1. Introduction

1.1 Some Characteristic of Fluids

One of the ways we can distinguish a fluid from a solid is when acted on by a shearing stress.

$$\tau = \frac{F}{A} \tag{1.1}$$

Definition 1.1.1 — Shearing stress. A shearing stress τ (force per unit area) is created whenever a tangential force acts on a surface.

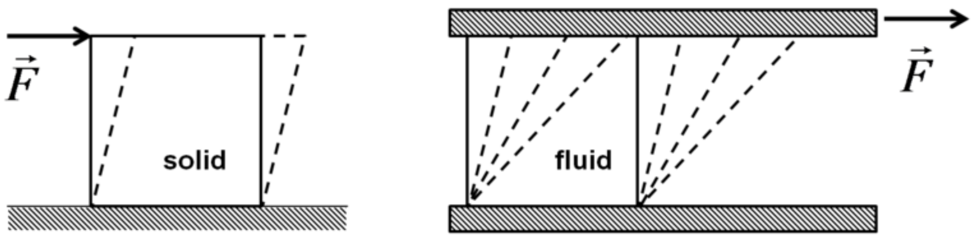


Figure 1.1: A solid and a fluid under application of a shear stress.

- When a shear stress is applied to the surface of a solid, the solid will deform a little, and then remain at rest—in its new distorted shape.
- When a shear stress is applied to the surface of the fluid, the fluid will continuously deform; it will set up some kind of flow pattern inside the container.

That is to say, fluid is any substance that deforms continuously when subjected to a shear stress, no matter how small.

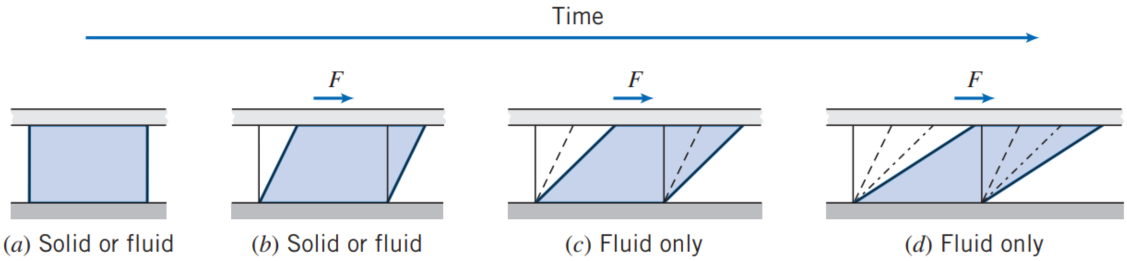


Figure 1.2: Difference in behavior of a solid and a fluid.

In a solid, the shear force may cause some initial displacement of one layer over another, but the material does not continue to move indefinitely and a position of stable equilibrium is reached.

We will expand more on this in the later sections of chapter 1, when talking about viscosity and the rate of deformation.

1.2 Dimensions, Dimensional Homogeneity, and Units

In our study of fluid mechanics, it is necessary to develop a system for describing these characteristics both qualitatively and quantitatively.

1.2.1 Systems of Dimension

The quantitative description is conveniently given in terms of certain primary quantities or basic dimensions, such as length, L, time, T, mass, M, and temperature, Θ.

In this course, we have two basic systems of dimensions, corresponding to the different ways of specifying the primary dimensions.

1. mass $[M]$, length $[L]$, time $[T]$, temperature $[\Theta]$
2. force $[F]$, length $[L]$, time $[T]$, temperature $[\Theta]$

Typically, we just refer to them as the MLT or FLT system. We can convert from one system to the other by using the following conversion, where

$$F \doteq MLT^{-2} \tag{1.2}$$

...or

$$M \doteq FL^{-1}T^{2} \tag{1.3}$$

Any valid equation that relates physical quantities must be dimensionally homogeneous.

Definition 1.2.1 — Dimensional homogeneous. The dimensions of the left side of the equation must be the same as those on the right side, and all additive separate terms must have the same dimensions.

1.2.2 Unit Systems

There's two systems that are commonly used in engineering.

Base Units	BG System		SI System	
	Name	Symbol	Name	Symbol
Mass	Slug	slug	Kilogram	kg
Force	Pound	lb	Newton	N
Length	Foot	ft	Meter	m
Time	Second	s	Second	s
Temperature	Rankine	°R	Kelvin	K

Table 1.1: Difference in British Gravitational (BG) System and International System (SI).

R For this course, we won't use the British Gravitational (BG) system and will only be using the International System (SI).

1.3 Measures of Fluid Mass and Weight

Density, specific volume, specific weight, and specific gravity are all terms that are incredibly common in the application of fluid mechanics. We'll briefly go over the definition and equation for each one.

1.3.1 Density

Density ρ is defined as mass per unit volume. It is typically used to characterize the mass of a fluid system.

$$\rho = \frac{m}{V} \qquad [\rho] = \frac{\mathrm{kg}}{\mathrm{m}^3} \tag{1.4}$$

R The square brackets are used to indicate the SI units for density.

The specific volume υ is the volume per unit mass, therefore is the reciprocal of the density.

$$\upsilon = \frac{1}{\rho} \qquad [\upsilon] = \frac{\mathrm{m}^3}{\mathrm{kg}} \tag{1.5}$$

Though, this property is not commonly used in fluid mechanics, but is used in thermodynamics.

1.3.2 Specific Weight

The specific weight γ of a fluid is another useful material property to characterize the mass of a fluid system. It is defined as the weight of a substance per unit volume.

$$\gamma = \rho g \qquad [\gamma] = \frac{\mathrm{N}}{\mathrm{m}^3} \tag{1.6}$$

. . . where g is the local acceleration of gravity.

1.3.3 Specific Gravity

An alternative way of expressing the density of a fluid is to compare it to an accepted reference value, typically the maximum density or specific weight of water taken at 4 °C.

$$\rho_{\mathrm{H_2O@4\,^\circ C}} = 1000\,\frac{\mathrm{kg}}{\mathrm{m}^3} \qquad \gamma_{\mathrm{H_2O@4\,^\circ C}} = 9807\,\frac{\mathrm{N}}{\mathrm{m}^3}$$

Thus, the specific gravity SG of a substance is expressed as

$$SG = \frac{\rho}{\rho_{\mathrm{H_2O@4\,^\circ C}}} = \frac{\gamma}{\gamma_{\mathrm{H_2O@4\,^\circ C}}} \tag{1.7}$$

R Often questions (like in Ch. 2) will not provide the density or specific weight of a fluid and instead will provide you the specific gravity and the reference value for water. You would then use this equation to find the fluid's density or specific weight.

1.4 Ideal Gas Law

In many problems it is necessary to bring into the analysis additional relations that describe the behavior of physical properties of fluids under given conditions.

The ideal gas equation of states gases are highly compressible in comparison to liquids, with changes in gas density directly related to changes in pressure and temperature.

$$\rho = \frac{p}{RT} \tag{1.8}$$

...where p is the absolute pressure, ρ is the density, T is the absolute temperature and R is gas constant.

R The gas constant R depends on the particular gas and is related to the molecular weight of the gas.

1.4.1 Absolute Pressure and Gage Pressure

Pressure is a normal stress and hence has dimensions of force per unit area.

$$p = \frac{F}{A} \qquad [p] = \frac{\mathrm{N}}{\mathrm{m}^2} \tag{1.9}$$

...where

$$1\,\frac{\mathrm{N}}{\mathrm{m}^2} = 1\,\mathrm{Pa}$$

Abbreviated as Pa, pressures are commonly specified in pascals.

R If you noticed, shear stress τ is also force per unit area. The difference is that pressure is normal stress (i.e. perpendicular to the surface), whereas shear stress is tangential stress.

There two types of definition for pressure:

- absolute pressure and ...
- gage pressure

R Note that the pressure p in the ideal gas law must be expressed as an absolute pressure.

It is important to remember that pressure values must be stated with respect to a reference level. If the reference level is a vacuum, pressures are termed absolute.

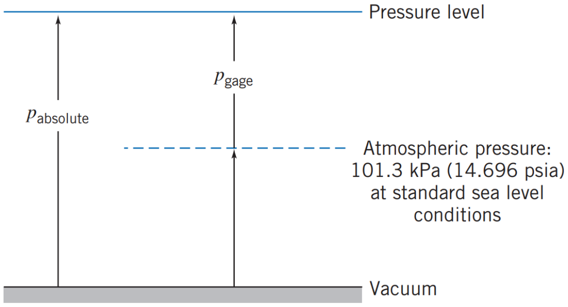


Figure 1.3: Absolute and gage pressures, showing reference levels.

