Intro to General Relativity: Geodesic Eqn and Einstein's Eqn

Daniela Breitman

February 22, 2021

Lecture of May 20, 2020

1 What are theories made of?

A classical filed theory has two components:

1. **Equation of motion (EoM):** it's the field equation. It determines how the field behaves in the presence of a source. e.g. in Newton's case $\nabla^2\Phi = 4\pi G\rho$
2. **A force equation:** it determines the motion of objects in the presence of a field. e.g. in Newton's case $F = ma = m\nabla\Phi$.

The fields are functions of spacetime i.e. $\Phi(t, \vec{x})$. The force law tells us how the motion deviates from straight lines ($a = 0$) in the presence of a field. The key point in general relativity is that gravity is not due to a field (a function of spacetime), but rather due to a **feature of spacetime itself**: the curvature. So in general relativity, we associate $\Phi \to g_{\mu\nu}$, the metric tensor. In general relativity:

1. The field equation is Einstein's equation, which tells us how spacetime curves in the presence of matter:
$$\nabla^2\Phi = 4\phi G\rho \to G_{\mu\nu} = 8\pi G T_{\mu\nu}$$
2. The force equation is the geodesic equation, which describes how objects move in a curved spacetime:
$$F = ma = m\nabla\Phi \to \ddot{x}^\mu + \Gamma^\mu_{\nu\rho}\dot{x}^\nu\dot{x}^\rho = 0$$

2 The Geodesic Equation

2.1 But First: A bit of Classical Mechanics

Definition 2.1. The **Lagrangian** L is defined as $L = K - V$, where K is kinetic energy, and V is potential energy.

The Lagrangian satisfies the **Euler-Lagrange (E-L) equation**:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{\partial L}{\partial x}$$

Let's explain the background behind this fact: To understand what

This proof comes from Introduction to Classical Mechanics by David Morin. Note that it's a great reference for all classical mechanics!

the E-L equation means and where it came from, we need the principle of stationary action. Consider a functional $S[x(t)]$, where $x(t)$ is some path between two endpoints t_1, t_2. The question we ask ourselves is what $x(t)$ yields a **stationary** value of the action S, i.e. a local minimum, maximum, or saddle point. To answer, we present the following theorem:

Theorem 2.1. *If the function $x(t)$ yields a stationary value of S, then* $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{\partial L}{\partial x}$.

Proof. Key idea: if $x(t)$ yields a stationary value of S, then a perturbation of $x(t)$ will yield the same S up to first order.And so, consider a perturbation around the $x(t)$: $x_a(t) = x(t) + a \cdot x\beta(t), a \in \mathbb{R}$, and β keeps the endpoints fixed i.e. $\beta(t_1) = \beta(t_2) = 0$.
No change in S at first order in $a \implies \frac{\partial S[x_a(t)]}{\partial a} = 0$:

$$\begin{aligned} \frac{\partial S[x_a(t)]}{\partial a} &= \frac{\partial}{\partial a}\int_{t_1}^{t_2} L dt = \int_{t_1}^{t_2} \frac{\partial L}{\partial a} dt \\ \text{Apply Chain Rule } &= \int_{t_1}^{t_2}\left(\frac{\partial L}{\partial x_a}\frac{\partial x_a}{\partial a} + \frac{\partial L}{\partial \dot{x}_a}\frac{\partial \dot{x}_a}{\partial a}\right) dt = \int_{t_1}^{t_2}\left(\frac{\partial L}{\partial x_a}\beta + \frac{\partial L}{\partial \dot{x}_a}\dot{\beta}\right) dt \\ \text{By Parts 2nd term } &= \int_{t_1}^{t_2}\left(\frac{\partial L}{\partial x_a} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_a}\right)\right)\beta dt + \frac{\partial L}{\partial x_a}\beta\Big|_{t_1}^{t_2} \end{aligned}$$

The last term evaluates to 0, since $\beta(t_1) = \beta(t_2) = 0$, and our requirement that $\frac{\partial S[x_a(t)]}{\partial a} = 0$ for all $\beta(t)$ yields the Euler-Lagrange equation:

$$\frac{\partial L}{\partial x_a} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_a}\right) \tag{1}$$

□

The E-L equation is equivalent to $F = ma$.

Definition 2.2. Principle of stationary action: the path of a particle is the one that yields a stationary action.

Note that this is a **local** property.

2.2 *The Geodesic Equation*

Let us now write the action of general relativity. Recall that the line element is $ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu$, so

$$S = \int_{\lambda_0}^{\lambda_1} ds = \int_{\lambda_0}^{\lambda_1} \sqrt{g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu} d\lambda$$

Definition 2.3. A path which extremises $S[x^\mu(\lambda)]$ is a **geodesic**.

Definition 2.4. The length S of this path is the **geodesic distance** between $x^\mu(\lambda_0)$ and $x^\mu(\lambda_1)$.

- If S is real, then the points are space-like separated.
- If S is imaginary, then the points are time-like separated.
- If S is 0, the points are null/light-like separated.

