ISIS-1104-01 Matemática Estructural y Lógica
MidTerm 2
Date: October 20, 2016

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- Está permitido el uso de las hojas de teoremas publicadas en sicua+.
- Está prohibido el uso de cualquier dispositivo electrónico.
- El intercambio de información con otro estudiante está terminantemente prohibido.
- Cualquier irregularidad con respecto a estas reglas podría ser considerada fraude.
- Responda el examen en los espacios proporcionados. No se aceptarán hojas adicionales.
- No olvide marcar el examen antes de entregarlo.

IMPORTANTE: Soy consciente de que cualquier tipo de fraude en los exámenes es considerado como una falta grave en la Universidad. Al firmar y entregar este examen doy expreso testimonio de que este trabajo fue desarrollado de acuerdo con las normas establecidas. Del mismo modo, aseguro que no participé en ningún tipo de fraude.

Nombre	Carné
Firma	Fecha

NO ESCRIBIR NADA BAJO ESTA LÍNEA

1.1	5%	
1.2	15%	
1.3	20%	
2.1.1	10%	
2.1.2	10%	
2.2	20%	
3	20%	
Total	100%	

1 [40%] Sets

We define this new binary operator over sets:

$$(A \odot B) = (\sim (A \setminus B))$$

Prove or refute the following statements. Remember U is the Universe.

1.1 [5%] $(A \odot B) = (\sim A \ \cup \ B)$

$$\begin{array}{ll}
 & A \odot B \\
= & \qquad \langle \text{ Definition of } \odot \text{ } \rangle \\
 & \sim (A \setminus B) \\
= & \qquad \langle \text{ Definition of } \setminus \text{ } \rangle \\
 & \sim (A \cap \sim B) \\
= & \qquad \langle \text{ de Morgan } \rangle \\
 & \sim A \cup \sim\sim B \\
= & \qquad \langle \text{ Double } \sim \text{ } \rangle \\
 & \sim A \cup B
\end{array}$$

1.2 [15%] $((A \odot B) = U) \equiv (A \subseteq B)$

Using exercise 1.1 what we have to prove is: $((\sim A \cup B) = U) \equiv (A \subseteq B)$

Applying the metaTheorem we have to show: $((\neg P_A \vee P_B) \equiv true) \equiv (P_A \Rightarrow P_B)$

$$\begin{array}{ll}
 & (\neg P_A \vee P_B) \equiv true \\
= & \qquad \langle \text{ Identity } \rangle \\
 & (\neg P_A \vee P_B) \\
= & \qquad \langle \text{ Definition of } \Rightarrow \text{ } \rangle \\
 & P_A \Rightarrow P_B
\end{array}$$

1.3 [20%] $((A \odot B) \ \cap \ (B \odot C)) \subseteq (A \odot C)$

Using the metatheorem and exercise 1, what we have to prove is:

$$((\neg P_A \vee P_B) \wedge (\neg P_B \vee P_C)) \Rightarrow (\neg P_A \vee P_B)$$

$$\begin{array}{ll}
 & (\neg P_A \vee P_B) \wedge (\neg P_B \vee P_C) \\
\Rightarrow & \qquad \langle \text{ Resolution } \rangle \\
 & (\neg P_A \vee P_C)
\end{array}$$

2 Relations[40%]

2.1 [20%] Modeling with n-ary relations

Given the following sets and relations. You may assume all names (people, songs, albums, labels) are unique.

$Singers$**:** Set singers.

$Composers$**:** Set of composers.

$Songs$**:** Set of Songs.

$Labels$**:** Set of record labels.

$Albums$**:** Set of albums (album ids)

$AlbumLabel$**:** A relation between $Albums$ and $RecordLabels$, where $(a, rl) \in AlbumLabel$ if the album a was produced under label $rl\cdot$

$Compose$**:** A relation among $Composers$ and $Songs$, where $(c, s) \in Compose$ if composer c wrote song c.

$Prod$**:** A relation among singers, songs and albums where $(v, s, a) \in Prod$ if album a contains song s sung by singer v.

Define the following:

2.1.1 [10%] A relation R1 among singers, composers, and songs where $(v, c, s) \in R1$ if singer v has interpreted (sung) song s by composer c

Let: $R = Join_1(Composers, Prod_{\langle songs, singers, albums\rangle})$

$$R1 = R_{\langle singer, composer, song\rangle})$$

2.1.2 [10%] The set of singers that have performed songs that he/she composed herself under the label SonyMusic.

Using R from the previous exercise:

$$(S_{\{RecordLabel=sonyMusic\}}(Join_1(S_{\{singer=composer\}}(R), AlbumLabel)))_{\langle singers\rangle}$$

2.2 [25%] Binary relations: properties

Given the following relation between Pairs of integers numbers:

$$R : (\mathbb{Z} \times \mathbb{Z}) \times (\mathbb{Z} \times \mathbb{Z})$$
$$(x, y)\ R\ (a, b) \equiv (|x - y| \leq |a - b|)$$

We use $|x|$ to denote the absolute value of $|x|$: $|x| = x$ if $x \geq 0$; $|x| = -x$ if $x < 0$

Is this relation:

1. reflexive? YES
2. symmetric? NO
3. antisymmetric? NO
4. transitive? YES

Prove that your answers are correct.

