Linear Programming Algorithms

Sample Solved Exercises

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Exercise 1
If we express the linear program below, in standard from describe the various elements of this matrix formulation, respectively, n, m, A, b and c? Provide two feasible solutions to this problem and indicate the value of the objective function for each one.

Objective function:

$$\max\ 2x_1 - 3x_2 + 3x_3$$

subject to:

$$\begin{array}{rrrrr} x_1 & x_2 & -x_3 & \leq & 7 \\ -x_1 & -x_2 & +x_3 & \leq & -7 \\ -x_1 & -2x_2 & +2x_3 & \leq & 4 \end{array}$$

and

$$x_1, x_2, x_3 \geq 0$$

Solution:
For this linear program, already in standard form, we have that n = 3, m = 3, c^T = [+2, -3, +3], b = [+7, -7, +4], and

$$A = \begin{array}{rrr} +1 & +1 & -1 \\ -1 & -1 & +1 \\ -1 & -2 & +2 \end{array}$$

Feasible solution 1: $x_1 = 5, x_2 = 2, x_3 = 0,\ z = 4$
Feasible solution 2: $x_1 = 7, x_2 = 0, x_3 = 0,\ z = 14$

Exercise 2
Convert the LP formulation in Exercise 1 in Slack Form and derive N, B, A, b, c and v.

Solution:
v = 0, c^T = [+2,-3,+3] b = [+7,-7,+4], N = (1, 2, 3), B = (4, 5, 6) and

$$A = \begin{matrix} -1 & -1 & +1 \\ +1 & +1 & -1 \\ +1 & +2 & -2 \end{matrix}$$

As the LP formulation of this problem in Slack form is:

max $z = 2x_1 - 3x_2 + 3x_3$

subject to:

$$\begin{aligned} x_4 &= 7 - x_1 - x_2 + x_3 \\ x_5 &= -7 + x_1 + x_2 - x_3 \\ x_6 &= 4 - x_1 + x_2 - 2x_3 \end{aligned}$$

$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$

Exercise 3.
Convert the following linear program into Standard form.

$\min\ 2x_1 + 7x_2$

subject to:

$$\begin{array}{llllr} x_1 & & & = & 7 \\ 3x_1 & +x_2 & & \geq & 24 \\ & x_2 & & \geq & 0 \\ & & x_3 & \leq & 0 \end{array}$$

$x_1, x_2, x_3 \geq 0$

Solution:
We begin by inserting two variables for each of the three variables, x_1 and x_3, respectively x_{10}, x_{30}, x_{11} and x_{31}. Each of these pairs of variables is designed to cover the negative and positive sets of values of the original variables. As such as have the intermediate linear programming formulation where we have also converted a minimization problem into a maximization problem.

$\max - 2(x_{10} - x_{11}) -\ 7x_2$

$$\begin{array}{llllr} x_{10} - x_{11} & & & = & 7 \\ 3x_{10} - x_{11} & +x_{20} & & \geq & 24 \\ & x_{20} & & \geq & 0 \\ & & x_{30} - x_{31} & \leq & 0 \end{array}$$

$x_{10}, x_{11}, x_2, x_{30}, x_{31} \geq 0$

We now simply invert the direction of the inequalities and fold the equality constraint into two inequality constraints yielding the linear programming problem in the standard form as shown below.

$\max - 2(x_{10} - x_{11}) -\ 7x_2$

$$\begin{array}{llllr} -x_{10} + x_{11} & & & \leq & -7 \\ x_{10} - x_{11} & & & \leq & 7 \\ -3x_{10} + 3x_{11} & -x_2 & & \leq & -24 \\ & -x_2 & & \leq & 0 \\ & & x_{30} - x_{31} & \leq & 0 \end{array}$$

$x_{10}, x_{11}, x_2, x_{30}, x_{31} \geq 0$

Exercise 4
Convert the following linear program into slack form

$\max\ 2x_1 - 6x_3$

subject to:

$$
\begin{array}{rrrcr}
x_1 & +x_2 & -x_3 & \leq & 7 \\
3x_1 & -x_2 & & \geq & 8 \\
-x_1 & +2x_2 & +2x_3 & \geq & 0
\end{array}
$$

$x_1, x_2, x_3 \geq 0$

Solution:
The problem is almost in standard form. We simply need to change the direction of the inequalities and then introduce the slack variables.

