ISIS-1104-09 Matemï¿½tica Estructural y Lï¿½gica
MidTerm 1
Date: February 28, 2019

- This exam is closed book , closed notes.
- You are allowed to use the cheat-sheet that was handed out and left on Sicua+ for downloading.
- The use of any electronic device is prohibitted.
- There should be no communication among students.
- Answer in the spaces provided; no additional sheets of paper will be accepted.
- Do not forget to write you name and studient id number on the exam before handing it in.

Name	StudentId #

Do not write below ithis line

1.1	10 %	
1.2	20 %	
1.3	20 %	
2.1	15 %	
2.2	20 %	
3.1	15 %	
3.2	**** 15 %	
Total	115 %	

1. Propositional Calculus

Suppose we add the following function to the propositional calculus:

$$foo(X, Y, Z) \equiv ((X \Rightarrow Y) \Rightarrow Z) \tag{1}$$

1.1. Prove or refute:

$$foo(X, Y, Z) \equiv (X \Rightarrow (Y \wedge Z))$$

This is false. Take

X= true

Y= false

Z= false

				A		B	
X	Y	Z	$X \Rightarrow Y$	$(X \Rightarrow Y) \Rightarrow Z$	$Y \wedge Z$	$X \Rightarrow (Y \wedge Z)$	$A \equiv B$
true	false	false	false	true	false	false	false

1.2. Prove or refute:

$$foo(X, Y, \neg Y) \quad \equiv \quad \neg Y$$

$$\begin{array}{ll}
 & foo(X, Y, \neg Y) \\
= & \qquad \langle \text{ Defintion of } foo \ \rangle \\
 & (\ X \ \Rightarrow \ Y\ ) \Rightarrow \ \neg Y \\
= & \qquad \langle \text{ Defintion of } \Rightarrow \ \rangle \\
 & (\neg X \ \vee \ Y) \Rightarrow \ \neg Y \\
= & \qquad \langle \text{ Defintion of } \Rightarrow \ \rangle \\
 & \neg(\neg X \ \vee \ Y) \vee \neg Y \\
= & \qquad \langle \text{ De Morgan and double negation } \rangle \\
 & (X \ \wedge \ \neg Y) \vee \ \neg Y \\
= & \qquad \langle \text{ Absorption } \rangle \\
 & \neg Y
\end{array}$$

1.3. Prove or refute the validity of the following inference rule:

$$\frac{foo(X, Y, Z)}{X \ \vee \ Z}$$

$$
\begin{array}{ll}
 & foo(X, Y, Z) \\
= & \quad \langle \text{ Def. } foo \ \rangle \\
 & (X \ \Rightarrow \ Y) \ \Rightarrow \ Z \\
= & \quad \langle \text{ Defintion of } \Rightarrow \ \rangle \\
 & (\neg X \ \vee \ Y) \ \Rightarrow \ Z \\
= & \quad \langle \text{ Defintion of } \Rightarrow \ \rangle \\
 & \neg(\neg X \ \vee \ Y) \ \vee \ Z \\
= & \quad \langle \text{ De Morgan and double negation } \rangle \\
 & (X \ \wedge \ Y) \ \vee \ Z \\
\Rightarrow & \quad \langle \text{ Simplification } \rangle \\
 & X \ \vee \ Z
\end{array}
$$

2. Deduction in the propositional calculus

We have three boxes labeled A, B and C. There is a prize in at least one of the boxes, and at least one of the boxes does not have a prize. Someone has written the following statement on box A: "If there is a prize here, then there is no prize in box C". We do not know whether this statement is true or false, but we do know that if it is true, then there is a prize in A and there is a prize in B. We also know that if there is a prize in A then the statement is false.

For each box, mark the correct option with an X.

	Prize	No prize	It cannot be determined with the information
Box A	X		
Box B		X	
Box C	X		

2.1. Modeling

Given the variables listed below to represent the facts of the problem, model the hypotheses and your conclusion regarding what boxes have prizes and what boxes do not.

A: There is a prize in box A.

B: There is a prize in box B.

C: There is a prize in box C.

This is the statement: $A \Rightarrow \neg C$. The statement is not a hypothesis.

These are the hypotheses:

1. There is a prize in at least one of the boxes: $A \wedge B \wedge C$

2. at least one of the boxes does not have a prize: $\neg A \wedge \neg B \wedge \neg C$

3. if the statment is true, then there is a prize in A and there is a prize in B: $(A \Rightarrow \neg C) \Rightarrow (A \wedge B)$

4. if there is a prize in A then the statement is false $A \Rightarrow \neg(A \Rightarrow \neg C)$

The conclusion: $A \wedge \neg B \wedge C$

2.2. Proofs in Propositional Calculus

Formally prove your conclusion.

