ISIS-1104-01 Matemática Estructural y Lógica
Mid Term 3
Fecha: December 5, 2016

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- Está permintido el uso de las "cheat sheets" que se encuentran en sicua+.
- Está prohibido el uso de cualquier otro material como cuadernos, libros o fotocopias.
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- El intercambio de información relevante a esta prueba con otro estudiante está terminantemente prohibido.
- Cualquier irregularidad con respecto a estas reglas podría ser considerada fraude.
- Responda el examen en los espacios proporcionados. No se aceptarán hojas adicionales.
- No olvide marcar el examen antes de entregarlo.

IMPORTANTE: Soy consciente de que cualquier tipo de fraude en los exámenes es considerado como una falta grave en la Universidad. Al firmar y entregar este examen doy expreso testimonio de que este trabajo fue desarrollado de acuerdo con las normas establecidas. Del mismo modo, aseguro que no participé en ningún tipo de fraude.

Nombre	Carné
Firma	Fecha

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4.	10 %	***

1. Integers

Do not use induction for any of the problems in this section.

1.1. Prove the following statement:

$$((m \mid a) \wedge (m \nmid (a+b))) \Rightarrow (m \nmid b)$$

We will prove it by contradiction:

	Expresión	Justificación
1	$m \mid a$	Hypothesis
2	$m \nmid (a+b)$	Hypothesis
3	$m \mid b$	Assumption
4	$m \mid a+b$	Theorem $m \mid a \wedge m \mid a \Rightarrow m \mid (a+b)$
5	$False$	Contradiction (2,4)

1.2. Prove the following statement:

For any integers: a, b, c, d, n, with $n > 0$, if $(a \equiv_n b)$ and $((a+c) \equiv_n d)$ then $((b+c) \equiv_n d)$.

	Expresión	Justificación
1	$a \equiv_n b$	Hypothesis
2	$(a+c) \equiv_n d$	Hypothesis
3	$-a \equiv_n -b$	Th. $(a \equiv_n b) \Rightarrow (c \cdot a \equiv_n c \cdot b)$ (2)
4	$(a+c-a) \equiv_n d-b$	Th $(a \equiv_n b) \wedge (c \equiv_n d) \Rightarrow (a+c \equiv_n d+b)$ (1,3)
5	$(c) \equiv_n d-b$	Arithmetic (4)
6	$n \mid ((d-b)-c)$	Def. $. \equiv_n .$ (5)
7	$n \mid (d-(b+c))$	Arithmetic (6)
8	$((b+c) \equiv_n d)$	Def. $. \equiv_n .$ (7)

2. Induction on Natural Numbers

2.1. F_N: We define: Fibonacci: F_n

Basis case 0: $F_0 = 0$

Basis case 1: $F_1 = 1$

Inductive case: $F_{n+1} = F_n + F_{n-1}$ for $n > 0$

Prove that for all $n \geq 1$: $(F_{n+1} \cdot F_{n-1}) - F_n^2 = (-1)^n$

We only need one basis case because we will use just the previous value in the inductive case.

Basis Case $n = 1$: $(F_{1+1} \cdot F_{1-1}) - F_1^2 = (-1)^1$ Applying arithmetic: $(F_2 \cdot F_0) - F_1^2 = -1$

$$
\begin{aligned}
& (F_2 \cdot F_0) - F_1^2 \\
= & \quad \langle\ F_0 = 0\ \rangle \\
& F_2 \cdot 0 - F_1^2 \\
= & \quad \langle\ \text{Arithmetic}\ \rangle \\
& -(F_1{}^2) \\
= & \quad \langle\ F_1 = 1\ \rangle \\
& -(1^2) \\
= & \quad \langle\ \text{Arithmetic}\ \rangle \\
& -1
\end{aligned}
$$

Inductive Case: $((F_{k+1} \cdot F_{k-1}) - F_k^2 = (-1)^k) \Rightarrow ((F_{(k+1)+1} \cdot F_{(k+1)-1}) - F_{(k+1)}^2 = (-1)^{k+1})$ Applying arithmetic: $((F_{k+1} \cdot F_{k-1}) - F_k^2 = (-1)^k) \Rightarrow ((F_{k+2} \cdot F_k - F_{(k+1)}^2 = (-1)^{k+1})$