The absolute pressure can be obtained from the gage pressure by adding the value of the atmospheric pressure, while the gage pressure is the pressure relative to atmospheric pressure.

R Pressure is a particularly important fluid characteristic, and it will be discussed more fully in the next chapter.

1.5 Viscosity

To measure the fluidity of a fluid, we need to define another property. Consider the following scenario.

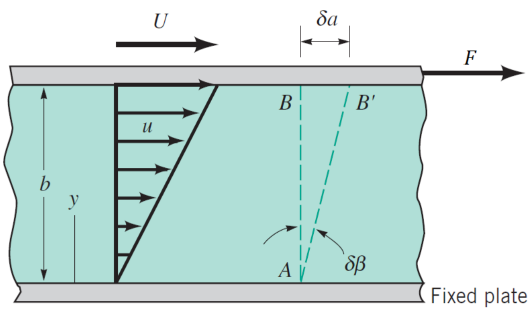


Figure 1.4: Behaviour of fluid placed between two parallel plates.

Suppose a force F is applied to the upper plate so that it is dragged across the fluid at a constant velocity U. How would the fluid react?

It is sometimes useful to think of fluid as flowing in layers, with each layers moving at a different velocity.

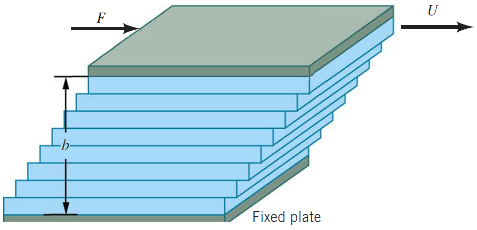


Figure 1.5: Layers of fluid.

When two layers are moving relative to one another because they are flowing at different velocity, a shear stress develop between them. The effect of this shear stress is more obvious in flow close to the wall.

R This is a very important observation in fluid mechanics, referred to as the no-slip condition, where fluid “sticks” to the solid boundaries.

The fluid between the two plates moves with velocity $u = u(y)$ that would be found to vary linearly.

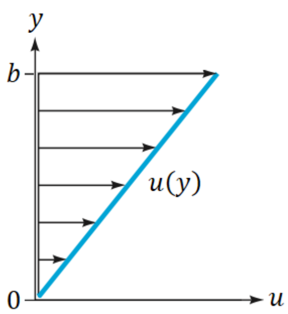


Figure 1.6: The velocity $u(y)$ varies linearly from 0 at the fixed plate to U at the moving plate.

The velocity of the fluid that is closer towards the moving plate will have a higher velocity compared to the fluid in contact with the bottom fixed plate.

1.5.1 Rate of Deformation

A velocity gradient is developed between the plates shown in fig. 1.6. In this particular case, the velocity gradient is a constant.

$$\frac{du}{dy} = \frac{U}{b} \tag{1.10}$$

Refer back to fig. 1.4. The rate of shearing strain $\dot{\gamma}$ refers to the amount of deformation perpendicular to a given line.

1. We’ll use A and B as our reference line to denote the rate of shearing strain.
2. Due to the shear stress τ, it will result in a small displacement δa from B to B', which will rotate through an angle $\delta\beta$.

3. Because this is happening at a very small scale, we can make an assumption, that is

$$\tan(\delta\beta) \approx \delta\beta = \frac{\delta a}{b}$$

4. Given the velocity of the moving plate is U, so for a small time increment δt, we can say the displacement is

$$\delta a = U\,\delta t$$

...then we can redefine the angle $\delta\beta$ in which it rotates as a function of time

$$\delta\beta = \frac{U\,\delta t}{b} \tag{1.11}$$

5. We define the rate of shearing strain as the rate of which $\delta\beta$ is changing as $\delta t \to 0$.

$$\dot{\gamma} = \lim_{\delta t \to 0} \frac{\delta\beta}{\delta t}$$

Substituting in eq. (1.11), we get

$$\dot{\gamma} = \frac{U\cancel{\delta t}}{b}\frac{1}{\cancel{\delta t}} = \frac{U}{b} = \frac{du}{dy}$$

...which in this instance is equal to the velocity gradient.

An observation can be made that the shearing stress τ is proportional to the rate of shearing strain $\dot{\gamma}$, that is

$$\tau \propto \dot{\gamma} \qquad\qquad \tau \propto \frac{du}{dy}$$

The constant of proportionality is the absolute viscosity μ. Thus, it can be related with a relationship of the form

$$\tau = \mu\frac{du}{dy} \tag{1.12}$$

As mentioned prior, the constant velocity gradient only applies to a particular case, which are for Newtonian fluids.

Definition 1.5.1 — Newtonian fluids. Fluids (like water, air, gasoline under normal condition) for which the shearing stress is linearly related to the rate of shearing strain.

R Fluids for which the shearing stress is not linearly related to the rate of shearing strain are non-Newtonian fluids, though we will only be concerned with Newtonian fluids.

Viscosity has the dimension of

$$[\mu] = \frac{\mathrm{N\,s}}{\mathrm{m}^2}$$

In fluid mechanics, the ratio of absolute viscosity μ to density ρ often arises. The ratio is called kinematic viscosity v.

$$v = \frac{\mu}{\rho} \qquad [v] = \frac{\mathrm{m}^2}{\mathrm{s}} \tag{1.13}$$

2. Fluid Statics

In Chapter 1, we defined a fluid as any substance that flows (continuously deforms) when it experiences a shear stress; hence for a static fluid (or one undergoing "rigid-body" motion) only normal stress is present—in other words, pressure.

2.1 Pressure at a Point

Our goal for this section is to obtain an equation for computing the pressure at a point from some arbitrary location within a fluid mass.

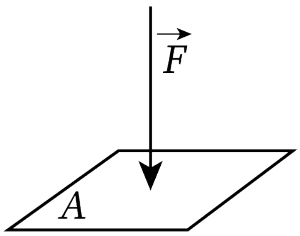


Figure 2.1: The perpendicular force acting on area A.

Pressure is normal stress, meaning it acts perpendicular to the surface. So it creates a perpendicular force, as pressure is force per unit area. Refer to eq. (1.9).

As will see in the next page, pressure at any point in a fluid is the same in all directions. This can be demonstrated by considering a small wedge-shaped fluid element in equilibrium, as shown in fig. 2.2.

In the figure below, δx, δy, and δz represent the length for its respective coordinate.

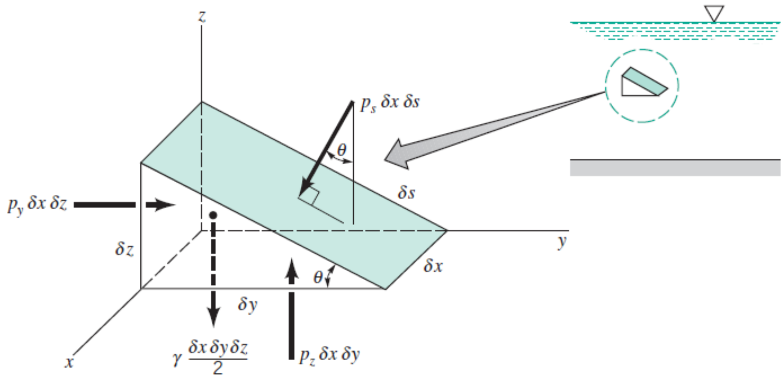


Figure 2.2: Forces on an arbitrary wedged-shaped element of fluid.

Using Newton's second law, $\sum \vec{F} = m\vec{a}$, then total forces acting in the x, y, and z-direction are

$$\sum \vec{F}_x = 0 \qquad \sum \vec{F}_y = 0 \qquad \sum \vec{F}_z = 0$$

R The fluid is static (i.e. in rest), so the acceleration $\vec{a}$ is zero.

Since pressure is force per unit area, the forces acting on the y-direction can be obtained by taking the pressure and area

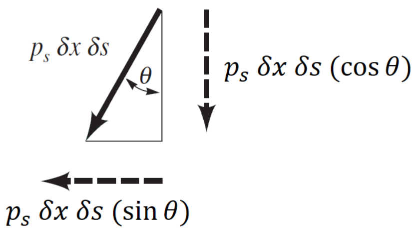


Figure 2.3: Forces in the y and z-direction.

