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Equations for calculation of International Standard Atmosphere and associated off-standard atmospheres

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EQUATIONS FOR CALCULATION OF INTERNATIONAL STANDARD ATMOSPHERE AND ASSOCIATED OFF-STANDARD ATMOSPHERES

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EQUATIONS FOR CALCULATION OF INTERNATIONAL STANDARD ATMOSPHERE AND ASSOCIATED OFF-STANDARD ATMOSPHERES

1. NOTATION AND DEFINITION

1.1 Notation

		SI	*British*
a	speed of sound	m/s, kn	ft/s, kn
g	acceleration due to gravity, see Section 8	m/s^2	ft/s^2
g_0	standard sea level value of g, see Table 11.1	m/s^2	ft/s^2
H	geopotential height	m	ft
H_b	geopotential height at base of layer in model atmosphere	m	ft
H_p	pressure height, (see Section 4.5)	m	ft
H_{p_b}	pressure height at base of layer in model atmosphere	m	ft
L	temperature gradient, dT/dH	K/m	K/ft
l	characteristic length	m	ft
M	Mach number, V/a	–	–
m	molar mass (see Section 1.2)	kg/kmol	(see Section 1.2)
p	pressure	N/m^2	lbf/ft^2
p_b	pressure at base of layer in model atmosphere	N/m^2	lbf/ft^2
q^*	kinetic pressure for $M = 1$ $(= \rho a^2/2)$	N/m^2	lbf/ft^2
$\mathfrak{R}$	universal gas constant (see Section 1.2)	N m/kmol K	(see Section 1.2)
R	gas constant for air, $\mathfrak{R}/m$	N m/kg K	lbf ft/slug K
Re^*	Reynolds number for $M = 1$ $(= \rho a l/\mu)$	–	–
r'_{e_ϕ}	fictitious Earth radius used in relating H and Z, see Table 11.1 for value in ISA and Section 8 for general case	m	ft
S	Sutherland coefficient (see Section 6.2)	K	K

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T	temperature	K	K
T_b	temperature at base of layer in model atmosphere	K	K
V	true airspeed	m/s, kn	ft/s, kn
V_e	equivalent airspeed $(= V\sqrt{\sigma} = Ma\sqrt{\sigma})$	m/s, kn	ft/s, kn
V_e^*	equivalent airspeed for $M = 1$ $(= a\sqrt{\sigma})$	m/s, kn	ft/s, kn
Z	geometric height	m	ft
β_s	Sutherland coefficient (see Section 6.2)	N s/m^{2}K^{½}	lbf s/ft^{2}K^{½}
γ	ratio of specific heat capacities of air (= 1.4 in this Item)	–	–
ΔT	temperature increment	K	K
δ	relative pressure, p/p_0	–	–
θ	relative temperature, T/T_0	–	–
ϕ	geographic latitude, positive in northern hemisphere, negative in southern hemisphere	deg	deg
μ	dynamic viscosity	N s/m^2	lbf s/ft^2
ρ	density	kg/m^3	slug/ft^3
σ	relative density, ρ/ρ_0	–	–

Subscripts

H_p	denotes value at constant pressure height
sl	denotes value at sea level
std	denotes value in International Standard Atmosphere, ISA
Z	denotes value at geometric height Z
ϕ	denotes value at geographic latitude ϕ
0	denotes value at sea level in International Standard Atmosphere, ISA – see Table 11.1

1.2 Molar Mass and Gas Constants

In Derivation 1 the term "molar mass" is used to denote the quantity which used to be called "kilogram molecule". The molar mass, m, is expressed in kilograms per kilo mole (kg/kmol). The mole is a base SI unit for the *amount of a substance* and is defined as the amount of substance of a system which contains as many elementary entities as there are carbon atoms in 0.012 kg of carbon 12. These elementary entities must be specified and may be atoms, molecules, ions, electrons, other particles or specified groups of particles. In defining properties of model atmospheres, in particular the International Standard Atmosphere, a mixture of gases is taken which typifies the composition of clean dry air near sea level. The resulting mean air molar mass is assumed to remain constant throughout the height range considered. For consistency with the notation just described, the universal gas constant, $\Re$, takes units N m/kmol K while the gas constant for air, $R(= \Re/m)$ takes the familiar units of N m/kg K.

