ISIS-1104-07 Matemï¿½tica Estructural y Lï¿½gica
MidTerm 2
Date: October 18, 2018

- This exam is closed book, closed notes.
- You are allowed to use the cheat-sheet that was handed out and left on Sicua+ for downloading.
- The use of any electronic device is prohibitted.
- There should be no communication among students.
- Answer in the spaces provided; no additional sheets of paper will be accepted.
- Do not forget to write you name and studient id number on the exam before handing it in.

Name	StudentId #

Do not write below ithis line

1.1	10 %	
1.2	15 %	
1.3	20 %	
2.	20 %	
3.1	10 %	
3.2	15 %	
4 ***	10 %	
Total	110 %	

1. Sets

Suppose we add the following function to the set calculus: (remember: I use notatiom $\sim S$ to denote a set's complement S^c

$$foo(A, B, C) = (\sim A \ \cup \ B) \ \cap \ (A \ \cup \ C) \tag{1}$$

1.1. Prove or refute:

$$foo(A, \sim A,\ B) = B \setminus A$$

$foo(A, \sim A,\ B)$
$=$ $\langle$ Definition foo $\rangle$
$(\sim A \ \cup \ \sim A) \ \cap \ (B \ \cup \ A)$
$=$ $\langle$ Idempotency - $\cup$ $\rangle$
$\sim A \ \cap \ (B \ \cup \ A)$
$=$ $\langle \sim \ \cap / \cup$ - Absorption $\rangle$
$\sim A \ \cap \ B$
$=$ $\langle$ Commutativity of $\cap$ $\rangle$
$B \ \cap \ \sim A$
$=$ $\langle$ Definition $\setminus$ $\rangle$
$B \setminus A$

1.2. Prove or refute:

$$foo(A \cap B, A, B) = B$$

$foo(A \cap B, A, B)$

$=$ $\langle$ Definition foo $\rangle$

$(\sim (A \cap B) \cup A) \cap ((A \cap B) \cup B)$

$=$ $\langle$ de Morgan $\rangle$

$(\sim A \cup \sim B) \cup A) \cap ((A \cap B) \cup B)$

$=$ $\langle$ Associativity and commutativity of $\cup$ $\rangle$

$(A \cup \sim A) \cup \sim B) \cap ((A \cap B) \cup B)$

$=$ $\langle$ Negation $\cup$ $\rangle$

$(U \cup \sim B) \cap ((A \cap B) \cup B)$

$=$ $\langle$ Domination $\cup$ $\rangle$

$U \cap ((A \cap B) \cup B)$

$=$ $\langle$ Identity $\cap$ $\rangle$

$((A \cap B) \cup B)$

$=$ $\langle$ Absbsorption $\cup$ / $\cap$ $\rangle$

B

1.3. Prove or refute the soundness of the following inference rule:

$$\frac{B \subseteq C}{foo(A, B, B) = B \ \cap \ C}$$

It may be useful to use this lemma, that we proved in class:

$$\frac{B \subseteq C}{B \ \cap \ C = B}$$

Since, $B \subseteq C$, by the lemma we can state that $B = B \ \cap \ C$. We will use $B = B \ \cap \ C$ as a hypotheses to prove $foo(A, B, B) = B \ \cap \ C$.

$$\begin{array}{ll}
 & foo(A, B, B) \\
= & \quad \langle \text{ Definition } foo \ \rangle \\
 & (\sim A \ \cup \ B) \ \cap \ (A \ \cup \ B) \\
= & \quad \langle \ \text{ Distributivity } \cup/\cap \ \rangle \\
 & (\sim A \ \cap \ A) \ \cup \ B \\
= & \quad \langle \text{ Negataion } \rangle \\
 & \emptyset \ \cup \ B \\
= & \quad \langle \text{ Identity } \rangle \\
 & B \\
= & \quad \langle \text{ Hypothesis } B \subseteq C \text{ with lemma } \ \rangle \\
 & B \ \cap \ C
\end{array}$$

2. Relations

We define the following relation between pairs of integers: $R : (\mathbb{Z} \times \mathbb{Z}) \leftrightarrow (\mathbb{Z} \times \mathbb{Z})$

$$(x, y)\ R\ (a, b) \equiv (x \leq y) \wedge (a \leq b) \wedge (\exists n : \mathbb{Z} | (x \leq n \leq y) \wedge (a \leq n \leq b))$$

Example:

- $(1, 6)\ R\ (4, 8)$ (Works with $n = 5$)
- $(1, 6)\ R\ (6, 8)$ (Works with $n = 6$)
- $\neg((1, 6)\ R\ (16, 38))$ (there is no integer that is both between 1 and 6 as well as between 16 and 18)
- $\neg((6, 1)\ R\ (4, 8))$ ($(6 \leq 1)$ is false)

Is this relation an equivalence relation? It is not an equivalence relation, for it is not transitive nor reflexive.