$\max\ 2x_1 - 6x_3$

subject to:

$$
\begin{array}{rrrcr}
x_1 & +x_2 & -x_3 & \leq & 7 \\
-3x_1 & +x_2 & & \leq & -8 \\
x_1 & -2x_2 & -2x_3 & \leq & 0
\end{array}
$$

$x_1, x_2, x_3 \geq 0$

Now we introduce the slack variables respectively x_4, x_5 and x_6, and get the linear program in the Slack Form as shown below. For this formulation the basic variables are the variables x_4, x_5 and x_6, and the non-basic variables are the variables x_1, x_2 and x_3.

$\max\ z = 2x_1 - 6x_3$

subject to:

$$
\begin{array}{rrrrr}
x_4 = & 7 & -x_1 & -x_2 & +x_3 \\
x_5 = & -8 & 3x_1 & -x_2 & \\
x_6 = & 0 & -x_1 & +2x_2 & +2x_3
\end{array}
$$

$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$

Exercise 5.
Show that the following linear program is infeasible.

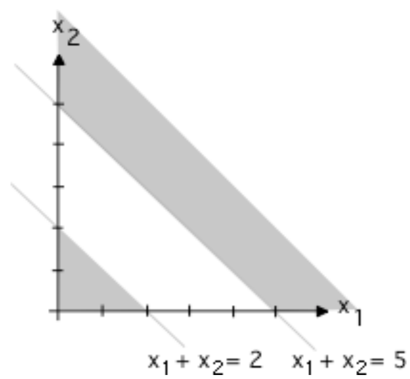
max $3x_1 - 2x_2$

subject to:

$$\begin{array}{rrcl} x_1 & +x_2 & \leq & 2 \\ -2x_1 & -2x_2 & \leq & 10 \end{array}$$

$$x_1, x_2 \geq 0$$

Solution:
There are essentially two ways of showing that this problem is infeasible, namely an algebraic and a geometric approach. We show here both for illustration purposes. In the geometric approach, which is simple given that we are dealing with two variables only we can see that the domain of feasibility is empty. The first constraint implies that all feasible points are below the line $x_1 + x_2 = 2$ whereas the second constraint imposes that the feasible points are above the line $x_1 + x_2 = 5$. The intersection of the two regions is thus empty as the figure below illustrates.



In the algebraic approach we attempt to solve the linear program and, in this case, should arrive at an infeasible solution. We begin by converting the problem into the slack form as shown below.

max $3x_1 - 2x_2$

subject to:

$$\begin{array}{rrrr} x_3 = & 2 & -x_1 & -x_2 \\ x_4 = & -10 & +2x_1 & +2x_2 \end{array}$$

$$x_1, x_2, x_3, x_4, \geq 0$$

The issue with this problem is that its basic initial solution is infeasible. For $x_1 = 0$ and $x_2 = 0$ we obtain an infeasible solution as the value of x_4 would be negative.

As such we need to create an auxiliary problem and solve it introducing an auxiliary x_0 variable and changing the objective function z to be $-x_0$. This step yields the formulation below for which we now need to find the optimal solution. The first pivot to an unfeasible solution will yield the linear program on the in the middle corresponding to the x_4 leaving and x_0 entering the set of basic variables.

