigma_k^{(p)}(T_n(b)) \le \|b_+ - c_n\|_W \|\chi_{-k}\|_W \|b_-\|_W = O(\|b_+ - c_n\|_W).$$

From Lemma 1.17 we know that $\|b_+^{-1} - c_n^{-1}\|_W = O(e^{-\alpha n})$. This implies that $\|b_+ - c_n\|_W = O(e^{-\alpha n})$ and so gives (9.2).

We now prove inequality (9.3). This time we assume without loss of generality that $k = -j < 0$, since for $k = 0$ the assertion follows from Theorems 3.7 and 9.2 and for $k > 0$ we may pass to adjoints. We have $b = c\chi_j$, where c has no zeros on $\mathbf{T}$ and the winding number of c is zero. As $\|T(\chi_{-j})\|_p = 1$, we deduce from Theorem 9.3 that

$$\begin{aligned} \sigma_{j+1}^{(p)}(T_n(b)) &= \sigma_{j+1}^{(p)}(T_n(c\chi_j)) = \sigma_{j+1}^{(p)}(T_n(c\chi_j)) \, \|T(\chi_{-j})\|_p \\ &\ge \sigma_{j+1}^{(p)}(T_n(c\chi_j) T_n(\chi_{-j})) = \sigma_{j+1}^{(p)}(T_n(c) - P_n H(c\chi_j) H(\chi_j) P_n) \end{aligned}$$

(recall Proposition 3.10 for the last equality). Obviously, $\dim \operatorname{Im} H(\chi_j) = j$. Consequently, $H_j := P_n H(c\chi_j) H(\chi_j) P_n \in \mathcal{F}_j^{(n)}$ and hence

$$\begin{aligned} \sigma_{j+1}^{(p)}(T_n(c) - H_j) &= a_{n-j-1}^{(p)}(T_n(c) - H_j) \\ &= \min\left\{ \|T_n(c) - H_j - K_{n-j-1}\|_p : K_{n-j-1} \in \mathcal{F}_{n-j-1}^{(n)} \right\} \\ &\ge \min\left\{ \|T_n(c) - L_{n-1}\|_p : L_{n-1} \in \mathcal{P}_{n-1}^{(n)} \right\} \\ &= a_{n-1}^{(p)}(T_n(c)) = \sigma_1^{(p)}(T_n(c)). \end{aligned}$$

Since $T(c)$ is invertible, Theorems 3.7 and 9.2 yield that

$$\liminf_{n\to\infty} \sigma_1^{(p)}(T_n(c)) = \liminf_{n\to\infty} \|T_n^{-1}(c)\|_p^{-1} = d > 0. \qquad \square$$

9.3 Singular Values of Circulant Matrices

Throughout the rest of this chapter we restrict ourselves to the case $p = 2$.

Let b be a Laurent polynomial,

$$b(t) = \sum_{j=-r}^{r} b_j t^j \quad (t \in \mathbf{T}). \tag{9.4}$$

In (9.4) we do not require that both the coefficients b_{-r} and b_r are nonzero. For $n \geq 2r+1$, we define the circulant matrix $C_n(b)$ as in Section 2.1. We have

$$C_n(b) - T_n(b) = \begin{pmatrix} O_{(n-r)\times(n-r)} & D_r \\ E_r & O_{r\times r} \end{pmatrix}, \tag{9.5}$$

where O denotes the zero matrix and

$$D_r = \begin{pmatrix} b_r & b_{r-1} & \dots & b_1 \\ 0 & b_r & \dots & b_2 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & b_r \end{pmatrix}, \quad E_r = \begin{pmatrix} b_{-r} & 0 & \dots & 0 \\ b_{-r+1} & b_{-r} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ b_{-1} & b_{-3} & \dots & b_{-r} \end{pmatrix}.$$

Proposition 9.5. *The singular values of $C_n(b)$ are*

$$|b(1)|, |b(\omega_n)|, \dots, |b(\omega_n^{n-1})|,$$

where $\omega_n := \exp(2\pi i/n)$.

Proof. Clearly, $C_n^*(b) = C_n(\overline{b})$. From (2.6) we infer that $C_n^*(b)C_n(b) = C_n(\overline{b}b) = C_n(|b|^2)$. Thus, by Proposition 2.1, the eigenvalues of $C_n^*(b)C_n(b)$ are just $|b(\omega_n^j)|^2$ $(j = 0, 1, \dots, n-1)$. Formula (9.1) completes the proof. $\square$

Theorem 9.6. *Let b be a Laurent polynomial of the form* (9.4) *and suppose $|b|$ is not constant. Put*

$$m = \min_{t\in\mathbf{T}} |b(t)|, \quad M = \max_{t\in\mathbf{T}} |b(t)|,$$

denote by $\alpha \in \mathbf{N}$ the maximal order of the zeros of $|b| - m$ on $\mathbf{T}$, and let $\beta \in \mathbf{N}$ be the maximal order of the zeros of $M - |b|$ on $\mathbf{T}$. Then for each $k \in \mathbf{N}$ and all sufficiently large n,

$$m \leq \sigma_k(C_n(b)) \leq m + E_k \frac{1}{n^\alpha}, \quad M - D_k \frac{1}{n^\beta} \leq \sigma_{n-k}(C_n(b)) \leq M,$$

where $E_k, D_k \in (0, \infty)$ are constants independent of n.

Proof. Put $f(\theta) = |b(e^{i\theta})| - m$ for $\theta \in [0, 2\pi)$ and let f have a zero of the maximal order α at $\theta_0 \in [0, 2\pi)$. By Proposition 9.5, the singular values of $C_n(b)$ $(n \geq 2r+1)$ are $f(2\pi j/n) + m$ $(j = 0, 1, \dots, n-1)$. Let $U_{k,n}$ be the segment $[\theta_0, \theta_0 + 4\pi k/n]$ and denote by $j_1 < \cdots < j_q$ the numbers j_μ for which $2\pi j_\mu/n$ belongs to $U_{k,n}$. If n is large enough, then f is strictly monotonically increasing on $U_{k,n}$ and q is approximately equal to $2k$. Hence $\sigma_k(C_n(b)) \le f(2\pi j_q/n) + m$. It follows that

$$0 \le \sigma_k(C_n(b)) - m \le f\left(\frac{2\pi j_q}{n}\right) \le f\left(\theta_0 + \frac{4\pi k}{n}\right) \le E\left(\frac{4\pi k}{n}\right)^\alpha = E_k \frac{1}{n^\alpha}.$$

The estimate for $\sigma_{n-k}(C_n(b))$ can be shown analogously. $\square$

9.4 Extreme Singular Values

Theorem 9.4 leaves us with the case where $T(b)$ is not Fredholm. In this section we show that in that case $\sigma_k(T_n(b))$ goes to zero as $n \to \infty$ for each fixed k. This will be done with the help of Theorem 9.6 and the following well-known interlacing result for singular values.

Theorem 9.7. *Let A be a complex $n \times n$ matrix and let $B = P_{n-1} A P_{n-1}$ be the $(n-1)\times(n-1)$ principal submatrix. Then*

$$\begin{array}{ccccc} & & \sigma_1(A) & \le & \sigma_3(B) \\ \sigma_2(B) & \le & \sigma_2(A) & \le & \sigma_4(B) \\ & & \cdots & & \\ \sigma_{n-3}(B) & \le & \sigma_{n-3}(A) & \le & \sigma_{n-1}(B) \\ \sigma_{n-2}(B) & \le & \sigma_{n-2}(A) & \le & \sigma_n(B) \\ \sigma_{n-1}(B) & \le & \sigma_{n-1}(A). & & \end{array}$$

Here is the desired result on the lower singular values.

Theorem 9.8. *Let b be a nonconstant Laurent polynomial and suppose $T(b)$ is not Fredholm. Let $\alpha \in \mathbf{N}$ be the maximal order of the zeros of $|b|$ on $\mathbf{T}$. Then for each natural number $k \ge 1$, $\sigma_k(T_n(b)) = O(1/n^\alpha)$ as $n \to \infty$.*

Proof. First notice that, by Theorem 1.9, $|b|$ does have zeros on $\mathbf{T}$ if $T(b)$ is not Fredholm.

Let b be as in (9.4) and $n \ge 2r+1$. From (9.5) we know that $T_n(b)$ can be successively extended to $C_{n+r}(b)$ by adding one row and one column in each step. We have $\sigma_k(T_n(b)) \le \sigma_{k+1}(T_n(b))$. Since $k+1 \ge 2$, we can r times employ Theorem 9.7 to get $\sigma_k(T_n(b)) \le \sigma_{k+1}(C_{n+r}(b))$ for all sufficiently large n, and Theorem 9.6 with $m = 0$ implies that $\sigma_{k+1}(C_{n+r}(b)) = O(1/n^\alpha)$. $\square$

Note that Theorem 9.8 together with the equality $\|T_n^{-1}(b)\|_2 = 1/\sigma_1(T_n(b))$ yields another proof of Corollary 4.12.

The following result shows that the upper singular values $\sigma_{n-k}(T_n(b))$ always approach $\|T(b)\|_2 = \|b\|_\infty$ as $n \to \infty$, independently of whether $T(b)$ is Fredholm or not.

Theorem 9.9. *Let b be a Laurent polynomial and suppose the modulus $|b|$ is not constant on* $\mathbf{T}$. *Denote by $\beta \in \mathbf{N}$ the maximal order of the zeros of $\|b\|_\infty - |b|$ on the unit circle* $\mathbf{T}$. *Then for each $k \geq 0$,*

$$\|b\|_\infty - D_k \frac{1}{n^\beta} \leq \sigma_{n-k}(T_n(b)) \leq \|b\|_\infty$$

with some constant $D_k \in (0, \infty)$ independent of n.

Proof. If n is large enough then, by (9.5) and Theorem 9.7,

$$\sigma_{n-k}(T_n(b)) \geq \sigma_{n-k-2r}(C_{n+r}(b)).$$

The assertion is therefore immediate from Theorem 9.6. □

What happens if $|b|$ is constant? By Proposition 5.6, this occurs if and only if $b(t) = \gamma t^m$ ($t \in \mathbf{T}$) with $\gamma \in \mathbf{C}$ and $m \in \mathbf{Z}$. If $\gamma = 0$, then all singular values of $T_n(b)$ are zero, and if $\gamma \neq 0$, it is easy to see that $|m|$ singular values of $T_n(b)$ are zero and that the $n - |m|$ remaining singular values are equal to $|\gamma|$.

9.5 The Limiting Set

The objective of this section is the determination of the limiting sets $\liminf \Sigma(B_n)$ and $\limsup \Sigma(B_n)$ in the case where B_n is Toeplitz-like. Recall that the set $\Sigma(B)$ of the singular values of a Hilbert space operator is defined as the set of all $\sigma \in [0, \infty)$ for which $\sigma^2 \in \operatorname{sp} B^*B$.

Lemma 9.10. *Let $B_n = T_n(b) + P_n K P_n + W_n L W_n + C_n$, where b is a Laurent polynomial, K and L are matrices with only a finite number of nonzero entries, and $\|C_n\|_2 \to 0$ as $n \to \infty$. Put $B = T(b) + K$ and $\widetilde{B} = T(\widetilde{b}) + L$. Then*

$$\liminf_{n\to\infty} \operatorname{sp}(B_n) \subset \limsup_{n\to\infty} \operatorname{sp}(B_n) \subset \operatorname{sp} B \cup \operatorname{sp} \widetilde{B}, \tag{9.6}$$

and if, in addition, the matrices B_n are all Hermitian, then

$$\liminf_{n\to\infty} \operatorname{sp}(B_n) = \limsup_{n\to\infty} \operatorname{sp}(B_n) = \operatorname{sp} B \cup \operatorname{sp} \widetilde{B}. \tag{9.7}$$

Proof. Let $\lambda \notin \operatorname{sp} B \cup \operatorname{sp} \widetilde{B}$. Then $B - \lambda I$ and $(B - \lambda I)^\sim = \widetilde{B} - \lambda I$ are invertible, and hence Theorem 3.13 implies that there are numbers n_0 and $M \in (0, \infty)$ such that $\|(B_n - \lambda I)^{-1}\|_2 \leq M$ for all $n \geq n_0$. It follows that the spectral radius of $(B_n - \lambda I)^{-1}$ is at most M, which gives $U_{1/M}(0) \cap \operatorname{sp}(B_n - \lambda I) = \emptyset$ for $n \geq n_0$, where $U_\delta(\mu) := \{\lambda \in \mathbf{C} : |\lambda - \mu| < \delta\}$. Hence $U_{1/M}(\lambda) \cap \operatorname{sp} B_n = \emptyset$ for $n \geq n_0$ and thus $\lambda \notin \limsup \operatorname{sp} B_n$. This completes the proof of (9.6).