...which is

$$p_y \delta x \delta z - p_s \delta x \delta s (\sin\theta) = 0 \tag{2.1}$$

If you notice, it follows from the geometry that is

$$\delta y = \delta s(\cos\theta) \qquad \delta z = \delta s(\sin\theta)$$

...then we can rewrite eq. (2.1) as

$$p_y \cancel{\delta x} \left[\cancel{\delta s(\sin\theta)}\right] - p_s \cancel{\delta x \delta s(\sin\theta)} = 0$$

Simplifying the equation, we get

$$p_y = p_s \tag{2.2}$$

If we were to do the same thing, but for the forces acting on the z-direction, we would end up with

$$p_z = p_s \tag{2.3}$$

2.1.1 Pascal's Law

The result we have obtained is more commonly known as Pascal's law.

> **Definition 2.1.1 — Pascal's law.** Pressure at a point in a fluid at rest, or in motion (i.e. whole fluid in motion), is independent of direction as long as there is no shearing stress present.

The pressure is the same on the side as it is on the bottom.

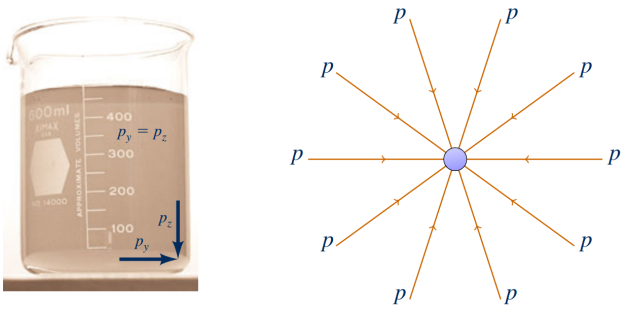


Figure 2.4: The pressure at a point in a fluid at rest is independent of direction.

As demonstrated from eqs. (2.2) and (2.3), it can be shown that

$$p = p_s = p_y = p_z \tag{2.4}$$

2.2 Basic Equation for Pressure Field

As we established, pressure has the same magnitude in all directions, so our next goal is to determine how pressure varies horizontally and vertically (i.e. the x, y, and z-direction).

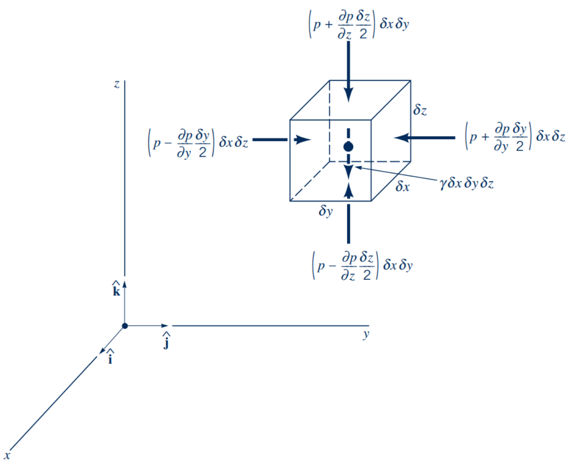


Figure 2.5: Surface and body forces acting on small fluid element.

The textbook and slides goes in-depth in the derivation, to which if you are interested in the theory behind it, refer to those in your own discretion.

R I will only briefly skim over it, as the key takeaway will be discussed in the next section, which we'll use to solve different types of pressure-related problems.

The general equation of motion of fluid in which there is no shearing stress is

$$-\nabla p - \gamma\hat{\mathbf{k}} = \rho\vec{a} \tag{2.5}$$

A few pointers regarding this equation:

- As a recap from MTH312, we use the ∇ operator to indicate the gradient of a function, which in this case is the pressure gradient; the change in pressure measured across a given distance.

$$\nabla p = \frac{\partial p}{\partial x}\hat{\mathbf{i}} + \frac{\partial p}{\partial y}\hat{\mathbf{j}} + \frac{\partial p}{\partial z}\hat{\mathbf{k}} \tag{2.6}$$

- The specific gravity γ only affects the z-direction (i.e. the upward and vertical direction), thus $\hat{\mathbf{k}}$ is used to indicate it.
- Lastly, it is set equal to the density ρ and acceleration $\vec{a}$ of the fluid.

2.3 Pressure Variation in a Fluid at Rest

Since the fluid is at rest, $\vec{a} = 0$, eq. (2.5) can be rewritten as

$$-\nabla p - \gamma\hat{\mathbf{k}} = 0$$

. . . or in component form

$$-\frac{\partial p}{\partial x}\hat{\mathbf{i}} - \frac{\partial p}{\partial y}\hat{\mathbf{j}} - \frac{\partial p}{\partial z}\hat{\mathbf{k}} - \gamma\hat{\mathbf{k}} = 0$$

. . . which can rewritten to describe the pressure variation in each of the three coordinate directions

$$\frac{\partial p}{\partial x} = 0 \qquad \frac{\partial p}{\partial y} = 0 \qquad \frac{\partial p}{\partial z} = -\gamma \tag{2.7}$$

As shown, the pressure is independent of coordinates x and y; it depends on z alone.

R This is an important part of the chapter, as it plays a key part in how we can solve the pressures at different levels in a static fluid.

Since p is a function of a single variable z, a total derivative may be used instead of a partial derivative.

$$\frac{dp}{dz} = -\gamma = -\rho g \tag{2.8}$$

Although ρg may be defined as the specific weight γ, it may help to write it as ρg in eq. (2.8) to emphasize that both ρ and g must be considered variables.

As an example of how pressure varies horizontally and vertically, refer to the figure below.

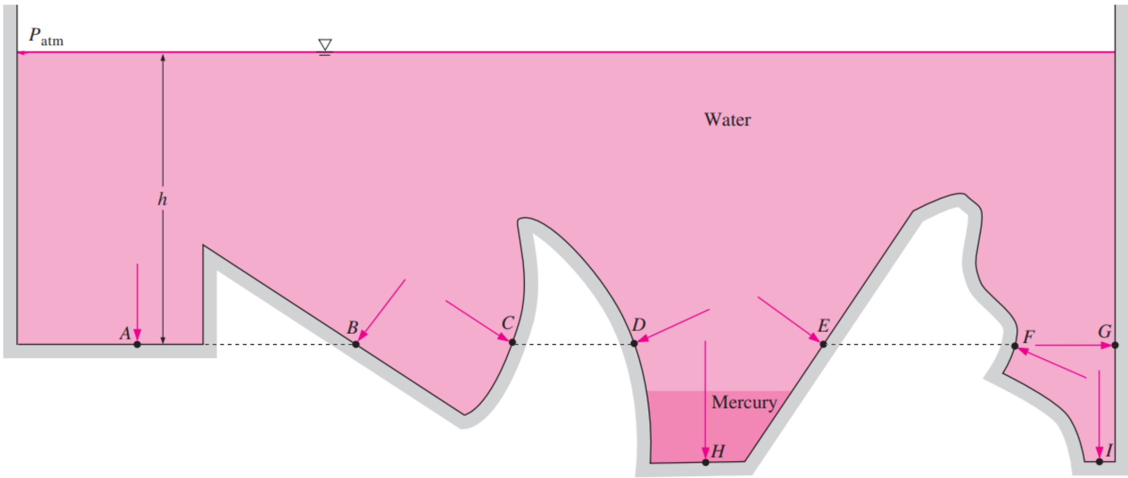


Figure 2.6: The pressure of a given fluid.

Pressure in a fluid at rest is independent of the shape of the container. It changes with the vertical distance, but remains constant in other directions. Therefore, we can say

$$p_A = p_B = p_C = p_D = p_E = p_F = p_G \neq p_I$$

... and note that

$$p_H \neq p_I$$

... since these two points cannot be interconnected by the same fluid (i.e. mercury and water), despite having the same vertical distance.

Now comes the question, of how exactly can we calculate the pressure between two points given the elevation between them. The answer depends on the type of fluid in question, which can either be incompressible or compressible fluid.

2.3.1 Incompressible Fluid

Let's first consider the idea of what incompressible fluids are; in general, a fluid with constant density

$$\frac{dp}{dz} = -\rho g = \text{constant}$$

... which allows us to derive the equation for the pressure distribution by

$$p_1 - p_2 = \gamma h \tag{2.9}$$

... or

$$p_1 = p_2 + \gamma h \tag{2.10}$$

... where p_1 and p_2 are the pressures at the distance h, which is the depth of fluid measured downward from the location of p_2. Refer to fig. 2.7.

