Contents

MATH 691: Spectral Geometry

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February 22, 2021

Supervised by D. Jakobson. Based on *Analysis on Manifolds via the Laplacian* by Yaiza Canzani, and many others.

1 Class Notes

1.1 Weak Sequential Convergence in L^p

Definition 1.1. Let X be a normed linear space. A sequence $\{f_n\}$ in X **converges weakly** in X to f if $\lim_{n\to\infty} T(f_n) = T(f)$ for all $T \in X^*$.

Note that weak sequential limits are unique. We can show this easily by contradiction.
Note that strong convergence ($\lim_{n\to\infty} ||f_n - f|| = 0$) implies weak convergence, but not the converse: $|T(f_n) - T(f)| = |T(f_n - f)| \leq ||T||_* \cdot ||f_n - f||$ for all $T \in X^*$.

Proposition 1.1. *Let E be a measurable set, $1 \leq p < \infty$, and q the conjugate of p. Then (f_n) converges weakly to $f \in L^p(E)$ if and only if $\lim_{n\to\infty} \int_E g \cdot f_n = \int_E g \cdot f \;\; \forall g \in L^q(E)$.*

Proof. This is a restatement of the Riesz Representation theorem using weak convergence. □

Note that this definition also holds for $p = \infty$, but the name changes:

Definition 1.2. Let $p = \infty$. Let E be a measurable set. We say that a sequence $(f_n)_{n\in\mathbb{N}} \subset L^\infty(E)$ **converges weakly - *** to $f \in L^\infty(E)$ if

$$\lim_{n\to\infty} \int_E f_n \cdot g dx = \int_E f \cdot g dx \;\; \forall g \in L^1(E)$$

Exercises in MATH 691 paper notebook.

1.2 Implications in Spectral Geometry

1.2.1 Riemann-Lebesgue Lemma

Lemma 1.2. *(Riemann-Lebesgue)*

1. *Let E be a measurable set. Let $I = [-\pi, \pi]$, and $1 < p < \infty$. Define $f_n(x) = \sin(nx)$ for $n \in \mathbb{N}, x \in I$. $|f_n| \leq 1$ on I for every n and is*

therefore a bounded sequence in $L^p(I)$. Then, by the preceding corollary, the sequence (f_n) converges weakly in $L^p(I)$ to 0 if and only if

$$\lim_{n\to\infty} \int_{-\pi}^{x} \sin(nt)dt = 0 \ \ \forall x \in I.$$

Notice, however, that for each n,

$$\int_{-\pi}^{\pi} |sin(nt)|^2 dt = \int_{-\pi}^{\pi} \sin^2(nt)dt = \pi$$

This means that no subsequence of (f_n) converges strongly to 0 in $L^2(I)$. This also holds for any p, We can also apply the BCT, and obtain that there is no subsequence that converges a.e to 0 in $L^p(I)$.

2. *(General Statement): Let $f \in L^1$. Then*

$$\lim_{n\to\pm\infty} \int f(x)\cos(nx)dx = 0$$
$$\lim_{n\to\pm\infty} \int f(x)\sin(nx)dx = 0$$
$$\lim_{n\to\pm\infty} \int f(x)e^{inx}dx = 0$$

Proof. Hint: approximate any function by simple function, let $n \to \infty$ and use density argument. Proof in three steps:

1. Case $f(x) = \chi_{(a,b)}$:

$$\lim_{n\to\pm\infty} \int f(x)e^{inx}dx = \lim_{n\to\pm\infty} \int_a^b e^{inx}dx = \left.\frac{e^{inx}}{in}\right|_a^b = \lim_{n\to\pm\infty} \frac{e^{inb} - e^{ina}}{in} = 0$$

2. Case $f(x)$ simple function.
   We use previous case: every simple function can be written as a linear combination (finite sum) of characteristic functions.

3. $f(x) \in L^1$ arbitrary.
   Simple functions are dense in L^1, therefore $\exists \tilde{f} \in L^1$ simple such that $\int |f - \tilde{f}| < \varepsilon$. Since $\tilde{f}(x)$ is simple, by Step 2,

$$\lim_{n\to\pm\infty} \int \tilde{f}(x)e^{inx}dx = 0 \implies \left| \int \tilde{f}(x)e^{inx}dx \right| < \varepsilon$$

   for some $\varepsilon > 0$ for n large enough. Then by the triangle inequality and previous steps:

$$\begin{aligned}
\left| \int \Big(f(x) - \tilde{f}(x) + \tilde{f}(x)\Big)e^{inx}dx \right| &= \left| \int \Big(f(x) - \tilde{f}(x)\Big)e^{inx}dx + \int_I \tilde{f}(x)e^{inx}dx \right| \\
&\leq \left| \int \Big(f(x) - \tilde{f}(x)\Big)e^{inx}dx \right| + \left| \int \tilde{f}(x)e^{inx}dx \right| \\
&\leq \int \Big|f(x) - \tilde{f}(x)\Big|dx + \int |\tilde{f}(x)e^{inx}|dx \\
&\leq 2\varepsilon
\end{aligned}$$

□

1.3 Quantum Limits on Flat Tori

We are interested in finiding nontrivial limits of $|\phi_\lambda|^2$ on $\mathbb{T}^k, k = 2$.

1.4 Quantum Unique Ergodicity Theorem on the Circle

(Eigenfunctions become uniformly distributed as $n \to \infty$). Add notes from ppt + this link

Definition 1.3. A **lattice point** is a point with integer coordinates.

Definition 1.4. The **multiplicity** of an eigenvalue λ on the Laplacian $\triangle_{T^k}$ is the number of lattice points on a circle of radius $\sqrt{\lambda}$. With the Pythagorean theorem in mind, we can also define multiplicity as the number of ways of writing λ as a sum of squares. Note that for $k \geq 2$, the multiplicity is unbounded.

We want to find nontrivial limits of $|\phi_\lambda|^2$ as $\lambda \to \infty$ on $\mathbb{T}^k$, i.e nontrivial quantum limits on the flat torus.

Definition 1.5. A **quantum limit** is a (weak) limit as $\lambda_j \to \infty$ of $|\phi_{\lambda_j}(x)|^2 dx$.

Lemma 1.3. *Let $k = 2$. Let $v_1, v_2, v_3 \in \mathbb{Z}^2$ be nonzero chords of a circle connecting lattice points. Suppose there are infinitely many possible concentric circles with parallel translates of v_1, v_2, v_3 (i.e they still have integer coordinates). Then two of v_j have the same length.*

This result comes from discrete solutions to a system of coupled Pell's equations:

Definition 1.6. A **Pell's equation** is an equation of type $ax^2 - by^2 = c$, where $a, b, c \in \mathbb{Z}$ are strictly positive. We look for integer solutions $x, y \in \mathbb{Z}$ i.e lattice points on a hyperbola.

