SIGGRAPH ASIA 2024 TOKYO 東京

Conference | 3–6 December 2024
Exhibition | 4–6 December 2024
Venue | Tokyo International Forum, Japan

Mesh Quality Meets The Virtual Element Method

Sponsored by acm

Organized by koelnmesse

D. Cabiddu

T. Sorgente

Consiglio Nazionale delle Ricerche

imati

+

Politecnico di Torino

Dipartimento di Scienze Matematiche "G. L. Lagrange"

1859

special thanks to
prof G. Manzini
and S. Berrone

S. Biasotti

M. Spagnuolo

F. Vicini

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Course Schedule

Introduction (20 min)

a) Motivations (M. Spagnuolo)

b) PEMesh Software (D. Cabiddu)

Part I - Mesh Quality (70 min)

a) Background and Notations (T. Sorgente)

b) Element Quality Indicators (T. Sorgente)

c) Mesh Quality Indicators (S. Biasotti)

d) PEMesh Exercise on Mesh Quality (D. Cabiddu)

Part II - The Virtual Element Method (70 min)

a) Background and Notations (F. Vicini)

b) VEM vs FEM (F. Vicini)

c) VEM Definition (F. Vicini)

Break (15 min)

d) Mesh Requirements for the VEM (T. Sorgente)

e) Mesh Quality Indicator for the VEM (T. Sorgente)

f) PEMesh Exercise on VEM (D. Cabiddu)

Part III - Optimizing Mesh Quality (40 min)

a) How to Get a Quality Mesh (S. Biasotti)

b) Mesh Optimization Driven by Quality Indicators (S. Biasotti)

c) PEMesh Exercise on Optimization (D. Cabiddu)

Conclusions (10 min)

a) Perspectives (M. Spagnuolo)

b) Q&A

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Introduction

Why mesh quality and VEM?

- numerical simulations play a crucial role in Computer Graphics – *realistic animation effects*
- the "*digital twin*" hype brings simulations among the topmost enabling technologies of the future, beyond Computer Graphics
- FE methods are well studied – *theory and numerics*
- geometry processing reached impressive maturity
- computing power is not any longer a problem

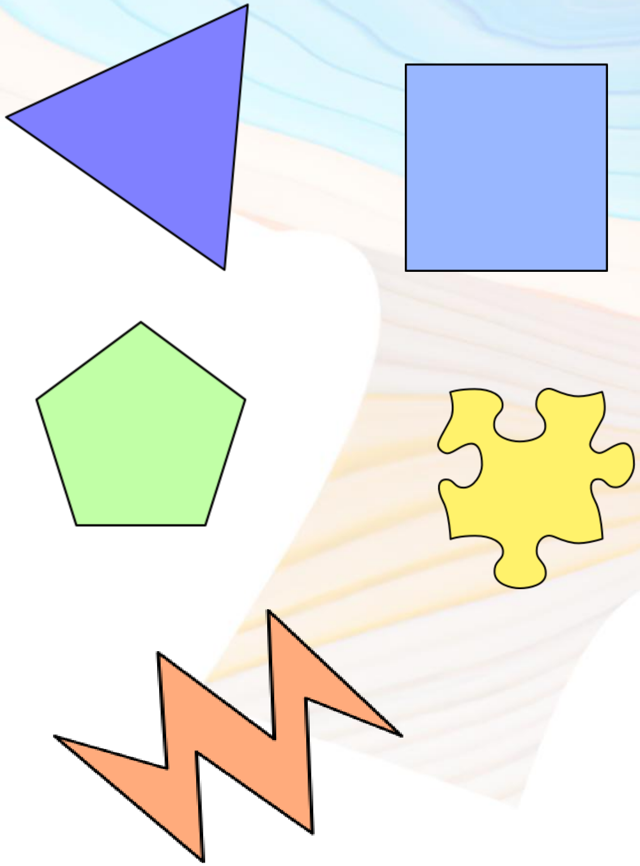
ACM ToG: 158 papers mention *physical simulation* as keyword in the period 2003-2023, of which 50 in the years 2021-2023

yet, meshing still takes a considerable amount of time in the FE pipeline

...meshing still takes a considerable amount of time in the FE pipeline...

Why mesh quality and VEM?

- meshing come into play for the discretization of the physical domain where the simulation runs
  - *element type*: triangles, tets, quads, hex, polygons, polyhedral,...
  - *mesh*: pure, structured/unstructured, conforming,...
- mesh properties influence numerical simulations
  - error estimates and convergence rates
- different characteristics of the domain may suggest the use of different mesh elements and strategies
- different properties of the physical problem to solve may drive the choice of the meshing approach



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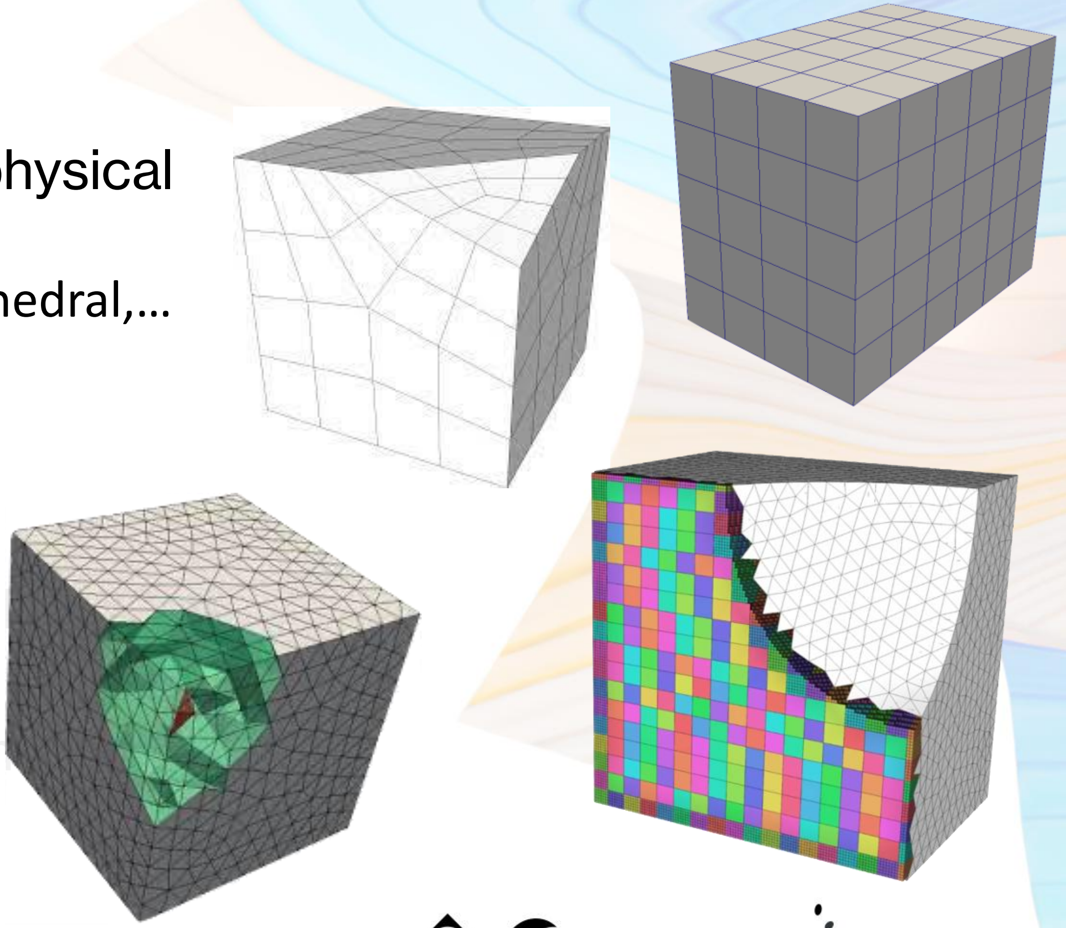

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A Large-Scale Comparison of Tetrahedral and Hexahedral Elements for Solving Elliptic PDEs with the Finite Element Method
T. Schneider, Y. Hu, X. Gao, J. Dumas, D. Zorin, and D. Panozzo.
ACM Trans. Graph. 41(3), 23, June 2022

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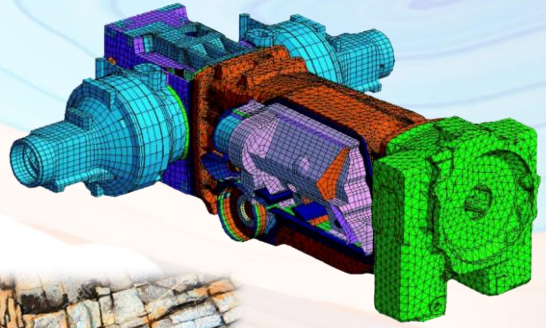
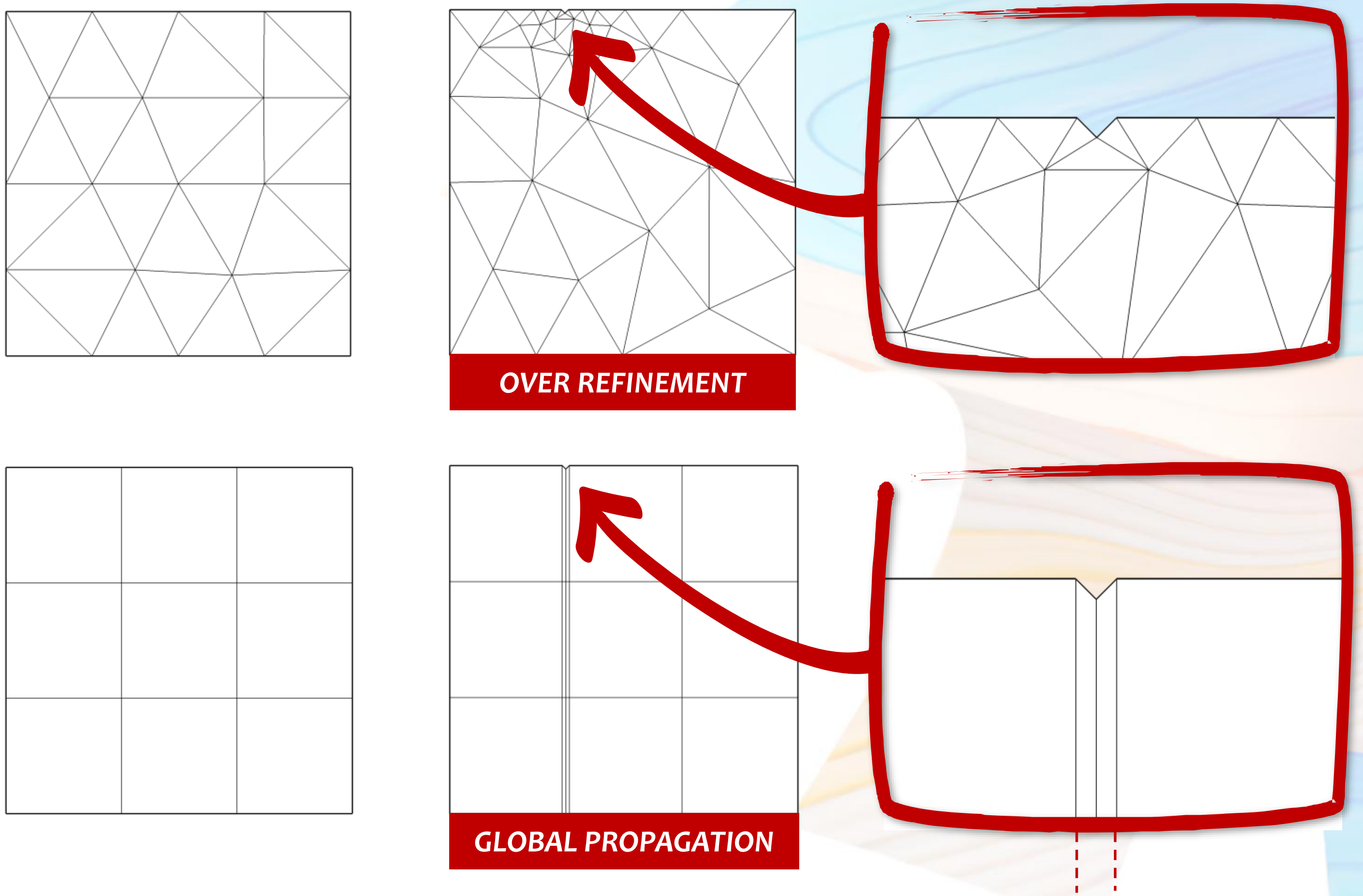


Polytopal elements and flexibility



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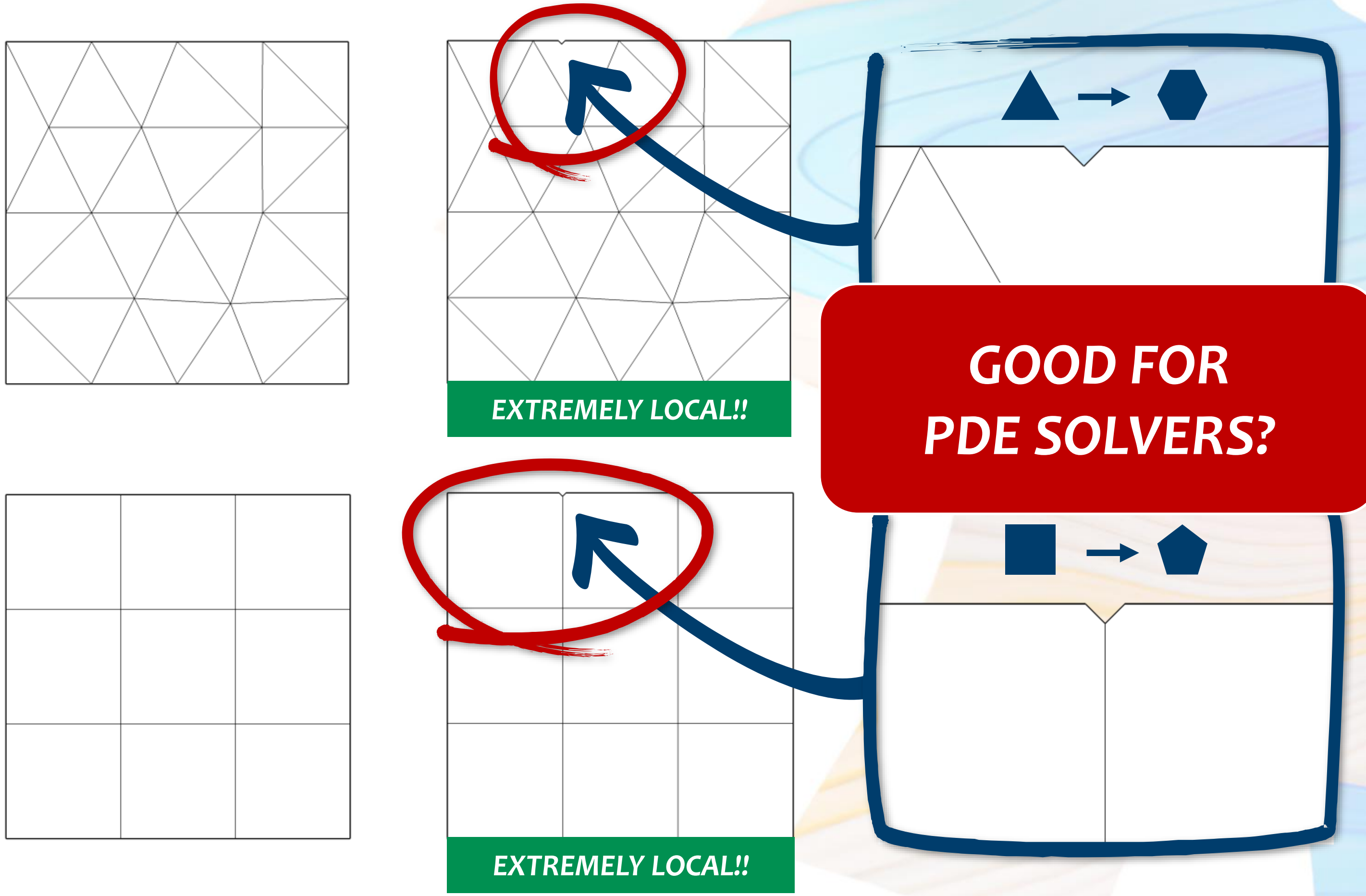

European Research Council
Established by the European Commission

CHANGE - 2016 / 2021
New CHallenges for (adaptive) PDE solvers: the interplay of ANalysis and GEometry

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↑

Polyhedral Element Methods

↓

…domain-aware and PDE-aware meshing …

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Polyhedral Element Methods

...domain-aware and PDE-aware meshing ...

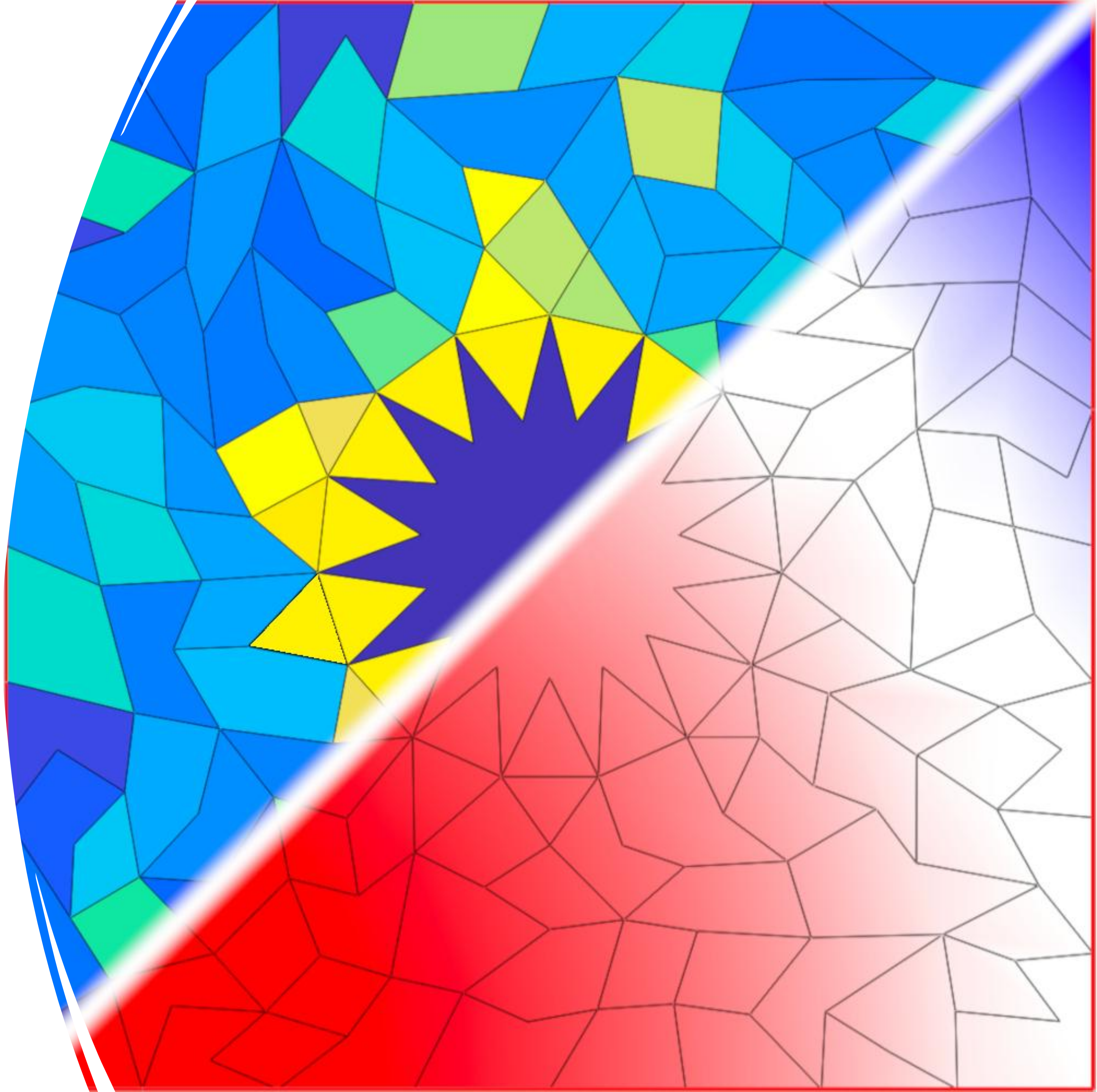
RAISE

Finanziato dall'Unione europea
NextGenerationEU

Robotics and AI for Socio-economic empowerment

PEMesh time!