So what do geodesics look like?

$$ds = \sqrt{g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}d\lambda \implies \frac{ds}{d\lambda} = \sqrt{g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}$$

S is extremised under perturbations $x^\mu \to x^\mu + \delta x^\mu$, so we can apply the E-L equation here:

The metric $g_{\mu\nu}$ only depends on space time i.e. only on x. It does not depend on $\dot{x}$ explicitly.
Notice the notation: $\frac{\partial g_{\mu\nu}}{\partial x^\rho} \equiv g_{\mu\nu,\rho}$

$$\frac{d}{d\lambda}\left(\frac{\partial L}{\partial \dot{x}^\beta}\right) - \frac{\partial L}{\partial x^\beta} = 0$$

$$\frac{d}{d\lambda}\left(\frac{1}{2ds/d\lambda}\left(\cancel{\dot{x}^\mu\dot{x}^\nu\frac{\partial g_{\mu\nu}}{\partial \dot{x}^\beta}} + g_{\mu\nu}\dot{x}^\mu\frac{\partial \dot{x}^\nu}{\dot{x}^\beta} + g_{\mu\nu}\dot{x}^\nu\frac{\partial \dot{x}^\mu}{\dot{x}^\beta}\right)\right) - \frac{\dot{x}^\mu\dot{x}^\nu\frac{\partial g_{\mu\nu}}{\partial x^\beta}}{2ds/d\lambda} = 0$$

$$\frac{d}{d\lambda}\left(\frac{1}{2ds/d\lambda}\left(g_{\mu\nu}\dot{x}^\mu\delta^\nu_\beta + g_{\mu\nu}\dot{x}^\nu\delta^\mu_\beta\right)\right) - \frac{\dot{x}^\mu\dot{x}^\nu g_{\mu\nu,\beta}}{2ds/d\lambda} = 0$$

$$\frac{d}{d\lambda}\left(\frac{1}{2ds/d\lambda}\left(g_{\mu\beta}\dot{x}^\beta + g_{\beta\nu}\dot{x}^\beta\right)\right) - \frac{\dot{x}^\mu\dot{x}^\nu g_{\mu\nu,\beta}}{2ds/d\lambda} = 0 \quad \text{Relabel indices:}$$

$$\frac{d}{d\lambda}\left(\frac{\cancel{2}g_{\mu\beta}\dot{x}^\beta}{\cancel{2}ds/d\lambda}\right) - \frac{\dot{x}^\alpha\dot{x}^\beta g_{\alpha\beta,\mu}}{2ds/d\lambda} = 0$$

We relabel indices that are contracted i.e. summed over (e.g in the negative term of the E-L eqn) so that they match the contracted indices in the first term. We continue applying the chain rule:

$$\frac{d}{d\lambda}\left(\frac{g_{\mu\beta}\dot{x}^\beta}{ds/d\lambda}\right) - \frac{\dot{x}^\alpha\dot{x}^\beta g_{\alpha\beta,\mu}}{2ds/d\lambda} = 0$$

$$-g_{\mu\beta}\dot{x}^\beta\frac{\partial^2 s/\partial\lambda^2}{(\partial s/\partial\lambda)^2} + \frac{1}{ds/d\lambda}\left(g_{\mu\beta}\ddot{x}^\beta + \frac{dg_{\mu\beta}}{d\lambda}\dot{x}^\beta - \dot{x}^\alpha\dot{x}^\beta\frac{g_{\alpha\beta,\mu}}{2}\right) = 0$$

$$-g_{\mu\beta}\dot{x}^\beta\frac{\partial^2 s/\partial\lambda^2}{(\partial s/\partial\lambda)^2} + \frac{1}{ds/d\lambda}\left(g_{\mu\beta}\ddot{x}^\beta + \frac{\partial g_{\mu\beta}}{\partial x^\alpha}\frac{\partial x^\alpha}{\partial\lambda}\dot{x}^\beta - \dot{x}^\alpha\dot{x}^\beta\frac{g_{\alpha\beta,\mu}}{2}\right) = 0$$

$$-g_{\mu\beta}\dot{x}^\beta\frac{\partial^2 s/\partial\lambda^2}{(\partial s/\partial\lambda)^2} + \frac{1}{ds/d\lambda}\left(g_{\mu\beta}\ddot{x}^\beta + \dot{x}^\alpha\dot{x}^\beta\left(g_{\mu\beta,\alpha} - \frac{g_{\alpha\beta,\mu}}{2}\right)\right) = 0$$

We now use a little relabelling trick: in the last term, the indices α, β are the same i.e. changing them doesn't change anything, so we can write: $g_{\mu\beta,\alpha} = 1/2(g_{\mu\beta,\alpha} + g_{\mu\alpha,\beta})$.

$$-g_{\mu\beta}\dot{x}^{\beta}\frac{\partial^2 s/\partial\lambda^2}{(\partial s/\partial\lambda)^2} + \frac{1}{ds/d\lambda}\left(g_{\mu\beta}\ddot{x}^{\beta} + \dot{x}^{\alpha}\dot{x}^{\beta}\left(\frac{1}{2}g_{\mu\beta,\alpha} + \frac{1}{2}g_{\mu\alpha,\beta} - \frac{g_{\alpha\beta,\mu}}{2}\right)\right) = 0 \text{ Relabel:}$$

