Contents

MATH 480: Measures and Distributions

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Supervised by J. Vetois. Based on Real Analysis - Royden and Fitzpatrick. This course serves as an extension of the MATH 454 material into the material of MATH 455. It also serves as the basics for the study of distributions in Analysis by Loss and Lieb.

1 Chapter 1 - Measure and Integration

1.1 Introduction

Integration is one of the most important tools in analysis. Integration was first defined by the Riemann integral which, in spite of being convenient for many uses, is lacking for modern analysis. **The issue with the Riemann integral is that the class of Riemann integrable functions is not closed under the process of taking pointwise limits of sequences.** Consider an n-dimensional graph. We generally expect the integral to be the $n+1$-dimensional volume under the graph. **The question we are asking is how to define this volume.** The Riemann integral uses $n+1$-dimensional cubic towers with a base $\times$ height approach, where the height is a value of the function over that cube. **The issue is that the definition of this height is sometimes impossible to define given a sufficiently discontinuous function.** The Lebesgue integral solves this problem by computing the integral by instead calculating the n-dimensional volume where the function is greater than some number y. This volume is a well-behaved function of y which can then be Riemann integrated.

A more concrete example to illustrate this difference:

Suppose we want to calculate the volume of a mountain.

- The Riemann way:
  Partition the base of the mountain into $1m^2$ squares, and get the height of the mountain at the centre of each square. The volume of the mountain is the sum of all heights $\times 1m^2$.

- The Lebesgue way:
  Draw contours of the mountain that are $1m$ apart in altitude. Compute the area of each contour. The Lebesgue integral will be the sum of all the areas $\times 1m$.

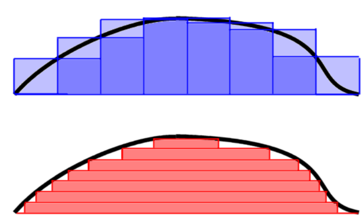


Figure 1: Riemann partition in blue, Lebesgue partition in red.

1.2 Basics of Measure Theory

In order to define the measure of a set, we first need to define what a measureable set is.

Definition 1.1. A distinguished collection, Σ, of subsets of a universal set Ω (e.g $\mathbb{R}^n$), is a **sigma-algebra** if:

1. If $A \in \Sigma$, then $A^c \in \Sigma$, where $A^c = \Omega \setminus A$.
2. If $A_1, A_2, \ldots \in \Sigma$, then their (countable) union $\bigcup_{i=1}^{\infty} A_i \in \Sigma$.
3. $\Omega \in \Sigma$.

Question 1.1. *How come 1. + 2. doesn't imply 3.? 3* $\iff \Omega \neq \emptyset$.

Remark 1. These imply that $\emptyset \in \Sigma$, and that Σ is also closed under countable intersections. Also $A_1 \setminus A_2 \in \Sigma$.

Remark 2. Any family $\mathcal{F}$ of subsets of Ω can be extended to a sigma-algebra by just adding all the other subsets of Ω. However,

Definition 1.2. A **measureable set** is an element of a sigma algebra.

Definition 1.3. A **measure** m, defined on a sigma-algebra Σ, is a function $m : \Sigma \to \mathbb{R} \bigcup \{\infty\}$ such that $m(\emptyset) = 0$ and countable additivity holds.

Definition 1.4. Countable additivity: If $A_1, A_2, \ldots$ is a sequence of disjoint sets in Σ, then

$$m\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} m(A_i) \tag{1}$$

Remark 3. Countable additivity is an **essential** property. Finite additivity is a special case of this requirement, and is not enough for integration theory.

Some important consequences of countable additivity:

$$m(A) \leq m(B) \qquad \text{if } A \subset B \tag{2}$$

$$\lim_{j\to\infty} m(A_j) = m\left(\bigcup_{i=1}^{\infty} A_i\right) \qquad \text{if } A_1 \subset A_2 \subset \cdots \tag{3}$$

$$\lim_{j\to\infty} m(A_j) = m\left(\bigcap_{i=1}^{\infty} A_i\right) \qquad \text{if } A_1 \supset A_2 \supset \cdots \text{ and } m(A_1) < \infty \tag{4}$$

2 *General Measure Spaces: Properties and Construction*

2.1 *Measures and Measurable sets.*

Definition 2.1. A **measurable space** $(X, \mathcal{M})$ consists of a set X and a σ-algebra $\mathcal{M}$ of the subsets of X. A set is **measurable** only if it belongs to $\mathcal{M}$.

For any set X, we define $\mathcal{M} = 2^X$ to be the collection of all subsets of X. We define a measure by first setting the measure of a finite set to simply be the number of elements in the set, and the measure of an infinite set to be ∞. We call this primitive measure the **counting measure** on X.

Definition 2.2. A **measure** μ on a measurable space $(X, \mathcal{M})$ is an extended real-valued nonnegative set function $\mu : \mathcal{M} \to [0, \infty]$ such that:

- $\mu(\emptyset) = 0$
- For any countable disjoint collection of measurable sets $\{E_j\}_{j\in\mathbb{N}}$,

$$\mu\Big(\bigcup_{j\in\mathbb{N}} E_j\Big) = \sum_{j\in\mathbb{N}} \mu(E_j) \text{ (countable additivity).}$$

Examples of measure spaces:

- ($\mathbb{R}$, Lebesgue measurable sets, Lebesgue measure)
- ($\mathbb{R}$, Borel sets, Lebesgue measure)

Notice that for any measure space $(X, \mathcal{M}, \mu)$, if $X' \in \mathcal{M}$, then $(X', \mathcal{M}', \mu')$ is also a measure space, where $\mathcal{M}'$ is the collection of all the subsets of $\mathcal{M} \subseteq X'$, and μ' is the restriction of μ to $\mathcal{M}'$.

Proposition 2.1. *Let $(X, \mathcal{M}, \mu)$ be a measure space.*

- *(Finite Additivity) For any finite disjoint collection $\{E_k\}_{k=1}^n$ of measurable sets $\mu\Big(\bigcup_{k=1}^n E_k\Big) = \sum_{k=1}^n \mu(E_k)$.*
- *(Monotonicity) If A and B are measurable sets and $A \subseteq B$ then $\mu(A) \leq \mu(B)$.*
- *(Excision) If A and B are measurable sets and $A \subseteq B$ with $\mu(A) < \infty$, then $m(B \setminus A) = \mu(B) - \mu(A)$*
- *(Countable Monotonicity) For any countable collection of measurable sets $\{E_k\}_{k\in\mathbb{N}}$ that covers a measurable set E, $\mu(E) \leq \sum_{k\in\mathbb{N}} \mu(E_k)$.*

Proof.
- From countable additivity: for any countable disjoint collection of measurable sets $\{E_j\}_{j\in\mathbb{N}}$, $\mu\Big(\bigcup_{j\in\mathbb{N}} E_j\Big) = \sum_{j\in\mathbb{N}} \mu(E_j)$. Let $E_j = \emptyset \forall j \geq n$, and we obtain finite additivity.
- Finite additivity implies monotonicity.
- Finite additivity implies excision as well.
- To show countable monotonicity, we first use monotonicity, then countable additivity, and then monotonicity again. We define a set $\{G_k\}_{k\in\mathbb{N}}$ such that $G_1 = E_1$, and then $G_k = E_k \setminus \left(\bigcup_{i=1}^{k-1} E_i\right) \forall k \geq 2$. This construction guarantees that the G_ks are disjoint, that $\bigcup G_k = \bigcup E_k$, and that $G_k \subseteq E_k$. We then obtain: $\mu(E) \leq \mu\left(\bigcup_{k=1}^\infty E_k\right) = \mu\left(\bigcup_{k=1}^\infty G_k\right) = \sum_{k=1}^\infty \mu(G_k) \leq \sum_{k=1}^\infty \mu(E_k)$.

Notice how countable monotonicity regroups countable additivity and monotonicity. $\square$

$\{E_k\}$ is ascending $\iff$ $E_k \subseteq E_{k+1}$. Similarly, descending is when $E_{k+1} \subseteq E_k$.

Proposition 2.2. ***Continuity of measure*** *Let $(X, \mathcal{M}, \mu)$ be a measure space.*

1. *If $\{A_k\}_{k\in\mathbb{N}}$ is an ascending sequence of measurable sets, then $\mu\big(\bigcup_{k=1}^{\infty} A_k\big) = \lim_{k\to\infty} \mu(A_k)$.*
2. *If $\{B_k\}_{k\in\mathbb{N}}$ is a descending sequence of measurable sets for which $\mu(B_1) < \infty$, then $\mu\big(\bigcap_{k=1}^{\infty} B_k\big) = \lim_{k\to\infty} \mu(B_k)$.*

Proof. See MATH 454 notes for the detailed proof. General steps are: break up nested sets into disjoint sets to be able to apply the properties above. (For first statement, there are two cases, one for finite and infinite $m(A_k)$.) First apply countable additivity and excision property. Insert limit by changing countable additivity to finite additivity. For the second statement, recall De Morgan's law that $\bigcap_k B_k = B_1 \setminus \big(\bigcup_k (B_1 \setminus B_k)\big)$, which is essentially the first statement. □

Definition 2.3. For a measure space $(X, \mathcal{M}, \mu)$, and a measurable subset E of X, let P be a statement. P holds **almost everywhere** in E if $\mu(\{x \in E : P(x) \text{ is False }\}) = 0$.

Lemma 2.3. ***Borel-Cantelli Lemma*** *Let $(X, \mathcal{M}, \mu)$ be a measure space and $\{E_k\}_{k\in\mathbb{N}}$ a countable collection of measurable sets for which $\sum_{k\in\mathbb{N}} \mu(E_k) < \infty$, then almost all $x \in X$ belong to at most a finite number of the E_ks.*

This is the classic proof technique for 'almost everywhere' statements. Only show that the measure of the set for which the statement doesn't hold is 0.

Proof. We simply need to show that the measure of the set for which this statement is false is 0. We therefore consider the set of $x \in X$ that belong to an infinite number of the E_ks: $\bigcap_{n=1}^{\infty}\big(\bigcup_{k=n}^{\infty} E_k\big)$. Since the union forms a descending sequence, we may apply continuity of measure, case 2, and finally countable monotonicity.

$$\bigcap_{n=1}^{\infty}\left(\bigcup_{k=n}^{\infty} E_k\right) = \lim_{n\to\infty} \mu\Big(\bigcup_{k=n}^{\infty} E_k\Big) \le \lim_{n\to\infty} \sum_{k=n}^{\infty} \mu(E_k) = 0$$

□

The counting measure on an uncountable set is not σ-finite.(σ-finite: if X is a countable union of measurable sets of finite measure.

Definition 2.4. A measure space $(X, \mathcal{M}, \mu)$ is **complete** if $\mathcal{M}$ contains all subsets of sets of measure zero, i.e if $E \in \mathcal{M}$ and $\mu(E) = 0$, then every subset of E also belongs to $\mathcal{M}$.

Lebesgue measure on the real line is complete. However, the Lebesgue measure on the real line, when restricted to the σ-algebra of Borel sets, is not complete. (We showed that the Cantor set, a Borel set of Lebesgue measure 0, contains a subset that is not Borel.)

The following proposition says that every measure space can be completed.

Proposition 2.4. *Let $(X, \mathcal{M}, \mu)$ be a measure space. Define $\mathcal{M}_0$ to be the collection of subsets E of X of the form $E = A \bigcup B$ where $B \in \mathcal{M}$ and $A \subseteq C$ for some $C \in \mathcal{M}$ for which $\mu(C) = 0$. For such a set E define $\mu_0(E) = \mu(B)$, then $\mathcal{M}_0$ is a σ-algebra that contains $\mathcal{M}$, mu_0 is a measure that extends μ, and $(X, \mathcal{M}_0, \mu_0)$ is a complete measure space.*

Proof. We're simply adding all the subsets of sets of measure 0 to $\mathcal{M}$. □

Let μ, η be measures defined on the same measurable space $(X, \mathcal{M})$. For $E \in \mathcal{M}$, define $\nu(E) = \max\{\mu(E), \eta(E)\}$. Is ν a measure on $(X, \mathcal{M})$? Check if definition of measure applies:

- $\nu(\emptyset) = 0$: $\nu(\emptyset) = \max\{\mu(\emptyset), \eta(\emptyset)\} = \max\{\emptyset, \emptyset\} = \emptyset$.
- For any countable disjoint collection of measurable sets $\{E_j\}_{j \in \mathbb{N}}$, $\nu(\bigcup_j E_j) = \sum_j \nu(E_j)$:

  Is it true that $\sum_j \nu(\bigcup_j E_j) = \max\{\sum_j \mu(E_j), \sum_j \eta(E_j)\}$?