What it is & How to get it



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PEMesh

A Graphical Framework to Study the Correlation between Geometric Design and Simulation. [Cabiddu et al., 2022]

Geometry Design of 2D Polygonal Meshes

Element/Mesh Quality Indicators *(see Part I)*

PDE Solver (not included) *(see Part II)*

Correlation Quality Indicators <-> Solver Performances

https://github.com/DanielaCabiddu/PEMesh

PEMesh.2 – New Release

NEW

A Graphical Framework to Study the Correlation between Geometric Design and Simulation. [Cabiddu et al., 2022]

Geometry Design of 2D Polygonal Meshes

Element/Mesh Quality Indicators *(see Part I)*

PDE (VEM) Solver *(see Part II)*

Correlation Quality Indicators <-> Solver Performances

Mesh Optimization *(see Part III)*

cinolib

METIS

https://github.com/DanielaCabiddu/PEMesh.2

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Part I - Mesh Quality

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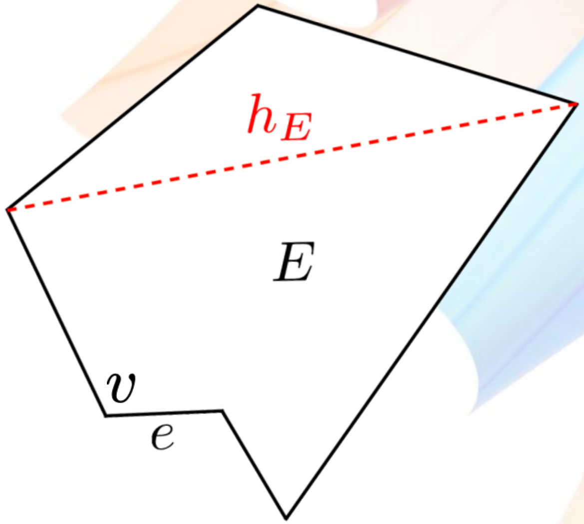
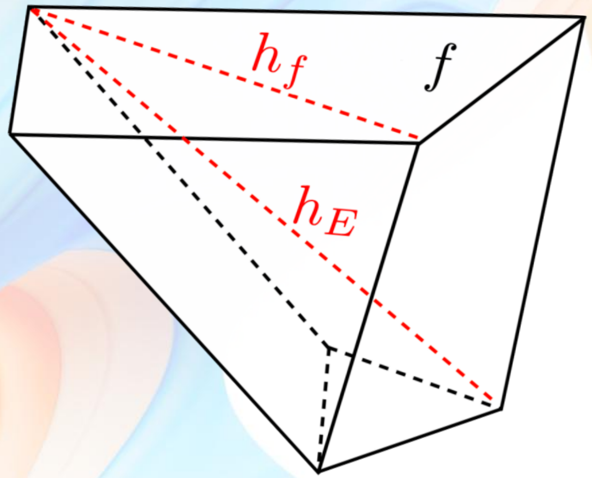
Mesh Generalities

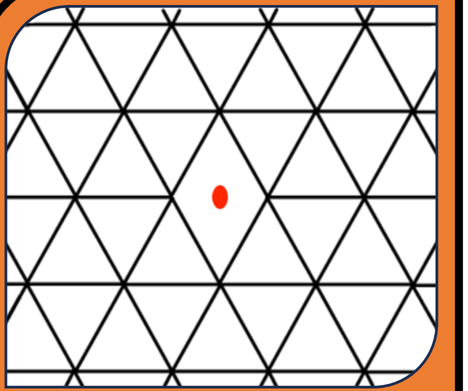




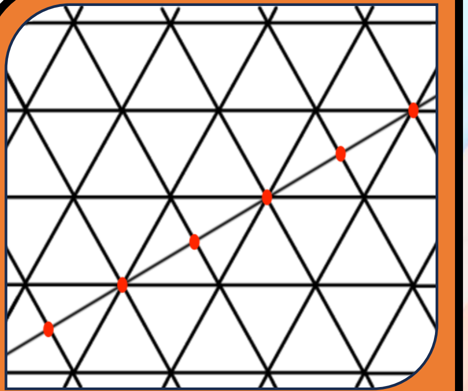
mesh Ω_h: discrete approximation of a domain $\Omega \subset \mathbb{R}^d$, partitioned into a finite collection of non-overlapping closed cells with diameter h

cell E: subset of $\mathbb{R}^d$ with no holes and no self-intersections, made of **faces** f, **edges** e, and **nodes** v

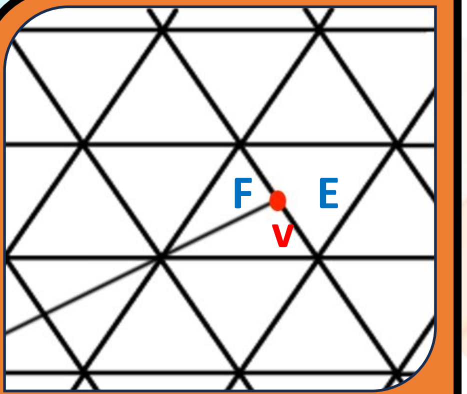
diameter h: maximum point-to-point Euclidean distance in the cell/face



pure: all elements have the same number of edges/faces

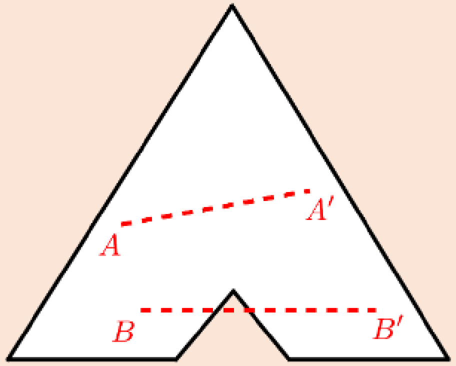


structured: every internal node has the same valence (#incident edges)

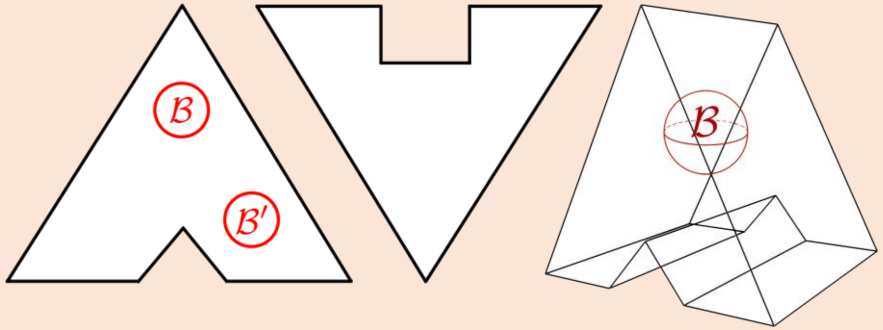


conforming: two adjacent cells only share a node or a whole edge/face

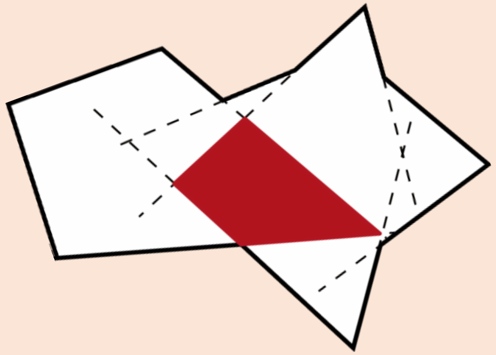
Polytopes Generalities



convexity: all points are visible from each other

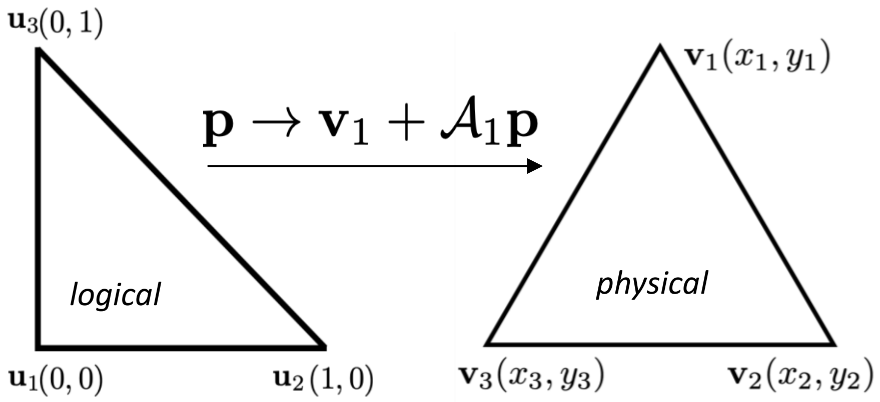


star-shapedness: exists a ball from which all points are visible



kernel: set of the points that "see" the whole polytope

Spectral Operators



$$\mathcal{A}_1 = \begin{bmatrix} x_2 - x_1 & x_3 - x_1 \\ y_2 - y_1 & y_3 - y_1 \end{bmatrix}$$

$$\mathcal{A}_i = \begin{bmatrix} x_{i+1} - x_i & x_{i+2} - x_i \\ y_{i+1} - y_i & y_{i+2} - y_i \end{bmatrix}$$

$$\mathcal{J} = \det \mathcal{A}_i$$

Jacobian matrix $\boldsymbol{A_i}$: the columns are the Jacobian of the affine map w.r.t. the logical variables
Jacobian $\boldsymbol{J}$: determinant of any A_i

$$\tilde{\mathcal{A}} = \mathcal{A}_i \mathcal{W}_i^{-1}$$

$$\kappa(\tilde{\mathcal{A}}) = \|\tilde{\mathcal{A}}\|_p \, \|\tilde{\mathcal{A}}^{-1}\|_p$$

Jacobian-based operators: condition number, trace, metric tensors $A^T A$…

other polys

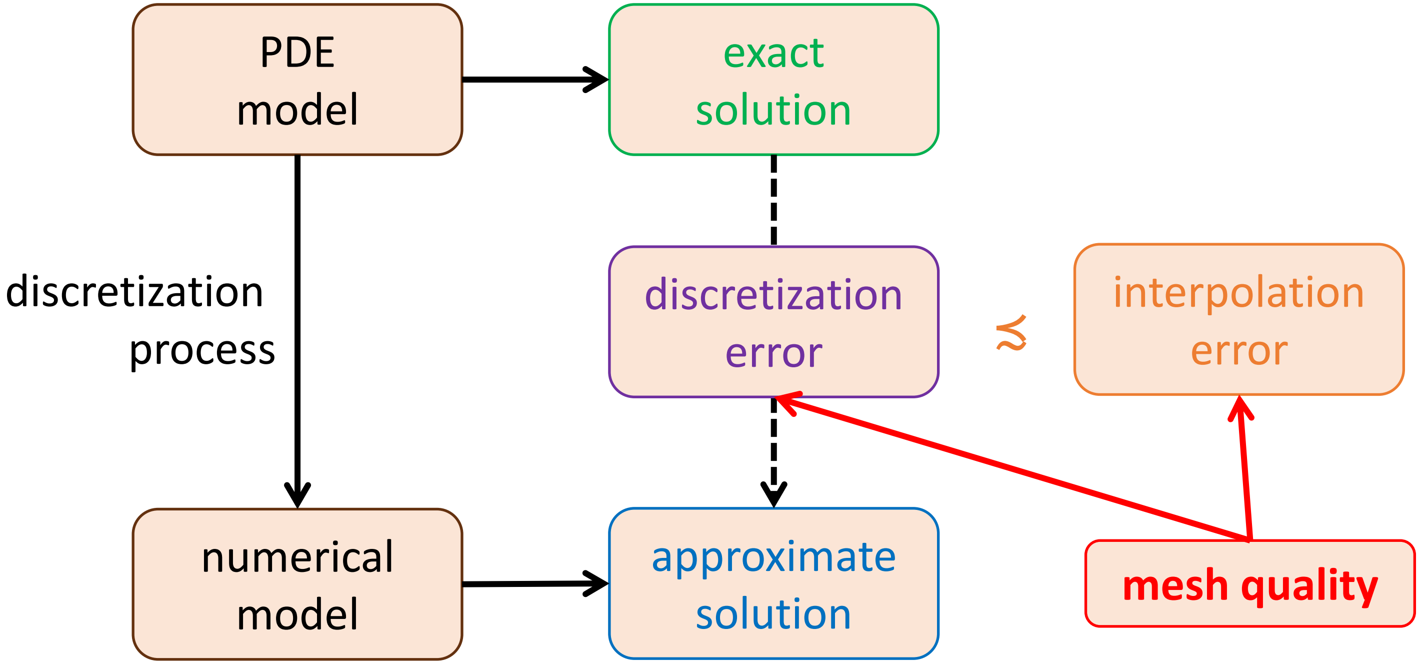
- tets: analogue
- polygons: min/max/sum/ave
- polyhedra: only if valence 3

Quality Indicator

a function defined over a mesh, capable of giving insights on the accuracy and the convergence speed of a PDE simulation on that mesh before solving the numerical problem

- *indicator*, *measure* (positive + bounded), or *metric* (positive + triangle inequality)?
- **good properties:** universality, degeneracies detection, existence of an ideal element, boundedness, invariance wrt size, orientation, translations, nodes indexing, …

Mesh Quality and the Finite Element Method



$$\|u - u_h\|_{1,\Omega} \leq C\, h\, |u|_{2,\Omega}$$

$$|u - u_h|_{1,\Omega} \leq C |u - u_I|_{1,\Omega}$$

[Zlàmal 1968]: for triangular meshes,

$$C = C(\theta_{\max})$$

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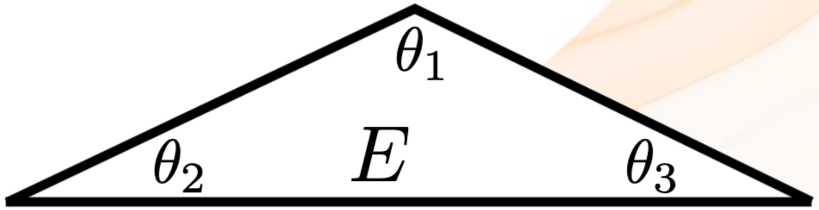


[Zlàmal 1968]: for triangular meshes,

$$C = C(\theta_{\max})$$

An Example

bad angle condition

$$\lambda_{\max}(K^E) \to \infty$$

↓

unbounded eigenvalue spectrum

$$\lambda_{\max}(K) \to \infty$$

↓

high condition number

$$\kappa(K) \to \infty$$

↓

poor convergence rate & numerical accuracy

$$K^E = \frac{1}{2}\begin{bmatrix} \cot\theta_2 + \cot\theta_3 & -\cot\theta_3 & -\cot\theta_2 \\ -\cot\theta_3 & \cot\theta_1 + \cot\theta_3 & -\cot\theta_1 \\ -\cot\theta_2 & -\cot\theta_1 & \cot\theta_1 + \cot\theta_2 \end{bmatrix}$$

$$\min_E |E| \lesssim \lambda_{\min}(K) \lesssim \max_E |E|$$

$$\max_E \lambda_{\max}(K^E) \le \lambda_{\max}(K) \le m \max_E \lambda_{\max}(K^E)$$

[Fried 1972]

$$\kappa(K) := \lambda_{\max}(K)/\lambda_{\min}(K)$$

Element Quality Indicators

defined element-wise and averaged over the mesh

element types
- tris and tets
- quads and hexes
- polytopes

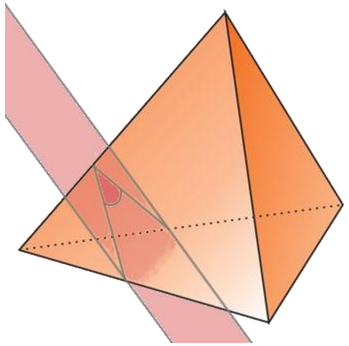
quality indicators classes
- aspect ratio
- skewness
- interpolation
- warping
- taper
- mean ratio
- shape regularity

interpolation error: based on geometric entities like edges, areas, volumes, diameters

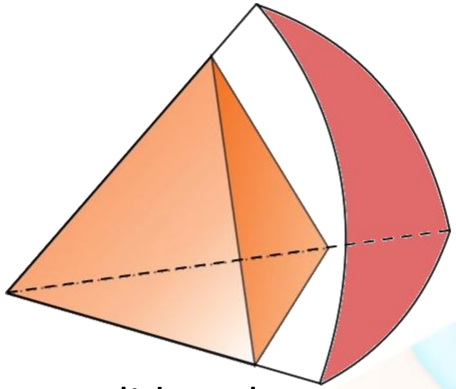
discretization error: based on algebraic entities like matrix conditioning, eigenvalues, norms

[Sorgente et al, A Survey of Indicators for Mesh Quality Assessment, 2023]

Tris and Tets

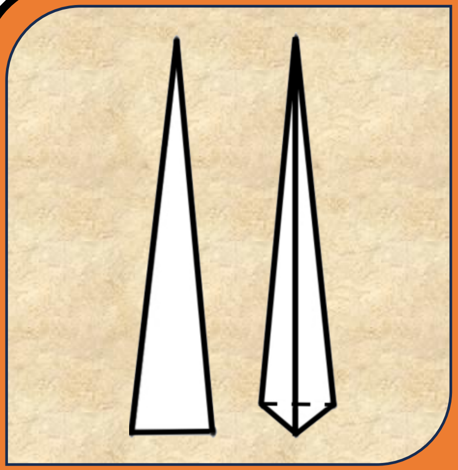


dihedral angle

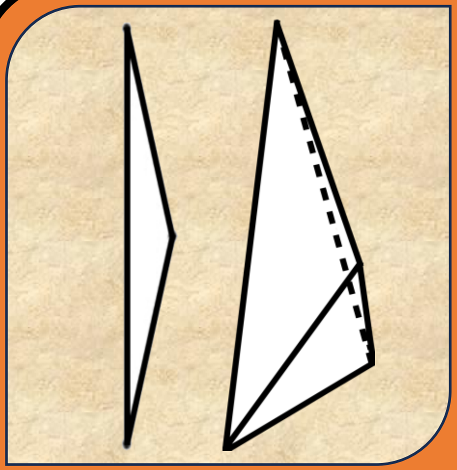


solid angle

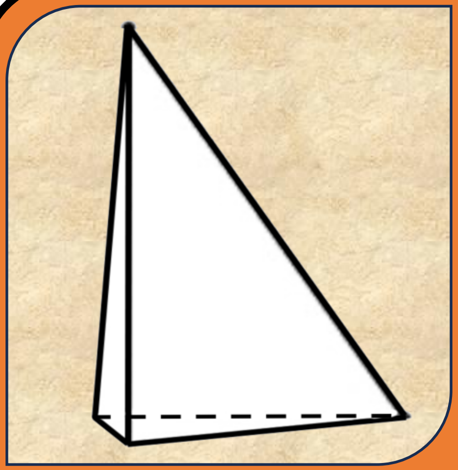
quality: deviation from an equilateral tri/tet



needle
1 small (solid) angle



flat / sliver
1 large (solid) angle



wedge
1 small face

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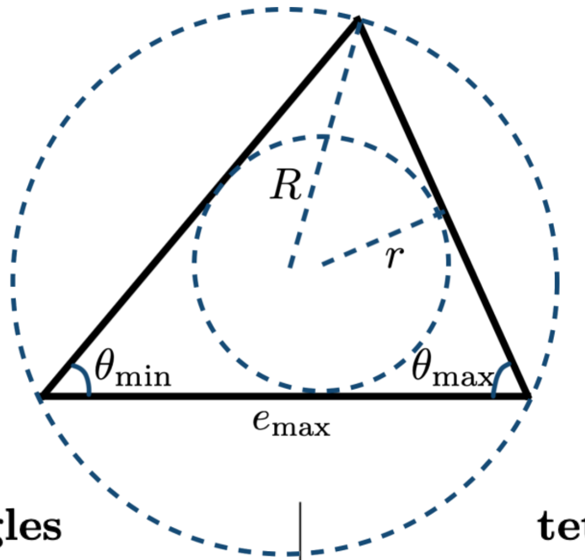
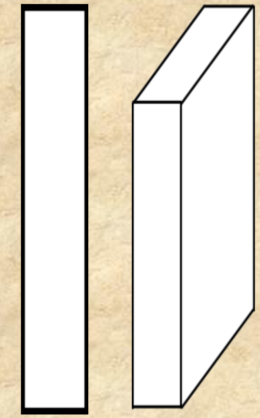


beware of false friends (e.g. edge ratio and dihedral angle)
radius ratio, mean ratio, sine of solid angle, Frobenius ratio are equivalent

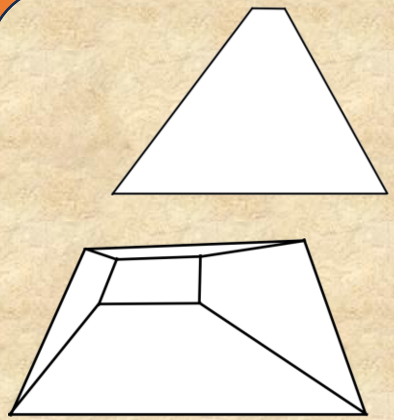
	triangles	tetrahedra	references
aspect ratio	$\frac{r}{R}, \frac{r}{e_{\max}}, \frac{e_{\max}}{R}$	$\frac{r}{R}, \frac{r}{e_{\max}}, \frac{e_{\max}}{R}$	Liu 94, Pébay 03, Dompierre 98
skewness	$\sin(\theta_{\min}), \sin(\theta_{\max}), \frac{\sin(\theta_{\min})}{\sin(\theta_{\max})}$	$\min_i \{\sin(\theta_i/2)\}$	Liu 94, Field 00
interpolation	$\frac{A}{R^2}, \frac{A}{(e_1\ e_2\ e_3)^{2/3}}$	$\frac{V}{R^3}, \frac{V \sum_i A_i}{\sum_{i,j} A_i A_j e_{ij}}$	Shewchuk 02
mean ratio	$\frac{\mathcal{J}}{\mathrm{trace}(\tilde{A}^T\tilde{A})}, \frac{2}{\kappa(\tilde{A})}$	$\frac{3\mathcal{J}^{2/3}}{\mathrm{trace}(\tilde{A}^T\tilde{A})}, \frac{3}{\kappa(\tilde{A})}$	Dompierre 98, Knupp 01, Pébay 03
shape regularity	$\min_i \left\{\frac{\mathcal{J}}{e_{i-1}\ e_i}\right\}, \frac{3A}{\sum_{i=1}^{3} e_i^2}$	$\min_i \left\{\frac{\mathcal{J}}{e_{i-1}\ e_i\ e_{i+1}}\right\}, \frac{4V}{(\sum_i A_i^2)^{3/4}}$	Shewchuk 02, Knupp 00

quality: deviation from a square/cube

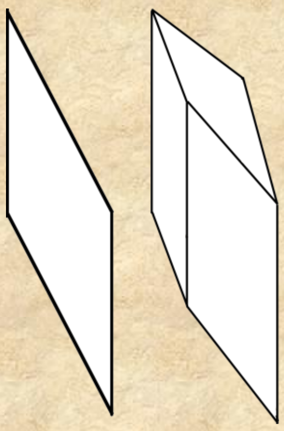
Quads and Hexes



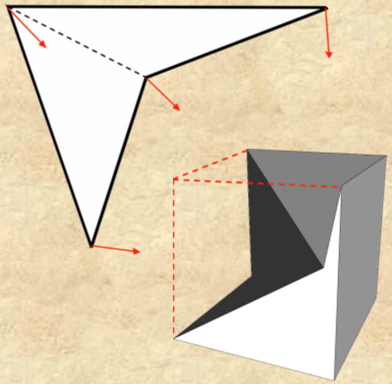
low aspect ratio
≠ consecutive edges



high taper
≠ opposite edges



skewed
1 small (solid) angle



warped
non-planar (faces)

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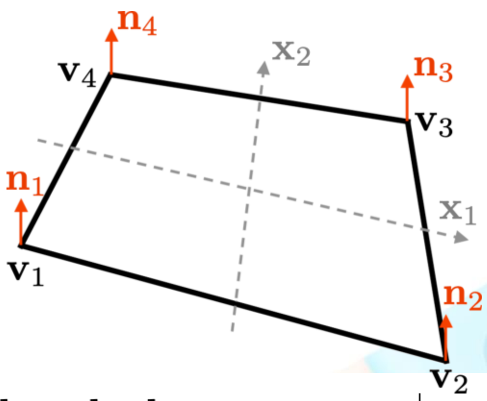
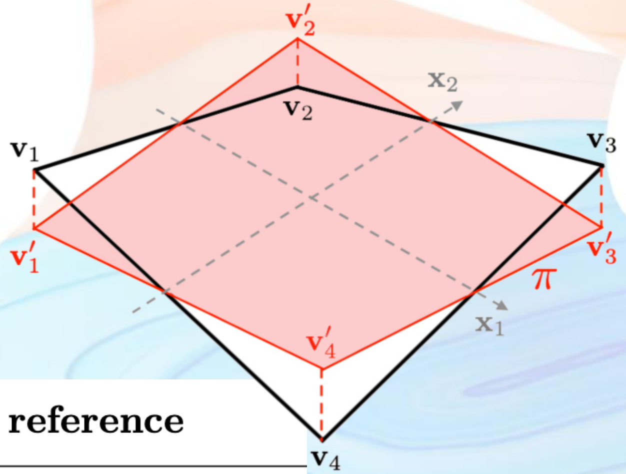
n_i normal versors
x_j principal axes
$\boldsymbol{\pi}$ projection plane