$$\max\ z = -x_0$$

subject to:

$$\begin{aligned} x_3 &= 2 - x_1 - x_2 + x_0 \\ x_4 &= -10 + 2x_1 + 2x_2 + x_0 \\ x_0, x_1, x_2, x_3, x_4, &\geq 0 \end{aligned}$$

$$\max\ z = 2x_1 + 2x_2 - x_4 - 10$$

subject to:

$$\begin{aligned} x_3 &= 12 - 3x_1 - 3x_2 + x_4 \\ x_4 &= 10 - 2x_1 - 2x_2 + x_4 \\ x_0, x_1, x_2, x_3, x_4, &\geq 0 \end{aligned}$$

$$\max\ z = -\frac{2}{3}x_3 - \frac{1}{3}x_4 - 2$$

subject to:

$$\begin{aligned} x_1 &= 4 - x_2 - \tfrac{1}{3}x_3 + \tfrac{1}{3}x_4 \\ x_0 &= 2 + \tfrac{2}{3}x_3 + \tfrac{1}{3}x_4 \\ x_0, x_1, x_2, x_3, x_4, &\geq 0 \end{aligned}$$

We now rewrite x_1 as a function of x_3 to increase the value of the z function as much as possible. This results in the linear program (right-hand-side). In this problem all coefficients of variables in z are negative and as such we cannot improve the z function any further. This means that the solution $x_3 =$ 0, x_4= 0, x_1 = 4 and x_0 = 2 is the optimal solution to this auxiliary problem. As x_0 is not zero the original linear problem is infeasible.

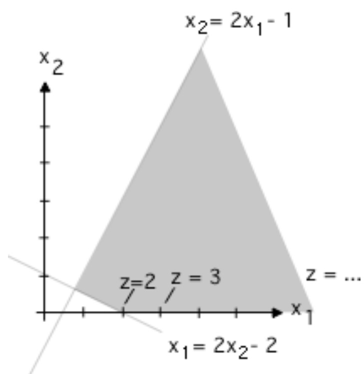
Exercise 6
Show that the following linear program is unbounded.

max $x_1 - x_2$

subject to:

$$\begin{aligned} -2x_1 \quad +x_2 \quad &\le -1 \\ -x_1 \quad -2x_2 \quad &\le -2 \\ x_1, x_2 &\ge 0 \end{aligned}$$

Solution:
There are essentially two ways of showing that this problem is unbounded, namely an algebraic and a geometric approach. We show here both for illustration purposes. In the geometric approach, which is simple given that we are dealing with two variables only we can see that the domain of feasibility is unbounded and that the objective function z can be made as large as possible.



Algebraically, we first convert the problem to slack form as shown below.

max $z = x_1 - x_2$

subject to:

$$\begin{aligned} x_3 &= -1 \quad +2x_1 \quad -x_2 \\ x_4 &= -2 \quad +x_1 \quad +x_2 \\ x_1, x_2, x_3, x_4 &\ge 0 \end{aligned}$$

max $z = 2 - 2x_2 + x_4$

subject to:

$$\begin{aligned} x_3 &= 3 \quad -3x_2 \quad +2x_4 \\ x_1 &= 2 \quad -x_2 \quad +x_4 \\ x_1, x_2, x_3, x_4 &\ge 0 \end{aligned}$$

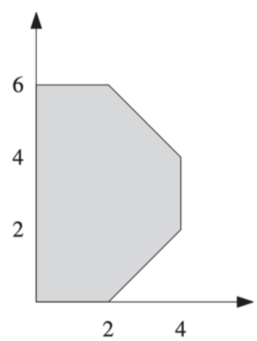
Unfortunately, the basic solution of this problem is infeasible. We try to find another solution, say for $x_1 = 2$, and all other variables to 0. This corresponds to having x_1 expressed as a function of the other variables, yielding the formulation on the right-hand-side above.

Now as can be seen here the coefficient of x_4 in the objective function z is positive. As it is also positive in the expressions of x_1 and x_3. This means that we can make x_4 as large as we want without violating the constraints on any of the variables, since we can compensate for the increase of x_4 by increasing x_2. The original problem is thus unbounded.

Exercise 7
Given the LP problem below, already in Standard Form, determine its feasible region and show the various pivot steps of the use of the Simplex algorithm to find the optimal solution, should it exist. At each pivot step, explain the decision of the choice of variables.

$$\begin{aligned} \text{maximize:} \quad & z = 4x_1 + x_2 \\ \text{subject to:} \quad & x_2 \leq 6 \\ & x_1 + x_2 \leq 8 \\ & x_1 \leq 4 \\ & x_1 - x_2 \leq 2 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Solution:
The various constraints on x_1, x_2 define the region shown below where we have indicated the various space-delimiting planes (lines in this specific case).



