ISIS-1104-07 Matemática Estructural y Lógica
MidTerm 3
Date: May 10, 2018

- This exam is closed book , closed notes.
- You are allowed to use the cheat-sheet that was handed out and left on Sicua+ for downloading.
- The use of any electronic device is prohibitted.
- There should be no communication among students.
- Answer in the spaces provided; no additional sheets of paper will be accepted.
- Do not forget to write you name and studient id number on the exam before handing it in.

Name	StudentId #

Do not write below ithis line

1.	25 %	
2	20 %	
3.	30 %	
4.	15 %	
5.	10 %	

Exercise 1. Prove the following statement, without using induction:

$$(gcd(n,m)=1) \wedge (a \ \equiv_n \ b) \wedge (a \ \equiv_m \ b) \Rightarrow (a \ \equiv_{m \cdot n} \ b)$$

There are two possible solutions

Solution 1:

	Expression	Justification
1	$gcd(n,m)=1$	Hypothesis
2	$a \ \equiv_n \ b$	Hypothesis
3	$a \ \equiv_m \ b$	Hypothesis
4	$n \cdot \hat{s} + m \cdot \hat{t} = 1$	Def. of gcd. with linear combinations (1)
5	$n \mid (b-a)$	Def. of . $\equiv_z$. (2)
6	$m \mid (b-a)$	Def. of . $\equiv_z$. (3)
7	$n \cdot \hat{x} = (b-a)$	Def od . | . (5)
8	$m \cdot \hat{y} = (b-a)$	Def od . | . (5)
9	$n \cdot \hat{s} \cdot (b-a) + m \cdot \hat{t} \cdot (b-a) = (b-a)$	Multiply be $b-a$ (4)
10	$n \cdot \hat{s} \cdot m \cdot \hat{y} + m \cdot \hat{t} \cdot (b-a) = (b-a)$	Replace (8,9) (the first (b-a))
11	$n \cdot \hat{s} \cdot m \cdot \hat{y} + m \cdot \hat{t} \cdot n \cdot \hat{x} = (b-a)$	Replace (7,10)
12	$n \cdot m \cdot (\hat{s} \cdot \hat{y} + \hat{t} \cdot \hat{x}) = (b-a)$	Aritmetic (11)
13	$n \cdot m \mid (b-a)$	Def. of . | . (12)
14	$a \ \equiv_{n \cdot m} \ b$	Def. of . $\equiv_z$. (14)

Solution 2: (This was a much better apporach, and most students attempted to prove it like this)

	Expression	Justification
1	$gcd(n,m)=1$	Hypothesis
2	$a \ \equiv_n \ b$	Hypothesis
3	$a \ \equiv_m \ b$	Hypothesis
4	$a \ \equiv_{lcm(n,m)} \ b$	Th: $(a \ \equiv_n \ b) \wedge (a \ \equiv_m \ b) \Rightarrow a \ \equiv_{lcm(n,m)} \ b$
5	$gcd(n,m) \cdot lcm(n,m) = n \cdot m$	Theorem
6	$lcm(n,m) = n \cdot m$	Replace (1,4) and Arithmetic
7	$a \ \equiv_{n \cdot m} \ b$	Replace (6,4)

Exercise 2. Prove using induction over n:

$$(\forall n : \mathbb{N} | n > 1 : n! < n^n)$$

Basis Case ($n = 2$): $2! < 2^2$

$$\begin{array}{ll} & 2! < 2^2 \\ = & \quad \langle \text{ Def factorial and } x^n \rangle \\ & 2 < 4 \\ = & \quad \langle \text{ Arithmetic } \rangle \\ & \mathit{true} \end{array}$$

Inductive Case: $k! < k^k \Rightarrow (k+1)! < (k+1)^{(k+1)}$

I.H. $k! < k^k$

Prove: $(k+1)! < (k+1)^{(k+1)}$

	Expression	Justification
1	$k! < k^k$	H.I.
2	$(k+1) \cdot k! < (k+1) \cdot (k^k)$	Aritmetic, multiply by $k+1$, $k+1 > 0$, (1)
3	$k^k < (k+1)^k$	Arithmetic: since $k > 1$
4	$(k+1) \cdot (k^k) < (k+1) \cdot ((k+1)^k)$	Arithmetic multiply by $k+1$, (3) since $k+1 > 0$
5	$(k+1) \cdot k! < (k+1) \cdot (k+1)^{(k)}$	Transitivity (2,4)
6	$(k+1)! < (k+1)^{(k+1)}$	Def factorial and x^n (5)

Exercise 3. We define the function value from sequences of digits to integers as follows: $value : Seq_{\{0,\ldots,9\}} \to \mathbb{Z}$

Basis Case: $value(\epsilon) = 0$

Inductive Case: $value(d \triangleleft S) = d \cdot 10^{Long(S)} + value(S)$

Prove the following statement using structural induction over S**:**

$$value(S \triangleright d) = value(S) \cdot 10 + d$$

You may use the following lemma, without proving it.