As you can see, the pressure must increase with depth to "hold up" the fluid above it.

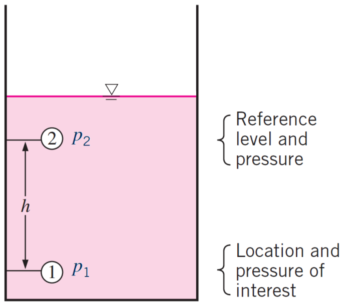


Figure 2.7: Pressure in a liquid at rest increases linearly with depth.

This type of pressure distribution is commonly called a hydrostatic distribution. It can also be observed that

$$h = \frac{p_1 - p_2}{\gamma} \tag{2.11}$$

. . . which in this case, h is referred to as the pressure head. Devices used for this purpose are called manometers, which will cover in a bit.

Free Surface

In some scenarios, there is often a free surface which we can use as a reference plane and it means that the pressure there is atmospheric pressure, p_{atm}.

R The free surface is indicated by an upside-down triangle $\underline{\nabla}$, which you can see referenced in fig. 2.6 and fig. 2.7.

We'll denote the reference pressure as p_0, it follows that the pressure p at any depth h below free surface is

$$p = p_0 + \gamma h \tag{2.12}$$

. . . and if we consider $p_0 = 0$, then p is the gage pressure.

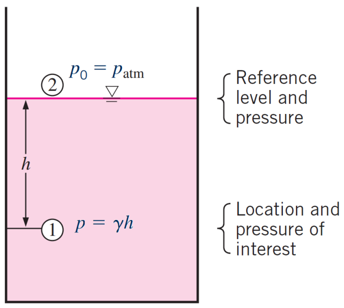


Figure 2.8: Pressure relative to atmospheric pressure.

R Whenever a question asks to determine the gage pressure, the pressure at free surface is set to 0 (i.e. atmospheric pressure). It might make more sense if you refer back to fig. 1.3.

2.3.2 Compressible Fluid

With compressible fluids (i.e. gasses), the pressure distribution becomes a bit more complicated, as density can change significantly with changes in pressure and temperature.

R Most of the questions in this chapter uses incompressible fluids, but nonetheless this part will still be included since it's introduced in the textbook and slides.

As you can see in fig. 2.9, the temperature can either remain constant or change with altitude.

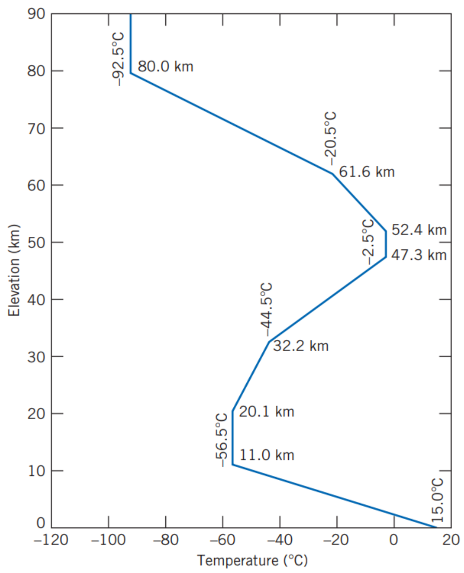


Figure 2.9: Temperature variation with altitude in the U.S. Standard Atmosphere.

When the temperature is constant, T_0 (i.e. vertical), the equation yields

$$p_2 = p_1 \cdot \exp\left[-\frac{g(z_2 - z_1)}{RT_0}\right] \tag{2.13}$$

...and when the temperature varies with altitude, $T = az + b$ (i.e. slope), the equation yields

$$\frac{p_2}{p_1} = \left[\frac{T_2}{T_1}\right]^{-\frac{g}{aR}} \tag{2.14}$$

2.4 Measurement of Pressure

As is noted briefly in Chapter 1 and shown in fig. 1.3, the pressure at a point within a fluid mass will be designated as either an absolute pressure or a gage pressure.

- Absolute pressure is measured relative to a perfect vacuum (absolute zero pressure).
- Gage pressure is measured relative to the local atmospheric pressure.

Note that absolute pressures are always positive, but gage pressures can be either positive or negative.

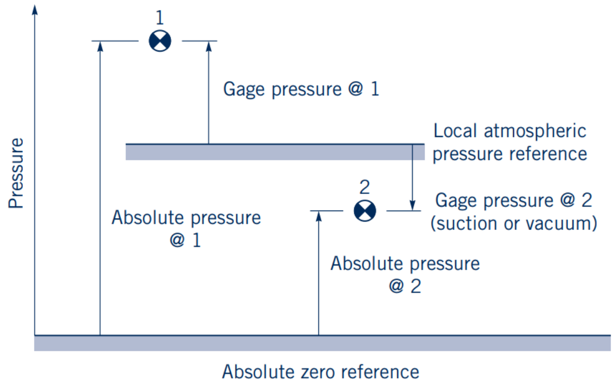


Figure 2.10: Graphical representation of gage and absolute pressure.

For example. the gage pressure @ 1 is positive, whereas the gage pressure @ 2 is negative.

2.4.1 Barometer

One of the ways we can measure the atmospheric pressure is by using a scientific instrument called a barometer.

Definition 2.4.1 — Barometer. A glass of mercury, closed in one end with the open end immersed in a container of mercury.

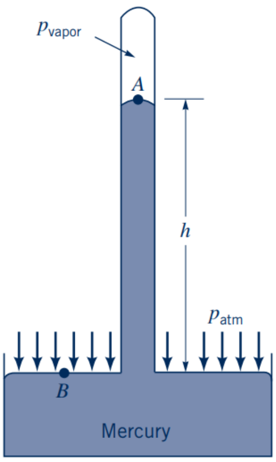


Figure 2.11: Mercury barometer.

From eq. (2.10) and fig. 2.11, we can calculate the atmospheric pressure p_{atm} by

$$p_{\text{atm}} = p_{\text{vapor}} + \gamma h$$

. . . and since p_{vapor} is very small, we make an approximation, so that

$$p_{\text{atm}} \approx \gamma h$$

R There is no pressure at p_A, exerted on the top of the liquid because of the vacuum, so the atmospheric pressure p_{atm} is fluid proportional to depth h.

2.5 Manometry

Many of the problem-solving questions you'll see relating to chapter 2 involves the use of a manometers.

Definition 2.5.1 — Manometer. A pressure-measuring devices that involves the use of liquid columns in vertical or inclined tubes.

For this section, we'll cover three common types of manometer:

1. Piezometer Tube
2. U-tube Manometer
3. Inclined-tube Manometer

Make sure you are familiar with eq. (2.10) (i.e. hydrostatic distribution), as we'll be using that equation repetitively throughout.

2.5.1 Piezometer Tube

The piezometer tube is a very simple and accurate pressure-measuring device. But, it has several disadvantages; limited to pressure being greater than atmospheric pressure and must be relatively small.

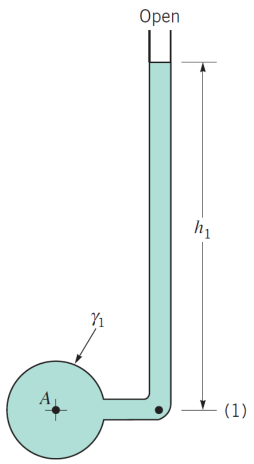


Figure 2.12: Piezometer tube.

Since points A and (1) are at the same elevation, we can say $p_A = p_1$. From eq. (2.12), application of this equation to the piezometer tube indicates that the pressure can be determined by

$$p_A = \gamma_1 h_1$$

...where p_A is the gage pressure at point A.

R Many of these problems are concerned with the gage pressure at specific point. So at free surface, we denote the pressure to be $p_{atm} = 0$. Refer back to fig. 2.8.

2.5.2 U-Tube Manometer

As the name suggests, a U-tube manometer consists of a tube formed into the shape of a U.

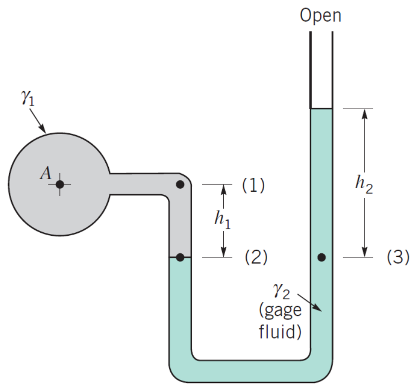


Figure 2.13: Simple U-tube manometer.