Now suppose that $B_n = B_n^*$ for all n. Then B and $\widetilde{B}$ are selfadjoint and all spectra occurring in (9.7) are subsets of the real line. We are left to show that if $\lambda \in \mathbf{R}$ and $\lambda \notin \liminf \operatorname{sp} B_n$, then $\lambda \notin \operatorname{sp} B \cup \operatorname{sp} \widetilde{B}$. But if λ is real and not in $\liminf \operatorname{sp} B_n$, then there exists a $\delta > 0$ such that $U_\delta(\lambda) \cap \operatorname{sp} B_{n_k} = \emptyset$ for infinitely many n_k, that is, $U_\delta(0) \cap \operatorname{sp}(B_{n_k} - \lambda I) = \emptyset$

for infinitely many n_k. As $B_{n_k} - \lambda I$ is Hermitian, the spectral radius and the norm of the operator $(B_{n_k} - \lambda I)^{-1}$ coincide, which gives $\|(B_{n_k} - \lambda I)^{-1}\|_2 < 1/\delta$ for infinitely many n_k. It follows that $\{B_{n_k} - \lambda I\}$ and thus also $\{W_{n_k}(B_{n_k} - \lambda I)W_{n_k}\}$ are stable. Lemma 3.4 now shows that $B - \lambda I$ and $\widetilde{B} - \lambda I$ are invertible. This proves (9.7). □

Corollary 9.11. *Let B_n, B, $\widetilde{B}$ be as in Lemma* 9.10. *Then*

$$\liminf_{n\to\infty} \Sigma(B_n) = \limsup_{n\to\infty} \Sigma(B_n) = \Sigma(B) \cup \Sigma(\widetilde{B}). \tag{9.8}$$

In particular, for every Laurent polynomial b,

$$\liminf_{n\to\infty} \Sigma(T_n(b)) = \limsup_{n\to\infty} \Sigma(T_n(b)) = \Sigma(T(b)) \cup \Sigma(T(\widetilde{b})). \tag{9.9}$$

Proof. We have $B_n^* B_n = T_n(\overline{b}\, b) + P_n X P_n + W_n Y W_n + D_n$, where X and Y have only a finitely many nonzero entries and $\|D_n\|_2 \to 0$ as $n \to \infty$. Equalities (9.8) are therefore straightforward from (9.7). □

Remark 9.12. Let $V : \ell^2 \to \ell^2$ be the map given by $(Vx)_j = \overline{x_j}$. Since

$$\operatorname{sp} VAV = \operatorname{sp} A, \quad \operatorname{sp}(A^*A) \cup \{0\} = \operatorname{sp}(AA^*) \cup \{0\} \tag{9.10}$$

for every $A \in \mathcal{B}(\ell^2)$ and $VT(b)V = T(\widetilde{\overline{b}})$, we obtain

$$\begin{aligned}(\Sigma(T(\overline{b})))^2 &= \operatorname{sp} T(b)T(\overline{b}) = \operatorname{sp} VT(b)T(\overline{b})V \\ &= \operatorname{sp} VT(b)VVT(\overline{b})V = \operatorname{sp} T(\widetilde{\overline{b}})T(\widetilde{b}) = (\Sigma(T(\widetilde{b})))^2,\end{aligned}$$

that is, $\Sigma(T(\widetilde{b})) = \Sigma(T(\overline{b}))$. This and the second equality of (9.10) imply that

$$\Sigma(T(b)) \cup \{0\} = \Sigma(T(\widetilde{b})) \cup \{0\}.$$

However, in general the sets $\Sigma(T(b))$ and $\Sigma(T(\widetilde{b}))$ need not coincide: If $b(t) = t$, then $T^*(b)T(b) = \operatorname{diag}(1, 1, 1, \dots)$, $T^*(\widetilde{b})T(\widetilde{b}) = \operatorname{diag}(0, 1, 1, \dots)$, whence $\Sigma(T(b)) = \{1\}$ and $\Sigma(T(\widetilde{b})) = \{0, 1\}$. □

The set $\Sigma(T(b))$ is available in special cases only. Sometimes the following is useful.

Proposition 9.13. *If b is a Laurent polynomial, then*

$$[\min|b|, \max|b|] \subset \Sigma(T(b)) \subset [0, \max|b|].$$

Proof. From Propositions 1.2 and 1.3 and Corollary 1.10 we see that there is a compact operator K such that

$$\begin{aligned}(\Sigma(T(b)))^2 &= \operatorname{sp} T(\overline{b})T(b) = \operatorname{sp}(T(|b|^2) + K) \\ &\supset \operatorname{sp}_{\mathrm{ess}}(T(|b|^2) + K) = \operatorname{sp}_{\mathrm{ess}} T(|b|^2) = [\min|b|^2, \max|b|^2],\end{aligned}$$

and obviously,

$$(\Sigma(T(b)))^2 = \operatorname{sp} T(\overline{b})T(b) \subset [0, \|T(b)\|_2^2] = [0, \max |b|^2]. \qquad \square$$

Thus, if $T(b)$ is not Fredholm, which is equivalent to the equality $\min |b| = 0$, then

$$\Sigma(T(b)) \cup \Sigma(T(\widetilde{b})) = [0, \max |b|].$$

However, if $T(b)$ is Fredholm, the question of finding

$$\Big(\Sigma(T(b)) \cup \Sigma(T(\widetilde{b}))\Big) \cap [0, \min |b|]$$

is difficult.

9.6 The Limiting Measure

The purpose of this section is to show that if b is a Laurent polynomial, then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f(\sigma_k(T_n(b))) = \frac{1}{2\pi} \int_0^{2\pi} f(|b(e^{i\theta})|)d\theta \tag{9.11}$$

for every compactly supported function $f : \mathbf{R} \to \mathbf{C}$ of bounded variation. Formula (9.11) is the **Avram-Parter theorem**. The approach of this section is due to Zizler, Zuidwijk, Taylor, and Arimoto [303].

Functions of bounded variation. Let $f : \mathbf{R} \to \mathbf{C}$ be a function with compact support. The function f is said to have *bounded variation on a segment* $[a, b] \subset \mathbf{R}$, $f \in BV[a, b]$, if there exists a constant $V \in [0, \infty)$ such that

$$\sum_{j=1}^{m} |f(x_j) - f(x_{j-1})| \le V \tag{9.12}$$

for every partition $a = x_0 < x_1 < \cdots < x_m = b$ of $[a, b]$. The minimal V for which (9.12) is true for every partition of the segment $[a, b]$ is called the *total variation of f on* $[a, b]$ and is denoted by $V_{[a,b]}(f)$. We let BV stand for the set of all functions $f : \mathbf{R} \to \mathbf{C}$ that have compact support and are of bounded variation on each segment $[a, b] \subset \mathbf{R}$. Such functions are simply referred to as *functions of bounded variation*.

If f is compactly supported and continuously differentiable, then f is clearly BV and $V_{[a,b]}(f) \le \|f'\|_\infty(b - a)$. The characteristic function χ_E of a finite interval E is also of bounded variation and $V_{[a,b]}(\chi_E) = 2$ whenever $[a, b] \supset E$.

If $f \in BV$ and $a \le x \le y \le b$, then

$$|f(y) - f(x)| \le V_{[a,b]}(f); \tag{9.13}$$

indeed, by the definition of $V_{[a,b]}(f)$, we even have

$$|f(x) - f(a)| + |f(y) - f(x)| + |f(b) - f(x)| \le V_{[a,b]}(f).$$

We begin with a result on the singular values of matrices, large blocks of which coincide.

Theorem 9.14. *Let r and n be natural numbers such that $1 \le r < n$, let $K = \{k_1, \dots, k_r\}$ be a subset of $\{1, 2, \dots, n\}$, and put $L = \{1, 2, \dots, n\} \setminus K$. Suppose A and A' are two complex $n \times n$ matrices whose jk entries coincide for all $(j, k) \in L \times L$. If $f \in BV$ and $[a, b]$ is any segment that contains all singular values of A and A', then*

$$\sum_{k=1}^{n} \left| f(\sigma_k(A)) - f(\sigma_k(A')) \right| \le 3r V_{[a,b]}(f).$$

Proof. Suppose first that $r = 1$. We can without loss of generality assume that $K = \{n\}$ (the general case can be reduced to this case by permutation similarity). Let $A = (a_{jk})_{j,k=1}^{n}$ and define $B = (a_{jk})_{j,k=1}^{n-1}$. Applying Theorem 9.7 to the pairs (A, B) and (A', B), we get

$$\begin{aligned}
\sigma_1(A), \sigma_1(A') &\in [a, \sigma_3(B)],\\
\sigma_2(A), \sigma_2(A') &\in [\sigma_2(B), \sigma_4(B)],\\
&\dots\\
\sigma_{n-2}(A), \sigma_{n-2}(A') &\in [\sigma_{n-2}(B), b],\\
\sigma_{n-1}(A), \sigma_{n-1}(A') &\in [\sigma_{n-1}(B), b],\\
\sigma_n(A), \sigma_n(A') &\in [\sigma_{n-1}(B), b].
\end{aligned}$$

This in conjunction with (9.13) and the abbreviation $\sigma_j(B) := \sigma_j$ gives

$$\begin{aligned}
&\sum_{k=1}^{n} \left| f(\sigma_k(A)) - f(\sigma_k(A')) \right|\\
&\le V_{[a,\sigma_3]}(f) + V_{[\sigma_2,\sigma_4]}(f) + \dots + V_{[\sigma_{n-2},b]}(f) + V_{[\sigma_{n-1},b]}(f) + V_{[\sigma_{n-1},b]}(f).
\end{aligned}$$

Since each point of $[a, b]$ is covered by at most three of the segments occurring in the last sum, it follows that this sum is at most $3V_{[a,b]}(f)$, which completes the proof for $r = 1$.

Now let $r > 1$. Again we may assume that $K = \{n - r + 1, \dots, n\}$. Define $n \times n$ matrices $A^{(0)}, A^{(1)}, \dots, A^{(r)}$ so that $A^{(0)} = A$, $A^{(r)} = A'$, and the pairs $A^{(\nu-1)}, A^{(\nu)}$ $(\nu = 1, \dots, r)$ are as in the case $r = 1$ considered above. This can be achieved by setting, for $\nu = 0, \dots, r$,

$$a_{jk}^{(\nu)} = \begin{cases} a_{jk} & \text{for} \quad 1 \le j, k \le n - \nu, \\ a'_{jk} & \text{for} \quad n - \nu < j \le n \ \text{ or } \ n - \nu < k \le n. \end{cases}$$

Let $[c, d] \supset [a, b]$ by any segment which contains the singular values of all $A^{(\nu)}$ and define $\widetilde{f} : \mathbf{R} \to \mathbf{C}$ by

$$\widetilde{f} = \begin{cases} 0 & \text{for} \quad x \in (-\infty, a) \cup (b, \infty), \\ f(x) & \text{for} \quad x \in [a, b]. \end{cases}$$

Clearly, $\widetilde{f} \in BV$. From what was proved for $r = 1$ we obtain

$$\sum_{k=1}^{n} \big| f(\sigma_k(A)) - f(\sigma_k(A')) \big| = \sum_{k=1}^{n} \big| \widetilde{f}(\sigma_k(A)) - \widetilde{f}(\sigma_k(A^{(r)})) \big|$$
$$\leq \sum_{\nu=1}^{r} \sum_{k=1}^{n} \big| \widetilde{f}(\sigma_k(A^{(\nu-1)})) - \widetilde{f}(\sigma_k(A^{(\nu)})) \big| \leq \sum_{\nu=1}^{r} 3V_{[c,d]}(\widetilde{f}) = 3r V_{[a,b]}(f). \qquad \square$$

Theorem 9.15. *Let $b(t) = \sum_{j=-r}^{r} b_j t^j$ ($t \in \mathbf{T}$) be a Laurent polynomial and let $f \in BV$. If $[c, d]$ is any segment that contains $[0, \|b\|_\infty]$, then for all $n \geq 1$*

$$\left| \sum_{k=1}^{n} f(\sigma_k(T_n(b))) - \frac{n}{2\pi} \int_0^{2\pi} f(|b(e^{i\theta})|) d\theta \right| \leq 7r V_{[c,d]}(f).$$

Proof. Suppose first that $|b|$ is constant on $\mathbf{T}$. As observed in the end of Section 9.4, in that case $b(t) = \gamma t^m$, $|m|$ singular values of $T_n(b)$ are zero and $n - |m|$ singular values are equal to $|\gamma|$. Hence

$$\sum_{k=1}^{n} f(\sigma_k(T_n(b))) = |m| f(0) + (n - |m|) f(|\gamma|), \quad \frac{n}{2\pi} \int_0^{2\pi} f(|b(e^{i\theta})|) d\theta = nf(|\gamma|),$$

and the assertion amounts to the inequality $|m| \, |f(0) - f(|\gamma|)| \leq 7|m| V_{[c,d]}(f)$, which is certainly true because $|f(0) - f(|\gamma|)| \leq V_{[c,d]}(f)$ by virtue of (9.13).