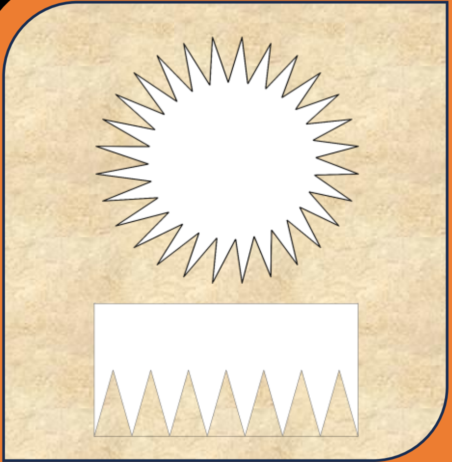




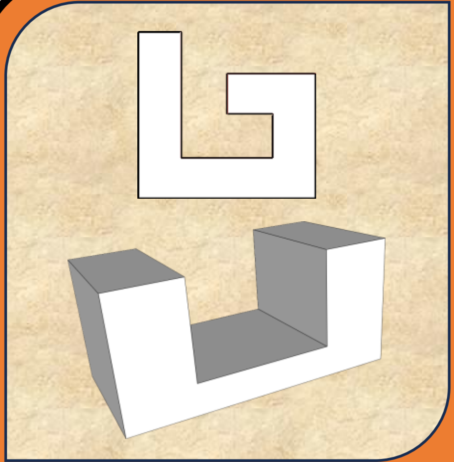
	quadrangles	**hexahedra**	**reference**
warping	$1 - \min\left\{(\mathbf{n}_1 \cdot \mathbf{n}_3)^3, (\mathbf{n}_2 \cdot \mathbf{n}_4)^3\right\}$	sum/average/min/max of the faces	Stimpson 07
aspect ratio	$\max\left\{\frac{\lVert\mathbf{x}_1\rVert}{\lVert\mathbf{x}_2\rVert}, \frac{\lVert\mathbf{x}_2\rVert}{\lVert\mathbf{x}_1\rVert}\right\}, \ \left(1 - \left\lvert 1 - \frac{\theta_1 + \theta_3}{\pi}\right\rvert\right)^{-1}$	sum/average/min/max of the faces	Field 00, Pébay 04
skewness	$\left\lvert\frac{\mathbf{x}_1}{\lVert\mathbf{x}_1\rVert} \cdot \frac{\mathbf{x}_2}{\lVert\mathbf{x}_2\rVert}\right\rvert, \ \frac{4}{\sum_{i=1}^{4} 1/\sin(\theta_i)}$	sum/average/min/max of the faces	Robinson 87, Knupp 03
taper	$\frac{\lVert\mathbf{x}_{12}\rVert}{\min\{\lVert\mathbf{x}_1\rVert, \lVert\mathbf{x}_2\rVert\}}$	sum/average/min/max of the faces	Robinson 94
interpolation	$\max_i\left\{\frac{(e_i^2 - e_{i+1}^2)^2 + 4(e_i\ e_{i+1})^2}{2\lVert e_{i-1} \times e_i\rVert^2}\right\}$	sum/average/min/max of the faces	Oddy 88
mean ratio	$\min_i\left\{\frac{2}{\kappa(\mathcal{A}_i)}\right\}, \ \frac{36}{\sum_i \kappa(\mathcal{A}_i)^2}$	$\min_i\left\{\frac{3}{\kappa(\mathcal{A}_i)}\right\}, \ \frac{72}{\sum_i \kappa(\mathcal{A}_i)^2}$	Knupp 01-03
shape regularity	$\min_i\left\{\frac{\mathcal{J}_i}{e_{i-1}\ e_i}\right\}, \ \frac{\min_i\{\mathcal{J}_i\}}{\max_i\{\mathcal{J}_i\}}$	$\min_i\left\{\frac{\mathcal{J}_i}{e_{i-1}\ e_i\ e_{i+1}}\right\}, \ \frac{\min_i\{\mathcal{J}_i\}}{\max_i\{\mathcal{J}_i\}}$	Knupp 00

quality: deviation from..?

Polytopes Quality



high number of edges /faces



not star-shaped empty kernel

???

???

???

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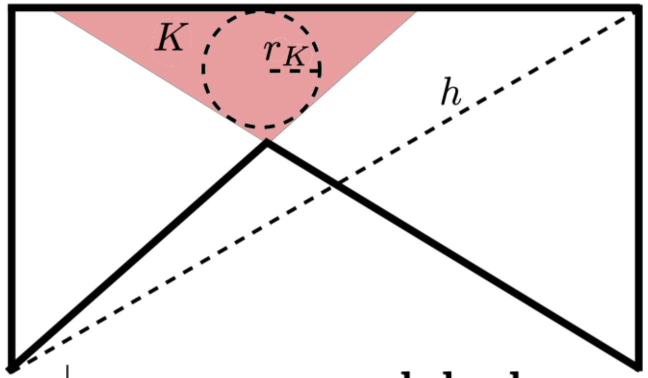


	polygons	polyhedra	reference
warping	$\max_i \left\{ \sin^{-1}\left(\frac{\lVert \mathbf{v}_i - \mathbf{v}'_i \rVert}{l} \right) \right\}$	$\max_i \left\{ \sin^{-1}\left(\frac{\lVert \mathbf{v}_i - \mathbf{v}'_i \rVert}{l} \right) \right\}$	Stimpson 07
aspect ratio	$\frac{r}{R}, \frac{2r}{h}, \frac{h}{2R}, \frac{r_K}{R}, \frac{A_K}{A}, \frac{A}{P^2}$	$\frac{r}{R}, \frac{2r}{h}, \frac{h}{2R}, \frac{r_K}{R}, \frac{V_K}{V}, \frac{V}{P^3}$	Gillette 12, Attene 21
skewness	$\theta_{\min}, \theta_{\max}, \frac{\theta_{\min}}{\theta_{\max}}$	$\min_{i=1,\dots,n} \{\theta_i\}$	Robinson 87, Knupp 03
interpolation	$\sqrt{\frac{\varrho_1\varrho_2 + \varrho_1\varrho_3 + \varrho_1\varrho_4}{3}}$ [Part II]	$\sqrt{\frac{\varrho_1\varrho_2 + \varrho_1\varrho_3}{2}}$ [Part II]	Sorgente 22-23
mean ratio	$\min_i \left\{ \frac{2}{\kappa(\mathcal{A}_i)} \right\}$	$\min_i \left\{ \frac{3}{\kappa(\mathcal{A}_i)} \right\}$	Knupp 07-12
shape regularity	$\min_i \left\{ \frac{\mathcal{J}_i}{e_{i-1}\ e_i} \right\}, \frac{\int_E \lVert x - \tilde{x} \rVert^2 \mathrm{dx}}{2\left(\int_E \mathrm{dx}\right)^2}$	$\min_i \left\{ \frac{\mathcal{J}_i}{e_{i-1}\ e_i\ e_{i+1}} \right\}, \frac{\int_E \lVert x - \tilde{x} \rVert^2 \mathrm{dx}}{3\left(\int_E \mathrm{dx}\right)^{5/2}}$	Knupp 00, Wang 17

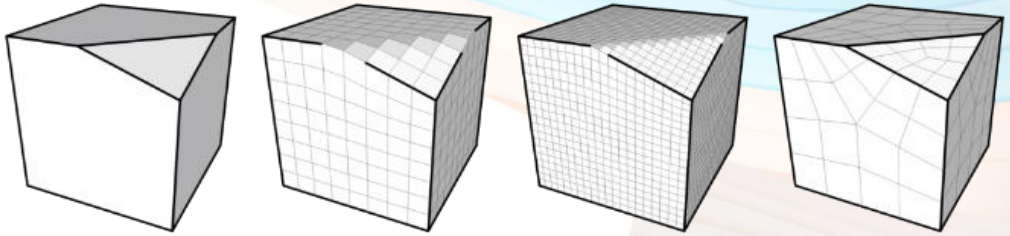
nesse

Mesh Quality Indicators: Consistency

geometry of the domain
- **max distance** between a domain point and the nearest node
- **alignment** to features lines (sharp creases)

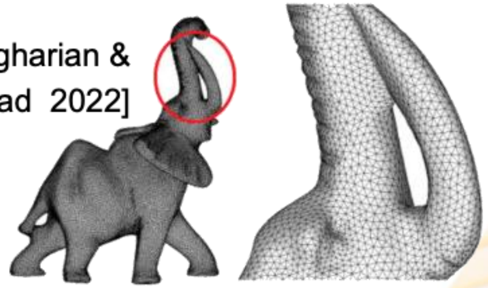
topology of the domain
- consistency rules (**Euler relation**)
- **number** of defects, holes, handles, connected components

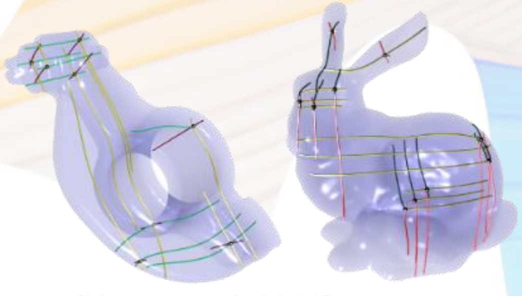
solution of the physical problem
- **alignment** to preferred directions (streamlines, vector fields, interior layers)
- adaptive **density** in regions of rapid variations (high gradients)



[Livesu et al. 2020]

[Asgharian & Ebrahimnezhad 2022]

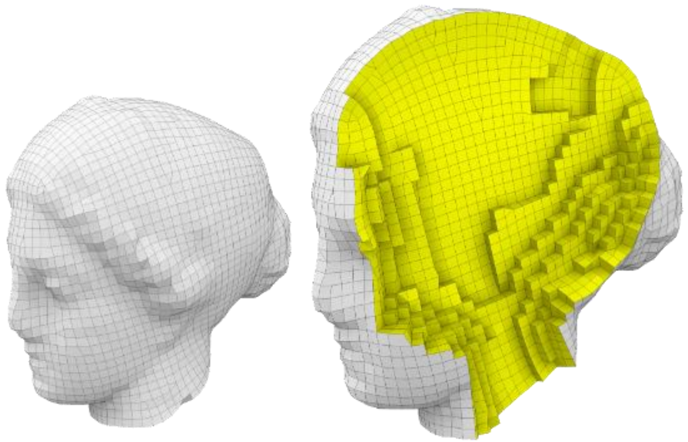




[Huang et al. 2011]

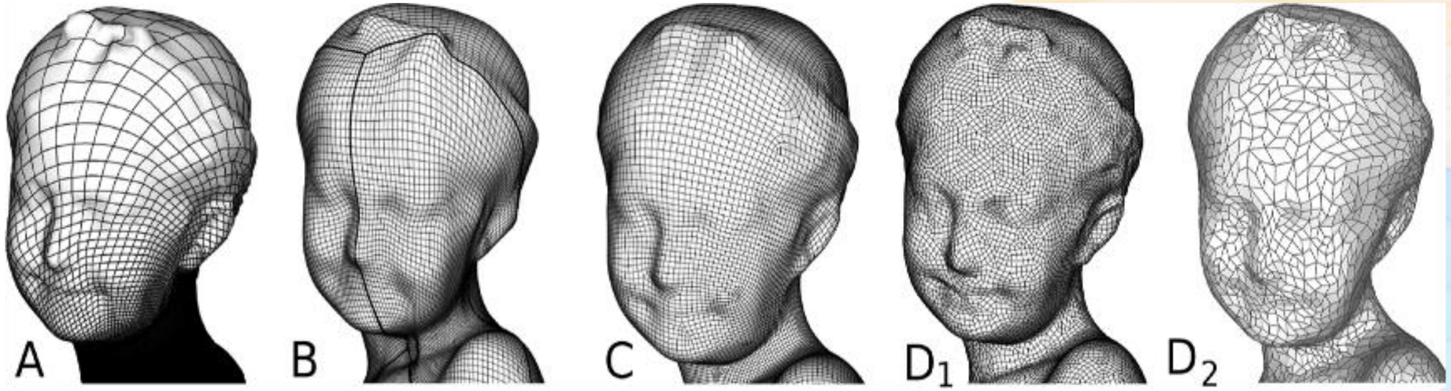
Mesh Quality Indicators: Structure

conformity
- **n.** of T-junctions
- n. of non-standard elements



[Gao et al. 2017]

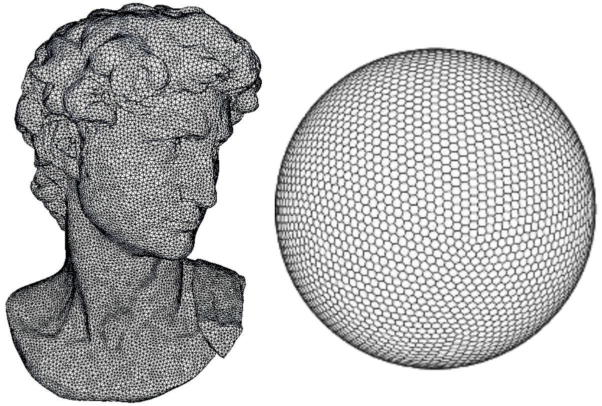
regularity (n. of singular nodes)
- regular/structured
- semi/block-regular
- valence semi-regular
- irregular/unstructured



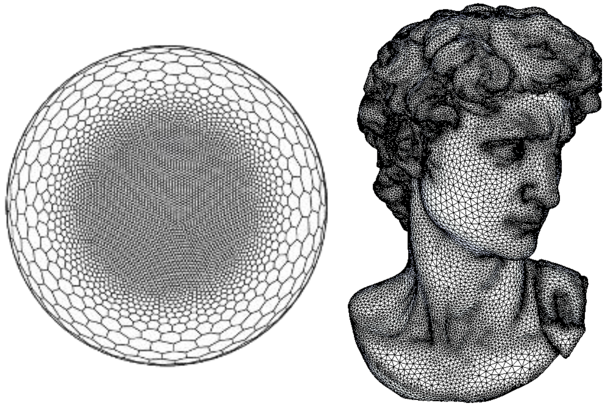
[Bommes et al. 2013]

Mesh Quality Indicators: Distribution

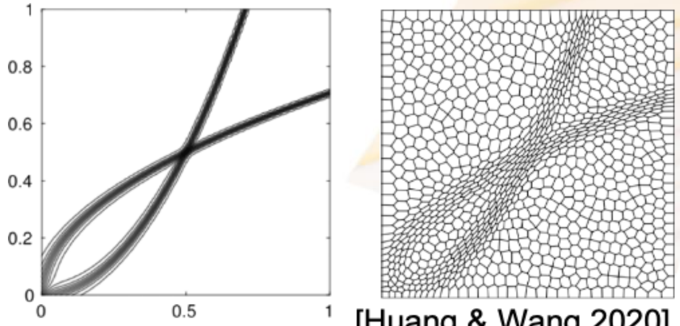
uniform
the element size (area, edge length, volumes, sum of the faces areas) must be similar across the mesh



adaptive
smooth size change, neighboring elements cannot have too different sizes (mesh gradation control)



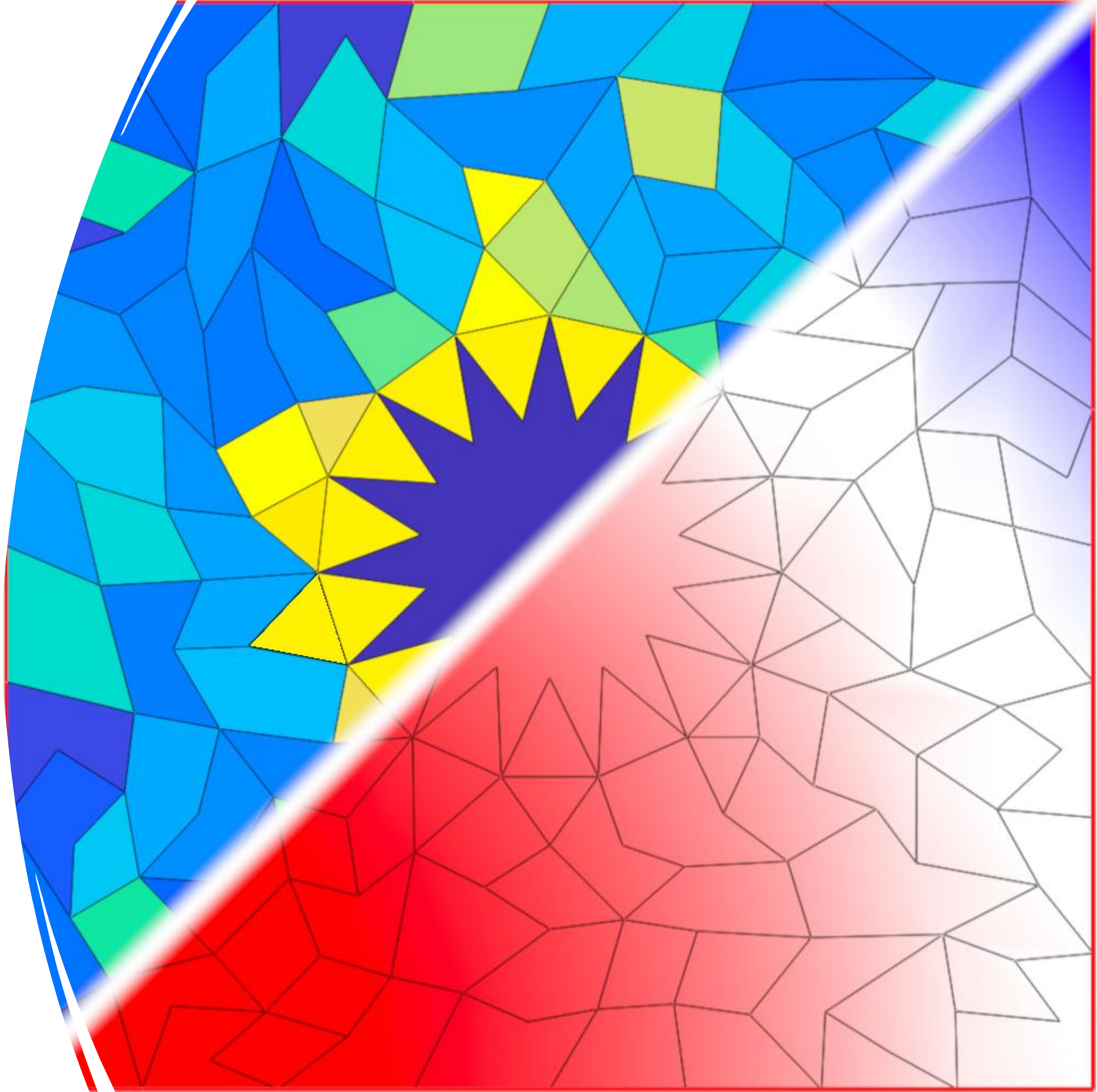
anisotropic
mesh generation guided by a prescribed (problem-dependent) anisotropy field



[Huang & Wang 2020]

PEMesh time!

Mesh Design & Quality Indicators



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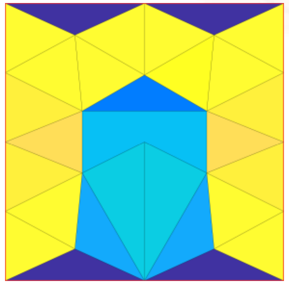
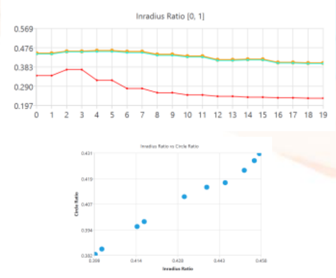
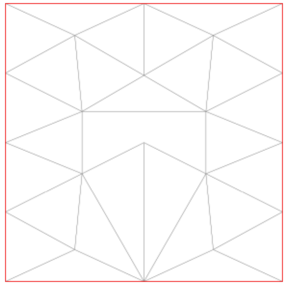
PEMesh - Demo

Parametric Meshes

- From parametric polygons







Element Quality Indicators

- Several way to visualize & analyze them

PEMesh - Background

A parametric polygon

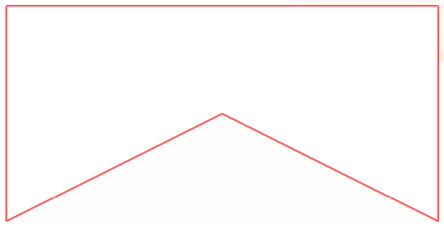
- **has a benign baseline configuration $P(0)$**

  no or little critical geometrical features

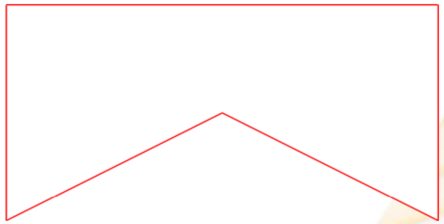
- **is progressively subject to a deformation, controlled by the parameter t**

  deformations introduces critical pathological geometrical features

[Attene et al. 2021]



$P(0)$

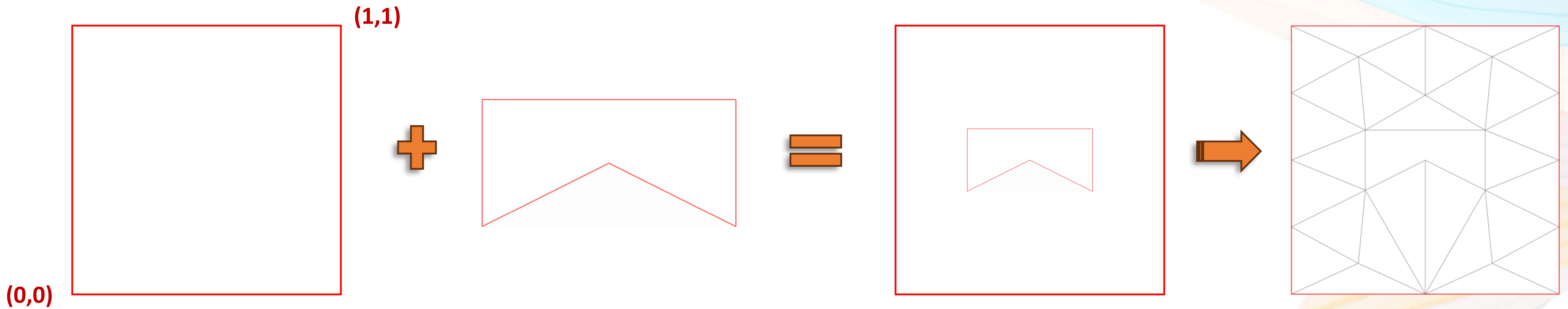


$t_i = \frac{1}{i},$

where $i = 1, \dots, N$ (# deformations)