$\implies \ddot{x}^{\nu} + \dot{x}^{\nu}\dot{x}^{\rho}\Gamma^{\mu}_{\nu\rho} = f(\lambda)\dot{x}^{\mu}$, where $f(\lambda) = \left(\frac{\partial^2 s}{\partial\lambda^2}\right)\left(\frac{\partial s}{\partial\lambda}\right)^{-1}$, and

Definition 2.5. The **Christoffel symbols** are defined as
$\boxed{\Gamma^{\mu}_{\nu\rho} = \frac{1}{2}g^{\mu\sigma}(g_{\sigma\nu,\rho} + g_{\sigma\rho,\nu} - g_{\nu\rho,\sigma})}$.

Note that the Christoffel symbol is **not a tensor**. You can check that by using the definition of tensor:

Definition 2.6. A rank (k, l) tensor transoforms as:

$$T^{\mu_1',\cdots,\mu_k'}{}_{\nu_1',\cdots\nu_k'} = \left(\frac{\partial x^{\mu_1'}}{\partial x^{\mu_1}}\right)\cdots\left(\frac{\partial x^{\mu_k'}}{\partial x^{\mu_k}}\right)\left(\frac{\partial x^{\nu_1'}}{\partial x^{\nu_1}}\right)\cdots\left(\frac{\partial x^{\nu_l'}}{\partial x^{\nu_l}}\right)T^{\mu_1,\cdots,\mu_k}{}_{\nu_1,\cdots\nu_k}$$

Note than λ is an arbitrary parameter. For any **non-null** geodesic, we can choose λ such that $f(\lambda) = 0$, such a λ is called an **affine parameter**. It simplifies the geodesic equation to:

$$\ddot{x}^{\nu} + \dot{x}^{\nu}\dot{x}^{\rho}\Gamma^{\mu}_{\nu\rho} = 0 \tag{2}$$

Exercise 2.1. If λ is an affine parameter, we can obtain the geodesic equation 2 by varying the much simpler action

$$\tilde{S}[x^{\mu}(\lambda)] = \int g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu}d\lambda.$$

Basically just repeat proof from above with $L = g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu}$. This implies that a very fast way to compute Christoffel symbols is by writing down the E-L equations for the Lagrangian without the square root and read off the Christoffel symbols from the resulting geodesic equation.

Exercise 2.2. The metric on the two-sphere S^2 is $ds^2 = d\theta^2 + \sin^2\theta d\phi^2$. Compute the Christoffel symbols. Show that the lines of constant longitude (ϕ = constant) are geodesics. Show that the only line of constant latitude (θ = constant) which is a geodesic is the equator $\theta = \pi/2$.

Exercise 2.3. Consider the Schwarzchild metric, which is of the form $ds^2 = -e^{2\alpha(r)}dt^2 + e^{2\beta(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$. Compute the Christoffel symbols. Note that many will be zero! You can do this either using the definition or using the trick with the E-L equation (the latter will be much faster).

3 Einstein's Equation

Now that we have a force equation, we proceed to derive the field equation: Einstein's equation.

3.1 But First: Some Tensors

Recall the covariant derivative: It is a $(1,1)$-tensor that is:

- Linear: $\nabla_\mu(v^\rho + w^\rho) = \nabla_\mu v^\rho + \nabla_\mu w^\rho$
- Obeys product (Leibniz) rule: $\nabla_\mu(fv^\rho) = (\partial_\mu f)v^\rho + f(\nabla_\mu v^\rho)$

So to satisfy the above conditions, we define the covariant derivative as $\nabla_\mu v^\rho + \Gamma^\rho_{\mu\nu} v^\nu$, where v^ρ is a $(1,0)$-tensor. More generally,

We define the covariant derivative of a scalar $\nabla_\mu f = \partial_\mu f = \frac{\partial f}{\partial x^\mu}$.

Definition 3.1. We define the **covariant derivative** of a $(k,l)-$tensor as

$$\nabla_\alpha\left(T^{\mu_1,\cdots,\mu_k}{}_{\nu_1,\cdots,\nu_l}\right) = \partial_\alpha T^{\mu_1,\cdots,\mu_k}{}_{\nu_1,\cdots,\nu_l} + \Gamma_\alpha{}^{\mu_1}{}_\sigma T^{\sigma,\cdots,\mu_k}{}_{\nu_1,\cdots,\nu_l} + \cdots - \Gamma_\alpha{}^\rho{}_{\nu_1} T^{\mu_1,\cdots,\mu_k}{}_{\rho,\cdots,\nu_l} - \cdots$$

Definition 3.2. The **Riemann curvature tensor** $R^\rho{}_{\sigma\mu\nu}$ is defined by

$$R^\rho{}_{\sigma\mu\nu} = \partial_\mu \Gamma_\nu{}^\rho{}_\sigma - \partial_\nu \Gamma_\mu{}^\rho{}_\sigma + \Gamma_\mu{}^\rho{}_\lambda \Gamma_\nu{}^\lambda{}_\sigma - \Gamma_\nu{}^\rho{}_\lambda \Gamma_\mu{}^\lambda{}_\sigma$$

