M_W from G_F at $\mathcal{O}(\alpha)$ and $\mathcal{O}(\alpha\alpha_s)$

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Abstract

We describe the calculation of the W-mass M_W from the Fermi-constant G_F utilizing muon decay. These notes closely follow the german-language summary in [1].

1 Definitions and Setup

The decay $\mu^- \to e^- \nu_\mu \overline{nu}_e$ is described with the phenomenological 4-Fermi Lagrangian

$$\mathcal{L}_F = \frac{4G_F}{\sqrt{2}}(\overline{\nu}_e \gamma^\alpha P_L l_e)(\bar{l}_\mu \gamma_\alpha P_L \nu_\mu). \tag{1}$$

This Lagrangian represents a leading order effective theory to the full electroweak interaction, after the heavy W boson has been integrated out.
To leading order we can calculate the decay width for the reaction $\mu^- \to \nu_\mu e^- \overline{\nu}_e$ by integrating the Born-level matrix element over the entire allowed phase space. We find to leading order, but retaining the full electron mass dependence:

$$\Gamma_\mu^{\mathrm{LO}} = \frac{G_F^2 m_\mu^5}{192\pi^3} F\left(\frac{m_e^2}{m_\mu^2}\right) \qquad \text{with} \qquad F(x) = 1 - 8x + 8x^3 - x^4 - 12x^2 \log x. \tag{2}$$

If one were to calculate higher orders from expanding the W propagator of the process one would find additional terms modifying the result

$$\Gamma_\mu^{\mathrm{Born}} = \left(1 + \frac{3}{5}\frac{m_\mu^2}{M_W^4} + ...\right)\Gamma_\mu^{\mathrm{LO}}. \tag{3}$$

These terms are usually contained in the definition of the Fermi constant, though they represent SM corrections beyond 4-Fermi theory. Additionally one includes pure QED corrections to process when defining the Fermi constant. This we explain in the next section.

2 QED Corrections to the Fermi Model

The pure QED corrections to muon decay in the 4-Fermi model are virtual loop corrections as well as the real radiation of additional photons off the muon or electron legs. The corrections factorize and we can write the QED contributions as a correction factor

$$\Gamma_\mu^{\mathrm{LO}} = \frac{G_F^2 m_\mu^5}{192\pi^3} F\left(\frac{m_e^2}{m_\mu^2}\right)(1+\Delta q). \tag{4}$$

This serves as the definition of the Fermi constant.
To next-to-leading order one finds the corrections to be [2–4]

$$\Delta q^{(\alpha)} = \frac{\alpha(m_\mu^2)}{2\pi}\left(\frac{25}{4} - \pi^2\right). \tag{5}$$

Through the necessary regularization procedure one introduces an unphysical running of the electromagnetic coupling constant. The NLO beta function therefor connects the coupling at the muon scale to the coupling a zero momentum (so-called Thompson limit)

$$\alpha(m_\mu^2) = \frac{\alpha}{1 - \frac{\alpha}{3\pi}\log\frac{m_\mu^2}{m_e^2}}. \tag{6}$$

Higher order corrections are also known and are consequently included in the definition of G_F [5, 6]. One has to distinguish between purely photonic corrections and corrections with closed fermion loops, in which only both leptons and hadrons are allowed to contribute. At NNLO we find

$$\Delta q^{(\alpha^2)} = -\left(\frac{\alpha(m_\mu^2)}{\pi}\right)^2 (6.700 \pm 0.002), \tag{7}$$

with the corresponding 2-loop running of α.

