ISIS-1104-05 Matemática Estructural y Lógica
MidTerm 3
Date: May 15, 2017

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- Está prohibido el uso de cualquier dispositivo electrónico.
- El intercambio de información con otro estudiante está terminantemente prohibido.
- Cualquier irregularidad con respecto a estas reglas podría ser considerada fraude.
- Responda el examen en los espacios proporcionados. No se aceptarán hojas adicionales.
- No olvide marcar el examen antes de entregarlo.
- Las preguntas son en inglés, pero si lo desea, puede responder en español.

IMPORTANTE: Soy consciente de que cualquier tipo de fraude en los exámenes es considerado como una falta grave en la Universidad. Al firmar y entregar este examen doy expreso testimonio de que este trabajo fue desarrollado de acuerdo con las normas establecidas. Del mismo modo, aseguro que no participé en ningún tipo de fraude.

Nombre	Carné
Firma	Fecha

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1.	25 %	
2	30 %	
3.	30 %	
4.	15 %	

1. Integers

Do not use induction for any of the problems in this section.

1.1. Prove the following statement:

$$(a+b) \ \equiv_n \ (a \ mod \ n) + (b \ mod \ n)$$

We will use the following Lemma, which is true by the definition of "|".

$$n \mid n \cdot c$$

$$
\begin{array}{ll}
 & (a+b) \ \equiv_n \ (a \ mod \ n) + (b \ mod \ n) \\
= & \qquad \langle \text{ Division Algorithm } \rangle \\
 & ((n \cdot (a \div n) + (a \ mod \ n)) + (n \cdot (b \div n) + (b \ mod \ n))) \ \equiv_n \ (a \ mod \ n) + (b \ mod \ n) \\
= & \qquad \langle \text{ Definition of } x \ \equiv_n \ y \ \rangle \\
= & \qquad \langle \text{ Division Algorithm } \rangle \\
 & n \mid ((a \ mod \ n) + (b \ mod \ n)) - ((n \cdot (a \div n) + (a \ mod \ n)) + (n \cdot (b \div n) + (b \ mod \ n))) \\
= & \qquad \langle \text{ Arithmetic } \rangle \\
 & n \mid -n \cdot ((a \div n) + n \cdot (b \div n)) \\
= & \qquad \langle \text{ Arithmetic } \rangle \\
 & n \mid n \cdot (-(a \div n) - (b \div n)) \\
= & \qquad \langle \text{ Lemma } \rangle \\
 & true
\end{array}
$$

1.2. Use the fundamental theorem of arithmetic to prove the following statement:

If n is a perfect square then $2 \cdot n$ is not a perfect square.

We will use the following lema:

Lemma: if n is a perfect square then the exponents in its decomposition into prime factors are all even. Formally $\overline{n} = \langle 2 \cdot p_1, 2 \cdot p_2, \ldots 2 \cdot p_m \rangle$

Now using the Lemma:

- Let n be a perfect square
- then by the lemma, $\overline{n} = \langle 2 \cdot p_1, 2 \cdot p_2, \ldots 2 \cdot p_m \rangle$
- $\overline{2 \cdot n} = \langle 2 \cdot p_1 + 1, 2 \cdot p_2, \ldots 2 \cdot p_m \rangle$

Since the decomposition into prime factors is unique. Therefore, $\langle 2 \cdot p_1 + 1, 2 \cdot p_2, \ldots 2 \cdot p_m \rangle$ represents the only decomposition. 2's exponent $2 \cdot p_1 + 1$ is not even. Then by the lemma (using the contrapositive) $2 \cdot n$ is not a perfect square.

Now we prove the Lemma.

- If n is a perfect square then there is an integer m such that $n = m \cdot m$
- By the Fundamental Theorem of Arithmetic, m has a unique decomposition into prime factors. $\overline{m} = \langle p_1, p_2, \ldots p_m \rangle$.
- Using properties of exponents: $\overline{m} = \langle 2 \cdot p_1, 2 \cdot p_2, \ldots 2 \cdot p_m \rangle$.

