ISIS-1104-01 Matemática Estructural y Lógica
MidTerm 2
Date: April 3, 2017

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- Está permitido el uso de las hojas de teoremas publicadas en sicua+.
- Está prohibido el uso de cualquier dispositivo electrónico.
- El intercambio de información con otro estudiante está terminantemente prohibido.
- Cualquier irregularidad con respecto a estas reglas podría ser considerada fraude.
- Responda el examen en los espacios proporcionados. No se aceptarán hojas adicionales.
- No olvide marcar el examen antes de entregarlo.

Nombre	Carné

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1.1	15%	
1.2	10%	
1.3	20%	
2.1.1	10%	
2.1.2	10%	
2.2.1	5%	
2.2.2	20%	
3	20%	
Total	110%	

1 [45%] Sets

We define a new set operator and a new set function:

$$A \oplus B = (A \cap B) \cup (\sim A \cap \sim B)$$
$$only2(A, B, C) = ((A \cap B) \cup (A \cap C) \cup (B \cap C)) \setminus (A \cap B \cap C)$$

Prove or refute the following statements. Remember U is the Universe.

1.1 [15%] $\sim (A \oplus B) = (A \cup B) \setminus (A \cap B)$

$$\begin{array}{ll}
 & \sim (A \oplus B) \\
= & \quad \langle \text{ Definition } \oplus \rangle \\
 & \sim ((A \cap B) \cup (\sim A \cap \sim B)) \\
= & \quad \langle \text{ De Morgan } \rangle \\
 & \sim (A \cap B) \cap \sim (\sim A \cap \sim B) \\
= & \quad \langle \text{ De Morgan } \rangle \\
 & \sim (A \cap B) \cap (\sim\sim A \cup \sim\sim B) \\
= & \quad \langle \text{ Double } \sim \rangle \\
 & \sim (A \cap B) \cap (A \cup B) \\
= & \quad \langle \text{ Commutativity and Def. of } \setminus \rangle \\
 & (A \cup B) \setminus (A \cap B)
\end{array}$$

1.2 [10%] $only2(U, B, C) = \sim (B \oplus C)$

$$\begin{array}{ll}
 & only2(U, B, C) \\
= & \quad \langle \text{ Defintion of } only2 \rangle \\
 & ((U \cap B) \cup (U \cap C) \cup (B \cap C)) \setminus (U \cap B \cap C) \\
= & \quad \langle \text{ Identity of } \cap \text{ 3 times } \rangle \\
 & ((B \cup C \cup (B \cap C)) \setminus (B \cap C)) \\
= & \quad \langle \text{ Absorption } \rangle \\
 & ((B \cup C) \setminus (B \cap C)) \\
= & \quad \langle \text{ Exercise 1.1 } \rangle \\
 & \sim (B \oplus C)
\end{array}$$

1.3 [15%] $((A \subseteq B) \wedge (B \subseteq C)) \Rightarrow (only2(A, B, C) = (B \setminus A))$

Hints:

1. $X \subseteq Y \Rightarrow X \cap Y = X$
2. $X \subseteq Y \Rightarrow X \cup Y = Y$

Hypothesis 1: $(A \subseteq B)$

Hypothesus 2: $(B \subseteq C)$

Prove: $only2(A, B, C) = (B \setminus A)$

$$
\begin{aligned}
& only2(A, B, C) \\
= & \quad \langle \text{ Defintion of } only2 \ \rangle \\
& ((A \cap B) \cup (A \cap C) \cup (B \cap C)) \setminus (A \cap B \cap C) \\
= & \quad \langle \text{ Hyp 2. and Hint 1, twice } \rangle \\
& ((A \cap B) \cup (A \cap C) \cup B) \setminus (A \cap B) \\
= & \quad \langle \text{ Hyp 2. and Hint 1, twice } \rangle \\
& (A \cup (A \cap C) \cup B) \setminus A \\
= & \quad \langle \text{ Absorption } \rangle \\
& (A \cup B) \setminus A \\
= & \quad \langle \text{ Hypothesis 1 and Hint 2 } \rangle \\
& B \setminus A
\end{aligned}
$$

2 Relations[45%]

2.1 [20%] Modeling with n-ary relations

Given the following relations:

CourseTaken A relation that relates students, courses, terms, sections and grades: $(s, c, sem, sec, g) \in CourseTaken$ if student s was enrolled in course c for term sem in section sec and obtained grade g.

CourseP Between courses and degree programs: $(c, p) \in CursoP$ if course c is a course that belongs to program p.

ProgS Between programs and students: $(p, s) \in ProgS$ if student s is enrolled in program p.

2.1.1 [10%]

Define S, the set of students, not enrolled in program ISC that took a course course of the ISC program and got a perfect grade (5.0).

$(S_{\{g=5, sp\neq ISC, cp=ISC\}}(Join_1(Join_1(ProgS, CourseTaken)_{\langle prog_s, student, coure\rangle}, CourseP))_{\langle student\rangle}$

2.1.2 [10%]

Define $RankC$, a relation between students: $(s, t) \in RankC$ if s and t took the same course (any term, any section) and s got a better grade than t.