When working in British units, the method adopted in this Item is to accept that m and $\Re$ are defined in SI units only and to convert the gas constant for air, R, into the equivalent British units (lbf ft/slug K).

2. INTRODUCTION

This Item describes the "atmospheric model" and gives equations and numerical data required to evaluate properties of the International Standard Atmosphere (Derivation 1) and off-standard atmospheres which differ in a prescribed way from the Standard. The basic equations for temperature, T, pressure, p, and density ρ, are given in Sections 3 to 5 while the defining constants and temperature gradients are given in Tables 11.1 and 11.2. Table 11.3 gives numerical expressions for T, p and ρ while Table 11.4 gives the reverse expressions for determining pressure height for a given pressure, and the difference between pressure height and geopotential height for specified off-standard atmospheres. The relationships between pressure, pressure height and geopotential height are considered in Section 7 while the relationship between geopotential height and geometric height is presented in Section 8. Expressions for speed of sound and dynamic viscosity are given in Section 6.

For direct reference to tabulated values of T, θ, $\sqrt{\theta}$, p, δ, $\sqrt{\delta}$, ρ, σ, $\sqrt{\sigma}$, a, V_e^*, q^*, and Re^*/l for heights up to 50 km, see Item Nos 68046 (Reference 5) and 72018 (Reference 6). For information on the atmosphere for heights above 50 km, see Derivation 1 (up to 80 km), Reference 9 and Aerodynamics Data Item No. 77021 (Reference 10) (up to 1 000 km). For information on non-standard (or other) atmospheres, see Reference 11.

3. THE ATMOSPHERIC MODEL

The atmospheric model described here has been widely adopted. The atmosphere consists of air which is considered to be a perfect gas and thus local values of pressure, density and temperature are related by the perfect gas equation

$$p = \frac{\rho \Re T}{m} = \rho R T . \tag{3.1}$$

The atmosphere is assumed to be static with respect to the earth and so the hydrostatic equation must be satisfied,

$$\frac{\mathrm{d}p}{\mathrm{d}Z} = -\rho g . \tag{3.2}$$

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Equation (3.2) provides a relationship between pressure, density and geometric height, Z, but in considering pressure distribution in the atmosphere, it is convenient to work in terms of geopotential height, H.

Geopotential height is the (geometric) height in a uniform gravitational field which gives the same potential energy as exists at the point under consideration in the actual, variable gravity field. Consideration of the work done in moving between two geopotential surfaces separated by a distance Z leads to the relationship

$$H = \frac{1}{g_{sl}} \int_0^Z g \,\mathrm{d}Z . \tag{3.3}$$

Equation (3.3) relates the geopotential height in a uniform gravitational field in which the acceleration due to gravity is taken as equal to the sea-level value, g_{sl}, to the actual variation of g with Z. For the purposes of atmospheric modelling and aircraft performance work, g_{sl} in Equation (3.3) is taken as the standard value, g_0, and the resulting measurement of H is called the standard geopotential height (throughout this Item, "geopotential height" implies, "standard geopotential height"). The accepted value of g_0 is given in Table 11.1 and reference to the Lambert equation for the variation of g_{sl} with latitude (see Section 8, Equation (8.2)) shows that this value of g_0 corresponds to a geographic latitude of 45.5425 degrees. It follows from Equation (3.3) that a given value of geopotential height corresponds to slightly different geometric heights at different latitudes – Section 8 gives information on this relationship.