PEMesh - Background

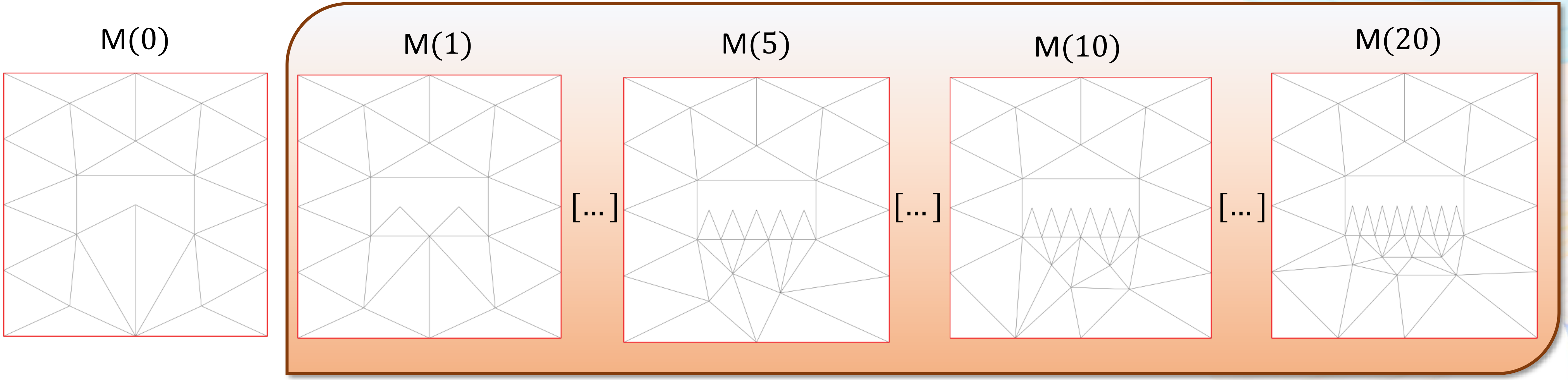
Mesh Generation



[Attene et al. 2021]

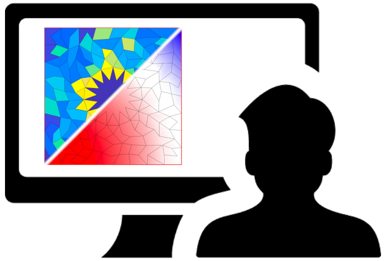
PEMesh - Background

Parametric dataset



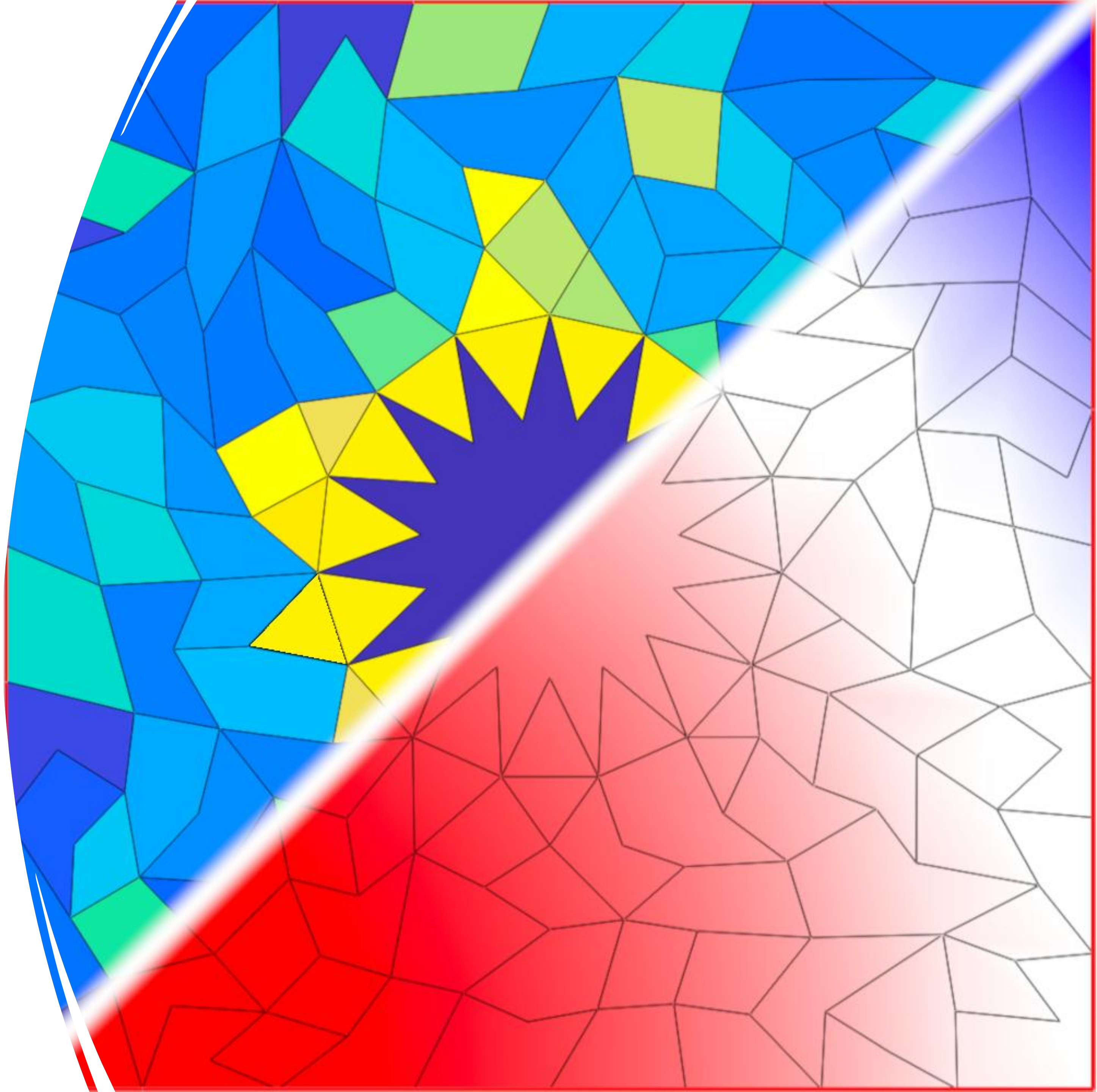
[Attene et al. 2021]

For each $t_i = \frac{1}{i}$, $with\ i = 1, \dots, N$



PEMesh time!

Mesh Design & Quality Indicators



Part II - The Virtual Element Method

Course Schedule

Introduction (20 min)

a) Motivations (M. Spagnuolo)

b) PEMesh Software (D. Cabiddu)

Part I - Mesh Quality (70 min)

a) Background and Notations (T. Sorgente)

b) Element Quality Indicators (T. Sorgente)

c) Mesh Quality Indicators (S. Biasotti)

d) PEMesh Exercise on Mesh Quality (D. Cabiddu)

Part II - The Virtual Element Method (70 min)

a) Background and Notations (F. Vicini)

b) VEM vs FEM (F. Vicini)

c) VEM Definition (F. Vicini)

Break (15 min)

d) Mesh Requirements for the VEM (T. Sorgente)

e) Mesh Quality Indicator for the VEM (T. Sorgente)

f) PEMesh Exercise on VEM (D. Cabiddu)

Part III - Optimizing Mesh Quality (40 min)

a) How to Get a Quality Mesh (S. Biasotti)

b) Mesh Optimization Driven by Quality Indicators (S. Biasotti)

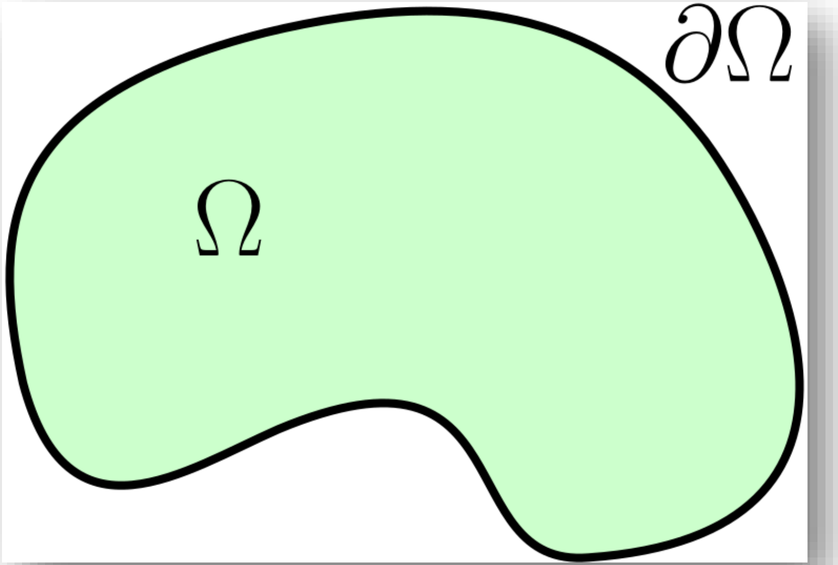
c) PEMesh Exercise on Optimization (D. Cabiddu)

Conclusions (10 min)

a) Perspectives (M. Spagnuolo)

b) Q&A

GOAL - Solve PDE



Elliptic Problem - Strong Formulation

Find u s.t.

$$\begin{cases} -\nabla \cdot (\mathbb{K}\nabla u) + \mathbf{b} \cdot \nabla u + cu = f & \text{on } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$$

FOCUS: Poisson Equation + Zero boundary conditions

$$\begin{cases} -\Delta u = f & \text{on } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

Elliptic Problem - Weak Formulation

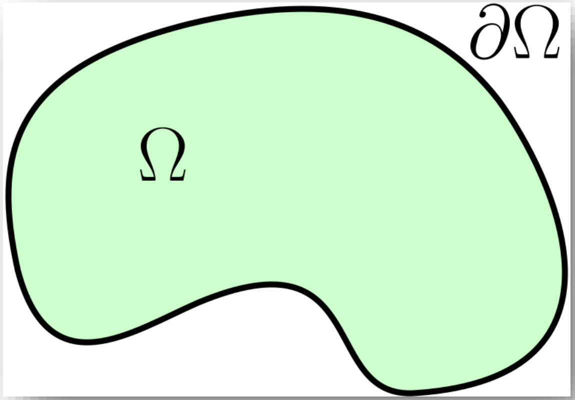
Find $u = (u_0 + u_D)$ s.t. $u_D|_{\partial\Omega} = g \in V^D$, $u_0 \in V$ and $\forall v \in V$

$$\int_\Omega (\mathbb{K}\nabla u_0)\cdot\nabla v + \mathbf{b}\cdot\nabla u_0 v + c u_0 v = \int_\Omega f v - (\mathbb{K}\nabla u_D)\cdot\nabla v$$

FOCUS: Poisson Equation + Zero boundary conditions

Find $u = u_0 \in V^D \equiv V := H^1_0(\Omega)$ *s.t.*

$$a(u,v) := \int_\Omega \nabla u \cdot \nabla v = \int_\Omega f v := F(v) \quad \forall v \in V$$



Unique solution [Lax-Milgram]

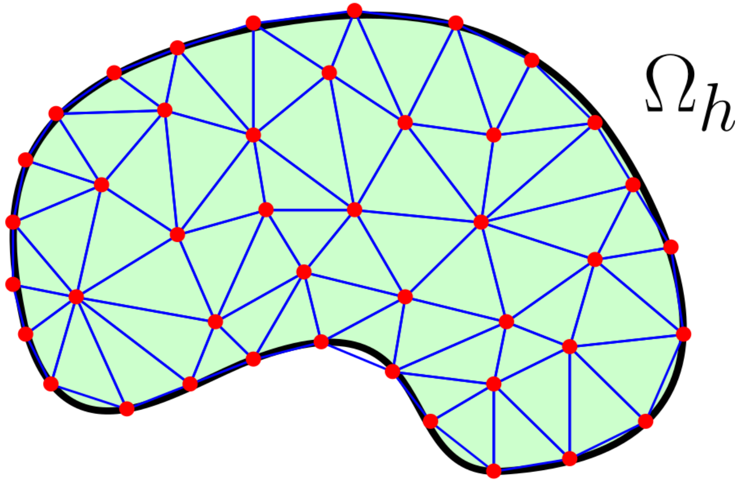
$u \in V$ exists and is unique $\Leftrightarrow$

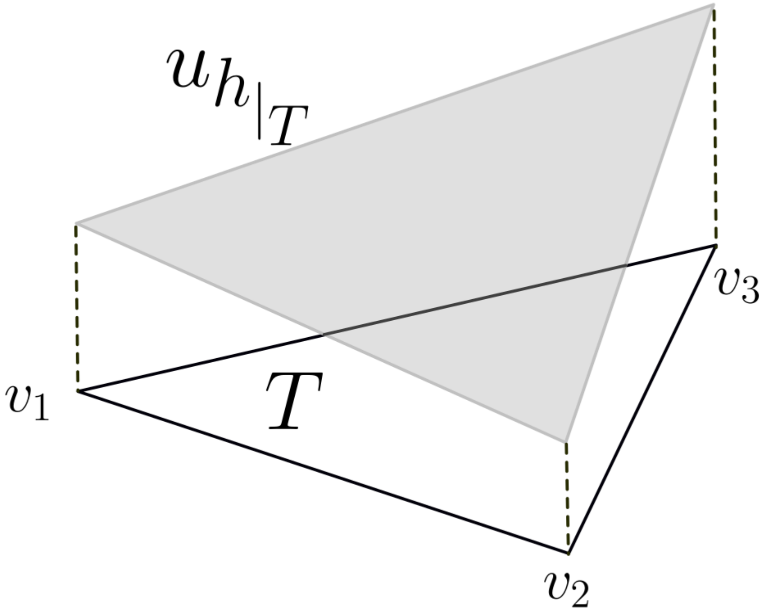
- ✓ a and F are continous
- ✓ a is coercive

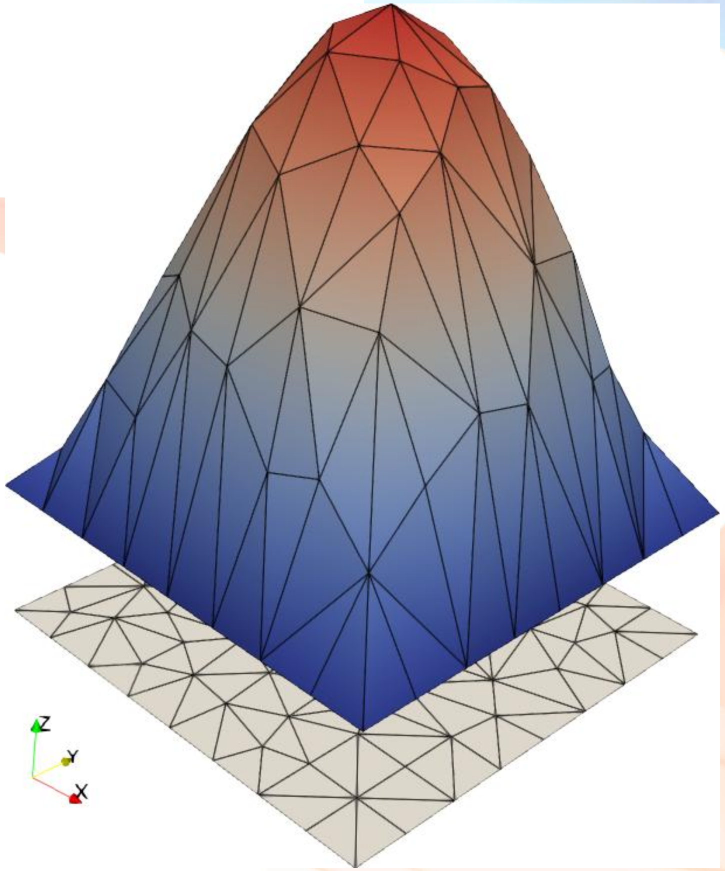
Finite Element Method (FEM)

u_h **Piecewise Polynomial** approximation on triangles $T \in \Omega_h$, $u_{h_{|T}} \in V_k^T \equiv \mathbb{P}_k(T)$

- ✓ local on the mesh element
- ✓ optimal error estimates







FEM Space

$V_h := \{u_h \in C^0(\Omega_h) : u_{h_{|T}} \in V_k^T, \ \forall T \in \Omega_h\} \cap V$ of dimension $N_h < +\infty$

Discrete Problem

We choose a basis $\{\varphi_i\}_{i=1}^{N_h}$ and we obtain:

Find $u_h \in V_h, u_h = \sum_{i=1}^{N_h} \varphi_i u_i$ *s.t.*

$$a(u_h, \varphi_j) := \int_\Omega \sum_{i=1}^{N_h} u_i \nabla\varphi_i \cdot \nabla\varphi_j = \int_\Omega f\varphi_j := F(\varphi_j) \quad \forall\varphi_j \in V_h$$

Unique solution [Lax-Milgram]

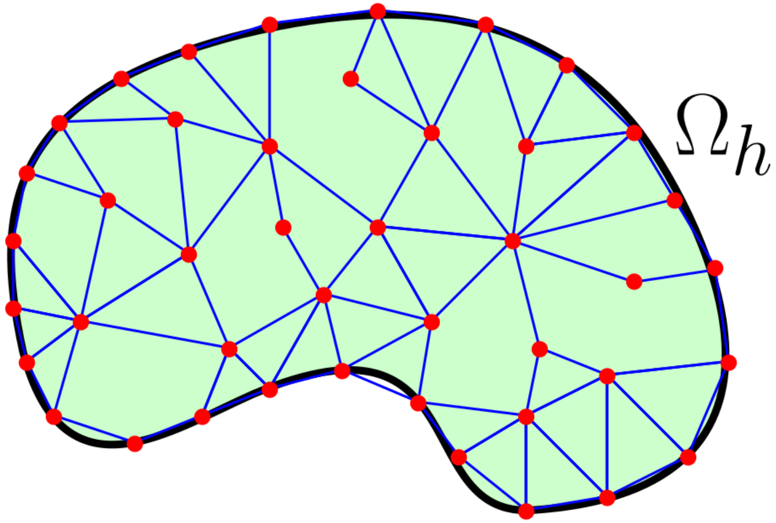
- ✓ $u_h \in V_h$ exists and is unique
- ✓ optimal error estimates

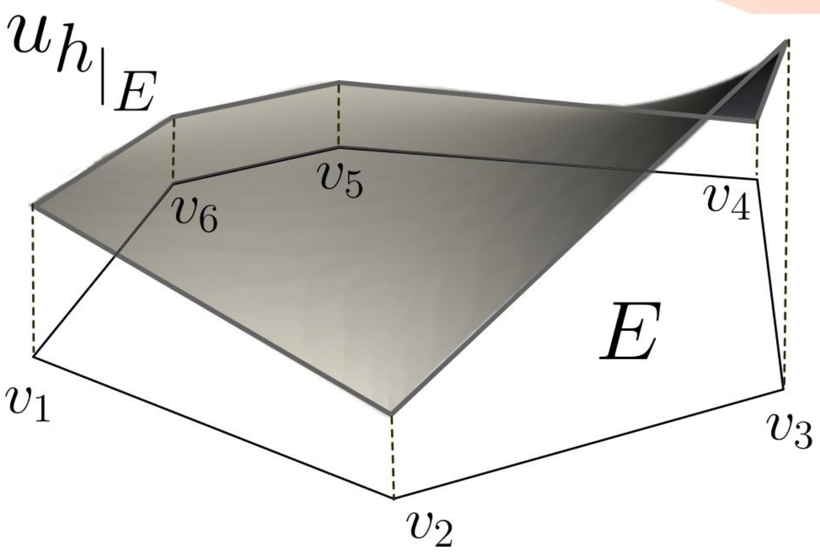
$$\begin{aligned} ||u - u_h||_{L^2(\Omega)} &\leq C_1 h^s |u|_{H^r(\Omega)} \quad s := \min\{k+1, r\} \\ ||u - u_h||_{H^1(\Omega)} &\leq C_2 h^{s-1} |u|_{H^r(\Omega)} \quad s := \min\{k+1, r\} \end{aligned}$$

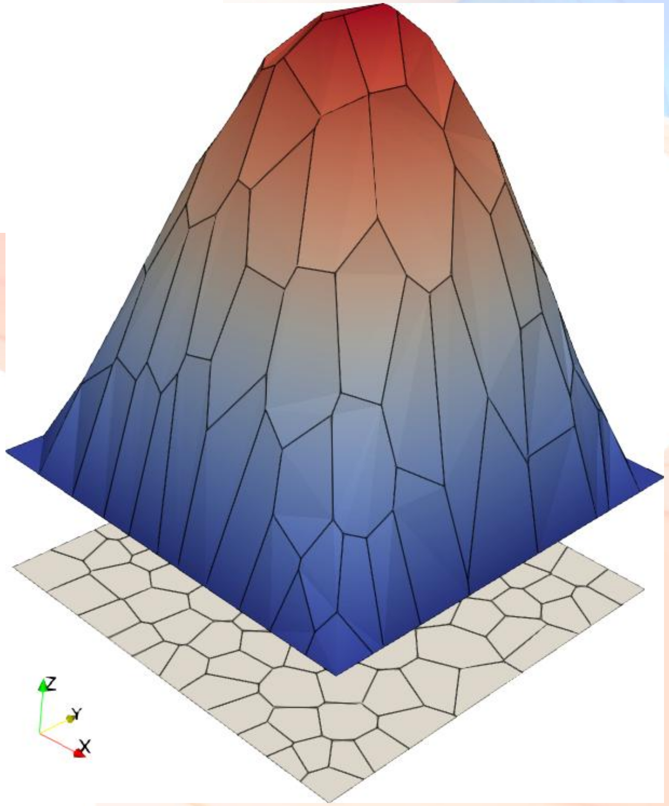
Virtual Element Method (VEM)

u_h **Piecewise** approximation on **GENERIC POLYGONS** $E \in \Omega_h$, $u_{h|_E} \in V_k^E \supseteq \mathbb{P}_k(E)$

- ✓ local on the mesh element
- ✓ optimal error







"*The virtual element method*" - Beirão Da Veiga, Brezzi, Marini, Russo - https://doi.org/10.1017/S0962492922000095

Polygonal methods

NOTE: VEM are not the only polygonal method

1. Mimetic Finite Differences;
2. Finite Volumes;
3. Discontinous Galerkin and Hybrid High-Order (Cangiani, Dong, Di Pietro, Ern,...);
4. Weak Galerkin (Mu, Wang, Ye,...);
5. ...