3 Matching Fermi-theory to the SM and M_W

In an effort to extract the value of M_W from the much more precise measurement of G_F we have to match the effective 4-Fermi theory (G_F in this case would be considered a so-called Wilson coefficient) to the full electroweak SM.
A lowest order analysis of the muon decay process in the SM enables us to compare coefficients and determine

$$\frac{G_F}{\sqrt{2}} = \frac{\alpha\pi}{2 s_W^2 M_W^2}. \tag{8}$$

This relation can however be improved upon through the introduction of higher order electroweak (starting at 2-loop also QCD) corrections, which would also be integrated out in

4-Fermi theory. The QED corrections considered earlier would also appear in the full SM. Since they have however been included in the 4-Fermi calculation one needs to carefully combine the two.
When matching the SM to the 4-Fermi theory we consider all external momenta of the process to be identically zero. This enables us to write all radiative corrections in one factor multiplying the Born-level expression

$$\frac{G_F}{\sqrt{2}} = \frac{\alpha\pi}{2s_W^2 M_W^2}(1+\Delta r). \tag{9}$$

All terms that are neglected in doing so are suppressed by M_W and are therefor not playing a role. When doing these calculations one usually also neglects all fermion masses running in the loops as well as the light quark masses.
This enables us to invert the above relation and solve for M_W

$$M_W^2 = M_Z^2\left(\frac{1}{2}+\sqrt{\frac{1}{4}-\frac{\alpha\pi}{\sqrt{2}G_F M_Z^2}(1+\Delta r(M_W^2,...))}\right). \tag{10}$$

Note that Δr is also dependent on M_W. The above relation can then be solved for the "true" value of M_W iteratively. The solution converges since M_W is an attractive fixed point of the relation.

4 Corrections with a closed Fermion loop

Corrections to Δr with a closed Fermion loop (schematically called $N_f\alpha$) represent a meaningful, gauge invariant subset of the full 1-loop corrections. At this order only two contributions exist: The Fermionic Self-Energy type correction to the W-propagator mediating the decay ((A) in Fig. 1) and the Counterterms deriving from closed-Fermion-loop gauge boson self-energies (the external Wavefunction renormalization is purely Bosonic in nature). Since all internal field renormalizations cancel in the final result we only have to consider the contributions stemming from the W-mass renormalization δM_W^2, the mixing angle renormalization δs_W as well as the charge Renormalization δZ_e.

At one loop order the gauge boson mass renormalizations are defined via the complex pole of their propagators. We simply quote the result here from the literature [7]

$$\delta M_W^2 = \Re\left\{\Sigma_{WW}^{(1)}(M_W^2)\right\} \quad \text{and} \quad \delta M_Z^2 = \Re\left\{\Sigma_{ZZ}^{(1)}(M_Z^2)\right\} \tag{11}$$

for the transversal parts of the self-energies.
The on-shell mixing angle $s_W^2 = 1 - c_W^2$ is not an independent parameter and is renormalized

$$s_W^2 = 1 - \frac{M_W^2}{M_Z^2} \quad \text{such that} \quad \delta s_W = \frac{c_W^2}{2s_W}\left[\frac{\delta M_Z^2}{M_Z^2} - \frac{\delta M_W^2}{M_W^2}\right]. \tag{12}$$

The renormalization of the electric charge is more complicated and described in the following.

4.1 Charge Renormalization and $\Delta\alpha$

The charge renormalization is defined via the photon vacuum polarization and the transversal part of the $\gamma - Z$ mixing via

$$\delta Z_e = \frac{1}{2}\Pi_{\gamma\gamma}(0) - \frac{s_W}{c_W}\frac{\Sigma_{\gamma Z}(0)}{M_Z^2} \quad \text{with} \quad \Pi_{\gamma\gamma}(p^2) = \frac{\Sigma_{\gamma\gamma(p^2)}}{p^2}. \tag{13}$$

In the limit of vanishing Fermion masses the photon vacuum polarization diverges. Though we can include the lepton masses to regulate some of these divergencies, this is not possible in the case of closed quark loops. At zero momentum transfer QCD-confinement effects kick in and the masses of the light quarks are not defined satisfyingly enough. Contributions from bosonic loops are well-behaved.
We therefor split the photon vacuum polarization into its components according to

$$\Pi_{\gamma\gamma}(0) = \Pi_{\gamma\gamma}^{\text{top}}(0) + \Pi_{\gamma\gamma}^{\text{bos}}(0) + \Pi_{\gamma\gamma}^{\text{light}}(0). \tag{14}$$