Note that $R^\rho{}_{\sigma\mu\nu}$ is a $(1,3)-$tensor, even though it is made up of Christoffel symbols which are not tensors. In fact, the Christoffel symbol is exactly what it needs to be in order to make $R^\rho{}_{\sigma\mu\nu}$ a tensor since in this exact combination, all non-tensorial terms cancel. Some properties of the Riemann curvature tensor:

- $R_{\rho\sigma\mu\nu} = -R_{\rho\sigma\nu\mu}$
- $R_{\rho\sigma\mu\nu} = -R_{\sigma\rho\mu\nu}$
- $R_{\rho\sigma\mu\nu} = R_{\mu\nu\rho\sigma}$
- $R_{[\rho\sigma\mu\nu]} = 0$
- Bianchi Identity: $\nabla_{[\lambda} R_{\rho\sigma]\mu\nu} = 0$

(You should read about these properties in more detail in Carroll.)

Definition 3.3. We can take the trace of the Riemann tensor to obtain the **Ricci tensor**

$$R_{\mu\nu} = R^\rho{}_{\mu\rho\nu}$$

$R_{\mu\nu} = 0$ is the vacuum Einstein equation.

Definition 3.4. We can contract the indices of the Ricci tensor to obtain the **Ricci scalar** $R = g^{\mu\nu} R_{\mu\nu}$.

Definition 3.5. The **Einstein tensor** is $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}$.

Notice that the Bianchi identity implies that $\nabla^{\mu} G_{\mu\nu} = 0$ i.e. the Einstein tensor is conserved. (c.f. continuity equation in electromagnetism).

3.2 *Einstein's Equation*

How can we formulate the laws of physics in curved spacetime? More specifically, how do we translate special relativity into general relativity. Recall the equivalence principle: near any point in spacetime, we can find an inertial coordinate system such that $g_{\mu\nu} \approx \eta_{\mu\nu} + \cdots$ and $\Gamma_{\rho}{}^{\mu}{}_{\nu} = 0 + \cdots$, so the natural way to proceed is:

1. Start with a law of physics in special relativity.
2. Interpret it as a tensorial law in curved spacetime, just written in inertial coordinates around some point.
3. This tensor equation is the correct law of physics in curved spacetime.

Essentially, we 'covariantise' every EoM: $\eta_{\mu\nu} \to g_{\mu\nu}$, and $\partial_{\mu} \to \nabla_{\mu}$. This idea is called the **principle of minimal coupling**. Recall the Newtonian field equation: $\nabla^2 \Phi = 4\pi G \rho$. For $v \varsigma < 1$, we expect the GR EoM to reduce to this form. Recall also that for intuition, we can associate $\Phi \tilde{g}_{\mu\nu}$ and $F = -\nabla \Phi = \Gamma$, the Christoffel symbol. We start with the $\rho = 0$ case: $\nabla^2 \Phi = 0 \implies$ we want a tensor with second order derivatives of the metric. The only tensor which is a second derivative of the metric is the Riemann curvature tensor. Well, naturally $R^{\rho}{}_{\sigma\mu\nu} = 0$ is an option, but this implies that spacetime must be flat (since it requires all Christoffel symbols to be 0). Our next option is to let its trace $R_{\mu\nu} = 0 \equiv G_{\mu\nu} = 0$. This is an equation for 10 unknowns. So in general, we look for an equation of the form $G_{\mu\nu} =$ 'stuff'.
There are ten symmetries of $\mathbb{R}^{(3,1)}$:

- Four translations
- Three rotations
- Three Lorentz boosts.

These ten symmetries packed together form the **stress tensor** $T_{\mu\nu}$. Notice also, that like $G_{\mu\nu}$, $T_{\mu\nu}$

For example, T_{00} is energy density, and T_{10} is momentum.

- Symmetric $T_{\mu\nu} = T_{\nu\mu}$
- Conserved $\nabla^{\mu} T_{\mu\nu} = 0$

So the general form of Einstein's equation is $G_{\mu\nu} = T_{\mu\nu} \cdot$ cst. Using our requirement that in the non-relativistic case, this EoM must reduce to the Newtonian case, we obtain that this constant is $8\pi G$, and so we obtain Einstein's equation:

$$G_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{3}$$

Lecture of May 27,2020

4 FRW Universe

Cosmological models are based on the idea that we are not special, i.e. that the universe looks the same everywhere. This idea is translated into two precise mathematical properties:

Definition 4.1. A manifold $\mathcal{M}$ is **isotropic**, if for all pairs of vectors v^{μ}, w^{μ} defined at a point, there exists an isometry which takes v to w. This means that the universe looks the same in any direction you look.

Definition 4.2. Homogeneity is the property that the spacetime metric is the same everywhere on the manifold.