  Obviously not, because the maximum is not additive i.e $\max(A \bigcup B) \neq \max(A) + \max(B)$. More specifically, if we let $A = \{1\}, B = \{2\}$, we obtain 2 on the left and 3 on the right.

2.2 *Signed Measures: the Hahn Decomposition*

We can construct a new measure on a measure space $(X, \mathcal{M})$ from two other measures on that same space simply by taking a linear combination of the two. However, if we took negative coefficients in our linear combination, what would happen? For example, given μ_1, μ_2 on $(X, \mathcal{M})$, let $\nu(S) = \alpha\mu_1(S) - \beta\mu_2(S) \forall S \in \mathcal{M}$. The first obvious consequences of such a definition are that ν is not always ≥ 0. Moreover, ν is undefined if both $\mu_1 = \mu_2 = \infty$.

Definition 2.5. A **signed measure** ν on the measurable space $(X, \mathcal{M})$ is an extended real-valued set function $\nu : \mathcal{M} \to [-\infty, \infty]$ with the following properties:

1. ν assumes at most one of the values $\pm\infty$.
2. $\nu(\emptyset) = 0$.
3. Countable additivity: for any countable collection of disjoint measurable sets, $\nu\left(\bigcup_k^\infty E_k\right) = \sum_k^\infty \nu(E_k)$, where the sum converges absolutely in the union is finite.

A measure is then a special case of a signed measure.

Definitions 2.6. A set S is **positive** wrt ν if S is measurable and for ever measurable subset E of S, $\nu(E) \geq 0$. Similarly, a set is **negative**

if it is measurable and every one of its measurable subsets has nonpositive measure. A measurable set S is **null** wrt ν if every one of its subsets has measure zero. (Note that a set of measure zero may not necessarily be null as it might have subsets of nonzero measures that cancel).

Since a signed measure is finite, for measurable sets A, B, we have that if $A \subseteq B$ and $|\nu(B)| < \infty$, then $|\nu(A)| < \infty$.

Proposition 2.5. *Let ν be a signed measure on the measurable space $(X, \mathcal{M})$, then every measurable subset of a positive set is also positive and the union of a countable collection of positive sets is positive.*

Proof.
- Every measurable subset of a positive set is also positive: True by the definition of positive set.
- The union of a countable collection of positive sets is positive: True by the definition of positive set and countable additivity (construct disjoint sets).

□

Lemma 2.6. ***Hahn's Lemma***
Let ν be a signed measure on $(X, \mathcal{M})$ and E a measurable set such that $) < \nu(E) < \infty$, then there is a measurable set $A \subseteq E$ that is positive and of positive measure.

Proof. We must construct a positive set $A \subseteq E$. We do so by removing all the sets of negative measure in E and calling that set A. Let m_1 be the smallest $m_1 \in \mathbb{N}$ such that there exists a measurable set $E_1 \subseteq E$ with $\nu(E_1) < -1/m_1$. We then choose a sequence of natural numbers $m_1, m_2, \ldots, m_n$ such that for every m_i, we associate a measurable set $E_i \subseteq E$ with $\nu(E_i) < -1/m_i$. Then if the process is finite, we have constructed the set A which must be positive by construction. However, if the process does not terminate, we define $A = E \setminus \bigcup_{k=1}^{\infty} E_k$ such that $E = A \bigcup \left(\bigcup_{k=1}^{\infty} E_k \right)$ is a disjoint decomposition of E. Since $\bigcup_k E_k$ is a measurable subset of E, and $|\nu(E)| < \infty$ (by finiteness of ν + countable additivity), we get $-\infty < \nu\left(\bigcup_{k=1}^{\infty} E_k \right) = \sum_{k=1}^{\infty} \nu(E_k) \leq \sum_{k=1}^{\infty} -1/m_k$, and $\lim_{k \to \infty} m_k = \infty$. In other words, we weed out all possible nonpositive sets in E to yield the positive set A. We can see that A is positive by taking a set $B \subseteq A$. Its measure $\nu(B) \geq -1/(m_k - 1) \to 0 \implies \nu(B) \geq 0$ as required. We must still show that $\nu(A) > 0$, but this is follows from finite additivity as $A = E \setminus (E \setminus A)$ and $\nu(E) > 0$, and $\nu(E \setminus A) < 0$. □

Theorem 2.7. ***The Hahn Decomposition Theorem***
Let ν be a signed measure on $(X, \mathcal{M})$, then there is a positive set A and a negative set B for which $X = A \bigcup B$ and $A \bigcap B = \emptyset$.

Proof. We proceed by constructing a positive set A by regrouping all subsets of positive measure, and then we obtain the negative set by simply subtracting A from X.
Recall from the definition of a signed measure that it can only assume one of $\pm\infty$. WLOG, suppose $\nu < \infty$. Then let $\mathbb{P}$ be the collection of all positive subsets of X and $\lambda = \sup\{\nu(E) | E \in \mathbb{P}\}$. Since $\emptyset \in \mathbb{P}$, then $\lambda \geq 0$. Let $\{A_k\}_{k \in \mathbb{N}}$ be a countable collection of positive sets for which $\lambda = \lim_{k \to \infty} \nu(A_k)$. We then let $A = \bigcup_k A_k$. We show that $\nu(A) = \lambda$:

- $\nu(A) \leq \lambda$: By the previous proposition, A is positive and so $\nu(A) \leq \lambda$.
- $\nu(A) \geq \lambda$: for all $k \in \mathbb{N}$, $A \setminus A_k \subseteq A \implies \nu(A \setminus A_k) \geq 0 \implies \nu(A) = \nu(A_k) + \nu(A \setminus A_k) \geq \nu(A_k)$.

Thus $\nu(A) = \lambda < \infty$ since $\nu < \infty$. Lastly, we let $B = X \setminus A$. To show B is negative, we use contradiction. Suppose it is not negative, then $\exists E \subseteq B$ such that $\nu(E) \geq 0$. Apply Hahn's Lemma on E: there must exist $E_0 \subseteq E$ that is positive and of positive measure. Then $A \bigcup E_0$ is also a positive set. But then $\nu(A \bigcup E_0) = \nu(A) + \nu(E_0) > \lambda$ which is a contradiction to the choice of λ. □

Decomposing a measure space $(X, \mathcal{M})$ into the union of two disjoint sets $X = A \bigcup B$, where one is positive and the other negative is called a **Hahn decomposition** for ν. The previous theorem tells us that a Hahn decomposition exists for every signed measure (albeit not unique: we can remove a null set E from A and paste it onto B to obtain a different Hahn decomposition). This is one way of decomposing a signed measure space. Notice, however, that we may go about it differently as well. For example, by decomposing the signed measure itself into two measures. This is the Jordan decomposition. Given a Hahn decomposition for ν, we define two measures ν^+, ν^- such that $\nu = \nu^+ - \nu^-$ and $\nu^+(E) = \nu(E \bigcap A)$ and $\nu^-(E) = -\nu(E \bigcap B)$.

Definition 2.7. Two measures ν_1, ν_2 on $(X, \mathcal{M})$ are **mutually singular** ($\nu_1 \perp \nu_2$) if there are disjoint measurable sets A and B with $X = A \bigcup B$ for which $\nu_1(A) = \nu_2(B) = 0$.

The measures ν^+ and ν^- are mutually singular.

Theorem 2.8. ***The Jordan Decomposition Theorem***
Let ν be a signed measure on $(X, \mathcal{M})$, then there are two mutually singular measures ν^+ and ν^- on $(X, \mathcal{M})$ for which $\nu = \nu^+ - \nu^-$. Moreover, there is only one such pair of mutually singular measures.

Remark 4. We proved existence above by using the Hahn decomposition.

Show that if E is any measurable set, then

$$-\nu^-(E) \leq \nu(E) \leq \nu^+(E) \text{ and } |\nu(E)| \leq |\nu|(E)$$

- $-\nu^-(E) \leq \nu(E) \leq \nu^+(E)$: we previously showed that for a measure ν, we can define two (nonnegative) measures ν^+, ν^- such that $\nu(E) = \nu^+(E) - \nu^-(E)$ for any measurable set E. We then have that clearly $\nu(E) \leq \nu^+(E)$. Multiplying the equality by -1, we obtain that $-\nu(E) = \nu^-(E) - \nu^+(E)$, which also implies that $-\nu(E) \leq \nu^-(E)$. Multiply this by -1 again: $\nu(E) \geq -\nu^-(E)$. Putting these two together, we get the desired result that $-\nu(E) \leq \nu(E) \leq \nu^+(E)$.

- $|\nu(E)| \leq |\nu|(E)$: $|\nu|(E) = \nu^+(E) + \nu^-(E)$. On the other hand, $|\nu(E)| = |\nu^+(E) - \nu^-(E)|$. By the triangle inequality, we have that $|\nu^+(E) - \nu^-(E)| \leq |\nu^+(E)| + |\nu^-(E)| \implies |\nu(E)| \leq |\nu|(E)$.

Recall that $|\mu| = \mu^+ + \mu^-$.

Show that if ν_1 and ν_2 are any two finite signed measures, then so is $\alpha\nu_1 + \beta\nu_2$, where α and β are real numbers. Show that $|\alpha\nu| = |\alpha||\nu|$ and that $|\nu_1 + \nu_2| \leq |\nu_1| + |\nu_2|$. ($\nu \leq \mu \implies \nu(E) \leq \mu(E) \forall$ measurable sets E)

- $\mu = \alpha\nu_1 + \beta\nu_2$ is a measure. (In comparison to the first exercise with max of two measures, here μ is most certainly a measure since countable additivity is preserved by +.)
  - $\mu(\emptyset) = \emptyset$: $\mu(\emptyset) = \alpha\nu_1(\emptyset) + \beta\nu_2(\emptyset) = \alpha \cdot 0 + \beta \cdot 0 = 0$.
  - Countable additivity: given a countable disjoint collection of measurable sets $\{E_j\}_{j\in\mathbb{N}}$, $\mu(\bigcup_j E_j) = \alpha\nu_1(\bigcup_j E_j) + \beta\nu_2(\bigcup_j E_j) = \alpha\sum_j \nu_1(E_j) + \beta\sum_j \nu_2(E_j) = \sum_j \left(\alpha\nu_1(E_j) + \beta\nu_2(E_j)\right) = \sum_j \mu(E_j)$

- $|\alpha\nu| = |\alpha||\nu|$: Let $\nu = \nu^+ - \nu^-$, then $|\alpha\nu| = |\alpha\nu^+ - \alpha\nu^-| = |\alpha|\nu^+ + |\alpha|\nu^- = |\alpha|(\nu^+ + \nu^-) = |\alpha||\nu|$

- $|\nu_1 + \nu_2| \leq |\nu_1| + |\nu_2|$: First, note that this must be true for all measurable sets E since $\nu(E) \in \mathbb{R}$, and the triangle inequality holds in $\mathbb{R}$. More specifically, notice that $|\nu_1 + \nu_2| = |\nu_1^+ - \nu_1^- + \nu_2^+ - \nu_2^-| \leq \nu_1^+ + \nu_1^- + \nu_2^+ + \nu_2^- = |\nu_1| + |\nu_2|$.

3 *The Caratheodory Measure Induced by an Outer Measure*

We explore the general concept of an outer measure and of measurability of a set with respect to an outer measure. Caratheodory came up with a strategy to construct the Lebesgue measure on the real line.

We will show that this strategy works in general as well.

Definition 3.1. A set function $\mu : S \to [0, \infty]$ defined on a collection $\mathcal{S}$ of subsets of a set X is **countably monotone** if whenever a set $E \in \mathcal{S}$ is covered by a countable collection $\{E_k\}_{k\in\mathbb{N}}$ in $\mathcal{S}$, then $\mu(E) \leq \sum_{k=1}^{\infty}\mu(E_k)$. If moreover $\emptyset \in \mathcal{S}$ and $\mu(\emptyset) = 0$, then μ is **finitely monotone**.