Now suppose that $|b|$ is not constant on $\mathbf{T}$. Define $C_n(b)$ as in Section 2.1 for $n \geq 2r+1$ and put $C_n(b) = T_n(b)$ for $n \leq 2r$. The singular values of $C_n(b)$ and $T_n(b)$ are all contained in $[0, \|b\|_\infty]$. If $n \geq 2r + 1$, then (9.5) implies that $T_n(b)$ and $C_n(b)$ differ only in the last r columns and rows. Consequently, by Theorem 9.14,

$$\left| \sum_{k=1}^{n} f(\sigma_k(T_n(b))) - \sum_{k=1}^{n} f(\sigma_k(C_n(b))) \right| \leq 3r V_{[c,d]}(f). \tag{9.14}$$

Put $h(\theta) = f(|b(e^{i\theta})|)$. By Proposition 9.5,

$$\sum_{k=1}^{n} f(\sigma_k(C_n(b))) = \sum_{k=0}^{n-1} h\left(\frac{2\pi k}{n}\right),$$

which gives

$$\left| \sum_{k=1}^{n} f(\sigma_k(C_n(b))) - \frac{n}{2\pi} \int_0^{2\pi} h(\theta) d\theta \right|$$
$$= \left| \frac{n}{2\pi} \sum_{k=0}^{n-1} \int_{2\pi k/n}^{2\pi(k+1)/n} \left(h\left(\frac{2\pi k}{n}\right) - h(\theta) \right) d\theta \right|$$

$$\leq \frac{n}{2\pi}\sum_{k=0}^{n-1}\int_{2\pi k/n}^{2\pi(k+1)/n} V_{[2\pi k/n,2\pi(k+1)/n]}(h)d\theta \quad \text{(recall (9.13))}$$
$$= \sum_{k=0}^{n-1} V_{[2\pi k/n,2\pi(k+1)/n]}(h) = V_{[0,2\pi]}(h). \tag{9.15}$$

Now let $u(\theta) = |b(e^{i\theta})|^2$. By assumption, u is a nonconstant and nonnegative Laurent polynomial of degree at most $2r$. Thus, u has at least 2 and at most $4r$ local extrema in $[0, 2\pi)$. Let $\theta_1 < \theta_2 < \cdots < \theta_\ell$ denote the local extrema. As $|b|$ is monotonous on $[\theta_j, \theta_{j+1}]$ ($\theta_{\ell+1} := \theta_1 + 2\pi$), we get

$$\begin{aligned}
&V_{[0,2\pi]}(h) = V_{[\theta_1,\theta_1+2\pi]}(f \circ |b|)\\
&= V_{[\theta_1,\theta_2]}(f \circ |b|) + V_{[\theta_2,\theta_3]}(f \circ |b|) + \cdots + V_{[\theta_\ell,\theta_1+2\pi]}(f \circ |b|)\\
&\leq V_{[c,d]}(f) + V_{[c,d]}(f) + \cdots + V_{[c,d]}(f) = \ell V_{[c,d]}(f) \leq 4r V_{[c,d]}(f).
\end{aligned} \tag{9.16}$$

Combining (9.14), (9.15), and (9.16) we arrive at the assertion. □

Corollary 9.16. *Let b be a Laurent polynomial and let $f : \mathbf{R} \to \mathbf{C}$ be a function with compact support. If f is continuous or of bounded variation, then*

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} f(\sigma_k(T_n(b))) = \frac{1}{2\pi}\int_0^{2\pi} f(|b(e^{i\theta})|)d\theta. \tag{9.17}$$

Proof. For $f \in BV$, the assertion is immediate from Theorem 9.15. So suppose f is continuous. Then f can be uniformly approximated by compactly supported functions of bounded variation (e.g., by continuously differentiable functions) f_m. Given $\varepsilon > 0$, there is an m_0 such that $|f(x) - f_{m_0}(x)| \leq \varepsilon$ for $x \in \mathbf{R}$. It follows that

$$\left|\frac{1}{n}\sum_{k=1}^{n} f(\sigma_k(T_n(b))) - \frac{1}{n}\sum_{k=1}^{n} f_{m_0}(\sigma_k(T_n(b)))\right|$$
$$\leq \frac{1}{n}\sum_{k=1}^{n}\big| f(\sigma_k(T_n(b))) - f_{m_0}(\sigma_k(T_n(b)))\big| \leq \frac{1}{n} n\varepsilon = \varepsilon,$$
$$\left|\frac{1}{2\pi}\int_0^{2\pi} f(|b(e^{i\theta})|)d\theta - \frac{1}{2\pi}\int_0^{2\pi} f_{m_0}(|b(e^{i\theta})|)d\theta\right|$$
$$\leq \frac{1}{2\pi}\int_0^{2\pi}\big| f(|b(e^{i\theta})|) - f_{m_0}(|b(e^{i\theta})|)\big|\, d\theta \leq \frac{1}{2\pi} 2\pi\varepsilon = \varepsilon,$$

and as

$$\left|\frac{1}{n}\sum_{k=1}^{n} f_{m_0}(\sigma_k(T_n(b))) - \frac{1}{2\pi}\int_0^{2\pi} f_{m_0}(|b(e^{i\theta})|)d\theta\right| < \varepsilon$$

for all sufficiently large n due to Theorem 9.15, we get

$$\left|\frac{1}{n}\sum_{k=1}^{n} f(\sigma_k(T_n(b))) - \frac{1}{2\pi}\int_0^{2\pi} f(|b(e^{i\theta})|)d\theta\right| < 3\varepsilon$$

whenever n is large enough. This implies (9.17). □

Let E be a (Lebesgue) measurable subset of $\mathbf{R}$. Given $n \in \mathbf{N}$, we denote by $N_n(E)$ the number of singular values of $T_n(b)$ in E (multiplicities taken into account):

$$N_n(E) = \sum_{k=1}^{n} \chi_E(\sigma_k(T_n(b))).$$

We define the measure μ_n by

$$\mu_n(E) = \frac{1}{n} N_n(E)$$

and we let μ denote the measure given by

$$\mu(E) = \frac{1}{2\pi} \int_0^{2\pi} \chi_E(|b(e^{i\theta})|)d\theta = \frac{1}{2\pi} \Big| \{t \in \mathbf{T} : |b(t)| \in E\} \Big|,$$

where $|\cdot|$ stands for the Lebesgue measure on $\mathbf{T}$.

Corollary 9.17. *If b is a Laurent polynomial, then the measures μ_n converge weakly to the measure μ, that is,*

$$\int_{\mathbf{R}} f d\mu_n \to \int_{\mathbf{R}} f d\mu$$

for every compactly supported continuous function $f : \mathbf{R} \to \mathbf{C}$.

Proof. Since

$$\int_{\mathbf{R}} f d\mu_n = \frac{1}{n} \sum_{k=1}^{n} f(\sigma_k(T_n(b))), \quad \int_{\mathbf{R}} f d\mu = \frac{1}{2\pi} \int_0^{2\pi} f(|b(e^{i\theta})|)d\theta,$$

this is a straightforward consequence of Corollary 9.16. □

Obviously, all singular values of $T_n(b)$ lie in $[0, \max|b|]$.

Corollary 9.18. *Let b is a Laurent polynomial of the form $b(t) = \sum_{j=-r}^{r} b_j t^j$ $(t \in \mathbf{T})$. If $E \subset \mathbf{R}$ is any segment, then*

$$|\, N_n(E) - n\mu(E) \,| \le 14r \;\; \textit{for all} \;\; n \ge 1, \tag{9.18}$$

and if $E = [\min|b|, \max|b|]$, then even

$$|\, N_n(E) - n \,| \le 7r \;\; \textit{for all} \;\; n \ge 1. \tag{9.19}$$

Proof. Theorem 9.15 with $f = \chi_E$ and $[c, d] = [0, \max|b|]$ gives $|N_n(E) - n\mu(E)| \le 7r V_{[c,d]}(\chi_E)$. Since $V_{[c,d]}(\chi_E) \le 2$, we get (9.18). If $E = [\min|b|, \max|b|]$, then $\mu(E) = 1$ and $V_{[c,d]}(\chi_E) \le 1$. This yields (9.19). □

Our next objective is an improvement of estimate (9.19). For this purpose we need the following analogue of Theorem 9.7.

Theorem 9.19 (Cauchy's interlacing theorem). *Let A be a Hermitian $n \times n$ matrix and let $B = P_{n-1}AP_{n-1}$ be the $(n-1) \times (n-1)$ principal submatrix. Then*

$$\begin{array}{ccccc} & & \lambda_1(A) & \leq & \lambda_1(B) \\ \lambda_1(B) & \leq & \lambda_2(A) & \leq & \lambda_2(B) \\ & & \dots & & \\ \lambda_{n-2}(B) & \leq & \lambda_{n-1}(A) & \leq & \lambda_{n-1}(B) \\ \lambda_{n-1}(B) & \leq & \lambda_n(A). & & \end{array}$$

Theorem 9.20. *Let $b(t) = \sum_{j=-s}^{r} b_j t^j$ $(t \in \mathbf{T})$ with $r, s \geq 0$. Then*

$$N_n([0, \min|b|)) \leq r + s \quad \textit{for all} \quad n \geq 1. \tag{9.20}$$

Proof. If $n \leq r + s$, then (9.20) is trivial. So let $n \geq r + s + 1$. We have $T_n(\overline{b})T_n(b) = T_n(|b|^2) - P_nK_sP_n - W_nL_rW_n$, where K_s and L_r are infinite matrices whose entries outside the upper-left $s \times s$ and $r \times r$ blocks, respectively, vanish (see the beginning of the proof of Theorem 5.8). Thus, we may think of $T_n(\overline{b})T_n(b)$ as resulting from $T_{n-r-s}(|b|^2)$ by $r + s$ times adding a row and a column. On $r + s$ times employing Theorem 9.19, we get $\lambda_1(T_{n-r-s}(|b|^2)) \leq \lambda_{r+s+1}(T_n(\overline{b})T_n(b))$. As $\lambda_1(T_{n-r-s}(|b|^2) \geq \min|b|^2$ by Corollary 4.28, it follows that $\lambda_{r+s+1}(T_n(\overline{b})T_n(b)) \geq \min|b|^2$. Consequently, at most $r + s$ eigenvalues of $T_n(\overline{b})T_n(b)$ are located in $[0, \min|b|^2)$. This is equivalent to saying that at most $r + s$ singular values of $T_n(b)$ lie in the set $[0, \min|b|)$. □

Let $b(t) = \sum_{j=-s}^{r} b_j t^j$ $(t \in \mathbf{T})$ with $r, s \geq 0$ and suppose $\min|b| > 0$. Denote by k the winding number of b about the origin. Since

$$b(t) = t^{-s}(b_{-s} + b_{-s+1}t + \cdots + b_r t^{r+s})$$

and since k is the difference of the number of zeros and the number of poles of $b(z)$ in the unit disk, we see that $|k| \leq \max(r, s)$. Form Theorem 9.4 we know that if n is sufficiently large, then at least $|k|$ singular values of $T_n(b)$ lie in $[0, \min|b|)$, and Theorem 9.20 shows that, for every $n \geq 1$, at most $r + s$ singular values of $T_n(b)$ are contained in $[0, \min|b|)$.

9.7 Proper Clusters

Let E be a subset of $\mathbf{R}$ and denote by $\gamma_n(E)$ the number of the singular values of $T_n(b)$ (multiplicities taken into account) that do not belong to E. Thus, with $N_n(E)$ as in Section 9.6, $\gamma_n(E) = n - N_n(E)$. For $\varepsilon > 0$, put $U_\varepsilon(E) = \{\lambda \in \mathbf{R} : \text{dist}(\lambda, E) < \varepsilon\}$. Tyrtyshnikov calls E a *cluster* and a *proper cluster* for $\Sigma(T_n(b))$ if, respectively, $\gamma_n(U_\varepsilon(E)) = o(n)$ and $\gamma_n(U_\varepsilon(E)) = O(1)$ for each $\varepsilon > 0$. Put $\mathcal{R}(|b|) = [\min|b|, \max|b|]$.

Theorem 9.21. *Let b be a Laurent polynomial of degree r. Then $\gamma_n(\mathcal{R}(|b|)) \leq 7r$ and hence $\mathcal{R}(|b|)$ is a proper cluster for $\Sigma(T_n(b))$. If E is a subset of $\mathcal{R}(|b|)$ and the closure of E is properly contained in $\mathcal{R}(|b|)$, then E is not a cluster for $\Sigma(T_n(b))$.*

Proof. Formula (9.19) is equivalent to the inequality $\gamma_n(\mathcal{R}(|b|)) \leq 7r$. As

$$\gamma_n(U_\varepsilon(\mathcal{R}(|b|))) \leq \gamma_n(\mathcal{R}(|b|)),$$

it follows that $\mathcal{R}(|b|)$ is a proper cluster. Now let $E \subset \mathcal{R}(|b|)$ and suppose $\mathcal{R}(|b|) \setminus E$ contains some interval (c, d) with $c < d$. The $\mathcal{R}(|b|) \setminus U_\varepsilon(E)$ also contains some interval $(c_\varepsilon, d_\varepsilon)$ with $c_\varepsilon < d_\varepsilon$ if only $\varepsilon > 0$ is sufficiently small. Clearly,

$$\mu((c_\varepsilon, d_\varepsilon)) = \frac{1}{2\pi} \,|\{t \in \mathbf{T} : |b(t)| \in (c_\varepsilon, d_\varepsilon)\}\,| =: \delta_\varepsilon > 0.$$

From formula (9.18) we therefore obtain that

$$\begin{aligned}\gamma_n(U_\varepsilon(E)) &= N_n(\mathcal{R}(|b|) \setminus U_\varepsilon(E)) \ge N_n((c_\varepsilon, d_\varepsilon)) \\ &\ge n\mu((c_\varepsilon, d_\varepsilon)) - 14r = n\delta_\varepsilon - 14r,\end{aligned}$$

which shows that $\gamma_n(U_\varepsilon(E))/n$ does not converge to zero. Thus, E cannot be a cluster for $\Sigma(T_n(b))$. □

9.8 Norm of Matrix Times Random Vector

Let A_n be a real $n \times n$ matrix and let $\sigma_1 \le \sigma_2 \le \cdots \le \sigma_n$ be the singular values of A_n. We have $\|A_n x\|_2 \le \|A_n\|_2$ for every unit vector $x \in \mathbf{R}^n$, and the set $\{\|A_n x\|_2/\|A_n\|_2 : \|x\|_2 = 1\}$ coincides with the segment $[\sigma_1/\sigma_n, 1]$. The purpose of this section is to show that for a randomly chosen unit vector x the value of $\|A_n x\|_2^2/\|A_n\|_2^2$ typically lies near

$$\frac{1}{\sigma_n^2} \frac{\sigma_1^2 + \cdots + \sigma_n^2}{n}. \tag{9.21}$$

Notice that $\sigma_n = \|A_n\|_2$ and that $\sigma_1^2 + \cdots + \sigma_n^2 = \|A_n\|_{\mathrm{F}}^2$, where $\|A_n\|_{\mathrm{F}}$ is the Frobenius (or Hilbert-Schmidt norm). Thus, if $\|A_n\|_2 = 1$, then for a typical unit vector x the value of $\|A_n x\|_2^2$ is close to $\|A_n\|_{\mathrm{F}}^2/n$.