Proof. A system of two simultaneous Pell's equations can only have finitely many solutions. A system of coupled Pell's equations may look something like this: $\begin{cases} a_1x^2 - by^2 = c_1 \\ a_2x^2 - b_2z^2 = c_2 \end{cases}$ which amounts to finding solutions on two hyperbolas simultaneously. □

Theorem 1.4. *Quantum limits are always absolutely continuous wrt dx.*

Proof. Proof due to Bourgain. See in paper. (decompose measure into abs cont + singular part...) □

Theorem 1.5. *Let $k = 2$. Suppose $|\phi_{\lambda_j}(x)|^2 \to f(x)dx$ on $\mathbb{T}^2$. Then $f(x)$ is a trigonometric polynomial such that $\exists 0 < a < b < \infty$ with $f(x) = 1 + \sum_{\tau \in \mathbb{Z}^2 : ||\tau||^2 = a} c_\tau e^{i\langle \tau, x\rangle} + \sum_{\tau \in \mathbb{Z}^2 : ||\tau||^2 = b} c_\tau e^{i\langle \tau, x\rangle}$, i.e all non-zero frequencies lie on at most two circles.*

Proof. By contradiction, suppose there exist $0 < a < b < c$ such that $f(x) = 1 + \sum c_\tau e^{i\langle \tau, x\rangle}$ for $||\tau_1||^2 = a, ||\tau_2||^2 = b, ||\tau_3||^2 = c$. i.e on a circle of radius $\sqrt{\lambda_k}$, we have $||v_1||^2 = a, ||v_2||^2 = b, ||v_3||^2 = c$. However, this statement contradicts the previous lemma. □

2 Introduction to Hyperbolic Geometry and Geodesic Flows

The notes in this section are based off Chapters 1 and 3 of *Fuchsian groups, geodesic flows on surfaces of constant negative curvature and symbolic coding of geodesics* by Svetlana Katok. Basic definitions and theorems come from *Riemannian Geometry* by Do Carmo.

2.1 Hyperbolic Geometry

Consider a hyperboloid with curvature -1 and a pseudo-metric in $\mathbb{R}^3$:

$$ds_h^2 = dx_1^2 + dx_2^2 - dx_3^2$$

This metric corresponds to the bilinear symmetric form of signature (2,1):

$$(x, y)_{(2,1)} = x_1 y_1 + x_2 y_2 - x_3 y_3$$

We can then define the upper sheet of the hyperboloid using

$$H^2 = \{(x_1, x_2, x_3 > 0) \in \mathbb{R}^3 : x_1^2 + x_2^2 - x_3^2 = -1\}$$

In comparison, note that the Euclidean metric corresponds to standard inner product.

This represents all points on the surface of a hyperboloid embedded in $\mathbb{R}^3$. Note that all points outside the hyperboloid have $(x, x)_(2, 1) > -1$, and all points inside the hyperboloid have $(x, x)_(2, 1) < -1$. To be able to measure lengths of curves, areas of domains, and angles between two curves on a surface, which then leads to the definition of geodesics, we need to equip the tangent space at a point p on a manifold $\mathcal{M}$ with an inner product that varies smoothly as we vary p on $\mathcal{M}$ i.e. a Riemannian metric. It is then crucial to show that the pseudo-metric ds_h^2 induces a Riemannian metric on H^2. First, recall the definition of a Riemannian metric:

Definition 2.1. A **Riemannian metric** on a differentiable manifold $\mathcal{M}$ of dimension n is a correspondence which associates to each point $p \in \mathcal{M}$ an inner product $\langle\ ,\ \rangle_p$ (i.e. a symmetric, bilinear, positive-definite form) on the tangent space $T_p\mathcal{M}$, which varies differentiably in the following sense: If $x : U \subset R^n \to \mathcal{M}$ is a coordinate system around p with $x(x_1, x_2, \cdots, x_n) = q \in x(U)$ and $\frac{\partial}{\partial x_i}(q) = (0, \cdots, 1, \cdots, 0)$, then $\left\langle \frac{\partial}{\partial x_i}(q), \frac{\partial}{\partial x_j}(q) \right\rangle_q = g_{ij}(x_1, \cdots, x_n)$ is a differentiable function on U. Clearly, this definition is independent of coordinate system.

If the metric tensor is positive semidefinite then we get a semi-Riemannian manifold or pseudo-Riemannian manifold.

To show that ds_h^2 is a Riemannian metric, it thus remains to show that it's positive-definite. Let $x = (x_1, x_2, x_3) \in H^2$, and define $x^\perp = \{y \in \mathbb{R}^2 : (y, x)_{(2,1)} = 0\}$, which is a plane passing through the origin. $x + x^\perp$ is thus a plane passing through x.

Proposition 2.1. $(,)_{(2,1)}$ *restricted to* $x^\perp$ *is positive-definite.*

Proof.
- We first show that the tangent plane to the hyperboloid at a point x, T_xH^2, is $x + x^\perp$.
  - $T_xH^2 \subseteq x + x^\perp$:
    The upper sheet of the hyperboloid is given by $x_3 = \sqrt{x_1^2 + x_2^2 + 1}$.
    We then have that for $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, we have tangent basis vectors $\frac{\partial x}{\partial x_1} = \left(1, 0, \frac{\partial x_3}{\partial x_1}\right)$, and similarly for x_2, $\frac{\partial x}{\partial x_2} = \left(0, 1, \frac{\partial x_3}{\partial x_2}\right)$, then any tangent vector $v \in T_xH^2$ at x is a linear combination of these basis vectors:
    $$v - x = a\left(1, 0, \frac{\partial x_3}{\partial x_1}\right) + b\left(0, 1, \frac{\partial x_3}{\partial x_2}\right) = \left(a, b, a\frac{\partial x_3}{\partial x_1} + b\frac{\partial x_3}{\partial x_2}\right) = \left(a, b, \frac{ax_1 + bx_2}{x_3}\right),$$
    since $\frac{\partial x_3}{\partial x_1} = \frac{2x_1}{2\sqrt{x_1^2+x_2^2+1}} = \frac{x_1}{x_3}$, and similarly $\frac{\partial x_3}{\partial x_2} = \frac{x_2}{x_3}$.
    Then for all $v \in T_xH^2$, $(v - x, x)_{(2,1)} = ax_1 + bx_1 - (ax_1 + bx_2) = 0$, and hence $T_xH^2 \subseteq x + x^\perp$.
  - $x + x^\perp \subseteq T_xH^2$: Obvious.
- We now conclude by showing that for $v \in x^\perp$, $(v, v)_{(2,1)} > 0$.
  Let $v \in x^\perp$, so that $x + v$ is outside of the hyperboloid, so $(x + v, x + v)_{(2,1)} > -1$. Then
  $$-1 < (x + v, x + v)_{(2,1)} = (x, x)_{(2,1)} + 2(x, v)_{(2,1)} + (v, v)_{(2,1)} = -1 + (v, v)_{(2,1)}$$
  and thus $(v, v)_{(2,1)} > 0$ for all $v \in x^\perp$.

□

There are many different ways to define hyperbolic space using the general hyperboloid model described above.

Exercises

1. Prove that the metric in the Poincare disk model is given by $ds^2 = \frac{4(d\eta_1^2+d\eta_2^2)}{(1-(d\eta_1^2+d\eta_2^2))^2}$
2. Prove that the upper-half plane model is given by $ds^2 = \frac{d\eta_1^2+d\eta_2^2}{d\eta_3^2}$

Let $\mathcal{H} = \{z \in \mathbb{C} : \text{Im}\,(z) > 0\}$ be the upper half-plane.

Definition 2.2. $\mathcal{H}$ together with the metric $ds^2 = \frac{dx^2+dy^2}{y^2}$ are called **hyperbolic** or **Lobachevski** plane.

The geodesics (the shortest curves wrt this metric) will be straight lines and semicircles orthogonal to the real line $\text{Im}\,(z) = 0$. Using this, we can show that any two points in $\mathcal{H}$ can be joined by a unique geodesic, and that from any point in $\mathcal{H}$ one can draw a geodesic in any direction.

We will now define some key geometric quantities in the hyperbolic plane: angle between two geodesics and the distance between two points along a geodesic.