To apply the Simplex algorithm, we first rewrite the linear program into the initial Slack form, introducing the slack variables x3 , x4 , x5 and x6. This Slack form is as shown below.

$$\begin{aligned} \text{maximize:} \quad & z = 4x_1 + x_2 \\ \text{subject to:} \quad & x_3 = 6 - x_2 \\ & x_4 = 8 - x_1 - x_2 \\ & x_5 = 4 - x_1 \\ & x_6 = 2 - x_1 + x_2 \\ & x_1, x_2, x_3, x_4, x_5, x_6 \geq 0 \end{aligned}$$

Free Variables: $F = \{1, 2\}$
Basic Variables: $B = \{3, 4, 5, 6\}$
Basic Solution: $(0, 0, 6, 8, 4, 2)$
Objective Value: $c_* = 0$

Next, we look at the objective function and notice that increasing either x1 or x2 will increase the objective value. We choose to raise the value of x1 as it has the largest coefficient. This greedy strategy does not necessarily improve runtime, by the way, but it is nevertheless often a useful choice in practice. While keeping x2 fixed at zero, we increase x1 as far as possible. The nonnegativity constraints of the basic variables impose the constraints x1 ≤ ∞ from x3, x1 ≤ 8 from x4, x1 ≤ 4 from x5 and x1 ≤ 2 from x6. The tightest of these constraints is x1 ≤ 2. So, we pivot x1 and x6, setting x1 = 2 and x6 = 0 in the basic solution. This produces a new but equivalent slack form.

$$\begin{aligned} \text{maximize:} \quad & z = 8 + 5x_2 - 4x_6 \\ \text{subject to:} \quad & x_1 = 2 + x_2 - x_6 \\ & x_3 = 6 - x_2 \\ & x_4 = 6 - 2x_2 + x_6 \\ & x_5 = 2 - x_2 + x_6 \\ & x_1, x_2, x_3, x_4, x_5, x_6 \geq 0 \end{aligned}$$

Free Variables: $F = \{2, 6\}$
Basic Variables: $B = \{1, 3, 4, 5\}$
Basic Solution: $(2, 0, 6, 6, 2, 0)$
Objective Value: $c_* = 8$

Since the coefficient on x6 is negative, raising it from zero will only decrease the objective function. For this reason, we keep x6 fixed at zero and raise x2 until we hit the first constraint. The constraints on x2 are: x2 ≤ ∞, x2 ≤ 6, x2 ≤ 3, and x2 ≤ 2. The tightest of these constraints is x2 ≤ 2 coming from the basic variable x5. So, we set x2 = 2 and x5 = 0 by pivoting x2 and x5. This yields a new Slack form.

$$\begin{aligned} \text{maximize:} \quad & z = 18 - 5x_5 + x_6 \\ \text{subject to:} \quad & x_1 = 4 - x_5 \\ & x_2 = 2 - x_5 + x_6 \\ & x_3 = 4 + +x_5 - x_6 \\ & x_4 = 2 + 2x_5 - x_6 \\ & x_1, x_2, x_3, x_4, x_5, x_6 \geq 0 \end{aligned}$$

Free Variables: $F = \{5, 6\}$
Basic Variables: $B = \{1, 2, 3, 4\}$
Basic Solution: $(4, 2, 4, 2, 0, 0)$
Objective Value: $c_* = 18$

In this Slack form, the coefficient of x5 is negative. So, we keep x5 fixed at zero and increase x6. The tightest constraint is x6 ≤ 2 coming from the basic variable x4. So, we pivot x4 and x6, yielding the next slack form.