We need $\mu(\emptyset) = 0$ so that we can set $E_k = \emptyset$ for $k > n$.

Definition 3.2. A set function $\mu^* : 2^X \to [0, \infty]$ is an **outer measure** if $\mu^*(\emptyset) = 0$ and μ^* is countably monotone.

We previously showed that monotonicity + countable additivity $\implies$ countable monotonicity.

Just like in our study of Lebesgue measure on $\mathbb{R}$, we define measurable subsets with respect to an outer measure:

Definition 3.3. For an outer measure $\mu^* : 2^X \to [0, \infty]$, a set $E \subseteq X$ is **measurable wrt** μ^* if for every set $A \subseteq X$, $\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c)$.

Just like for Lebesgue measurability, the monotonicity of the outer measure implies that we only need to show that $\mu^*(A) \geq \mu^*(A \cap E) + \mu^*(A \cap E^c)$ holds to have $E \subseteq X$ measurable.

Proposition 3.1. *The union of a finite collection of measurable sets is measurable.*

Proof. Idea: Show for two sets with monotonicity and set identities, and use induction. See MATH 454 notes for details. □

Proposition 3.2. *Let $A \subseteq X$ and $\{E_k\}_{k=1}^n$ be a finite disjoint collection of measurable sets. Then $\mu^*\bigg(A \cap \big(\cup_{k=1}^n E_k\big)\bigg) = \sum_{k=1}^n \mu^*(A \cap E_k)$ i.e the restriction of μ^* to the collection $\{E_k\}$ is finitely additive.*

Proof. By induction. □

Proposition 3.3. *The union of a countable collection of measurable sets is measurable.*

Proof. After rewriting it as a collection of disjoint sets, let $E = \cup_{k=1}^{\infty} E_k$ and $F_n = \cup_{k=1}^{n} E_k$, then by monotonicity $\mu^*(A) = \mu^*(A \cap F_n) + \mu^*(A \cap F_n^c) \geq \mu^*(A \cap F_n) + \mu^*(A \cap E^c)$, then given the fact that a finite collection of measurable sets is measurable: $\mu^*(A) \geq \sum_{k=1}^n \mu^*(A \cap E_k) + \mu^*(A \cap E^c)$, but the left side is independent of n, so we can let $n \to \infty$: $\mu^*(A) \geq \sum_{k=1}^{\infty} \mu^*(A \cap E_k) + \mu^*(A \cap E^c)$, and countable monotonicity gives $\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c)$ as desired. □

Theorem 3.4. *Let μ^* be an outer measure on 2^X, then the collection $\mathcal{M}$ of sets that are measurable wrt μ^* is a $\sigma-$algebra. If $\overline{\mu}$ is the restriction of μ^* to $\mathcal{M}$, then $(X, \mathcal{M}, \overline{\mu})$ is a complete measure space.*

4 The Construction of Outer Measures

To define the Lebesgue measure, we constructed Lebesgue outer measure on subsets of the real line by first defining a set function that assigns length to a bounded interval. We then defined the outer measure of a set to be the infimum of sums of lengths of countable collections of bounded intervals that cover the set. We then claim that if a set is Lebesgue measurable then its Lebesgue measure is equal to its outer measure. This construction works in general as well.

Theorem 4.1. *Let $\mathcal{S}$ be a collection of subsets of a set X and $\mu : \mathcal{S} \to [0, \infty]$ a set function. Define $\mu^*(\emptyset) = 0$, and for $E \subseteq X, E \neq 0$, define*

$$\mu^*(E) = \inf\left\{ \sum_{k=1}^{\infty} \mu(E_k) : E_k \textit{ open, bounded and } E \subseteq \bigcup_{k \in \mathbb{N}} E_k \right\}$$

$\mu^* : 2^X \to [0, \infty]$ is the outer measure induced by μ.

Proof. To check that this is an outer measure, we must check that countable monotonicity holds, since we define $\mu^*(\emptyset) = 0$. We have two cases to consider:

- $\mu^*(E_k) = \infty$ for some k: $\mu^*(E) \leq \infty$.
- $\mu^*(E_k) < \infty$: By the definition of infimum, $\forall \varepsilon > 0 \exists \{E_{ik}\}_{i=1}^{\infty}$ such that $E_k \subseteq \cup_{i \in \mathbb{N}} E_{ik}$ and $\sum_{i=1}^{\infty} \mu(E_{ik} < \mu^*(E_k) + \varepsilon 2^{-k}$. Then

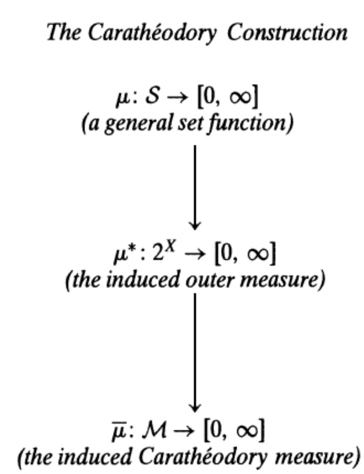
$$\begin{aligned} \mu^*(E) &\leq \sum_{k=1}^{\infty} \sum_{i=1}^{\infty} \mu(E_{ik}) \\ &\leq \sum_{k=1}^{\infty} \mu^*(E_k) + \varepsilon 2^{-k} \end{aligned}$$

As we let $\varepsilon \to 0$, we obtain countable monotonicity.

□

$\inf\{\emptyset\} = \infty$ so if $E \subseteq X$ cannot be covered by a countable collection of sets in $\mathcal{S}$ has outer measure ∞.

Definition 4.1. Let $\mathcal{S}$ be a collection of subsets of X, and $\mu : \mathcal{S} \to [0, \infty]$ a set function with outer measure induced by μ. The restriction of μ^* to the $\sigma-$algebra $\mathcal{M}$ of μ^*-measurable sets is the **Caratheodory measure induced by** μ, denoted $\overline{\mu}$.



In MATH 454, we showed that a Lebesgue measurable set A can be approximated by F_σ and G_δ sets whose measure is equal to $m(A)$. We now prove a similar result for general measures. In general, sets that are countable unions of sets in some collection $\mathcal{S}$ are called $\mathcal{S}_\sigma$, and those that are countable intersections $\mathcal{S}_{\sigma\delta}$.

G_δ is the real line equivalent of $\mathcal{S}_{\sigma\delta}$.

Proposition 4.2. *(Outer Approximation) Let $\mu : \mathcal{S} \to [0, \infty]$ be a set function defined on a collection $\mathcal{S}$ of subsets of a set X and $\overline{\mu} : \mathcal{M} \to [0, \infty]$ the Caratheodory measure induced by μ. Let $E \subseteq X, \mu^*(E) < \infty$, then $\exists A \subseteq X$ such that*

$$A \in \mathcal{S}_{\sigma\delta}, E \subseteq A, \text{ and } \mu^*(E) = \mu^*(A).$$

Moreover, if E and every set in $\mathcal{S}$ are measurable wrt μ^* then so is A and $\overline{\mu}(A \setminus E) = 0$.

Proof. Let $\epsilon > 0$. By the definition of outer measure, there is a countable collection of open bounded sets E_k which covers A such that $\sum_{k=1}^{\infty} \mu^*(E_k) < \mu(E) + \epsilon$. Let $\mathcal{O}_\epsilon = \bigcup_{k=1}^{\infty} E_k$. Then $A \subseteq \mathcal{O}_\epsilon$ and:

$$\mu(\mathcal{O}_\epsilon) \leq \sum_{k=1}^{\infty} \mu^*(E_k) < \mu(E) + \epsilon$$

Then for E bounded, excision yields that: $\mu(\mathcal{O} \setminus E) \leq \epsilon$. Finally, define $A = \cap_{k=1}^{\infty} A_{1/k}$, then $A \in \mathcal{S}_{\sigma\delta}$ and $E \subseteq A$. Monotonicity yields that $\mu^*(E) \leq \mu^*(A) \leq \mu^*(A_{1/k} \leq \mu^*(E) + \frac{1}{k}$ for all $k \in \mathbb{N}$, and thus $\mu^*(A) = \mu^*(E)$. Excision property yields that $\mu^*(A \setminus E) = 0$. ☐

On the collection $\mathcal{S}$ of all subsets of $\mathbb{R}$, define the set function $\mu : \mathcal{S} \to \mathbb{R}$ by setting $\mu(A)$ to be the number of integers in A. Determine the outer measure μ^* induced by μ and the $\sigma-$algebra of measurable sets.

- What is the outer measure induced by μ?
  Since for $E \subseteq \mathbb{R}$,

  $$\mu^*(E) = \inf \left\{ \sum_{k=1}^{\infty} \mu(E_k) : E_k \subseteq \mathbb{R} \text{ open, bounded, and } E \subseteq \bigcup_{k=1}^{\infty} E_k \right\}$$

  If there is an integer in some $E \subseteq \mathbb{R}$, then $\mu^*(E)$ will be 1 since μ returns the number of integers in an interval. Thus in general, we have that $\mu^* : E \to \mathbb{N} \cup \{0\}$, where $E \subseteq \mathbb{R}$.

- What is the $\sigma-$algebra of measurable sets.
  We want to find all sets A such that $\mu^*(B) = \mu^*(B \setminus A) + \mu^*(B \cap A)$ holds for all $B \subseteq \mathbb{R}$. The $\sigma-$algebra of measurable sets must be $\mathcal{S}$, the collection of all subsets of $\mathbb{R}$ since the number of integers in a set is preserved by set operations i.e if B has n integers, and they have k integers in common, then

  $$\begin{aligned} \mu^*(B) &= \mu^*(B \setminus A) + \mu^*(B \cap A) \\ n &= (n-k) + k \\ n &= n \end{aligned}$$

Let $\mathcal{S}$ be a collection of subsets of X and $\mu : \mathcal{S} \to [0, \infty]$ a set function. Is every set in $\mathcal{S}$ measurable with respect to the outer measure induced by μ?
A set A is measurable with respect to the outer measure induced by μ if for any set $B \subseteq X, \mu^*(B) = \mu^*(B \setminus A) + \mu^*(B \cap A)$.
So in the context of Lebesgue measure, this question is equivalent to asking is every set measurable, and the answer is no. There exist non-measurable sets that do not satisfy this equality like the Vitali set. The set of all cosets of $\mathbb{Q}$ in $\mathbb{R}$ is the quotient group $\mathbb{R}/\mathbb{Q}$. Every coset has a nonempty intersection with the unit interval. The Vitali set is a choice set of representatives of each coset in the unit interval. It is non-measurable because by translation invariariance we expect the measure of the set to be between 1 and 3, since it is a subset of translates of the unit interval, but this cannot be true by countable additivity. Thus if we let for example $A = [0, 1]$, then $\mu^*(A \cap V) + \mu^*(A \setminus V) = \mu^*(V) + \mu^*(A \setminus V) > \mu^*(A) = 1$.

5 *The Caratheodory-Hahn Theorem: The Extension of a Premeasure to a Measure*

Given a set function $\mu : \mathcal{S} \to \overline{\mathbb{R}}$ defined on a nonempty collection $\mathcal{S}$ of subsets of a set X, we have defined conditions for a set to be measurable in terms of the outer measure, but it yet remains for us to define the relation between the measure and the outer measure of a measurable set. In the case of Lebesgue measure, we had that if we find a set to be measurable with the outer measure, then we claim its Lebesgue measure is equal to its outer measure. But in a more general case, it remains to find the properties of μ and $\mathcal{S}$ such that the Caratheodory measure $\overline{\mu}$ induced by μ is an extension of μ. Pictorially, we want to find the conditions such that the dashed arrow in Figure 2 holds.

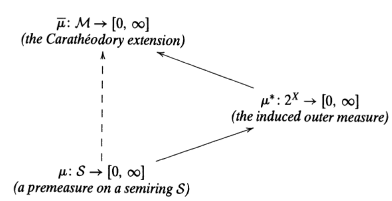


Figure 2: The Caratheodory Construction extends a premeasure on a semiring to a measure.