The Riemannian metric $ds^2 = \frac{dx^2+dy^2}{y^2}$ is induced by an inner product on $T_z\mathcal{H}$ for a given $z \in \mathcal{H}$: let $z_1, z_2 \in T_z\mathcal{H}$, then $\langle z_1, z_2\rangle = \frac{(z_1,z_2)}{\textbf{Im}\,(z)^2}$, where (z_1, z_2) is Euclidean inner product.

Definition 2.3. The angle between two geodesics in $\mathcal{H}$ at their intersection point z is the angle between their tangents in $T_z\mathcal{H}$

$$\cos\phi = \frac{\langle z_1, z_2\rangle}{||z_1|| \cdot ||z_2||} = \frac{(z_1, z_2)}{|z_1| \cdot |z_2|},$$

where $|.|$ is Euclidean norm, and $||.||$ is norm corresponding to $\langle , \rangle$. So the angles are the same whether we use $\langle , \rangle$ or $(,)$.

It is important to note that the first four axioms of Euclid hold for this geometry, but the fifth of Euclid's Elements, the axiom of parallels, does not hold: let $z \in \mathcal{H}$, and L be a geodesic in $\mathcal{H}$ with $z \notin L$, then there is more than one geodesic passing through the point z that does not intersect L. This means that the geometry in $\mathcal{H}$ is non-Euclidean.

To measure the distance between two points in $\mathcal{H}$ we use the hyperbolic metric $ds^2 = \frac{dx^2+dy^2}{y^2}$, just like in the Euclidean case. Let I be the unit interval, and $\ell : I \to \mathcal{H}$ be a piecewise differentiable curve parametrised by $t \in I$, then its length is

$$h(\ell) = \int_0^1 \frac{\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}}{y(t)} dt$$

Definition 2.4. The **hyperbolic distance** between two points $w, z \in \mathcal{H}$ is $\rho(w, z) = \inf h(\ell)$, where the infimum is taken over all piecewise differentiable curves connecting z and w.

Proposition 2.2. *ρ is a metric, i.e. nonnegative, symmetric, and satisfies the triangle inequality.*

We will now start building tools to formally prove that the geodesics of hyperbolic space are semicircles and half lines. The first concept we need is the Mobius transformation.

Consider the Special Linear group of two by two matrices with unit determinant, $SL(2, \mathbb{R})$. It acts on $\mathcal{H}$ via Mobius transformations:

Definition 2.5. Let $z \in \mathcal{H}$. For each $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{R})$, the **Mobius transformation** maps

$$T_g(z) = \frac{az + b}{cz + d}$$

Proposition 2.3. *$f(z) \in Aut\ (\mathcal{H}) \iff f(z)$ is a Mobius transformation.*

Proof. • $(\implies)$:

• $(\impliedby)$:

∎

Theorem 2.4. *The group $PSL(2, \mathbb{R})$ acts on $\mathcal{H}$ by homeomorphisms.*

Definition 2.6. Let $(X, d_x), (Y, d_y)$ be metric spaces. A map $f : X \to Y$ is an **isometry** if $d_y(f(a), f(b)) = d_x(a, b)$ i.e. it preserves distances.

Isometries obviously form a group, call it $\mathrm{Isom}(\mathcal{H})$.

Theorem 2.5. *Mobius transofrmations are isometries, and so $PSL(2, \mathbb{R}) \subseteq$ Isom$(\mathcal{H})$.*

Theorem 2.6. *The geodesics in $\mathcal{H}$ are semicircles and lines orthogonal to the real axis.*

Theorem 2.7. *Any isometry of $\mathcal{H}$ maps geodesics to geodesics.*

A few interesting results about isometries:

Theorem 2.8. *The group Isom$(\mathcal{H})$ is generated by the Mobius transformations from $PSL(2, \mathbb{R})$ together with the transformation $z \mapsto -\bar{z}$. The group $PSL(2, \mathbb{R})$ is a subgroup of Isom$(\mathcal{H})$ of index 2.*

The transformations in $PSL(2, \mathbb{R})$ are **orientation-preserving** isometries, and all others are **orientation-reversing** isometries. Let T be a Mobius transformation, and let DT be its differential map at a point $z \in \mathcal{H}$ $DT : T_z\mathcal{H} \to T_{T(z)}\mathcal{H}$

Theorem 2.9. *Let $T \in PSL(2,\mathbb{R})$, then T preserves the norm in the tangent space at all points.*

Corollary 2.10. *A Mobius transformation is conformal at every point.*

Definition 2.7. Let $A \subset \mathcal{H}$, then we define the **hyperbolic area** of A

$$\mu(A) = \int_A \frac{dx\,dy}{y^2},$$

if this integral exists.

Theorem 2.11. *Hyperbolic area is invariant under all Mobius transformations $T \in PSL(2,\mathbb{R})$, i.e. if $\mu(A)$ exists, then $\mu(A) = \mu(T(A))$*

Theorem 2.12. *(Gauss-Bonnet) Let Δ be a hyperbolic triangle with angles α, β, γ, then $\mu(\Delta) = \pi - \alpha - \beta - \gamma$.*

2.2 Geodesic Flow and Ergodic Theory

Definition 2.8. The **geodesic flow** $\{\tilde{\phi}^t\}$ on $\mathcal{H}$ is a local $\mathbb{R}-$action on the unit tangent bundle $S\mathcal{H}$ which moves a tangent vector along the geodesic defined by this vector with unit speed.

Let $\Gamma \subset SL(2,\mathbb{R})$ be a lattice subgroup and $M = \Gamma \setminus \mathcal{H}$ be the hyperbolic subgroup of finite order. The unit tangent bundle $S\mathcal{H} \simeq PSL(2,\mathbb{R})$, which implies that $SM = \Gamma \setminus S\mathcal{H} \simeq \Gamma \setminus PSL(2,\mathbb{R})$.

With this identification, the geodesic flow $g_t : SM \to SM$ is identified with the $PSL(2,\mathbb{R})-$action on $\mathcal{H}$ by Mobius transformations corresponds to left multiplications, and the geodesic flow corresponds to right multiplication by a one-parameter subgroup a_t:

$$\begin{aligned} g_t : &\Gamma \setminus PSL(2,\mathbb{R}) \to \Gamma \setminus PSL(2,\mathbb{R}) \\ &x \mapsto x \cdot \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix} = x \cdot a_t \end{aligned}$$

Theorem 2.13. *The geodesic flow $\{\tilde{\phi}^t\}$ on the unit tangent bundle SM of a complete Riemannian manifold M preserves a smooth measure called Liouville measure: $\mu = \frac{dx\,dy}{y^2} d\theta$, where $d\theta$ is Lebesgue measure on S^1. If M is compact then the Liouville measure is finite and can hence be normalized.*

Proof. Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R})$, $g : (z,v) \mapsto \left(\frac{az+b}{cz+d}, \frac{v}{(cz+d)^2}\right)$. it is easy to check that the Liouville measure is left $SL(2,\mathbb{R})$-invariant (i.e. it defines a left invariant measure on $PSL(2,\mathbb{R})$. We can show that the Liouville is right-invariant as well from the uniqueness of invariant measure on $PSL(2,\mathbb{R})$ □

Restatement: g_t preserves Liouville measure.

Let $X = \Gamma \setminus SM$.

Definition 2.9. 1. g_t is **ergodic** if for every measurable $A \subset X$, which is g_t-invariant, we have $\mu(A) = 0$ or 1.

2. g_t is **mixing** if for every measurable $A, B \subset X$, $\mu(A \cap g_t^{-1}B) \to \mu(A)\mu(B)$.