Take the time to think how these two properties are independent i.e. you could have one but not the other. However, note that if a space is isotropic everywhere (i.e. not only at a single point), then it must be homogeneous. More importantly, if space is isotropic at one point and homogeneous everywhere, then it will be homogeneous everywhere. Since cosmological surveys do indeed observe that the Universe is isotropic from Earth, and we are not special (i.e. spacetime is the same everywhere), then it is very justified to assume that the Universe is spatially isotropic and homogeneous in our calculations.
These two properties are very useful because they imply maximal symmetry i.e. the space has its maximal number of Killing vectors.

Recall that v^{μ} is a **Killing vector** if it solves the Killing equation $\nabla_{\mu} v_{\nu} + \nabla_{\nu} v_{\mu} = 0$ i.e. its Lie derivative $\pounds_{v} g_{\mu\nu} = 0$. Every Killing vector stands for a symmetry.

4.1 RW Metric

We proceed to derive the RW metric for a spatially homogeneous and isotropic Universe which evolves in time. The metric thus looks like:

$$ds^2 = -dt^2 + a^2(t) d\sigma^2,$$

where $a(t)$ is the **scale factor**.
$d\sigma^2$ is the spatial component of the metric, and so we ask what it must look like so that it is maximally symmetric.
Well, it happens that there are only three symmetric spacetimes in four dimensions, namely flat space with zero curvature, de Sitter spacetime with positive curvature, and anti-de Sitter spacetime with negative curvature.
We therefore have that the **spatial** part of the metric i.e. $d\sigma^2$ is one of these three possible spaces in 3-D i.e. $\mathbb{S}^3, \mathbb{R}^3, \mathbb{H}^3$:

Definition 4.3. The **Robertson-Walker metric** is

$$ds^2 = -dt^2 + a^2(t)d\sigma_\kappa^2, \text{ where}$$

$$d\sigma_\kappa^2 = \begin{cases} ds^2_{\mathbb{S}^3} = d\chi^2 + \sin^2\chi d\Omega^2 & \kappa = +1 \\ ds^2_{\mathbb{R}^3} = d\chi^2 + \chi^2 d\Omega^2 & \kappa = 0 \\ ds^2_{\mathbb{H}^3} = d\chi^2 + \sinh^2\chi d\Omega^2 & \kappa = -1 \end{cases}$$

Note that $d\Omega^2 = d\theta^2 + \cos^2\theta d\phi^2$

Note that $ds^2_{\mathbb{R}^3}$ is just $dx^2 + dy^2 + dz^2$ in spherical coordinates.

In Carroll, they derive the metric differently, but obtain a more general form, where $d\chi = \frac{dr^2}{\sqrt{1-\kappa r^2}}$.

- Case $\kappa = +1$: spatial slices are spheres $\implies$ closed universe i.e. start at a point, walk in a 'straight' line, and you come back to the same place.

- Case $\kappa = -1$: spatial slices are hyperbolas $\implies$ open universe i.e. start at a point, walk in a 'straight' line, and you never come back to the starting point.

- Case $\kappa = 0$: spatial slices are flat $\implies$ flat universe.

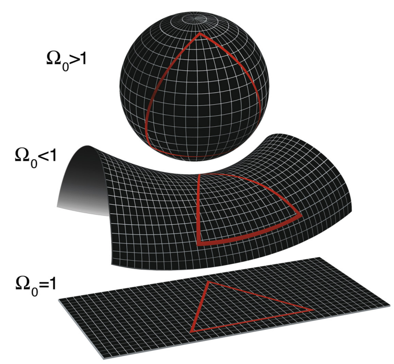

Figure 1: Three possible geometries for a maximally symmetric FRW universe with positive, negative, and zero curvature (top to bottom)

4.2 *Friedmann Equations*

The Friedmann equations give the conditions such that the RW-metric satisfies the Einstein equation. To derive the Friedmann equations, we simply write down Einstein's equation using the RW-metric. This means calculating $G_{\mu\nu}$, which in turn means calculating the Ricci tensor and scalar, and the Christoffel symbols for the Riemann tensor. For the stress tensor, we will work under the assumption that all matter is a perfect fluid, meaning that $T_{00} = \rho(t)$, and $T_{ij} = p(t)g_{ij}$.
Let's solve the entire problem for the $\kappa = 0$, and to practice, you can redo this entire process for the $\kappa \neq 0$ case.
so the metric we will consider in the following sections is:

$$ds^2 = -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2)$$

4.2.1 *Christoffel Symbols*

We will use the Euler-Lagrange method. Let's start by writing down the action:

$$S = \int ds = \int d\lambda \sqrt{-\dot{t}^2 + a^2(t)\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right)},$$

where λ is an affine parameter. So applying the result from Exercise 2.1, we can use Lagrangian $L = -\dot{t}^2 + a^2(t)\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right)$ to get the Christoffel symbols.