2. Induction on Natural Numbers

2.1. F_N: We define: Fibonacci: F_n

Basis case 0: $F_0 = 0$

Basis case 1: $F_1 = 1$

Inductive case: $F_{n+1} = F_n + F_{n-1}$ for $n > 0$

Prove the following statement: For $n > 0$:

$$(\Sigma\ i \mid 1\ \leq i\ \leq\ 2n\ :\ F_i \cdot (-1)^i)\ =\ F_{(2n-1)} - 1$$

Hint: Note that for the basis case n=1, you are adding two terms: 1 and 2. For the inductive case, two terms must be splitted off. ($i := 2k+1$, $i := 2k+2$).

Basis Case: $n = 1$ $(\Sigma\ i \mid 1\ \leq i\ \leq\ 2 \cdot 1\ :\ F_i \cdot (-1)^i)\ =\ F_{(2 \cdot 1-1)} - 1$

Which is equivalent to:$(\Sigma\ i \mid 1\ \leq i\ \leq\ 2\ :\ F_i \cdot (-1)^i)\ =\ F_1 - 1$

Using $F_1 = 1$: $(\Sigma\ i \mid 1\ \leq i\ \leq\ 2\ :\ F_i \cdot (-1)^i)\ =\ 0$

$$\begin{array}{ll}
 & (\Sigma\ i \mid 1\ \leq i\ \leq\ 2\ :\ F_i \cdot (-1)^i) \\
= & \quad \langle \text{ Split-off term } \rangle \\
 & (\Sigma\ i \mid 1\ \leq i\ \leq\ 1\ :\ F_i \cdot (-1)^i) + F_2 \cdot (-1)^2 \\
= & \quad \langle \text{ Arithmetic } \rangle \\
 & (\Sigma\ i \mid i\ =\ 1\ :\ F_i \cdot (-1)^i) + F_2 \cdot (-1)^2 \\
= & \quad \langle \text{ One Point } \rangle \\
 & F_1 \cdot (-1)^1 + F_2 \cdot (-1)^2 \\
= & \quad \langle \text{ Arithmetic } \rangle \\
 & -F_1 + F_2 \\
= & \quad \langle \text{ Def. } F_n \rangle \\
 & -F_1 + F_1 + F_0 \\
= & \quad \langle \text{ Arithmetic } \rangle \\
 & F_0 \\
= & \quad \langle \text{ Def. } F_0 \rangle \\
 & 0
\end{array}$$

Inductive Case: I.H.: $(\Sigma\ i \mid 1\ \leq i\ \leq\ 2 \cdot k\ :\ F_i \cdot (-1)^i)\ =\ F_{(2 \cdot k-1)} - 1$

Prove: $(\Sigma\ i \mid 1\ \leq i\ \leq\ 2 \cdot (k+1)\ :\ F_i \cdot (-1)^i)\ =\ F_{(2 \cdot (k+1)-1)} - 1$

Which is: $(\Sigma\ i \mid 1\ \leq i\ \leq\ 2 \cdot k + 2\ :\ F_i \cdot (-1)^i)\ =\ F_{2 \cdot k+1} - 1$

$$\begin{array}{ll}
 & (\Sigma\ i \mid 1\ \leq i\ \leq\ 2 \cdot k + 2\ :\ F_i \cdot (-1)^i) \\
= & \quad \langle \text{ Split-off Term } \rangle \\
 & (\Sigma\ i \mid 1\ \leq i\ \leq\ 2 \cdot k + 1\ :\ F_i \cdot (-1)^i) + F_{2k+2} \cdot (-1)^{2k+2} \\
= & \quad \langle \text{ Split-off Term } \rangle \\
 & (\Sigma\ i \mid 1\ \leq i\ \leq\ 2 \cdot k + 1\ :\ F_i \cdot (-1)^i) + F_{2k+1} \cdot (-1)^{2k+1} + F_{2k+2} \cdot (-1)^{2k+2} \\
= & \quad \langle \text{ I.H. } \rangle \\
 & F_{(2 \cdot k-1)} - 1 + F_{2k+1} \cdot (-1)^{2k+1} + F_{2k+2} \cdot (-1)^{2k+2} \\
= & \quad \langle \text{ Arithmetic } \rangle
\end{array}$$

$$
\begin{array}{ll}
 & F_{(2\cdot k-1)} - 1 - F_{2k+1} + F_{2k+2} \\
= & \quad \langle \text{ Def. } F_n \text{ twice } \rangle \\
 & F_{(2\cdot k-1)} - 1 - (F_{2k} + F_{2k-1}) + (F_{2k+1} + F_{2k}) \\
= & \quad \langle \text{ Arithmetic } \rangle \\
 & F_{(2\cdot k-1)} - 1 - F_{2k} - F_{2k-1} + F_{2k+1} + F_{2k} \\
= & \quad \langle \text{ Arithmetic } \rangle \\
 & F_{2k+1} - 1
\end{array}
$$

