ISIS-1104-01 Matemática Estructural y Lógica
Parcial 3
Fecha: May 5, 2016

- Esta prueba es INDIVIDUAL.
- Está permitido el uso de las hojas de fórmulas.
- Está prohibido el uso de cualquier otro material como cuadernos, libros o fotocopias.
- Está prohibido el uso de cualquier dispositivo electrónico.
- El intercambio de información con otro estudiante está terminantemente prohibido.
- Cualquier irregularidad con respecto a estas reglas podría ser considerada fraude.
- Responda el examen en los espacios proporcionados. No se aceptarán hojas adicionales.
- No olvide marcar el examen antes de entregarlo.

IMPORTANTE: Soy consciente de que cualquier tipo de fraude en los exámenes es considerado como una falta grave en la Universidad. Al firmar y entregar este examen doy expreso testimonio de que este trabajo fue desarrollado de acuerdo con las normas establecidas. Del mismo modo, aseguro que no participé en ningún tipo de fraude.

Nombre	Carné
Firma	Fecha

NO ESCRIBIR NADA BAJO ESTA LÍNEA

1	20%	
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5	10%	
Total	100%	

Exercise 1. Using induction on n prove the following statement: $(\Sigma\ i \mid 1 \le i \le n\ :\ \frac{1}{2^i}\ ) = \frac{2^n-1}{2^n}$

Basis Case: $n = 1$ $(\Sigma\ i \mid 1 \le i \le 1\ :\ \frac{1}{2^i}\ ) = \frac{2^1-1}{2^1}$ By arithmetic:

$$(\Sigma\ i \mid 1 \le i \le 1\ :\ \frac{1}{2^i}\ ) = \frac{1}{2}$$

$$\begin{array}{ll}
 & (\Sigma\ i \mid \boxed{1 \le i \le 1}\ :\ \frac{1}{2^i}\ ) \\
= & \quad \langle \text{ Arithmetic } \rangle \\
 & (\Sigma\ i \mid i = 1\ :\ \frac{1}{2^i}\ ) \\
= & \quad \langle \text{ One Point } \rangle \\
 & \frac{1}{2^1} \\
= & \quad \langle \text{ Arithmetic } \rangle \\
 & \frac{1}{2}
\end{array}$$

Inductive Case: I.H.: $(\Sigma\ i \mid 1 \le i \le k\ :\ \frac{1}{2^i}\ ) = \frac{2^k-1}{2^k}$

Prove: $(\Sigma\ i \mid 1 \le i \le k+1\ :\ \frac{1}{2^i}\ ) = \frac{2^{k+1}-1}{2^{k+1}}$

$$\begin{array}{ll}
 & (\Sigma\ i \mid 1 \le i \le k+1\ :\ \frac{1}{2^i}\ ) \\
= & \quad \langle \text{ Split-off term } \rangle \\
 & \boxed{(\Sigma\ i \mid 1 \le i \le k\ :\ \frac{1}{2^i}\ )} + \frac{1}{2^{k+1}} \\
= & \quad \langle \text{ I.H. } \rangle \\
 & \boxed{\frac{2^k-1}{2^k}} + \frac{1}{2^{k+1}} \\
= & \quad \langle \text{ Arithmetic: multiply by } \frac{2}{2} \rangle \\
 & \boxed{\frac{2\cdot(2^k-1)}{2\cdot 2^k}} + \frac{1}{2^{k+1}} \\
= & \quad \langle \text{ Arithmetic } \rangle \\
 & \frac{2^{k+1}-2}{2^{k+1}} + \frac{1}{2^{k+1}} \\
= & \quad \langle \text{ Arithmetic } \rangle \\
 & \frac{2^{k+1}-2+1}{2^{k+1}} \\
= & \quad \langle \text{ Arithmetic } \rangle \\
 & \frac{2^{k+1}-1}{2^{k+1}}
\end{array}$$

Exercise 2. Using the recursive definition of the Fibonacci sequence:

Base Case$_0$: $F_0 = 0$

Base Case$_1$: $F_1 = 1$

Inductive Case: $F_{k+1} = F_k + F_{k-1}$

Using induction on n prove that following statement is true for $n \geq 1$:

$$gcd(F_n, F_{n+1}) = 1$$

Basis Case: $n = 1$

$gcd(F_1, F_2)$
$= \quad \langle$ Def. F $\rangle$
$gcd(1, 1)$
$= \quad \langle$ Theo: $gcd(x, 1) = x$ $\rangle$
1