The top quark is heavy enough to neglect confinement effects and directly calculate the one-loop vacuum polarization [8]. Analogously we find the bosonic piece.

$$\begin{aligned}
\Pi_{\gamma\gamma}^{\text{top}}(0) =& \frac{2\alpha}{9\pi}(D-2)\frac{A_0(m_t^2)}{m_t^2} \\
\Pi_{\gamma\gamma}^{\text{bos}}(0) =& \frac{\alpha}{24\pi}(D-2)(D-13-12c_W^2-12s_W^2)\frac{A_0(M_W^2)}{M_W^2}
\end{aligned} \tag{15}$$

To include the contributions from the light fermions we use a little trick [9]. We can calculate the the vacuum polarization at the Z-scale ($p^2 = M_Z^2$), which is high enough to consider the quarks asymptotically free - the UV behavior is identical to the the one at zero momentum. Therefor the difference between the two is UV-finite and we can write the renormalized photon vacuum polarization for light quark loops as

$$\hat{\Pi}_{\gamma\gamma}^{\text{light}}(M_Z^2) = \Pi_{\gamma\gamma}^{\text{light}}(M_Z^2) - \Pi_{\gamma\gamma}^{\text{light}}(0). \tag{16}$$

This expression can now be solved for the desired photon vacuum polarization at zero momentum. The renormalized photon vacuum polarization that corresponds to the finite difference between $\Pi_{\gamma\gamma}^{\text{light}}(M_Z^2)$ and $\Pi_{\gamma\gamma}^{\text{light}}(0)$ is directly responsible for a shift of the electromagnetic fine structure constant according to

$$\alpha(M_Z^2) = \alpha(0)(1+\Delta\alpha) + \mathcal{O}(\alpha^3) \quad \text{with} \quad \Delta\alpha = -\hat{\Pi}_{\gamma\gamma}^{\text{light}}(M_Z^2). \tag{17}$$

All that's left to do is to determine the numerical value of this shift $\Delta\alpha$. In the case of the leptons, as mentioned at the beginning, we can straightforwardly calculate the diagrams with lepton masses in place. This contribution amounts, at 1-loop, to

$$\hat{\Pi}_{\gamma\gamma}^{\text{lep}}(M_Z^2) = \sum_l \frac{\alpha}{3\pi}\left(\frac{5}{3} + \log\frac{m_l^2}{M_Z^2}\right) + \mathcal{O}\left(\frac{m_l^2}{M_Z^2}\right) = -0.031419. \tag{18}$$

Some of the higher orders are in the literature [10].
The hadronic contribution on the other hand is extracted from experimentally measured scattering cross sections. Via the optical theorem we can determine

$$\hat{\Pi}^{\text{had}}_{\gamma\gamma}(M_Z^2) = \frac{\alpha}{3\pi} M_Z^2 \int_{4m_\pi^2}^{\infty} ds \frac{R_\gamma(s)}{s(s - M_Z^2 - i\epsilon)} \tag{19}$$

where the R-ratio is defined as

$$R_\gamma(s) = \frac{\sigma(e^+e^- \to \gamma \to \text{had})}{\sigma(e^+e^- \to \gamma \to \mu^+\mu^-)}. \tag{20}$$

In the non-perturbative, low-energy regime the ratio is directly measured where at higher energies QCD becomes asymptotically free and we can use perturbation theory. Results and precise definitions can be found in the literature [11]. This however is not a unique procedure. Using the perturbative calculations down to the τ-scale is in good agreement with this approach [12–14].
Using the above described approach we find the shift of the electromagnetic fine structure constant to be

$$\Delta\alpha = 0.05954 \pm 0.00065 \quad \text{for} \quad M_Z^2 = 91.1867. \tag{21}$$

5 Electroweak/Bosonic corrections to Δr

The purely Bosonic corrections to Δr are divided into different categories and are described in the following.