Theorem 2.14. *The geodesic flow is mixing.*

Mean Ergodic thm = Birkhoff Ergodic Thm.

Theorem 2.15. *(Mean Ergodic Thm) For all $\phi \in L^2(X)$, we have that $\left\| \frac{1}{T} \int_0^T \phi(g_t \cdot x)dt - \int_X \phi d\mu \right\| \xrightarrow{T\to\infty} 0$ (Follows from mixing.)*

Theorem 2.16. *(Pointwise Ergodic Thm) For all $\phi \in L^2(X)$, and for almost every $x \in X$), we have that $\frac{1}{T}\int_0^T \phi(g_t \cdot x)dt \xrightarrow{T\to\infty} \int_X \phi d\mu$*

Let $X = \Gamma \setminus SL(2, \mathbb{R})$

Theorem 2.17. *(Unique Ergodicity) Suppose X is compact, then for all $x \in X$, and for all $f \in \mathbb{C}(X)$: $\frac{1}{T}\int_0^T f(h_s(x))dx \to \int_X f d\mu$, where h_s is the horocycle flow.*

3 Geodesic Flows

The notes in this section are based off *Introduction to Modern Dynamical Systems* by A. Katok and B. Hasselblatt, and what I learned in my Classical Mechanics courses.

3.1 Basic Concepts

In this section, we study geodesic flows from the perspective of Lagrangian and Hamiltonian mechanics.

Definition 3.1. Let (M, g) be a Riemannian manifold with Riemaniam metric $g_x(\cdot,\cdot)$, and define the Lagrangian $L(x, v) = \frac{1}{2}g_x(v, v)$. The restriction of this Lagrangian system on the unit tangent bundle SM is the **geodesic flow** of the Riemannian manifold (M, g).

For a given Lagrangian L, we have

$$p_i = \frac{\partial L}{\partial \dot{x}^i}.$$

When a Riemannian manifold possesses a lot of isometries the geodesic flow can be described without explicitly solving the Lagrange equation. We present a useful lemma to capture this fact before considering some simple examples.

Lemma 3.1. *Let M be a Riemannian manifold and Γ a group of isometries. Suppose that Γ is transitive on unit vectors i.e. if $v, v' \in SM$ then there exists $\phi \in \Gamma$ such that $\phi(v) = v'$. If $\mathbb{C}$ is a nonempty family of unit-speed curves such that for $c \in \mathbb{C}$, $c : \mathbb{R} \to M$ with the properties:*

1. *if $\phi \in \Gamma$, and $c \in \mathbb{C}$, then $\phi \circ c \in \mathbb{C}$*
2. *if $c, c' \in \mathbb{C}$, then there exists $\phi_{c,c'} \in \Gamma$ such that $\phi_{c,c'} \circ c = c'$*
3. *if $c \in \mathbb{C}$, then there exists $\phi_c \in \Gamma$ such that $Fix(\phi_c) = c(\mathbb{R})$*

then $\mathbb{C}$ is the collection of unit-speed geodesics of M.

Proof. Coming soon... □

Examples 3.1. By the above lemma:

1. Sphere: the geodesics on S^2 are the great circles parametrized with unit speed.
2. $\mathbb{R}^2$: the geodesics of $\mathbb{R}^2$ as the straight lines.
3. Flat torus $\mathbb{T}^2 = \mathbb{R}^2 \setminus \mathbb{Z}^2$: the geodesics on the flat torus are the projections of straight lines.
4. Hyperbolic plane: We showed in the previous section that the geodesics of the hyperbolic plane are lines orthogonal to the real line and semicircles.

An alternative model of the hyperbolic plane is the Poincare Disk, which maps the upper half-plane $\mathcal{H}$ onto the open unit disk $\mathcal{D}$ with $f : z \mapsto \frac{z-i}{z+i}$ using the following metric: $\langle v, w \rangle = \langle Df^{-1}v, Df^{-1}w \rangle$ on $\mathcal{D}$. Since this metric is nothing but a push forward the hyperbolic Riemannian metric $\langle \cdot, \cdot \rangle$ on $\mathcal{H}$ onto the unit disk, f is an isometry. f being an isometry in turn implies that it maps geodesics onto geodesics and preserves angles. It thus follows that the geodesics in the Poincare disk are diameters and arcs of circles perpendicular to S^1. Using the Poincare disk, we can construct compact manifolds that are locally isometric to the hyperbolic plane. For example, we can construct a geodesic octagon on the Poicare disk and identify the opposite sides together. This identification space is a surface of genus 2. By replacing these eight octagon sides of the geodesic octagon by $4g$ sides, where $g \geq 2$ is the genus, we obtain a metric locally isomorphic to that of $\mathcal{H}$ on an orientable surface of genus g.

We now study the dynamics of the geodesic flow on compact hyperbolic surfaces, which will be much more complicated than the ones in the previous examples. The behaviours that make these dynamics so much more complicated are the following: density of closed orbits, topological transitivity, and ergodicity wrt Liouville measure.

Theorem 3.2. *Let Γ be a discrete group of fixed-point-free isometries of $\mathcal{D}$ such that $M = \Gamma \setminus \mathcal{D}$ is compact. Then the periodic orbits of the geodesics flow on SM are dense in SM.*

Proof. □

Theorem 3.3. *Let Γ be a discrete group of fixed-point-free isometries of $\mathcal{D}$ such that $M = \Gamma \setminus \mathcal{D}$ is compact. Then the geodesic flow on SM has an orbit which is dense in SM, that is, it is topologically transitive.*

Proof. □

We now prove a stronger result, namely, we show that the geodesic flow on a compact factor of H is ergodic with respect to the natural smooth invariant measure

Theorem 3.4. *The Liouville measure for the geodesic flow on a compact connected factor of $\mathcal{H}$ is ergodic.*

Proof. □

Definition 3.2. Two variables, say p_x and x are a **canonical pair** if their Poisson bracket $\{x, p_x\} = 1$.

A Hamiltonian $H(p, x)$ is a function of generalised coordinates x_i and canonical momenta p_i. For a given Lagrangian L, we have

$$p_i = \frac{\partial L}{\partial \dot{x}^i}.$$

Definition 3.3. The **Hamiltonian** is defined as follows:

$$H(p, x) = p^i \dot{x}_i - L(x, \dot{x})$$

Using the above definition, we derive the Hamiltonian equations:

$$\frac{\partial H}{\partial p} = \dot{x} + p\frac{\partial \dot{x}}{\partial p} - \frac{\partial L}{\partial \dot{x}}\frac{\partial \dot{x}}{\partial p} = \dot{x}$$

since $\frac{\partial L}{\partial \dot{x}} = p$ Similarly, we obtain that

$$\frac{\partial H}{\partial x} = -\frac{\partial L}{\partial x} = -\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = -\dot{p}$$

again using $\frac{\partial L}{\partial \dot{x}} = p$. If we now interpret the Hamiltonian as only the kinetic energy:

$$H = \frac{1}{2}g^{ij}p_i p_j = \frac{1}{2}g_{ij}\dot{x}^i\dot{x}^j$$

Definition 3.4. On the cotangent bundle T^*M with the form $\omega = dp^i \wedge dx_i$, this system of equations is called the **geodesic flow** of the Riemannian manfiold (M, g).

The geodesic flow preserves the length of tangent vectors because the total energy is given by $\frac{1}{2}g_x(v,v)$. Since the Hamiltonian $H = K + V = \frac{1}{2}k_x(v,v) + V(x)$ is invariant in time, the geodesic flow on any compact manifold is a complete flow.