I.H.: $(F_{k+1} \cdot F_{k-1}) - F_k^2 = (-1)^k$
Prove: $(F_{k+2} \cdot F_k - F_{(k+1)}^2 = (-1)^{k+1}$

$$
\begin{aligned}
& F_{k+2} \cdot F_k - F_{(k+1)}^2 \\
= & \quad \langle\ \text{Definition of } F_n\ \rangle \\
& (F_{k+1} + F_k) \cdot F_k - F_{(k+1)}^2 \\
= & \quad \langle\ \text{Aritmetic}\ \rangle \\
& F_{k+1} \cdot F_k + F_k^2 - F_{(k+1)}^2 \\
= & \quad \langle\ \text{Aritmetic}\ \rangle \\
& F_{k+1} \cdot (F_k - F_{(k+1)}) + F_k^2 \\
= & \quad \langle\ \text{Definition of } F_n\ \rangle \\
& F_{k+1} \cdot (F_k - (F_k + F_{k-1}) + F_k^2 \\
= & \quad \langle\ \text{Aritmetic}\ \rangle \\
& F_{k+1} \cdot (-F_{k-1}) + F_k^2 \\
= & \quad \langle\ \text{Aritmetic}\ \rangle \\
& -1 \cdot (F_{k+1} \cdot (F_{k-1}) - F_k^2 \\
= & \quad \langle\ \text{I.H.}\ \rangle \\
& -1 \cdot (-1)^k \\
= & \quad \langle\ \text{Arithmetic}\ \rangle \\
& (-1)^{k+1}
\end{aligned}
$$

2.2. H_N

Given the following recursive definition:

Basis case 1: $H_1 = 1$

Inductive case: $H_{n+1} = H_n + \frac{1}{n+1}$, for $n \ > \ 1$

Use induction to prove that for $n \geq 1$: $(+i|1 \leq i \leq n : H_i) = (n+1) \cdot H_n - n$
Hint: Note that $H_n = H_{n+1} - \frac{1}{n+1}$ for $n \geq 1$

Basis Case (n = 1): $(+i|1 \leq i \leq 1 : H_i) = (1+1) \cdot H_1 - 1$

$$
\begin{array}{ll}
 & (+i|\ 1 \leq i \leq 1 : H_i) = (1+1) \cdot H_1 - 1 \\
= & \qquad \langle \text{ Arithmetic } \rangle \\
 & (+i|i = 1 : H_i) = (1+1) \cdot H_1 - 1 \\
= & \qquad \langle \text{ 1-point rule } \rangle \\
 & H_1 = (1+1) \cdot H_1 - 1 \\
= & \qquad \langle \text{ Basic definiton } H_i \ \rangle \\
 & 1 = (1+1) \cdot 1 - 1 \\
= & \qquad \langle \text{ Arithmetic } \rangle \\
 & true
\end{array}
$$

Inductive Case:

$$(+i|1 \leq i \leq k : H_i) = (k+1) \cdot H_k - k \Rightarrow (+i|1 \leq i \leq k+1 : H_i) = (k+1+1) \cdot H_k - (k+1)$$

Which by arithmetic is:

$$(+i|1 \leq i \leq k : H_i) = (k+1) \cdot H_k - k \Rightarrow (+i|1 \leq i \leq k+1 : H_i) = (k+2) \cdot H_k - (k+1)$$

I.H.: $(+i|1 \leq i \leq k : H_i) = (k+1) \cdot H_k - k$

Prove: $(+i|1 \leq i \leq k+1 : H_i) = (k+2) \cdot H_k - (k+1)$

$$
\begin{array}{ll}
 & (+i|1 \leq i \leq k+1 : H_i) \\
= & \qquad \langle \text{ Split-off term } \rangle \\
 & (+i|1 \leq i \leq k : H_i) + H_{k+1} \\
= & \qquad \langle \text{ H.I. } \rangle \\
 & (k+1) \cdot H_k - k + H_{k+1} \\
= & \qquad \langle \text{ Hint } \rangle \\
 & (k+1) \cdot (H_{k+1} - \frac{1}{k+1}) - k + H_{k+1} \\
= & \qquad \langle \text{ Arithmetic } \rangle \\
 & (k+1) \cdot H_{k+1} - (k+1) \cdot \frac{1}{k+1} - k + H_{k+1} \\
= & \qquad \langle \text{ Arithmetic } \rangle \\
 & (k+2) \cdot H_{k+1} - (k+1) \cdot \frac{1}{k+1} - k \\
= & \qquad \langle \text{ Arithmetic } \rangle \\
 & (k+2) \cdot H_{k+1} - 1 - k \\
= & \qquad \langle \text{ Arithmetic } \rangle \\
 & (k+2) \cdot H_{k+1} - (k+1)
\end{array}
$$

3. Structural Induction

We can define a well formed formula of sums of x's (wff_{sx}) inductively as follows.

