ISIS-1104-05 Matemática Estructural y Lógica
MidTerm 1
Date: February 2, 2017

- Esta prueba es INDIVIDUAL.
- Sólo está permitido el uso de las hojas de fórmulas publicadas en Sicua+.
- Está prohibido el uso de cualquier dispositivo electrónico.
- El intercambio de información con otro estudiante está terminantemente prohibido.
- Cualquier irregularidad con respecto a estas reglas podría ser considerada fraude.
- Responda el examen en los espacios proporcionados. No se aceptarán hojas adicionales.
- No olvide marcar el examen antes de entregarlo.
- Las preguntas son en inglés, pero si lo desea, puede responder en español.

IMPORTANTE: Soy consciente de que cualquier tipo de fraude en los exámenes es considerado como una falta grave en la Universidad. Al firmar y entregar este examen doy expreso testimonio de que este trabajo fue desarrollado de acuerdo con las normas establecidas. Del mismo modo, aseguro que no participé en ningún tipo de fraude.

Nombre	Carné
Firma	Fecha

NO ESCRIBIR NADA BAJO ESTA LÍNEA

1.1	10 %	
1.2	20 %	
2.1	15 %	
2.2	20 %	
3.1	15 %	
3.2	20 %	
Total	100 %	

1. [30 %] Propositional Calculus

Suppose we add the following function to the propositional calculus:

$$2of3(P, Q, R) \equiv ((P \ \wedge \ Q) \vee (P \ \wedge \ R) \vee (Q \ \wedge \ R)) \wedge \neg(P \wedge Q \wedge R)) \tag{1}$$

1.1. [10 %] Prove or refute: $2of3(true, q, r) \equiv (q \not\equiv r)$

$2of3(true, q, r)$

$= \quad \langle \text{ Def. } 2of3 \ \rangle$

$(\, (true \ \wedge \ q) \vee (true \ \wedge \ r) \vee (q \ \wedge \ r)) \wedge \neg \, (true \wedge q \wedge r)$

$= \quad \langle \text{ Identity of } \wedge \ \rangle$

$(q \vee r \vee (q \ \wedge \ r)\,) \wedge \neg(q \wedge r)$

$= \quad \langle \text{ Absorption } \rangle$

$(q \vee r) \wedge \neg(q \wedge r)$

$= \quad \langle \text{ Def. } \not\equiv \ \rangle$

$q \not\equiv r$

(q ∨ r ∨ (q ∧ r))

1.2. [20 %] Prove or refute the validity of the following inference rule:

$$\begin{array}{l} 2of3(p,q,r) \\ p \\ r \Rightarrow q \\ \hline \neg r \wedge q \end{array}$$

	Expression	Justification
1	$2of3(p,q,r)$	Hypothesis
2	p	Hypothesis
3	$r \Rightarrow q$	Hypothesis
4	$r \not\equiv q$	Ex. 1.1 (1,2)
5	$\neg(r \equiv q)$	Def. $\not\equiv$ (4)
6	$\neg r \equiv q$	$\neg \equiv$ (5)
7	$\neg r \vee q$	Def. $\Rightarrow$ (3)
8	$q \vee q$	Leibniz (6, 7)
9	q	Idempotency (8)
10	$\neg r$	Leibniz (9,6)
11	$\neg r \wedge q$	Composition (8,9)

2. Deduction in the propositional calculus [35 %]

We have four boxes each with a different color: white, red,blue, orange. There may be a prize in one or more of these boxes.

We know the following facts:

1. There are exactly two boxes among the boxes that are not white (red,blue, orange) that do not have a prize.

   $20f3(\neg R, \neg B, \neg O)$

2. If the blue box has a prize then the red box has a prize $B \Rightarrow R$

3. If the orange box does not have a prize then the red box does not have a prize

   $\neg O \Rightarrow \neg R$

4. The orange box has a prize if and only if the white box has a prize. $O \equiv W$

What boxes have a prize and what boxes do not have a prize? You must answer the question "prize or not?" for each box.