2.2. Prove: $(\Sigma\ i \mid 0 \leq i \leq n\ :\ \binom{n}{i}\ ) = 2^n$

Use the following hints. You do not need to apply the definition of $\binom{n}{m}$. Only use the hints!

1. $\binom{n}{n} = 1$ for any n, therefore $\binom{n}{n} = \binom{n+1}{n+1}$
2. $\binom{n}{0} = 1$ for any n, therefore $\binom{n}{0} = \binom{n+1}{0}$
3. if $0 < m \leq n$ then $\binom{n+1}{m} = \binom{n}{m} + \binom{n}{m-1}$.

You may also use these rules for sumations:

Change of dummy: $(\Sigma\ i \mid 1\ \leq i\ \leq\ k\ : E(i-1)) = (\Sigma\ i \mid 0\ \leq i\ \leq\ k-1\ : E(i))$

Distributivity: $(\Sigma\ i\ |R : E + T) = (\Sigma\ i\ |R : E) + (\Sigma\ i\ |R : T)$

Basis Case: $(\Sigma\ i \mid 0 \leq i \leq 0\ :\ \binom{0}{i}\ ) = 2^0$ By arithmetic: $(\Sigma\ i \mid 0 \leq i \leq 0\ :\ \binom{0}{i}\ ) = 1$

$$
\begin{array}{ll}
 & (\Sigma\ i \mid 0 \leq i \leq 0\ :\ \binom{0}{i}\ ) \\
= & \qquad \langle \text{ Arithmetic } \rangle \\
 & (\Sigma\ i \mid i = 0\ :\ \binom{0}{i}\ ) \\
= & \qquad \langle \text{ 1-point } \rangle \\
 & \binom{0}{0} \\
= & \qquad \langle \text{ Hint 1 } \rangle \\
 & 1
\end{array}
$$

Inductive Case I.H.: $(\Sigma\ i \mid 0 \leq i \leq k\ :\ \binom{k}{i}\ ) = 2^k$

Prove: $(\Sigma\ i \mid 0 \leq i \leq k+1\ :\ \binom{k+1}{i}\ ) = 2^{k+1}$

$$
\begin{array}{ll}
 & (\Sigma\ i \mid 0 \leq i \leq k+1\ :\ \binom{k+1}{i}\ ) \\
= & \qquad \langle \text{ Split off term } \rangle \\
 & (\Sigma\ i \mid 0 \leq i \leq k\ :\ \binom{k+1}{i}\ ) + \binom{k+1}{k+1} \\
= & \qquad \langle \text{ Split off term } \rangle \\
 & \binom{k+1}{0} + (\Sigma\ i \mid 1 \leq i \leq k\ :\ \binom{k+1}{i}\ ) + \binom{k+1}{k+1} \\
= & \qquad \langle \text{ Hint 3, } 1 \leq i \leq k\ \rangle \\
 & \binom{k+1}{0} + (\Sigma\ i \mid 1 \leq i \leq k\ :\ \binom{k}{i} + \binom{k}{i-1}\ ) + \binom{k+1}{k+1} \\
= & \qquad \langle \text{ Distributivity } \rangle \\
 & \binom{k+1}{0} + (\Sigma\ i \mid 1 \leq i \leq k\ :\ \binom{k}{i}\ ) + (\Sigma\ i \mid 1 \leq i \leq k\ :\ \binom{k}{i-1}\ ) + \binom{k+1}{k+1} \\
= & \qquad \langle \text{ Dummy Renaming } \rangle \\
 & \binom{k+1}{0} + (\Sigma\ i \mid 1 \leq i \leq k\ :\ \binom{k}{i}\ ) + (\Sigma\ i \mid 0 \leq i \leq k-1\ :\ \binom{k}{i}\ ) + \binom{k+1}{k+1} \\
= & \qquad \langle \text{ Hints 1 and 2 } \rangle \\
 & \binom{k}{0} + (\Sigma\ i \mid 1 \leq i \leq k\ :\ \binom{k}{i}\ ) + (\Sigma\ i \mid 0 \leq i \leq k-1\ :\ \binom{k}{i}\ ) + \binom{k}{k} \\
= & \qquad \langle \text{ Slit-off term (or add first term) } \rangle \\
 & (\Sigma\ i \mid 0 \leq i \leq k\ :\ \binom{k}{i}\ ) + (\Sigma\ i \mid 0 \leq i \leq k-1\ :\ \binom{k}{i}\ ) + \binom{k}{k} \\
= & \qquad \langle \text{ Split-off term (or add last term) } \rangle \\
 & (\Sigma\ i \mid 0 \leq i \leq k\ :\ \binom{k}{i}\ ) + (\Sigma\ i \mid 0 \leq i \leq k\ :\ \binom{k}{i}\ ) \\
= & \qquad \langle \text{ Arithmetic } \rangle \\
 & 2 \cdot (\Sigma\ i \mid 0 \leq i \leq k\ :\ \binom{k}{i}\ )
\end{array}
$$