To obtain a relationship between standard geopotential height and pressure, Equations (3.1), (3.2) and (3.3) are combined to give,

$$\frac{\mathrm{d}p}{\mathrm{d}H} = -g_0 \frac{mp}{\Re T} . \tag{3.4}$$

To define a particular model atmosphere (as in Section 3.1) the usual procedure is to substitute into Equation (3.4) a particular variation of temperature with height and to deduce the corresponding variation of pressure with height. The temperature-height profile is derived from experimental results, frequently obtained over large geographic areas and/or time periods, and may consist of numerical data or be an analytical expression. It should however be noted that the variety of temperature profiles that may be used in conjunction with Equation (3.4) is restricted to those that produce a vertically stable atmosphere (see Reference 2).

The variation of temperature with height may be expressed as

$$T = f(H) \tag{3.5}$$

and if Equations (3.4) and (3.5) are combined the following relationship between pressure and height is obtained

$$\frac{\mathrm{d}p}{p} = -\frac{g_0}{\Re} \int \frac{m \,\mathrm{d}H}{f(H)} . \tag{3.6}$$

The integration of Equation (3.6) depends on the form of $f(H)$ and the variation, if any, of m with H.

3.1 Atmospheres with Linear or Zero Change of *T* with *H* and Constant Molar Mass

If a model atmosphere is assumed to consist of a number of layers within each of which temperature varies linearly with height or is constant, Equation (3.5) can be expressed as,

$$T = T_b + L(H - H_b)\,. \tag{3.7}$$

In Equation (3.7), T_b is the temperature at height H_b and L is the temperature gradient at heights between H_b and H_{b+1}.

For heights up to about 80 km the mean molar mass of air can be assumed constant (see Section 1.2 and References 1, 9, 13) and equal to the sea-level value m_0. When Equation (3.7) is substituted into Equation (3.6) together with the assumed constant relationship $R = \Re/m_0$ two solution for pressure ratio are obtained,

$$\frac{p}{p_b} = \left(\frac{T}{T_b}\right)^{-g_0/RL} \tag{3.8}$$

for a layer where $L \neq 0$ and

$$\frac{p}{p_b} = \exp\left(-\frac{g_0(H - H_b)}{RT_b}\right), \tag{3.9}$$

for a layer where $L = 0$.

Values of *p* at any height may be derived from these ratios by use of multipliers which relate conditions at the bases of successive layers. The general expression for the pressure at any height is then of the form,

$$p = \left(\frac{p}{p_b}\right)\left(\frac{p_b}{p_{b-1}}\right)\ldots\ldots\ldots\left(\frac{p_2}{p_1}\right)p_1, \tag{3.10}$$

where p_1 is the pressure at the base of the first layer – usually taken as mean sea level. Note that for heights below sea level, properties should be calculated as for the first layer above sea level.

Atmospheric density, ρ, is calculated most readily using Equation (3.1) with values of T and p from Equations (3.7) to (3.10). However, comparable expressions to Equations (3.8) and (3.9) can be derived[†], to give

$$\frac{\rho}{\rho_b} = \frac{p}{p_b}\,\frac{T_b}{T} = \left(\frac{T}{T_b}\right)^{-(1 + g_0/RL)} \tag{3.11}$$

for a layer where $L \neq 0$ and

$$\frac{\rho}{\rho_b} = \frac{p}{p_b} = \exp\left(-\frac{g_0(H - H_b)}{RT_b}\right) \tag{3.12}$$

for a layer where $L = 0$.

[†] Equations (3.11) and (3.12) do not apply to the “off-standard” atmospheres of Section 5, see footnote to Section 5.1.

4. THE INTERNATIONAL STANDARD ATMOSPHERE

The International Standard Atmosphere (Derivation 1) is one of the main applications of the atmospheric model described in Section 3.1. Calculation methods for values of temperature, pressure and density in the ISA, using Equations (3.7) to (3.10) and (3.1), and numerical expressions based on them, are described in Sections 4.1 to 4.4. Calculation methods for other air properties are given in Section 6. Tabulated values of properties of the ISA are given in Data Item Nos 68046 (Reference 5) and 72018 (Reference 6).

