Universidad de los Andes

ISIS-1104-05 Matemática Estructural y Lógica
Parcial 1
Fecha: Septiembre 12, 2016

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- Las preguntas son en inglés, pero si lo desea, puede responder en español.

IMPORTANTE: Soy consciente de que cualquier tipo de fraude en los exámenes es considerado como una falta grave en la Universidad. Al firmar y entregar este examen doy expreso testimonio de que este trabajo fue desarrollado de acuerdo con las normas establecidas. Del mismo modo, aseguro que no participé en ningún tipo de fraude.

Nombre	Carné
Firma	Fecha

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1.1	15 %	
1.2	15 %	
2.1	15 %	
2.2	20 %	
3.1	15 %	
3.2	20 %	
Total	100 %	

1. [30 %] Propositional Calculus

Suppose we add the following function to the propositional calculus:

$$choose2(P, Q, R, S) \equiv ((P \not\equiv Q) \wedge (R \not\equiv S)) \tag{1}$$

1.1. [15 %] Prove or refute: $choose2(p, false, true, q) \equiv (p \wedge \neg q)$

$$
\begin{array}{ll}
 & choose2(p, false, true, q) \\
= & \quad \langle \text{ Def. of choose2 } \rangle \\
 & (\boxed{p \not\equiv false}) \wedge (\boxed{true \not\equiv q}) \\
= & \quad \langle \text{ Def. of } \not\equiv \text{ twice } \rangle \\
 & \boxed{\neg(p \equiv false)} \wedge \boxed{\neg(true \equiv q)} \\
= & \quad \langle \text{ Negation of } \equiv \text{ twice } \rangle \\
 & (p \equiv \boxed{\neg false}) \wedge (true \equiv \neg q) \\
= & \quad \langle \text{ Negation of } false \ \rangle \\
 & \boxed{(p \equiv true)} \wedge \boxed{(true \equiv \neg q)} \\
= & \quad \langle \text{ Identity twice } \rangle \\
 & p \wedge \neg q
\end{array}
$$

1.2. [15 %] Prove or refute the validity of the following inference rule:

$$
\begin{array}{l}
choose2(p, q, r, s) \\
p \\
r \Rightarrow q \\
\hline
s
\end{array}
$$

	Expresión	Justificación
1	$choose2(p, q, r, s)$	Hypothesis
2	p	Hypothesis
3	$r \Rightarrow q$	Hypothesis
4	$(p \not\equiv q) \wedge (r \not\equiv s)$	Def. Choose2
5	$p \not\equiv q$	Simplification (4)
6	$r \not\equiv s$	Simplification (4)
7	$\neg(p \equiv q)$	Definition $\not\equiv$ (5)
8	$p \equiv \neg q$	Negation $\equiv$ (7)
9	$\neg q$	Deduction $\equiv$ (8,2)
10	$\neg r$	Modus Tollens (9,3)
11	$\neg(r \equiv s)$	Definition $\not\equiv$ (6)
12	$\neg r \equiv s$	Negation $\equiv$ (11)
13	s	Deduction $\equiv$ (10,12)

This would have been much easier if we have the following inference rules, which validity is easy ro prove.

$$\frac{\begin{array}{l} p \not\equiv q \\ p \end{array}}{\neg q}$$

$$\begin{array}{ll} & (p \not\equiv q) \wedge p \\ = & \quad \langle \text{ Definition of } \not\equiv \rangle \\ & \neg(p \equiv q) \wedge p \\ = & \quad \langle \text{ negation of } \equiv \rangle \\ & (p \equiv \neg q) \wedge p \\ \Rightarrow & \quad \langle \equiv\text{-Deduction } \rangle \\ & negq \end{array}$$

$$\frac{\begin{array}{l} p \not\equiv q \\ \neg p \end{array}}{q}$$

$$\begin{array}{ll} & (p \not\equiv q) \wedge \neg p \\ = & \quad \langle \text{ Definition of } \not\equiv \rangle \\ & \neg(p \equiv q) \wedge p \\ = & \quad \langle \text{ negation of } \equiv \rangle \\ & (\neg p \equiv q) \wedge \neg p \\ \Rightarrow & \quad \langle \equiv\text{-Deduction } \rangle \\ & q \end{array}$$

Now the proof with these two lemmas which we can call $\not\equiv$-deduction.