$= \quad \langle \text{ I.H. } \rangle$

$2 \cdot (2^k)$

$= \quad \langle \text{ Arithmetic } \rangle$

2^{k+1}

3. Structural Induction

For these exercises we will use sequences. Please refer to the attached document. You may use any definition and any of the theorems that appear as practice exercises.

We define the following new functions for sequences.

$copyR(S)$**:** Concatenate a list with its reverse:

Basis Case: $copyR(\epsilon) = \epsilon$

Recursive Case: $copyR(x \triangleleft S) = x \triangleleft copyR(S) \triangleright x$

$DelX$**:** Delete all occurrences of X from a sequence.

Basis Case: $DelX(x, \epsilon) = \epsilon$

Recursive Case: $DelX(x, y \triangleleft S) = \begin{cases} DelX(x, S) & \text{if } x = y \\ y \triangleleft DelX(x, S) & \text{otw} \end{cases}$

3.1. Using Structural Induction Prove

$$copyR(S) = S@rev(S)$$

Basis Case: $copyR(\epsilon) = \epsilon@rev(\epsilon)$

$$\begin{aligned}
& copyR(\epsilon) = \epsilon@rev(\epsilon) \\
= & \quad \langle \text{ Def. } copyR \rangle \\
& \epsilon = \epsilon@rev(\epsilon) \\
= & \quad \langle \text{ Def. } @ \rangle \\
& \epsilon = rev(\epsilon) \\
= & \quad \langle \text{ Def. } rev \rangle \\
& True
\end{aligned}$$

Inductive Case: H.I. $copyR(S) = S@rev(S)$

Demostrar: $copyR(x \triangleleft S) = (x \triangleleft S)@rev(x \triangleleft S)$

$$\begin{aligned}
& (x \triangleleft S)@rev(x \triangleleft S) \\
= & \quad \langle \text{ DEf CopyR } \rangle \\
& (x \triangleleft S)@(rev(S) \triangleright x) \\
= & \quad \langle \text{ Theorem 4 } \rangle \\
& ((x \triangleleft S)@rev(S)) \triangleright x \\
= & \quad \langle \text{ Def } @ \rangle \\
& x \triangleleft (S@rev(S)) \triangleright x \\
= & \quad \langle \text{ I.H. } \rangle \\
& x \triangleleft copyR(S) \triangleright x \\
= & \quad \langle \text{ Def CopyR } \rangle \\
& copyR(x \triangleleft S)
\end{aligned}$$