$$\begin{aligned} \text{maximize:} \quad & z = 20 - 3x_5 - x_4 \\ \text{subject to:} \quad & x_1 = 4 - x_5 \\ & x_2 = 4 - x_4 + x_5 \\ & x_3 = 2 + x_4 - x_5 \\ & x_6 = 2 + 2x_5 - x_4 \\ & x_1, x_2, x_3, x_4, x_5, x_6 \geq 0 \end{aligned}$$

Free Variables: $F = \{4, 5\}$
Basic Variables: $B = \{1, 2, 3, 6\}$
Basic Solution: $(4, 4, 2, 0, 0, 2)$
Objective Value: $c_* = 20$

Now something interesting has happened! Both of the variables in the objective function have negative coefficients, so increasing either of them would decrease the objective function. So, we conclude that we must be at the optimum and stop. From the basic solution we see that x1 = 4, x2 = 4 and c∗ = 20, which is the solution to our original problem.

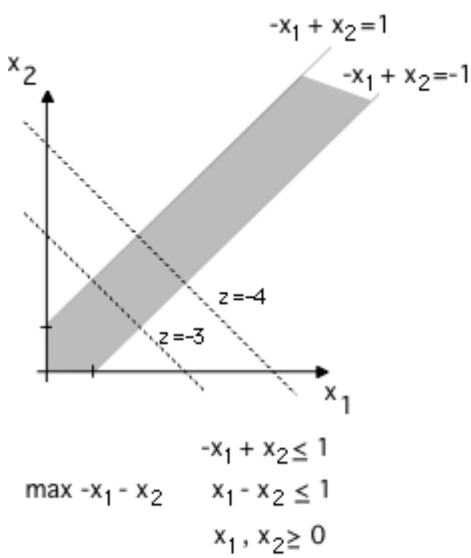
Exercise 8
Suppose that we have a general linear program with n variables and m constraints, and suppose that we convert it to the standard form. Give an upper bound on the number of variables and constraints in the resulting linear program. What about a conversion to the slack form, does the bound change?

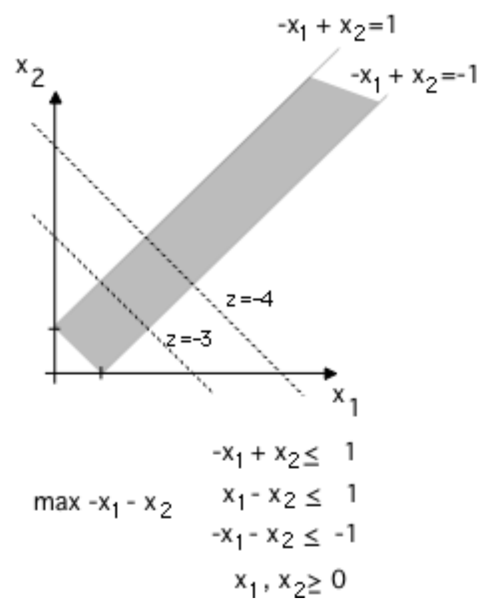
Solution:
In the case of n variables all of which are unconstrained, we need to include an additional variable leading to a bounded problem with 2n variables. As to the constraints the worst possible setting is when the constraints are of equality type. In this case we will replace a single equality constraint with two constrains thus yielding a linear problem with 2m inequality constraints.

When converting to the Slack form we have to include an additional slack variable for each of the now 2m constraints thus leading to a linear program with 2(n+m) variables and 2m constraints.

Exercise 9
Give an example of a liner program for which the feasible region is not bounded but the optimal objective value is finite. Can you also find a linear program that has a feasible region that is unbounded but has an infinite number of solutions with a finite value?

Solution:
A possible solution (among many) is depicted below. Here the feasible region is unbounded but because the z function is maximized at the origin, there is a single optimal value for the objective function.





Exercise 10
Solve the following linear program using the Simplex algorithm:

$\max\ 18x_1 + 12.5x_2$

subject to:

$$\begin{aligned} x_1 + x_2 &\le 20 \\ x_1 &\le 12 \\ x_2 &\le 16 \\ x_1, x_2 &\ge 0 \end{aligned}$$

Solution:
We begin by converting the linear program into Slack form as shown on the left below. Next, we pick x_1 as the leaving variable as it is the one with the largest positive coefficient in z. As to the most limiting variables we have x_4 given that it imposed the strictest constrains to the growth of x_1. This step is shown in the middle linear program. In the next step we pick x_2 as the leaving variables and x_3 as the entering variable yielding the linear program on the right.