- For t: Apply E-L equation:

$$\begin{aligned} 0 &= \frac{\partial}{\partial t}\Big(-\dot{t}^2 + a^2(t)\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right)\Big) - \frac{d}{d\lambda}\left(\frac{\partial}{\partial \dot{t}}\Big(-\dot{t}^2 + a^2(t)\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right)\Big)\right) \\ &= 2a(t)\frac{\partial a}{\partial t}\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) + 2\ddot{t} \\ \implies & \ddot{t} + a(t)\frac{\partial a}{\partial t}\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) = 0 \end{aligned}$$

- For x:

$$\begin{aligned} 0 &= \frac{\partial}{\partial x}\Big(-\dot{t}^2 + a^2(t)\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right)\Big) - \frac{d}{d\lambda}\left(\frac{\partial}{\partial \dot{x}}\Big(-\dot{t}^2 + a^2(t)\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right)\Big)\right) \\ &= 0 - \frac{d}{d\lambda}\left(2a^2(t)\dot{x}\right) \\ &= -4a(t)\frac{\partial a}{\partial t}\dot{t}\dot{x} - 2a^2(t)\ddot{x} \\ \implies & \ddot{x} + \frac{2}{a(t)}\frac{\partial a}{\partial t}\dot{t}\dot{x} = 0 \end{aligned}$$

- For y, z: same as for x.

We read Christoffel symbols off these geodesic equations:

$$\boxed{\begin{array}{l} \Gamma_{i\ i}^{\ t} = \frac{\partial a}{\partial t}a(t) \qquad \Gamma_{t\ i}^{\ i} = \Gamma_{i\ t}^{\ i} = \frac{\partial a}{\partial t}\frac{1}{a(t)}, \qquad i \in \{1,2,3\}, \\ \text{All other Christoffel symbols are 0.} \end{array}}$$

4.2.2 *Riemann Tensor*

Recall the definition of the Riemann tensor in terms of derivatives of Christoffel symbols:

$$R^{\rho}{}_{\sigma\mu\nu} = \partial_\mu \Gamma_{\nu\ \sigma}^{\ \rho} - \partial_\nu \Gamma_{\mu\ \sigma}^{\ \rho} + \Gamma_{\mu\ \lambda}^{\ \rho}\Gamma_{\nu\ \sigma}^{\ \lambda} - \Gamma_{\nu\ \lambda}^{\ \rho}\Gamma_{\mu\ \sigma}^{\ \lambda}$$

To spare you a lot of trial and error, I will only write down the non-zero terms. So it turns out there are only two 'kinds' of non-zero

In general, one can think of some arguments to show that some Riemann tensor terms are zero. In this case, I didn't really have any idea how to know a priori that some terms are zero. I just calculated all of them.

terms:

$$R^i{}_{jij} = \partial_i \Gamma_j{}^i{}_j - \partial_j \Gamma_i{}^i{}_j + \Gamma_i{}^i{}_\lambda \Gamma_j{}^\lambda{}_j - \Gamma_j{}^i{}_\lambda \Gamma_i{}^\lambda{}_j = \Gamma_i{}^i{}_\lambda \Gamma_j{}^\lambda{}_j = \left(\frac{\partial a}{\partial t}\right)^2$$

$$R^t{}_{iti} = \partial_t \Gamma_i{}^t{}_i - \partial_i \Gamma_t{}^t{}_i + \Gamma_t{}^t{}_\lambda \Gamma_i{}^\lambda{}_i - \Gamma_i{}^t{}_\lambda \Gamma_t{}^\lambda{}_i = \partial_t \Gamma_i{}^t{}_i - \Gamma_i{}^t{}_\lambda \Gamma_t{}^\lambda{}_i = \frac{\partial^2 a}{\partial t^2} a(t)$$

$$\boxed{R^t{}_{jtj} = \tfrac{\partial^2 a}{\partial t^2} a(t); \qquad R^i{}_{jij} = \left(\tfrac{\partial a}{\partial t}\right)^2, \text{ where } i, j \in \{1, 2, 3\}.}$$

If anyone needs more details or explanation for the algebra here, do not hesitate to email me!

4.2.3 *Ricci Tensor and Scalar*

The Ricci tensor is the trace of the Riemann tensor:

- $R_{tt} = R^\rho{}_{t\rho t} = 3R^j{}_{tjt} = -\frac{3}{a(t)} \frac{\partial^2 a}{\partial t^2}$
- $R_{ii} = R^\rho{}_{i\rho i} = \frac{\partial^2 a}{\partial t^2} a(t) + 2\left(\frac{\partial a}{\partial t}\right)^2$
- All other terms are zero.

The Ricci scalar then:

$$R = g^{\mu\nu} R_{\mu\nu} = g^{tt} R_{tt} + g^{ii} R_{ii} = \frac{6}{a(t)} \frac{\partial^2 a}{\partial t^2} + \frac{6}{a^2(t)} \left(\frac{\partial a}{\partial t}\right)^2$$

4.2.4 *Einstein Tensor*

We finally compiled all the ingredients to compute the Einstein tensor $G_{\mu\nu}$:

- $G_{tt} = R_{tt} - \frac{1}{2} R g_{tt} = R_{tt} - \frac{1}{2} R = \frac{3}{a^2(t)} \left(\frac{\partial a}{\partial t}\right)^2$
- $G_{ii} = R_{ii} - \frac{1}{2} R g_{ii} = R_{ii} - \frac{1}{2} R a^2(t) = -2 \frac{\partial^2 a}{\partial t} a(t) - \left(\frac{\partial a}{\partial t}\right)^2$
- All other terms are zero.