	Expresión	Justificación
1	$choose2(p, q, r, s)$	Hypothesis
2	p	Hypothesis
3	$r \Rightarrow q$	Hypothesis
4	$(p \not\equiv q) \wedge (r \not\equiv s)$	Def. Choose2
5	$p \not\equiv q$	Simplification (4)
6	$r \not\equiv s$	Simplification (4)
7	$\neg q$	Deduction $\not\equiv$ (5,2)
8	$\neg r$	Modus Tollens (7,3)
9	s	Deduction $\not\equiv$ (8,6)

2. Deduction in the propositional calculus [35 %]

There was a robbery in the bank and we want to find who is guilty. We have the following facts.

1. The suspects are Adam, Bret, Charlie and David. $A \vee B \vee C \vee D$

2. We know that there exactly two people were needed to commit the robbery: one to drive and the other to open the safe. Therefore, the guilty pair has to have one that can drive and one that can open the safe. $choose2(A, C, B, D)$

   - Adam and Charlie can drive but cannot open safes
   - Bret and David can open safes but cannot drive

3. If Bret is guilty then Adam and Charlie are both innocent. $B \Rightarrow \neg A \wedge \neg C$

4. David is innocent if Bret and Charlie are both innocent. $\neg B \wedge \neg C \Rightarrow \neg D$

2.1. Modeling [15 %]

Using only the variables listed below to represent the facts of the problem, model the hypotheses and your conclusion regarding who is guilty and who is innocent.

A: Adam is guilty

B: Bret is guilty

C: Charlie is guilty

D: David is guilty

1. If Bret were guilty then both drivers (Adam and Charlie) would be innocent, thus contradicting the fact that the criminals needed a diver.
2. So Bret is innocent. Since we also need safe breaker, then David must be guilty.
3. Using the contrapositive, we know that either Bret or Charlie are guilty. Since we know by (1) that Bret is innocent, Charlie must be guilty.
4. Since there ws only one driver, Adam must be innocent.

2.2. Proofs in Propositional Calculus [20 %]

Formally prove your conclusion.

First we prove a lemma: $\neg B$.

	Expresión	Justificación
1	B	Assumption
2	$B \Rightarrow \neg A \wedge \neg C$	Hypothesis
3	$\neg A \wedge \neg C$	M.P. (1,2)
4	$choose2(A, C, B, D)$	hypothesis
5	$(A \not\equiv C) \wedge (B \not\equiv D)$	Def. Choose (4)
6	$(A \not\equiv C)$	Simplification (5)
7	$(A \vee C) \wedge \not(A \wedge C)$	Definition $\not\equiv$ (6)
8	$A \vee C$	Simplification (7)
9	$\neg(\neg A \vee \neg C)$	de Morgan and Double negation (8)
10	$False$	Contradiction (2,9)

Since we had assumed B, we can conclude $\neg B$

Now, knowing $\neg B$ we can prove the that Charlie must be guilty.

	Expresión	Justificación
1	$\neg B$	Lemma
2	$choose2(A, C, B, D)$	hypothesis
3	$(A \not\equiv C) \wedge (B \not\equiv D)$	Def. Choose2 (2)
4	$(B \not\equiv D)$	Simplification (3)
5	$\neg(B \equiv D)$	Definition $\not\equiv$ (4)
6	$\neg B \equiv D$	Negation $\equiv$ (5)
7	D	Leibniz (1,6)
8	$\neg B \wedge \neg C \Rightarrow \neg D$	Hypothesis
9	$\neg\neg D$	Double negation (7)
10	$\neg(\neg B \wedge \neg C)$	Modus Tollens (8, 9)
11	$\neg\neg B \vee \neg\neg C)$	De Morgan (10)
12	$\neg\neg C$	Disjunctive Syllogism (11,1)
13	C	Double Negation (12)
14	$(C \not\equiv A)$	Simplification (3)
15	$(C \vee A) \wedge \neg(C \wedge A)$	Definition $\not\equiv$ (14)
16	$\neg(C \wedge A)$	Simplification (15)
17	$\neg C \vee \neg A$	De Morgan (17)
18	$\neg A$	Disjunctive Syllogism (17,12)
19	$\neg B \wedge D \wedge C \wedge \neg A$	Conjunction (1,7,13,18)

Using $\not\equiv$ deduction, this proof would have been much shorter.