$(S_{\{g_1>g_2\}}Join_1(CourseTaken_{\langle s_1, g_1, c\rangle}, CourseTaken_{\langle c, s_2, g_2\rangle}))_{\langle s_1, s_2\rangle}$

2.2 [25%] Binary relations: properties

Given the following binary relation between integer numbers:

$$(a\ F\ b) \equiv (\exists q\ :\ \mathbb{Z}\ |(a-b) = 10 \cdot q)$$

2.2.1 [5%] Give 5 examples of pairs of integers x, y such that $(x\ F\ y)$

- $(10\ F\ 20)$ in this case $q = -1$
- $(15\ F\ 5)$ in this case $q = 1$
- $(67\ F\ 37)$ in this case $q = 3$
- $(64\ F\ 104)$ in this case $q = -4$
- $(100\ F\ 100)$ in this case $q = 0$

2.2.2 [20%]

Is this relation:

- An equivalence relation? (Reflexive, symmetric, transitive) It is an equivalence relatiion.

  Reflexive let x be any integer. We must find an integer $\hat{q}$ such that $10 \cdot \hat{q} = x - x$. This integer is 0. Therefore, the relation is reflexive.

  Symmetric: $(a\ F\ b) \Rightarrow (b\ F\ a)$

  $$\begin{array}{ll} & (a\ F\ b) \\ = & \quad \langle \text{ Def. F. } \rangle \\ & (\exists q\ :\ \mathbb{Z}\ |(a-b) = 10 \cdot q) \\ = & \quad \langle \text{ } \exists\text{-instantiation } \rangle \\ & (a-b) = 10 \cdot \hat{q} \\ = & \quad \langle \text{ Multiply by (-1) } \rangle \\ & (b-q) = 10 \cdot -\hat{q} \\ = & \quad \langle \text{ } \exists\text{-generalization (if } \hat{q} \text{ is an integer, so is } -\hat{q}) \rangle \\ & (\exists q\ :\ \mathbb{Z}\ |(b-a) = 10 \cdot q) \\ = & \quad \langle \text{ Def. F. } \rangle \\ & (b\ F\ a) \end{array}$$

  TRansitive: $(a\ F\ b) \wedge (b\ F\ c) \Rightarrow (a\ F\ c)$

	Expresión	Justificación
1	$(a\ F\ b)$	Hypothesis
2	$(b\ F\ c)$	Hypothesis
3	$(\exists q\ :\ \mathbb{Z}\ \|(a-b) = 10 \cdot q)$	Definition of F (1)
4	$(a-b) = 10 \cdot \hat{q}$	$\exists$-instantiation (3)
5	$(\exists q\ :\ \mathbb{Z}\ \|(b-c) = 10 \cdot r)$	Definition of F (2)
6	$(b-c) = 10 \cdot \hat{r}$	$\exists$-instantiation (5)
7	$(a-b) + (b-c) = 10 \cdot \hat{q} + 10 \cdot \hat{r}$	Add (4,6)
8	$(a-c) = 10 \cdot (\hat{q} + \hat{r})$	Arithmetic (7)
9	$(\exists q\ :\ \mathbb{Z}\ \|(a-c) = 10 \cdot q)$	$\exists$-generalization (8)(if $\hat{r}$ and $\hat{q}$ are integers, so is $(\hat{q} + \hat{r})$)
10	$(a\ F\ c)$	Definition of F

- A partial order? (Reflexive, antisymmetric, transitive) It is not a partial order because it is not antisymmetric.
  - $(5\ F\ 10)$ because there is an integer (-1) such that $5 \cdot (-1) = (5 - 10)$.
  - $(5\ F\ 10)$ because there is an integer (1) such that $5 \cdot (1) = (10 - 15)$.
  - However, $5 \neq 10$
- None of the above

Formally prove that your answer is correct.

3 Functions [20%]

Given the following function from pairs of integers to integers:

$$\texttt{foo} : (\mathbb{Z} \times \mathbb{Z}) \rightarrow \mathbb{Z}$$

$$foo(n, m) = 2 \cdot n + 3 \cdot m$$

Is this function:

Onto?
- Yes.
- Let $x \in \mathbb{Z}$, we have to find values of a and b such that: $foo(a, b) = z$
- Let $a = -x$ and $b = x$.

$$\begin{array}{ll} & foo(a, b) \\ = & \qquad \langle \text{ Def. } a,\ b \ \rangle \\ & foo(-z, z) \\ = & \qquad \langle \text{ Def. } foo \ \rangle \\ & 2 \cdot (-z) + 3 \cdot z \\ = & \qquad \langle \text{ Arithmetic } \rangle \\ & z \end{array}$$

One to One? No. For example: $foo(0, 2) = 6$ and $foo(3, 0) = 6$ but $(0, 2) \neq (3, 0)$

Prove that your answers are correct.