4.1 Tables 11.1 and 11.2

Table 11.1 lists the defining constants and sea-level values for the Standard Atmosphere in both SI and British units while Table 11.2 lists the value of height, temperature and pressure at the base of each layer and the lapse rate in each layer for heights up to 50 km. The results obtained when these data are substituted in Equations (3.8) to (3.10) for pressure and Equation (3.1) for density should agree with those given in Derivation 1 to six significant figures (but see Section 4.4). Note that in Table 11.2 the values of pressure other than p_0 are redundant but the use of these values (taken from Derivation 1) has been found to produce better agreement of the calculated atmospheric properties with those given in Derivation 1 than if sequentially computed values are used.

4.2 Table 11.3

Table 11.3 presents numerical expressions for temperature, pressure and density in the Standard Atmosphere. The constants in these expressions have been derived to give agreement with the atmospheric properties given in Derivation 1 to six significant figures (but see Section 4.4). The constants given in British units are direct conversions of those in SI units (see Reference 8).

4.3 Table 11.4

Table 11.4 includes numerical expressions for calculating the height in the Standard Atmosphere from a given value of pressure.

4.4 Accuracy of Calculated Atmospheric Properties and Comparison with Other Standard Atmospheres

The calculated atmospheric properties obtained from the methods given in this Item should be in agreement with the values given to six significant figures in Derivation 1. In practice, however, only the temperature-height relationship is exact and discrepancies of one digit in the sixth figure may arise in the cases of pressure and density. The precise reasons for these discrepancies are not known but, when this degree of correspondence of results is sought, the results can be affected by the number of digits used in the computation, the order in which operations are performed, the accuracy of the particular calculator and the point at which conversion from one set of units to another is made. Discrepancies must also be expected between values obtained by rounding and those obtained by truncation to the required number of significant figures.

For most practical purposes the data obtained from the methods given here are the same as those given in References 3, 4 and 9. However, small differences in the sixth and occasionally in the fifth significant figure can arise between the calculated values and the values tabulated in References 3, 4 and 9. These discrepancies may be attributed both to the sources mentioned above and to some small differences in the defining constants. In References 3, 4 and 9 the value of m_0 does not include the last digit of the value given in Table 11.1. Also, in References 3, 4 and 9 the “British units” versions use a value of $g_0 = 32.174\,1$ ft/s^2 whereas the converted SI value would be $g_0 = 32.174\,049$ ft/s^2 .

The tabulated values of properties of the Standard Atmosphere in Item Nos 68046 (Reference 5) and 72018 (Reference 6) are quoted to five significant figures and are generally in agreement with values obtained from the methods given here when rounded to the same number of figures.

4.5 Pressure Height

The pressure height of a point in any atmosphere is the (geopotential) height in the Standard Atmosphere giving the same pressure, *i.e*. for a given value of pressure, $H_p = H_{std}$. The pressure-geopotential height relationship of the ISA (see Section 7.2) is used as the calibration law for altimeters so that a reading of such an instrument in any atmospheric condition gives pressure height – provided it is set to show zero at $p = p_0$. All references to heights in the ISA in Tables 11.3 and 11.4 are in terms of pressure height.

5. OFF-STANDARD ATMOSPHERES

5.1 Definition

The temperature profiles that define these atmospheres are obtained by adding a constant increment to the temperature at each height in the Standard Atmosphere. This temperature increment is here referred to *constant pressure height* and denoted ΔT_{H_p}. Equation (3.7) then becomes

$$T_{H_p} = T_{std} + \Delta T_{H_p} = T_{b_{std}} + \Delta T_{H_p} + L_{std}(H_p - H_{p_b}) , \tag{5.1}$$

where the subscript "*std*" denotes a value in ISA conditions (see Table 11.2 for L_{std} and $T_{b_{std}}$ values).

