Universidad de los Andes

ISIS-1104-07 Matemática Estructural y Lógica
MidTerm 1
Date: September 13, 2018

- This exam is closed book, closed notes.
- You are allowed to use the cheat-sheet that was handed out and left on Sicua+ for downloading.
- The use of any electronic device is prohibitted.
- There should be no communication among students.
- Answer in the spaces provided; no additional sheets of paper will be accepted.
- Do not forget to write you name and studient id number on the exam before handing it in.

Name	StudentId #

Do not write below ithis line

1.1	20 %	
1.2	25 %	
2.1	15 %	
2.2	25 %	
3.1	15 %	
3.2	15 %	
Total	115 %	

1. Propositional Calculus

Suppose we add the following function to the propositional calculus:

$$imp(p, q, r) \equiv (p \Rightarrow q) \wedge (\neg p \Rightarrow r) \tag{1}$$

1.1. Prove or refute:

$$imp(p, \neg p,\ r) \equiv (\neg p\ \wedge\ r)$$

$$
\begin{aligned}
& imp(p, \neg p,\ r) \\
= & \qquad \langle \text{ Definition } \star \rangle \\
& (\boxed{p \Rightarrow \neg p}) \wedge (\neg p \Rightarrow r) \\
= & \qquad \langle \text{ Def. } \Rightarrow \rangle \\
& (\boxed{\neg p \vee \neg p}) \wedge (\neg p \Rightarrow r) \\
= & \qquad \langle \text{ Idempotency - } \vee \rangle \\
& \neg p \wedge (\boxed{\neg p \Rightarrow r}) \\
= & \qquad \langle \text{ Def. } \Rightarrow \rangle \\
& \neg p \wedge (\boxed{\neg\neg p} \vee r) \\
= & \qquad \langle \text{ Double } \neg \rangle \\
& \neg p \wedge (p \vee r) \\
= & \qquad \langle \neg - \wedge\text{- Absorption } \rangle \\
& \neg p \wedge r
\end{aligned}
$$

1.2. Prove or refute the soundness of the following inference rule:

$$\frac{\begin{array}{c} imp(p, q, q) \\ q \Rightarrow r \end{array}}{q \wedge r}$$

We will use the following hypotheses

1. $mp(p, q, q)$
2. $q \Rightarrow r$

$$\begin{array}{ll}
\multicolumn{2}{l}{Hip: \quad 1} \\
 & imp(p, q, q) \\
= & \qquad \langle \text{ Definition } \star \rangle \\
 & (p \Rightarrow q) \wedge (\neg p \Rightarrow q) \\
= & \qquad \langle \text{ Right distributivity } \Rightarrow \rangle \\
 & (p \vee \neg p) \Rightarrow q \\
= & \qquad \langle \text{ Excluded middle } \rangle \\
 & true \Rightarrow q \\
= & \qquad \langle \text{ Left identity - } \Rightarrow \rangle \\
 & q
\end{array}$$

So now we have Lemma 1: q

$$\begin{array}{ll}
 & \qquad \langle \text{ Lemma 1 } \rangle \\
 & q \\
\Rightarrow & \qquad \langle \text{ Modus Ponens with hypothesis 2 } \rangle \\
 & r \\
= & \qquad \langle \text{ iComposition with Lemma 1 } \rangle \\
 & r \wedge q
\end{array}$$

2. Deduction in the propositional calculus

We have three boxes, labeled A, B, and C, that may or may not have a prize.

The boxes are accurately marked as follows:

Box A: If C does not have a prize then there is there is a prize here. $\neg C \Rightarrow A$

Box B: If C has a prize then there is a prize here. $C \Rightarrow B$

Box C: There is a prize here if there is a prize in A. $A \Rightarrow C$

For each box, mark the correct option with an X.

	Prize	No prize	It cannot be determined with the information
Box A			X
Box B	X		
Box C	X		

B and C have prizes $B \wedge C$ It cannot dteremine whether or not A has a prize.

2.1. Modeling

Given the variables listed below to represent the facts of the problem, model the hypotheses and your conclusion regarding what boxes have prizes.

A: There is a prize in box A.

B: There is a prize in box B.

C: There is a prize in box C.

2.2. Proofs in Propositional Calculus

Formally prove your conclusion.

	Expression	Justification
1	$\neg C \;\Rightarrow\; A$	Hypothesis
2	$C \;\Rightarrow\; B$	Hypothesis
3	$A \;\Rightarrow\; C$	Hypothesis
4	$\neg A \Rightarrow C$	Contrapositive (1)
5	$(A \Rightarrow C) \wedge (\neg A \Rightarrow C)$	Composition (3,4)
6	$imp(A, C, C)$	Def $\star$ (5)
7	$(C \wedge B)$	Problem 1.2 (6,2)

3. Predicate calculus

Exploring the galaxy you are sent to explore planet MEL-201820-S7. You know the following facts:

1. There are two types of aliens, alpha and beta. An alien (any alien) can by an alpha a beta or both.

   $(\forall\ x \mid\ \ :\ a(x) \vee bx\ )$

2. Red aliens are alphas

   $(\forall\ x \mid r(x)\ :\ a(x)\ )$

3. An alien (any alien) can fly if and only if it has wings.

   $(\forall\ x \mid\ \ :\ f(x) \equiv w(x)\ )$

4. If there is an alpha that can fly, then all alphas can fly.

   $(\exists\ x \mid a(x)\ :\ f(x)\ ) \Rightarrow (\forall\ x \mid a(x)\ :\ f(x)\ )$