$\max\ z = 18x_1 + 12.5x_2$

subject to:

$$\begin{aligned} x_3 &= 20 - x_1 - x_2 \\ x_4 &= 12 - x_1 \\ x_5 &= 16 - x_2 \\ x_1, x_2, x_3, x_4, x_5 &\ge 0 \end{aligned}$$

$\max\ z = 216 + 12.5x_2 - 18x_4$

subject to:

$$\begin{aligned} x_3 &= 8 - x_2 - x_4 \\ x_4 &= 12 - x_4 \\ x_5 &= 16 - x_2 \\ x_1, x_2, x_3, x_4, x_5 &\ge 0 \end{aligned}$$

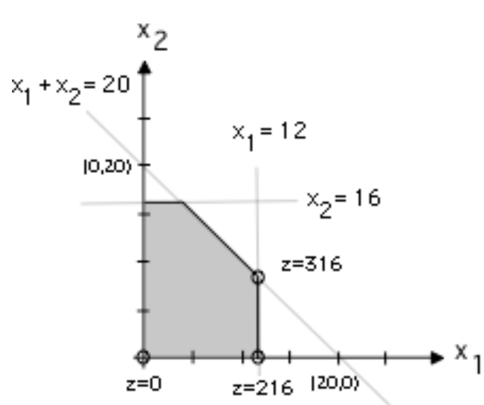
$\max\ z = 316 - 12.5x_3 - 30.5x_4$

subject to:

$$\begin{aligned} x_3 &= 8 - x_3 - x_4 \\ x_4 &= 12 - x_4 \\ x_5 &= 8 + x_3 + x_4 \\ x_1, x_2, x_3, x_4, x_5 &\ge 0 \end{aligned}$$

In this last linear program, we verify that the coefficients of x_3 and x_4 are both negative in the objective function z. As such we cannot improve the value of z and the optimal solution to the original linear program is thus $x_1 = 12$, $x_2 = 8$, $x_3 = 0$, $x_4 = 0$, $x_5 = 8$, with the optimal value of 316 for the objective function z. Notice also, that in this particular case we have an infinite number of optimal points (any combination of x1 and x2 that satisfy the constraint $x_1 + x_2 = 20$.

Graphically, the Simplex algorithm has explored a range of corner solutions as depicted in the figure below where we show for each corner point the value of the objective function.



Exercise 11
Solve the following linear program using the SIMPLEX algorithm:

$\max -5x_1 - 3x_2$

subject to:

$$\begin{aligned} x_1 - x_2 &\leq 1 \\ 2x_1 + x_2 &\leq 2 \\ x_1, x_2 &\geq 0 \end{aligned}$$

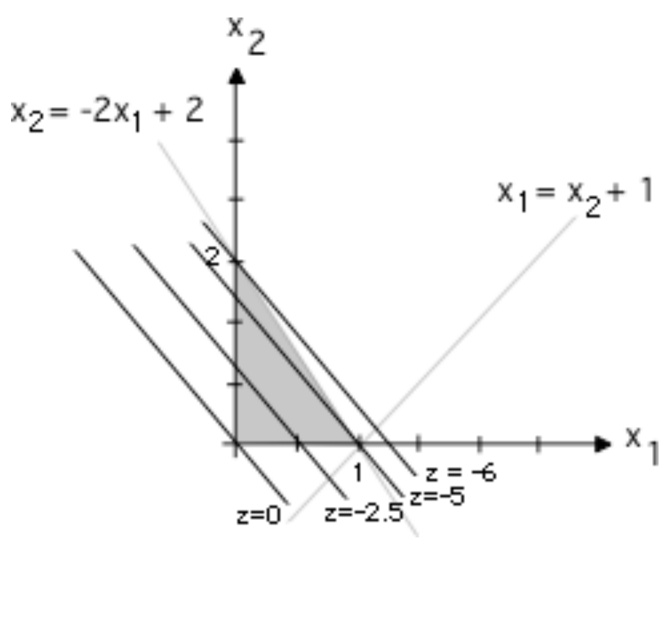
Solution:
We begin by converting the linear program into slack form as shown on the left below.