Lemma 1: $A \wedge C$

$$
\begin{array}{ll}
 & \quad \langle \text{ Hypothesis 3 } \rangle \\
 & (A \Rightarrow \neg C) \Rightarrow (A \wedge B) \\
= & \quad \langle \text{ Defintion of foo } \rangle \\
 & foo(A, \neg C, (A \wedge B)) \\
\Rightarrow & \quad \langle \text{ Problem 1.3 } \rangle \\
 & A \vee (A \wedge B) \\
= & \quad \langle \text{ Absorption } \rangle \\
 & A \\
\Rightarrow & \quad \langle \text{ Hypothesis 4 } \rangle \\
 & \neg(A \Rightarrow \neg C) \\
= & \quad \langle \text{ Definition of } \Rightarrow \rangle \\
 & \neg(\neg A \vee \neg C) \\
= & \quad \langle \text{ De Morgan and double negation } \rangle \\
 & A \wedge C
\end{array}
$$

Lemma 3: $\neg B$

$$
\begin{array}{ll}
 & \quad \langle \text{ Lemma 1 } \rangle \\
 & A \wedge C \\
= & \quad \langle \text{ Disjunctive Syllogism with hyphothesis 2 } \rangle \\
 & \neg B
\end{array}
$$

Now with the two lemmas we can conclude: $A \wedge \neg B \wedge C$

Without using foo

Lemma 1: The statement is false:

Assume the statement is true

$$
\begin{array}{ll}
 & \quad \langle \text{ The Statement is true } \rangle \\
 & A \Rightarrow \neg C \\
\Rightarrow & \quad \langle \text{ Hypothesis 3 } \rangle \\
 & A \wedge B \\
\Rightarrow & \quad \langle \text{ Simplification } \rangle \\
 & A \\
\Rightarrow & \quad \langle \text{ Hypothesis 4 } \rangle \\
 & \neg(A \Rightarrow \neg C)
\end{array}
$$

Since we assumed that the statement was true, and proved it was false, we can conclude that the statement is false:

Lemma 2: $A \ \wedge \ C$

$$
\begin{array}{ll}
 & \qquad \langle \text{ Lemma 1 } \rangle \\
 & \neg(A \Rightarrow \neg C) \\
= & \qquad \langle \text{ Definition of } \Rightarrow \rangle \\
 & \neg(\neg A \vee \neg C) \\
= & \qquad \langle \text{ De Morgan and double negation } \rangle \\
 & A \ \wedge \ C
\end{array}
$$

Lemma 3: $\neg B$

$$
\begin{array}{ll}
 & \qquad \langle \text{ Lemma 2 } \rangle \\
 & A \ \wedge \ C \\
= & \qquad \langle \text{ Disjunctive Syllogism with hyphothesis 2 } \rangle \\
 & \neg B
\end{array}
$$

Now with Lemma 2 and LEmma 3 we can conclude: $A \ \wedge \ \neg B \ \wedge \ C$

3. Predicate calculus

1. Boxes contain chocolates, truffles, or hard candy. Boxes may contain more than one type of candy.

$$(\forall\ x : Boxes \mid\ \ :\ ch(x)\ \vee\ t(x)\ \vee\ c(x)\ )$$

2. Boxes are blue or red; a box cannot be blue and red.

$$(\forall\ x : Boxes \mid\ \ :\ r(x) \not\equiv b(x)\ )$$

or

$$(\forall\ x : Boxes \mid\ \ :\ \neg r(x) \equiv b(x)\ )$$

3. Chocolates are always placed in blue boxes.

$$(\forall\ x : Boxes \mid ch(x)\ :\ b(x)\ )$$

4. Two boxes can be packaged together. When this happens, one and exactly one of the two boxes contains chocolates.

$$(\forall\ x, y : Boxes \mid p(x, y)\ :\ ch(x) \not\equiv ch(y)\ )$$

or also

$$(\forall\ x, y : Boxes \mid p(x, y)\ :\ \neg ch(x) \equiv ch(y)\ )$$

or

$$(\forall\ x, y : Boxes \mid p(x, y)\ :\ ch(x) \equiv \neg ch(y)\ )$$

You find a package with two boxes labeled A and B. You only know that the box labeled A is red. You conclude that the other box has to be blue.

$$p(A, B) \wedge r(A) \Rightarrow b(B)$$

Our premisses are then:

1. $p(A, B)$
2. $r(A)$

3.1. Modeling

Model the problem, using the following predicates:

- b(x) : box labeled x is blue.
- r(x) : box labeled x is red.
- ch(x) : box labeled x contains chocolates.
- t(x) : box labeled x contains truffles.
- c(x): box labeled x contains hard candy.
- p(x,y): boxes x and y are packaged toghether

Write you answers above, beside each statement,

3.2. Deduction in the predicate calculus

Formally prove that you can reach the conclusion from the hypotheses and the premises.

$$
\begin{array}{ll}
 & p(A,B) \wedge r(A) \\
\Rightarrow & \quad \langle \text{ Deduction with } \not\equiv \text{ hyp 2 } \rangle \\
 & p(A,B) \wedge \neg b(A) \\
\Rightarrow & \quad \langle \text{ Universal Modus Tollens hyp 3 } \rangle \\
 & p(A,B) \wedge \neg ch(A) \\
\Rightarrow & \quad \langle \text{ Modus Ponens hyp 4 } \rangle \\
 & (\neg ch(A) \equiv ch(B)) \wedge \neg ch(A) \\
\Rightarrow & \quad \langle \text{ Leibniz } \rangle \\
 & ch(B) \\
\Rightarrow & \quad \langle \text{ Universal Modus Ponens hyp 3 } \rangle \\
 & b(B)
\end{array}
$$