4.2.5 *Friedmann Equations*

In the FRW universe, we model matter as a perfect fluid i.e. $T_{tt} = \rho(t)$ and $T_{ii} = p g_{ii} = p(t) a^2(t)$. From now on, we write $\frac{\partial a}{\partial t} = \dot{a}$. Applying Einstein's equation:

$$G_{tt} = \frac{3}{a^2} \dot{a}^2 = 8\pi G T_{tt} = 8\pi G \rho(t)$$

$$\boxed{\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3} G\rho(t)}$$

And now we get the second equation:

$$G_{ii} = -2\ddot{a}a(t) - \dot{a}^2 = 8\pi G T_{ii} = 8\pi G p(t) a^2(t)$$

We divide both sides by $a^2(t)$:

$$-2\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} = 8\pi G p(t)$$

Use the first Friedmann equation we got above:

$$-2\frac{\ddot{a}}{a} - \frac{8}{3}\pi G\rho = 8\pi G p(t)$$

$$\boxed{\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\big(\rho(t) + 3p(t)\big)}$$

Exercise 4.1. Redo the above steps for the most general RW metric:

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-\kappa r^2} + r^2 d^2\Omega\right]$$

You will obtain almost the same Friedmann equation:

$$\boxed{\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G\rho}{3} - \frac{\kappa}{a^2}}$$

The second equation is exactly the same as in the $\kappa = 0$ case. For some hints for the Christoffel symbols, and the other tensors, see the end of section 8.2 in Carroll.

The left hand side of the first Friedmann equation ($\dot{a}/a$) corresponds to the rate of expansion of the universe, which we call the Hubble parameter.

Definition 4.4. The **Hubble parameter** $H \equiv \frac{\dot{a}}{a}$.

The rate of expansion of the universe is one of the key quantities needed to test a cosmological theory. Astronomers can measure the Hubble parameter today i.e. $H(t_0) = H_0$ also known as the Hubble constant. We also define the critical density:

Definition 4.5. The **critical density** is $\rho_{crit} = \frac{3H^2}{8\pi G}$.

Now, our first Friedmann equation becomes:

$$H^2 = \frac{8\pi G\rho}{3} - \frac{\kappa}{a^2} \tag{4}$$

$$\implies 1 = \frac{\rho}{\rho_{crit}} - \frac{\kappa}{a^2 H^2} \tag{5}$$

This motivates us to define the dimensionless density parameter:

Definition 4.6. The **density parameter** Ω_c of some constituent c is $\Omega_c = \frac{\rho_c}{\rho_{crit}}$, ρ_c is the energy density of c.

We then obtain the final form of the Friedmann equation:

$$\Omega - 1 = \frac{\kappa}{a^2 H^2} \tag{6}$$

This equation then tells us that the curvature of the universe is determined by Ω:

- $\rho > \rho_{crit} \iff \Omega > 1 \iff \kappa > 0 \iff$ closed universe
- $\rho = \rho_{crit} \iff \Omega = 1 \iff \kappa = 0 \iff$ flat universe
- $\rho < \rho_{crit} \iff \Omega < 1 \iff \kappa < 0 \iff$ open universe

We thus have that calculating the Hubble parameter gives a value for the critical energy density, which then allows us to determine the curvature and therefore the ultimate fate of the universe. Because of this, calculating the Hubble parameter is a very serious endeavour in cosmology but is also somewhat controversial since different measurements have been in disagreement for a long time.

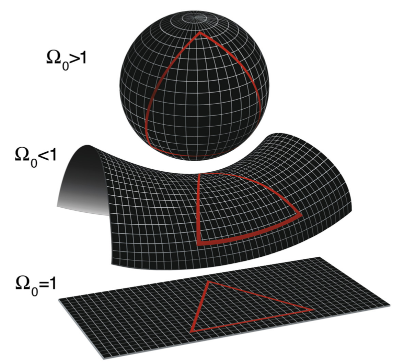

Figure 2: Three possible geometries for a maximally symmetric FRW universe with positive, negative, and zero curvature (top to bottom)

5 *The Cosmological Constant*

Definition 5.1. The **vacuum energy** is the energy density of empty space.

More details on this in Carroll Ch.4.

The vacuum energy density has been added to Einstein's equation because the absolute value of gravitational energy matters, not only the difference between states (like for the potential: you can add any constant to the potential and the motion is unchanged, so only the difference actually matters!). This means that we can assign a value of gravitational energy to empty space. An important property that we want this vacuum energy to have is that it is isotropic and Lorentz invariant. Einstein wanted to construct a model for a static universe, and for the static universe model to solve Einstein's equation, the introduction of the **cosmological constant** Λ was necessary, which turns out to be equivalent to adding vacuum energy. This cosmological constant modifies the Einstein equation to the following form:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{7}$$

This form is equivalent to adding a vacuum energy density $\rho_{vac} = \frac{\Lambda}{8\pi G}$.