5.1 Self-Energy Corrections

Bosonic corrections to the W-propagator are analogously calculated to the Fermionic ones. Example diagrams are shown in Fig. 1. Here it is easiest to employ a projector, designed to extract the correctly normalized transversal part of the Self-Energy

$$\Sigma_T(p^2) = \frac{1}{D-1}\left(\eta^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}\right)\Sigma^{\mu\nu}(p^2). \tag{22}$$

This procedure works at arbitrary loop order. Note though that any of the irreducible contributions include the Counterterm insertions of lower loop diagrams.
For the vacuum Self-energy, with the momentum $p^2 = 0$ (as is the case for the W-propagator in our approximation) the projector simplifies to

$$\Sigma_T = \frac{1}{D}\Sigma^{\mu\nu}\eta_{\mu\nu}. \tag{23}$$

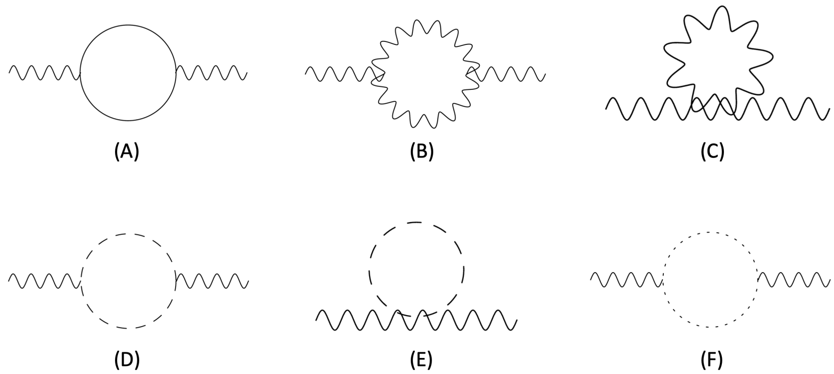

Figure 1: Schematic representation of possible Self-Energies. Contributions: (A) Fermionic, (B) and (C) Bosonic self-coupling, (D) and (E) Goldstone and (F) Ghosts.

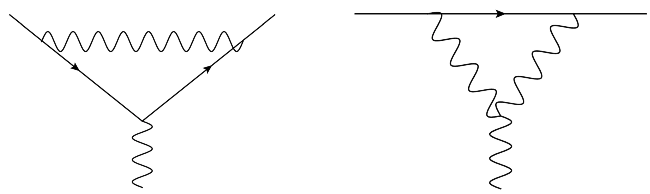

Figure 2: Schematic representation of possible Vertex corrections. No scalar contributions since all external fermions are massless.

5.2 Vertex Corrections

The class of weak Vertex corrections are shown in Fig. 2. Similarly to before we can reduce the FeynArts output for the Vertex function $\Gamma^{\mu}_{Wl\nu} = \Delta V_{\text{Vertex}} \frac{e}{\sqrt{2}s_W} \gamma^{\mu} P_L$ and find the Vertex contributions via a projector

$$\Delta V_{\text{Vertex}} = -\frac{s_W}{\sqrt{2}eD} \text{Tr}[\Gamma^{\mu} P_L \gamma_{\mu}]. \tag{24}$$

Note that here we have contributions from both the electron and muon lines, as well as the corresponding Counterterms.

5.3 Box Corrections

Finally the electroweak box corrections are shown in Fig. 3. Again we can decompose the irreducible box diagrams into form factors along their spinor chains

$$\begin{aligned} \mathcal{M}_{\text{Box}} &= \mathcal{M}_{\text{Born}} \Delta r \\ &= C \, (\overline{u} \Gamma^{\mu}_1 u) \, (\overline{u} \Gamma_{2\mu} u) \end{aligned} \tag{25}$$

where due to helicity arguments the form factors inside the spinor chains have to have the structure $\Gamma^{\mu}_i = c_i \gamma^{\mu} P_L$.