Proposition 5.1. *Let $\mathcal{S}$ be a collection of subsets of a set X and $\mu : \mathcal{S} \to [0, \infty]$ a set function. For the Caratheodory measure induced by μ to be an extension of μ, it is necessary that μ be both finitely additive and countable monotone and, if $\emptyset \in \mathcal{S}$, then $\mu(\emptyset) = 0$.*

Definition 5.1. Let $\mathcal{S}$ be a collection of subsets of a set X and $\mu : \mathcal{S} \to [0, \infty]$ a set function. Then μ is a **premeasure** if it is both finitely additive and countable monotone and, if $\emptyset \in \mathcal{S}$, then $\mu(\emptyset) = 0$

Notice that being a premeasure is a necessary but not sufficient condition for the Caratheodory measure induced by μ to be an extension of μ. However, if we impose some additional conditions on $\mathcal{S}$, this necessary condition becomes sufficient.

Definition 5.2. A collection $\mathcal{S}$ of subsets of X is **closed with respect to the formation of relative complements** $\iff$ A and $B \in \mathcal{S} \implies$ the relative complement $A \setminus B \in \mathcal{S}$. The collection S is **closed with respect to the formation of finite intersections** $\iff$ A and $B \in \mathcal{S} \implies A \cap B \in \mathcal{S}$.

Notice that closed with respect to the formation of relative complements $\implies$ closed with respect to the formation of finite intersections by defnition of complement.

Also notice that we still have $\emptyset \in \mathcal{S} \because A \setminus A = \emptyset \in \mathcal{S}$.

Theorem 5.2. *Let $\mu : \mathcal{S} \to [0, \infty]$ be a premeasure on a collection of sets $\mathcal{S}$ subset of X. If μ is closed wrt to the formation of relative complements, then the Caratheodory measure induced by μ, $\overline{\mu} : \mathcal{M} \to [0, \infty]$, is the Caratheodory extension of μ.*

Proof. We want to show $\mu(A) = \overline{\mu}(A)\ \forall A \in \mathcal{S}$.

- We first show $\mu(A) = \mu^*(A) \forall A \in \mathcal{S}$. This amounts to showing that A is measurable wrt μ^*. Let $E \subseteq X$ be any set with $m^*(E) < \infty$. We want to show that $\mu^*(E) \geq \mu^*(E \cap A) + \mu^*(E \setminus A) = \mu^*(E \cap A) + \mu^*(E \cap A^c)$ By the definition of outer measure, we have that for all $\epsilon > 0$, there exists an open cover $\{E_k\}_{k=1}^{\infty} \in \mathcal{S}$ such that $E \subseteq \bigcup_{k \in \mathbb{N}} E_k$, and $\mu^*(E) + \epsilon \geq \sum_{k=1}^{\infty} \mu^*(E_k)$
  $\mathcal{S}$ closed wrt formation of relative complements $\implies \forall k,\ E_k \cap A, E_k \cap A^c \in \mathcal{S}$. Since a premeasure is finitely additive, we have:
  $$\mu(E_k) = \mu(E_k \cap A) + \mu(E_k \cap A^c)$$
  Sum this over k: $\sum_{k=1}^{\infty} \mu(E_k) = \sum_{k=1}^{\infty} \mu(E_k \cap A) + \sum_{k=1}^{\infty} \mu(E_k \cap A^c)$
  Since $\sum_{k=1}^{\infty} \mu(E_k \cap A) \geq m^*(E \cap A)$ and $\sum_{k=1}^{\infty} \mu(E_k \cap A^c) \geq m^*(E \cap A^c)$ We obtain that indeed $\mu^*(E) \geq \mu^*(E \cap A) + \mu^*(E \setminus A)$
- Show $\mu^*(A) = \overline{\mu}(A)$. Since we now know that every set in $\mathcal{S}$ is measurable, then we also have that $\mu^*(A) = \overline{\mu}(A)$ for all $A \in \mathcal{S}$.
- Hence $\mu(A) = \mu^*(A) = \overline{\mu}(A)$

□

Many premeasures such as the length ℓ introduced in Lebesgue measure are defined on collections of sets that are not closed wrt formation of relative complements. This is why we need to introduce the semiring. A semiring $\mathcal{S}$ has the property that every premeasure defined on it has a unique extension to a premeasure on a collection of sets that is closed wrt the formation of relative complements. This is precisely what we are looking for: premeasures on semirings are extended by their induced Caratheodory measure.

Definition 5.3. A nonempty collection $\mathcal{S}$ of subsets of a set X is a **semiring** $\iff$ A and $B \in \mathcal{S} \implies A \cap B \in S$ and there exists a finite disjoint collection $\{C_k\}_{k=1}^n \in \mathcal{S}$ such that $A \setminus B = \bigcup_{k=1}^n C_k$.

Proposition 5.3. *Let $\mathcal{S}$ be a semiring of subsets of a set X. Define $\mathcal{S}'$ to be the collection of unions of finite disjoint collections of sets in $\mathcal{S}$. Then $\mathcal{S}'$ is closed wrt the formation of relative complements. Furthermore, any premeasure on $\mathcal{S}$ has a unique extension to a premeasure on $\mathcal{S}'$.*

Proof.
- We show that $\mathcal{S}'$ is closed wrt the formation of relative complements. Immediate from definition of $\mathcal{S}'$ and semiring.
- We show that the premeasure on $\mathcal{S}$ has a unique extension to a premeasure on $\mathcal{S}'$. We proceed by contradiction.

□

Theorem 5.4. ***(Caratheodory-Hahn Theorem)*** *Let $\mu : \mathcal{S} \to [0, \infty]$ be a premeasure on a semiring $\mathcal{S}$ of subsets of X. Then the Caratheodory measure $\overline{\mu}$ induced by μ is an extension of μ. Furthermore, if μ is σ-finite, then so is $\overline{\mu}$ and $\overline{\mu}$ is the unique measure on the σ-algebra of μ^*-measurable sets that extends μ.*

Proof. Notice that by the previous proposition and theorem, we have that the Caratheodory measure $\overline{\mu}$ induced by μ is an extension of μ. It only remains to show uniqueness. As usual, we proceed by contradiction. Suppose there exists another measure $\tilde{\mu}$ that extends μ. Then we want to show that $\overline{\mu}(E) = \tilde{\mu}(E)$. □

If we start with an outer measure μ^* on 2^X and form the induced measure $\overline{\mu}$ on the μ^*-measurable sets, we can view $\overline{\mu}$ as a set function and denote by μ^+ the outer measure induced by $\overline{\mu}$.

- Show that for each set $E \subseteq X$ we have $\mu^+(E) \geq \mu^*(E)$.
  Let $\mathcal{M}$ denote the σ-algebra of all μ^*-measurable sets. Since μ^* is an extension of $\overline{\mu}$,

$$\mu^+(E) = \inf\left\{\sum_{k=1}^{\infty} \overline{\mu}(E_k) : E_k \in \mathcal{M}, E \subseteq \bigcup_{k=1}^{\infty} E_k\right\}$$
$$= \inf\left\{\sum_{k=1}^{\infty} \mu^*(E_k) : E_k \in \mathcal{M}, E \subseteq \bigcup_{k=1}^{\infty} E_k\right\}$$

$$\forall k \in \mathbb{N} \ \exists \{E_k\}_{k\in\mathbb{N}}\ s.t\ E \subseteq \bigcup_{k\in\mathbb{N}} E_k \text{ and } \mu^+(E) + 2^{-k} \geq \sum_{k=1}^{\infty} \mu^*(E_k)$$
$$\geq \sum_{k=1}^{\infty} \mu^*(E_k) - 2^{-k}$$
$$\geq \mu^*(E)$$

Since $\mu^*(E)$ is the infimum of all such sums.

- For a given set E, show that $\mu^+(E) = \mu^*(E)$ if and only if there is a μ^*-measurable set $A \supseteq E$ with $\mu^*(A) = \mu^*(E)$.
  - ($\implies$) : Assume $\mu^+(E) = \mu^*(E)$ i.e E is μ^+-measurable. Again since μ^* is an extension of $\overline{\mu}$:

$$\mu^+(E) = \inf\left\{\sum_{k=1}^{\infty} \overline{\mu}(E_k) : E_k \in \mathcal{M}, E \subseteq \bigcup_{k=1}^{\infty} E_k\right\}$$
$$= \inf\left\{\sum_{k=1}^{\infty} \mu^*(E_k) : E_k \in \mathcal{M}, E \subseteq \bigcup_{k=1}^{\infty} E_k\right\}$$

    We need to prove inner approximation by F_σ sets. The proof is exactly the same as in MATH 454: We first prove outer approximation by open sets and G_δ sets as in Proposition 4.2. Then, we take the complement to obtain inner approximation: Let $A = \cup_{k=1}^{\infty} F_k$, where $F_k \subseteq E$ and $\mu^*(E \setminus f_k) \leq 1/k$. We thus have that $\mu^*(A) = \mu^*(E)$ with $A \subseteq E$ and A is μ^*-measurable.
  - ($\impliedby$) : By definition,

$$\mu^+(E) = \inf\left\{\sum_{k=1}^{\infty} \overline{\mu}(E_k) : E_k \in \mathcal{M}, E \subseteq \bigcup_{k=1}^{\infty} E_k\right\}$$
$$= \inf\left\{\sum_{k=1}^{\infty} \mu^*(E_k) : E_k \in \mathcal{M}, E \subseteq \bigcup_{k=1}^{\infty} E_k\right\}$$

    By assumption, we know that there exists $A \subseteq E$ such that $\mu^*(E) = \mu^*(A)$, where A is μ^*-measurable. It follows that $\mu^+(E) = \mu^*(E)$.

6 Integration Over General Measure Spaces

6.1 Radon-Nikodym Theorem

Theorem 6.1. *Let $(X, \mathcal{M}, \mu)$ be a $\sigma-$finite measure space and ν a $\sigma-$finite measure defined on the measurable space $(X, \mathcal{M})$ that is absolutely continuous with respect to μ. Then there is a nonnegative function f on X that is measurable with respect to $\mathcal{M}$ for which*

$$\int_X f d\mu > 0 \text{ and } \forall E \in \mathcal{M}, \int_E f d\mu = \nu(E)$$

We call $f = \frac{d\nu}{d\mu}$ the Radon-Nikodym derivative.

Note that the Radon-Nikodym derivative only exists if the second measure is absolutely continuous with respect to the first. The Radon-Nikodym derivative is useful because it gives a representation of a measure as the integral of a function—that function being the derivative (almost everywhere).

We know, in general, that for any measurable set A, $\mu(A) = \int_A \chi_A d\mu$. Now the theorem says that if we have a nonnegative function f on X, then we can define a new measure ν such that $\nu(A) \equiv \int_A f d\mu$, where f is the Radon-Nikodym derivative.

In the special case of a discrete probability measure, the Radon-Nikodym derivative with respect to counting measure is just the probability mass function.

Corollary 6.2. *Let $(X, \mathcal{M}, \mu)$ be a $\sigma-$finite measure space and ν a finite signed measure on the measurable space $(X, \mathcal{M})$ that is absolutely continuous wrt μ. Then there is a function f that is integrable over X wrt μ and $\nu(E) = \int_E f d\nu \ \forall E \in \mathcal{M}$*

Let $X = [0,1]$, $\mathcal{M}$ the collection of Lebesgue measurable subsets of $[0,1]$, and take ν to be Lebesgue measure and μ the counting measure of $\mathcal{M}$. Show that ν is finite and absolutely continuous with respect to μ, but there is no (nonnegative) function f for which $\nu(E) = \int_E f d\mu$ for all $E \in \mathcal{M}$.

Proof. By contradiction, suppose such f exists and $\nu(E) = \int_E f d\mu$ for all $E \in \mathcal{M}$. Then for $E = [0,1]$, we have that

$$\nu(E) = \nu([0,1]) = 1 = \int_{[0,1]} f d\mu$$

However, if we let $E = \{x\} \in \mathcal{M}$, where $x \in [0,1]$, then we have that

$$\nu(\{x\}) = 0 = \int_{\{x\}} f d\mu = f(x)\mu(\{x\}) = f(x) \cdot 1 = f(x)$$

for all $x \in [0,1] \implies f(x) = 0 \ \forall x \in [0,1]$.
But then if we take $E = [0,1]$ again, then $\mu(E) = 1 = \int_{[0,1]} f d\mu = 0$, which is a contradiction. □

7 L^p Spaces

7.1 Normed Linear Spaces

The space of functions whose pth power is integrable is an L^p space. More precisely, if E is a measurable set and $p \in \mathbb{N}$, $L^p(E)$ is the set of all functions f such that $|f|^p$ is integrable over E. More precisely, let $\sim$ be an equivalence relation defined as follows: $f \sim g$ if $f = g$ almost everywhere in E, then:

Definition 7.1. Let $(X, \mathcal{M}, \mu)$ be a measure space and $p \in \mathbb{N}$. The space $L^p(X)$ is the set of equivalence classes of measurable functions for which

$$\int_E |f|^p < \infty$$

So L^1 is the collection of all integrable functions.
A **functional** is a real-valued function that has a linear function space as its domain.

