AXIOMS AND THEOREMS

Name	Rule
Double Negation	$\neg\neg p = p$
Definition of $false$	$false = \neg true$
Negation of $false$	$\neg false = true$

Table 1: Equivalences for False / True and Double Negation

Name	Op	Rule	Op	Rule
Conmutativity	$\vee$	$p \vee q \equiv q \vee p$	$\wedge$	$p \wedge q \equiv q \wedge p$
Asociativity	$\vee$	$(p \vee q) \vee r \equiv p \vee (q \vee r)$	$\wedge$	$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$
Identity	$\vee$	$p \vee false \equiv p$	$\wedge$	$p \wedge true \equiv p$
Dominance	$\vee$	$p \vee true \equiv true$	$\wedge$	$p \wedge false \equiv false$
Idempotence	$\vee$	$p \vee p \equiv p$	$\wedge$	$p \wedge p \equiv p$
Distributivity	$\vee/\wedge$	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$	$\wedge/\vee$	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
de Morgan	$\neg\vee$	$\neg(p \vee q) \equiv \neg p \wedge \neg q$	$\neg\wedge$	$\neg(p \wedge q) \equiv \neg p \vee \neg q$
Absorption	$\vee/\wedge$	$p \vee (p \wedge q) \equiv p$	$\wedge/\vee$	$p \wedge (p \vee q) \equiv p$
Absorption-$\neg$	$\vee/\wedge$	$\neg p \vee (p \wedge q) \equiv \neg p \vee q$	$\wedge/\vee$	$\neg p \wedge (p \vee q) \equiv \neg p \wedge q$
Negation	$\vee$	$p \vee \neg p \equiv true$	$\wedge$	$p \wedge \neg p \equiv false$

Table 2: Equivalences of $\vee$ y of $\wedge$

Negation of $\vee$ is called *Excluded Middle*. Negation of $\wedge$ is called *Contradiction*.

Rule	Name
$p \Rightarrow q \equiv \neg p \vee q$	Definition of $\Rightarrow$
$p \Rightarrow q \equiv \neg q \Rightarrow \neg p$	Contrapositive
$true \Rightarrow p \equiv p$	Left Identity of $\Rightarrow$
$p \Rightarrow false \equiv \neg p$	Right Negation of $\Rightarrow$
$false \Rightarrow p \equiv true$	Left False of $\Rightarrow$
$p \Rightarrow true \equiv true$	Right Zero of $\Rightarrow$
$p \Rightarrow p \equiv true$	Reflexivity $\Rightarrow$
$p \vee q \equiv \neg p \Rightarrow q$	Definition of $\vee$ with $\Rightarrow$
$p \wedge q \equiv \neg(p \Rightarrow \neg q)$	Definition of $\wedge$ with $\Rightarrow$
$\neg(p \Rightarrow q) \equiv p \wedge \neg q$	Negation of $\Rightarrow$
$(p \Rightarrow q) \wedge (p \Rightarrow r) \equiv (p \Rightarrow (q \wedge r))$	Left Distributivity $\Rightarrow /\wedge$
$(p \Rightarrow q) \vee (p \Rightarrow r) \equiv (p \Rightarrow (q \vee r))$	Left Distributivity $\Rightarrow /\vee$
$(p \Rightarrow r) \wedge (q \Rightarrow r) \equiv (p \vee q) \Rightarrow r$	Right Distributivity $\Rightarrow /\wedge$
$(p \Rightarrow r) \vee (q \Rightarrow r) \equiv (p \wedge q) \Rightarrow r$	Right Distributivity $\Rightarrow /\wedge$
$p \Rightarrow (q \Rightarrow r) \equiv (p \wedge q) \Rightarrow r$	Left Associativity of $\Rightarrow$