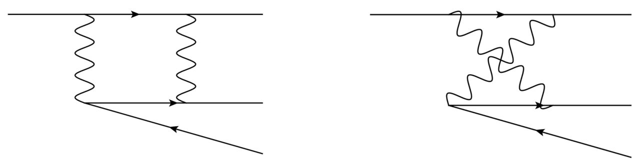

Figure 3: Schematic representation of possible Box corrections. No scalar contributions since all external fermions are massless.

This can be inverted to find Δr_{Box}:

$$\Delta r_{\text{Box}} = \frac{C}{D} \frac{s_W^2 M_W^2}{2e^2} \text{Tr}[\Gamma_1^\mu P_L \gamma^\nu] \text{Tr}[\Gamma_{2\mu} P_L \gamma_\nu]. \tag{26}$$

Explicitly one finds (ignoring diagrams containing photons for now)

$$\Delta r_{\text{Box}} = \frac{\alpha}{4\pi} \frac{5 - 10 s_W^2 + 2 s_W^4}{2 s_W^2} \log \frac{M_Z^2}{M_W^2} \tag{27}$$

which is finite without the inclusion of Counterterms.

6 QED-like corrections to Δr

The QED-like corrections to Δr are topologically different between SM and 4-Fermi theory. We therefor have to include the difference between the diagrams in Fig. 4. The IR structure

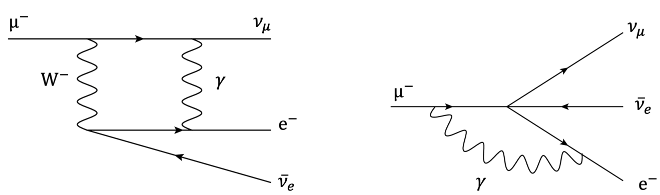

Figure 4: QED corrections in the full SM and the 4-Fermi theory.

is the same in the SM and 4-Fermi calculation so that the corresponding poles cancel. There is however a non-trivial UV divergence stemming from the external Wavefunction renormalization of the 4-Fermi model, that is not completely compensated ($B_0(0,0,0)$ function).
Two account for this contribution one can do one of two things: Due to issues dimensional regularization exhibits in this case one can abandon it altogether and use e.g. Pauli-Villars regularization (introduction of a heavy photon partner of mass Λ that compensates the divergence) to find

$$\Delta r = \frac{\alpha}{4\pi} \log \frac{M_W^2}{\Lambda^2}. \tag{28}$$

To account for the difference in regularization schemes we can then subtract the external Wavefunction renormalization calculated in Pauli-Villars regularization and add the same terms calculated in dimensional regularization on top.
Alternatively one can perform the entire calculation dimensional regularization. To this end a modified Chisholm identity has to be utilized

$$\gamma^\mu\gamma^\nu\gamma^\rho = (-i\epsilon^{\mu\nu\rho\sigma}\gamma_\sigma\gamma_5 + g^{\mu\nu}\gamma^\rho + g^{\rho\nu}\gamma^\mu - g^{\mu\rho}\gamma^\nu)\left(1+\frac{3}{4}(D-4)\right). \tag{29}$$

This, together with the non-compensated pole, gives a finite contribution to Δr.
Both methods lead to the same result

$$\Delta r_{\mathrm{QED}} = \frac{\alpha}{4\pi}\left[-\frac{1}{\epsilon}+\frac{1}{2}+\log\frac{M_W^2}{\mu^2}\right]. \tag{30}$$

The remaining pole cancels once all contributions are added up.