Moreover with VEM

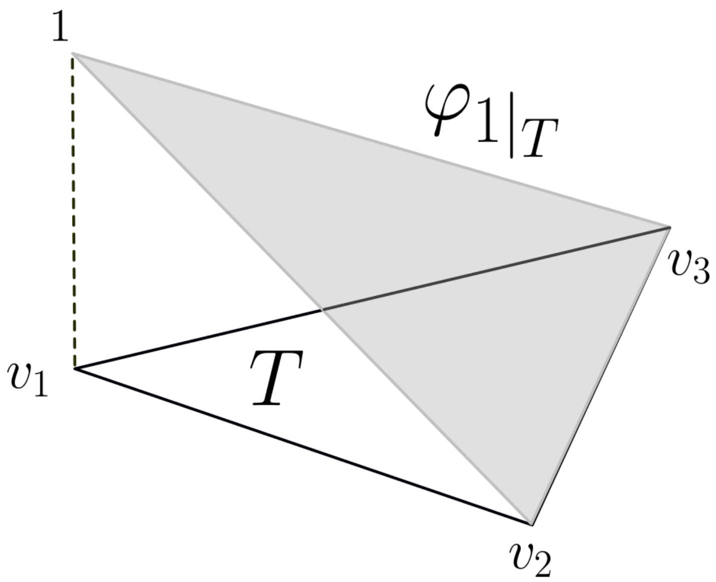
- Easy coupling with **FEM**;
- Possibility to **adapt the space** with the problem (Stokes, Elasticity,...);

FEM vs VEM

FEM Basis Functions

- are KNOWN in closed form

$$\varphi_i|_{\bar{T}} \in \mathbb{P}_k(T)$$



VEM Basis Functions

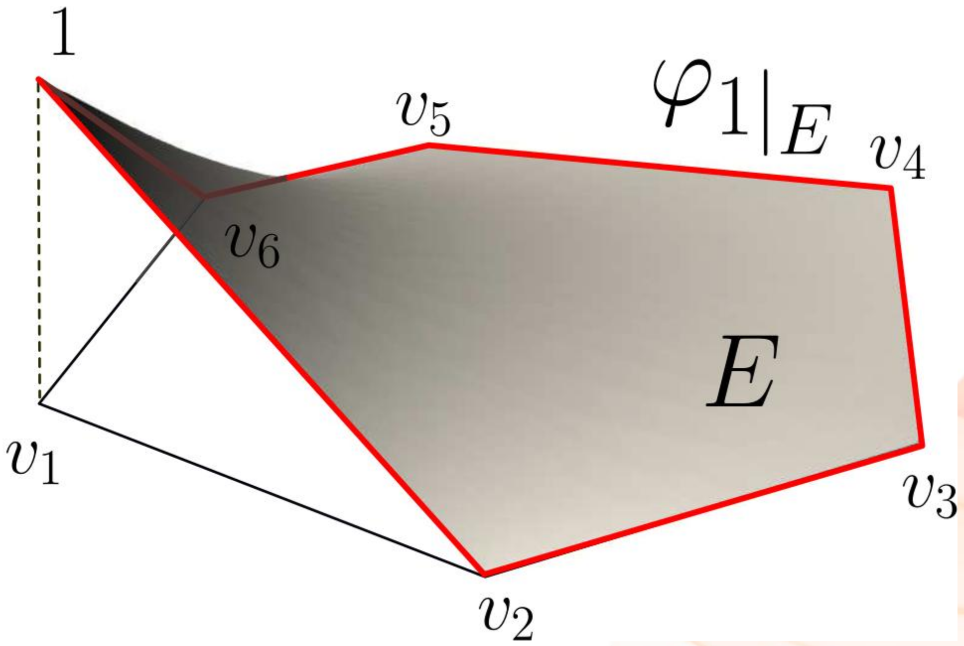
- are KNOWN in closed form on edges

$$\varphi_i|_{\partial E} \in B_k(\partial E) := \{v \in C^0 : (\partial E),\ v_{|_e} \in \mathbb{P}_k(e)\ \forall e \in \partial E\}$$

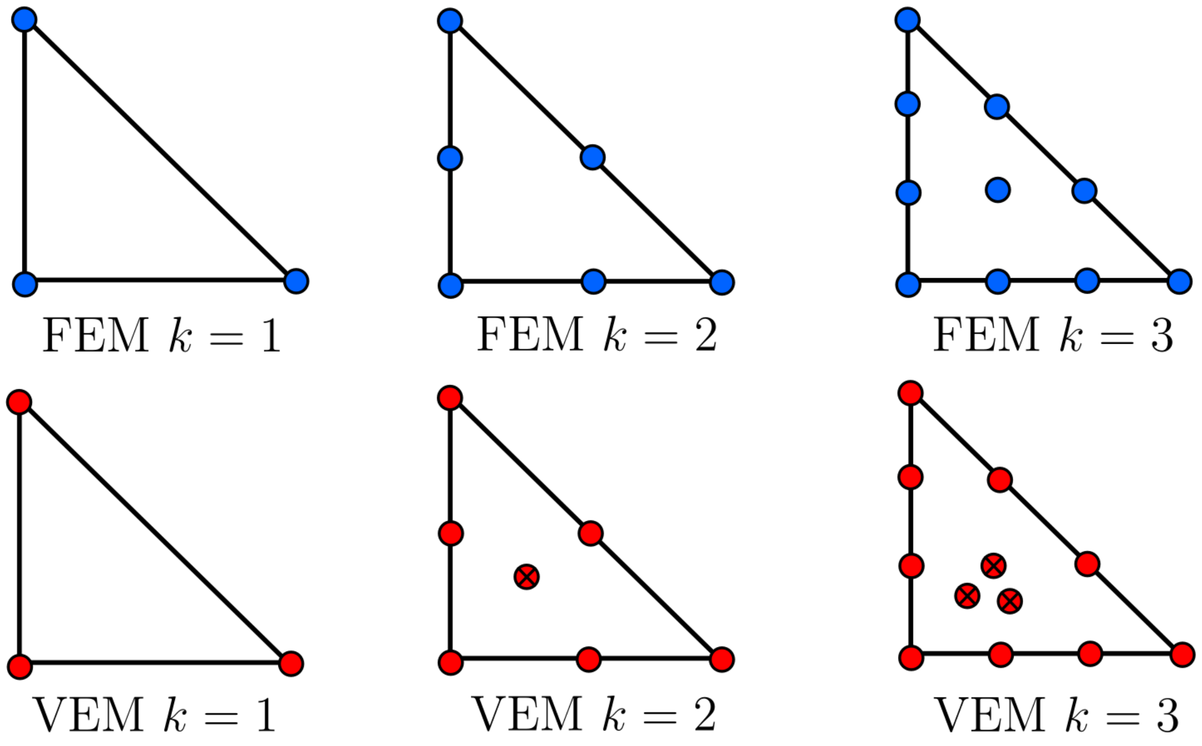
- are NOT KNOWN internally:

$$\varphi_i|_E \text{ is VIRTUAL}$$

- are in general NON LINEAR



FEM vs VEM



FEM Local Space

Generated by basis function φ_i evaluations on points.

VEM Local Space

Generated by basis function φ_i evaluations on border points, but **what about** internally?

NOTE: How to compare FEM and VEM when $k > 1$?

VEM space

Discrete Problem

We choose a basis $\{\varphi_i\}_{i=1}^{N_h}$ of $V_h \subset V$ and we obtain:

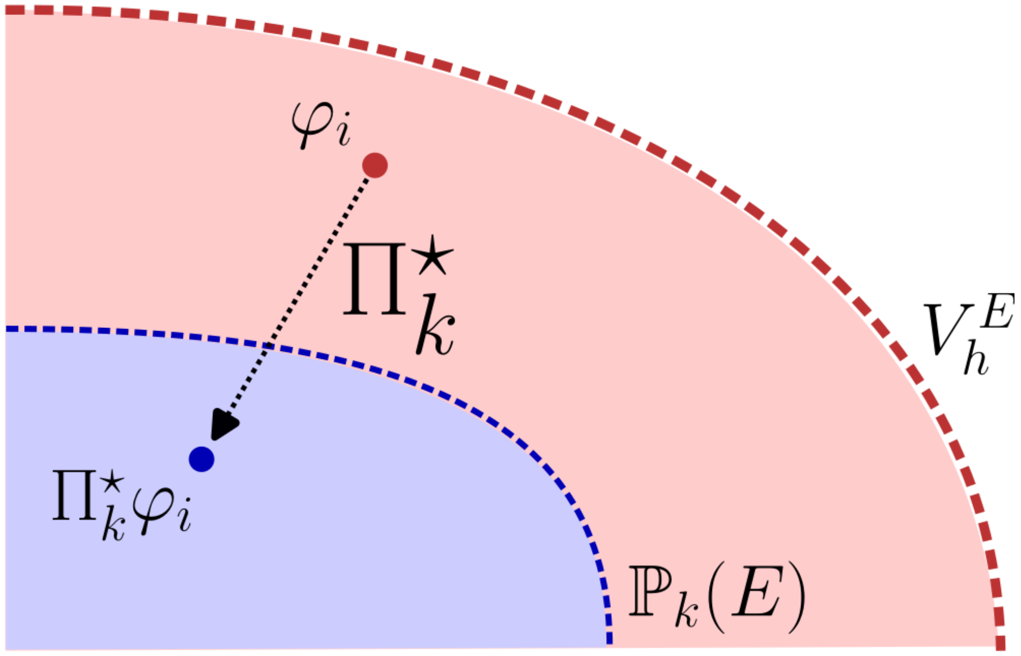
Find $u_h \in V_h, u_h = \sum_{i=1}^{N_h} \varphi_i u_i$ *s.t.*

$$a(u_h, \varphi_j) := \int_\Omega \sum_{i=1}^{N_h} u_i \nabla \varphi_i \cdot \nabla \varphi_j = \int_\Omega f \varphi_j := F(\varphi_j) \quad \forall \varphi_j$$

Questions:

1. **WHO** are V_k^E and V_h?
2. **HOW** to evaluate φ_i for a and f on VEM?

Project the unknown in something known...



New COMPUTABLE forms

Projector for $a(\cdot,\cdot)$

$$\boxed{\Pi_k^\nabla : V_h^E \subset H^1(E) \to \mathbb{P}_k(E)}$$

$$\int_E \nabla \Pi_k^\nabla \varphi_i \cdot \nabla p = \int_E \nabla \varphi_i \cdot \nabla p \quad \forall p \in \mathbb{P}_k(E)$$

Projector for $f(\cdot)$

$$\boxed{\Pi_k^0 : V_h^E \subset L^2(E) \to \mathbb{P}_k(E)}$$

$$\int_E p \Pi_k^0 \varphi_i = \int_E p \varphi_i \quad \forall p \in \mathbb{P}_k(E)$$

$$\boxed{a_h(\varphi_i, \varphi_j) := \int_\Omega \nabla \Pi_k^\nabla \varphi_i \cdot \nabla \Pi_k^\nabla \varphi_j} \quad \boxed{F_h(\varphi_j) := \int_\Omega f \Pi_k^0 \varphi_j}$$

VEM space

Π_k^∇ Computation

We need to compute

$$\boxed{\int_E \nabla \Pi_k^\nabla \varphi_i \cdot \nabla p = \int_E \nabla \varphi_i \cdot \nabla p \quad \forall p \in \mathbb{P}_k(E)}$$

We use **by parts** formula:

$$\int_E \nabla \varphi_i \cdot \nabla p = \boxed{\int_{\partial E} \varphi_i \nabla p \cdot \mathbf{n}} - \boxed{\int_E \varphi_i \Delta p}$$

Computation $k = 1$

✓ $\varphi_i|_{\partial E}$ KNOWN

✓ $\Delta p = 0$

Computation $k > 1$

✓ $\varphi_i|_{\partial E}$ KNOWN

X $\varphi_i|_E$ UNKNOWN, $\Delta p \in \mathbb{P}_{k-2}(E)$

VEM space

Π_k^0 Computation

We need to compute

$$\boxed{\int_E p\Pi_k^0\varphi_i = \int_E p\varphi_i \quad \forall p \in \mathbb{P}_k(E)}$$

again the problem is:

$$\boxed{\int_E p\varphi_i}$$

Computation $k = 1$

X $\varphi_i|_E$ UNKNOWN

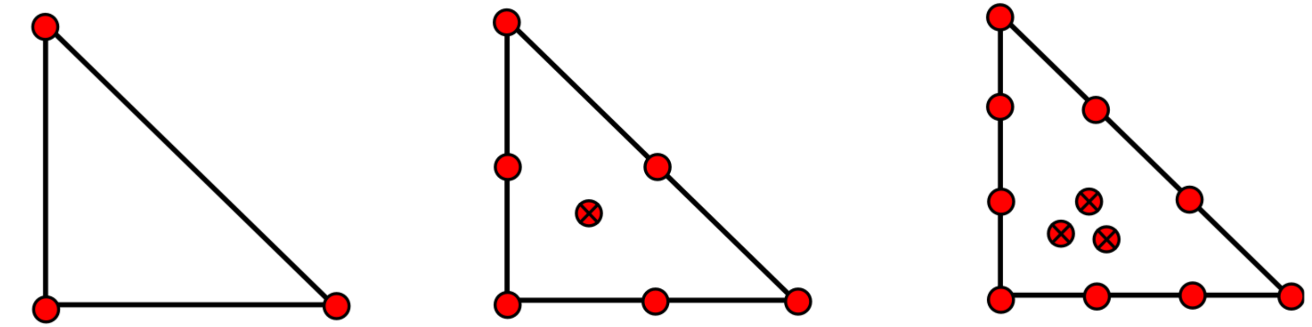
Computation $k > 1$

X $\varphi_i|_E$ UNKNOWN

VEM Space and DOFS

We require to **KNOW**:

1. N evaluation of φ_i at N vertices of E
2. $N(k-1)$ evaluation of φ_i at ∂E
3. $\frac{1}{2}k(k-1)$ internal **moments** $\frac{1}{|E|}\int_E \varphi_i m, \forall m \in \mathbb{P}_{k-2}(E)$
4. $$\int_E \varphi_i \tilde{p} = \int_E \Pi_k^0 \varphi_i \tilde{p} = \int_E \Pi_k^\nabla \varphi_i \tilde{p} \quad \forall \tilde{p} \in \mathbb{P}_{k-1}^{\text{hom}}(E) \cup \mathbb{P}_k^{\text{hom}}(E)$$



VEM $k=1$ VEM $k=2$ VEM $k=3$

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Π_k^∇ Π_k^0 Computation

1. N evaluation of φ_i at N vertices of E
2. $N(k-1)$ evaluation of φ_i at ∂E
3. $\frac{1}{2}k(k-1)$ internal **moments** $\frac{1}{|E|}\int_E \varphi_i m,\ \forall m \in \mathbb{P}_{k-2}(E)$
4. $$\int_E \varphi_i \tilde{p} = \int_E \Pi_k^0 \varphi_i \tilde{p} = \int_E \Pi_k^\nabla \varphi_i \tilde{p} \quad \forall \tilde{p} \in \mathbb{P}_{k-1}^{\text{hom}}(E) \cup \mathbb{P}_k^{\text{hom}}(E)$$

Now we can compute:

$$\boxed{\int_{\partial E} \varphi_i \nabla p \cdot \mathbf{n}} \qquad \boxed{\int_E \varphi_i \Delta p} \qquad \boxed{\int_E p\varphi_i}$$

✓ thanks to 1 and 2 ✓ thanks to 3 ✓ thanks to 4

VEM space

- **FOCUS**: Poisson Equation $-\Delta u = f$
- We want: $V_k^E \supseteq \mathbb{P}_k(E) \Rightarrow \Delta u \in \mathbb{P}_{k-2}(E)$

$$\tilde{V}_k^E := \left\{ v \in C^0(E) : v_{|_e} \in \mathbb{P}_k(e)\ \forall e \subset \partial E,\ \Delta v \in \mathbb{P}_{k-2}(E) \right\}$$

VEM (enhanced) Space:

$$\boxed{\begin{aligned} V_k^E := \Big\{ v \in C^0(E) : v_{|_e} \in \mathbb{P}_k(e)\ \forall e \subset \partial E,\ \Delta v \in \mathbb{P}_k(E), \\ \int_E (v - \Pi_k^\nabla v) p = 0\ \forall p \in \mathbb{P}_{k-1}^{\text{hom}}(E) \cup \mathbb{P}_k^{\text{hom}}(E) \Big\} \end{aligned}}$$

$$\boxed{V_h := \{ u_h \in C^0(\Omega_h) : u_{h|_E} \in V_k^E,\ \forall E \in \Omega_h \} \cap V}$$

VEM Operators

$$\boxed{a_h(\varphi_i, \varphi_j) := \int_\Omega \nabla \Pi_k^\nabla \varphi_i \cdot \nabla \Pi_k^\nabla \varphi_j} \quad \boxed{F_h(\varphi_j) := \int_\Omega f \Pi_k^0 \varphi_j}$$

Unique solution [Lax-Milgram]...**not yet**

X $u_h \in V_h$ exists and is unique

We miss the **stability** property of a_h

$$\alpha_* a^E(v_h, v_h) \leq a_h^E(v_h, v_h) \leq \alpha^* a^E(v_h, v_h) \quad \forall v_h \in V_k^E$$

The new COMPUTABLE and STABLE form

$$a_h^E(\varphi_i, \varphi_j) := \int_E \nabla \Pi_k^\nabla \varphi_i \cdot \nabla \Pi_k^\nabla \varphi_j + \boxed{S^E\left((I - \Pi_k^\nabla)\varphi_i, (I - \Pi_k^\nabla)\varphi_j\right)}$$

Stabilization

We choose S^E such that $\forall v_h \in V_k^E$ where $\boxed{\Pi_k^\nabla v_h = 0}$:

$$\boxed{c_1 a^E(v_h, v_h) \leq S_h^E(v_h, v_h) \leq c_2 a^E(v_h, v_h)}$$

Different alternatives:

1.
$$S_h^E(v_h, w_h) := \sum_{i=1}^{\#\text{dofs}} \text{dof}_i(v_h)\text{dof}_i(w_h)$$

2.
$$S_h^E(v_h, w_h) := \frac{1}{h_E} \int_{\partial E} v_h w_h$$

VEM Space - Summary

$V_h := \{u_h \in C^0(\Omega_h) : u_{h_{|T}} \in V_k^E,\ \forall E \in \Omega_h\} \cap V$ of dimension $N_h < +\infty$

Discrete Problem

We choose a basis $\{\varphi_i\}_{i=1}^{N_h}$ and we obtain:

Find $u_h \in V_h, u_h = \sum_{i=1}^{N_h} \varphi_i u_i$ *s.t.*

$$a(u_h, \varphi_j) := \sum_{E \in \Omega_h} \int_E \nabla \Pi_k^\nabla \varphi_i \cdot \nabla \Pi_k^\nabla \varphi_j + S^E\left((I - \Pi_k^\nabla)\varphi_i, (I - \Pi_k^\nabla)\varphi_j\right) =$$
$$= \sum_{E \in \Omega_h} \int_E f \Pi_k^0 \varphi_j := F(\varphi_j) \quad \forall \varphi_j \in V_h$$

Unique solution [Lax-Milgram]

- ✓ $u_h \in V_h$ exists and is unique
- ✓ optimal error estimates

$$||u - u_h||_{L^2(\Omega)} \leq C_1 h^s |u|_{H^r(\Omega)} \quad s := \min\{k+1, r\}$$
$$||u - u_h||_{H^1(\Omega)} \leq C_2 h^{s-1} |u|_{H^r(\Omega)} \quad s := \min\{k+1, r\}$$

VEM error estimation

Error numerical measures

- L^2-Error :=

$$\frac{\sum_{E\in\Omega_h} ||u - \Pi_k^0 u_h||_{0,E}}{||u||_{0,\Omega}}$$

- H^1-Error :=

$$\frac{\sum_{E\in\Omega_h} ||\nabla u - \nabla\Pi_k^\nabla u_h||_{0,E}}{|u|_{1,\Omega}}$$

- condition number $\kappa(u_h)$ is the smallest value such that:

$$\frac{||\mathbb{A}u_h - \mathbb{A}\hat{u}_h||_2}{||\mathbb{A}u_h||_2} \leq \kappa(u_h)\frac{||u_h - \hat{u}_h||_2}{||u_h||_2}$$

- residual $r(u_h)$:=

$$\frac{||\mathbb{A}u_h - b||_2}{||b||_2}$$

VEM Stabilization

alternatives

- *"A stabilization-free virtual element method based on divergence-free projections"* - Berrone, Borio, Marcon - https://doi.org/10.1016/j.cma.2024.116885
- *"Reduced basis stabilization and post-processing for the virtual element method"* - Credali, Bertoluzza, Prada - https://doi.org/10.1016/j.cma.2023.116693
- *"When rational functions meet virtual elements: the lightning virtual element method"* - Trezzi, Zerbinati - https://doi.org/10.1007/s10092-024-00585-1
- *"The lowest-order Neural Approximated Virtual Element Method on polygonal elements"* (NAVEM) - Berrone, Pintore, Teora - https://doi.org/10.48550/arXiv.2409.15917

VEM Implementation
Libraries

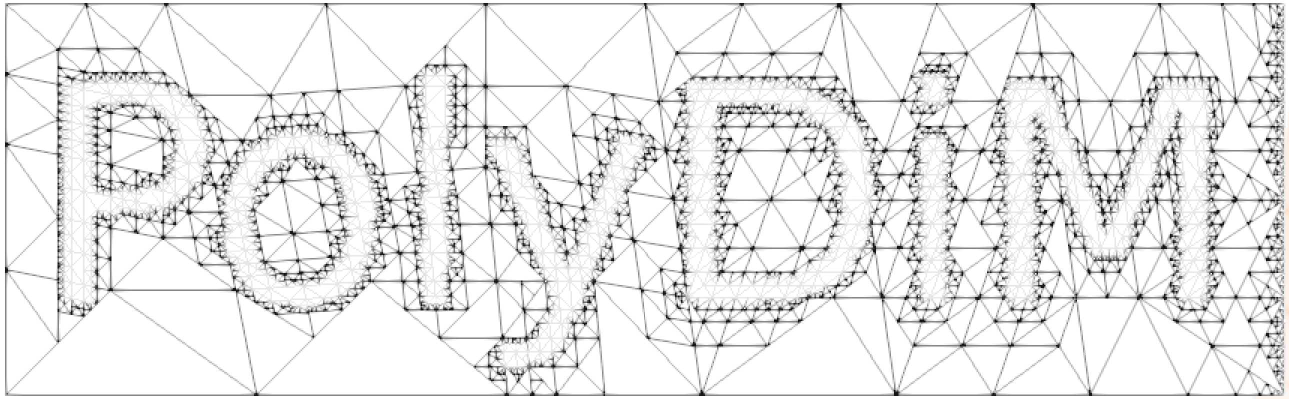
- **DUNE VEM** - https://dune-project.org/sphinx/content/sphinx/dune-fem/vemdemo_descr.html

Virtual Element Methods: the DUNE-VEM module

This module is based on dune-fem and provides implementation for the Virtual Element Method.