Prove that your answer is correct.

It is not an equivalence relation, for it is not transitive nor reflexive.

Note that $(1, 6)\ R\ (5, 8)$ and $(5, 8)\ R\ (7, 10)$ For the first relation, there are, in fact, two values of n that work: 5 and 6. For the second relation, the values that would work are 6 and 8. But: $\neg((1, 6)R(7, 10))$ because there are no integers that are both between 1 and 6 and between 7 and 10.

It is not reflexive either. For it to be reflexive, for any pair of integers (x, y), $(x, y)R(x, y)$. However, $\neg(3, 2)R(3, 2)$, for it is not true that $(3 < 2)$!.

Either of the two previous proofs would have been enough to prove that the relation is not an equivalnce relation.

However, the relation is symmetric. This is to say

$$(x, y)R(a, b) \Rightarrow (a, b)R(x, y)$$

$$\begin{array}{ll}
 & (x, y)R(a, b) \\
= & \quad \langle \text{ Definition of } R \rangle \\
 & (x \leq y) \wedge (a \leq b) \wedge (\exists n : \mathbb{Z} | (x \leq n \leq y) \wedge (a \leq n \leq b)) \\
= & \quad \langle \text{ Commutativity of } \wedge \text{ twice } \rangle \\
 & (a \leq b) \wedge (x \leq y) \wedge (\exists n : \mathbb{Z} | (a \leq n \leq b) \wedge (x \leq n \leq y)) \\
= & \quad \langle \text{ Definition of } R \rangle \\
 & (a, b)R(x, y)
\end{array}$$

3. Functions

We define function *goo* from positive integers to positive integers as follows:

$$goo : \mathbb{Z}^+ \rightarrow \mathbb{Z}^+$$

$$goo(x) = 7 \cdot x + x \cdot (-1)^{2 \cdot x+1} \tag{2}$$

We define function *hoo* from integers to integers as follows:

$$hoo : \mathbb{Z} \rightarrow \mathbb{Z}$$

$$hoo(x) = x + (-1)^x \tag{3}$$

Remember:

- $(-1)^n = 1$ is n is even and $(-1)^n = -1$ if n is odd.
- $2n$ denotes an even number
- $2n + 1$ denotes an odd number

3.1. Is goo?

One to One? YES

Onto? NO

Prove that your answers are correct. We begin by rewriting the defintion of *goo*

$$\begin{array}{ll} & goo(x) \\ = & \quad \langle \text{ Defintion } \rangle \\ & 7 \cdot x + x \cdot (-1)^{2 \cdot x+1} \\ = & \quad \langle \text{ Hints } \rangle \\ & 7 \cdot x + x \cdot (-1) \\ = & \quad \langle \text{ Ariitmetic } \rangle \\ & 6 \cdot x \end{array}$$

So:

$$goo(x) = 6x$$

To prove that it is 1-1, we have to prove:

$$goo(x) = goo(y) \Rightarrow x = y$$

goo(x)=goo(y)
$=$ $\langle$ Redefinition of *goo* $\rangle$
6x=6y
$=$ $\langle$ Arithmetic $\rangle$
x=y

To prove that *goo* is not onto, we have to find a positive integer z, such that there is no integer x that satisfies: $goo(x) = z$.

This is quite easy as all values of *goo* are multiples of 6. Take $z = 5$. We have find a number x such that $goo(x) = 5$.

goo(x)=5
$=$ $\langle$ Redefinition of *goo* $\rangle$
6x=5
$=$ $\langle$ Arithmetic $\rangle$
$x = \frac{5}{6}$

Since $\frac{5}{6}$ is not an integer, then there is no integer x, such that $goo(x) = 5$. Therefore, *goo* is not onto.

3.2. Is hoo?

One to One? YES

Onto? YES

Prove that your answers are correct.