Table 3: $\Rightarrow$ Equivalences

Rule	Name
$p \equiv q \equiv (p \Rightarrow q) \wedge (q \Rightarrow p)$	Definition$_1$ of $\equiv$
$p \not\equiv q \equiv \neg(p \equiv q)$	Definition$_1$ $\not\equiv$
$p \equiv q \equiv (p \wedge q) \vee (\neg q \wedge \neg p)$	Definition$_2$ of $\equiv$
$(p \equiv q) \equiv (q \equiv p)$	Commutativity of $\equiv$
$((p \equiv q) \equiv r) \equiv (p \equiv (q \equiv r))$	Associativity of $\equiv$
$p \equiv p \equiv true$	Identity
$p \equiv \neg p \equiv false$	Definition$_2$ of false
$\neg(p \equiv q) \equiv \neg p \equiv q$	Negation$_1$ $\equiv$
$\neg(p \equiv q) \equiv p \equiv \neg q$	Negation$_2$ $\equiv$
$p \equiv \neg q \equiv \neg p \equiv q$	Negation$_3$ $\equiv$
$p \not\equiv q \equiv (p \vee q) \wedge \neg(p \wedge q)$	Definition$_2$ $\not\equiv$
$(p \not\equiv q) \equiv (q \not\equiv p)$	Commutativity of $\not\equiv$
$((p \not\equiv q) \not\equiv r) \equiv (p \not\equiv (q \not\equiv r))$	Associativity of $\not\equiv$
$r \vee (p \equiv q) \equiv (r \vee p) \equiv (r \vee q)$	Distrib $\vee / \equiv$
$r \wedge (p \equiv q) \equiv (r \wedge p) \equiv (r \wedge q) \equiv r$	Distrib $\wedge / \equiv$
$p \wedge q \equiv p \vee q \equiv p \equiv q$	Golden Rule

Table 4: Equivalence Laws for $\equiv$

Rule	Name
$\dfrac{X = Y \quad E[z := X]}{E[z := Y]}$	Leibniz
$\dfrac{p \quad p \Rightarrow q}{q}$	Modus ponens
$\dfrac{\neg q \quad p \Rightarrow q}{\neg p}$	Modus tollens
$\dfrac{p \Rightarrow q \quad q \Rightarrow r}{p \Rightarrow r}$	Transitivity
$\dfrac{p \vee q \quad \neg q}{p}$	Disjunctive Syllogism
$\dfrac{p}{p \vee q}$	Addition
$\dfrac{p \wedge q}{p}$	Simplification
$\dfrac{p \quad q}{p \wedge q}$	Conjunction
$\dfrac{p \vee q \quad \neg p \vee r}{q \vee r}$	Resolution

Table 5: Rules of Inference 1

Rule	Name
$\dfrac{p \equiv q}{p \Rightarrow q}$	$\equiv$ Simplification$_1$
$\dfrac{p \equiv q}{q \Rightarrow p}$	$\equiv$ Simplification$_2$
$\dfrac{p \equiv q}{\neg p \Rightarrow \neg q}$	$\equiv$ Simplification$_3$
$\dfrac{p \equiv q}{\neg q \Rightarrow \neg p}$	$\equiv$ Simplification$_4$
$\dfrac{p \quad p \equiv q}{q}$	$\equiv$ Deduction$_1$
$\dfrac{q \quad p \equiv q}{p}$	$\equiv$ Deduction$_2$
$\dfrac{\neg p \quad p \equiv q}{\neg q}$	$\equiv$ Deduction$_3$
$\dfrac{\neg q \quad p \equiv q}{\neg p}$	$\equiv$ Deduction$_3$

Table 6: Rules of Inference 2

Name	Rule
Substitution	$\dfrac{true}{(\star x\|R:S)[z:=E] = (\star x\|R[z:=E]:S[z:=E])}$ x doesn't appear in z, nor in E
Replace in Range	$\dfrac{\begin{array}{c}(\forall x\| : P = Q) \\ (\star x\|R[z:=P]:S)\end{array}}{(\star x\|R[z:=Q]:S)}$
Replace in Body	$\dfrac{\begin{array}{c}(\forall x\|R : P = Q) \\ (\star x\|R:S[z:=P])\end{array}}{(\star x\|R:S[z:=Q])}$

Table 7: Substitutions in Quantified Expressions

Name	Rule
Empty (false) range	$(\star x\|false : S) = u$
One-point rule	$(\star x\|x = E : S) = S[x := E]$
Distributivity	$(\star x\|P : Q) \star (\star x\|P : R) = (\star x\|P : Q \star R)$
Range split	$(\star\ x\ \|R : P) \star (\star\ x\ \|S : P) = (\star\ x\ \|R \vee S : P) \star (\star\ x\ \|R \wedge S : P)$
Disjunctive range split	$\neg(R \wedge S) \Rightarrow (\star\ x\ \|R : P) \star (\star\ x\ \|S : P) = (\star\ x\ \|R \vee S : P)$
Idempotent range split ($\star$ idempotent)	$(\star\ x\ \|R : P) \star (\star\ x\ \|S : P) = (\star\ x\ \|R \vee S : P)$
Interchange of dummies (y doesn't appear in R; x doesn't appear in Q)	$(\star x\|R : (\star y\|Q : E)) = (\star y\|Q : (\star x\|R : E))$
Nesting (y doesn't appear in R)	$(\star x, y\|R \wedge Q : E) = (\oplus x\|R : (\oplus y\|Q : E))$
Dummy renaming (y doesn't appear in R nor in E)	$(\star x\|R : E) = (\star y\|R[x := y] : E[x := y])$
Change of dummy (y doesn't appear in R nor in E; f has an inverse function)	$(\star x\|R : E) = (\star y\|R[x := f.y] : E[x := f.y])$