We can start by measuring the gage pressure at point (3), which is

$$p_3 = \gamma_2 h_2$$

If you notice, the points (2) and (3) are at the same elevation (and interconnected by the same fluid), so we can say $p_2 = p_3$. Then we can measure the gage pressure at point (1) by

$$p_1 = \gamma_2 h_2 - \gamma_1 h_1$$

Likewise, points A and (1) are at the same elevation, so we can say $p_A = p_1$. Therefore, the pressure p_A can be rewritten as

$$p_A = \gamma_2 h_2 - \gamma_1 h_1$$

Difference in Pressure Between Two Points

The U-tube manometer is also widely used to measure the difference in pressure between two containers or two points in a given system.

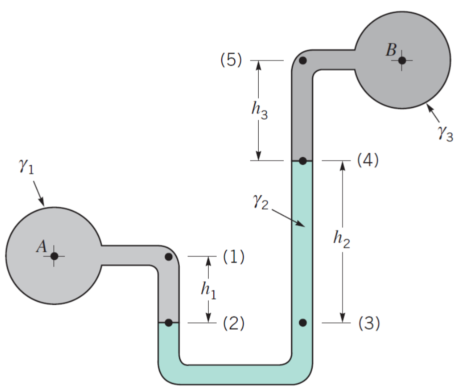


Figure 2.14: Differential U-tube manometer.

In fig. 2.14, our goal is to determine the pressure difference between point A and point B, or in other words, find $p_A - p_B$. Note that we don't have a free surface in this scenario, as everything is closed off.

> **R** The process to solve them are fairly straightforward, as long as you understand that pressure increases downward and decreases upward in a given fluid.

Since the points A and (1) are at the same elevation, then $p_A = p_1$. As we move to point (2), the pressure increases by $\gamma_1 h_1$.

$$p_2 = p_A + \gamma_1 h_1$$

Likewise, the pressure at points (2) and (3) are equal, $p_2 = p_3$.

$$p_3 = p_A + \gamma_1 h_1$$

As we move upward to point (4), the pressure decreases by $\gamma_2 h_2$.

$$p_4 = p_3 - \gamma_2 h_2$$

. . . or

$$p_4 = p_A + \gamma_1 h_1 - \gamma_2 h_2$$

Similarly, as we move upwards to point (5), the pressure decreases by $\gamma_3 h_3$.

$$p_5 = p_A + \gamma_1 h_1 - \gamma_2 h_2 - \gamma_3 h_3$$

Finally, the points B and (5) are at the same elevation, then $p_B = p_5$.

$$p_B = p_A + \gamma_1 h_1 - \gamma_2 h_2 - \gamma_3 h_3$$

. . . and the difference between the pressure at points A and B is

$$p_A - p_B = \gamma_2 h_2 + \gamma_3 h_3 - \gamma_1 h_1$$

2.5.3 Inclined-Tube Manometer

Lastly, the inclined-tube manometer allows us to accurately measure small pressure difference by placing one leg of the manometer at an angle θ.

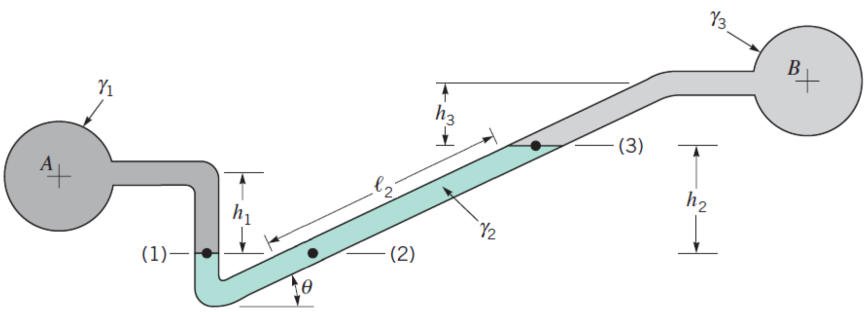


Figure 2.15: Inclined-tube manometer.

> **R** The image is modified a bit, to explain certain parts of the equation, so it is not exactly the same as the one shown in the slides/textbook.

The process is also quite similar to the one we previously covered with the U-tube manometer, but now we also have to account for the height displacement in regards to θ.

By now you should at least know that any two points at the same elevation are equal, regardless of shape, as long as they are interconnected by the same fluid.

Starting from point A, as we move downward to point (1), the pressure increases by $\gamma_1 h_1$.

$$p_1 = p_A + \gamma_1 h_1$$

Basic trigonometry allows us to calculate the height at which the inclined raises from point (2) to point (3), which is by

$$h_2 = \gamma_2 \ell_2 \sin\theta$$

...then as we move upwards to point (3), the pressure decreases by $\gamma_2 \ell_2 \sin\theta$.

$$p_3 = p_A + \gamma_1 h_1 - \gamma_2 \ell_2 \sin\theta$$

Finally, as we move upwards to point B, the pressure decreases by $\gamma_3 h_3$.

$$p_B = p_A + \gamma_1 h_1 - \gamma_2 \ell_2 \sin\theta - \gamma_3 h_3$$

...and the difference between the pressure at points A and B is

$$p_A - p_B = \gamma_2 \ell_2 \sin\theta + \gamma_3 h_3 - \gamma_1 h_1$$

Differences in Gas Pressures

The inclined-tube manometer is often used to measure small differences in gas pressures so that if pipes A and B contain a gas, then

$$p_A - p_B = \gamma_2 \ell_2 \sin\theta$$

...or

$$\ell_2 = \frac{p_A - p_B}{\gamma_2 \sin\theta}$$

When working with gas, we can neglect the pressure difference from h_1 and h_3, as the change in pressure for a gas is almost negligible.

R Notice if you placed your finger at one point move downwards, you can barely feel any difference in pressure. Oppose to if you're in a swimming pool, you can feel the difference if you move downwards from the surface level.

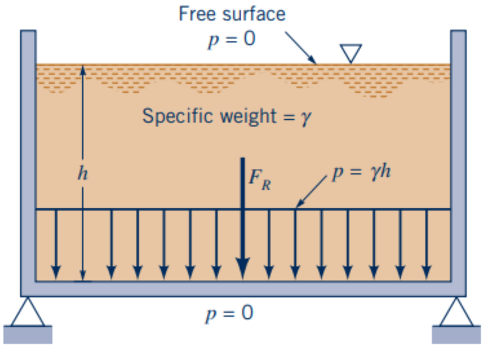
2.6 Hydrostatic Force on a Plane Surface

Now that we have determined how the pressure varies in a static fluid, we can examine the force on a surface submerged in a liquid.

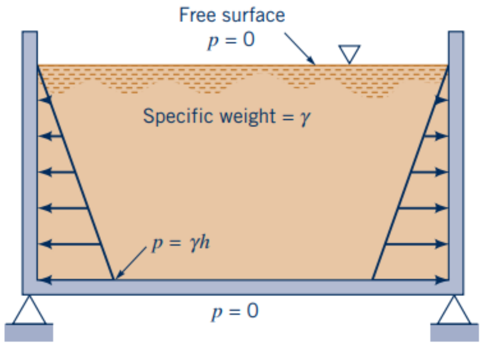
R Another section which consists of fairly lengthy derivation, but main takeaway is knowing how to apply the equation $F_R = \gamma h_c A$.

Our goal is to determine the resulting force and the center of action (i.e. where the force is being applied).

To make it easier to understand, we can break it down into two scenarios: (1) Horizontal surfaces and (2) Vertical or inclined surfaces.



(a) Pressure on tank bottom.



(b) Pressure on tank ends.

Figure 2.16: Resultant force acting on a tank.

- When we have a horizontal surface (i.e. fig. 2.16a), we have a uniform distribution, since for all points at the surface, the height remains the same.
- In comparison to a vertical surface (i.e. fig. 2.16b), where the pressure changes linearly as the pressure increases with height.

It is important that you understand how the distribution of pressure varies, as it will be useful in understanding where the force is being applied. There's two important "properties" you would need to keep in the back of your mind, as we go through the derivation:

1. The force must be perpendicular to the surface. It may help if you refer back to fig. 2.1.
2. The resultant force passes through the center of pressure, which is the center of action. Denoted as F_R, you can see where the resultant force is acting on the tank bottom in fig. 2.16a.