Obviously, in the case where A_n is a large Toeplitz matrix, the expression (9.21) can be tackled by the Avram-Parter formula (9.11).

Let $\mathbf{B}_n = \{x \in \mathbf{R}^n : \|x\|_2 \le 1\}$ and $\mathbf{S}_{n-1} = \{x \in \mathbf{R}^n : \|x\|_2 = 1\}$. For a given real $n \times n$ matrix A_n, we consider the random variable

$$X_n(x) = \frac{\|A_n x\|_2}{\|A_n\|_2},$$

where x is uniformly distributed on $\mathbf{S}_{n-1}$. For $k \in \mathbf{N}$, the expectation of X_n^k is

$$EX_n^k = \frac{1}{|\mathbf{S}_{n-1}|} \int_{\mathbf{S}_{n-1}} \frac{\|A_n x\|_2^k}{\|A_n\|_2^k}\, d\sigma(x),$$

where $d\sigma$ is the surface measure on $\mathbf{S}_{n-1}$. The variance of X_n^k is

$$\sigma^2 X_n^k = E\left(X_n^k - EX_n^k\right)^2 = EX_n^{2k} - \left(EX_n^k\right)^2.$$

Lemma 9.22. *For every natural number k,*

$$\frac{1}{|\mathbf{S}_{n-1}|} \int_{\mathbf{S}_{n-1}} \frac{\|A_n x\|_2^k}{\|A_n\|_2^k}\, d\sigma(x) = \frac{1}{|\mathbf{B}_n|} \int_{\mathbf{B}_n} \frac{\|A_n x\|_2^k}{\|A_n\|_2^k \|x\|_2^k}\, dx.$$

Proof. Using spherical coordinates, $x = rx'$ with $x' \in \mathbf{S}_{n-1}$, we get

$$\int_{\mathbf{B}_n} \frac{\|A_n x\|_2^k}{\|x\|_2^k} dx = \int_0^1 \int_{\mathbf{S}_{n-1}} \frac{r^k \|A_n x'\|_2^k}{r^k} r^{n-1}\, d\sigma(x') dr = \frac{1}{n} \int_{\mathbf{S}_{n-1}} \|A_n x'\|_2^k\, d\sigma(x'),$$

and since

$$|\mathbf{S}_{n-1}| = \frac{2\pi^{n/2}}{\Gamma(n/2)} \quad \text{and} \quad |\mathbf{B}_n| = \frac{\pi^{n/2}}{\Gamma(n/2+1)} \tag{9.22}$$

and thus $|\mathbf{S}_{n-1}|/n = |\mathbf{B}_n|$, the assertion follows. □

Theorem 9.23. *If $A_n \neq 0$, then*

$$E X_n^2 = \frac{1}{\sigma_n^2} \frac{\sigma_1^2 + \cdots + \sigma_n^2}{n}, \tag{9.23}$$

$$\sigma^2 X_n^2 = \frac{2}{n+2} \frac{1}{\sigma_n^4} \left(\frac{\sigma_1^4 + \cdots + \sigma_n^4}{n} - \left(\frac{\sigma_1^2 + \cdots + \sigma_n^2}{n} \right)^2 \right). \tag{9.24}$$

Proof. Let $A_n = U_n D_n V_n$ be the singular value decomposition. Thus, U_n and V_n are orthogonal matrices and $D_n = \operatorname{diag}(\sigma_1, \ldots, \sigma_n)$. By Lemma 9.22,

$$\begin{aligned} E X_n^2 &= \frac{1}{|\mathbf{B}_n|} \int_{\mathbf{B}_n} \frac{\|U_n D_n V_n x\|_2^2}{\|U_n D_n V_n\|_2^2\, \|x\|_2^2}\, dx \\ &= \frac{1}{|\mathbf{B}_n|} \int_{\mathbf{B}_n} \frac{\|D_n V_n x\|_2^2}{\|D_n\|_2^2\, \|V_n x\|_2^2}\, dx = \frac{1}{|\mathbf{B}_n|} \int_{\mathbf{B}_n} \frac{\|D_n x\|_2^2}{\|D_n\|_2^2\, \|x\|_2^2}\, dx \\ &= \frac{1}{|\mathbf{B}_n|} \int_{\mathbf{B}_n} \frac{\sigma_1^2 x_1^2 + \cdots + \sigma_n^2 x_n^2}{\sigma_n^2 (x_1^2 + \cdots + x_n^2)}\, dx_1 \ldots dx_n. \end{aligned} \tag{9.25}$$

A formula by Liouville states that if $\lambda < (p_1 + \cdots + p_n)/2$, then

$$\begin{aligned} &\underset{\substack{x_1, \ldots, x_n \geq 0 \\ x_1^2 + \cdots + x_n^2 \leq 1}}{\int \cdots \int} \frac{x_1^{p_1 - 1} \ldots x_n^{p_n - 1}}{(x_1^2 + \cdots + x_n^2)^\lambda}\, dx_1 \ldots dx_n \\ &= \frac{1}{2^n \left(\dfrac{p_1 + \cdots + p_n}{2} - \lambda \right)} \frac{\Gamma\left(\dfrac{p_1}{2}\right) \ldots \Gamma\left(\dfrac{p_n}{2}\right)}{\Gamma\left(\dfrac{p_1 + \cdots + p_n}{2}\right)} \end{aligned} \tag{9.26}$$

(see, e.g., [120, No. 676.14]). From (9.22) and (9.26) we infer that

$$\begin{aligned} &\frac{1}{|\mathbf{B}_n|} \int_{\mathbf{B}_n} \frac{x_j^2}{x_1^2 + \cdots + x_n^2}\, dx \\ &= \frac{\Gamma\left(\dfrac{n}{2} + 1\right)}{\pi^{n/2}} \frac{2^n}{2^n \left(\dfrac{n-1}{2} + \dfrac{3}{2} - 1 \right)} \frac{\Gamma\left(\dfrac{1}{2}\right)^{n-1} \Gamma\left(\dfrac{3}{2}\right)}{\Gamma\left(\dfrac{n-1}{2} + \dfrac{3}{2}\right)} = \frac{1}{n}. \end{aligned}$$

This together with (9.25) gives (9.23). In analogy to (9.25),

$$EX_n^4 = \frac{1}{|\mathbf{B}_n|}\int_{\mathbf{B}_n} \frac{(\sigma_1^2 x_1^2 + \cdots + \sigma_n^2 x_n^2)^2}{\sigma_n^4 (x_1^2 + \cdots + x_n^2)^2}\, dx_1 \ldots dx_n. \tag{9.27}$$

From (9.26) we obtain

$$\frac{1}{|\mathbf{B}_n|}\int_{\mathbf{B}_n} \frac{x_j^4}{(x_1^2 + \cdots + x_n^2)^2}\, dx$$

$$= \frac{\Gamma\left(\frac{n}{2}+1\right)}{\pi^{n/2}} \frac{2^n}{2^n\left(\frac{n-1}{2}+\frac{5}{2}-2\right)} \frac{\Gamma\left(\frac{1}{2}\right)^{n-1}\Gamma\left(\frac{5}{2}\right)}{\Gamma\left(\frac{n-1}{2}+\frac{5}{2}\right)} = \frac{3}{n(n+2)},$$

$$\frac{1}{|\mathbf{B}_n|}\int_{\mathbf{B}_n} \frac{x_j^2 x_k^2}{(x_1^2 + \cdots + x_n^2)^2}\, dx$$

$$= \frac{\Gamma\left(\frac{n}{2}+1\right)}{\pi^{n/2}} \frac{2^n}{2^n\left(\frac{n-2}{2}+\frac{3}{2}+\frac{3}{2}-2\right)} \frac{\Gamma\left(\frac{1}{2}\right)^{n-2}\Gamma\left(\frac{3}{2}\right)^2}{\Gamma\left(\frac{n-2}{2}++\frac{3}{2}+\frac{3}{2}\right)} = \frac{1}{n(n+2)},$$

whence, by (9.27),

$$\begin{aligned} EX_n^4 &= \sum_{j=1}^{n} \frac{\sigma_j^4}{\sigma_n^4}\frac{3}{n(n+2)} + 2\sum_{j<k}\frac{\sigma_j^2\sigma_k^2}{\sigma_n^4}\frac{1}{n(n+2)} \\ &= \frac{1}{n(n+2)}\frac{1}{\sigma_n^4}\left(2(\sigma_1^4 + \cdots + \sigma_n^4) + (\sigma_1^2 + \cdots + \sigma_n^2)^2\right). \end{aligned} \tag{9.28}$$

Since $\sigma^2 X_n^2 = EX_n^4 - (EX_n^2)^2$, formula (9.24) follows from (9.23) and (9.28). □

From (9.24) we see that always $\sigma^2 X_n^2 \le 2/(n+2)$. Thus, by Chebyshev's inequality,

$$P\left(\left|X_n^2 - \frac{1}{\sigma_n^2}\frac{\sigma_1^2 + \cdots + \sigma_n^2}{n}\right| \ge \varepsilon\right) \le \frac{2}{(n+2)\varepsilon^2}$$

for each $\varepsilon > 0$. This reveals that for large n the values of $\|A_n x\|_2^2/(\|A_n\|_2^2\,\|x\|_2^2)$ cluster around

$$\frac{1}{\sigma_n^2}\frac{\sigma_1^2 + \cdots + \sigma_n^2}{n}.$$

Notice also that $\sigma^2 X_n^2$ can be written as

$$\sigma^2 X_n^2 = \frac{2}{n+2}\frac{1}{\sigma_n^4}\sum_{i<j}\left(\frac{\sigma_j^2 - \sigma_i^2}{n}\right)^2.$$

Figures 9.2 to 9.4 illustrate the phenomenon by two examples.

Obvious modifications of the proof of Theorem 9.23 show that Theorem 9.23 remains true for complex matrices on $\mathbf{C}^n$ with the ℓ^2 norm.

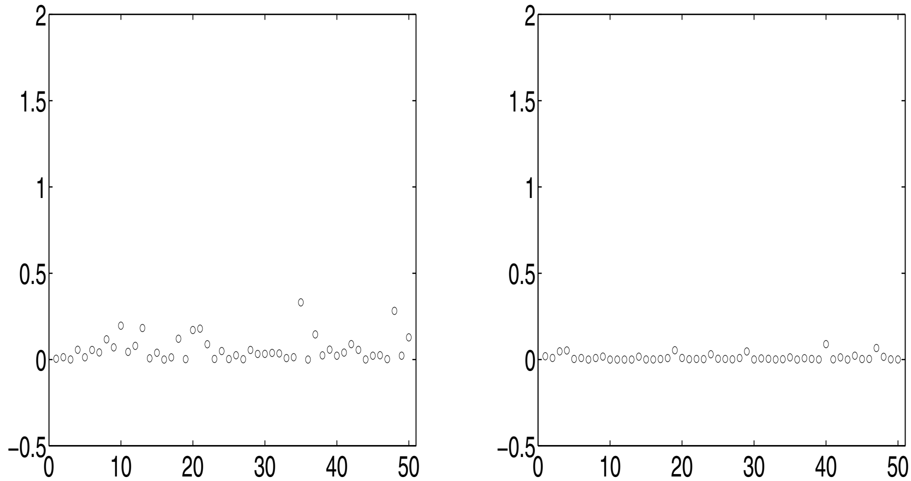

Figure 9.2. *Let A_n be the $n \times n$ matrix all entries of which are 1. We see the values $\|A_n x\|_2^2/\|A_n\|_2^2$ for 50 vectors x that were randomly drawn from the unit sphere of $\mathbf{R}^n$ with the uniform distribution. Note that the expected value of $\|A_n x\|_2^2/\|A_n\|_2^2$ is $1/n$ and that the variance is less than $2/n^2$. The n is 20 in the left picture and 100 in the right.*

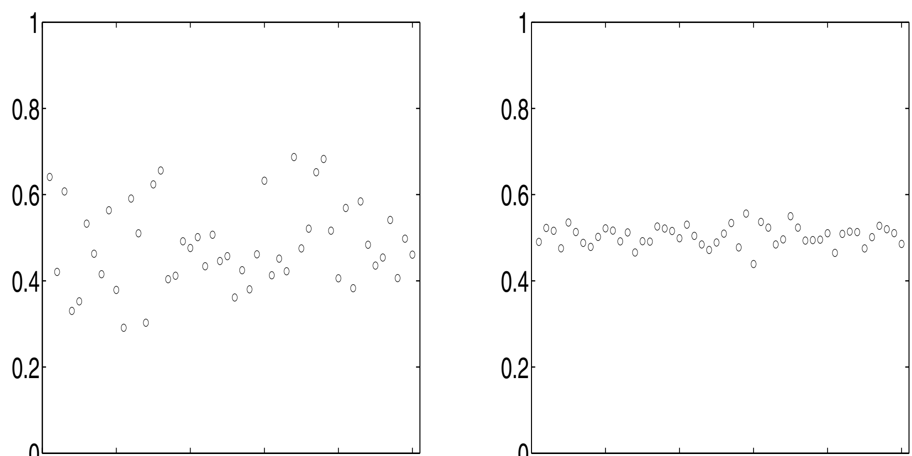

Figure 9.3. *Let $b(t) = t + t^{-1}$. The pictures show $\|T_n(b)x\|_2^2/\|T_n(b)\|_2^2$ for 50 vectors x that were randomly drawn from the unit sphere of $\mathbf{R}^n$ with the uniform distribution. We have $n = 30$ in the left picture and $n = 600$ in the right. The expected value for $\|T_n(b)x\|_2^2/\|T_n(b)\|_2^2$ converges to 0.5 as $n \to \infty$.*