$\max\ z = -5x_1 - 3x_2$

subject to:

$$\begin{aligned} x_3 &= 1 - x_1 + x_2 \\ x_4 &= 2 - 2x_1 - x_2 \\ x_1, x_2, x_3, x_4 &\geq 0 \end{aligned}$$

The initial basic solution $x_1 = 0$, $x_2 = 0$, $x_3 = 1$, $x_4 = 2$ corresponding to the values of the objective function z of 0 is the optimal solution as in this form all non-basic variables have negative coefficients in the objective function z. Geometrically, the feasible region is as shown in the figure below where we show the curves of iso-value for the objective function z. As can be seen the optimal point in the feasible region occurs at the origin.



Exercise 12
Consider the LP problem below. Show that it has an infeasible initial solution and set up a feasible solution to the corresponding auxiliary problem that once solved would provide the initial solution to this original LP problem. Do not solve the original LP problem.

min $z = x_1 + x_2 + x_3$

subject to:

$$
\begin{aligned}
2x_1 + 7.5x_2 + 3x_3 &\geq 10 \\
20x_1 + 5x_2 + 10x_3 &\geq 30 \\
x_1, x_2, x_3 &\geq 0
\end{aligned}
$$

Solution:
We begin by converting the linear program into Standard and then Slack form as shown below.

max $z = -x_1 - x_2 - x_3$

subject to:

$$
\begin{aligned}
-2x_1 - 7.5x_2 - 3x_3 &\leq -10 \\
-20x_1 - 5x_2 - 10x_3 &\leq -30 \\
x_1, x_2, x_3 &\geq 0
\end{aligned}
$$

max $z = -x_1 - x_2 - x_3$

subject to:

$$
\begin{aligned}
x_4 &= -10 + 2x_1 + 7.5\,x_2 + 3x_3 \\
x_5 &= -30 + 20x_1 + 5x_2 + 10x_3 \\
x_1, x_2, &x_3, x_4, x_5 \geq 0
\end{aligned}
$$

For this LP problem the initial solution is not feasible as $x_1 = x_2 = x_3 = 0$ and z = 0 leads to $x_3, x_4, x_5 < 0$. As such, we need to define an auxiliary LP problem L_{aux} and find its optimal solution which will be the initial feasible solution of this original LP problem, should it have a solution. The auxiliary problem is as shown below:

max $z = -x_0$

subject to:

$$
\begin{aligned}
x_4 &= -10 + 2x_1 + 7.5\,x_2 + 3x_3 + x_0 \\
x_5 &= -30 + 20x_1 + 5x_2 + 10x_3 + x_0 \\
x_0, x_1, x_2, &x_3, x_4, x_5 \geq 0
\end{aligned}
$$

Since this problem also has a n infeasible initial solution (as $x_4, x_5 < 0$, the all the other variables set to 0), we pivot and are guaranteed to have a feasible initial solution. We select x_5 as the variable that enters to basis and x_0 as the variable that leaves the basis since b_i= 30 associated with x_5 is the smallest element in b. This leads to the LP formulation:

max $z = -30 + 20x_1 + 5x_2 + 10x_3 - x_5$

subject to:

$$
\begin{aligned}
x_0 &= 30 - 20x_1 - 5\,x_2 - 10x_3 + x_5 \\
x_4 &= 20 - 18x_1 + 2.5x_2 - 7x_3 + x_5 \\
x_0, x_1, x_2, &x_3, x_4, x_5 \geq 0
\end{aligned}
$$

which now has a feasible initial solution (30, 0, 0, 0, 20, 0) and z = -30. The solution to this LP problem would then be the initial feasible solution for the original LP problem.