- **POLYDIM** - (to appear in January 2025) - https://github.com/AURION-Polito

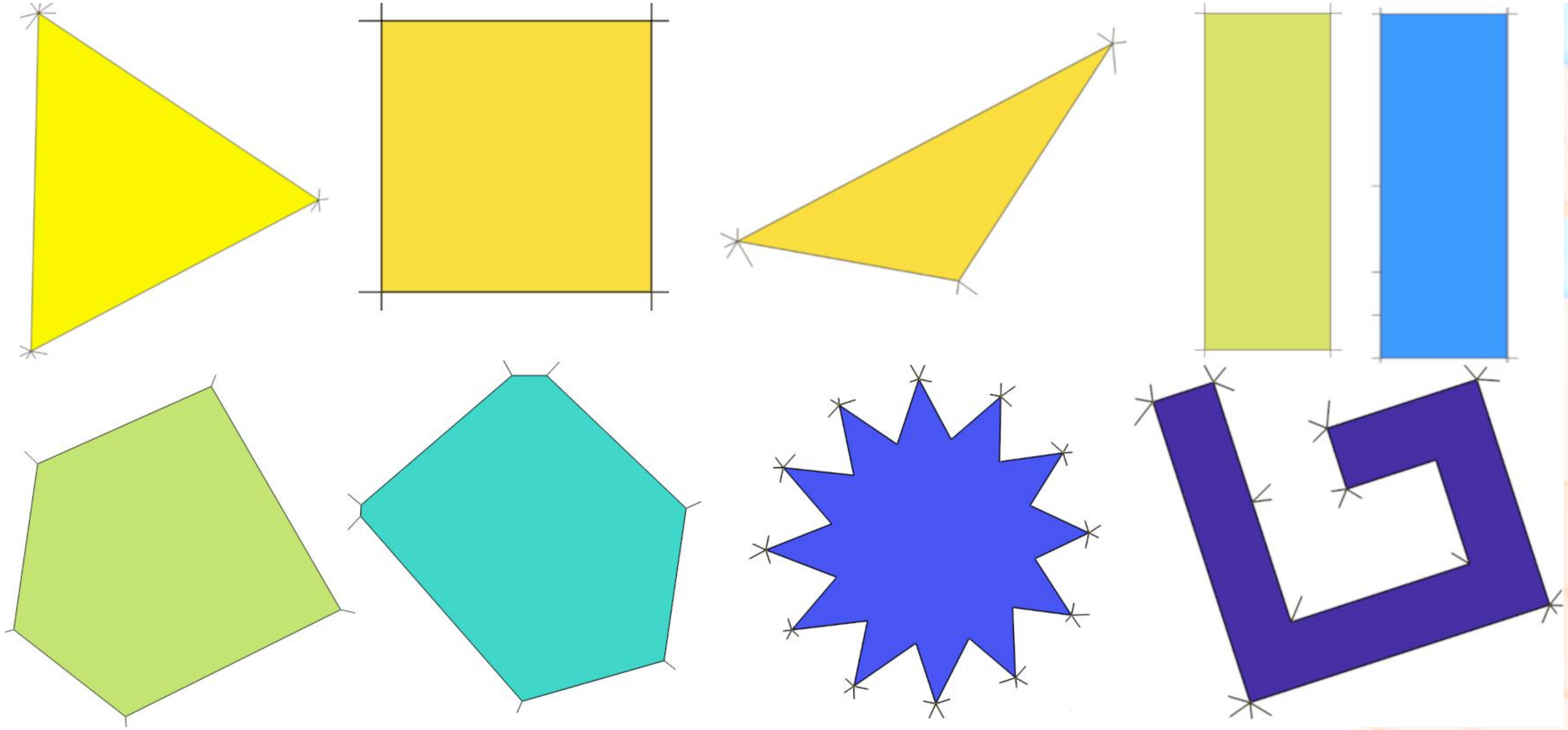
POLYDIM



Final questions

?? are VEM possible on **EVERY polygons**?

?? if yes... is the VEM solution **GOOD** on every polygons?

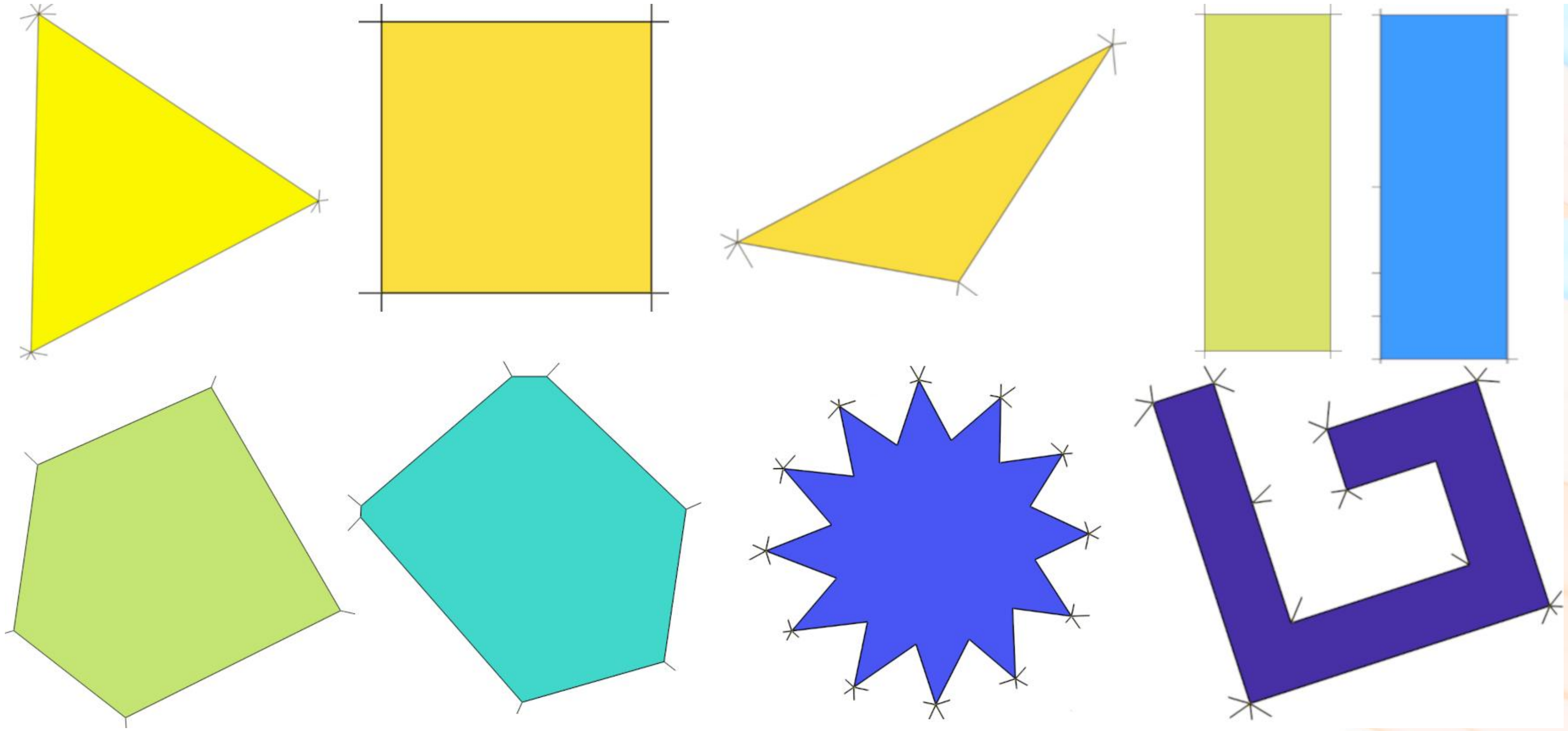


Break (15 min)

Final questions

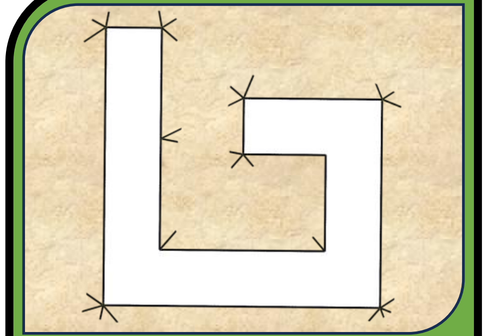
?? are VEM possible on **EVERY polygons**?

?? if yes... is the VEM solution **GOOD** on every polygons?

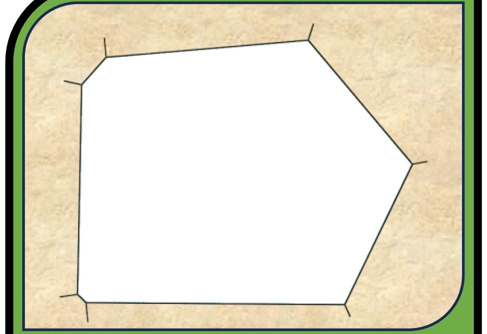


VEM Geometrical Assumptions

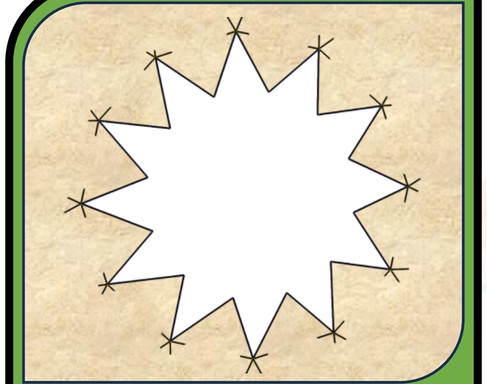
[Sorgente et al, VEM and the Mesh, 2021]



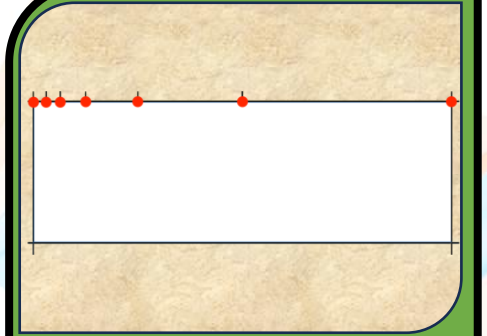
G1: every element is star-shaped with respect to a disc with bounded radius



G2: for every element, the ratio between each edge and the diameter is bounded



G3: the number of edges of every element is bounded



G4: the boundary sub-mesh of every element is quasi-uniform

Theorem: if the mesh satisfies G1+G2, or G1+G3, or G1+G4, then the VEM will converge optimally

and if not??

VEM Quality Indicators

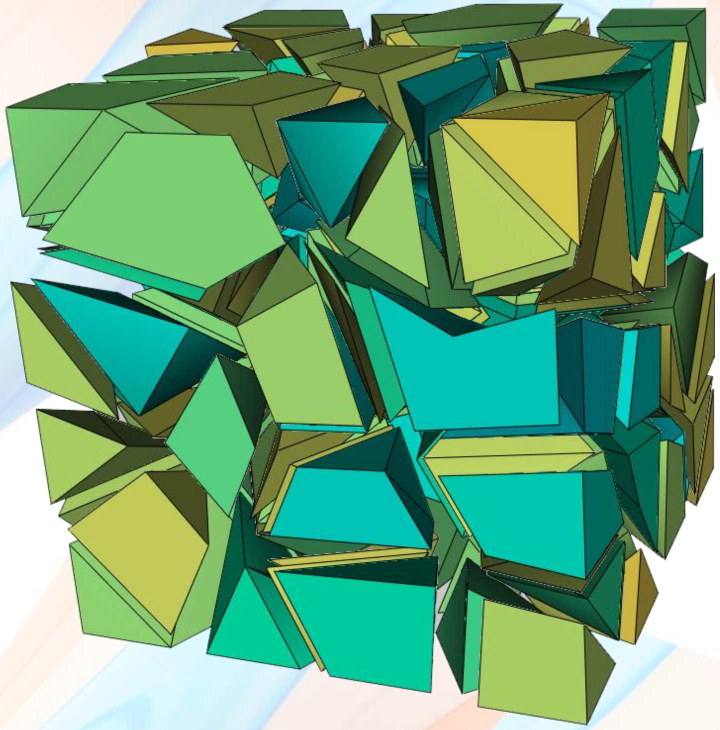
from each geometrical assumption $\boldsymbol{G_i}$ we derive a scalar indicator $\varrho_i: \Omega_h \rightarrow [0,1]$

$$\mathbf{G1} \rightarrow \varrho_1(E) = \frac{A_K}{A} = \begin{cases} 1 & \text{if } E \text{ is convex} \\ \in (0,1) & \text{if } E \text{ is concave and star-shaped} \\ 0 & \text{if } E \text{ is not star-shaped} \end{cases}$$

$$\mathbf{G2} \rightarrow \varrho_2(E) = \frac{\min\left(\sqrt{A}, e_{\min}\right)}{h}$$

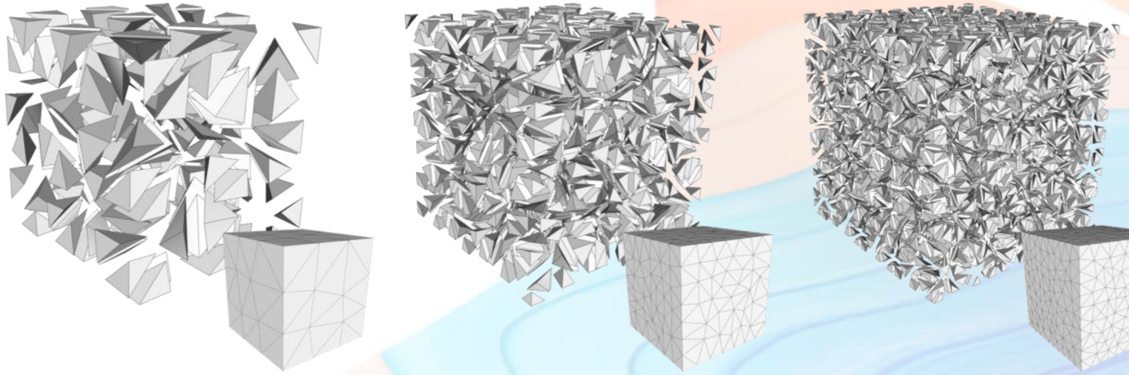
$$\mathbf{G3} \rightarrow \varrho_3(E) = \frac{3}{\#\{e \in \partial E\}} = \begin{cases} 1 & \text{if } E \text{ is a triangle} \\ \in (0,1) & \text{otherwise} \end{cases}$$

$$\mathbf{G4} \rightarrow \varrho_4(E) = \min_i \left\{ \frac{\min_{\partial E_i} e}{\max_{\partial E_i} e} \right\}$$

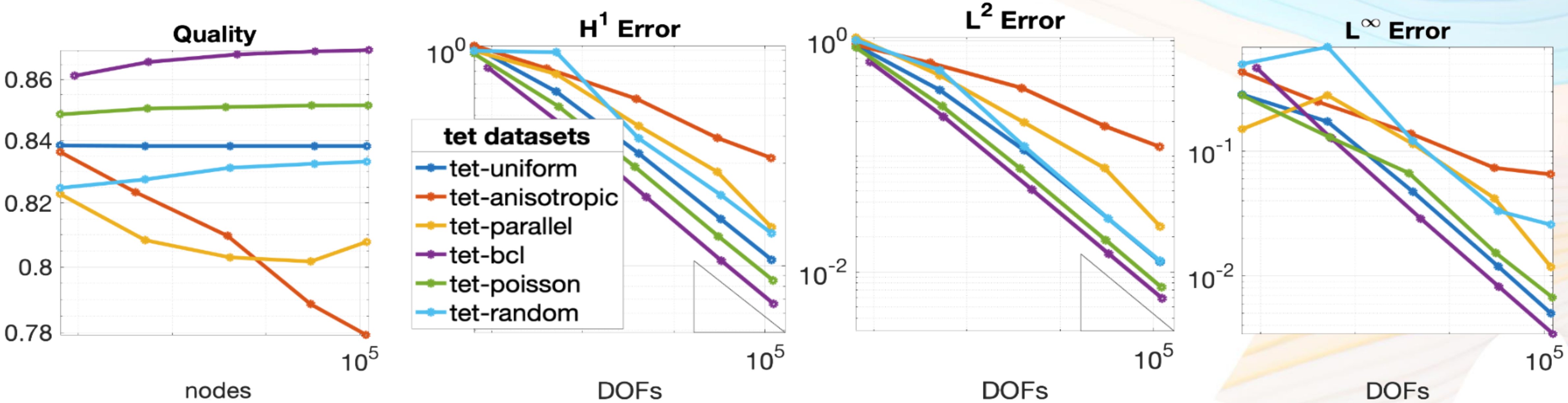


the single indicators are combined in an element quality indicator ϱ

$$\varrho(E) = \sqrt{\frac{\varrho_1(E)[\varrho_2(E) + \varrho_3(E) + \varrho_4(E)]}{3}}$$



Indicator / Performance Correlation

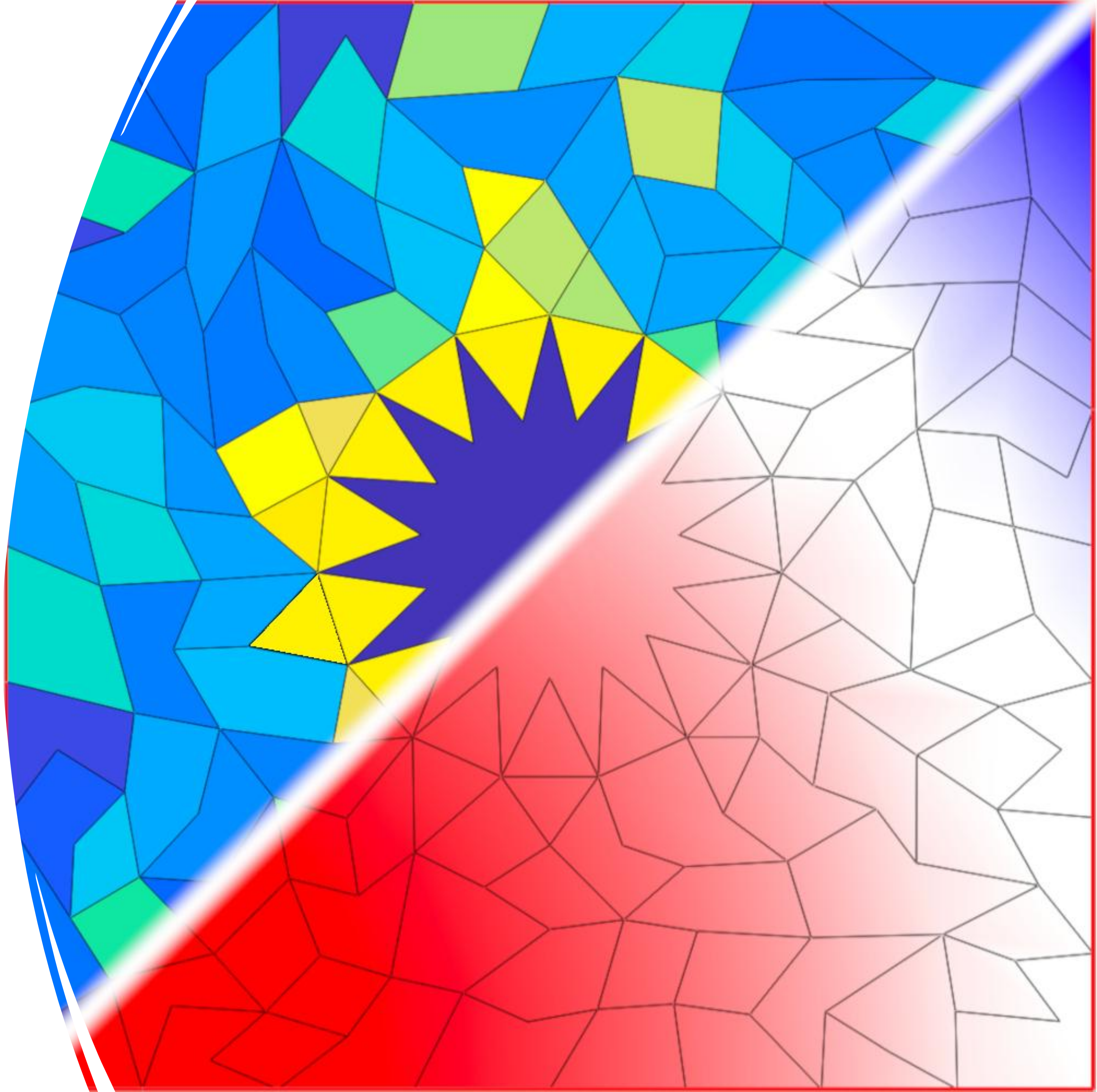


value ⟷ approximation error
slope ⟷ convergence rate

[Sorgente et al, The Role of Mesh Quality and Mesh Quality Indicators in the VEM, 2022]
[Sorgente et al, Polyhedral Mesh Quality Indicator for the VEM, 2022]

PEMesh time!