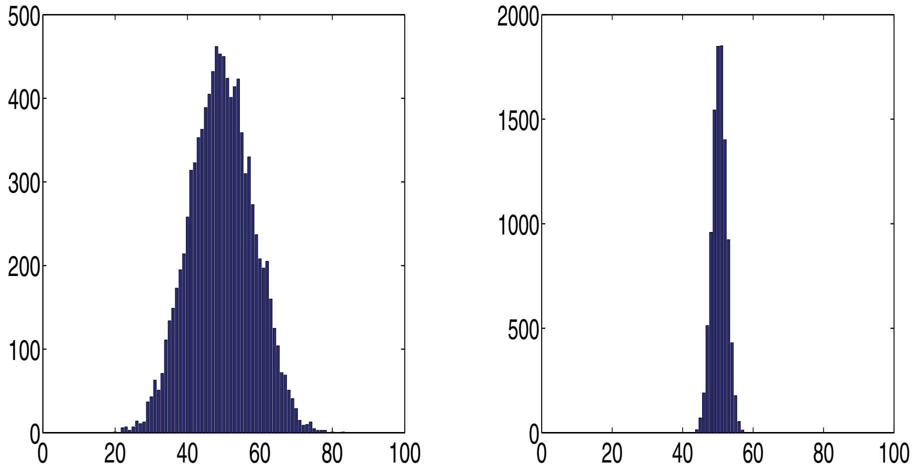

Figure 9.4. *The symbol is again $b(t) = t + t^{-1}$. The pictures show the distribution of $100\,\|T_n(b)x\|_2^2/\|T_n(b)\|_2^2$ for 10000 vectors x that were randomly drawn from the unit sphere of $\mathbf{R}^n$ with the uniform distribution. In the left picture we took $n = 30$ and in the right $n = 600$. Notice the different scales of the vertical axes.*

9.9 The Case of Toeplitz and Circulant Matrices

We need one more simple auxilary result.

Lemma 9.24. *Let $EX_n^2 = \mu_n^2$ and suppose $\mu_n \to \mu$ as $n \to \infty$. If $\varepsilon > 0$ and $|\mu_n - \mu| < \varepsilon$, then*

$$P(|X_n - \mu| \geq \varepsilon) \leq \frac{\sigma^2 X_n^2}{\mu_n^2(\varepsilon - |\mu_n - \mu|)^2}.$$

Proof. We have

$$\begin{aligned}
P(|X_n - \mu| \geq \varepsilon) &\leq P\Big(|X_n - \mu_n| \geq \varepsilon - |\mu_n - \mu|\Big) \\
&\leq P\Big(|X_n - \mu_n|(X_n + \mu_n) \geq \mu_n(\varepsilon - |\mu_n - \mu|)\Big) \\
&= P\Big(|X_n^2 - \mu_n^2| \geq \mu_n(\varepsilon - |\mu_n - \mu|)\Big),
\end{aligned}$$

and the assertion is now immediate from Chebyshev's inequality. □

Now let b be a Laurent polynomial and let $\sigma_1(T_n(b)) \leq \cdots \leq \sigma_n(T_n(b))$ be the singular values of $T_n(b)$. We abbreviate $\sigma_j(T_n(b))$ to σ_j. The Avram-Parter formula (9.11) tells us that

$$\lim_{n\to\infty} \frac{f(\sigma_1) + \cdots + f(\sigma_n)}{n} = \frac{1}{2\pi}\int_0^{2\pi} f(|b(e^{i\theta})|)d\theta \tag{9.29}$$

for every compactly supported function $f : \mathbf{R} \to \mathbf{C}$ with bounded variation. In particular,

$$\lim_{n\to\infty} \frac{\sigma_1^k + \cdots + \sigma_n^k}{n} = \|b\|_k^k := \frac{1}{2\pi} \int_0^{2\pi} |b(e^{i\theta})|^k d\theta \tag{9.30}$$

for every natural number k. Moreover, if $T(b)$ is invertible, then $\sigma_1 = \sigma_1(T_n(b))$ stays away from zero as $n \to \infty$ (Theorem 3.7), and hence (9.29) with $f(s)$ equal to a negative integral power of s times the characteristic function of $[m, M]$ for appropriate $0 < m < M$ shows that (9.30) is true for every integer k.

Theorem 9.25. *If $|b|$ is not constant on the unit circle $\mathbf{T}$, then for each $\varepsilon > 0$ there is an n_0 such that*

$$P\left(\left| \frac{\|T_n(b)x\|_2}{\|T_n(b)\|_2 \, \|x\|_2} - \frac{\|b\|_2}{\|b\|_\infty} \right| \ge \varepsilon \right) \le \frac{3}{n+2} \frac{1}{\varepsilon^2} \frac{\|b\|_4^4 - \|b\|_2^2}{\|b\|_2^2 \, \|b\|_\infty^2} \tag{9.31}$$

for all $n \ge n_0$. If $|b|$ is constant throughout $\mathbf{T}$, then

$$P\left(\frac{\|T_n(b)x\|_2}{\|T_n(b)\|_2 \, \|x\|_2} \le 1 - \varepsilon \right) = o\left(\frac{1}{n}\right) \tag{9.32}$$

for each $\varepsilon > 0$.

Proof. Put

$$\mu_n = \frac{1}{\sigma_n} \sqrt{\frac{\sigma_1^2 + \cdots + \sigma_n^2}{n}}, \quad \mu = \frac{\|b\|_2}{\|b\|_\infty}.$$

Suppose first that $|b|$ is not constant. Then $\|b\|_4 > \|b\|_2$. From (9.30) we know that $\mu_n \to \mu$. Moreover, (9.30) and Theorem 9.23 imply that

$$\frac{n+2}{2} \sigma^2 X_n^2 \to \frac{1}{\|b\|_\infty^4} \left(\|b\|_4^4 - \|b\|_2^4 \right).$$

Thus, Lemma 9.24 shows that

$$P(|X_n - \mu| \ge \varepsilon) \le \frac{3}{n+2} \frac{1}{\|b\|_\infty^4} \left(\|b\|_4^4 - \|b\|_2^4 \right) \frac{1}{\mu^2 \varepsilon^2}$$

for all sufficiently large n, which is (9.31). On the other hand, if $|b|$ is constant, we infer from (9.30) and Theorem 9.23 that

$$\mu_n \to 1 \quad \text{and} \quad \frac{n+2}{2} \sigma^2 X_n^2 = o(1),$$

whence, by Lemma 9.24,

$$P(X_n \le 1 - \varepsilon) \le \frac{3}{n+2} \, o(1) \, \frac{1}{\varepsilon^2} = o\left(\frac{1}{n}\right),$$

which is (9.32). ◻

We now turn to inverses of Toeplitz matrices.

Theorem 9.26. *If b has no zeros on $\mathbf{T}$ and $\operatorname{wind} b = 0$, then*

$$P\left(\left| \frac{\|T_n^{-1}(b)x\|_2}{\|T_n^{-1}(b)\|_2 \, \|x\|_2} - \frac{\|b^{-1}\|_2}{\|T^{-1}(b)\|_2} \right| \ge \varepsilon \right) = O\left(\frac{1}{n}\right)$$

for each $\varepsilon > 0$.

Proof. The singular values of $T_n^{-1}(b)$ are $1/\sigma_j$ $(j = 1, \dots, n)$. Thus, by Theorem 9.23,

$$EX_n^2 = \frac{1}{n}\,\sigma_1^2\left(\frac{1}{\sigma_1^2} + \cdots + \frac{1}{\sigma_n^2}\right),$$

$$\sigma^2 X_n^2 = \frac{2}{n(n+2)}\,\sigma_1^4\left(\frac{1}{\sigma_1^4} + \cdots + \frac{1}{\sigma_n^4} - \frac{1}{n}\left(\frac{1}{\sigma_1^2} + \cdots + \frac{1}{\sigma_n^2}\right)^2\right).$$

Since $T(b)$ is invertible, we can invoke formula (9.30) with $k = -2$ and the first of formulas (6.16) to deduce that

$$\mu_n^2 := EX_n^2 \to \|T^{-1}(b)\|^{-2}\|b^{-1}\|_2^2 =: \mu^2.$$

As always $\sigma^2 X_n^2 \le 2/(n+2)$, we obtain from Lemma 9.24 that

$$P(|X_n - \mu| \ge \varepsilon) \le \frac{3}{n+2}\,\frac{1}{\mu^2}\,\frac{1}{\varepsilon^2} = O\left(\frac{1}{n}\right). \qquad ◻$$

If $|\operatorname{wind} b| = k \ge 1$, then $T_n(b)$ need not be invertible for all sufficiently large dimensions n. We therefore consider the Moore-Penrose inverse $T_n^+(b)$, which coincides with $T_n^{-1}(b)$ in the case of invertibility.

Theorem 9.27. *If b has no zeros on $\mathbf{T}$ and $|\operatorname{wind} b| \ge 1$, then for each $\varepsilon > 0$,*

$$P\left(\frac{\|T_n^+(b)x\|_2}{\|T_n^+(b)\|_2 \, \|x\|_2} \ge \varepsilon \right) = O\left(\frac{1}{n^2}\right).$$

Proof. Theorem 9.4 tells us that if $|\operatorname{wind} b| = k \ge 1$, then k singular values of $T_n(b)$ converge to zero with exponential speed, $\sigma_\ell \le Ce^{-\gamma n}$ $(\gamma > 0)$ for $\ell \le k$, while the remaining singular values stay away from zero, $\sigma_\ell \ge \lambda > 0$ for $\ell \ge k+1$. Thus, the singular values of $T_n^+(b)$ are

$$\underbrace{0, \dots, 0}_{j}, \frac{1}{\sigma_n}, \dots, \frac{1}{\sigma_{j+1}}, \frac{1}{\sigma_j}$$

with $0 < \sigma_j \le \sigma_{j+1} \le \cdots \le \sigma_n$ and $j \le k$, and from Theorem 9.23 we infer that

$$\begin{aligned} EX_n^2 &= \frac{1}{n}\,\sigma_j^2 \left(\frac{1}{\sigma_j^2} + \cdots + \frac{1}{\sigma_n^2} \right) \\ &= \frac{1}{n} \left(\frac{\sigma_j^2}{\sigma_j^2} + \cdots + \frac{\sigma_j^2}{\sigma_k^2} \right) + \frac{\sigma_j^2}{n} \left(\frac{1}{\sigma_{k+1}^2} + \cdots + \frac{1}{\sigma_n^2} \right) \\ &\le \frac{1}{n}\,(k-j+1) + \frac{C^2 e^{-2\gamma n}}{n}\,\frac{n-k}{\lambda^2} \\ &\le \frac{k}{n} + \frac{C^2 e^{-2\gamma n}}{\lambda^2} \le \frac{k+1}{n} \end{aligned}$$

for all sufficiently large n. Also by Theorem 9.23,

$$\sigma^2 X_n^2 = \frac{2}{n(n+2)}\,\sigma_j^4 \left(\frac{1}{\sigma_j^4} + \cdots + \frac{1}{\sigma_n^4} - \frac{1}{n} \left(\frac{1}{\sigma_j^2} + \cdots + \frac{1}{\sigma_n^2} \right)^2 \right),$$

and, analogously,

$$\begin{aligned} \sigma_j^4 \left(\frac{1}{\sigma_j^4} + \cdots + \frac{1}{\sigma_n^4} \right) &\le k + \frac{C^4 e^{-4\gamma n}(n-k)}{\lambda^4} \le k+1, \\ \sigma_j^4 \left(\frac{1}{\sigma_j^2} + \cdots + \frac{1}{\sigma_n^2} \right)^2 &\le \left(k + \frac{C^2 e^{-2\gamma n}(n-k)}{\lambda^2} \right)^2 \le (k+1)^2, \end{aligned}$$

which gives $\sigma^2 X_n^2 = O(1/n^2)$. If $\varepsilon > 0$, then

$$P(X_n \ge \varepsilon) = P(X_n^2 \ge \varepsilon^2) \le P\left(X_n^2 \ge \frac{k+1}{n} + \frac{\varepsilon^2}{2} \right)$$

for all sufficiently large n, and thus,

$$\begin{aligned} P(X_n \ge \varepsilon) &\le P\left(X_n^2 \ge EX_n^2 + \frac{\varepsilon^2}{2} \right) \\ &\le P\left(|X_n^2 - EX_n^2| \ge \frac{\varepsilon^2}{2} \right) \le \frac{4}{\varepsilon^4}\,\sigma^2 X_n^2 = O\left(\frac{1}{n^2} \right). \qquad \square \end{aligned}$$

Define the circulant matrices $C_n(b)$ as in Section 2.1. The singular values of $C_n(b)$ are $|b(\omega_n^j)|$ $(j = 0, \ldots, n-1)$, where $\omega_n = e^{2\pi i/n}$. The only Laurent polynomials b of constant modulus are $b(t) = \alpha t^k$ $(t \in \mathbf{T})$ with $\alpha \in \mathbf{C}$; in this case $\|C_n(b)x\|_2 = |\alpha|\,\|x\|_2$ for all x.