Any transformation preserving $\omega = dp^i \wedge dx_i$, preserves the Hamiltonian form of the equations. These transformations are called canonical transformations.

A key property of Hamiltonian (and Lagrangian) dynamics is that they preserve a canonically defined volume form. Since Hamilton's equations are divergence free, they preserve volume in phase space, which leads us to the following fact:

Proposition 3.5. *The geodesic flow on the unit tangent bundle SM of a complete Riemannian manifold M preserves a smooth measure called the Liouville measure. If M is compact then the Liouville measure is finite and can therefore be normalized.*

Over the next few pages, we will build up toward the general form of this result, also known as the Liouville theorem.

$\flat : TM \to T^*M$ such that $X^\flat = g_{ij}X^ie^j = X_je^j$ i.e. lowering an index. Similarly, $\omega^\sharp : T^*M \to TM$ such that $\omega^\sharp = g^{ij}w_ie_j = w^je_j$ i.e. raising an index.

Definition 3.5. Let E be a linear space. A 2-tensor $\alpha E \times E \to \mathbb{R}$ is said to be **nondegenerate** if $\alpha^\flat : v \mapsto \alpha(v,\cdot)$ is an isomorphism from E to its dual space E^*. It is **antisymmetric** if $\alpha(v,w) = -\alpha(w,v)$. A nondegenerate antisymmetric 2-from is a **symplectic** vector space.
If $(E,\alpha),(F,\beta)$ are symplectic vector spaces, then an invertible map $T : E \to F$ is **symplectic** if $T^*\beta = \alpha$.

Definition 3.6. Let (M,ω) be a symplectic manifold, and $H : M \to \mathbb{R}$ a smooth function. Then the vector field $X_H = dH^\sharp$is the **Hamiltonian vector field** associated with H. The flow ϕ^t with $\dot{\phi}^t = X_H$ is called the **Hamiltonian flow** of H.

We can easily check that this formulation yields the same Hamiltonian equations as before.

Proposition 3.6. *(Liouville Theorem) Hamiltonian flows are symplectic and hence volume preserving.*

Proof. □

Note that there are symplectic flows that are not Hamiltonian i.e the converse of Liouville's theorem is not true.

4 *Integrable Geodesic Flows*

In this section, we discuss integrable geodesic flows in more detail following *Integrable geodesic flows on Riemannian manifolds: Construction and Obstructions* by Bolsinov and Jovanovic. This section contains all the topics I will discuss in my talk.

4.1 *Introduction*

Geodesics are one of the key concepts in physics because they describe the trajectory of an object with no external forces i.e. no acceleration. On a plane, it is natural to us that such an object will move in a straight line, which is just the shortest path between two points in a plane. However, if the space is more complicated, the need for a more precise definition of a geodesic arises. It turns out that any geodesic satisfies the following equation:

Note that we assume Einstein summation notation everywhere.

$$\ddot{x}^\mu + \Gamma^\mu_{\nu\rho}\dot{x}^\nu\dot{x}^\rho = 0, \tag{1}$$

where $\Gamma^\mu_{\nu\rho}$ are the Christoffel symbols and the derivatives are taken with respect to some affine parameter. We can rewrite this equation in a coordinate-independent form using the covariant derivative ∇:

$$v^\mu \nabla_\mu v^\mu = 0, \tag{2}$$

where v^μ is a tangent vector of some curve C that is being parallel transported along it. Since this equation is coordinate-independent, it is only influenced by the topology of the space. This, in turn, implies that if we transform the space without affecting its geometry i.e. by isometry, then the geodesics are unaffected.

Recall that a Hamiltonian $H(p, x)$ is a function of generalised coordinates x_i and canonical momenta p_i. We can then rewrite the geodesic equation in the Hamiltonian form:

Let $\mathcal{M}$ be a smooth manifold with metric $g = (g_{ij})$. Let $x^1, \cdots, x^n$ be an arbitrary local coordinate system. We convert velocities $\dot{x}^i$ to momenta p_j using $p_j = g_{ij}\dot{x}^i$. Here p_j and x^i form a canonical pair.

The geodesic equations are then simply the Hamiltonian equations:

$$\frac{dx^i}{dt} = \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial x^i}, \tag{3}$$

where the Hamiltonian H on $T^*\mathcal{M}$ is the kinetic energy:

$$H = \frac{1}{2}g^{ij}p_i p_j = \frac{1}{2}g_{ij}\dot{x}^i\dot{x}^j \tag{4}$$

Here g^{ij} are the coefficients such that $g^{ik}g_{ij} = \delta^k_j$, and p and p are the standard variables in the cotangent bundle $T^*\mathcal{M}$, where $x =$

$(x^1, \cdots, x^n)$ are the coordinates of points on $\mathcal{M}$ and $p = (p_1, \cdots, p_n)$ are the coordinates of a covector in the cotangent space $T^*_x\mathcal{M}$ with respect to the basis $dx^2, \cdots, dx^n$. The equations of motion can be recovered as the flow along the Hamiltonian vector field associated to H via the standard Poisson brackets in $T^*\mathcal{M}$:

$$\frac{dx}{dt} = \{x, H\}, \quad \frac{dp}{dt} = \{p, H\} \tag{5}$$

A Riemannian manifold $(\mathcal{M}, g)$ with this system of Hamiltonian equations on the cotangent bundle $T^*\mathcal{M}$ with nondegenerate antisymmetric form $\omega = dp_i \wedge dx^i$ is called the **geodesic flow**. In other words, the geodesic flow is the one-parameter group of of diffeomorphisms defined by this system of equations.

For a very long time, the main purpose of classical mechanics was to just explicitly solve equations of motion (EoMs) in order to understand the underlying dynamics. This was the main motivation for calculating the integrals of various Hamiltonian systems, because the orbits are determined by the integrals only if enough of them are known. For a $2k$ dimensional phase space, we needed $2k - 1$ independent integrals, so a joint level set is one-dimensional and thus determines an orbit. This was how things were before Poincare's time. After, it turned out that due to the symplectic structure of the Hamiltonian equations, it suffices to have only n independent (first) integrals, but they must be in involution, i.e. with pairwise vanishing Poisson brackets, to be able to solve the EoMs. These systems are called **completely integrable**. We begin by presenting some simple examples.

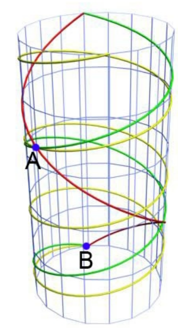

Figure 1: Two geodesics connecting two points on a cylinder.

1. S^2: The geodesics of a sphere are great circles. They are all closed and of the same length, so geodesic flow is simple.

2. Surface of revolution: First recall that a surface of revolution is made by rotating some curve around an axis called the axis of revolution. Some common examples are cones and cylinders. Usually, the geodesics of such surfaces move around the axis of revolution while oscillating along this axis. For the cylinder, for example, geodesics are spirals around its surface as seen in Figure 1. For some cases, like for the cylinder, geodesics are not closed. When this occurs, we notice that the closure of the geodesic is an annulus-like region on the surface. The dynamics are regular and are simply a superposition of rotation and oscillation. For more complicated surfaces, such as a three-axial ellipsoid, the situation is a little more complicated since there are now two kinds of annulus-like regions inside which geodesics move. In spite of these complications, the behaviour of geodesics is still regular. In contrast to the ellipsoid, we can also consider an arbitrary

closed surface with no symmetries whatsoever. In such cases, the behaviour of geodesics may still preserve some integrability, but also may contain some chaotic parts. Such systems are called Kolmogorov-Arnold-Moser (KAM) systems.