Definition 7.2. Let X be a linear space. A real-valued functional $||\cdot||$ on X is a **norm** if for all $f, g \in X$ and $\alpha \in \mathbb{R}$:

- (Triangle inequality) $$||f+g|| \leq ||f|| + ||g||$$
- (Positive Homogeneity) $$||\alpha f|| = |\alpha| ||f||$$
- (Nonnegativity) $$||f|| = 0 \iff f = 0$$

Definition 7.3. Let $(X, \mathcal{M}, \mu)$ be a measure space and $p \in \mathbb{N}$. The space ℓ^p is the set of real sequences for which

$$\sum_{k=1}^{\infty} |a_k|^p < \infty$$

We can check that ℓ^p is a normed linear space.

7.2 Young, Holder, and Minkowski Inequalities

All of the results in this subsection were proved in Analysis 2, hence we omit the proofs.

Definition 7.4. Let E be a measurable set and $p \in \mathbb{N}$, and $f \in L^p(E)$, then the **p-norm** as

$$||f||_p = \left(\int_E |f|^p \right)^{1/p}$$

Definition 7.5. The **conjugate** of a number $p \in \mathbb{N}$ is the unique number $q \in \mathbb{N}$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Note that the conjugate of 1 is ∞ and vice-versa.

Lemma 7.1. *(Young's Inequality) For $p \in \mathbb{N}$, p, q conjugate, and any $a, b > 0$,*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

Recall Choksi's neat proof using convexity

Theorem 7.2. *(Holder's Inequality) Let E be a measurable set, $p \in \mathbb{N}$, and q the conjugate of p. If $f \in L^p(E)$ and $g \in L^q(E)$, then their product $f \cdot g$ is integrable over E, and*

$$\int_E |f \cdot g| \leq ||f||_p \cdot ||g||_q$$

Moreover, if $f \neq 0$, the conjugate of f, $f^* = ||f||_p^{p-1} \cdot sgn(f) \cdot |f|^{p-1}$ belongs to $L^q(X, \mu)$, then

$$\int_E f \cdot f^* = ||f||_p \text{ and } ||f^*||_q = 1$$

Theorem 7.3. *(Minkowski's Inequality) Let E be a measurable set, and $p \in \mathbb{N}$. If $f, g \in L^p(E)$, then $f + g \in L^p(E)$ and*

$$||f + g||_p \leq ||f||_p + ||g||_p$$

Proof. Using the fact that $||f + g||_p = \int_E (f + g) \cdot (f + g)^*$ from Holder's Inequality, apply linearity of integral. □

Special case of Holder's inequality for $p = q = 2$.

Theorem 7.4. *(Cauchy-Schwarz Inequality) Let E be a measurable set and f and g measurable functions on E for which f^2 and g^2 are integrable over E, then their product $f \cdot g$ is also integrable over E with*

$$\left(\int_E |fg| \right)^2 \leq \int_E f^2 \cdot \int_E g^2$$

Corollary 7.5. *Let E be measurable set and $1 < p < \infty$. If there exists a family of functions $\mathcal{F}$ in $L^p(E)$ that is bounded in $L^p(E)$ in the sense that there is a constant M for which $||f||_p \leq M \forall f \in \mathcal{F}$, then the family $\mathcal{F}$ is uniformly integrable over E.*

Corollary 7.6. *Let E be a measurable set of finite measure and $1 \leq p_1 < p_2 \leq \infty$, then $L^{p_2}(E) \subseteq L^{p_1}(E)$. Moreover $||f||_{p_1} \leq c||f||_{p_2} \forall f \in L^{p_2}(E)$, where $c = \big(m(E)\big)^{\frac{p_2 - p_1}{p_1 p_2}}$ if $p_2 < \infty$ and $c = \big(m(E)\big)^{1/p_1}$ if $p_2 = \infty$.*

This result follows from the Holder inequality.

Example 7.1. Counterexample that shows that this claim fails if E is not of finite measure: $f(x) = 1/x \in L^2([1, \infty))$, but is clearly not in $L^1([1, \infty))$.

Here is a nicer reformulation of the previous corollary:

Theorem 7.7. *Let $(X, \mathcal{B}, m)$ be a σ-finite measure space, where m is a non-negative measure.Then TFAE:*

1. *$L^p(X, \mathcal{B}, m) \supset L^q(X, \mathcal{B}, m)$ for some p, q such that $1 \leq p < q < \infty$*

2. *$m(X) < \infty$*

3. *$L^p(X, \mathcal{B}, m) \supset L^q(X, \mathcal{B}, m)$ for all p, q such that $1 \leq p < q < \infty$*

For $1 \leq p < \infty$, q the conjugate of p, and $f \in L^p(E)$, show that $f = 0$ if and only if $\int_E f \cdot g = 0 \ \forall g \in L^q(E)$

- ($\Longrightarrow$) : Apply Holder's inequality: for all $g \in L^q(E)$ $\int_E |f \cdot g| \leq ||f||_p \cdot ||g||_q = 0 \cdot ||g||_q = 0$ by the nonnegativity of the norm of f.

- ($\Longleftarrow$) : By contradiction, suppose $f > 0$ or $f < 0$ on a set of positive measure in E. WLOG suppose it is the latter. Then define $A = \{x \in E : f(x) < 0\}$ and $A_k = \{x \in A : f(x) < -1/k\}$. Then $A_k = f^{-1}(-\infty, -1/k)$ all measurable and $A = \bigcup_{k \in \mathbb{N}} A_k$ so there is some A_k measurable with $m(A_k) > 0$. We obtain that $\int_{A_k} f \cdot g < \int_{A_k} (-1/k) g = -1/k \underbrace{\int_{A_k} g}_{\text{finite}}$ for some $g \neq 0 \in L^q(E)$, and thus we obtain a contradiction that $\int_E f \cdot g \neq 0$. We can repeat this exact same proof for the case $f > 0$ by changing $<$ to $>$ and again obtain that $\int_E f \cdot g \neq 0$.

(Riesz) For $1 < p < \infty$ show that if the absolutely continuous function F on $I = [a, b]$ is the indefinite integral of an $L^p([a, b])$ function, then there is a constant $M > 0$ such that for any partition $a = x_0 < x_1 < \cdots < x_N = b$

$$\sum_{k=1}^{N} \frac{|F(x_k) - F(x_{k-1})|^p}{|x_k - x_{k-1}|^{p-1}} \leq M$$

- F absolutely continuous on $[a, b] \iff \forall \epsilon > 0 \exists \delta > 0$ such that for any partition $a = x_0 < \cdots < x_N = b$, if $\sum_{k=1}^{N} |x_k - x_{k-1}| < \delta$ then $\sum_{k=1}^{N} |F(x_k) - F(x_{k-1})| < \epsilon$

- Since F is the integral of an L^p function, and F is absolutely continuous:

$$\begin{aligned}
\sum_{k=1}^{N} \frac{|F(x_k) - F(x_{k-1})|^p}{|x_k - x_{k-1}|^{p-1}} &= \sum_{k=1}^{N} \frac{|\int_a^{x_k} f - \int_a^{x_{k-1}} f|^p}{|x_k - x_{k-1}|^{p-1}} \\
\text{Linearity} \implies &= \sum_{k=1}^{N} \frac{|\int_{x_{k-1}}^{x_k} f|^p}{|x_k - x_{k-1}|^{p-1}} \\
&\leq \sum_{k=1}^{N} \frac{(\int_{x_{k-1}}^{x_k} |f \cdot 1|)^p}{|x_k - x_{k-1}|^{p-1}} \\
f \in L^p([a,b]) + \mathit{Holder} \implies &\leq \sum_{k=1}^{N} \frac{\left(||f||_p \cdot ||\chi_{(x_{k-1},x_k)}||_q\right)^p}{|x_k - x_{k-1}|^{p-1}} \\
&= \sum_{k=1}^{N} \frac{\left((\int_a^b |f|^p)^{1/p} (\int_{x_{k-1}}^{x_k} 1^q)^{1/q}\right)^p}{|x_k - x_{k-1}|^{p-1}} \\
q \text{ conjugate of } p \therefore 1/q = 1 - 1/p \implies &= \sum_{k=1}^{N} \frac{\int_a^b |f|^p \cdot |x_k - x_{k-1}|^{(1-1/p)p}}{|x_k - x_{k-1}|^{p-1}} \\
&= \sum_{k=1}^{N} \frac{\int_a^b |f|^p \cdot |x_k - x_{k-1}|^{p-1}}{|x_k - x_{k-1}|^{p-1}} \\
\text{cancellation and } |x_k - x_{k-1}| > 0 \implies &= \sum_{k=1}^{N} \int_a^b |f|^p \\
f \in L^p([a,b]) \text{ integrable} \implies &\leq c \sum_{k=1}^{N} = c \cdot (N-1) \equiv M
\end{aligned}$$

7.3 Completeness of L^p: Riesz-Fischer Theorem

Definition 7.6. A sequence $\{f_n\}$ in a linear space X normed by $|| \cdot ||$ **converges to** f **in** X if $\lim_{n\to\infty} ||f_n - f|| = 0$. We may also write $\{f_n\} \to f$ in X or $\lim_{n\to\infty} f_n = f$.

Definition 7.7. A sequence $\{f_n\}$ in a linear space X normed by $||\cdot||$ is **Cauchy** in X if $\forall \epsilon > 0\ \exists N \in \mathbb{N}$ such that $||f_n - f_m|| < \epsilon\ \forall m, n \geq N$

Definition 7.8. A normed linear space X is **complete** if every Cauchy sequence in X converges to a function in X. A complete normed linear space is a **Banach space**.

Proposition 7.8. *Let X be a normed linear space. Then every convergent sequence in X is Cauchy. Moreover, if a Cauchy sequence has a convergent subsequence in X then it converges in X.*

Proof. Let $\{f_n\} \to f$

- $\{f_n\}$ is clearly Cauchy by the triangle inequality.
- Suppose we have convergent subsequence $\{f_{n_k}\} \to f$ in X. Since f Cauchy, $\forall\ \epsilon > 0\ \exists N \in \mathbb{N}$ such that $||f_n - f_m|| < \epsilon/2\ \forall m, n \geq N$. Since $\{f_{n_k}\} \to f$ in X, we take $n_k > N$ and $||f_{n_k} - f|| < \epsilon/2$. We apply the triangle inequality to obtain that $||f_n - f|| \leq \epsilon$ and thus that $\{f_n\} \to f$ in X.

□

Definition 7.9. Let X be a normed linear space. A sequence $\{f_n\}$ in X is **rapidly Cauchy** if there exists a convergent sequence of positive numbers $\sum_{k=1}^{\infty} \epsilon_k$ such that $||f_{k+1} - f_k|| \leq \epsilon_k^2\ \forall k$

Proposition 7.9. *Let X be a normed linear space. Then every rapidly Cauchy sequence in X is Cauchy. Furthermore, every Cauchy sequence has a rapidly Cauchy subsequence.*

Theorem 7.10. *Let E be a measurable set and $1 \leq p < \infty$. Then every rapidly Cauchy sequence in $L^p(E)$ converges both wrt the $L^p(E)$ norm and pointwise a.e. on E to a function in $L^p(E)$.*

Theorem 7.11. *(Riesz-Fischer) Let E be a measurable set and $1 \leq p < \infty$. Then $L^p(E)$ is a Banach space. Moreover, if $\{f_n\} \to f$ in $L^p(E)$, then a subsequence of $\{f_n\}$ converges pointwise a.e on E to f.*

Theorem 7.12. *Let E be a measurable set and $1 \leq p < \infty$. Suppose $\{f_n\}$ is a sequence in $L^p(E)$ that converges pointwise a.e on E $f \in L^p(E)$. Then $\{f_n\} \to f$ in $L^p(E) \iff \lim_{n\to\infty} \int_E |f_n|^p = \int_E |f|^p$.*

(The L^p Dominated Convergence Theorem) Let $\{f_n\}$ be a sequence of measurable functions that converges pointwise a.e on E to f. For $1 \leq p < \infty$ suppose there is a function $g \in L^p(E)$ such that $|f_n| \leq g$ a.e in E $\forall n$. Prove that $\{f_n\} \to f$ in $L^p(E)$.