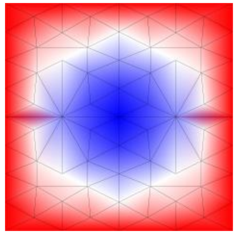
VEM Solver & Quality Indicators

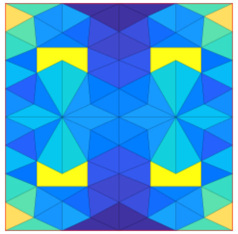


PEMesh - Demo

VEM Solver

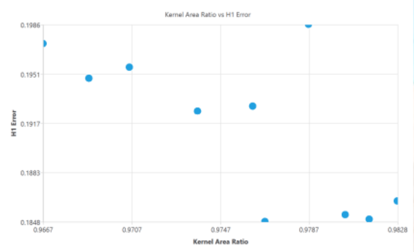
- Settings





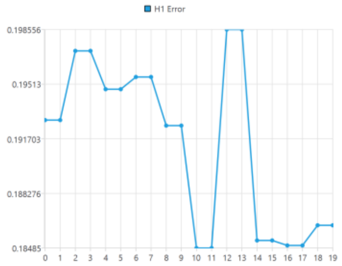
VEM Performance Analysis

- Errors
- Condition Number



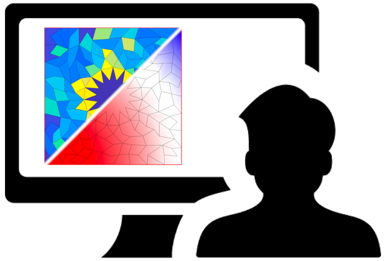
Graphical Inspection

- Solution
- Errors



Correlation between

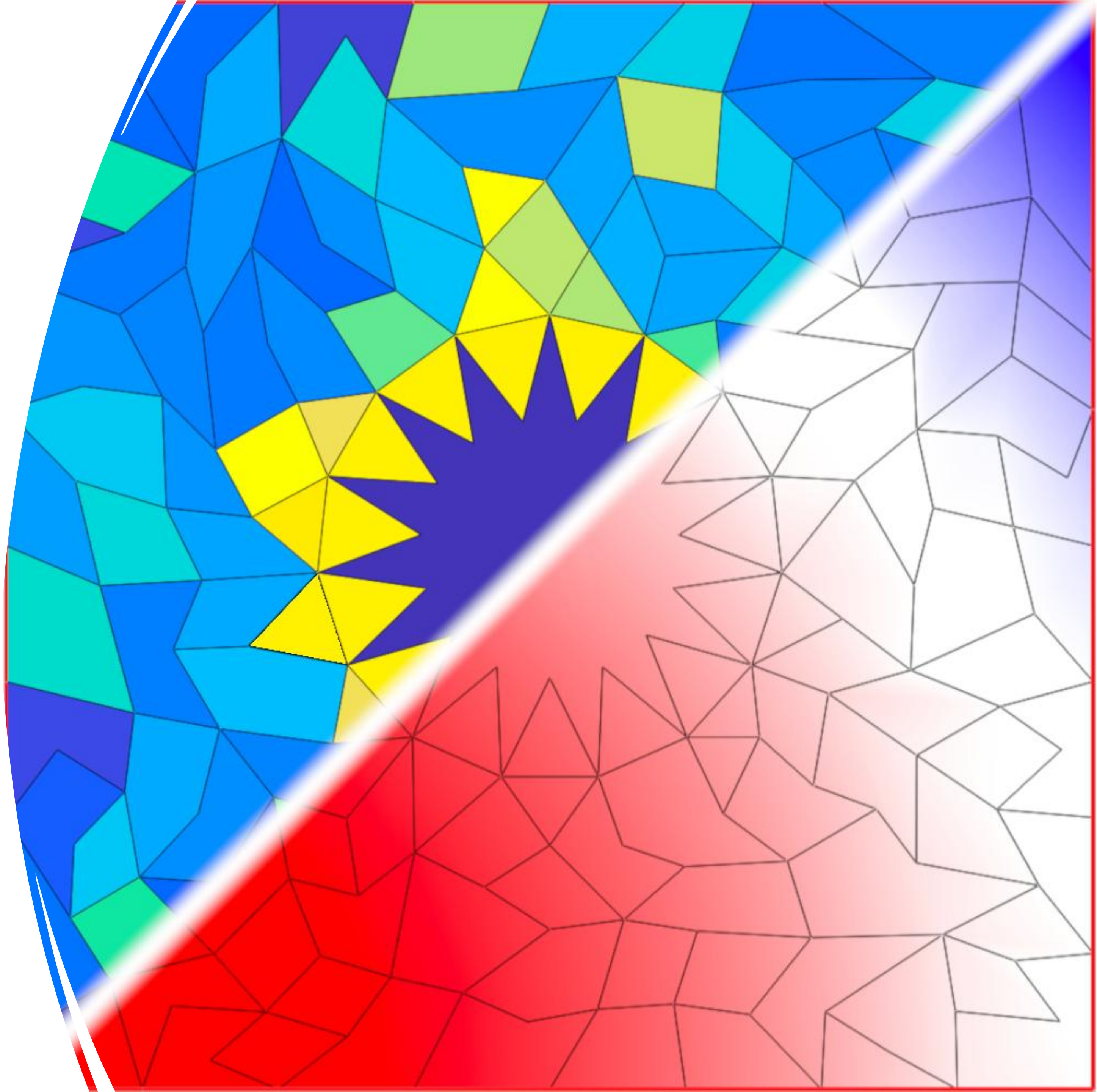
- VEM Performances
- Quality Indicators



PEMesh time!

VEM Solver & Quality Indicators

Docker must be running and NOT in “Save Resources” mode



Part III – Optimizing Mesh Quality

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Course Schedule

Introduction (20 min)

a) Motivations (M. Spagnuolo)

b) PEMesh Software (D. Cabiddu)

Part I - Mesh Quality (70 min)

a) Background and Notations (T. Sorgente)

b) Element Quality Indicators (T. Sorgente)

c) Mesh Quality Indicators (S. Biasotti)

d) PEMesh Exercise on Mesh Quality (D. Cabiddu)

Part II - The Virtual Element Method (70 min)

a) Background and Notations (F. Vicini)

b) VEM vs FEM (F. Vicini)

c) VEM Definition (F. Vicini)

Break (15 min)

d) Mesh Requirements for the VEM (T. Sorgente)

e) Mesh Quality Indicator for the VEM (T. Sorgente)

f) PEMesh Exercise on VEM (D. Cabiddu)

Part III - Optimizing Mesh Quality (40 min)

a) How to Get a Quality Mesh (S. Biasotti)

b) Mesh Optimization Driven by Quality Indicators (S. Biasotti)

c) PEMesh Exercise on Optimization (D. Cabiddu)

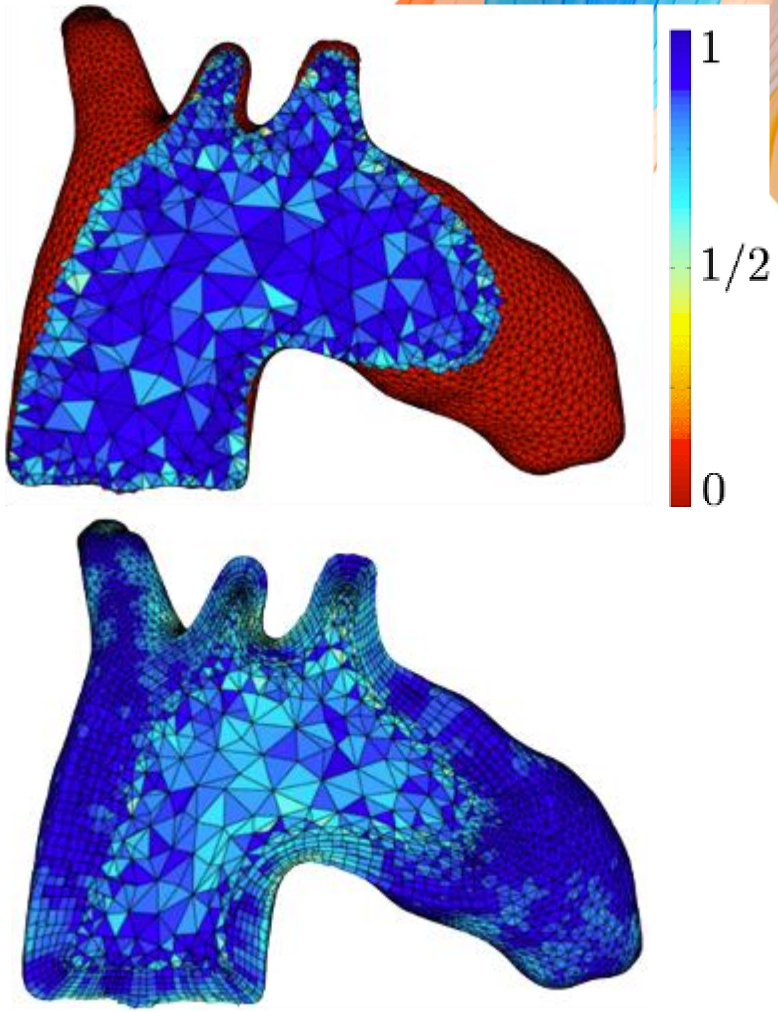
Conclusions (10 min)

a) Perspectives (M. Spagnuolo)

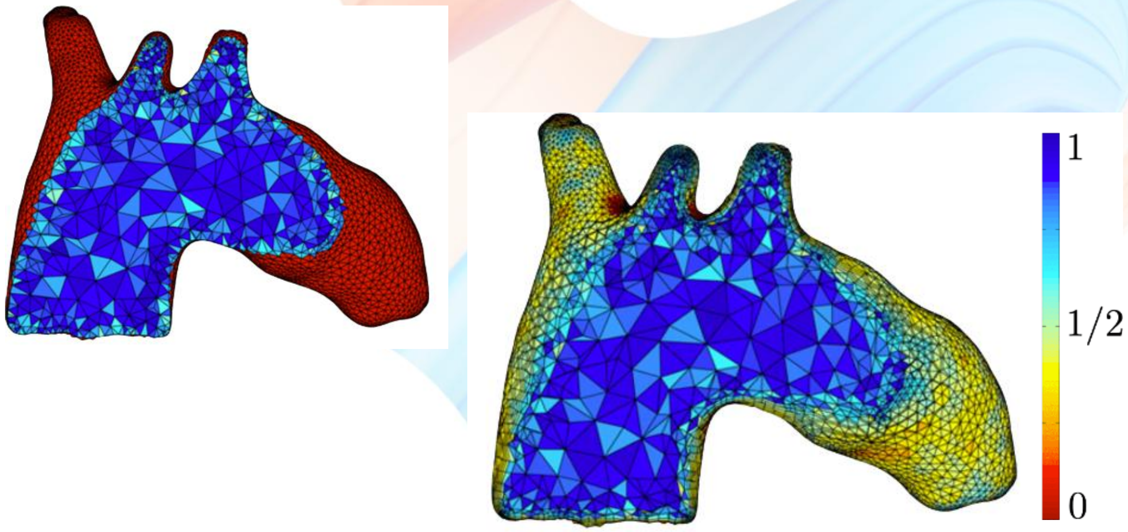
b) Q&A

Towards Mesh Quality

- mesh quality improvement is often treated as an optimization problem
- mesh optimization is often seen as a two-step process
    - **generation** of an initial mesh that might include poorly-shape elements
    - **optimization** step
- mesh repairing and denoising can be included in the optimization of a mesh



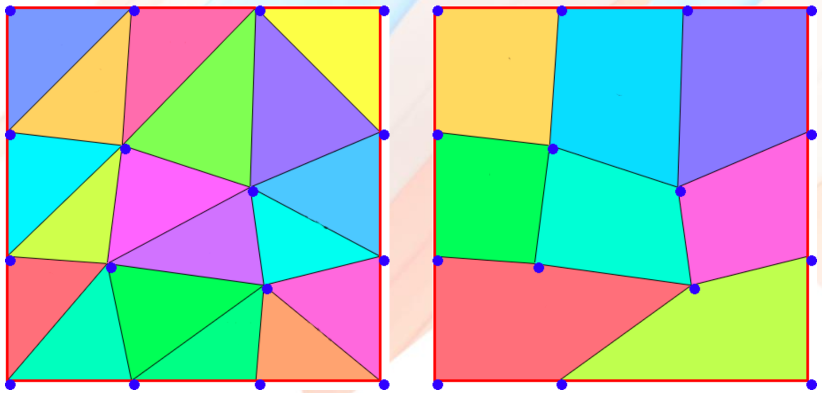
[Vartziotis & Wipper 2012]



[Vartziotis & Wipper 2012]

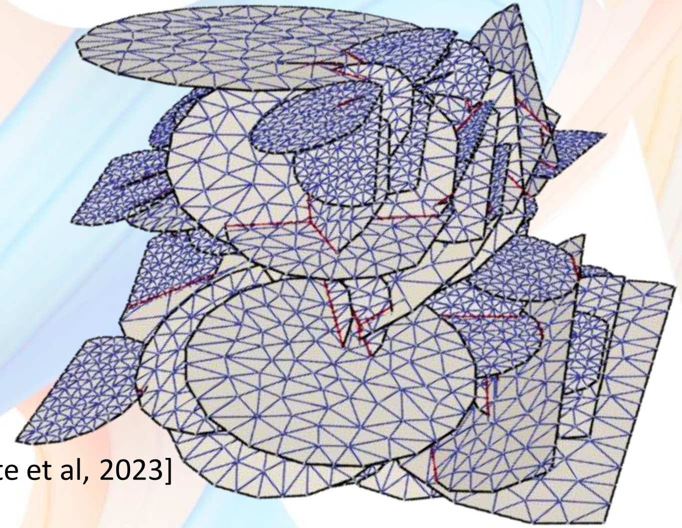
Strategies

- geometry-based operations modify the element shapes by shifting the nodal points (connectivity is unchanged)
  - Laplacian smoothing is a mesh optimization scheme which is quality-unaware
- topology-based operations modify the element connections among the nodal points (the position of the nodes does not change, nodes can be added or deleted)
- more complex but effective optimization schemes balance geometry-based and topology-based operations
- numerical solutions on complex geometries **need to balance accuracy** and **computational cost**



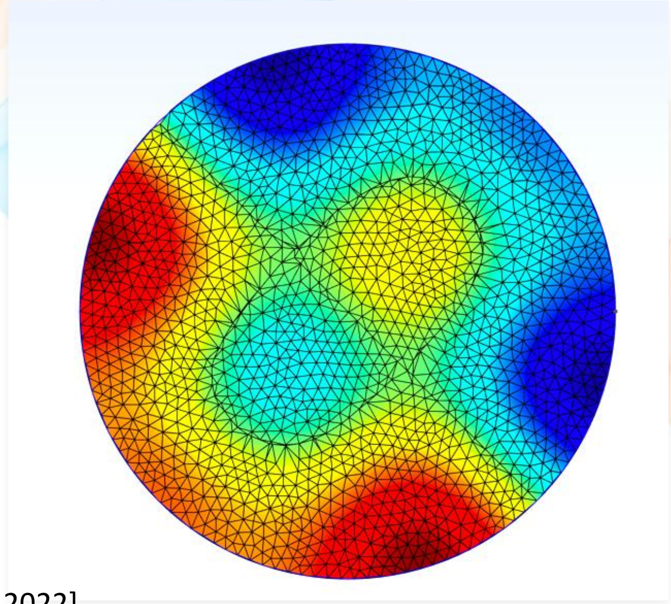
[Sorgente et al, 2023]

Adaptivity

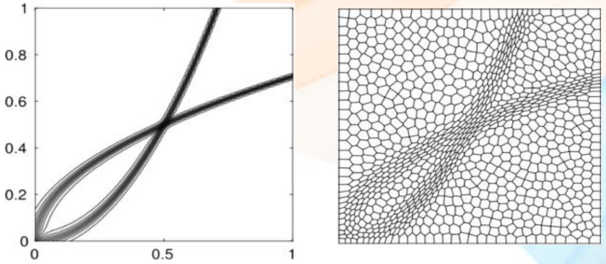

[Sorgente et al, 2023]

- adaptivity is a key point that guarantees **reliability** and **accuracy**
- adaptivity with respect to
  - **complex domains**
  - the **physical features/problems**
- generic polytopal meshes are
  - suited to handle complex geometries
  - able to feature complex geometries with (few) elements
- element refinement and agglomeration operations need to be carefully thought out
  - having full freedom over the type of polygonal elements might make complex the management of the mesh

https://www.x-mesh.eu/gallery/

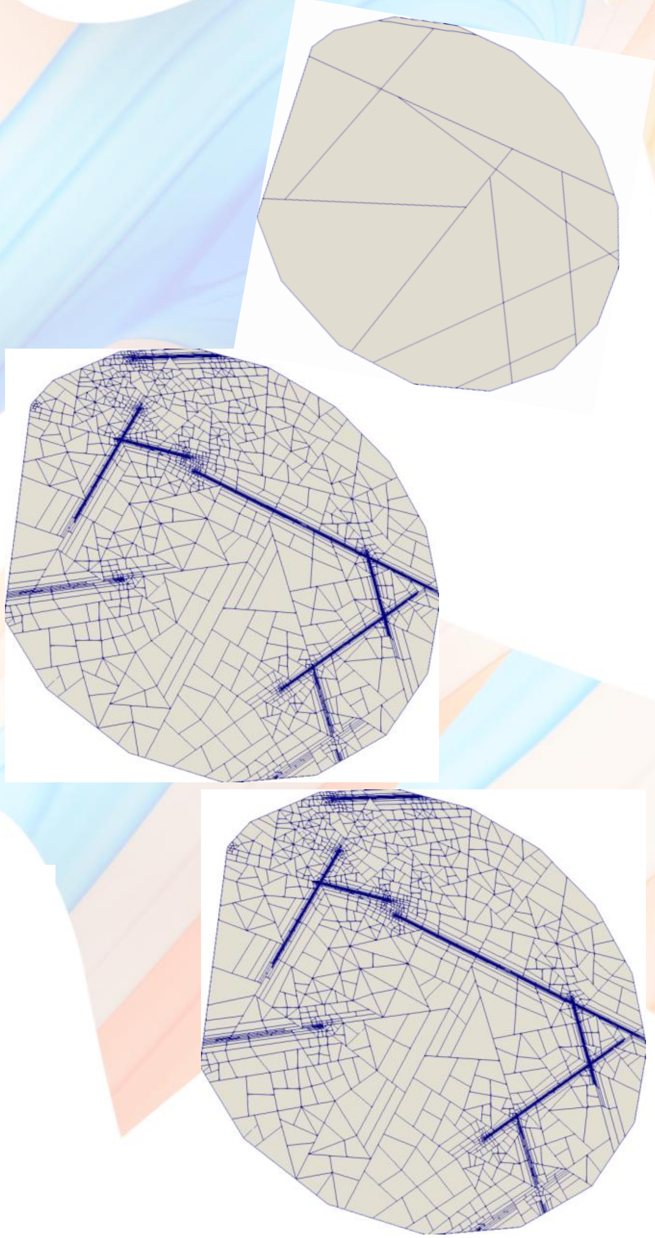

[Moes et al, 2022]
https://www.x-mesh.eu/gallery/

Refinement

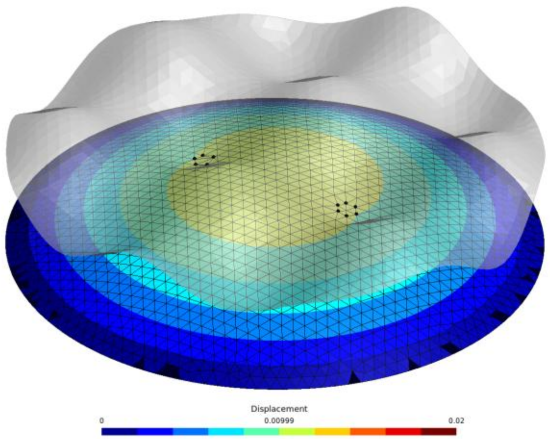


[Huang 2020]

- a **partition of the mesh elements** into smaller ones and the creation of a **finer grid**
- **domain-driven**
  - e.g. Discrete Fracture Networks (DFN): arbitrary intersections of planes in 3D space, arbitrarily oriented representing fracture in the underground rock formations
- **driven** by the geometry of the **solution**
  - anisotropic mesh refinement
  - adaptive mesh deformation

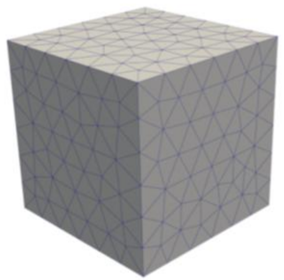


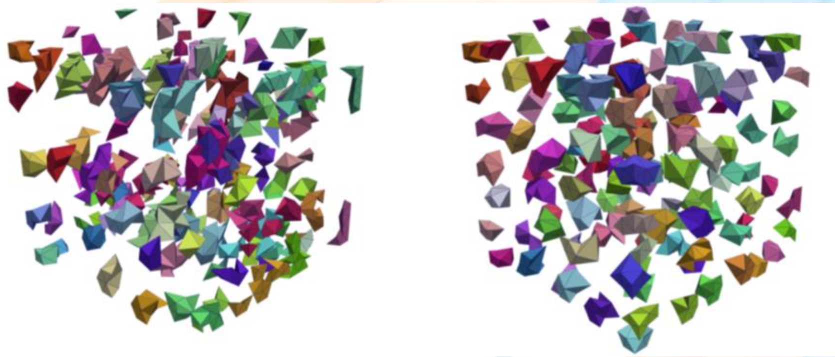
[Berrone et al, 2019]



[Moes et al, 2022]

Agglomeration

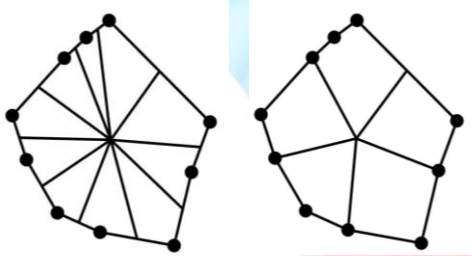




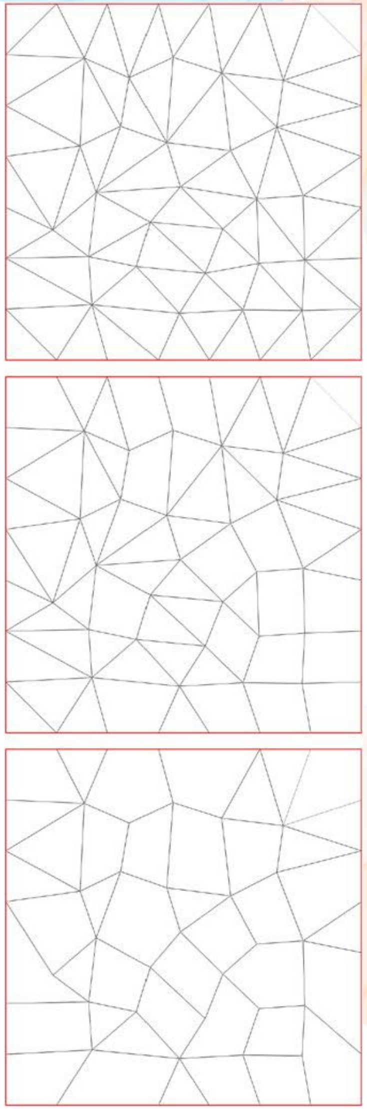
[Antonietti et al, 2024]

degrees of freedom (DOFs) optimization by **aggregating elements**

- **direct** methods
    - aggregation is driven by a global energy
    - local operations are decided based on the quality indicator of the aggregated polygons
    - the shape of the outcome polytopes is “free”, even non-convex
- **learning-based** methods
    - some good elements are selected as templates
    - the network is trained to recognize patterns of polytopes that resemble one of the templates
    - currently, only a limited number of convex elements is considered



[Antonietti et al, 2022]

