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Digital Signal Processing

May 17, 2021

Task 1

Design (according to your actual variant) a filter to process signals with a period of length $m = 1998$ that passes (without any change) frequencies $\frac{2k\pi}{m}$ for all $k \in [0 \ldots (m-1)]$ but $k = 22$ and $k = 3$. Explain the design algorithms and all design steps (providing references to the properties justifying the steps)

Solution

Let G_k be the frequency response of the filter we want to design g_n. G_k can be represented as follows: $G_k = \begin{cases} 0 & , if\ k = 22\ or\ k = 3 \\ 1 & , otherwise \end{cases}$
g_n is the inverse-DFT mapping defined as:

$$g_n = \frac{1}{m} \sum_{k=0}^{k=m-1} G_k W_m^{-kn},\ 0 \le k \le m-1,\ \text{where}\ W_m = e^{-\frac{2j\pi}{m}}$$

Since DFT is an orthogonal vector decomposition

$$\begin{aligned} g_n &= \frac{1}{m} \sum_{k=0}^{k=m-1} G_k e^{\frac{2jnk\pi}{m}} \\ &= \frac{1}{m} \sum_{k=0}^{k=m-1} G_k e^{\frac{2jnk\pi}{m}} \\ &= \begin{cases} \frac{1}{m}(1 + e^{\frac{2jn\pi}{m}} + \ldots + e^{\frac{2(m-1)jn\pi}{m}} - e^{\frac{44jn\pi}{m}} - e^{\frac{6jn\pi}{m}}) & \text{if } n \neq 0 \\ \frac{1}{m}(m-2) & \text{if } n = 0 \end{cases} \\ &= \begin{cases} -\frac{1}{m}(e^{\frac{44jn\pi}{m}} + e^{\frac{6jn\pi}{m}}) & \text{if } n \neq 0 \\ 1 - \frac{2}{m} & \text{if } n = 0 \end{cases} \end{aligned}$$

$$\textbf{Ans:}\quad g_n = \begin{cases} -\frac{1}{1998}(e^{\frac{22jn\pi}{999}} + e^{\frac{jn\pi}{333}}) & \text{if } n \neq 0 \\ 1 - \frac{1}{999} & \text{if } n = 0 \end{cases}$$

Task 2

Design (according to your actual variant) an ideal low-pass filter to process infinite discrete signals that passes (without any change) only frequencies in range $[-\frac{22}{1998}, +\frac{3}{1998}]$. Explain the design algorithms and all design steps (providing references to the properties justifying the steps)

Solution

Let $G(e^{j\omega})$ be the frequency response of the filter we want to design $g(n)$. $G(e^{j\omega})$ can be represented as follows(transforming it to the frequency response of the cardinal sine signal whose DTFT is known, we have the following):

$$G(e^{j\omega}) = \begin{cases} 1 & , -\frac{22}{1998} \leq \omega \leq \frac{3}{1998} \\ 0 & , \text{otherwise} \end{cases}$$

$$G(e^{j\omega}) = \sqrt{\frac{25/1998}{2\pi}} \begin{cases} \sqrt{\frac{2\pi}{25/1998}} & , -\frac{25}{3996} \leq \omega + \frac{19}{3996} \leq \frac{25}{3996} \\ 0 & , \text{otherwise} \end{cases}$$

$$\text{Hence} \quad \sqrt{\frac{25/1998}{2\pi}} e^{-\frac{19jn}{3996}} \sqrt{\frac{25/1998}{2\pi}} sinc \frac{25n/1998}{2} \xleftrightarrow{\text{DTFT}} G(e^{j\omega})$$

$$\textbf{Ans:} \quad g(n) = \frac{25}{3996\pi} e^{-\frac{19jn}{3996}} sinc \frac{25n}{3996}$$

Task 3

Validate (or refute and fix an error if any):

$$\sqrt{\frac{\omega_0}{2\pi}} sinc \frac{\omega_0 n}{2} \xleftrightarrow{\text{DTFT}} \begin{cases} \sqrt{\frac{2\pi}{\omega_0}} & if \ |\omega| \leq \frac{\omega_0}{2} \\ 0 & \text{otherwise} \end{cases} \quad \text{where (just for recall)} \quad sinc\ t = \begin{cases} \frac{\sin t}{t} & if \ t \neq 0 \\ 1 & if \ t = 0 \end{cases}$$

Solution

Let $X(e^{j\omega}) = \begin{cases} \sqrt{\frac{2\pi}{\omega_0}} & if\ |\omega| \leq \frac{\omega_0}{2} \\ 0 & \text{otherwise} \end{cases}$ the frequency response of signal $x[n]$. Using the Inverse DTFT $x[n]$ is obtained at follows.

$$x[n] = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(e^{j\omega})e^{j\omega n}d\omega = \frac{1}{2\pi}\int_{-\infty}^{-\frac{\omega_0}{2}} 0e^{j\omega n}d\omega + \frac{1}{2\pi}\int_{-\frac{\omega_0}{2}}^{+\frac{\omega_0}{2}} \sqrt{\frac{2\pi}{\omega_0}}e^{j\omega n}d\omega + \frac{1}{2\pi}\int_{+\frac{\omega_0}{2}}^{+\infty} 0e^{j\omega n}d\omega$$

$$= \sqrt{\frac{2\pi}{\omega_0}}\left.\frac{e^{j\omega n}}{2\pi jn}\right|_{-\frac{\omega_0}{2}}^{+\frac{\omega_0}{2}} = \sqrt{\frac{2\pi}{\omega_0}}\frac{e^{j\omega_0 n/2} - e^{-j\omega_0 n/2}}{2j\pi n} = \sqrt{\frac{2\pi}{\omega_0}}\frac{\sin\frac{\omega_0 n}{2}}{\pi n} = \frac{\omega_0}{2\pi}\sqrt{\frac{2\pi}{\omega_0}}\frac{\sin\frac{\omega_0 n}{2}}{\frac{\omega_0 n}{2}}$$

$$= \sqrt{\frac{\omega_0}{2\pi}}sinc\frac{\omega_0 n}{2}$$

Ans: The DTFT pair presented above is valid.