Inductive Case: I.H. $gcd(F_k, F_{k+1}) = 1$

Prove $gcd(F_{k+1}, F_{k+2}) = 1$

$gcd(F_{k+1}, F_{k+2})$
$= \quad \langle$ Def F $\rangle$
$gcd(F_{k+1}, F_{k+1} + F_k)$
$= \quad \langle$ Theo: $gcd(a, b) = gcd(a, a + b)$ $\rangle$
$gcd(F_{k+1}, F_k)$
$= \quad \langle$ Theo: $gcd(a, b) = gcd(b, a)$ $\rangle$
$gcd(F_k, F_{k+1})$
$= \quad \langle$ I.H. $\rangle$
1

Exercise 3. We modify our definition of sequences of elements of type T so that the basic constructor operation is append ($\triangleright$).

Basis Clause: The empty sequence represented by the symbol ϵ is a sequence.

Inductive Clause: If $n \in T$, and S is a sequence of T, $S \triangleright n$ (pronounced S append n) is a sequence and represents the sequence obtained by adding n at the end of sequence S.

In this case, this would be the proof pattern:

Basis Case: $P(\epsilon)$

Inductive Case: $P(S) \Rightarrow P(S \triangleright y)$

I.H. $P(S)$

Prove: $P(S \triangleright y)$

Given the following recursively defined functions:

$val : \ Seq_{\{0,\ldots,9\}} \to \mathbb{Z}$

Base Case: $val(\epsilon) = 0$

Inductive Case: $val(S \triangleright d) = 10 \cdot val(S) + d$

$foo : \ Seq_{\{0,\ldots,9\}} \to Seq_{\{0,\ldots,9\}}$

Base Case: $foo(\epsilon) = \epsilon \triangleright 1$

Inductive Case:

$$foo(S \triangleright d) = \begin{cases} S \triangleright (d+1) & \text{if } \ d < 9 \\ foo(S) \triangleright 0 & \text{otherwise} \end{cases}$$

Prove the following theorem, using structural induction. Remember you must use the new proof pattern.

$$val(foo(S)) = val(S) + 1$$

Hint: For the inductive case, use proof by cases (last element is 9 or last element is less than 9)

Basis Case: $val(foo(\epsilon)) = val(\epsilon) + 1$

$$\begin{array}{ll} & val(foo(\epsilon)) = val(\epsilon) + 1 \\ = & \quad \langle \text{ Basis def of } foo \ \rangle \\ & val(\epsilon \triangleright 1) = val(\epsilon) + 1 \\ = & \quad \langle \text{ Inductive def of } val \ \rangle \\ & 10 \cdot val(\epsilon) + 1) = val(\epsilon) + 1 \\ = & \quad \langle \text{ Basis def of } val\text{, twice } \rangle \\ & 10 \cdot 0 + 1 = 0 + 1 \\ = & \quad \langle \text{ Arithmetic } \rangle \\ & true \end{array}$$

Inductive Case: I.H. $val(foo(S)) = val(S) + 1$

Prove $val(foo(S \triangleright d)) = val(S \triangleright d) + 1$

Case 1 $d = 9$**:**

$$
\begin{array}{ll}
& val(\boxed{foo(S \triangleright 9)}) = val(S \triangleright 9) + 1 \\
= & \quad \langle \text{ Inductive def of } foo\text{, for } d = 9 \text{ } \rangle \\
& val(foo(S) \triangleright 0) = val(S \triangleright 9) + 1 \\
= & \quad \langle \text{ Inductive def of } val\text{, twice } \rangle \\
& 10 \cdot \boxed{val(foo(S))} + 0 = (10 \cdot val(S) + 9) + 1 \\
= & \quad \langle \text{ I.H. } \rangle \\
& 10 \cdot (val(S) + 1) + 0) = (10 \cdot val(S) + 9) + 1 \\
= & \quad \langle \text{ Arithmetic } \rangle \\
& 10 \cdot val(S) + 10 = (10 \cdot val(S) + 9) + 1 \\
= & \quad \langle \text{ Arithmetic } \rangle \\
& 10 \cdot val(S) + 10 = 10 \cdot val(S) + (9 + 1) \\
= & \quad \langle \text{ Arithmetic } \rangle \\
& true
\end{array}
$$

Case 2 $d < 9$**:**

$$
\begin{array}{ll}
& val(\boxed{foo(S \triangleright d)}) = val(S \triangleright d) + 1 \\
= & \quad \langle \text{ Inductive def of } foo\text{, for } d < 9 \text{ } \rangle \\
& val(S \triangleright (d + 1)) = val(S \triangleright d) + 1 \\
= & \quad \langle \text{ Inductive def of } val\text{, twice } \rangle \\
& 10 \cdot val(S) + (d + 1)) = (10 \cdot val(S) + d) + 1 \\
= & \quad \langle \text{ Arithmetic } \rangle \\
& true
\end{array}
$$

Exercise 4. For this exercise, we also use the modified definition of sequences.