- Lebesgue Dominated Convergence Theorem: Let $\{f_n\}$ be a sequence of measurable functions on $A \subseteq \mathbb{R}$ such that

  1. $\exists g$ integrable over A such that $\forall n \in \mathbb{N} |h_n| \leq g$ on A.
  2. $\exists h : A \to \overline{\mathbb{R}}$ such that $h_n \to h$ pointwise a.e in A,

  Then the functions h_n and h are integrable and $\lim_{n\to\infty} \int_A h_n = \int_A h$

- Apply DCT on $h_n = |f_n - f|^p$ dominated by $(2g)^p$ on measurable domain E.
- Then $\lim_{n\to\infty} \int_E h_n = \int_E h \implies \{h_n\} \to h$ in $L^p(E)$

For E measurable set and $1 \leq p < \infty$, assume $\{f_n\} \to f$ in $L^p(E)$. Show there is a subsequence $\{f_{n_k}\}$ and a function $g \in L^p(E)$ for which $|f_{n_k}| \leq g$ a.e on E and for all k.

- (Existence of subsequence): By the Riesz-Fischer Theorem, we know that if $\{f_n\} \to f$ in $L^p(E)$, then there exists a subsequence of $\{f_n\}$, call it $\{f_{n_k}\}_{n_k \in \mathbb{N}}$, that converges pointwise a.e on E to f. This implies that there exist $n_k \in \mathbb{N}$ such that $||f_{n_k} - f||_p \leq 2^{-k}$.
- (Existence of g): By the previous point, we know the distance between the subsequence and f, thus we define $g \equiv |f| + \sum_{k=1}^{\infty} |f_{n_k} - f|$.
- (g is an upper bound): $|f_{n_k}| = |f_{n_k} - f + f| \leq |f_{n_k} - f| + |f|$, and thus $|f_{n_k}| \leq g$.
- ($g \in L^p(E)$): Apply Minkowski's inequality: $||g||_p \leq ||f||_p + \sum_{k=1}^{\infty} ||f_{n_k} - f||_p = ||f||_p + \sum_{k=1}^{\infty} 2^{-k} = ||f||_p + 1 < \infty$, and thus $g \in L^p(E)$.

7.4 *Approximation and Separability*

Definition 7.10. Let X be a normed linear space. For $\mathcal{F} \subseteq G$ in X, $\mathcal{F}$ is **dense** in $\mathcal{G}$ if for every $g \in \mathcal{G}$ and $\epsilon > 0$, there exists $f \in \mathcal{F}$ such that $||f - g|| < \epsilon$

Remark 5.
- $\mathcal{F}$ dense in $\mathcal{G} \iff \forall g \in \mathcal{G} \exists \{f_n\} \in \mathcal{F}$ such that $\lim_{n\to\infty} f_n = g \in X$.
- For $\mathcal{F} \subseteq \mathcal{G} \subseteq \mathcal{H} \subseteq X$, if $\mathcal{F}$ dense in $\mathcal{G}$, and $\mathcal{G}$ dense in $\mathcal{H}$, then $\mathcal{F}$ dense in $\mathcal{H}$ (transitivity).

Proposition 7.13. *Let E be a measurable set and $1 \leq p \leq \infty$. Then the subspace of simple functions in $L^p(E)$ is dense in $L^p(E)$.*

Proof. Let $g \in L^p(E)$.

1. ($p = \infty$): Let $N \subseteq E$ be a subset such that $\mu(N) = 0$ and g is bounded on $E \setminus N$. We can apply the Simple Approximation Lemma there: on $E \setminus N$, there exist simple functions $(\phi_n)_{n\in\mathbb{N}}$ that converge uniformly to g i.e $||\phi_n - g||_\infty \to 0$, and therefore simple functions are dense in $L^\infty(E)$.
2. ($1 \leq p < \infty$): Since g is measurable but not necessarily bounded, we can apply the Simple Approximation Theorem. We then have

that there exists a sequence of simple functions $(\phi_n)_{n\in\mathbb{N}}$ such that ϕ_n converges pointwise to g on E and $|\phi_n| \leq |g|$ on E for all $n \in \mathbb{N}$. This implies that $\phi_n \in L^p(E)$ for all $n \in \mathbb{N}$. We conclude that

$$|\phi_n - g|^p \leq 2^p\left(|\phi_n|^p + |g|^p\right) \leq 2^{p+1}|g|^p < \infty \text{ on } E$$

since $g \in L^p(E)$. By the DCT, we can pull the limit inside the integral and thus $(\phi_n)_{n\in\mathbb{N}} \to g \in L^p(E)$.

□

Proposition 7.14. *Let $[a,b]$ be a closed bounded interval and $1 \leq p < \infty$. Then the subspace of step functions on $[a,b]$ is dense in $L^p([a,b])$.*

Proof. Since the step functions are dense in the simple functions, the result follows from the previous proposition. Note that the step functions are not dense in L^∞, but simple functions are. □

Definition 7.11. For a measurable subset $E \subseteq \mathbb{R}$, the linear space of continuous real-valued functions on E that vanish outside a bounded set is called $C_c(E)$.

Use Lusin's theorem to prove the following theorem:

Theorem 7.15. *Let E be a measurable set and $1 \leq p < \infty$ then $C_c(E)$ is dense in $L^p(E)$.*

- Lusin's theorem: Let $f : A \to \mathbb{R}$ be measurable, then $\forall \epsilon > 0 \exists F_\epsilon \subseteq A$ closed such that f is cont on F_ϵ and $m(A \setminus F_\epsilon) < \epsilon$. $\implies f \in C_c(E)$.
- $C_c(E)$ is the linear space of continuous real-valued functions with compact support.
- WTS $C_c(E)$ dense in $L^p(E) \iff \forall g \in L^p(E)$ and $\epsilon > 0 \exists f \in C_c(E)$ such that $||f - g|| < \epsilon$
- Let $g \in L^p(E)$, then by definition of norm:

$$||f-g||_p = \left(\int_E |f-g|\right)^{1/p}$$

By Lusin's theorem, $\exists F_\epsilon \subseteq E$ such that $g \in C_c(E)$ on F_ϵ and with $m(E \setminus F_\epsilon) < \epsilon/2$. Choose $f \in C_c(E)$ such that

$|f-g| < \epsilon/2$ on F_ϵ then:

$$||f-g||_p = \left(\int_{F_\epsilon} |f-g| + \int_{E \setminus F_\epsilon} |f-g| \right)^{1/p}$$
$$\leq \left(\overbrace{m(F_\epsilon)}^{\text{finite}} \underbrace{\sup_{F_\epsilon} |f-g|}_{<\epsilon/2} + \underbrace{m(E \setminus F_\epsilon)}_{<\epsilon/2} \overbrace{\sup_{E \setminus F_\epsilon} |f-g|}^{\text{finite } \because E \setminus F_\epsilon \text{ bded}} \right)^{1/p}$$
$$\leq \epsilon^{1/p} < \tilde{\epsilon}$$

8 L^p Spaces: Duality and Weak Convergence

8.1 Riesz Representation for the Dual of L^p

Note that this entire chapter carries over to general measure spaces (more or less as is) in Ch 19 of Royden and Fitzpatrick.

Definition 8.1. A **linear functional** on a linear space X is a real-valued function T on X such that for $g, h \in X$ and $\alpha, \beta \in \mathbb{R}$, $T(\alpha \cdot g + \beta \cdot h) = \alpha \cdot T(g) + \beta \cdot T(h)$

Example 8.1. For E measurable, $1 \leq p < \infty$, q conjugate of p, let $g \in L^q(E)$. We define the real-valued functional T on $L^p(E)$ as $T(f) = \int_E f \cdot g \ \forall f \in L^p(E)$.

Holder's inequality $\implies |T(f)| \leq ||g||_q \cdot ||f||_p \ \forall f \in L^p(E)$

Remark 6. Note that we can also define $T(f)$ in the following way: For a closed bounded interval $[a,b]$, let g be of bounded variation on $[a,b]$, and define $T(f)$ on $C([a,b])$ as $T(f) = \int_a^b f(x)dg(x)$ in the sense of Riemann-Stieltjes.

The Riemann-Stieltjes is a bit more general than the Riemann integral but less so than Lebesgue.

We can then do the entire proof of Riesz Representation using this definition instead. See this paper.

A good introduction to Riemann-Stieltjes integration can be found here.

Definition 8.2. A linear functional T on a normed linear space X is **bounded** if there exists an $M \geq 0$ such that $|T(f)| \leq M \cdot ||f|| \ \forall f \in X$. We then define the **norm of** T, $||T||_*$, as follows:

$$||T||_* = \inf\{M > 0 : |T(f)| \leq M \cdot ||f|| \ \forall f \in X\}$$

We can also show that an equivalent definition is the following:

$$||T||_* = \sup\{|T(f)| : f \in X, ||f|| \leq 1\}$$

Exercise 8.2. If T is a bounded linear functional and $\lim_{n\to\infty} f_n = f$ in X, then $\lim_{n\to\infty} T(f_n) = T(f)$.

Proof. T bounded $\implies$ $\exists M \geq 0$ such that $|T(f) \leq M \cdot ||f||$ for all $f \in X$. This definition also holds for $M = ||T||_*$, and so linearity of T gives us that $|T(f) - T(h)| \leq ||T||_* \cdot ||f - h||$ for all $f, h \in X$. Our result follows immediately from this fact. □

Proposition 8.1. *Let X be a normed linear space. Then the collection of bounded linear functionals on X is a linear space on which $||\cdot||_*$ is a norm. This normed linear space is called the* ***dual space*** *of X and is denoted X^**

Theorem 8.2. *(The Riesz Representation Theorem for the Dual of $L^p(E)$) Let E be a measurable set, $1 \leq p < \infty$, and q the conjugate of p. For each $g \in L^q(E)$, define the bounded linear functional R_g on $L^p(E)$ by $R_g(f) = \int_E g \cdot f$ for all $f \in L^p(E)$. Then, for each linear functional T on $L^p(E)$, there exists a unique function $g \in L^q(E)$ for which $R_g = T$ and $||T||_* = ||g||_q$.*

In short, the Riesz Representation Theorem tells us that every bounded linear functional on $L^p(E)$ is given by integration against a function in $L^q(E)$.

Proposition 8.1 and the Riesz Representation theorem combined show that the dual space of $L^p(E)$ (bounded linear operators) is isomorphic to $L^q(E)$ (sequences).

Remark 7. The dual space of $L^\infty(E)$ is not $L^1(E)$ (see Kantorovich Representation Thm).

Remark 8. $p = q = 2 \implies L^2(E)$ is its own dual space. L^2 is special in other ways: it is the only L^p space on which an inner product can be defined and is an example of a Hilbert space.

Example 8.3. Purpose of this theorem: ideas about finite dimensional vector spaces and linear maps can be carried over to infinite dimensional Hilbert spaces and continuous linear maps.

If a Hilbert space W has basis $\langle e_i \rangle$, then the dual has dual basis $\langle e_i, \cdot \rangle$, and the space of continuous maps $W \to V$ is isomorphic to $W^* \bigotimes V$.

9 Proof of Riesz Representation Theorem

Theorem 9.1. *(The Riesz Representation Theorem for the Dual of $L^p(E)$) Let E be a measurable set, $1 \leq p < \infty$, and q the conjugate of p. For each $g \in L^q(E)$, define the bounded linear functional R_g on $L^p(E)$ by $R_g(f) = \int_E g \cdot f$ for all $f \in L^p(E)$. Then, for each linear functional T on $L^p(E)$, there exists a unique function $g \in L^q(E)$ for which $R_g = T$ and $||T||_* = ||g||_q$, i.e T is an isometric isomorphism of $L^q(E)$ onto $L^p(E)$*

Note that this theorem does not extend to $p = \infty$.