	Expresión	Justificación
1	$\neg B$	Lemma
2	$choose2(A, C, B, D)$	hypothesis
3	$(A \not\equiv C) \wedge (B \not\equiv D)$	Def. Choose2 (2)
4	$(B \not\equiv D)$	Simplification (3)
5	D	$\not\equiv$ deduction(1,5)
6	$\neg B \wedge \neg C \Rightarrow \neg D$	Hypothesis
7	$\neg\neg D$	Double negation (7)
8	$\neg(\neg B \wedge \neg C)$	Modus Tollens (8, 9)
9	$\neg\neg B \vee \neg\neg C)$	De Morgan (10)
10	$\neg\neg C$	Disjunctive Syllogism (11,1)
11	C	Double Negation (12)
12	$(C \not\equiv A)$	Simplification (3)
13	$\neg A$	$\not\equiv$ deduction (13,14)
14	$\neg B \wedge D \wedge C \wedge \neg A$	Conjunction (1,7,13,18)

3. Predicate calculus [35 %]

You land on a distant planet. You do know the following:

1. There are three kinds of inhabitants: alpha, beta or delta. $(\forall\ h \mid \quad : \ alpha(h) \lor beta(h) \lor alpha(h)\ )$

2. Betas are green. $(\forall\ h \mid beta(h) \ : \ green(h)\ )$

3. Deltas are green $(\forall\ h \mid delta(h) \ : \ green(h)\ )$

4. Inhabitants can form couples.

   a) An Alpha's couple is never an Alpha. $(\forall\ h, g \mid alpha(h) \land couple(h, g) \ : \ \neg alpha(g)\ )$

   b) A Delta's couple is never an Alpha. $(\forall\ h, g \mid delta(h) \land couple(h, g) \ : \ \neg alpha(g)\ )$

You meet two inhabitants: A and B.

1. A is not green $\neg green(A)$

2. A and B are a couple $couple(A, B)$

This information is enough for us to determine the type of A and B. What are these types?

Since A is not green, it cannot be a beta nor a delta, then it must be an alpha. B and A are a couple. Since A is an alpha, B cannot be an alpha. B cannot be a delta either, because A is an alpha as well as B's couple, and since delta's couples are not alphas. Therefor B, must be a beta.

$$alpha(A) \land beta(B)$$

3.1. Modeling [15 %]

Model the problem, using the following predicates:

- alpha(d) : d is alpha
- beta(d) : d is beta
- delta(d) : d is delta
- green(d) : d is green
- couple(p,c): p and c are a couple

3.2. Deduction in the predicate calculus [20 %]

Formally prove that you can reach your conclusion from the hypotheses and the premises.

You may use this hypothesis for the first item.

$$(\forall\ h \mid\ \ :\ alpha(h)\ \vee\ beta(h)\ \vee\ delta(h)\ )$$

You may also use this hypothesis if you need it:

$$(\forall\ a, b \mid\ \ :\ couple(a, b)\ \equiv\ couple(b, a)\ )$$

	Expresión	Justificación
1	$(\forall\ h \mid\ \ :\ alpha(h) \vee beta(h) \vee alpha(h)\ )$	Hypothesis
2	$(\forall\ a, b \mid\ \ :\ couple(a, b)\ \equiv\ couple(b, a)\ )$	Hypothesis
3	$(\forall\ h \mid beta(h)\ :\ green(h)\ )$	Hypothesis
4	$(\forall\ h \mid delta(h)\ :\ green(h)\ )$	Hypothesis
5	$(\forall\ h, g \mid alpha(h) \wedge couple(h, g)\ :\ \neg alpha(g)\ )$	Hypothesis
6	$(\forall\ h, g \mid delta(h) \wedge couple(h, g)\ :\ \neg alpha(g)\ )$	Hypothesis
7	$\neg green(A)$	Premise
8	$couple(A, B)$	Premise
9	$\neg beta(A)$	Modus Tollens (7,3)
10	$\neg delta(A)$	Modus Tollens (7,4)
11	$alpha(A)$	∀-Disjunctive Syllogism (1,9,10)
12	$alpha(A) \wedge couple(A, B)$	Composition (8,11)
13	$\neg alpha(B)$	Modus Ponens (12,5)
14	$\neg\neg alpha(A)$	Double Negation (11)
15	$\neg(delta(B) \wedge couple(B, A))$	Modus Tollens (14,4)
16	$\neg delta(B) \vee \neg couple(B, A)$	De Morgan (15)
17	$couple(B, A)$	Leibniz (8,2)
18	$\neg\neg couple(B, A)$	Double Negation (17))
19	$\neg delta(B)$	Disjunctive Syllogism (18,16)
20	$beta(B)$	∀-Disjunctive Syllogism (13,19,1)
21	$beta(B) \wedge alpha(A)$	Composition (20,11)