3.2. Using structural inducton prove:

$$count(x, S) = long(S) - long(DelX(x, S))$$

Basis Case: $count(x, \epsilon) = long(\epsilon) - long(DelX(x, \epsilon))$

$$\begin{array}{ll}
 & count(x, \epsilon) = long(\epsilon) - long(DelX(x, \epsilon)) \\
= & \quad \langle \text{ Def. } DelX \rangle \\
 & count(x, \epsilon) = long(\epsilon) - long(\epsilon) \\
= & \quad \langle \text{ Def. } Long \rangle \\
 & count(x, \epsilon) = 0 - 0 \\
= & \quad \langle \text{ Arithmetic } \rangle \\
 & count(x, \epsilon) = 0 \\
= & \quad \langle \text{ Def. } count \rangle \\
 & true
\end{array}$$

Inductive Case: I.H. $count(x, S) = long(S) - long(DelX(x, S))$

Prove: $count(x, y \triangleright S) = long(y \triangleright S) - long(DelX(x, y \triangleright S))$

By cases:

$x = y$

$$\begin{array}{ll}
 & count(x, y \triangleright S) = long(y \triangleright S) - long(DelX(x, y \triangleright S)) \\
= & \quad \langle \text{ Def Count } \rangle \\
 & 1 + count(x, S) = long(y \triangleright S) - long(DelX(x, y \triangleright S)) \\
= & \quad \langle \text{ Def Del X } \rangle \\
 & 1 + count(x, S) = long(y \triangleright S) - long(DelX(S)) \\
= & \quad \langle \text{ Def } Long \rangle \\
 & 1 + count(x, S) = 1 + long(S) - long(DelX(S)) \\
= & \quad \langle \text{ Arithmetic } \rangle \\
 & count(x, S) = long(S) - long(DelX(S)) \\
= & \quad \langle \text{ I.H. } \rangle \\
 & true
\end{array}$$

$x \neq y$

$$\begin{array}{ll}
 & count(x, y \triangleright S) = long(y \triangleright S) - long(DelX(x, y \triangleright S)) \\
= & \quad \langle \text{ Def Count } \rangle \\
 & count(x, S) = long(y \triangleright S) - long(DelX(x, y \triangleright S)) \\
= & \quad \langle \text{ Def Del X } \rangle \\
 & 1 + count(x, S) = long(y \triangleright S) - long(y \triangleright DelX(S)) \\
= & \quad \langle \text{ Def } Long \text{ twice } \rangle \\
 & count(x, S) = 1 + long(S) - (1 + long(DelX(S))) \\
= & \quad \langle \text{ Arithmetic } \rangle \\
 & count(x, S) = long(S) - long(DelX(S)) \\
= & \quad \langle \text{ I.H. } \rangle \\
 & true
\end{array}$$

4. Counting

Suppose you have a deck of cards. Each card has a color: Black, White, Blue, Red and a value 1,2,3,4,5,6,7,8.

1. How many ways can you pick 5 cards such that 3 cards have the same value.

   a) Choose the number: $\binom{8}{1}$ which is 8

   b) Choose the colors of the three cards: $\binom{4}{3}$

   c) Choose the remaining 2 cards. These cards cannot contian a card which has the same number as the 3 cards so its $\binom{28}{2}$

   The final answer is

   $$8 \cdot \binom{4}{3} \cdot \binom{28}{2}$$

2. How many ways can you choose 5 cards such the values are consecutive and all cards are of the same color. We just have to choose the color: 4 and the lowest numbered card whicy can only be a number from 1 to 4. So the answer is 16.

3. How many ways can you choose 5 cards such that the values are consecutive and they do not have the same color.

   a) Choose the lowest number: 4 ways

   b) Choose the color of the lowest number: 4 ways

   c) Choose the color of the second to lowest number (the value is fixed): 4 ways

   d) Choose the color of the third to lowest number (the value is fixed): 4 ways

   e) Choose the color of the fourth to lowest number (the value is fixed): 4 ways

   f) Choose the color of the fifth to lowest number (the value is fixed): 4 ways

   Howerver, we have counted all runs. Not only the ones where cards are not all of the same color. So we must subtract the result of the previous question.

   $$4^6 - 16$$

4. How many ways can you choose five cards such that all cards have different value and there are no two cards from the five you choose whose numbers add up to 9. None. You can divide the 8 numbers into four groups:

   - 1 and 8
   - 2 and 7
   - 3 and 6
   - 4 and 5

   By the Pigeon Hole Principle, if you choose 5 cards at least two will belong to the same group. This means that at least two will add up to 9.