Note we can also begin by using L^1 functions (a slightly more general start of the proof since all step functions are in L^1) and extend to L^q.

9.1 *Uniqueness of $g \in L^q(E)$ such that $R_g = T$ and $||T||_* = ||g||_q$.*

By contradiction, suppose we have $g_1, g_2 \in L^q(E)$ with $R_{g_1} = R_{g_2} = T$ and $||T||_* = ||g_1||_q = ||g_2||_q$. We prove the following result:

Proposition 9.2. *Let E be a measurable set, $1 \leq p < \infty$, and q the conjugate of p. Let $g \in L^q(E)$. Define the functional T on $L^p(E)$ as $T(f) = \int_E g \cdot f$ for all $f \in L^p(E)$. Then T is a bounded linear functional and $||T||_* = ||g||_q$.*

Proof.
- $(\leq)$: Holder's inequality tells us that $|T(f)| \leq ||g||_q \cdot ||f||_p$ for all $f \in L^p(E)$. This tells us that T is a bounded linear functional and that $||T||_* \leq ||g||_q$
- $(\geq)$: In our statement of Holder's inequality, we show that the conjugate of $g \in L^q(E)$, g^*, belongs to $L^q(E)$ and that $g^* = ||g||_q^{q-1} \operatorname{sgn}(g)|g|^{q-1}$, belongs to $L^p(E)$. Moreover, $T(g^*) = ||g||_q$, and $||g^*||_p = 1$.

  By definition, $||T||_* = \sup\{T(f) | f \in X, ||f|| \leq 1\} \implies ||T||_* \geq ||g||_q$
  We can thus conclude that $||T||_* = ||g||_q$ □

However, Proposition 9.2 says that for every $g \in L^q(E)$, R_g is a bounded linear functional with $||R_g||_* = ||g||_q$. By linearity: $0 = R_{g_1} - R_{g_2} = R_{g_2 - g_1}$, and thus we have that $0 = ||g_1 - g_2|| \implies g_1 = g_2 = g$.

9.2 *Existence of $g \in L^q([a,b])$ such that $R_g = T$ and $||T||_* = ||g||_q$.*

We prove this for the case where $[a,b]$ is a closed, bounded interval.

9.2.1 *Proof for when f is a Step Function*

Let $F(x) = T(\chi_{[a,x)})$. From MATH 454, we know that this function must be absolutely continuous and that since $g = F'(x)$ is integrable over $[a,b]$: $F(x) = \int_0^x g \ \forall x \in [a,b]$, and thus for any interval $[c,d] \subseteq (a,b)$, $T(\chi_{[c,d]}) = F(d) - F(c) = \int_a^b g \cdot \chi_{[c,d)}$. Since $f \mapsto \int_a^b g \cdot f$ is linear on the space of step functions and so is T^2, we obtain that $T(f) = \int_a^b g \cdot f$ for all step functions f on $[a,b]$.

9.2.2 *Proof for when f is a Simple Function*

We approximate f by a sequence of step functions, and use DCT to extend our result to all simple functions:

We showed that step functions are dense in $L^p([a,b])$ in Proposition 7.14. Since simple functions are also dense in $L^P([a,b])$ (Proposition 7.13), then we know that if f is a simple function on $[a,b]$, there is a sequence of step functions $(\phi_n)_{n\in\mathbb{N}}$ that converges to f in $L^p([a,b])$ and is uniformly pointwise bounded in $[a,b]$. Since T is linear and bounded on $L^p([a,b])$, Exercise 8.2 tells us that $\lim_{n\to\infty} T(\phi_n) = T(f)$.

Since we have pointwise convergence and boundedness dominance of the ϕ_n (since they are integrable step functions), we can apply the Dominated Convergence Theorem and obtain that $\lim_{n\to\infty}\int_a^b g\cdot\phi_n = \int_a^b \lim_{n\to\infty} g\cdot f = \int_a^b g\cdot f$.

We conclude that $T(f) = \int_a^b g\cdot f$ for all simple functions f on $[a,b]$.

Moreover, since T is bounded, we know that

$$\int_a^b g\cdot f = |T(f)| \le ||T||_* \cdot ||f||_p$$

for all simple functions f on $[a,b]$.

It remains to show two things: that $g\in L^q([a,b])$, and that $||T||_* = ||g||_q$.

9.2.3 $g \in L^q([a,b])$

Let E be a measurable set and $1\le p<\infty$. Suppose that the function g is integrable over E and that there exists $M\ge 0$ such that

$$\left|\int_E g\cdot f\right| \le M||f||_p$$

for all simple functions $f\in L^p(E)$, then $g\in L^q(E)$, where q is the conjugate of p, and $||g||_q \le M$.

Proof. From MATH 454 and Chebyshev's inequality, we know that g integrable over E implies that $|g|<\infty$ a.e in E. We can therefore assume that g is finite on all of E (otherwise, use $E\setminus N$, $\mu(N)=0$. N is the set where g is not finite).

- $(p>1)$: Apply the Simple Approximation Theorem on $|g|$ which is nonnegative measurable. We then have that there exists a sequence of nonnegative simple functions $(\phi_n)_{n\in\mathbb{N}}$ such that $0\le\phi_n\le|g|$ and $\lim_{n\to\infty}\phi_n = |g|$ on E.

  Take a power of q: $(\phi_n^q)_{n\in\mathbb{N}}$ is a sequence of nonnegative measurable functions that converges pointwise to $|g|^q$ on E. WTS that $g\in L^q(E)$ (i.e $|g|^q$ is integrable over E) and that $||g||_q\le M$. To show both of these at once, Fatou's Lemma suggests that we prove that

  $$\int_E \phi_n^q \le M^q \ \forall n\in\mathbb{N}$$

  We begin with a fixed n and approximate ϕ_n^q on E:

  $$\phi_n^q = \phi_n\cdot\phi_n^{q-1} \le |g|\cdot\phi_n^{q-1} = g\cdot \operatorname{sgn}(g)\cdot\phi_n^{q-1} \equiv g\cdot f_n$$

Since $0 \leq \phi_n \leq g, g$ integrable on E, ϕ_n is also integrable on E. Moreover, $f_n \in L^p(E)$ since ϕ_n is simple.

Since $\left| \int_E g \cdot f \right| \leq M||f||_p$ for all simple functions $f \in L^p(E)$, we have that

$$\begin{aligned}
& \int_E \phi_n \leq \int_E g \cdot f_n \leq M||f_n||_p \\
& = M\left(\int_E |f_n|^p \right)^{1/p} = M\left(\int_E \phi_n^{p(q-1)} \right)^{1/p} \\
& = M\left(\int_E \phi_n^q \right)^{1/p} \\
|\phi|_n^q \text{ integrable} \quad & \implies \left(\int_E \phi_n^q \right)^{1-1/p} \leq M \\
& \implies \int_E \phi_n^q \leq M^q
\end{aligned}$$

- $(p = 1)$: Proceed by contradiction. If $|g(x)|$ is not $\leq M$ for almost every $x \in E$ (i.e M is not an essential upper bound for g), then there exists some $\epsilon > 0$ such that $E_\epsilon = \{x \in E : |g(x)| > M + \epsilon\}$ has nonzero measure. Recall that we also have that $| \int_E g \cdot f| \leq M||f||_p$ for every simple function $f \in L^p(E)$. But then if we let $f = \chi_{E'_\epsilon}$, $E'_\epsilon \subseteq E_\epsilon$ with $0 < \mu(E'_\epsilon) < \infty$, we obtain a contradiction to this condition.

$\square$

9.2.4 $||T||_* = ||g||_q$

Let T and S be bounded linear functionals on a normed linear space X. If $T = S$ on a dense subset D of X, then $T = S$.

Proof. Let $g \in X$. D dense in $X \implies \exists$ a sequence $(g_n)_{n\in\mathbb{N}}$ in D that converges to g in X. Exercise 8.2 implies that $\lim_{n\to\infty} S(g_n) = S(g)$ and $\lim_{n\to\infty} T(g_n) = T(g)$. Since $S(g_n) = T(g_n)$, then we conclude that $S(g) = T(g)$. $\square$

9.3 *For all $f \in L^p(E)$*

1. E is a closed, bounded interval.

   We proved this in the previous subsection (Subsection 9.2)

2. $E = \mathbb{R}$. We apply previous case using a sequence of nexted intervals $[-n, n]$:

   Let T be a bounded linear functional on $L^p(\mathbb{R})$. Fix n and define T_n, a linear functional on $L^p([-n, n])$, by $T_n(f) = T(\tilde{f})\ \forall f \in$

$L^p([-n,n])$, where $\tilde{f}$ is an extension of f to $\mathbb{R}$ that vanishes beyond $[-n,n]$. By this definition, $||f||_p = ||\tilde{f}||_p$, and therefore $|T_n(f)| \leq ||T||_*||f||_p \ \forall f \in L^p([-n,n])$ and thus $||T_n||_* \leq ||T||_*$. By the previous case, we have that $\exists g_n \in L^q([-n,n])$ such that $T_n(f) = \int_{-n}^n g_n \cdot f$ for all $f \in L^p([-n,n])$ and $||g_n||_q = ||T_n||_* \leq ||T||_*$.

Since $g_{n+1}|_{[-n,n]} = g_n$ a.e in $[-n,n]$ by uniqueness (see Section 9.1), we define g to be a measurable function on $\mathbb{R}$ such that $g = g_n$ a.e on $[-n,n]$ for each n. Then, since $T_n(f) = \int_{-n}^n g_n \cdot f$, we have that

$$T(f) = \int_{\mathbb{R}} g \cdot f \ \forall f \in L^p(\mathbb{R}) \text{ with compact support.}$$

Since $||T_n||_* \leq ||T||_*$, then

$$\int_{-n}^{n} |g|^q \leq (||T||_*)^q \ \forall n \in \mathbb{N}$$

We apply Fatou's Lemma to conclude that $g \in L^q(\mathbb{R})$.

From Section 9.2.4, we have that $R_g = T$ on all of $L^p(\mathbb{R})$ since $R_g = T$ on a dense subspace of $L^p(\mathbb{R})$, where the dense subspace is the subspace of all $L^p(\mathbb{R})$ functions that have compact support.

3. E is any measurable set.

   We apply the previous case by restriction of a function $f \in L^p(\mathbb{R})$ to some measurable set E.

   Define the linear functional $\tilde{T}$ on $L^p(\mathbb{R})$ as $\tilde{T}(f) = T(f|_E)$, and so $\tilde{T}$ is bounded. By the previous case, there exists $\tilde{g} \in L^q(\mathbb{R})$. Finally, define $g = \tilde{g}|_E$, and thus $T = R_g$.

9.4 *Proof of the Riesz Representation Theorem for the Dual of $L^p(X,\mu)$.*

Note that a lot of the steps and the tools used in this more general version are very similar to the ones in the above case. Moreover, note that the following proof is more of an outline than a rigorous proof (due to time constraint...).

Theorem 9.3. *Let $(X, \mathcal{M}, \mu)$ be a $\sigma-$finite measure space, $1 \leq p < \infty$, and q the conjugate of p. For $f \in L^q(X,\mu)$, define $T_f \in (L^p(X,\mu))^*$ by $T_f(g) = \int_X fg\, d\mu \ \forall g \in L^p(X,\mu)$.. Then T is an isometric isomorphism of $L^q(X,\mu)$ onto $(L^p(X,\mu))^*$.*

Proof. • $(p = 1)$: Good exercise to do (no time), but logic is same as $p > 1$.