R Don't confuse yourself with the center of pressure and the centroid of the surface. They are two different things.

For a horizontal surface which has a uniform distribution of pressure, the center of pressure would be located at the center of the surface (i.e. the centroid). However, for a vertical or inclined surfaces, this is not case.

R Since the pressure changes linearly as it increases with depth, the center of pressure will always be below the centroid. We'll go more in-depth in the following subsection.

For a horizontal surface, such as the bottom of a liquid filled tank, the magnitude of the resultant force F_R is simply

$$F_R = pA$$

...and for an open tank as shown in eq. (2.12), the pressure is

$$p = \gamma h$$

...which we can combine to say

$$F_R = \gamma h A$$

But how would this work for a non-horizontal surface, where the height can vary depending on the point being referenced to relative to the free surface.

Consider the following figure shown below.

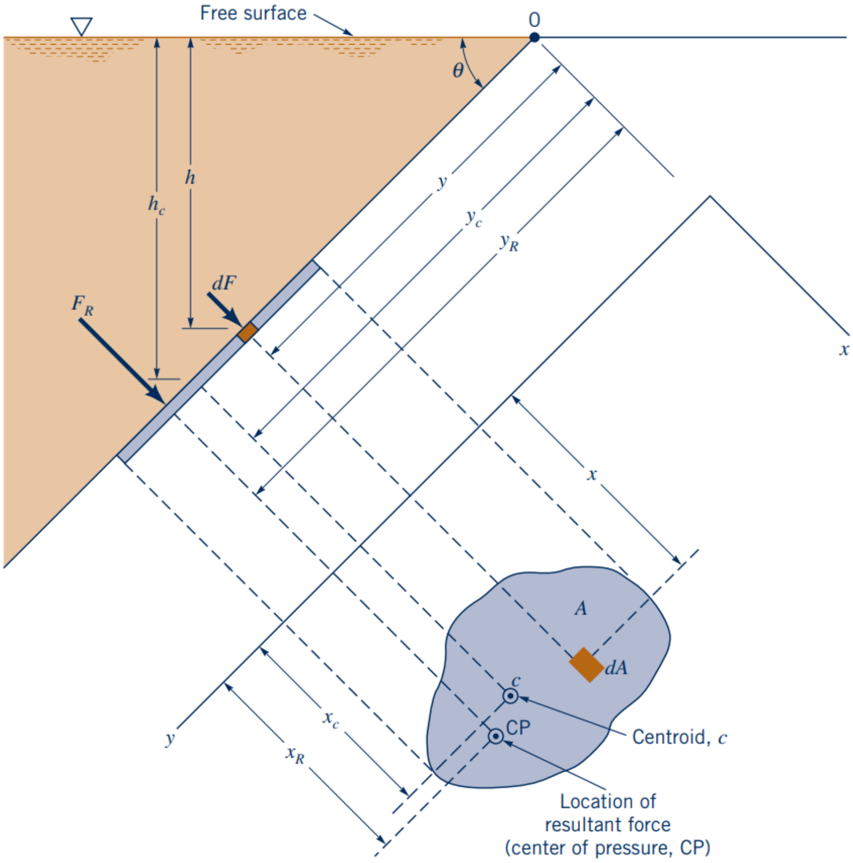


Figure 2.17: Notation for hydrostatic force on an inclined plane surface of arbitrary shape.

R The diagram can be fairly confusing as it displays two different perspective of the inclined plane, which can be hard to grasp what exactly it is showing. Refer to this YouTube clip.

The entirety of the derivation using fig. 2.17 can be summed to the following

$$F_R = \gamma h_c A \tag{2.15}$$

...where h_c is the vertical distance from the fluid surface to the centroid of the area. This is a general equation for calculating the resultant force, which can be used for horizontal, vertical, or inclined surface.

2.6.1 Location of Resultant Force

Going back to what we discussed in the beginning of this section, which is the center of action, it is possible to determine the exact location of the resultant force.

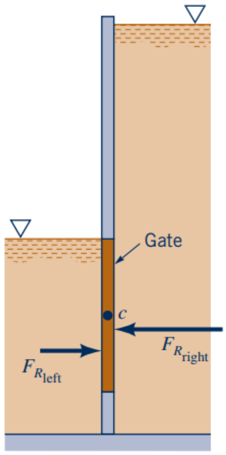


Figure 2.18: Resultant forces acting on the gate, where c is the centroid.

As we established, the center of pressure is always below the centroid of the surface for a vertical or inclined surface. So we are interested in figuring out how far it is from the centroid.

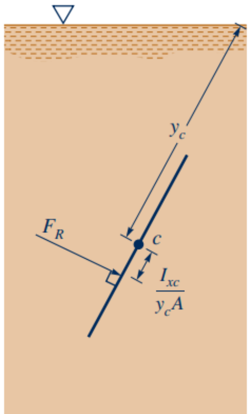


Figure 2.19: Distance from the centroid.

It might help if you refer back to fig. 2.17. The y coordinate of the resultant force can be determined by summation of moments around the x axis which is

$$y_R = \frac{I_{xc}}{y_c A} + y_c \tag{2.16}$$

...where I_{xc} is the second moment of the area about the x-axis passing through the centroid of the area and y_c is y coordinate of the centroid.

R I'll expand more on how we go about obtaining I_{xc} in the next page, but it's not as complicated as you may think from the description of it.

If we want to find the distance of the center of action from the centroid, as shown in fig. 2.19, we can simply rewrite the equation as

$$y_R - y_c = \frac{I_{xc}}{y_c A} \tag{2.17}$$

Likewise, we can also calculate the x coordinate of the resultant force which is

$$x_R = \frac{I_{xyc}}{y_c A} + x_c \tag{2.18}$$

...where I_{xyc} is the product of inertia with respect to an orthogonal coordinate system passing through the centroid and x_c is the x coordinate of the centroid.

R We don't really care that much about the x coordinate of the resultant force. Often the question relating to resultant force are interested in the y coordinate.

The centroidal coordinates and moments of inertia for some common areas are given by fig. 2.20.