- x is a wff_{sx}
- if α and β are wff_{sx}, $(\alpha + \beta)$ is also a wff_{sx}.

These are examples of wff_{sx}:

- `x`
- `(x + x)`
- `(x + (x + x))`

To prove that a property , P is true for all wff_{sx}, you have to use structural induction, and you must use the following pattern.

Basis Case: The property is *true* for `x`

Inductive Case: If the property is *true* for α and β then it is true for $(\alpha + \beta)$

> **I.H. 1:** $P(\alpha)$
> **I.H 2:** $P(\beta)$
> **Prove:** $P((\alpha + \beta))$

We define the following functions and predicates recursively for f, a wff_{sx}.

$Xs(f)$**:** Number of x's in f.

$$Xs(x) = 1$$
$$Xs(\alpha + \beta) = Xs(\alpha) + Xs(\beta)$$

$sums(f)$**:** Number of +'s in a f.

$$Sums(x) = 0$$
$$Sums(\alpha + \beta) = Sums(\alpha) + Sums(\beta) + 1$$

$D(x)$**:** Depth of a f.

$$D(x) = 0$$
$$D(\alpha + \beta) = max(D(\alpha), D(\beta)) + 1$$

$Balanced(f)$**:** Is f balanced?.

$$Balanced(x) = \text{true}$$
$$Balanced(\alpha + \beta) = Balanced(\alpha) \wedge Balanced(\beta) \wedge (D(\alpha) = D(\beta))$$

3.1. Using structural induction prove: $Xs(f) = Sums(f) + 1$

Basis Case: $Xs(\mathtt{x}) = Sums(\mathtt{x}) + 1$

$$
\begin{array}{ll}
 & Xs(\mathtt{x}) = Sums(\mathtt{x}) + 1 \\
= & \qquad \langle \text{ Def. Xs. } \rangle \\
 & 1 = Sums(\mathtt{x}) + 1 \\
= & \qquad \langle \text{ Arothmetic } \rangle \\
 & Sums(\mathtt{x}) = 0 \\
= & \qquad \langle \text{ Def. Sums } \rangle \\
 & True
\end{array}
$$

Inductive Case: $Xs(\alpha) = Sums(\alpha) + 1 \wedge Xs(\beta) = Sums(\beta) + 1 \Rightarrow Xs((\alpha + \beta)) = sums((\alpha + \beta)) + 1$

I.H. 1: $Xs(\alpha) = Sums(\alpha) + 1$

I.H. 2: $Xs(\beta) = Sums(\beta) + 1$

Prove: $Xs((\alpha + \beta)) = sums((\alpha + \beta)) + 1$

$$
\begin{array}{ll}
 & Xs((\alpha + \beta)) \\
= & \qquad \langle \text{ Def. } Xs \rangle \\
 & Xs(\alpha) + Xs(\beta) \\
= & \qquad \langle \text{ I.H. Twice } \rangle \\
 & Sums(\alpha) + 1 + Sums(\beta) + 1 \\
= & \qquad \langle \text{ Def. Sums } \rangle \\
 & Sums((\alpha + \beta)) + 1
\end{array}
$$

3.2. Using structural induction prove: $Blanced(f) \Rightarrow Xs(f) = 2^{D(f)}$

Basis Case: $Balanced(x) \Rightarrow Xs(x) = 2^{D(x)}$

$$
\begin{array}{ll}
 & Balanced(x) \Rightarrow Xs(x) = 2^{D(x)} \\
= & \qquad \langle \text{ Def. Balanced } \rangle \\
 & true \Rightarrow Xs(x) = 2^{D(x)} \\
= & \qquad \langle \text{ Left. Identity. of } \Rightarrow \rangle \\
 & Xs(x) = 2^{D(x)} \\
= & \qquad \langle \text{ Def. } Xs \text{ } \rangle \\
 & 1 = 2^{D(x)} \\
= & \qquad \langle \text{ Def. } D(x) \text{ } \rangle \\
 & 1 = 2^0 \\
= & \qquad \langle \text{ Arithmetic } \rangle \\
 & \text{TRUE}
\end{array}
$$

