Approximating functions with cubic singularities

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To approximate functions of the form

$$f(\sqrt[3]{t^2+\epsilon^2}, t), \qquad t \in [-1,1], \tag{1}$$

we set $y = t$ and recast f as a bivariate function $f(x,y)$ on the cubic curve γ, where

$$\gamma = \left\{(x,y) \ : \ y^2 = \phi(x) = x^3 - \epsilon^2, \qquad y \in [-1,1], \qquad x \in [\epsilon^{2/3}, \left(1+\epsilon^2\right)^{1/3}]\right\}, \tag{2}$$

and find its interpolant on γ. Let $(x_{k,N}, \pm y_{k,N}), k = 1, \ldots, N$ be $2N$ points on γ, where $y_{k,N} = \sqrt{\phi(x_{k,N})} \neq 0$, then from Theorem 3.2, the unique interpolant of f at the points $(x_{k,N}, \pm y_{k,N}), k = 1, \ldots, N$ is

$$L_N(f_e(x)) + yL_N(f_o(x)),$$

where

$$f_e = \frac{1}{2}\left[f(x, \sqrt{\phi(x)}) + f(x, -\sqrt{\phi(x)})\right], \qquad f_o = \frac{1}{2\sqrt{\phi(x)}}\left[f(x, \sqrt{\phi(x)}) - f(x, -\sqrt{\phi(x)})\right],$$

and $L_N(f_e(x))$, $L_N(f_o(x))$ are the Lagrange interpolating polynomials of f_e and f_o at $x_{k,N}, k = 1, \ldots, N$.

For comparison purposes with standard bases, we also approximate functions of the form (1) using algebraic Hermite–Padé (HP) approximation.

To compute an HP approximant, we require an orthogonal basis with respect to a discrete inner product. For concreteness, we choose the Chebyshev polynomials which are orthogonal with respect to the following discrete inner product:

$$\langle f, g\rangle = \frac{2}{N}\sum_{k=0}^{N-1} f(x_{k,N})g(x_{k,N}), \qquad x_{k,N} = \cos\left[(2k+1)\pi/(2N)\right]. \tag{3}$$

With this inner product, we approximate functions using polynomials of the form

$$p_j(x) = \sum_{k=0}^{d_j} \sqrt{w_k} c_k T_k(x), \qquad w_0 = \frac{1}{2}, w_k = 1, 1 \leq k \leq d_j,$$

so that (if $d_j \leq N-1$)

$$\|p_j\|^2 = \langle p_j, p_j\rangle = \|\boldsymbol{c}\|_2^2 = |c_0|^2 + \cdots + |c_{d_j}|^2. \tag{4}$$

Suppose we have the function values $f(x_{k,N}), k = 0, \ldots, N$. We want to find polynomials $p_0, \ldots, p_m$ of degrees $d_0, \ldots, d_m$ such that

$$\|p_0 + p_1 f + p_2 f^2 + \cdots + p_m f^m\| = \text{minimum}. \tag{5}$$

We assume some kind of normalization so that the trivial solution $p_0 = \ldots = p_m = 0$ is not admissible. Because of the isometry (4), (5) is a least squares problem whose solution can be computed with the SVD. If the number of unknown polynomial coefficients matches the number of points on the Chebyshev grid, we obtain the 'interpolation' case:

$$n := \sum_{j=0}^{m} d_j + m, \quad N = n \quad \Rightarrow \quad \|p_0 + p_1 f + p_2 f^2 + \cdots + p_m f^m\| = \text{minimum} = 0. \tag{6}$$

The HP approximant of $f(x)$, viz. $\psi(x)$, is the algebraic function defined by

$$p_0(x) + p_1(x)\psi(x) + p_2(x)\psi^2(x) + \cdots + p_m(x)\psi^m(x) = 0. \tag{7}$$

Note that if $m = 1$ and $p_1(x) = 1$ in (6), then the HP approximant, $\psi(x) = -p_0(x)$, is a polynomial interpolant of f on the grid; if $m = 1$, then the HP approximant, $\psi = -p_0(x)/p_1(x)$, is a rational interpolant of f (with poles in the complex x-plane). If $m \geq 2$, then for every x, $\psi(x)$ will generally be an m-valued approximant of f (with poles and algebraic branch points in the complex x-plane). We want to pick only one branch of the m-valued function ψ to approximate f. One way to do this is to solve (7) with Newton's method using a polynomial or rational approximant as first guess. We shall only consider 'diagonal' HP approximants, which are approximants for which all the polynomials have equal degrees ($d_0 = \cdots = d_m$).

As an example, we approximate

$$f(t) = \sin(10t + 20\sqrt[3]{t^2 + \epsilon^2}), \qquad t \in [-1, 1], \qquad \epsilon = 0.01.$$

by interpolating f at the points $(x_{k,N}, \pm\sqrt{\phi(x_{k,N})})$ on γ defined in (2), where $x_{k,N}$ are the Chebyshev points given in (3) and translated to the interval $[\epsilon^{2/3}, (1+\epsilon^2)^{1/3}]$. We also approximate f using HP approximants with $m = 0, 1, 2, 3$ (polynomial, rational, quadratic and cubic approximants). The figure shows that the interpolant on the cubic curve γ converges super-exponentially (since f is an entire function in x and y) and significantly faster the HP approximants (which in addition appear to have stability/ill-conditioning issues).