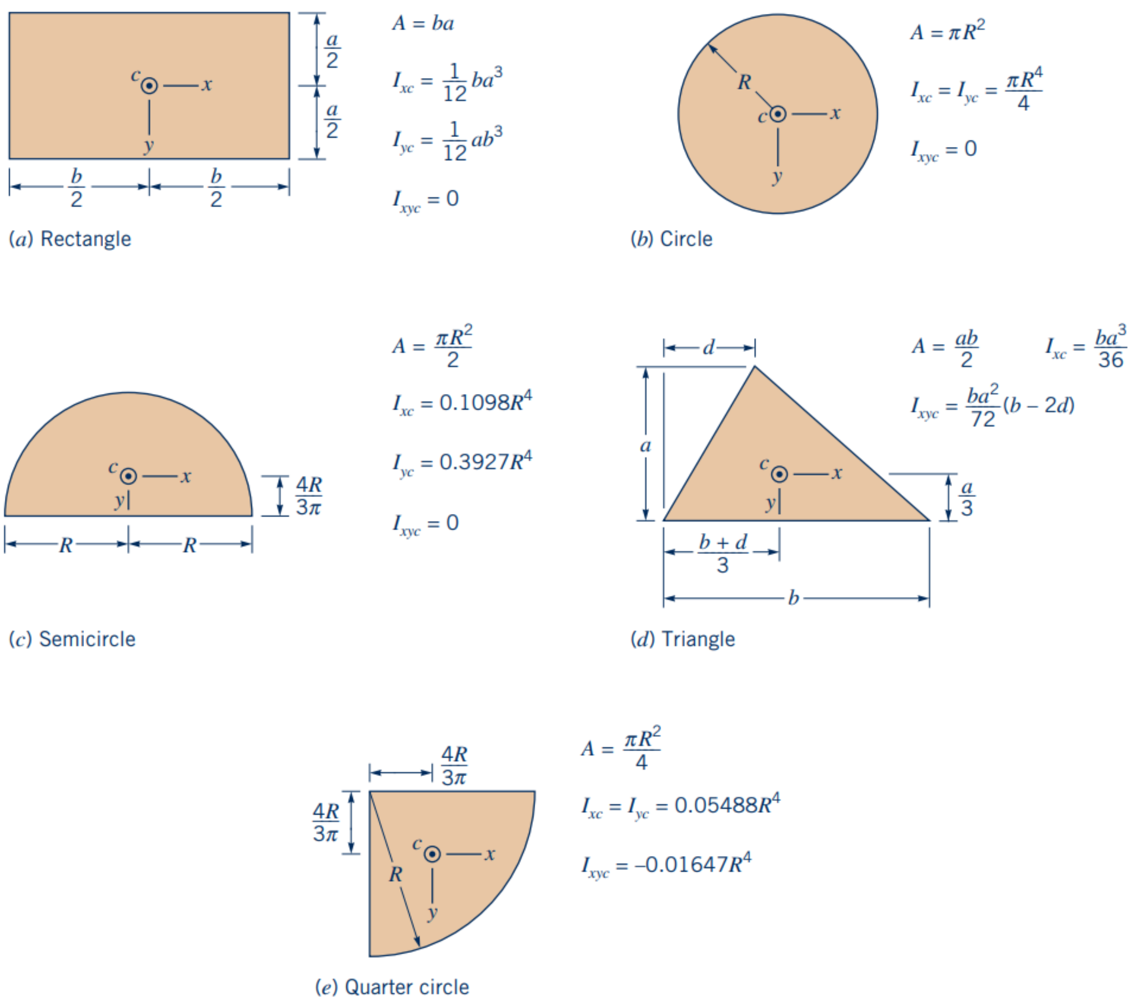


Figure 2.20: Geometric properties of some common shapes.

R In the test, you will be provided the respective properties according to the shape of the gate provided in the question.

The most confusing part which throws many people off is recognizing what parameters are your a, b and R, with respect to the x and y coordinates or measurement of the area. Refer to the problem in the next page.

Problem 2.1 A rectangular gate 6 m tall and 5 m wide in located in the side of an open tank as shown in fig. 2.21. Find the minimum applied force F to the gate to hold back a depth of water.

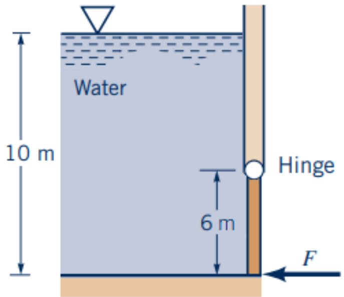


Figure 2.21: Diagram for Problem 2.1 (Not drawn to scale).

If we refer back to fig. 2.20, we are told the gate is rectangular, so we would use

$$I_{xc} = \frac{1}{12} b a^3$$

. . . but from the width and length, which one would be our b and a or x and y values respectively?

R You have to remember for the figures they show in these type of problems like in fig. 2.21, is that x is normal (i.e. perpendicular) to the gate. For reference, go back to fig. 2.17.

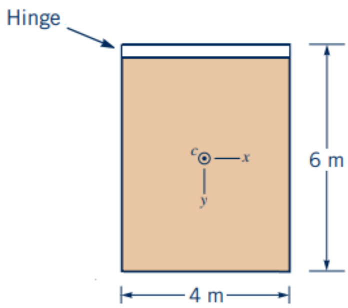


Figure 2.22: Side view of gate

It may help to draw out the side view of the gate if it is not otherwise provided, then we can say

$$I_{xc} = \frac{1}{12}(4\,\mathrm{m})(6\,\mathrm{m})^3 = 72\,\mathrm{m}^4$$

Now to solve the actual problem of finding the minimum applied force F, we have to do a short recap on what moment is.

2.6.2 Moment

The main concept of moment (similar to torque, which you maybe more familiar with) is a measure of its tendency to cause a body to rotate about a specific point, which in this case, is the hinge.

$$M = F \cdot r \tag{2.19}$$

. . . where r is the distance of the applied force from the axis of rotation.

R This is a very simplified explanation, but hopefully should be enough to understand what the problem is asking for.

With these types of problem, often the gate is connected to a hinge on one end and so we would have two opposing forces acting on the gate: (1) the resultant force F_R due to fluid and (2) the force F which we are looking to solve.

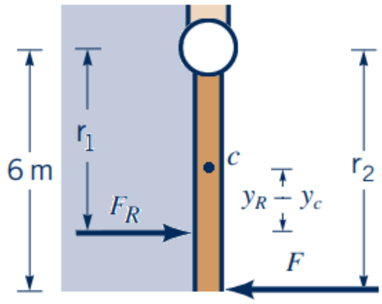


Figure 2.23: Forces acting on the gate.

To find the minimum applied force, we'll denote the sum of moment about the hinge to be

$$\sum M = F_R \cdot r_1 - F \cdot r_2 = 0$$

Again as recap, the center of action where the resultant force is being applied is always just below the centroid, which we can calculate using eq. (2.17), that is

$$y_R - y_c = \frac{I_{xc}}{y_c A} = \frac{72\,\mathrm{m}^4}{7\,\mathrm{m} \cdot 24\,\mathrm{m}^2} \approx 0.429\,\mathrm{m}$$

...then

$$r_1 = \frac{6\,\mathrm{m}}{2} + 0.429\,\mathrm{m} = 3.429\,\mathrm{m}$$

R The distance from the hinge to the resultant force is the distance from centroid (i.e. length divided by two) plus $y_r - y_c$.

The resultant force can easily be calculated by using eq. (2.15)

$$F_R = \gamma h_c A = (9807\,\mathrm{N\,m^{-3}})(7\,\mathrm{m})(24\,\mathrm{m}^2) \approx 164.76 \times 10^4\,\mathrm{N}$$

...then the minimum applied force F to the gate to hold back a depth of water is

$$F = \frac{F_R \cdot r_1}{r_2} = \frac{164.76 \times 10^4\,\mathrm{N} \cdot 3.429\,\mathrm{m}}{6\,\mathrm{m}} \approx 941.59 \times 10^3\,\mathrm{N}$$

2.7 Pressure Prism

In this section, we'll cover another method which we can use to calculate the resultant force acting on a surface similarly to previous one.

> **R** It is recommended that you the previous method, as this method only applies to limited cases, more specifically rectangular surfaces.

The whole idea revolves around using the volume of the pressure prism to determine the resultant force. The pressure prism is essentially the pressure distribution on a surface, which demonstrate how pressure varies in the surface.

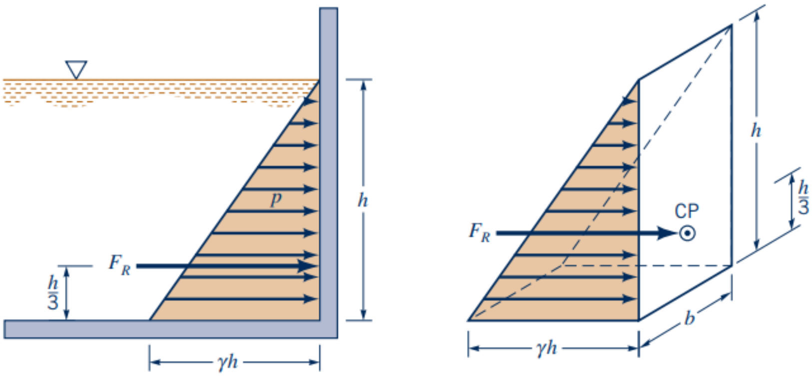


Figure 2.24: Pressure prism for vertical rectangular area.

For example, in a vertical surface, the pressure is equal to 0 at the upper surface and equal to γh at the bottom. Let the area of the rectangular surface be bh, then the volume of the pressure prism (i.e. the triangular prism) is

$$V = \frac{1}{2}(\gamma h)(bh) \tag{2.20}$$

If we were to use the previous method to calculate the resultant force, we will get

$$F_R = \gamma h_c A = \gamma\left(\frac{h}{2}\right)bh \tag{2.21}$$

Notice that eqs. (2.20) and (2.21) are equivalent of on another,

$$\frac{1}{2}(\gamma h)(bh) = \gamma\left(\frac{h}{2}\right)bh$$

. . . that is to say, that the resultant force acting on the rectangular surface is equal to the volume of the pressure prism.

2.7.1 Submerged Rectangular Plate

This same graphical approach can be used for plane rectangular surfaces that do not extend up to the fluid surface.