5.1 *What Our Universe is Made of*

Recall that the stress tensor is conserved i.e. $\nabla^\mu T_{\mu\nu} = 0$ (see 3.2). Also recall that

$$T_{\mu\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & & & \\ 0 & & g_{ij}P & \\ 0 & & & \end{pmatrix} \implies T^\mu{}_\nu = \Big(-\rho, P, P, P \Big) \cdot \mathbb{1} \tag{8}$$

for a fluid. Conservation of the stress tensor yields the following Ordinary Differential Equation (ODE):

$$\nabla_\mu T^\mu{}_0 = 0 \implies \dot{\rho} + 3\frac{\dot{a}}{a}(\rho + P) = 0 \tag{9}$$

Here we use $\rho = \sum_i \rho_i$, similarly $\Omega = \sum_j \Omega_j$, where every ρ_i is a noninteracting ('decoupled') constituent of the universe e.g. matter, radiation, etc.

Solving this ODE using the assumption that the **equation of state** $w \equiv \frac{P}{\rho}$ is linear:

$$\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + P) = 0 \tag{10}$$

$$\frac{\dot{\rho}}{\rho} + 3\frac{\dot{a}}{a}(1 + w) = 0 \tag{11}$$

$$\implies \rho = \rho_0 a^{-3(1+w)} \tag{12}$$

To describe the stress energy tensor in more broad terms i.e. without assuming a theory of matter, we need to come up with some generic conditions for ρ and P to satisfy. Such conditions are called **energy conditions**. See Carroll Ch. 4 for more details.

- Radiation: Recall your electrodynamics class, where you described $T_{\mu\nu}$ in terms of the field strength F:

$$T^{\mu\nu} = F^{\mu\sigma} F^{nu}_{\ \ \sigma} - \frac{1}{4} g^{\mu\nu} F^{\sigma\lambda}_{\ \ \sigma\lambda} \tag{13}$$

  Taking the trace $T^\mu{}_\mu$ and setting it equal to $-\rho + 3P$ yields an equation of state parameter $w_r = 1/3$. We thus have: $\rho_r = \rho_{r,0} a^{-4}$.

- Matter: Matter is pressureless $\implies w_m = 0 \implies \rho_m = \rho_{m,0} a^{-3}$

- Vacuum: Vacuum has an equation of state parameter $w_\Lambda = -1$, which is fixed by enforcing Lorentz invariance.

- Curvature: Recall our Friendmann equation:

$$1 = \frac{\rho}{\rho_{crit}} - \frac{\kappa}{a^2 H^2} \tag{14}$$

  How about we make this equation a bit more compact by treating the curvature as another energy density term. Of course, the curvature is not an *actual* energy density.

The Planck Collaboration has used the ΛCDM model to obtain cosmological parameters of $\Omega_m = 0.3175, \Omega_\Lambda = 0.6825$ today.

$$\rho_\kappa = -\frac{3\kappa}{8\pi G a^2} \implies \Omega_\kappa = -\frac{\kappa}{H^2 a^2} \tag{15}$$

Conveniently, our Friedmann equation is now:

$$1 = \Omega + \Omega_\kappa = \Omega_r + \Omega_m + \Omega_\Lambda + \Omega_\kappa \tag{16}$$

Looking at each of $\rho_r, \rho_m, \rho_\Lambda$, we can see that at different stages of the universe, different components dominated.

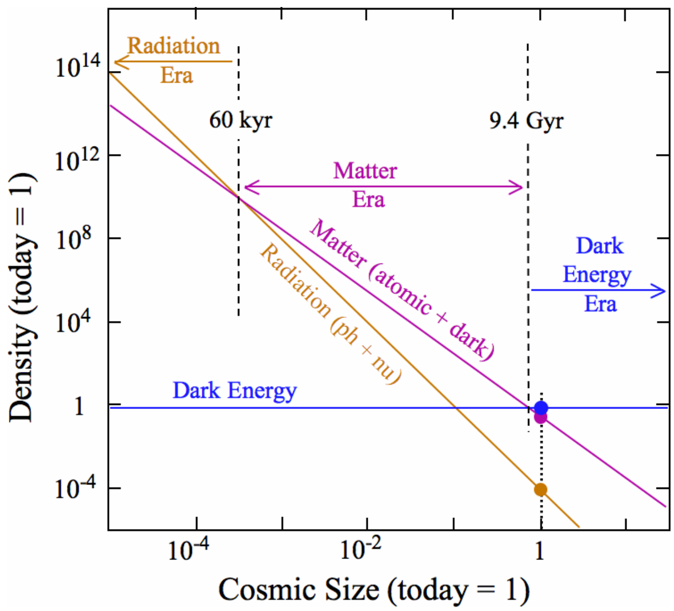

Figure 3: Radiation dominates the early universe, then matter takes over, and finally vacuum energy takes over.