Theorem 9.28. *If $|b|$ is not constant, then for each $\varepsilon > 0$ there exists an n_0 such that*

$$P\left(\left| \frac{\|C_n(b)x\|_2}{\|C_n(b)\|_2\,\|x\|_2} - \frac{\|b\|_2}{\|b\|_\infty} \right| \ge \varepsilon \right) \le \frac{3}{n+2}\,\frac{1}{\varepsilon^2}\,\frac{\|b\|_4^4 - \|b\|_2^2}{\|b\|_2^2\,\|b\|_\infty^2} \quad \textit{for all} \quad n \ge n_0.$$

Proof. The proof is analogous to the proof of (9.31). Note that now (9.30) amounts to the fact that the integral sum

$$\frac{\sigma_1^k + \cdots + \sigma_n^k}{n} = \sum_{j=0}^{n-1} |b(e^{2\pi i j/n})|^k \, \frac{1}{n}$$

converges to the Riemann integral

$$\int_0^1 |b(e^{2\pi i \theta})|^k d\theta = \int_0^{2\pi} |b(e^{i\theta})|^k \frac{d\theta}{2\pi} =: \|b\|_k^k.$$

Furthermore, it is obvious that $\sigma_n = \max |b(\omega_n^j)| \to \|b\|_\infty$. $\square$

If b has no zeros on $\mathbf{T}$, then $C_n^{-1}(b) = C_n(b^{-1})$, and hence Theorem 9.28 delivers

$$P\left(\left| \frac{\|C_n^{-1}(b)x\|_2}{\|C_n^{-1}(b)\|_2 \, \|x\|_2} - \frac{\|b^{-1}\|_2}{\|b^{-1}\|_\infty} \right| \geq \varepsilon \right) \leq \frac{3}{n+2} \, \frac{1}{\varepsilon^2} \, \frac{\|b^{-1}\|_4^4 - \|b^{-1}\|_2^2}{\|b^{-1}\|_2^2 \, \|b^{-1}\|_\infty^2} \tag{9.33}$$

for all sufficiently large n.

9.10 The Nearest Structured Matrix

We denote by $M_n(\mathbf{R})$ the linear space of all $n \times n$ matrices with real entries. Let $A_n \in M_n(\mathbf{R})$ and let $0 \leq \sigma_1 \leq \cdots \leq \sigma_n$ be the singular values of A_n. Suppose $\sigma_n > 0$. The random variable $X_n^2 = \|A_n x\|_2^2 / \|A_n\|_2^2$ assumes its values in [0, 1]. In this section we establish a few results on the distribution function of this random variable and we give a nice application to the problem of describing in probabilistic terms the distance of a matrix to the nearest matrix of a given structure. With notation as in the proof of Theorem 9.23,

$$E_\xi := \left\{ x \in S_{n-1} : \frac{\|A_n x\|_2^2}{\|A_n\|_2^2} < \xi \right\} = \left\{ x \in S_{n-1} : \frac{\|D_n V_n x\|_2^2}{\sigma_n^2} < \xi \right\}.$$

Put $G_\xi = \{x \in S_{n-1} : \|D_n x\|_2^2 / \sigma_n^2 < \xi\}$. Clearly, $G_\xi = V_n(E_\xi)$. Since V_n is an orthogonal matrix, it leaves the surface measure on S_{n-1} invariant. It follows that $|G_\xi| = |V_n(E_\xi)|$ and hence

$$F_n(\xi) := P(X_n^2 < \xi) = P\left(\frac{\|D_n x\|_2^2}{\sigma_n^2} < \xi \right) = P\left(\frac{\sigma_1^2 x_1^2 + \cdots + \sigma_n^2 x_n^2}{\sigma_n^2} < \xi \right). \tag{9.34}$$

This reveals first of all that the distribution function $F_n(\xi)$ depends only on the singular values of A_n. We let $f_n(\xi)$ stand for the density function corresponding to the distribution function $F_n(\xi)$.

A real-valued random variable X is said to be $B(\alpha, \beta)$ distributed on (a, b) if

$$P(c \leq X < d) = \int_c^d f(\xi) \, d\xi,$$

where the density function $f(\xi)$ is zero on $(-\infty, a]$ and $[b, \infty)$ and equals

$$\frac{(b-a)^{1-\alpha-\beta}}{B(\alpha, \beta)} \, (\xi - a)^{\alpha-1} (b - \xi)^{\beta-1}$$

on the interval (a, b). Here

$$B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha + \beta)}$$

is the common beta function and it is assumed that $\alpha > 0$ and $\beta > 0$.

We first consider 2×2 matrices, that is, we let $n = 2$. From (9.34) we infer that

$$F_2(\xi) = P\left(\frac{\sigma_1^2}{\sigma_2^2} x_1^2 + x_2^2 < \xi\right). \tag{9.35}$$

The constellation $\sigma_1 = \sigma_2$ is uninteresting, because $F_2(\xi) = 0$ for $\xi < 1$ and $F_2(\xi) = 1$ for $\xi \geq 1$ in this case.

Theorem 9.29. *If $\sigma_1 < \sigma_2$, then the random variable X_2^2 is subject to the $B(\frac{1}{2}, \frac{1}{2})$ distribution on $(\sigma_1^2/\sigma_2^2, 1)$.*

Proof. Put $\tau = \sigma_1/\sigma_2$. By (9.35), $F_2(\xi)$ is $\frac{1}{2\pi}$ times the length of the piece of the unit circle $x_1^2 + x_2^2 = 1$ that is contained in the interior of the ellipse $\tau^2 x_1^2 + x_2^2 = \xi$. This gives $F_2(\xi) = 0$ for $\xi \leq \tau^2$ and $F_2(\xi) = 1$ for $\xi \geq 1$. Thus, let $\xi \in (\tau^2, 1)$. Then the circle and the ellipse intersect at the four points

$$\left(\pm\sqrt{\frac{1-\xi}{1-\tau^2}}, \pm\sqrt{\frac{\xi-\tau^2}{1-\tau^2}}\right),$$

and consequently,

$$F_2(\xi) = \frac{2}{\pi} \arctan\sqrt{\frac{\xi - \tau^2}{1 - \xi}},$$

which implies that $F_2'(\xi)$ equals

$$\frac{1}{\pi}(\xi - \tau^2)^{-1/2}(1-\xi)^{-1/2} = \frac{1}{B(1/2, 1/2)}(\xi - \tau^2)^{-1/2}(1-\xi)^{-1/2}$$

and proves that X_2^2 has the $B(\frac{1}{2}, \frac{1}{2})$ distribution on $(\tau^2, 1)$. □

In the case $n \geq 3$, things are more involved. An idea of the variety of possible distribution functions is provided by the class of matrices whose singular values satisfy

$$0 = \sigma_1 = \cdots = \sigma_{n-2} < \sigma_{n-1} < \sigma_n.$$

For $y \in (0, 1)$, the complete elliptic integrals $\mathrm{K}(y)$ and $\mathrm{E}(y)$ are defined by

$$\mathrm{K}(y) = \int_0^{\pi/2} \frac{d\varphi}{\sqrt{1 - y^2 \sin^2\varphi}}, \qquad \mathrm{E}(y) = \int_0^{\pi/2} \sqrt{1 - y^2 \sin^2\varphi}\, d\varphi.$$

Put $\mu = \sigma_n/\sigma_{n-1}$. In [52], we showed that on $(0, 1/\mu^2)$ one has the following densities:

$$f_3(\xi) = \frac{\mu}{\pi\sqrt{1-\xi}}\,\mathrm{K}\left(\sqrt{\frac{\xi(\mu^2-1)}{1-\xi}}\right),$$

$$f_4(\xi) = \mu \quad \text{(uniform distribution)},$$

$$f_5(\xi) = \frac{3\mu\sqrt{1-\xi}}{\pi}\,\mathrm{E}\left(\sqrt{\frac{\xi(\mu^2-1)}{1-\xi}}\right),$$

$$f_6(\xi) = 2\mu - \mu(\mu^2+1)\xi,$$

and $f_7(\xi)$ equals

$$\frac{5\mu}{3\pi}\sqrt{1-\xi}\left((4-2\xi-2\xi\mu^2)\,\mathrm{E}\left(\sqrt{\frac{\xi(\mu^2-1)}{1-\xi}}\right) - (1-\xi\mu^2)\,\mathrm{K}\left(\sqrt{\frac{\xi(\mu^2-1)}{1-\xi}}\right)\right).$$

In some particular cases, one gets a complete answer. Here is an example.

Theorem 9.30. *Let $n \geq 3$. If $\sigma_1 = \cdots = \sigma_{n-m} = 0$ and $\sigma_{n-m+1} = \cdots = \sigma_n > 0$, then the random variable X_n^2 is $B(\frac{m}{2}, \frac{n-m}{2})$ distributed on* $(0, 1)$.

This can be proved by the argument of the proof of Theorem 9.29, the only difference being that now one has to compute some multidimensional integrals. A full proof is in [52].

Orthogonal projections have just the singular value pattern of Theorem 9.30. This leads to some pretty nice conclusions. Let E be an N-dimensional Euclidean space and let U be an m-dimensional linear subspace of E. We denote by $\mathcal{P}_U$ the orthogonal projection of E onto U. Then for $y \in E$, the element $\mathcal{P}_U y$ is the best approximation of y in U and we have $\|y\|^2 = \|\mathcal{P}_U y\|^2 + \|y - \mathcal{P}_U y\|^2$. The singular values of $\mathcal{P}_U$ are $N-m$ zeros and m units. Thus, Theorem 9.30 implies that if y is uniformly distributed on the unit sphere of E, then $\|\mathcal{P}_U y\|^2$ has the $B(\frac{m}{2}, \frac{N-m}{2})$ distribution on $(0, 1)$. In particular, if N is large, then $\mathcal{P}_U y$ lies with high probability close to the sphere of radius $\sqrt{\frac{m}{N}}$ and the squared distance $\|y - \mathcal{P}_U y\|^2$ clusters sharply around $1 - \frac{m}{N}$.

Now take $E = M_n(\mathbf{R})$. With the Frobenius norm $\|\cdot\|_{\mathrm{F}}$, E is an n^2-dimensional Euclidean space. Let $U = \mathrm{Str}_n(\mathbf{R})$ denote any class of structured matrices that form an m-dimensional linear subspace of $M_n(\mathbf{R})$. Examples include

the Toeplitz matrices, $\mathrm{Toep}_n(\mathbf{R})$

the Hankel matrices, $\mathrm{Hank}_n(\mathbf{R})$

the tridiagonal matrices, $\mathrm{Tridiag}_n(\mathbf{R})$

the tridiagonal Toeplitz matrices, $\mathrm{TridiagToep}_n(\mathbf{R})$

the symmetric matrices, $\mathrm{Symm}_n(\mathbf{R})$

the lower-triangular matrices, $\mathrm{Lowtriang}_n(\mathbf{R})$

the matrices with zero main diagonal, $\mathrm{Zerodiag}_n(\mathbf{R})$

the matrices with zero trace, $\mathrm{Zerotrace}_n(\mathbf{R})$.

The dimensions of these linear spaces are

$$\dim \mathrm{Toep}_n(\mathbf{R}) = 2n-1, \quad \dim \mathrm{Hank}_n(\mathbf{R}) = 2n-1,$$
$$\dim \mathrm{Tridiag}_n(\mathbf{R}) = 3n-2, \quad \dim \mathrm{TridiagToep}_n(\mathbf{R}) = 3,$$
$$\dim \mathrm{Symm}_n(\mathbf{R}) = \frac{n^2+n}{2}, \quad \dim \mathrm{Lowtriang}_n(\mathbf{R}) = \frac{n^2+n}{2},$$
$$\dim \mathrm{Zerodiag}_n(\mathbf{R}) = n^2-n, \quad \dim \mathrm{Zerotrace}_n(\mathbf{R}) = n^2-1.$$

Suppose n is large and $Y_n \in M_n(\mathbf{R})$ is uniformly distributed on the unit sphere on $M_n(\mathbf{R})$, $\|Y_n\|_{\mathrm{F}}^2 = 1$. Let $\mathcal{P}_{\mathrm{Str}} Y_n$ be the best approximation of Y_n by a matrix in $\mathrm{Str}_n(\mathbf{R})$. Notice that the determination of $\mathcal{P}_{\mathrm{Str}} Y_n$ is a least squares problem that can be easily solved. For instance, $\mathcal{P}_{\mathrm{Toep}} Y_n$ is the Toeplitz matrix whose kth diagonal, $k = -(n-1), \ldots, n-1$, is formed by the arithmetic mean of the numbers in the kth diagonal of Y_n. Recall that $\dim \mathrm{Str}_n(\mathbf{R}) = m$. From what was said in the preceding paragraph, we conclude that $\|\mathcal{P}_{\mathrm{Str}} Y_n\|_{\mathrm{F}}^2$ is $B(\frac{m}{2}, \frac{n^2-m}{2})$ distributed on $(0,1)$. For example, $\|\mathcal{P}_{\mathrm{Toep}} Y_n\|^2$ has the $B(\frac{2n-1}{2}, \frac{n^2-2n+1}{2})$ distribution on $(0,1)$. The expected value of the variable $\|Y_n - \mathcal{P}_{\mathrm{Toep}} Y_n\|^2$ is $1 - \frac{2}{n} + \frac{1}{n^2}$ and the variance does not exceed $\frac{4}{n^3}$. Hence, Chebyshev's inequality gives

$$P\left(1 - \frac{2}{n} + \frac{1}{n^2} - \frac{\varepsilon}{n} < \|Y_n - \mathcal{P}_{\mathrm{Toep}} Y_n\|^2 < 1 - \frac{2}{n} + \frac{1}{n^2} + \frac{\varepsilon}{n}\right) \geq 1 - \frac{4}{n\varepsilon^2}. \tag{9.36}$$

Consequently, $\mathcal{P}_{\mathrm{Toep}} Y_n$ is with high probability found near the sphere with the radius $\sqrt{\frac{2}{n} - \frac{1}{n^2}}$ and $\|Y_n - \mathcal{P}_{\mathrm{Toep}} Y_n\|_{\mathrm{F}}^2$ is tightly concentrated around $1 - \frac{2}{n} + \frac{1}{n^2}$.