Given the following recursively defined functions:

$sum : Seq_{\mathbb{Z}} \to \mathbb{Z}$

Base Case: $sum(\epsilon) = 0$

Inductive Case: $sum(S \triangleright d) = sum(S) + d$

$odds : Seq_{\mathbb{Z}} \to \mathbb{N}$

Base Case: $odds(\epsilon) = 0$

Inductive Case:

$$odds(S \triangleright d) = \begin{cases} odds(S) & \text{if } 2 \mid d \\ odds(S) + 1 & \text{otherwise} \end{cases}$$

Prove the following theorem, using structural induction. Remember: you must use the new proof pattern.

$$odds(S) \;\equiv_2\; sum(S)$$

Hint: For the inductive case, use proof by cases (last element is odd or last element is even)

Basis Case: $odds(\epsilon) \;\equiv_2\; sum(\epsilon)$

$$\begin{array}{ll} & odds(\epsilon) \;\equiv_2\; sum(\epsilon) \\ = & \quad \langle \text{ Basis definition of } odds \text{ } \rangle \\ & 0 \;\equiv_2\; \text{sum}(\epsilon) \\ = & \quad \langle \text{ Basis definition of } sum \text{ } \rangle \\ & 0 \;\equiv_2\; 0 \\ = & \quad \langle \text{ Theo. } (a \;\equiv_m\; a) \equiv true \text{ } \rangle \\ & true \end{array}$$

Inductive Case: I.H. $odds(S) \;\equiv_2\; sum(S)$

Prove $odds(S \triangleright d) \;\equiv_2\; sum(S \triangleright d)$

Case 1: d **is even** This means: $0 \;\equiv_2\; d$

	Expresión	Justificación
1	$0 \;\equiv_2\; d$	Case hypothesis
2	$odds(S) \;\equiv_2\; sum(S)$	I.H.
3	$odds(S) + 0 \;\equiv_2\; sum(S) + d$	Theo. $a \;\equiv_n\; b \wedge c \;\equiv_n\; d \Rightarrow a + c \;\equiv_n\; b + d$ (1,2)
4	$odds(S \triangleright d) \;\equiv_2\; sum(S) + d$	Inductive def. odds with case hypothesis (3)
5	$odds(S \triangleright d) \;\equiv_2\; sum(S \triangleright d)$	Inductive def. sum with case hypothesis (4)

Case 2: d **is odd** This means: $1 \;\equiv_2\; d$

	Expresión	Justificación
1	$1 \;\equiv_2\; d$	Case hypothesis
2	$odds(S) \;\equiv_2\; sum(S)$	I.H.
3	$odds(S) + 1 \;\equiv_2\; sum(S) + d$	Theo. $a \;\equiv_n\; b \wedge c \;\equiv_n\; d \Rightarrow a + c \;\equiv_n\; b + d$ (1,2)
4	$odds(S \triangleright d) \;\equiv_2\; sum(S) + d$	Inductive def. odds with case hypothesis (3)
5	$odds(S \triangleright d) \;\equiv_2\; sum(S \triangleright d)$	Inductive def. sum with case hypothesis (4)

Exercise 5. Counting Problems:

1. Suppose you have an organization with 5 women and 4 men.

   (a) How many committees of size 4 that have at least one woman can you form?

   - Total number of committees: $\binom{9}{4}$
   - Number of committees without any women: 1
   - Answer: $\binom{9}{4} - 1$

   (b) How many ways can you seat 5 people in a row so that no two women nor two men are seated side by side? Notice that in a row, the order does matter. The same people can form more than one row.

   woman - man - woman -man - woman OR man-woman-man-woman-man

   $$5 \cdot 4 \cdot 4 \cdot 3 \cdot 3 + 4 \cdot 5 \cdot 3 \cdot 4 \cdot 3$$

2. Give argument to prove that if you have a sequence of integers S such that

   $$long(S) > max_s(S) - min_s(S) + 1$$

   then there must be an element x of S such that $count(x, S) > 1$ If you have two numbers: a and b such that $a \leq b$, the number of distinct integers between a and b is $(b - a + 1)$. All numbers in the sequence are between $min_s(S)$ and $max_s(S)$. If there are more numbers than the possible values in the range, then by the *pigeonhole principle* there muust be a number that appears more than once in the sequence.