To prove that *hoo* is one-to-one we have to prove: $hoo(x) = hoo(y) \Rightarrow x = y$

We will prove this by cases:

Assume that x and y are both even

$$
\begin{aligned}
& hoo(x) = hoo(y) \\
= \quad & \langle \text{ Definition of } hoo \ \rangle \\
& x + (-1)^x = y + (-1)^y \\
= \quad & \langle \ x \text{ and } y \text{ are even } \rangle \\
& x + 1 = y + 1 \\
= \quad & \langle \text{ Arithmetic } \rangle \\
& x = y
\end{aligned}
$$

Assume that x and y are both odd

$$
\begin{aligned}
& hoo(x) = hoo(y) \\
= \quad & \langle \text{ Definition of } hoo \ \rangle \\
& x + (-1)^x = y + (-1)^y \\
= \quad & \langle \ x \text{ and } y \text{ are odd } \rangle \\
& x - 1 = y - 1 \\
= \quad & \langle \text{ Arithmetic } \rangle \\
& \text{x =y}
\end{aligned}
$$

Assume that x is even and y is odd We will show that this case is impossible:

$$
\begin{aligned}
& hoo(x) = hoo(y) \\
= \quad & \langle \text{ Definition of } hoo \ \rangle \\
& x + (-1)^x = y + (-1)^y \\
= \quad & \langle \ x \text{ is even and } y \text{ is odd } \rangle \\
& x + 1 = y - 1 \\
= \quad & \langle \text{ Arithmetic } \rangle \\
& x + 2 = y
\end{aligned}
$$

If x is even and $y = x + 2$ then y would also be even contradicting the assumptiom. Therefore, the assumption is impossible.

Assume that y is even and x is odd We will show that this case is impossible:

$$
\begin{array}{ll}
& hoo(x) = hoo(y) \\
= & \quad \langle \text{ Definition of } hoo \text{ } \rangle \\
& x + (-1)^x = y + (-1)^y \\
= & \quad \langle \text{ } y \text{ is even and } x \text{ is odd } \rangle \\
& x - 1 = y + 1 \\
= & \quad \langle \text{ Arithmetic } \rangle \\
& x = y + 2
\end{array}
$$

If y is even and $x = y + 2$ then x would also be even contradicting the assumptiom. Therefore, the assumption is impossible.

Now we must prove that hoo is onto. We must prove that for every integer z, there is an interger x, such that $hoo(x) = y$.

We will prove it my cases:

- z is even. This is to say $z = 2a$ for some integer a. We must find an x such that $hoo(x) = 2a$. Let $x = 2a + 1$.

$$
\begin{array}{ll}
& hoo(x) \\
= & \quad \langle \text{ Definition of } x \text{ } \rangle \\
& hoo(2a + 1) \\
= & \quad \langle \text{ Definition of } hoo \text{ } \rangle \\
& 2a + 1 + (-1)^{2a+1} \\
= & \quad \langle \text{ 2a+1 is odd } \rangle \\
& 2a + 1 - 1 \\
= & \quad \langle \text{ Arithmetic } \rangle \\
& 2a
\end{array}
$$

- z is odd. This is to say $z = 2a + 1$ for some integer a. We must find an x such that $hoo(x) = 2a + 1$. Let $x = 2a$.

$$
\begin{array}{ll}
& hoo(x) \\
= & \quad \langle \text{ Definition of } x \text{ } \rangle \\
& hoo(2a) \\
= & \quad \langle \text{ Definition of } hoo \text{ } \rangle \\
& 2a + (-1)^{2a} \\
= & \quad \langle \text{ 2a is even } \rangle \\
& 2a + 1
\end{array}
$$

4. [10 %] Extra credit

Given the following sets and relations:

*Actores***:** set of actors (male and female).

*Movies***:** Set of movies

*Studios***:** Set of movie studios

*ActsIn***:** a relation between *Actors* and *Movies*, where $(a, m) \in ActaIn$ if actor a had a role in movie m.

*Married***:** Between pairs of actors and dates where $(p, q, y) \in Married$ if p and q gor married on date y.

*Prod***:** Among*Movies*, *Studios*, and *dates* wheee $(p, s, y) \in Prod$ if movie p was produced my studio s on date y. y.

Define the following:

A relation R between pairs of actors and movies, where $(a, b, m) \in R$ if a and b acted in movie m before getting married.

We must define other relations:

ActedIn: A aealation between two actors and a movie where $(a1, a2, m) \in ActedIn$ if $a1$ and $a2$ acted together in movie m

$$ActedIn = S_{\{a1 \neq a2\}}(Join_1(ActsIn, ActsIn_{\langle movie, actor \rangle})_{\langle actor, actor, movie \rangle})$$

ActedTOn: A relation between two actors, a movie and a date where $(a1, a2, m) \in ActedTOn$ if $a1$ and $a2$ acted together in movie m on date d.

$$ActedIOn = Join_1(ActedIn, Prod)_{\langle movie, date, actor, actor \rangle}$$

R:

$$R = (S_{\{mdate > pdate\}}(Join_2(ActedTOn, Married)))_{\langle actor, actor, movie \rangle}$$