For the off-standard atmospheres the relationship between pressure height and pressure is identical, by definition, with that between geopotential height and pressure derived using Equations (3.8) to (3.10) with the constants given in Table 11.1. Values of density in the off-standard atmospheres are calculated using Equation (3.1) with values of temperature from Equation (5.1) and pressure from Equations (3.8) to (3.10). Alternatively, the expressions in Table 11.3 may be used. Values of density in off-standard atmospheres must *not* be calculated using Equations[†] (3.11) and (3.12).

Table 11.4 presents numerical expressions for calculating the pressure height and the temperature difference from the International Standard Atmosphere for given values of pressure and temperature.

Note that one effect of applying temperature increments at constant pressure height is that the resulting properties of off-standard atmospheres are obtained in terms of pressure height and, if required, the corresponding geopotential height must be calculated separately. For all off-standard atmospheres defined here, the relationships between H and H_p are derived in Section 7.3 while Table 11.4 gives numerical expressions from which values of $(H - H_p)$ may be calculated.

† If such expressions are required for off-standard atmospheres, their derivation must take account of the distinction between pressure height (desired as the independent variable) and geopotential height as used in the hydrostatic equation (Equation (3.4)).

6. OTHER AIR PROPERTIES

6.1 Speed of Sound, *a*

The values of a may be calculated from

$$a = (\gamma RT)^{½} = (1.4RT)^{½} = a_0(\theta)^{½}, \tag{6.1}$$

using values of R and a_0 from Table 11.1 and values of T or θ derived from Equation (3.7) or (5.1) or the numerical expressions of Table 11.3.

6.2 Dynamic Viscosity, μ

Dynamic viscosity is the value of the internal friction between two neighbouring layers of air moving at different speeds. Values may be calculated using the following expression which is based on kinetic theory,

$$\mu = \frac{\beta_s T^{3/2}}{T + S}, \tag{6.2}$$

where β_s and S are Sutherland's empirical coefficients. Thus, from Table 11.1

$$\mu = \frac{1.458 \times 10^{-6} T^{3/2}}{T + 110.4} \quad \frac{\text{N s}}{\text{m}^2} \tag{6.3}$$

or

$$\mu = \frac{30.4509 \times 10^{-9} T^{3/2}}{T + 110.4} \quad \frac{\text{lbf s}}{\text{ft}^2}. \tag{6.4}$$

Note that these expressions for μ are invalid for very high or very low temperatures and under conditions occurring at heights above 90 km. Values of μ at high temperatures are given in Aerodynamics Item No. 73017 (Reference 7).

7. RELATIONSHIPS BETWEEN PRESSURE, PRESSURE HEIGHT AND GEOPOTENTIAL HEIGHT

7.1 Introduction

The relationship between geopotential height and pressure (or pressure height) at a point of arbitrary temperature and pressure is obtained from the integration of Equation (3.6) from ground level to the point considered. To do this requires a knowledge of the ground-level pressure and the form of $f(H)$, so it is not possible to derive values of geopotential height without an adequate knowledge of the atmosphere as a whole.

7.2 The Standard Atmosphere

The relationship between geopotential height and pressure in the Standard Atmosphere†, and therefore the relationship between pressure height and pressure is, from Equations (3.7) to (3.9),

$$(H - H_b)_{std} = H_p - H_{p_b} = \frac{T_{b_{std}}}{L_{std}}\left[\left(\frac{p}{p_b}\right)^{-RL_{std}/g_o} - 1\right] \tag{7.1}$$

for $L_{std} \neq 0$ and

$$(H - H_b)_{std} = H_p - H_{p_b} = -\frac{RT_{b_{std}}}{g_0}\log_e\left[\frac{p}{p_b}\right] \tag{7.2}$$

for $L_{std} = 0$.