7 Final Result

If one combines all the previously described contributions one arrives at a compact expression for the complete 1-loop corrections to Δr [15, 16]:

$$\begin{aligned}\Delta r =& \Pi_{\gamma\gamma}(0) - \frac{c_W^2}{s_W^2}\left(\frac{\delta M_Z^2}{M_Z^2} - \frac{\delta M_W^2}{M_W^2}\right) + \frac{\Sigma_{WW}(0) - \delta M_W^2}{M_W^2} + 2\frac{c_W}{s_W}\frac{\Sigma_{\gamma Z}(0)}{M_Z^2}\\ &+ \frac{\alpha}{4\pi s_W^2}\left(6 + \frac{7-4s_W^2}{2s_W^2}\log c_W^2\right).\end{aligned} \tag{31}$$

Plugging numbers in for all constants, e.g.

$$\begin{aligned}G_F = 1.16637\times 10^{-5}\mathrm{GeV}^{-2} \quad \frac{1}{\alpha} = 137.03599976 \quad M_Z = 91.1527\mathrm{GeV}\\ \Delta\alpha = 0.05954 \quad m_H = 125.1\mathrm{GeV} \quad m_t = 174.3 \quad m_b = 4.7\mathrm{GeV}\end{aligned} \tag{32}$$

and iteratively solving for M_W as described in the beginning we can find a theoretical prediction for the W-Mass

$$M_W = 80.4437\mathrm{GeV}. \tag{33}$$

8 Higher Orders

In order to incorporate higher orders, it is instructive to have a compact formula that then can be expanded order by order in perturbation theory. The complete corrections to Δr can be written as

$$\begin{aligned}1+\Delta r =& \left(\frac{1+\delta Z_e}{1+\delta s_W/s_W}\sqrt{Z_W Z_L^l Z_L^\nu} + \Delta V_{\mathrm{Vertex}}\right)^2 \frac{M_W^2}{(M_W^2+\delta M_W^2)Z_W - \Sigma_W(0)} + \Delta r_{\mathrm{Box}}.\\ =& (1+\Delta r_{\mathrm{Vertex}})\frac{M_W^2}{M_W^2 - \hat{\Sigma}_W(0)} + \Delta r_{\mathrm{Box}}\end{aligned} \tag{34}$$

where $\hat{\Sigma}$ represents the renormalized, transversal Self-Energy.
This way one can focus on calculating the irreducible corrections to Self-Energy, Vertex and Box and not worry about double counting.

8.1 QCD corrections

The easiest higher order correction is coming in at order $\alpha\alpha_S$ and affects, much like the closed Fermion-loop corrections previously, only the Bosonic self-energies [17–19]. The schematic

Figure 5: Schematic representation of possible QCD corrections to the Self-Energies. The Fermionic loop is quarks only.

diagrams are shown in Fig. 5.
To include these into our calculation, it is worth noting that we define the 2-loop vacuum master integral as

$$T_{134}(m_1^2, m_3^2, m_4^2) = \frac{\mu^{4\epsilon}}{(i\pi^2)^2} \int \frac{d^D q_1 d^D q_2}{(q_1^2 - m_1^2)((q_1 - q_2)^2 - m_3^2)(q_2^2 - m_4^2)} \tag{35}$$

As well as the Self-energy master integral as

$$\begin{aligned} &T_{12345}(p^2, m_1^2, m_2^2, m_3^2, m_4^2, m_5^2) \\ &= \frac{\mu^{4\epsilon}}{(i\pi^2)^2} \int \frac{d^D q_1 d^D q_2}{(q_1^2 - m_1^2)((q_1 + p)^2 - m_2^2)((q_1 - q_2)^2 - m_3^2)(q_2^2 - m_4^2)((q_2 + p)^2 - m_5^2)} \end{aligned} \tag{36}$$

All other integrals can be reduced to these.
We include the QCD-corrections in our code.

8.2 2-loop and Beyond

Higher order EW corrections have been done completely at 2-loop (cf e.g. [20–22]), as well as mixed EW/QCD corrections at 3-loop [23, 24]. The complete EW 3-loop corrections are work in progress.

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