If the Orange box does not have a prize, then the Red one would not have a prize.By Modus Tollens, neither does the blue box. Contradicting the fact that exactly two boxes do not have a prize. Therefore, the orange box has a prize. To make 2of3 true, Blue and Red do not have prizes. White must have a prize because the white prize has a prize if and only if the orange one also has a prize.

This conclusion is the following:

$$\neg R \wedge \neg B \wedge O \wedge W$$

2.1. Modeling [15 %]

Using only the variables listed below to represent the facts of the problem, model the hypotheses and your conclusion regarding the contents of each box: prize or no prize.

W: The white box has a prize

R: The red box has a prize

B: The blue box has a prize

O: The orange box has a prize

2.2. Proofs in Propositional Calculus [20 %]

Formally prove your conclusion. Lemma 1: O

	Expression	Justification
1	$2of3(\neg R, \neg B, \neg O)$	Hypothesis
2	$\neg O \Rightarrow \neg R$	Hypothesis
3	$B \Rightarrow R$	Hypothesis
4	$\neg O$	Supposition
5	$\neg R \not\equiv \neg B$	Ex.1.1 (4,1)
6	$\neg(\neg R \equiv \neg B)$	Def. $\not\equiv$ (5)
7	$\neg R \equiv B$	Def. $\neg \equiv$ and double $\neg$ (6)
8	$\neg R$	M.P. (2, 4
9	$\neg B$	M.T. (8, 3)
10	B	Replace (7, 8)
11	$False$	Contradiction (9,10)

Since assuming $\neg O$ we arrived a contradiction, then we can conclude O

	Expression	Justification
1	$2of3(\neg R, \neg B, \neg O)$	Hypothesis
2	$((\neg R \ \wedge \ \neg B) \vee (\neg R \ \wedge \ \neg O) \vee (\neg B \ \wedge \ \neg O)) \wedge \neg(\neg R \wedge \neg B \wedge \neg O)$	Def 2of3 (1)
3	$((\neg R \ \wedge \ \neg B) \vee (\neg R \ \wedge \ \neg O) \vee (\neg B \ \wedge \ \neg O))$	Simplification (2)
4	O	Lemma 1
5	$((\neg R \ \wedge \ \neg B) \vee (\neg R \ \wedge \ \neg true) \vee (\neg B \ \wedge \ \neg true))$	Replacing (4,3)
6	$((\neg R \ \wedge \ \neg B) \vee (\neg R \ \wedge \ false) \vee (\neg B \ \wedge \ false))$	Def. false (5)
7	$(\neg R \ \wedge \ \neg B) \vee false \vee false$	Dominance $\wedge$ (6)
8	$\neg R \ \wedge \ \neg B$	Identity $\vee$ (7)
9	$O \equiv W$	Hypothesis
10	W	Leibniz (9, 4)
11	$O \wedge W \wedge \neg R \wedge \neg B$	Composition (4,8,10)

3. Predicate calculus [35 %]

You travel to a distant planet. Before arriving, you know the following:

1. There are two kinds of aliens: alphas or betas. Aliens cannot have two different types.

   $(\forall\ a\ |\ \ :\ alpha(a) \not\equiv \beta(a)\ )$

2. Aliens can be red or green. An alien is either red or green, but cannot be red and green.

   $(\forall\ a\ |\ \ :\ red(a) \not\equiv green(a)\ )$

3. All alphas are red or all alphas are green. $(\forall\ a\ |\ alpha(a)\ :\ red(a)\ ) \vee (\forall\ a\ |\ alpha(a)\ :\ green(a)\ )$

4. Betas have three eyes $(\forall\ a\ |\ beta(b)\ :\ three(b)\ )$

When you land on the planet, you are greeted by alien M.