It is Reflexive: We must prove $(x, y)R(x, y)$ for arbitrary x and y

$$\begin{aligned}
& (x, y)\ R\ (x, y) \\
= & \quad \langle \text{ Definition of } R \ \rangle \\
& |x - y| \leq |x - y| \\
= & \quad \langle \ \text{ Arithmetic: } X \leq X \ \rangle \\
& \mathit{true}
\end{aligned}$$

It is not symmetric: We must find values x, y, a and b such that $(x, y)\ R\ (a, b)$ but $(a, b)\ R\ (x, y)$ let $x = 5$, $y = 3$, $a = 6$, and $b = 3$: We can see that:

$$(5, 3)\ R\ (6, 3)$$

because

$$|5 - 3|\ \leq\ |6 - 3|$$

However:

$$\neg((6, 3)\ R\ (5, 3))$$

because

$$|6 - 3|\ >\ |5 - 3|$$

It is not antisymmetric We must find values x, y, a and b such that $(x, y)\ R\ (a, b)$ and $(a, b)\ R\ (x, y)$ but $(x, y)\ \neq\ (a, b)$ let $x = 2$, $y = 0$, $a = 3$, and $b = 1$: We can see that:

$$(2, 0)\ R\ (3, 1) \wedge (3, 1)\ R\ (2, 0)$$

because

$$|2 - 0|\ \leq\ |3 - 1| \wedge |3 - 1|\ \leq\ |2 - 0|$$

However:

$$(2, 0) \neq (3.1)$$

It is transitive We must prove that if we have $(x, y)R(z, w)$ and $(z, w)R(v, u)$ then $(x, y)R(v, u)$ for arbitrary: u, v, w, x, y, and z.

$$\begin{aligned}
& (x, y)\ R\ (z, w) \wedge (z, w)\ R\ (v, u) \\
= & \quad \langle \text{ Definition of } R \ \rangle \\
& |x - y| \leq |z - w| \wedge |z - w| \leq |v - u| \\
= & \quad \langle \ \text{ Arithmetic: } \leq \text{ is transitive } \rangle \\
& |x - y| \leq |v - u| \\
= & \quad \langle \text{ Definition of } R \ \rangle \\
& (x, y)\ R\ (v, u)
\end{aligned}$$

3 Functions [20%]: Given the following function from pairs of natural numbers to integer numbers:

$$\texttt{foo} : (\mathbb{N} \times \mathbb{N}) \to \mathbb{Z}$$
$$foo(n, m) = n + m \cdot (-1^m)$$

Is this function:

Onto? YES

One to One? No

Prove that your answers are correct. Remember $-1^m = 1$ if m is even and it is -1 if m is odd.

Onto? To prove that it is onto, we must prove: $(\forall\ z : \mathbb{Z} \mid\ \ :\ (\exists\ b, c : \mathbb{N} \mid\ \ :\ foo(a, b) = z\ )\ )$

By cases:

$z \geq 0$**:** choose: $n = z$, $m = 0$

$$\begin{aligned}
& goo(n, m) \\
= & \quad \langle\ n = z,\ m = 0\ \rangle \\
& goo(z, 0) \\
= & \quad \langle\ \text{def } goo\ \rangle \\
& z + 0 \cdot (-1^0) \\
= & \quad \langle\ \text{Arithmetic}\ \rangle \\
& z
\end{aligned}$$

$(z < 0) \wedge odd(-z)$ choose: $n = 0$, $m = -z$ It is important to point out that if $(z < 0)$ then $-z > 0$; therefore, $-z \in \mathbb{N}$

$$\begin{aligned}
& goo(n, m) \\
= & \quad \langle\ n = 0,\ m = -z\ \rangle \\
& goo(0, -z) \\
= & \quad \langle\ \text{def } goo\ \rangle \\
& 0 + (-z) \cdot (-1^{(-z)}) \\
= & \quad \langle\ \text{Arithmetic}\ \rangle \\
& (-z) \cdot (-1^{(-z)}) \\
= & \quad \langle\ \text{Arithmetic: (-z) is odd}\ \rangle \\
& (-z) \cdot (-1) \\
= & \quad \langle\ \text{Arithmetic}\ \rangle \\
& z
\end{aligned}$$

$(z < 0) \wedge even(-z)$ choose: $n = 1$ and $z = -z + 1$ It is important to point out that if $(z < 0)$ then $-z + 1 > 0$; therefore, $-z \in \mathbb{N}$. Also note that if $even(-z)$ then $odd(-z + 1)$

$$
\begin{array}{ll}
 & goo(n, m) \\
= & \quad \langle\ n = 1,\ m = -z + 1\ \rangle \\
 & goo(1, -z + 1) \\
= & \quad \langle\ \text{def } goo\ \rangle \\
 & 1 + (-z + 1) \cdot (-1^{(-z+1)}) \\
= & \quad \langle\ \text{Arithmetic: (-z+1) is odd}\ \rangle \\
 & 1 + (-z + 1) \cdot (-1) \\
= & \quad \langle\ \text{Arithmetic}\ \rangle \\
 & 1 + (z - 1) \\
= & \quad \langle\ \text{Arithmetic}\ \rangle \\
 & z
\end{array}
$$

Since for any z we found a and b, such that $foo(a, b) = z$, then foo is onto.

The function is not one-to-one Note that:

- $foo(2, 0) = 2$ and
- $foo(0, 2) = 2$
- but $(0, 2) \neq (2, 0)$

Therefore, it is not true that $(\forall\ x, y, a, b \mid foo(x, y) = foo(a, b)\ :\ (x, y) = (a, b)\ )$