$$Long(S \triangleright x) = 1 + Long(S)$$

Basis Case: $S = \epsilon$

$$value(\epsilon \triangleright d) = value(\epsilon) \cdot 10 + d$$

$$
\begin{aligned}
& value(\epsilon \triangleright d) = value(\epsilon) \cdot 10 + d \\
= & \quad \langle \text{ Def- } \triangleright \rangle \\
& value(d \triangleleft \epsilon) = value(\epsilon) \cdot 10 + d \\
= & \quad \langle \text{ Def- value } \rangle \\
& d \cdot 10^{Long(\epsilon)} + value(\epsilon) = value(\epsilon) \cdot 10 + d \\
= & \quad \langle \text{ Def- value } \rangle \\
& d \cdot 10^{Long(\epsilon)} + 0 = 0 \cdot 10 + d \\
= & \quad \langle \text{ Def- Long } \rangle \\
& d \cdot 10^{0} + 0 = 0 \cdot 10 + d \\
= & \quad \langle \text{ Arithmetic } \rangle \\
& True
\end{aligned}
$$

Inductive Case: I.H. $value(S \triangleright d) = value(S) \cdot 10 + d$

Prove $value((x \triangleleft S) \triangleright d) = value(x \triangleleft S) \cdot 10 + d$

$$
\begin{aligned}
& value((x \triangleleft S) \triangleright d) = value((x \triangleleft S)) \cdot 10 + d \\
= & \quad \langle \text{ Def } \triangleright \rangle \\
& value(x \triangleleft (S \triangleright d)) = value(x \triangleleft S) \cdot 10 + d \\
= & \quad \langle \text{ Def value } \rangle \\
& x \cdot 10^{Long(S \triangleright d)} + value(S \triangleright d) = (x \cdot 10^{Long(S)} + value(S)) \cdot 10 + d \\
= & \quad \langle \text{ Arithmetic } \rangle \\
& x \cdot 10^{Long(S \triangleright d)} + value(S \triangleright d) = x \cdot 10^{Long(S)+1} + value(S) \cdot 10 + d \\
= & \quad \langle \text{ Hint Arithmetic } \rangle \\
& value(S \triangleright d) = value(S) \cdot 10 + d \\
= & \quad \langle \text{ I.H. } \rangle \\
& \text{True}
\end{aligned}
$$

Exercise 4. Prove without using induction:

$$value(S \triangleright d) \;\equiv_5\; d$$

$$
\begin{array}{ll}
 & value(S \triangleright d) \;\equiv_5\; d \\
= & \qquad \langle \text{ Def. } \;\equiv_n \; \rangle \\
 & 5 \mid (d - value(S \triangleright d)\,) \\
= & \qquad \langle \text{ Previous exercise } \rangle \\
 & 5 \mid (d - value(S) \cdot 10 + d) \\
= & \qquad \langle \text{ Arithmetic } \rangle \\
 & 5 \mid \; - value(S) \cdot 10 \\
= & \qquad \langle \;\; 5 \mid 10 \; (10 = 2 \cdot 5) \text{ and } (a \mid b \Rightarrow a \mid c \cdot b) \; \rangle \\
 & true
\end{array}
$$

Exercise 5. Given the digits '0' to '9' and the following operators: $+$, $\div$, $\times$, $-$. We can form expressions of digits and operators. These expressions are just sequences of digits separted by operators thgat must start with a digit and end with a digit:

For example

- $0 + 1 - 2 + 9 - 8$ (this sequence has length 9)
- $0 + 1$ (this sequence has length 3)
- $1 + 0$ (this sequence has length 3)
- $2 + 6 \div 3 + 7 * 9 + 0$ (this sequence has length 11)

Answer the following questions.

1. How many sequences of length 11 such that all operators are different? (ignoring semantics, sequence $4 + 3$ is differnet from $3 + 4$ and $5 + 2$). A sequence of length 11 has 5 operators. Howwere, since we only have four operaators, by the pigeonhole principle, there is at least one operator tha appears more than once. Therefore, the number of sequences in which all operators are different is 0.

2. Ignoring semantincs, how many sequences of length 7 can you form in which all digits are different?

   10*4*9*4*8*4*7

   Or $\frac{10!}{(10-6)!} \cdot 4^3$

3. Suppose we can now use numbers, not just digits. We could have expressions of the form 45+78*9. We are only interested in expressiones onf the form N*M (a single multiplication), where N and M can only be one of these numbers: 2,3,5,7,11,13,17,19,23, and $N \neq M$. How many different values can you form. For instance you can form 6 with the following expression $2 * 6$ you can form 143 with $13 * 11$

   Since all numbers are prime then there is only one way to from each value. The answer si how many ways can we choose two from nine:

   $\binom{9}{2} = 36$