Definition 4.1. A geodesic flow is **completely Liouville integrable** if it admits n smooth functions $f_1, \cdots, f_n$ satisfying the following conditions:

1. $f_i(x, p)$ is an integral of the geodesic flow (constant along geodesic lines $(x(t), p(t))$.
2. $f_1, \cdots, f_n$ have pairwise vanishing Poisson brackets.
3. assuming $f_1, \cdots, f_n$ are real analytic, they must be functionally independent at one point at least.

Theorem 4.1. *(Liouville-Arnold Theorem) Suppose (M, ω) is a $2n-$dimensional symplectic manifold, $H = f_1, f_2, \cdots, f_n \in C^\infty(M), \{f_i, f_j\} = 0$ for $i, j = 1, \cdots, n$, and $x \in \mathbb{R}^n$ such that the differentials Df_i are pointwise linearly independent on $M_z = \{x \in M | f_i(x) = z_i, i\, 1, \cdots, n\}$, then:*

1. *M_z is a smooth Lagrangian submanifold invariant under the Hamiltonian flow ϕ_H^t of H.*
2. *If M_z is compact and connected then M_z is diffeomorphic to the $n-$torus $\mathbb{T}^n$*
3. *Via this diffeomorphism $\phi^t_{H\upharpoonright_{M_z}}$ is conjugate to a linear flow.*

In this work, we are interested in determining on which manifolds exist Riemannian metrics with integrable geodesic flows, essentially dividing all manifolds into two classes. This is a very big question for which we cannot expect a complete answer. There are two main ways to approach this problem at present, both of which we will describe in this work.

1. To determine whether integrable geodesic flow exists on a manifold (or some class of manifolds), there is no other way, at present, but to construct a specific example. We therefore consider new constructions of integrable Hamiltonian systems and study their geodesic flows.
2. The second method is to find topological obstructions in a manifold such that it cannot be integrable. This method comes down to adding to the class of manifolds which do not admit integrable geodesic flows.

4.2 Classical Examples

4.2.1 The Sphere S^2

A **first integral** is a non-constant function whose Lie derivative is 0 in the domain or iff Poisson bracket is 0.

The geodesics on a flat sphere $S^2 = \{x_1^2 + x_2^2 + x_3^2 = 1\}$ are great circles. Since great circles are closed, we have three independent directions along which to integrate instead of two, meaning we have three independent first integrals. These three integrals can be, for example, linear integrals corresponding to infinitesimal rotations of the sphere along each of the three axes. The integral corresponding to rotation about the third axis x_3 has the form $f_3 = p(k_\mu v^\mu)$, where $p \in T_x^* S^2$, and $k_\mu v^\mu = \frac{\partial}{\partial \phi}$ such that $\frac{\partial}{\partial \phi}(x_1, x_2, x_3) = (-x_2, x_3, 0)$. We can regroup these integrals into one vector integral $F(x, \dot{x}) = (f_1, f_2, f_3) = [x, \dot{x}]$, where $x \in S^2 \subset \mathbb{R}^3$ and $\dot{x} \in T_x S^2$ are vectors in $\mathbb{R}^3$. F is the vector orthogonal to the plane of the geodesic. Notice that this means that these three integrals do not commute with one another. However, every integral commutes with the Hamiltonian $H = \frac{1}{2}(f_1^2 + f_2^2 + f_3^2)$. Let's verify explicitly that this system is completely integrable:

Two of these three integrals stand for an explicit symmetry: rotational invariance, and time symmetry, as we will see in Schwarzschild solution discussion.

1. f_i is an integral i.e. constant along the geodesic line: $f_3 = p(k^\mu v_\mu) = p(\partial/\partial\phi)$, where k^μ is the Killing vector. By definition, the Killing vector satisfies the Killing equation i.e. $\pounds_v g_{\mu\nu} = 0$.

2. $\{f_i, f_j\} = 0$ for all pairs i, j: $\{f_i, f_j\} = 0$ since the f_i's are not functions of p_1, p_2, only of x_1, x_2.

3. The f_i are functionally independent: this follows from the fact that the Killing vectors are independent for every integral i.e. $k^\mu = (0, 0, 1)$ for f_3, but $k^\mu = (1, 0, 0)$ for f_1, since they correspond to different symmetries.

4.2.2 The Flat Torus $\mathbb{T}^2$

In this example, we consider the flat torus $\mathbb{T}^2$ with the Euclidean metric $ds^2 = dx_1^2 + dx_2^2$. In this case, the geodesics are quasiperiodic windings that we can write as $x_i(t) = c_i t$ for some parameter t, $i = 1, 2$, where $x_i \mod 2\pi$ are the standard angle coordinates on a torus. The first integrals are the momenta p_i, and the Hamiltionian is just $H(x, p) = \frac{1}{2}(p_1^2 + p_2^2)$. Let's verify explicitly that this system is completely integrable:

1. $f_i(x, p)$ is an integral i.e. constant along the geodesic line: $\pounds_{c_i t} p_i = \frac{dp_i}{dt} \propto \ddot{x} = 0$ since there is no acceleration.

2. $\{f_i, f_j\} = 0$ for all pairs i, j. $\{p_1, p_2\} = \frac{\partial p_1}{\partial x^1}\frac{\partial p_2}{\partial p_1} - \frac{\partial p_2}{\partial x^1}\frac{\partial p_1}{\partial p_1} + \frac{\partial p_1}{\partial x^2}\frac{\partial p_2}{\partial p_2} - \frac{\partial p_2}{\partial x^2}\frac{\partial p_1}{\partial p_2} = 0$

3. The f_i are functionally independent: p_1, p_2 are clearly independent.

This is essentially the same as the previous case with the sphere.

4.2.3 Metrics of Revolution

Theorem 4.2. *(Noether's Theorem) Let a Riemannian metric g_{ij} admit a one-parameter isometry group $\phi^s : \mathcal{M} \to \mathcal{M}$, then the corresponding geodesic flow has a linear integral of the form $f_\xi(x, p) = p(\xi(x))$, where $\xi(x) = \frac{d}{ds}\big|_{s=0}\phi^s(x)$ is the vector field associated with the one-parameter group ϕ^2.*

A more general restatement of this theorem is that if a metric admits an isometric group G, then the corresponding geodesic flow has an algebra of first integrals isomorphic to the Lie algebra of G. Using the Noether theorem above, we can show the following result:

Theorem 4.3. *(Clairaut) The geodesic flow on any surface of revolution has a non-trivial first integral independent of the total energy and is therefore integrable. This integral is called the Clairaut integral.*

Proof. • Existence of non-trivial Clairaut integral: Surfaces of revolution admit a one-parameter isometry group because of how they are built i.e. by rotating a curve around an axis, so this is just a case of Noether's theorem.

• The geodesic flow is integrable:

1. $f_i(x, p)$ is an integral i.e. constant along the geodesic line: By Noether's theorem.
2. $\{f_i, f_j\} = 0$ for all pairs i, j. By Noether's theorem, the integral has form $f_\xi(x, p) = p(\xi(x))$, where $\xi(x) = \frac{d}{ds}\big|_{s=0}\phi^s(x)$. This is again just like the sphere, where the argument is that the Poisson brackets are all 0 because momenta are independent.
3. The f_i are functionally independent: again, momenta are clearly independent.