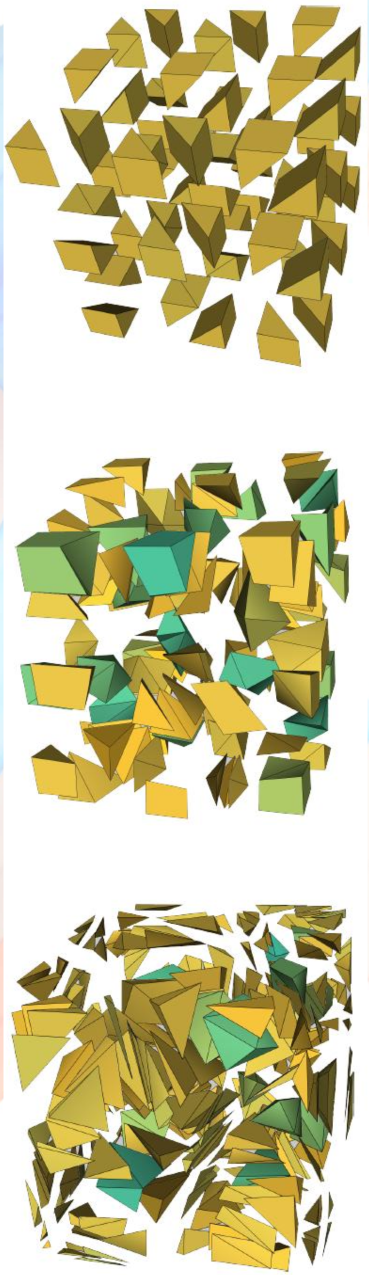


[Sorgente et al, 2023]

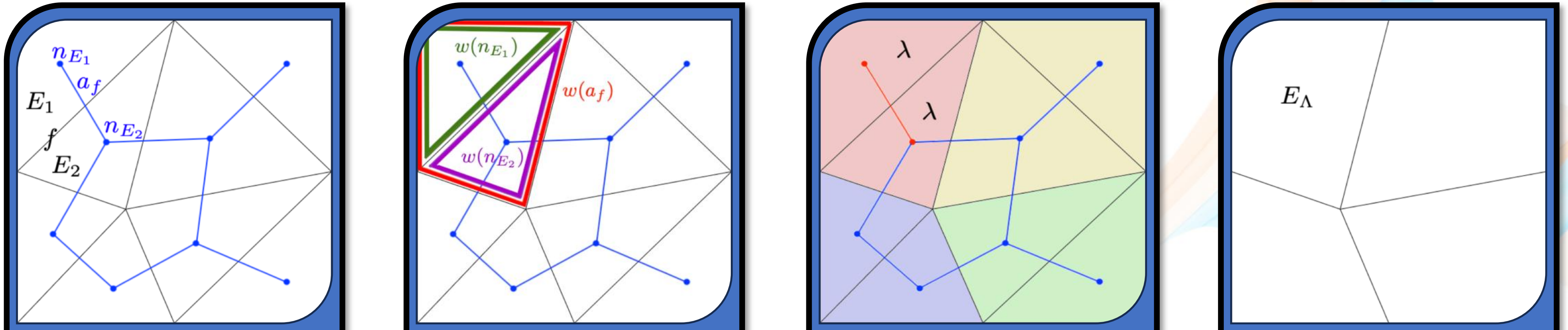
Optimizing With Quality Indicators

- element agglomeration can be driven by an energy functional
- optimization must balance the number of elements and their quality
- the optimization problem is re-framed as a graph partitioning problem, by encoding the connectivity structure of the mesh in a graph

mesh element	graph element	graph weight
cell E	node n_E	$w(n_E) = \varrho(E)$
internal edge (face) f shared by E_1 and E_2	arc a_f	$w(a_f) = \varrho(E_1 \cup E_2)$



Optimizing With Quality Indicators



1: construct the dual mesh graph

2: assign weights to graph nodes and arcs

3: partition the graph into a given n. of parts

4: agglomerate cells with the same label

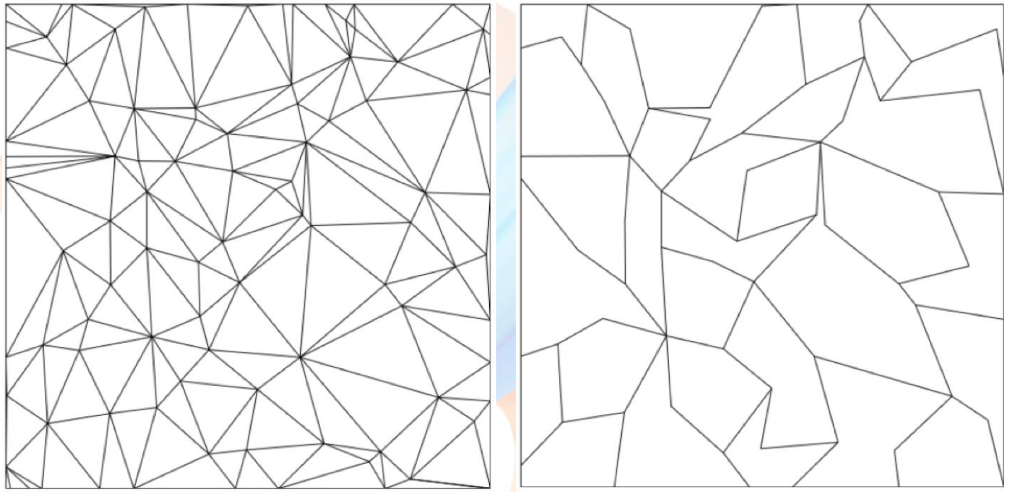
goal: optimize the size of a mesh with respect to its quality

Optimization in Action

- optimization parameter: $k \in [5,50]$
- n. of parts in the graph: $K = k\frac{\#P}{100}$

the optimized mesh has K cells, and we removed:

- $\#P - K$ cells
- at least $\#P - K$ edges (faces)
- some vertices if K is low



M M_{20}

- reduction of the DOFs involved in computations
- decrease of the global mesh quality decreases

example: $k = 20, K = 32$

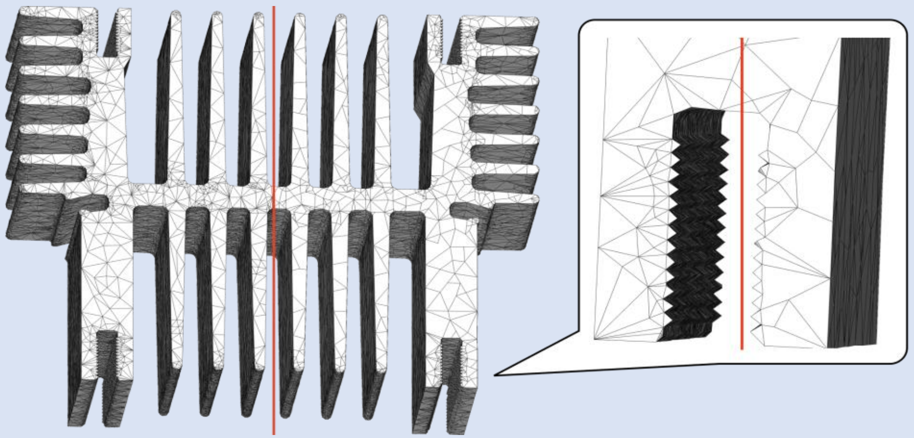
mesh	$\mathcal{T}_h$ #V	#E	#P	#DOFs $k=1$	$k=2$	$k=3$	ϱ
M	100	261	162	64	451	1000	0.90
M_{20}	70	101	32	50	163	308	0.62

the optimized mesh has the maximum quality we can achieve by removing that number of cells

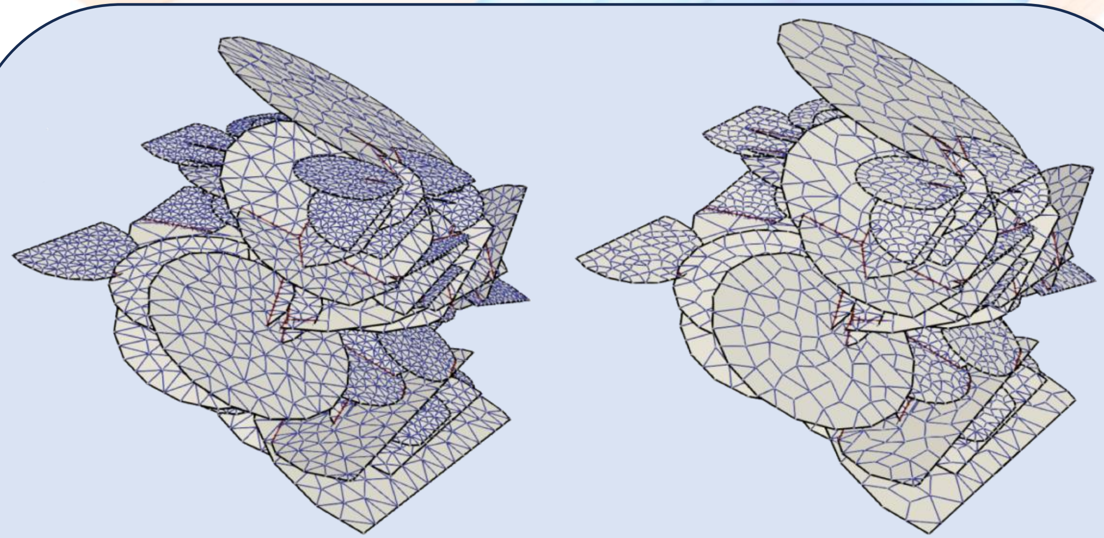
[Sorgente et al, 2024]

[Sorgente et al, Mesh Quality Agglomeration algorithm for the VEM applied to DFNs, 2023]
[Sorgente et al, Mesh Optimization for the VEM: How Small Can an Agglomerated Mesh Become?, 2024]

Applications



time-dependent problems: complex domain, different levels of detail, elements with variable size, variable n. of time steps

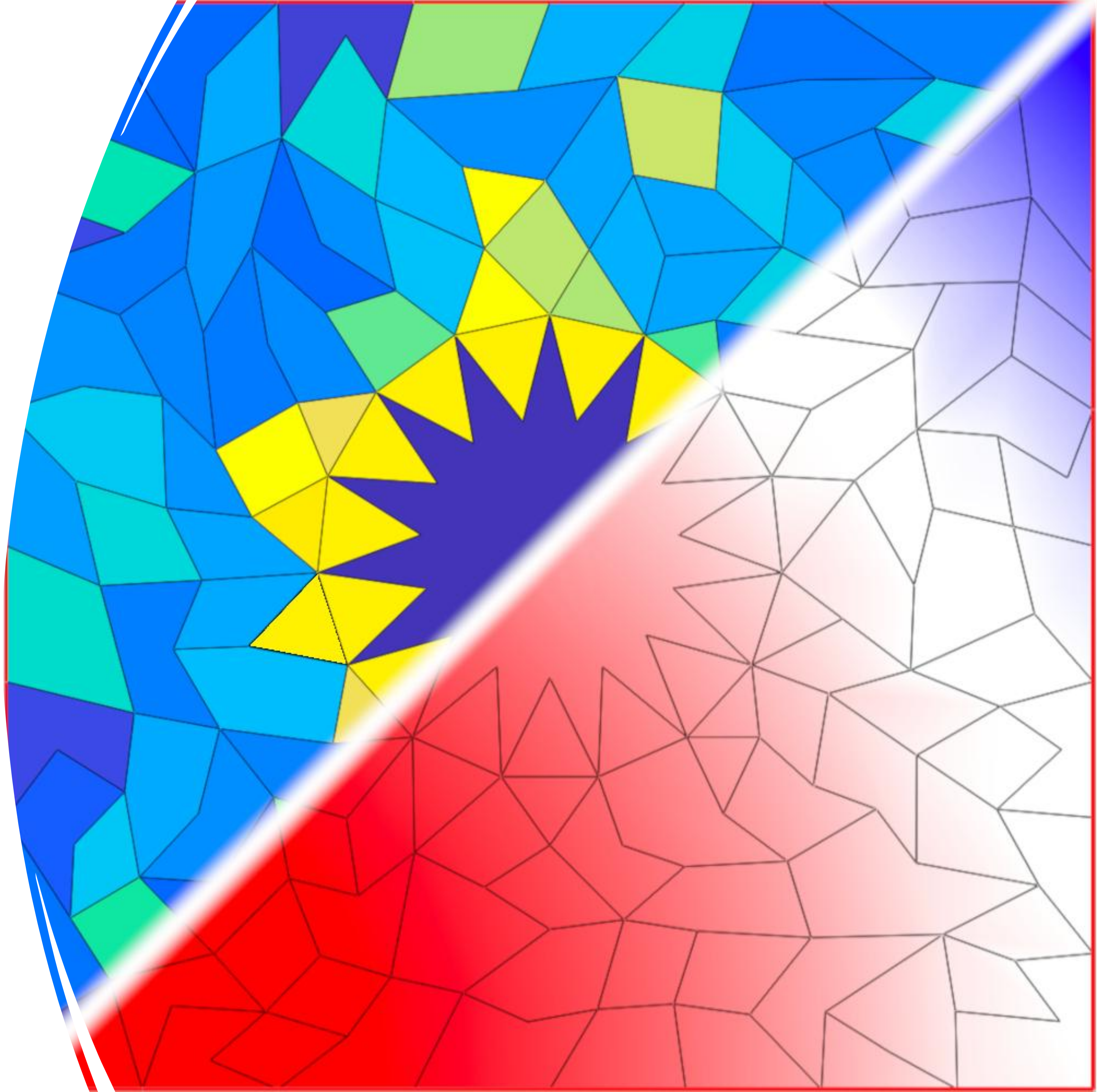


constrained problems: generate a global conforming mesh on the whole network, large n. of fractures with different sizes, intricate system of traces at different scales, elongated elements with aligned small edges

reduce the DOFs, preserve convergence rate, small error increase

PEMesh time!

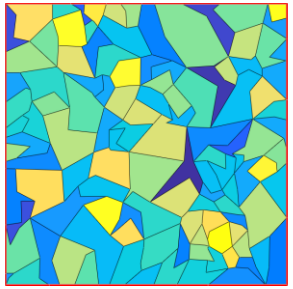
Mesh Optimization



PEMesh - Demo

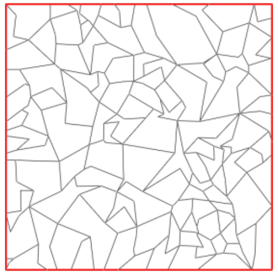
Dataset Generation by Optimization

- Case 1 : A single quality indicator (VEM)
- Case 2 : All quality indicators



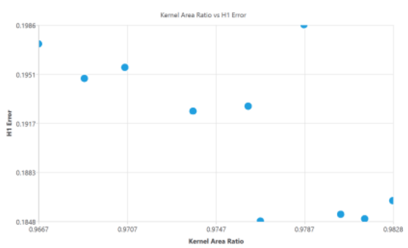
Correlation between

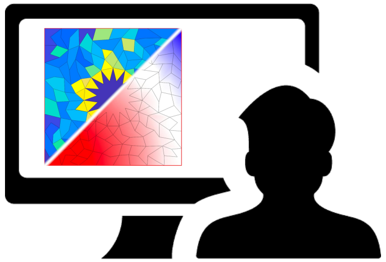
- VEM Performances
- Quality indicators



Quality Indicators

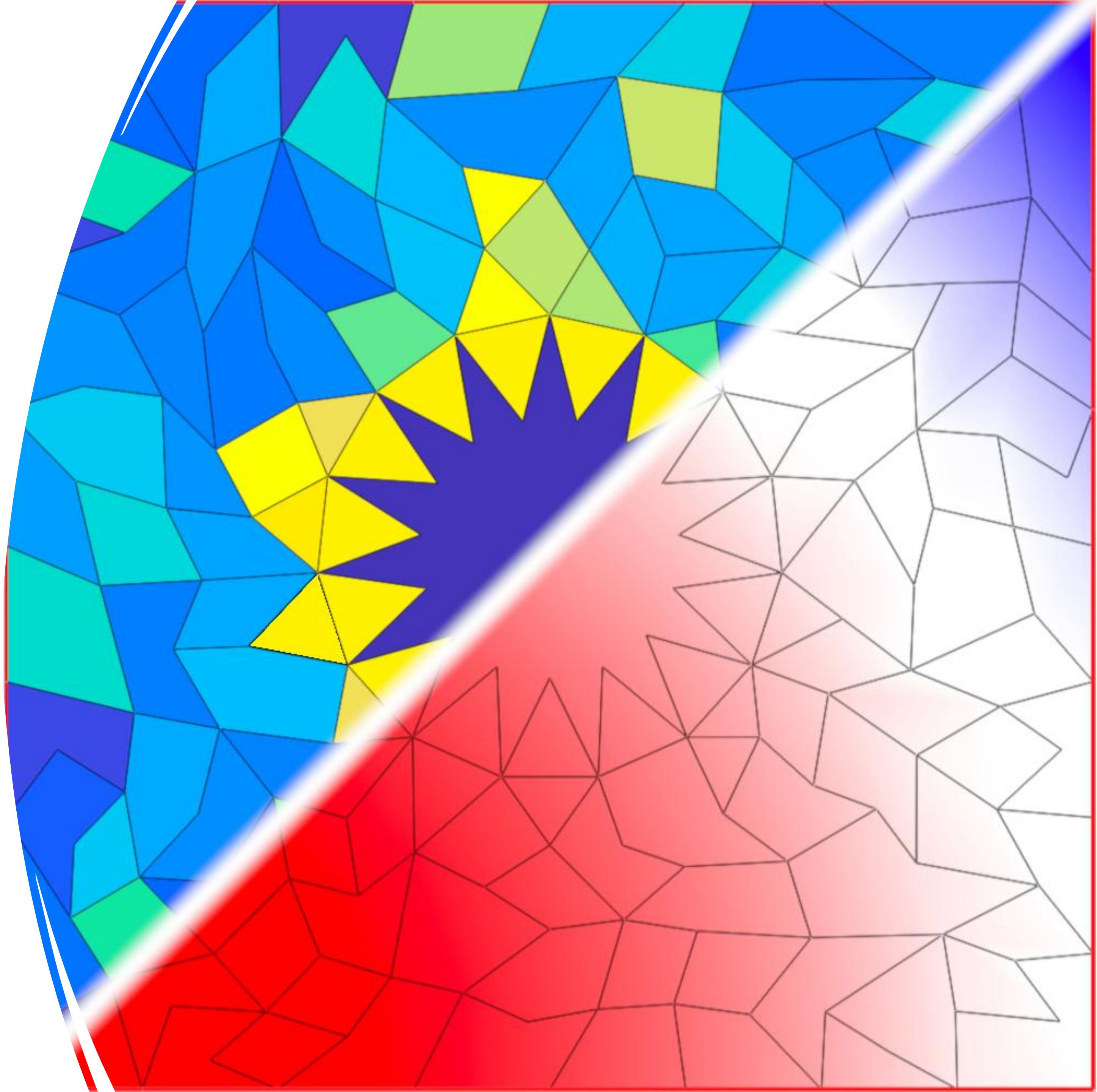
- Analysis



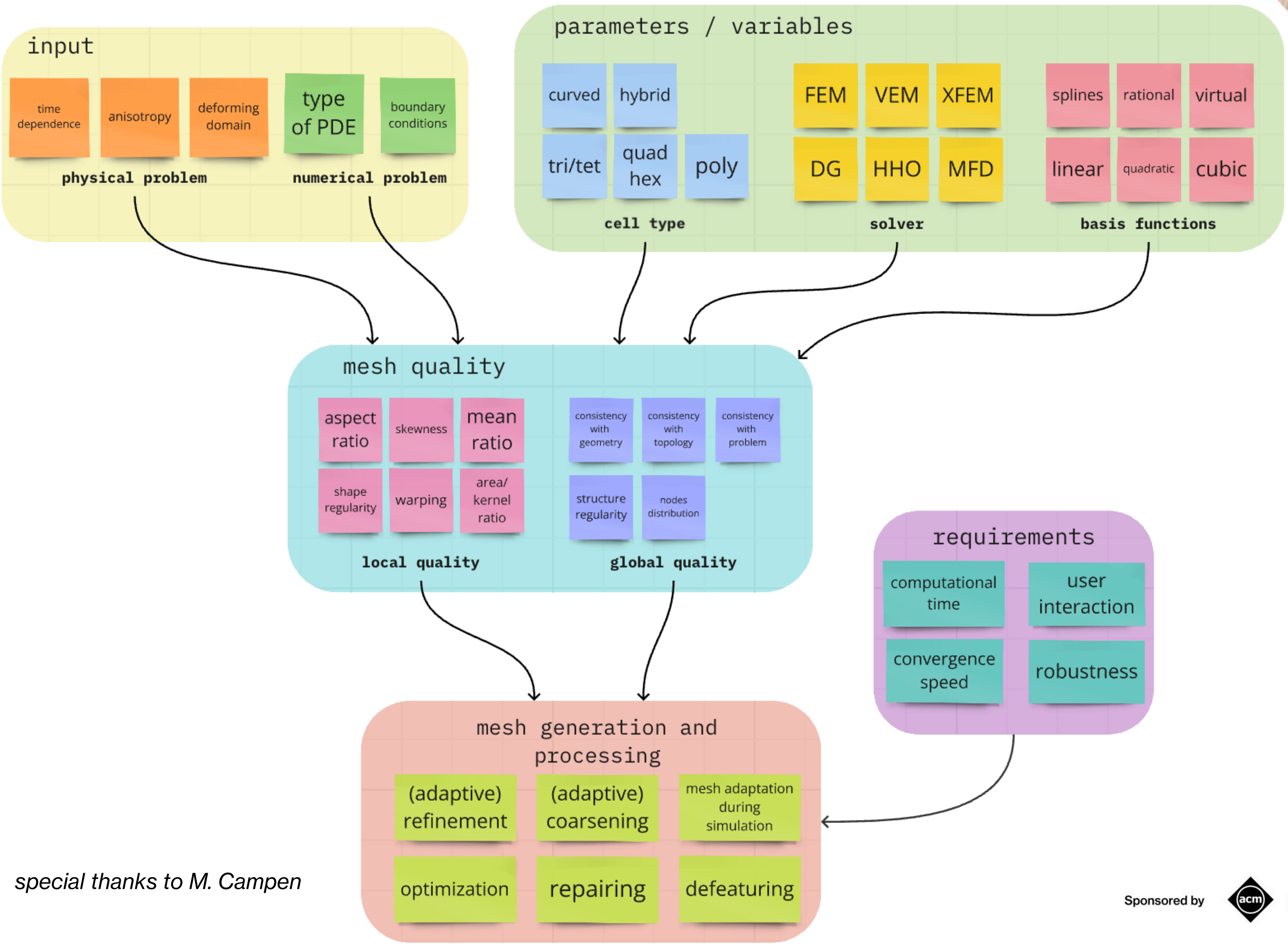


PEMesh time!

Mesh Optimization



Conclusions



special thanks to M. Campen

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thank you!

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We are hiring!

If you are interested, feel free to contact us.

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