Inductive Case: I.H. 1: $Balanced(\alpha) \Rightarrow Xs(\alpha) = 2^{D(\alpha)}$

I.H. 2: $Balanced(\beta) \Rightarrow Xs(\beta) = 2^{D(\beta)}$

Prove: $Balanced((\alpha+\beta)) \Rightarrow Xs((\alpha+\beta)) = 2^{D((\alpha+\beta))}$

	Expresión	Justificación
1	$Blanced(\alpha) \Rightarrow Xs(\alpha) = 2^{D(\alpha)}$	I.H.1
2	$Blanced(\beta) \Rightarrow Xs(\beta) = 2^{D(\beta)}$	I.H.2
3	$Balanced((\alpha+\beta))$	Premise
4	$Balanced(\alpha) \wedge balanced(\beta) \wedge D(\alpha) = D(\beta)$	Def. Balanced (3)
5	$Balanced(\alpha)$	Simp. (4)
6	$Balanced(\beta)$	Simp. (4)
7	$D(\alpha) = D(\beta)$	Simp. (4)
8	$Xs(\alpha) = 2^{D(\alpha)}$	Modus Ponens (5,1)
9	$Xs(\beta) = 2^{D(\beta)}$	Modus Ponens (6,2)

Now using the conclusions (8,9,7), we will prove: $Xs((\alpha+\beta)) = 2^{D((\alpha+\beta))}$

$$
\begin{array}{ll}
 & Xs((\alpha+\beta)) = 2^{D((\alpha+\beta))} \\
= & \qquad \langle \text{ Def. } Xs \text{ } \rangle \\
 & Xs(\alpha) + Xs(\beta) = 2^{D((\alpha+\beta))} \\
= & \qquad \langle \text{ Conclusions 8 and 9 } \rangle \\
 & 2^{D(\alpha)} + 2^{D(\beta)} = 2^{D((\alpha+\beta))} \\
= & \qquad \langle \text{ Def. D } \rangle \\
 & 2^{D(\alpha)} + 2^{D(\beta)} = 2^{max(D(\alpha),D(\beta))+1} \\
= & \qquad \langle \text{ Conclusion 7 } \rangle \\
 & 2^{D(\alpha)} + 2^{D(\alpha)} = 2^{max(D(\alpha),D(\alpha))+1} \\
= & \qquad \langle\ max(x,x) = x\ \rangle \\
 & 2^{D(\alpha)} + 2^{D(\alpha)} = 2^{D(\alpha)+1} \\
= & \qquad \langle\ A + A = 2 \cdot A\ \rangle
\end{array}
$$

$$2 \cdot (2^{D(\alpha)}) = 2^{D(\alpha)+1}$$

$= \quad \langle \text{ Arithmetic } \rangle$

$True$

4. Counting

Suppose you have an organization with 6 Americans, 6 French and 6 Colombians.
Suppose you have an organization with 6 Americans, 6 French and 6 Colombians.

1. How many different committees of 5 people can you form. We have 18 people (3*6) and we want to choose 5. The order does not matter.
   $\binom{18}{5}$

2. How many different committees of 5 people can you form in which there is at least one person from each nationality. We have to choose 1 American, 1 French and 1 Colombian. Each of these can be choses 6 different ways, The remaining 2 can be chose from the 15 that have n not been chosen, $6^3 \cdot \binom{15}{2}$

3. How many different ways can you seat 4 people in a row so that no two people of the same nationality are seated side by side. We have the following possible scenariosn:

   - The first three are af different nationalities and the last one is any one of the people not chosen of first two nationalities:
   $$18 \cdot 12 \cdot 6 \cdot 10$$
   - The first two are af different nationalities; the third one has the same nationality as the first one; The fouth one is of the same nationaloity as the second
   $$18 \cdot 12 \cdot 5 \cdot 5$$
   - The first two are af different nationality; the third one has the same nationality as the first one; The fouth one is neither of the same nationaloty of the first nor of the second
   $$18 \cdot 12 \cdot 5 \cdot 6$$

   The answeer would be the sum of these cases.
   $$18 \cdot 12 \cdot 6 \cdot 10 + 18 \cdot 12 \cdot 5 \cdot 5 + 18 \cdot 12 \cdot 5 \cdot 6$$

4. How many committees of 5 people can you form in which they are all of different nationalities. Zero. If you choose 5 people from a group with 3 different nationalities. By the Pigeon hole principle, at least two willbe of the same nationality/