□

In fact, we can also use this to prove an even more general result:

Theorem 4.4. *A surface admits a one-parameter group of isometries iff it is isometric to a surface of revolution (or part of such a surface).*

Proof. Coming soon... □

4.2.4 Liouville Metric

Theorem 4.5. *(Liouville) The geodesic flow of the metric* $ds^2 = \big(f(x_1) + g(x_2)\big)(dx_1^2 + dx_2^2)$ *admits the non-trivial quadratic integral of the form* $F(x,p) = \frac{g(x_2)p_1^2 - f(x_1)p_2^2}{f(x_1)+g(x_2)}$*, and is therefore integrable.*

We will show explicitly that this system is integrable:

1. $f_i(x,p)$ is an integral i.e. constant along the geodesic line: by Liouville theorem.
2. $\{f_i, f_j\} = 0$ for all pairs i, j: $\{F,F\} = \frac{\partial F}{\partial x^1}\frac{\partial F}{\partial p_1} - \frac{\partial F}{\partial x^1}\frac{\partial F}{\partial p_1} + \frac{\partial F}{\partial x^2}\frac{\partial F}{\partial p_2} - \frac{\partial F}{\partial x^2}\frac{\partial F}{\partial p_2} = 0$
3. The f_i are functionally independent: because there is only one integral.

4.3 Topological Obstructions to Integrability

In this section, we discuss surfaces that have certain properties that contradict integrability.

4.3.1 Two-Dimensional Surfaces

Theorem 4.6. *(Kozlov, 1979) Two-dimensional surfaces of genus* $g > 1$ *admit no analytically integrable geodesic flows.*

Although this theorem was proved using the properties of analycity, note that the analyticity condition actually can be weakened. The next natural question to ask is then whether we can loosen the analycity condition to $C^\infty-$smooth instead. Although the answer to this exact question is not yet known, here is a related result:

Theorem 4.7. *(Kolokol'tsov, 1982) Two-dimensional surfaces with genus* $g > 1$ *admit no geodesic flows integrable by means of an integral polynomial in momenta with smooth coefficients (no analycity condition here!).*

For this theorem, the proof uses the polynomial integral to construct a holomorphic form on the surface. By analysing the zeroes and poles, we obtain the genus of the surface. From this result, a natural question to ask is whether there exists an analog of this theorem in higher dimensions i.e. what are the topological barriers to polynomial integrability. It's easier to answer this question if we only consider linear and quadratic integrals because with some additional conditions, the flows admit spearation of variables on the configuration space. It will then only remain to identify singularities by studying the topology of the manifold. This approach was studied by Kiyohara, with no conclusive results yet.

4.3.2 Non Simply Connected Manifolds

Theorem 4.8. *(Taimanov,1987) If a geodesic flow on a closed manifold $\mathcal{M}$ is analytically integrable, then*

1. *the fundamental group of $\mathcal{M}$ is almost commutative (i.e. index of commutative subgroup is finite).*
2. *if $\dim H_1(\mathcal{M}, \mathbb{Q}) = d$, then $H^*(\mathcal{M}, \mathbb{Q})$ contains a subring isomorphic to the rational cohomology ring of the $d-$dimensional torus.*
3. *if $\dim H_1(\mathcal{M}, \mathbb{Q}) = n = \dim \mathcal{M}$, then the rational cohomology rings of $\mathcal{M}$ and of the $n-$dimensional torus are isomorphic.*

The proof of this statement is of purely topological nature. Note that the analyticity condition is not needed: Taimanov showed this result holds under the weaker assumption that the geodesic flow is geometrically simple. This implies that in a topological sense, the structure of the singular set where the first integrals are dependent is rather simple.

4.3.3 Topological Entropy

Topological entropy is a method suggested by Paternain in 1991 to find topological obstructions to integrability of geodesic flows. It is a property of a dynamic system on a compact manifold which essentially measures its level of chaos. As a general rule, integrable Hamiltonian systems have a topological entropy of zero, so we can start by estimating the topological entropy of a flow on a manifold using only topological information i.e. without even using the Riemannian metric. If the topology is complicated enough, the entropy of any flow will be automatically positive, and we're done. It remains to show the converse, namely that integrability of geodesic flow implies zero topological entropy, but before we do this, we will first explore the concept of topological entropy. To rigorously define topological entropy, we need a few preliminary concepts. Let F^t be a dynamical system on a compact manifold X. We consider F^t as the one-parameter group of diffeomorphisms. We want to approximate this system up to some $\varepsilon > 0$ on a segment $[0, T]$ with finitely many solutions i.e. let $x_1, \cdots, x_{N(\varepsilon,T)}$ be a finite number of points such that for all $y \in X$, there exists a point x_i such that $\text{dist}(F^t(y), F^t(x_i)) < \varepsilon$ for all $t \in [0, T]$, where dist is any metric compatible with X. To define the topological entropy, we take $N(\varepsilon, T)$ to be the minimal number of such x_i:

Definition 4.2. The **topological entropy** of the flow F^t, denoted $h_{top}(F^t)$ is:

$$h_{top}(F^t) = \lim_{\varepsilon \to 0} \limsup_{T \to \infty} \frac{\ln N(\varepsilon, T)}{T}.$$

We will now present results that describe the effects of the topology of a manifold on its topological entropy.

Theorem 4.9. *(Dinaburg, 1971) If the fundamental group $\pi_1(\mathcal{M})$ has exponential growth, then $h_{top} > 0$ for any smooth Riemannian metric on $\mathcal{M}$. In particular, the topological entropy of any geodesic flow on a 2-D surface with genus $g > 1$ is positive.*

Definition 4.3. A manifold $\mathcal{M}$ is **rationally elliptic** if its rational homotopy groups are all trivial after some dimension.

Theorem 4.10. *(Paternain, 1991) If a simply-connected manifold is not rationally elliptic, then $h_{top} > 0$ for any geodesic flow.*

Theorem 4.11. *(Babenko,1997) The topological entropy of a geodesic flow on a simply connected Riemannian manfiold $(\mathcal{M}, g)$ can be estimated:*

$$h_{top}(g) \geq D^h(\mathcal{M}, g)^{-1} \limsup_{n \to \infty} \Big(\frac{1}{n} \ln(\mathit{rank}\ \pi_n(\mathcal{M})) \Big),$$

where $D^h(\mathcal{M}, g)$ is the homology diameter of $(\mathcal{M}, g)$.

This theorem is very important because it allows us to conclude that the topological entropy is positive for a large class of manifolds: the limsup term is always positive, unless the manifold is rationally elliptic. Moreover, this formulation allows us to estimate h_{top} from below. We may now return to our problem in this section, namely showing that integrability of geodesic flow implies zero topological entropy. We will use the additional assumption that the first integrals are non-degenerate. For a singular point x, the integral f is non-degenerate at x if $\det\left(\frac{\partial^2 f}{\partial x^i \partial x^j}\right) \neq 0$. The integrals of a Hamiltonian system are non-degenerate if every point x in a symplectic manifold is non-degenerate. We then have the following result:

Theorem 4.12. *(Paternain, 1991) Let $(\mathcal{M}, g)$ be a smooth compact Riemannian manifold. Suppose that its geodesic flow is integrable and has a non-degenerate first integrals. Then the topological entropy of this geodesic flow is zero.*

It is important to note that this additional assumption is very strong and often does not hold in multi-dimensional cases, even for cases as nice as the n-dimensional ellipsoid ($n \geq 3$).