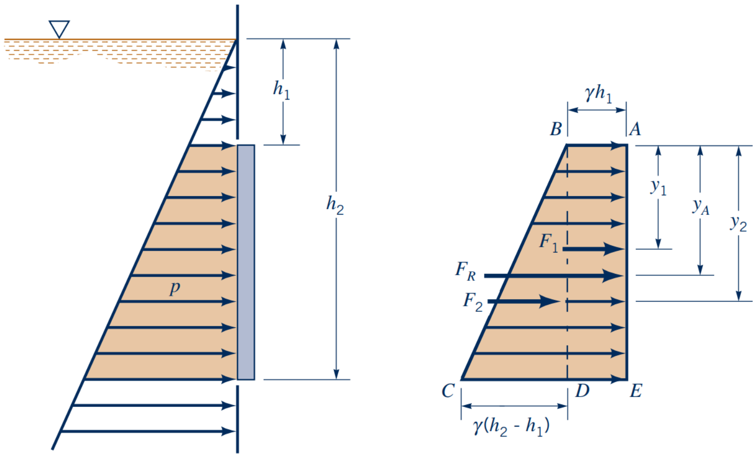


Figure 2.25: Graphical representation of hydrostatic forces on a vertical rectangular surface.

In this instance, the pressure prism forms a trapezoidal, which you may guess is a bit more complicated to find the volume. And so we decompose the pressure prism into two parts:

(i) A rectangular prism using points A, B, D, and E, which results in a force F_1.
(ii) A triangular prism using points B, C, and D, which results in a force F_2.

R I didn't write out the equation for the volume of the rectangular and triangular prism, cause it might just make it more confusing than it is. Refer to the MEC511 - Midterm Practice Solutions for an example.

You may have notice, by decomposing the trapezoid into two parts, it results in two forces due to the rectangular and triangular pressure prism. These two forces equal the resultant forces,

$$F_R = F_1 + F_2$$

. . . and likewise, the location can be determined by incorporating the height of which these forces occur int eh surface by

$$F_R y_A = F_1 y_1 + F_2 y_2$$

. . . where y_1 and y_2 can be determined by inspection:

- For a triangular pressure prism, the center of pressure occurs at $h/3$ from the base. Refer back to fig. 2.20(d) for a triangle.
- For a rectangular pressure prism, the center of pressure always occurs at the centroid, $h/2$.

R This is quite useful bit of info to remember and will save you a lot of hassle when problem-solving. Note this is only true for a rectangular surface.

3. Fluid Dynamics

3.1 Newton's Second Law

The main focus of this chapter revolves around Bernoulli's equation, which we can derive by using Newton's second law; determining the forces acting on a fluid particle and applying

$$\vec{F} = m\vec{a}$$

However, due to how it's derived, it has a few limitations, which will get to in a bit.

To apply Newton's second law to a fluid, we'll introduce two new concepts which go hand in hand, which are are steady flow and streamline.

- In steady flow, the velocity doesn't vary with time. Each particle slides along its path, and its velocity vector is everywhere tangent to the path
- Streamlines are the lines are tangent to the velocity vectors throughout the flow field; a path traced by a single particle in the fluid.

That is to say, each successive particle that passes through a given point, like (1) in fig. 3.1 will follow the same streamline.

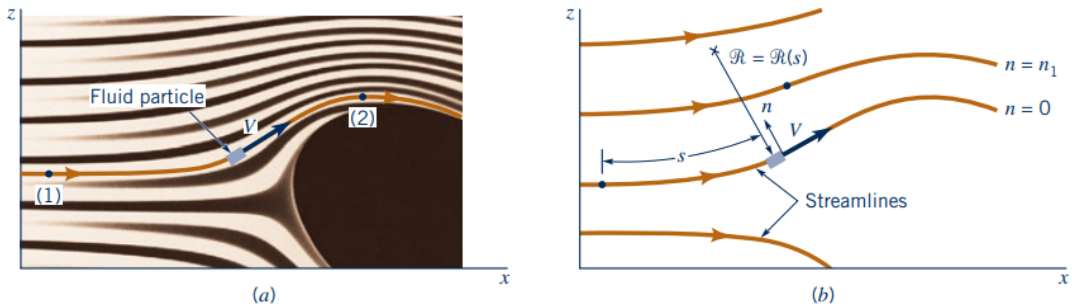


Figure 3.1: Flow of a fluid particle in motion.

In particular, there are two coordinate systems covered in the textbook, denoted by the subscript:

- s, the coordinate along the streamline and ...
- n, the coordinate normal to the streamline.

For many situations it is easiest to describe the flow in terms of the "streamline" coordinates based on the streamlines, which will be using throughout.

> R We only cover the forces acting along a streamline, so we don't need to worry about the coordinates normal to the streamline, thus will be skipped.

The components of acceleration in the s direction is given by:

$$a_s = V\frac{\delta V}{\delta s} \tag{3.1}$$

We will use the following in the next section to derive Bernoulli's equation.

3.2 F = ma Along a Streamline

For steady flow, the component of Newton's second law along the streamline direction, s, can be written as

$$\sum \delta F_s = \delta m a_s$$

... where δm is the particle mass and a_s is particle acceleration in the s direction, shown in eq. (3.1).

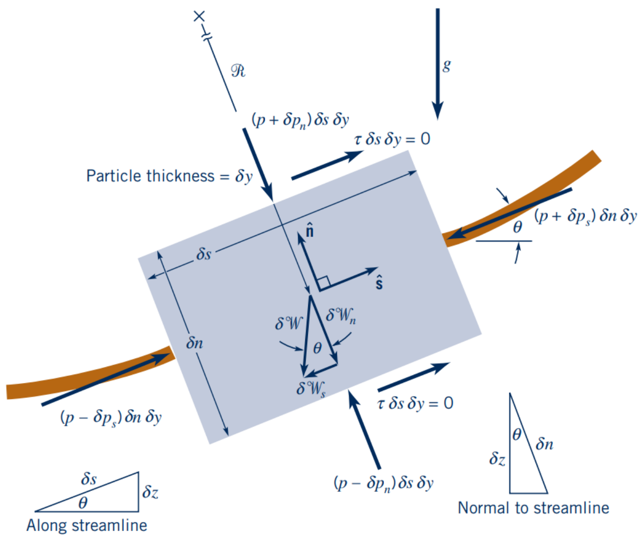


Figure 3.2: Free-body diagram of a fluid particle.

The textbook and slides demonstrates show the derivation using fig. 3.2, but in summary, Bernoulli's equation states

$$p + \frac{1}{2}\rho V^2 + \gamma z = \text{constant (along a streamline)} \tag{3.2}$$

... where p is the pressure, ρ is the density, V is the velocity, $\gamma = \rho g$ is specific weight, and z is the height above a reference level.

As mentioned prior, Bernoulli's equation has a few limitations, due to a few basic assumptions used in its derivation:

1. Steady flow; velocity doesn't vary with time
2. Inviscid flow; viscous effects are assumed negligible
3. Incompressible flow; density can be assumed to be constant

R There are equations for unsteady and compressible flow, but they are a bit more complicated and are not covered them in the span of this course.

3.3 Physical Interpretation

There are also various forms of Bernoulli's equation. The pressure form is shown in eq. (3.2), but it can also be presented in the head form,

$$\frac{p}{\gamma} + \frac{V^2}{2g} + z = \text{constant (along a streamline)} \tag{3.3}$$

. . . where each term has the units of length (i.e. in m) and represents a certain type of head:

- p/γ is the pressure head
- $V^2/2g$ is the velocity head
- z is the elevation head

Likewise, the same analysis can be made for the pressure form, which will discuss in the next section.

3.4 Static, Stagnation, Dynamic, and Total Pressure

In eq. (3.2), each term has the units in $\mathrm{N\,m^{-2}}$ or Pa, which represents a certain type of pressure:

- p is the static pressure
- $pV^2/2$ is the dynamic pressure
- γz is the hydrostatic pressure

R I'll go a bit in depth with how we measure the pressure in a fluid in motion, as it will help in the later parts when we talk about pitot tubes and how certain simplifications come about.

3.4.1 Static Pressure

We can measure the static pressure by drilling a hole perpendicular in a flat surface (fig. 3.3) and fasten a piezometer tube (fig. 3.4).

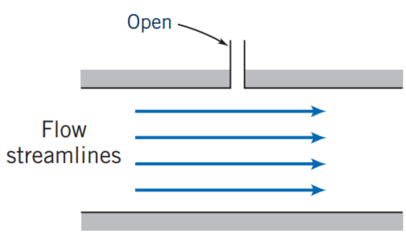


Figure 3.3: A hole perpendicular to the surface.