• $(p > 1)$:

  – $(\mu(X) < \infty)$: Let $S : L^p(X,\mu) \to \mathbb{R}$ be a bounded linear functional. Let $\mathcal{M}$ be the collection of ν-measurable sets, and define the set function $\nu(E) = S(\chi_E)$ for $E \in \mathcal{M}$, which makes sense since $\mu(X) < \infty$, and $\chi_E \in L^p(X, \mu)$. Check conditions to apply the corollary to the Radon-Nikodym Theorem (Corollary 6.2):

* WTS ν is a signed measure: Let $E = \cup_{n=1}^{\infty} E_n$ be a countable union of disjoint sets. Countable additivity of $\mu \implies$

  $$\mu(E) = \sum_{n=1}^{\infty} \mu(E_n) < \infty \implies \lim_{k\to\infty} \sum_{n=k+1}^{\infty} \mu(E_n) = 0$$

  Similarly for the $p-$norm, $\lim_{k\to\infty} \left(\sum_{n=k+1}^{\infty} \mu(E_n)\right)^{1/p} = 0$. Since S is both linear and continuous on $L^p(X, \mu)$, then

  $$S(\chi_E) = \sum_{n=1}^{\infty} S(\chi_{E_n}) \implies \nu(E) = \sum_{n=1}^{\infty} \nu(E_n)$$

  Let $s_n = \text{sgn}(S(\chi_{E_n}))$. Then $\sum_{n=1}^{\infty} S(s_n \cdot \chi_{E_n})$ is Cauchy and is therefore convergent so $\sum_{n=1}^{\infty} |\nu(E_n)| = \sum_{n=1}^{\infty} S(s_n \cdot \chi_{E_n})$ converges, which implies that ν is a signed measure.
* WTS ν is absolutely continuous wrt μ: if $\mu(E) = 0$ for $E \in \mathcal{M}$, then χ_E represents the zero element of $L^p(X, \mu)$, so by linearity of S, $\nu(E) = S(\chi_E) = 0$.
  Now, by the corollary to Radon-Nikodym, there exists a function f that is integrable over X such that

  $$S(\chi_E) = \nu(E) = \int_E f\, d\nu \ \forall E \in \mathcal{M}$$

Then we approximate using simple functions: since all simple functions belong to $L^p(X, \mu)$, then for a simple function ϕ, by the linearity of S and the linearity of integration $S(\phi) = \int_X f\phi d\mu$.
To show that $f \in L^q(E)$, we use without proof a result very similar to the one in section 9.2.3, which states:

Lemma 9.4. *Let $(X, \mathcal{M}, \mu)$ be a $\sigma-$finite measure space and $1 \leq p < \infty$. For an integrable function f over X, suppose there exists an $M \geq 0$ such that for every simple function g on X that vanishes outside of a set of finite measure*

$$\left| \int_X fg\, d\mu \right| \leq M \cdot ||g||_p$$

Then $f \in L^q(X, \mu)$, where q is the conjugate of p. Moreover, $||f||_q \leq M$.

And thus, since S is bounded on $L^p(X, \mu)$, $|S(g)| \leq ||S|| \cdot ||g||_p$ for every $g \in L^p(X, \mu)$, and thus

$$\left| \int_E f\phi\, d\mu \right| = |S(\phi)| \leq ||S|| \cdot ||\phi||_p \text{ for every } \phi \text{ simple.}$$

Thus, by the above lemma, $f \in L^q$. It remains to show that $S = T_f$. Since simple functions are dense in $L^p(X, \mu)$, and the functional $g \mapsto S(g) - \int_X fg\, d\mu \ \forall g \in L^p(X, \mu)$ vanishes on this space, we can conclude that $S - T_f$ vanishes on all of $L^p(X, \mu) \implies S = T_f$.

– (X is $\sigma-$finite):
  * Consider an ascending sequence $(X_n)_{n \in \mathbb{N}}$ of finite measurable sets with union X. Fix n. By the previous case, there exists $f_n \in L^q(X, \mu)$ such that $f_n = 0$ on $X \setminus X_n$, $\int_X |f_n|^q \, d\mu \leq ||S||^q$, and $S(g) = \int_{X_n} f_n g \, d\mu = \int_X f_n g \, d\mu$ if $g \in L^p(X, \mu)$ with $g = 0$ on $X \setminus X_n$. This f_n is uniquely determined (up to sets of measure 0). Notice that the same can be written for f_{n+1} restricted to X_n and thus we let $f(x) = f_n(x) \ \forall x \in X_n$. This means that f is measurable on X and $\lim_{n \to \infty} |f_n|^q = |f|^q$ a.e on X.
  * Since we have pointwise convergence, we can apply Fatou's Lemma to obtain an upper bound:

$$\int_X |f|^q d\mu \leq \liminf \int_X |f_n|^q d\mu \leq ||S||^q$$

    and therefore $f \in L^q$.
  * $\lim_{n \to \infty} g_n = g \in L^p(X, \mu)$: Let $g \in L^p(X, \mu)$. Let $g_n = \begin{cases} g & X_n \\ 0 X \setminus X_n \end{cases}$ for every n. By Holder's inequality, $|fg|$ is integrable over X and $|fg_n| \leq |fg|$ a.e on X, we can apply DCT and obtain that

$$\lim_{n \to \infty} fg_n \, d\mu \leq \int_X fg \, d\mu$$

    Since $\lim_{n \to \infty} |g_n - g|^p = 0$ a.e on X and $|g_n - g|^p \leq |g|^p$ a.e on X for every n, once more DCT allows us to conclude that $\lim_{n \to \infty} g_n = g \in L^p(X, \mu)$.
  * Since S is a continuous functional on $L^p(X, \mu), \lim_{n \to \infty} S(g_n) = S(g)$ and $S(g_n) = \int_{X_n} f_n g_n \, d\mu = \int_X fg_n \, d\mu$. This implies that finally $S(g) = \int_x fg \, d\mu = T_f$.

□

In light of the alternative method of proof of the Dual Representation theorem using the Riemann-Stieltjes integral (See this paper), I want to learn more about this type of integration, which I haven't seen in analysis 2 or any other course, in fact). To push myself to look through the material, I chose the following exercise that uses it:

Let f belong to $C[a,b]$. Show that there is a function g that is of bounded variation on $[a,b]$ for which

$$\int_a^b f dg = ||f||_{max} \text{ and } TV(g) = 1$$

Proof. First recall that $||f||_{max} = \max_{x\in[a,b]} |f(x)|$, and therefore there exists $y \in [a,b]$ such that $|f(y)| = ||f||_{max}$.

- Case $y \in (a,b]$: Recall the definition of the Riemann-Stieltjes integral:

  $$\int_a^b f(x)dg(x) = \lim_{N\to\infty} \sum_{n=1}^{N} f(x_n)\big(g(x_n) - g(x_{n-1})\big),$$

  for any partition $a = x_0 < x_1 < \cdots < x_N = b$.

  We want $\lim_{N\to\infty} \sum_{n=1}^{N} f(x_n)\big(g(x_n) - g(x_{n-1})\big) = |f(y)|$.

  1. Since the partition becomes finer in the limit, we can let $g(x) = \chi_{[y,y+1]}$, and thus $\lim_{N\to\infty} \sum_{n=1}^{N} f(x_n)\big(g(x_n) - g(x_{n-1})\big) = 0 + \cdots + 0 + f(y)\cdot(1-0) + f(y)\cdot(1-1) + \cdots = f(y)$.
  2. But since we want to have $|f(y)|$ instead, redefine $g(x) = \text{sgn}(f)\chi_{[y,y+1]}$, so that

     $$\int_a^b f(x)dg(x) = \text{sgn}(f(y))\cdot f(y) = |f(y)|$$

  3. $TV(g) = TV\big(\text{sgn}(f)\chi_{[y,y+1]}\big) = 1$ clearly.

- Case $y = a$: use the definition with the left end-points instead:

  $$\int_a^b f(x)dg(x) = \lim_{N\to\infty} \sum_{n=1}^{N} f(x_{n-1})\big(g(x_n) - g(x_{n-1})\big),$$

  for any partition $a = x_0 < x_1 < \cdots < x_N = b$.

  We can keep the same $g(x) = \text{sgn}(f)\chi_{[a,a+1]}$.

□

10 Weak Sequential Convergence in L^p

Definition 10.1. Let X be a normed linear space. A sequence $\{f_n\}$ in X **converges weakly** in X to f if $\lim_{n\to\infty} T(f_n) = T(f)$ for all $T \in X^*$.

Note that weak sequential limits are unique. We can show this easily by contradiction.
Note that strong convergence ($\lim_{n\to\infty} ||f_n - f|| = 0$) implies weak convergence, but not the converse: $|T(f_n) - T(f)| = |T(f_n - f)| \leq ||T||_* \cdot ||f_n - f||$ for all $T \in X^*$.

Proposition 10.1. *Let E be a measurable set, $1 \leq p < \infty$, and q the conjugate of p. Then (f_n) converges weakly to $f \in L^p(E)$ if and only if $\lim_{n\to\infty} \int_E g \cdot f_n = \int_E g \cdot f \ \ \forall g \in L^q(E)$.*

Proof. This is a restatement of the Riesz Representation theorem using weak convergence. □

Note that this definition also holds for $p = \infty$, but the name changes:

Definition 10.2. Let $p = \infty$. Let E be a measurable set. We say that a sequence $(f_n)_{n\in\mathbb{N}} \subset L^\infty(E)$ **converges weakly - *** to $f \in L^\infty(E)$ if

$$\lim_{n\to\infty} \int_E f_n \cdot g dx = \int_E f \cdot g dx \ \ \forall g \in L^1(E)$$

Let $[a, b]$ be a closed, bounded interval. Suppose $(f_n)_{n\in\mathbb{N}}$ converges weakly to f in $L^\infty([a, b])$. Show that

$$\lim_{n\to\infty} \int_a^x f_n = \int_a^x f \ \ \forall x \in [a, b]$$

Proof.
- A weakly converging sequence in L^∞ must be bounded.
- We apply the definition of weak - * convergence:

$$\begin{aligned}\lim_{n\to\infty} \int_a^x f_n(x) &= \lim_{n\to\infty} \int_a^b f_n(x)\chi_{[a,x]} \\ &= \int_a^b f(x)\chi_{[a,x]} \\ &= \int_a^x f(x)\end{aligned}$$

□

Note: I don't feel very confident about this solution as it seems too simple. I got the definition of weak-* convergence from this document. Please tell me if I used the definition wrong or misunderstood, and I will fix it. I did not have as much time as I wanted for this section...

Theorem 10.2. *Let E be a measurable set, and $1 \leq p < \infty$. Suppose (f_n) converges weakly to f in $L^p(E)$. Then (f_n) is bounded in $L^p(E)$ and $||f||_p \leq \liminf ||f_n||_p$.*

Note that not all of these results carry over exactly to $L^\infty(A)$ since $L^1(A)$ is not the dual of $L^\infty(A)$ (Kantorovitch thm). However, we can see $L^1(A)$ as a subset of the space of signed Radon measures on A with finite mass with the weak-* topology.

Corollary 10.3. *Let E be a measurable set, $1 \leq p < \infty$, and q the conjugate of p. Suppose (f_n) converges weakly to f in $L^p(E)$. Then*

$$\lim_{n\to\infty} \int_E g_n \cdot f = \int_E g \cdot f$$

10.1 Implications in Spectral Geometry

10.1.1 Riemann-Lebesgue Lemma

Lemma 10.4. *(Riemann-Lebesgue)*

1. *Let E be a measurable set. Let $I = [-\pi, \pi]$, and $1 < p < \infty$. Define $f_n(x) = \sin(nx)$ for $n \in \mathbb{N}, x \in I$. $|f_n| \leq 1$ on I for every n and is therefore a bounded sequence in $L^p(I)$. Then, by the preceding corollary, the sequence (f_n) converges weakly in $L^p(I)$ to 0 if and only if*

$$\lim_{n\to\infty} \int_{-\pi}^{x} \sin(nt)dt = 0 \quad \forall x \in I.$$

Notice, however, that for each n,

$$\int_{-\pi}^{\pi} |sin(nt)|^2 dt = \int_{-\pi}^{\pi} \sin^2(nt)dt = \pi$$

This means that no subsequence of (f_n) converges strongly to 0 in $L^2(I)$. This also holds for any p, We can also apply the BCT, and obtain that there is no subsequence that converges a.e to 0 in $L^p(I)$.

2. *(General Statement): Let f be Riemann integrable on $I = [a, b]$. Then*

$$\lim_{n\to\pm\infty} \int_I f(x)\cos(nx)dx = 0$$
$$\lim_{n\to\pm\infty} \int_I f(x)\sin(nx)dx = 0$$
$$\lim_{n\to\pm\infty} \int_I f(x)e^{inx}dx = 0$$

Proof. See proof here. □