We arrive at the conclusion that nearly all $n \times n$ matrices of Frobenius norm 1 *are at nearly the same distance to the set of all $n \times n$ Toeplitz matrices!*

This does not imply that the Toeplitz matrices are at the center of the universe. In fact, the conclusion is true for each of the classes $\mathrm{Str}_n(\mathbf{R})$ listed above. For instance, from Chebyshev's inequality we obtain

$$P\left(\frac{1}{2} - \frac{1}{2n} - \varepsilon < \|Y_n - \mathcal{P}_{\mathrm{Symm}} Y_n\|^2 < \frac{1}{2} - \frac{1}{2n} + \varepsilon\right) \geq 1 - \frac{1}{2n^2\varepsilon^2} \tag{9.37}$$

and

$$P\left(\frac{1}{n^2} - \frac{\varepsilon}{n^2} < \|Y_n - \mathcal{P}_{\mathrm{Zerotrace}} Y_n\|^2 < \frac{1}{n^2} + \frac{\varepsilon}{n^2}\right) \geq 1 - \frac{2}{n^2\varepsilon^2}.$$

If the expected value of $\|Y_n - \mathcal{P}_{\mathrm{Struct}} Y_n\|^2$ stays away from 0 and 1 as $n \to \infty$, we have much sharper estimates. Namely, Lemma 2.2 of [93] in conjunction with Theorem 9.30 implies that if X_n^2 has the $B(\frac{m}{2}, \frac{N-m}{2})$ distribution on $(0,1)$, then

$$P\left(X_n^2 \leq \sigma \frac{m}{N}\right) \leq \left(\sigma e^{1-\sigma}\right)^{m/2}, \quad P\left(X_n^2 \geq \tau \frac{m}{N}\right) \leq \left(\tau e^{1-\tau}\right)^{m/2} \tag{9.38}$$

for $0 < \sigma < 1 < \tau$. This yields, for example,

$$P\left(\sigma\left(\frac{1}{2} - \frac{1}{2n}\right) < \|Y_n - \mathcal{P}_{\mathrm{Symm}} Y_n\|_{\mathrm{F}}^2 < \tau\left(\frac{1}{2} - \frac{1}{2n}\right)\right)$$
$$\geq 1 - \left(\sigma e^{1-\sigma}\right)^{(n^2+n)/4} - \left(\tau e^{1-\tau}\right)^{(n^2+n)/4} \tag{9.39}$$

whenever $0 < \sigma < 1 < \tau$. Clearly, (9.39) is better than (9.37). On the other hand, let $\varepsilon > 0$ be small and choose τ such that $\tau\,(1 - \frac{2}{n} + \frac{1}{n^2}) = 1 - \frac{2}{n} + \frac{1}{n^2} + \frac{\varepsilon}{n}$. Then $(\tau e^{1-\tau})^{n-1/2} = 1 - \frac{\varepsilon^2}{2n} + O\,(\frac{1}{n^2})$, the O depending on ε, and hence (9.38) amounts to

$$P\left(\|Y_n - \mathcal{P}_{\mathrm{Toep}} Y_n\|^2 \geq 1 - \frac{2}{n} + \frac{1}{n^2} + \frac{\varepsilon}{n}\right) \leq 1 - \frac{\varepsilon^2}{2n} + O\left(\frac{1}{n^2}\right),$$

which is worse than the Chebyshev estimate (9.36).

Exercises

1. Let ℓ_n^p be the space $\mathbf{C}^n$ with the ℓ^p norm. Does every $n \times n$ matrix A_n have a representation $A_n = U_n S_n V_n$ where U_n and V_n induce invertible isometries on ℓ_n^p and S_n is a diagonal matrix?

2. Let $\{A_n\}$, $\{B_n\}$, $\{E_n\}$, $\{R_n\}$ be sequences of $n \times n$ matrices and suppose

$$A_n = B_n + E_n + R_n, \quad \|E_n\|_{\mathrm{F}}^2 = o(n), \quad \operatorname{rank} R_n = o(n).$$

Let $\lambda_j(C_n)$ and $\sigma_j(C_n)$ ($j = 1, \ldots, n$) denote the eigenvalues and singular values of an $n \times n$ matrix C_n.

(a) Prove that if, in addition, A_n, B_n, E_n, R_n are all Hermitian, then the eigenvalues of A_n and B_n are tied by the relation

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \big[\varphi(\lambda_j(A_n)) - \varphi(\lambda_j(B_n))\big] = 0$$

for every continuous function $\varphi : \mathbf{R} \to \mathbf{C}$ with compact support.

(b) Show that if C_n is an arbitrary $n \times n$ matrix and H_n denotes the Hermitian matrix $\left(\begin{smallmatrix} 0 & C_n \\ C_n^* & 0 \end{smallmatrix}\right)$, then

$$\{\lambda_1(H_n), \ldots, \lambda_{2n}(H_n)\} = \{\sigma_1(C_n), \ldots, \sigma_n(C_n), -\sigma_1(C_n), \ldots, -\sigma_n(C_n)\}.$$

(c) Deduce from (a) and (b) that the singular values of A_n and B_n satisfy

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \big[\varphi(\sigma_j(A_n)) - \varphi(\sigma_j(B_n))\big] = 0$$

for every compactly supported continuous function $\varphi : \mathbf{R} \to \mathbf{C}$.

3. Let $X_1, \ldots, X_n$ be independent random variables subject to the Gaussian normal distribution with mean 0 and variance 1. Show that

$$\frac{(X_1, \ldots, X_n)}{\sqrt{X_1^2 + \cdots + X_n^2}}$$

is uniformly distributed on the unit sphere $\mathbf{S}_{n-1}$.

4. Let A_n be the $n \times n$ matrix

$$A_n = \begin{pmatrix} 1 & 1 & \dots & 1 \\ 1 & 1 & \dots & 1 \\ \dots & \dots & \dots & \dots \\ 1 & 1 & \dots & 1 \end{pmatrix},$$

let $x = (x_1, \dots, x_n)$ be uniformly distributed on $\mathbf{S}_{n-1}$, and consider the random variable $X_n^2 = \|A_n x\|_2^2 / \|A_n\|_2^2$.

(a) Show that the inequality $\|A_n x\|_2^2 \le \|A_n\|_2^2 \|x\|_2^2$ is the inequality

$$(x_1 + \dots + x_n)^2 \le n\,(x_1^2 + \dots + x_n^2).$$

(b) Compute the singular values of A_n.

(c) Show that

$$E X_n^2 = \frac{1}{n}, \quad \sigma^2 X_n^2 = \frac{2}{n+2}\,\frac{1}{n}\left(1 - \frac{1}{n}\right).$$

Use Chebyshev's inequality to deduce that the inequality

$$(x_1 + \dots + x_n)^2 \le \frac{n}{2}\,(x_1^2 + \dots + x_n^2) \tag{9.40}$$

is true with probability of at least 90 % for $n \ge 18$ and with probability of at least 99 % for $n \ge 57$ and that the inequality

$$(x_1 + \dots + x_n)^2 \le \frac{n}{100}\,(x_1^2 + \dots + x_n^2) \tag{9.41}$$

is true with probability of at least 90 % for $n \ge 895$ and with probability of at least 99 % for $n \ge 2829$.

(d) Prove that X_n^2 is subject to the $B\,(\frac{1}{2}, \frac{n-1}{2})$ distribution on $(0, 1)$ and use this insight to show that (9.40) is true with probability of at least 90 % for $n \ge 6$ and with probability of at least 99 % for $n \ge 12$ and that (9.41) is true with probability of at least 90 % for $n \ge 271$ and with probability of at least 99 % for $n \ge 662$.

5. Every Hilbert space operator A with closed range has a well-defined Moore-Penrose inverse A^+. Let $A = T(a) : \ell^2 \to \ell^2$ with $a \in W$ and suppose $T(a)$ has closed range. Heinig and Hellinger [154] (also see [71]) showed that $T_n^+(a)$ converges strongly to $T^+(a)$ if and only if $a(t) \ne 0$ for $t \in \mathbf{T}$ and one of the following conditions is satisfied:

$$\operatorname{wind} a = 0, \tag{9.42}$$

$$\operatorname{wind} a > 0 \;\text{ and }\; (a^{-1})_{-m} = 0 \;\text{ for all sufficiently large }\; m, \tag{9.43}$$

$$\operatorname{wind} a < 0 \;\text{ and }\; (a^{-1})_{m} = 0 \;\text{ for all sufficiently large }\; m, \tag{9.44}$$

where $(a^{-1})_j$ denotes the jth Fourier coefficient of a^{-1}. Show that if $a \in \mathcal{P}$, then (9.43) or (9.44) are only possible if a is of the form $a(t) = t^k p_+(t)$ with $k \ge 1$ and a polynomial $p_+ \in \mathcal{P}^+$ such that $p_+(z) \ne 0$ for $|z| \le 1$ or $a(t) = t^{-k} p_-(t)$ with $k \ge 1$ and a polynomial $p_- \in \mathcal{P}^-$ such that $p_-(1/z) \ne 0$ for $|z| \le 1$.

6. To compute the numerical range $\mathcal{H}_2(A_n) = \{(A_n x, x) : \|x\|_2 = 1\}$ of an $n \times n$ matrix A_n one could try drawing a large number N of random vectors x_j from the uniform distribution on the unit sphere of $\mathbf{C}^n$ and plotting the superposition of the values $(A_n x_j, x_j)$ $(j = 1, \ldots, N)$. In the right picture of Figure 9.5 we see the result for $A_n = T_n(b)$ with b as in Figure 7.1, $n = 50$ and $N = 500$. As the numerical range contains at least the convex hull of the eigenvalues, which are shown in the left picture, we conclude that our experiment failed dramatically. Why did it fail?

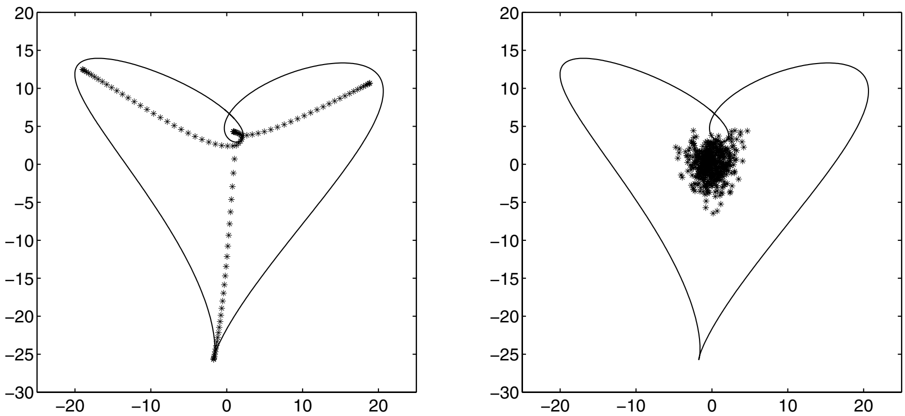

Figure 9.5. *We see the range $b(\mathbf{T})$ and the 50 eigenvalues of $T_{50}(b)$ (left) and the values $(T_{50}(b)x, x)$ for 500 vectors x drawn randomly from the uniform distribution on the unit sphere of $\mathbf{C}^{50}$ (right).*

Notes

Proofs of the theorems of Section 9.1 are in [27], [133], [166], for example.

The splitting phenomenon, Theorem 9.4, was discovered by Roch and Silbermann [224], [225]. They proved Theorem 9.4 for $p = 2$ and their proof is based on C^*-algebra techniques. In [35], another proof was given and the result was extended to $1 \le p \le \infty$. Estimate (9.2) was established in our paper [47]. The proof of Theorem 9.4 presented here is a combination of arguments of [35] and [48].

The idea of deriving results on Toeplitz matrices by comparing Toeplitz matrices with their circulant cousins has been developed by Shigeru Arimoto and coauthors in a series of papers since about 1985 which deal with problems of theoretical chemistry and in which sequences of banded circulant matrices are called alpha matrices. Independently, the same idea has emerged in papers by Beam and Warming [20], Tyrtyshnikov [284], and Serra Capizzano and Tilli [251]. Theorems 9.6 to 9.9 are from our book [48], but we are sure that versions of these theorems had been known earlier. We noticed in [48] that we found these theorems in our manuscripts but that we cannot remember whether we obtained them ourselves some time ago or whether we took them from somewhere.

Theorems 9.7 and 9.19 can be found in [27] and [167]. We thank Estelle Basor for pointing out that Theorem 9.7 was incorrectly stated in our previous book [48].

Formula (9.9) is Widom's [295] and its generalization (9.8) is due to Roch and Silbermann [222].

Formula (9.11), the so-called Avram-Parter theorem, was established by Parter [200] for symbols $b \in L^\infty$ which are locally normal, that is, which can be written as the product of a continuous and a real-valued function. Avram [10] proved (9.11) for general $b \in L^\infty$. We refer the reader to Sections 5.6 and 5.8 of [71] and to [301] for further generalizations of the Avram-Parter theorem. The elegant approach and the results of Section 9.6 are due to Zizler, Zuidwijk, Taylor, and Arimoto [303]. The notions of the cluster and the proper cluster were introduced by Tyrtyshnikov in [284].