In more recent works, it was shown that there are some assumptions on first integrals that cannot be entirely omitted i.e. topological entropy and "complexity" of the fundamental group are not obstructions to integrability of geodesic flows. For example, in 1998, Butler constructed an example that shows that the smooth analog to Taimanov's theorem 4.8 does not hold. This example is not geometrically simple because the base of the foliation into Liouville tori is not simply connected, and thus goes further to highlight the importance of the geometric simplicity assumption. This example is making us question which additional properties of the first integrals guarantee a topological entropy of zero.

4.4 *Geodesic Flows on Homogeneous Spaces*

Recall the definition of homogeneity:

Definition 4.4. Homogeneity is the statement that the metric is the same throughout the manifold, i.e. given any two points a and b in $\mathcal{M}$, there is an isometry that takes a to b.

To construct high dimensional integrable systems, we must use metrics with large symmetry groups because integrability is closely related to symmetry. In Schwarzschild solution of Einstein's equations in general relativity, for example, we deal with the group $SO(3)$, the group of symmetries of a rotating sphere, because spacetime is spherically symmetric in the Schwarzschild case. Another example, is the geodesic flow with left-invariant metric on $SO(3)$. The geodesic flow of both of these cases describes the orbital motion of a rigid body under its own inertia.

In this section, we will first mention some general results about integrable geodesic flows in homogeneous spaces and conclude with an in-depth discussion of geodesic flows in the Schwarzschild solution.

4.4.1 *General Results*

The geodesic flow with left-invariant metric on $SO(3)$ often reduces to the Euler equations. The Euler case can, however, be extended to multiple dimensions:

Theorem 4.13. *(Mishchenko, Fomenko, 1976) Every compact Lie group admits a family of left-invariant metrics with completely integrable geodesic flows.*

Definition 4.5. A **normal** $G-$invariant Riemannian metric on the homogeneous space G/H, where G is a compact group, is the metric induced from a bi-invariant metric on G.

Theorem 4.14. *(Thimm 1981, Mishchenko 1982) Geodesic flows of normal metrics on compact symmetric spaces $G \setminus H$ are completely integrable.*

Although there are many generalisations of this result, and many more other results to present, we will at present focus on a specific example of integrable geodesic flow on a homogeneous space in general relativity.

4.4.2 *Schwarzschild Solution*

Note that here we use units where the speed of light $c = 1$.

In this section, we will consider timelike geodesic flow in the Schwarzschild spacetime. The Schwarzschild solution is $ds^2 = -(1 - \frac{R_s}{r})dt^2 + (1 - \frac{R_s}{r})^{-1}dr^2 + r^2 d\Omega^2$, where $R_s = 2GM$ is the Schwarzschild radius, G is the gravitational constant, and $d\Omega^2 = d\theta^2 + \cos^2\theta d\phi^2$. This spacetime is spherically symmetric, and represents an empty universe with a static uncharged black hole of mass M at its center. We will use spherical coordinates where $x^\mu(\lambda) = (t(\lambda), r(\lambda), \theta(\lambda), \phi(\lambda))$, and λ is an affine parameter. Let's consider an object moving in the equatorial plane $\theta = 0$. The metric ds^2 has two generators $k^\mu \partial/\partial x^\mu = \partial/\partial t$, and $k^\mu \partial/\partial x^\mu = \partial/\partial \phi$, where k^μ are Killing vectors. If a metric has a Killing vector, then there is a coordinate system where the metric $g_{\mu\nu}$ is independent of one of the coordinates. This implies that every Killing vector corresponds to a symmetry, which, in turn, implies that we have a conserved quantity for every Killing vector.

- The time translation generator $\partial/\partial t$ corresponds to conservation of energy E: It has Killing vector $k^\mu = (1, 0, 0, 0)^t$. Using the metric, we can lower the index to get $k_\mu = k^{\mu\flat} = (-(1 - R_s/r), 0, 0, 0)^t$, which yields $E = -k_\mu \dot{x}^\mu = (1 - R/r)\dot{t}$, where $\dot{x}^\mu = (\dot{t}, \dot{r}, \dot{\theta}, \dot{\phi})$ as defined previously. So we have the equation

  $$\dot{t} = \frac{E}{1 - R_s/r},$$

  where E is a constant (the conserved energy).

- The rotation generator $\partial/\partial\phi$ corresponds to conservation of angular momentum L in x_3 direction: In this case, the Killing vector is $k^\mu = (0, 0, 0, 1)^t$, and so $k_\mu = (0, 0, 0, r^2)$. We let $L = k_\mu \dot{x}^\mu = r^2\dot{\phi}$, where L is the constant angular momentum which is conserved.

  $$\dot{\phi} = \frac{L}{r^2}$$

  Notice that this is the same angular momentum as in the Kepler problem.

Recall that the geodesic distance is

$$S = \int ds = \int_{\lambda_1}^{\lambda_2} \sqrt{g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu} d\lambda \implies \frac{ds}{d\lambda} = \sqrt{g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu}$$

Let τ be the proper time. Recall that $d\tau = i\, ds$, and so $-d\tau^2 = ds^2$, then from the geodesic distance, since we have an affine parameter λ, we obtain:

$$\begin{aligned}
&\left(\frac{ds}{d\lambda}\right)^2 = g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu = -\left(\frac{d\tau}{d\lambda}\right)^2 = \text{cst} \equiv -1 \\
\implies & -1 = -\left(1 - \frac{R_S}{r}\right)\dot{t}^2 + \left(1 - \frac{R_S}{r}\right)^{-1}\dot{r}^2 + r^2\dot{\phi}^2 \\
&\quad = -\frac{E^2}{1 - R_S/r} + \frac{\dot{r}^2}{1 - R_S/r} + \frac{L^2}{r} \\
&\implies \frac{1}{2}E^2 = \frac{1}{2}\dot{r}^2 + V_{eff}(r), \text{ where } V_{eff}(r) = \frac{1}{2} - \frac{R_S}{2r} + \frac{L^2}{2r^2} - \frac{R_S L^2}{2r^3}
\end{aligned}$$

This shows that geodesic flow in Schwarzschild spacetime is orbital. Using the effective potential $V_{eff}(r)$, we can solve for critical values of r. We obtain

$$r_c = \frac{L^2 \pm \sqrt{L^4 - 3R_S^2 L^2}}{R_S}.$$

These two solutions tell us that we don't have stable circular orbits for all values of L and r, unlike the Newtonian case. They also tell us that the orbits are not necessarily elliptical. This last fact is very important since it is one of the first experimental tests of general relativity: the precession of orbits. The first such measurement was made in 1919 for the perihelion of Mercury.

4.5 *Conclusion*

Geodesic flows are a description of geodesic motion in different kinds of multi-dimensional spaces that we call manifolds. For example, if we consider light, the geodesic flow will be the motion along null geodesics. They are extremely important in many fields, especially in physics, where we are often interested in understanding how and why light and celestial bodies travel along certain paths rather than others. In this work, we reviewed many general results about integrable systems. They make up the 'nicer' kind of geodesic flows, because, in principle, they can be solved more easily than general, non-integrable systems. Our initial goal is to divide all manifolds into two classes, where one is comprised of all the manifolds which admit integrable geodesic flows, and the other of all the manifolds which do not admit such flows. We looked at two general ways of performing this classification: by constructing an example, and by finding topological obstructions to integrability. Throughout, we presented many results without proof, and discussed their implications. In many cases, the minimal set of assumptions is not yet known. Lastly, we briefly discussed geodesic flows on homogeneous spaces, where, after a few general results, we dipped our toes in the vast sea of general relativity, where most classical examples are integrable flows.

Acknowledgements: many thanks to Dima Jacobson for teaching this fun course!