5. There are no betas that can fly. $\neg(\exists\ x \mid b(x)\ :\ f(x)\ )$

When you land on MEL-201828-S7 you are greeted by a red, winged alien who says "my name is K". $r(K)\ \wedge\ w(K)$

You conclude that an alien can be an alpha or a beta, but not both. $(\forall\ x \mid\ \ :\ a(x) \not\equiv b(x)\ )$

3.1. Modeling

Model the problem (hypotheses, premises and conclusion), using the following predicates:

- a(x) : x is an alpha.
- b(x) : x is a beta.
- r(x) : x is red.
- w(x): x has wings.
- f(x): x can fly.

Write you answers above, beside each statement.

3.2. Deduction in the predicate calculus: Formally prove that you can reach the conclusion from the hypotheses and the premises.

Ejercise 3.2.1. First prove that if an alien is not a beta, then it is an alpha, using only the first hypothesis.

$$
\begin{array}{ll}
 & \langle \text{ Hypothesis 2 } \rangle \\
 & (\forall\, x \mid \;:\; a(x) \vee b(x)\;) \\
= & \langle \text{ Double negation } \rangle \\
 & (\forall\, x \mid \;:\; a(x) \vee \neg\neg b(x)\;) \\
= & \langle \text{ Def. } \Rightarrow \rangle \\
 & (\forall\, x \mid \;:\; \neg b(x) \Rightarrow a(x)\;)
\end{array}
$$

Ejercise 3.2.2. Using the the hypotheses and premises, prove that if an alien is an alpha then it is not a beta.

$$
\begin{array}{ll}
 & \quad \langle \text{ Premis } \rangle \\
 & r(K) \wedge w(K) \\
\Rightarrow & \quad \langle \text{ $\forall$-Modus Ponens with hyporhesis 2 } \rangle \\
 & a(K) \wedge w(K) \\
\Rightarrow & \quad \langle \text{ Leibniz with hypotheis 3 } \rangle \\
 & a(K) \wedge f(K) \\
\Rightarrow & \quad \langle \text{ $\exists$-Generalization } \rangle \\
 & (\exists\, x \mid a(x) \;:\; f(x)\;) \\
\Rightarrow & \quad \langle \text{ Modus Ponns with hypothesis 4 } \rangle \\
 & (\forall\, x \mid a(x) \;:\; f(x)\;) \\
\Rightarrow & \quad \langle \text{ Compostion with hypothesis 5 } \rangle \\
 & (\forall\, x \mid a(x) \;:\; f(x)\;) \wedge \neg(\exists\, x \mid b(x) \;:\; f(x)\;) \\
= & \quad \langle \text{ Generalized De Morgan } \rangle \\
 & (\forall\, x \mid a(x) \;:\; f(x)\;) \wedge (\forall\, x \mid b(x) \;:\; \neg f(x)\;) \\
= & \quad \langle \text{ Trading } \rangle \\
 & (\forall\, x \mid a(x) \;:\; f(x)\;) \wedge (\forall\, x \mid \;:\; b(x) \Rightarrow \neg f(x)\;) \\
= & \quad \langle \text{ Contrapopsitive, double $\neg$ } \rangle \\
 & (\forall\, x \mid a(x) \;:\; f(x)\;) \wedge (\forall\, x \mid \;:\; f(x) \Rightarrow \neg b(x)\;) \\
= & \quad \langle \text{ Trading } \rangle \\
 & (\forall\, x \mid a(x) \;:\; f(x)\;) \wedge (\forall\, x \mid f(x) \;:\; \neg b(x)\;) \\
\Rightarrow & \quad \langle \text{ Transitiviy } \rangle \\
 & (\forall\, x \mid a(x) \;:\; \neg b(x)\;)
\end{array}
$$

Ejercise 3.2.3. Using the two previous results, prove the conclusion.

$$
\begin{array}{ll}
 & \quad \langle \text{ Previuous Results } \rangle \\
 & (\forall\ x \mid a(x)\ :\ \neg b(x)\ ) \wedge (\forall\ x \mid\ \ :\ \neg b(x) \Rightarrow a(x)\ ) \\
= & \quad \langle \text{ Trading } \rangle \\
 & (\forall\ x \mid\ \ :\ a(x) \Rightarrow \neg b(x)\ ) \wedge (\forall\ x \mid\ \ :\ \neg b(x) \Rightarrow a(x)\ ) \\
= & \quad \langle \text{ Distributivity } \rangle \\
 & (\forall\ x \mid\ \ :\ (a(x) \Rightarrow \neg b(x)) \wedge (\neg b(x) \Rightarrow a(x))\ ) \\
= & \quad \langle \text{ Definition } \equiv \rangle \\
 & (\forall\ x \mid\ \ :\ (a(x) \equiv \neg b(x))\ ) \\
= & \quad \langle\ \neg \equiv \rangle \\
 & (\forall\ x \mid\ \ :\ \neg(a(x) \equiv b(x))\ ) \\
= & \quad \langle \text{ Definition } \not\equiv \rangle \\
 & (\forall\ x \mid\ \ :\ (a(x) \not\equiv b(x))\ )
\end{array}
$$