Sections 9.8 to 9.10 are based on our paper [52].

A solution to Exercise 1 is in [71] (see also [250]). Exercise 2 is a result of Tyrtyshnikov [282], [284].

Further results: C^*-algebras I. The purpose of the following is to illustrate how a few simple C^*-algebra arguments yield part of the results established in the previous chapters very quickly. Of course, the C^* machinery forces us to limit ourselves to the case $p = 2$.

A *Banach algebra* is a complex Banach space $\mathcal{A}$ with an associative and distributive multiplication such that $\|ab\| \le \|a\| \, \|b\|$ for all $a, b \in \mathcal{A}$. If a Banach algebra has a unit element, which is usually denoted by e, 1, or I, it is referred to as a *unital Banach algebra*. A conjugate-linear map $a \mapsto a^*$ of a Banach algebra into itself is called an *involution* if $a^{**} = a$ and $(ab)^* = b^*a^*$ for all $a, b \in \mathcal{A}$. Finally, a C^*-algebra is a Banach algebra with an involution such that $\|a^*a\| = \|a\|^2$ for all $a \in \mathcal{A}$. Notice that C^*-algebras are especially nice Banach algebras, although the terminology suggests the contrary.

If H is a Hilbert space, then the sets $\mathcal{B}(H)$ and $\mathcal{K}(H)$ of all bounded and compact linear operators on H are C^*-algebras under the operator norm and passage to the Hermitian adjoint as involution. This is in particular the case for $\mathcal{B} := \mathcal{B}(\ell^2)$ and $\mathcal{K} := \mathcal{K}(\ell^2)$. Clearly, $\mathcal{B}$ is unital, but $\mathcal{K}$ has no unit element. The set $C := C(\mathbf{T})$ is a C^*-algebra under the norm $\|\cdot\|_\infty$ and the involution $a \mapsto \overline{a}$ (passage to the complex conjugate). The Wiener algebra W is a Banach algebra with an involution, $a \mapsto \overline{a}$, but it is not a C^*-algebra, because the equality $\|\overline{a}a\|_W = \|a\|_W^2$ is not satisfied for all $a \in W$. In the case $p = 2$, the Banach algebras $\mathbf{F}$ and $\mathbf{S}$ introduced in Section 3.4 are C^*-algebras. The involution is defined by $\{A_n\}^* := \{A_n^*\}$.

A subset of a C^*-algebra $\mathcal{A}$ that is itself a C^*-algebra is called a *C^*-subalgebra*. Let $\mathcal{A}$ be a C^*-algebra and let E be a subset of $\mathcal{A}$. The *C^*-algebra generated by E* is the smallest C^*-subalgebra of $\mathcal{A}$ that contains E. (In other words, the C^*-algebra generated by E is the intersection of all C^*-subalgebras of $\mathcal{A}$ that contain E.) If this C^*-algebra is $\mathcal{A}$ itself, one says that *$\mathcal{A}$ is generated by E*.

One of the many excellent properties of C^*-algebras is their *inverse closedness*. This means the following. If $\mathcal{A}_1$ is a unital C^*-algebra with the unit element e and $\mathcal{A}_2$ is a C^*-subalgebra of $\mathcal{A}_1$ which contains e, then every element $a \in \mathcal{A}_2$ that is invertible in $\mathcal{A}_1$ is automatically invertible in $\mathcal{A}_2$.

So far we have considered the Toeplitz operator $T(a)$ for $a \in W$ only. For $a \in C$ (or even $a \in L^\infty(\mathbf{T})$), this operator is also defined via the matrix $(a_{j-k})_{j,k=1}^\infty$ formed of the Fourier coefficients. It is not difficult to prove that $T(a)$ is bounded on ℓ^2 for $a \in C$ (or even $a \in L^\infty(\mathbf{T})$). If $b \in \mathcal{P}$, then the sequence $\{T_n(b)\}$ is an element of the C^*-algebra $\mathbf{F}$. Let $\mathbf{A}$ denote the C^*-algebra generated by $E = \{\{T_n(b)\} : b \in \mathcal{P}\}$ in $\mathbf{F}$. Let finally $\mathbf{G}$ be the set of all $\{A_n\} \in \mathbf{F}$ for which $\|A_n\|_2 \to 0$ as $n \to \infty$.

Theorem on A. *The C^*-algebra* $\mathbf{A}$ *is the set of all sequences* $\{A_n\}$ *of the form*

$$A_n = T_n(a) + P_n K P_n + W_n L W_n + C_n \tag{9.45}$$

with $a \in C$, $K \in \mathcal{K}$, $L \in \mathcal{K}$, $\{C_n\} \in \mathbf{G}$.

This theorem was established in [66] (proofs are also in [70, Proposition 7.27] and [62, Proposition 2.2]).

A *C^*-algebra homomorphism* is a map $f : \mathcal{A}_1 \to \mathcal{A}_2$ of a C^*-algebra $\mathcal{A}_1$ into a C^*-algebra $\mathcal{A}_2$ satisfying

$$f(\alpha a) = \alpha f(a), \quad f(a+b) = f(a) + f(b), \quad f(ab) = f(a)f(b), \quad f(a^*) = f(a)^*$$

for all $\alpha \in \mathbf{C}$, $a \in \mathcal{A}_1$, $b \in \mathcal{A}_1$. The set $\mathbf{G}$ is obviously a closed two-sided ideal of the C^*-algebra $\mathbf{A}$. Therefore the quotient algebra $\mathbf{A}/\mathbf{G}$ is a C^*-algebra with the usual quotient operations and the usual quotient norm. The sum $\mathcal{B} \oplus \mathcal{B}$ is the C^*-algebra of all ordered pairs $(A, B) \in \mathcal{B}^2$ with the natural operations and the norm $\|(A, B)\| := \max(\|A\|_2, \|B\|_2)$. Let $\{A_n\}$ be given by (9.45). Then

$$A_n \to A := T(a) + K \quad \text{and} \quad W_n A_n W_n \to \widetilde{A} := T(\widetilde{a}) + L$$

strongly as $n \to \infty$.

Theorem on A/G. *The map* Sym *defined by*

$$\mathrm{Sym} : \mathbf{A}/\mathbf{G} \to \mathcal{B} \oplus \mathcal{B}, \quad \{A_n\} + \mathbf{G} \mapsto (A, \widetilde{A})$$

is a C^*-algebra homomorphism that preserves spectra and norms.

It is easily verified that Sym is a C^*-algebra homomorphism. Since the only compact Toeplitz operator is the zero operator (this is Corollary 1.13 for $a \in C$), it follows that Sym is injective. As injective homomorphisms of unital C^*-algebras automatically preserve spectra and norms (which is another exquisite property of C^*-algebras that is not shared by general Banach algebras), we arrive at the conclusion of the theorem. This simple reasoning goes back to [66], [70, Theorem 7.11] and is explicit in [34].

Here are some immediate consequences of the theorem on $\mathbf{A}/\mathbf{G}$.

Consequence 1. A sequence $\{A_n\} \in \mathbf{A}$ is stable on ℓ^2 if and only if $A = T(a) + K$ and $\widetilde{A} = T(\widetilde{a}) + L$ are invertible on ℓ^2 (Theorem 3.13 for $p = 2$). Indeed, due to the inverse closedness of $\mathbf{A}/\mathbf{G}$ in $\mathbf{F}/\mathbf{G}$, the stability of $\{A_n\}$ is equivalent to the condition that 0 does not belong to the spectrum of $\{A_n\} + \mathbf{G}$ in $\mathbf{A}/\mathbf{G}$.

Consequence 2. If $\{A_n\} \in \mathbf{A}$ and both A and $\widetilde{A}$ are invertible on ℓ^2, then

$$A_n^{-1} = T_n(a^{-1}) + P_n X P_n + W_n Y W_n + D_n \tag{9.46}$$

with $X \in \mathcal{K}$, $Y \in \mathcal{K}$, $\{D_n\} \in \mathbf{G}$ for all sufficiently large n (recall Section 3.5). Indeed, the assumption implies that $(\{A_n\} + \mathbf{G})^{-1} \in \mathbf{A}/\mathbf{G}$ and the theorem on $\mathbf{A}$ therefore yields (9.46). Passing to the strong limit $n \to \infty$ in (9.46) we get

$$T^{-1}(a) = T(a^{-1}) + X \quad \text{and} \quad T^{-1}(\widetilde{a}) = T(\widetilde{a}^{-1}) + Y,$$

that is, we recover (3.15) and (3.16).

Consequence 3. If $\{A_n\} \in \mathbf{A}$, then

$$\lim_{n\to\infty} \|A_n\|_2 = \max(\|A\|_2, \|\widetilde{A}\|_2)$$

(Corollary 5.14 for $p = 2$). This follows from Theorem 3.1 and the equalities

$$\limsup_{n\to\infty} \|A_n\|_2 = \|\{A_n\} + \mathbf{G}\|_{\mathbf{A}/\mathbf{G}} = \|\mathrm{Sym}\,(\{A_n\} + \mathbf{G})\|_{\mathcal{B}\oplus\mathcal{B}} = \max(\|A\|_2, \|\widetilde{A}\|_2).$$

Consequence 4. If $\{A_n\} \in \mathbf{A}$ and A and $\widetilde{A}$ are invertible on ℓ^2, then

$$\lim_{n\to\infty} \|A_n^{-1}\|_2 = \max(\|A^{-1}\|_2, \|\widetilde{A}^{-1}\|_2)$$

(Theorem 6.3 for $p = 2$). To see this, combine Consequences 2 and 3.

Consequence 5. If $\{A_n\} \in \mathbf{A}$ and at least one of the operators A and $\widetilde{A}$ is nonzero, then

$$\lim_{n\to\infty} \kappa_2(A_n) = \max(\|A\|_2, \|\widetilde{A}\|_2)\ \max(\|A^{-1}\|_2, \|\widetilde{A}^{-1}\|_2)$$

(Corollary 6.4 for $p = 2$). This is straightforward from Consequences 3 and 4.

Consequence 6. If $\{A_n\} \in \mathbf{A}$ and $\varepsilon > 0$, then

$$\liminf_{n\to\infty} \mathrm{sp}_\varepsilon^{(2)} A_n = \limsup_{n\to\infty} \mathrm{sp}_\varepsilon^{(2)} A_n = \mathrm{sp}_\varepsilon^{(2)} A \cup \mathrm{sp}_\varepsilon^{(2)} \widetilde{A}$$

(generalization of Theorem 7.7 in the case $p = 2$). Once Consequence 4 is available, this can be proved by the argument of the proof of Theorem 7.7.

Consequence 7. If $\{A_n\} \in \mathbf{A}$ and $A_n^* = A_n$ for all n, then

$$\liminf_{n\to\infty} \mathrm{sp}\, A_n = \limsup_{n\to\infty} \mathrm{sp}\, A_n = \mathrm{sp}\, A \cup \mathrm{sp}\, \widetilde{A}$$

(Lemma 9.10). To see this, let $\lambda \notin \mathrm{sp}\, A \cup \mathrm{sp}\, \widetilde{A}$. Then $\{A_n - \lambda I\}$ is stable (Consequence 1) and hence the spectral radius of $(A_n - \lambda I)^{-1}$ remains bounded as $n \to \infty$. This implies that $\lambda \notin \limsup \mathrm{sp}\, A_n$. Conversely, let $\lambda \in \mathbf{R}$ and $\lambda \notin \liminf \mathrm{sp}\, A_n$. Then there exists a $\delta > 0$ such that $U_\delta(\lambda) \cap \mathrm{sp}\, A_n = \emptyset$ for infinitely many n, that is, $U_\delta(0) \cap \mathrm{sp}\,(A_n - \lambda I) = \emptyset$ for infinitely many n. As $A_n - \lambda I$ is Hermitian, the spectral radius and the norm of $(A_n - \lambda I)^{-1}$ coincide, which gives that $\|(A_n - \lambda I)^{-1}\|_2 \le 1/\delta$ for infinitely many n. Consequently, we arrive at a subsequence $\{n_k\}$ such that $\{A_{n_k} - \lambda I\}$ and thus also $\{W_{n_k}(A_{n_k} - \lambda I)W_{n_k}\}$ is stable. Lemma 3.4 now yields the invertibility of $A - \lambda I$ and $\widetilde{A} - \lambda I$.

Consequence 8. If $\{A_n\} \in \mathbf{A}$, then

$$\liminf_{n\to\infty} \Sigma(A_n) = \limsup_{n\to\infty} \Sigma(A_n) = \Sigma(A) \cup \Sigma(\widetilde{A})$$

(Corollary 9.11). Since $\{A_n^* A_n\} \in \mathbf{A}$, this follows from Consequence 7.

Summary. Thus, we have demonstrated that many sharp convergence results can be obtained very comfortably by working with appropriate C^*-algebras. For more details and for further developments of this idea we refer the reader to [36], [48], [223] and especially to Hagen, Roch, and Silbermann's monograph [149]. Another approach to questions of numerical analysis via C^*-algebras was worked out by Arveson [7], [8], [9]. Fragments of this approach will be outlined in the notes to Chapter 14.

We nevertheless want to emphasize that the C^*-algebra approach has its limitations. For example, it is restricted to Hilbert space operators. Moreover, refinements of the consequences cited above, such as estimates of the convergence speed, require hard analysis and hence the tools presented in the preceding chapters.