1. M does not have three eyes $\neg three(M)$

2. M is red $red(M)$

This information is enough for us to conclude that all alphas are red. $(\forall\ a\ |\ alpha(a)\ :\ red(a)\ )$

3.1. Modeling [15 %]

Model the problem, using the following predicates:

- alpha(d) : d is alpha
- beta(d) : d is beta
- red(d) : d is red
- green(d) : d is green
- three(p): p has three eyes

3.2. Deduction in the predicate calculus [20 %]

Formally prove that you can reach the conclusion from the hypotheses and the premises.

$$
\begin{array}{ll}
 & \langle \text{ Premise } \rangle \\
 & \neg three(M) \\
\Rightarrow & \langle \text{ }\forall\text{-Modus Ponens with Hypothesis (4) } \rangle \\
 & \neg beta(M) \\
\Rightarrow & \langle \not\equiv \text{ deduction with Hypothesis (1) } \rangle \\
 & alpha(M) \\
\Rightarrow & \langle \text{ Composition with Premise } \rangle \\
 & alpha(M) \wedge red(M) \\
\Rightarrow & \langle \text{ }\exists\text{-Generalization } \rangle \\
 & (\exists\, a \mid \ : alpha(a) \wedge red(a)\ ) \\
= & \langle \text{ Trading } \rangle \\
 & (\exists\, a \mid alpha(a)\ :\ red(a)\ ) \\
= & \langle \text{ Double Negation } \rangle \\
 & \neg\neg(\exists\, a \mid alpha(a)\ :\ red(a)\ ) \\
= & \langle \text{ Generalized de Morgan } \rangle \\
 & \neg(\forall\, a \mid alpha(a)\ :\ \neg red(a)\ ) \\
\Rightarrow & \langle \not\equiv \text{ deduction with Hypothesis (2) } \rangle \\
 & \neg(\forall\, a \mid alpha(a)\ :\ green(a)\ ) \\
\Rightarrow & \langle \text{ }\forall\text{-Disjunctive Syllogism with Hypothesis (3) } \rangle \\
 & (\forall\, a \mid alpha(a)\ :\ red(a)\ )
\end{array}
$$

Another way: Lemma:$\neg(\forall\, a \mid alpha(a)\ :\ green(a)\ )$ Proof of Lemma be contradiction: We Assume: $(\forall\, a \mid alpha(a)\ :\ green(a)\ )$

$$
\begin{array}{ll}
 & \langle \text{ Assumption } \rangle \\
 & \neg(\forall\, a \mid alpha(a)\ :\ green(a)\ ) \\
\Rightarrow & \langle \text{ }\forall\text{-Modus Ponens with Hypothesis (4) } \rangle \\
 & \neg beta(M) \\
\Rightarrow & \langle \not\equiv \text{ deduction with Hypothesis (1) } \rangle \\
 & alpha(M) \\
\Rightarrow & \langle \text{ Composition with Premise } \rangle \\
 & alpha(M) \wedge red(M) \\
\Rightarrow & \langle \text{ }\exists\text{-Generalization } \rangle \\
 & (\exists\, a \mid \ : alpha(a) \wedge red(a)\ ) \\
= & \langle \text{ Trading } \rangle \\
 & (\exists\, a \mid alpha(a)\ :\ red(a)\ ) \\
= & \langle \text{ Double Negation } \rangle \\
 & \neg\neg(\exists\, a \mid alpha(a)\ :\ red(a)\ ) \\
= & \langle \text{ Generalized de Morgan } \rangle \\
 & \neg(\forall\, a \mid alpha(a)\ :\ \neg red(a)\ ) \\
\Rightarrow & \langle \not\equiv \text{ deduction with Hypothesis (2) } \rangle \\
 & \neg(\forall\, a \mid alpha(a)\ :\ green(a)\ )
\end{array}
$$

$\Rightarrow \quad \langle$ $\forall$-Disjunctive Syllogism with Hypothesis (3) $\rangle$
$(\forall\ a \mid alpha(a)\ :\ red(a)\ )$

Now the proof:

$\langle$ Premise $\rangle$
$\neg(\forall\ a \mid alpha(a)\ :\ green(a)\ )$
$\Rightarrow \quad \langle$ $\forall$-Disjunctive Syllogism with Hypothesis (3) $\rangle$
$(\forall\ a \mid alpha(a)\ :\ red(a)\ )$