Table 8: Laws of Quantification
$\star$ is a commutative, associative operator, with an identity, u. We assume each quantification is defined.

Rule	Name
$(\forall x \mid R \lor Q : P) \equiv ((\forall x \mid R : P) \land (\forall x \mid Q : P))$	Range split $\forall$
$(\exists x \mid R \lor Q : P) \equiv ((\exists x \mid R : P) \lor (\exists x \mid Q : P))$	Range split $\exists$
$(\forall x \mid R : Q \land P) \equiv ((\forall x \mid R : Q) \land (\forall x \mid R : P))$	Distributivity $\forall$
$(\exists x \mid R : Q \lor P) \equiv ((\exists x \mid R : Q) \lor (\exists x \mid R : P))$	Distributivity $\exists$
x doesn't appear in P: $(\forall x \mid R : Q \lor P) \equiv (\forall x \mid R : Q) \lor P$	Distributivity $\lor/\forall$
x doesn't appear in P: $(\exists x \mid R : Q \land P) \equiv (\exists x \mid R : Q) \land P$	Distributivity $\land/\exists$
$(\forall x \mid R : P) \equiv (\forall x \mid : R \Rightarrow P)$	Trading-$\forall$
$(\exists x \mid R : P) \equiv \neg(\forall x \mid R : \neg P)$	Generalized de Morgan
$(\forall x \mid R : P) \equiv \neg(\exists x \mid R : \neg P)$	Generalized de Morgan$_2$
$\neg(\forall x \mid R : P) \equiv (\exists x \mid R : \neg P)$	Generalized de Morgan$_3$
$\neg(\exists x \mid R : P) \equiv (\forall x \mid R : \neg P)$	Generalized de Morgan$_4$
$(\exists x \mid R : P) \equiv (\exists x \mid : R \land P)$	Trading-$\exists$
$(\exists x \mid R \land Q : P) \equiv (\exists x \mid R : Q \land P)$	Trading-$\exists_1$
$(\forall x \mid R \land Q : P) \equiv (\forall x \mid R : Q \Rightarrow P)$	Trading-$\forall_1$
$(\forall x \mid Q : R \Rightarrow P) \equiv (\forall x \mid R : Q \Rightarrow P)$	Trading-$\forall_2$

Table 9: Equivalences for quantified logic expressions

Name	Rule	Condition
Universal Intantiation	$\dfrac{(\forall x \mid : P)}{P[x := c]}$	Any c
Universal Generalization	$\dfrac{P[x := c]}{(\forall x \mid : P)}$	for any c
Existential Instantiation	$\dfrac{(\exists x \mid : P)}{P[x := \hat{c}]}$	$\hat{c}$ is a specific value that makes P true
Existential Generalization	$\dfrac{P[x := c]}{(\exists x \mid : P)}$	c a value that makes P true
Universal Modus Ponens	$\dfrac{\begin{array}{c}(\forall x \mid P : Q)\\ P[x := c]\end{array}}{Q[x := c]}$	Any c
Universal Modus Tollens	$\dfrac{\begin{array}{c}(\forall x \mid P : Q)\\ \neg Q[x := c]\end{array}}{\neg P[x := c]}$	Any c
Universal Instantiation$_2$	$\dfrac{(\forall x \mid R : P)}{R[x := c] \Rightarrow P[x := c]}$	Any c
Universal Generalization$_2$	$\dfrac{R[x := c] \Rightarrow P[x := c]}{(\forall x \mid R : P)}$	c is an arbitrary element
Existential Instantiation$_2$	$\dfrac{(\exists x \mid R : P)}{R[x := \hat{c}] \wedge P[x := \hat{c}]}$	$\hat{c}$ a particular element that makes P and Q trues
Existential Generalization$_2$	$\dfrac{R[x := c] \wedge P[x := c]}{(\exists x \mid R : P)}$	c any element that makes R and C true

Table 10: Inference rules for quantified expressions