7.3 The Off-Standard Atmospheres

While it is convenient to have Off-Standard Atmospheres† (see Section 5) expressed in terms of pressure height, it can also be useful to know the corresponding geopotential height. A basic relationship between geopotential height and pressure is given in Equation (3.4) and for an off-standard atmosphere this becomes

$$\frac{\mathrm{d}p}{\mathrm{d}H} = \frac{-g_0 p}{(T_{std} + \Delta T_{H_p})R}, \tag{7.3}$$

while, for the Standard Atmosphere, it is

$$\frac{\mathrm{d}p}{\mathrm{d}H} = \frac{\mathrm{d}p}{\mathrm{d}H_p} = -\frac{g_0 p}{RT_{std}}. \tag{7.4}$$

Dividing Equation (7.4) by Equation (7.3) gives the relationship

$$\frac{\mathrm{d}H}{\mathrm{d}H_p} = \frac{T_{std} + \Delta T_{H_p}}{T_{std}}, \tag{7.5}$$

† The subscript "*std*" denotes values in the ISA (see Table 11.2 for the values of L_{std} and $T_{b_{std}}$) when these differ from values in the off-standard atmospheres; this subscript is unnecessary with pressure or pressure-related quantities which do not differ between Standard and off-standard conditions.

which, on substitution for T_{std} from Equation (3.7) gives the following expression for H in terms of H_p and ΔT_{H_p},

$$\int_{H_b}^{H} \mathrm{d}H = \int_{H_{p_b}}^{H_p} \frac{T_{b_{std}} + \Delta T_{H_p} + L_{std}(H_p - H_{p_b})}{T_{b_{std}} + L_{std}(H_p - H_{p_b})} \mathrm{d}H_p . \tag{7.6}$$

In the off-standard atmospheres described in Section 5 it is assumed that ΔT_{H_p} does not vary with H_p and so Equation (7.6) may be solved to give the following two solutions

$$H - H_b = (H_p - H_{p_b}) + \frac{\Delta T_{H_p}}{L_{std}} \log_e\left[\frac{T_{b_{std}} + L_{std}(H_p - H_{p_b})}{T_{b_{std}}}\right] \tag{7.7}$$

for a layer where $L_{std} \neq 0$ and

$$H - H_b = (H_p - H_{p_b})\left(\frac{T_{b_{std}} + \Delta T_{H_p}}{T_{b_{std}}}\right) \tag{7.8}$$

for a layer where $L_{std} = 0$.

Note that if the expressions for pressure from Equations (7.1) and (7.2) are substituted for $(H_p - H_{p_b})$ in Equations (7.7) and (7.8) the difference between geopotential and pressure heights is always

$$H - H_p = H_b - H_{p_b} - \frac{R}{g_0}\Delta T_{H_p} \log_e\left(\frac{p}{p_b}\right) . \tag{7.9}$$

Further, if the values of $(H_b - H_{p_b})$ for each layer are summed then, for any point in an off-standard atmosphere,

$$H - H_p = H_1 - \frac{R}{g_0}\Delta T_{H_p} \log_e\left(\frac{p}{p_0}\right) = H_1 - \frac{R}{g_0}\Delta T_{H_p} \log_e \delta . \tag{7.10}$$

In Equation (7.10) H_1 is the geopotential height corresponding to H_{p_1} (where $p = p_0$). In constructing the off-standard atmospheres it is usual to assume that the sea-level pressure is standard so that $H = 0$ when $H_p = 0$ and hence in Equations (7.7) to (7.10) $H_1 = 0$. Table 11.4 presents numerical expressions for $(H - H_p)$ based both on Equations (7.7) and (7.8) and Equation (7.10).

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8. RELATIONSHIP BETWEEN STANDARD GEOPOTENTIAL HEIGHT AND GEOMETRIC HEIGHT FOR ALL LATITUDES

In Section 3, the relationship between (standard) geopotential height and geometric height is defined implicitly for the International Standard Atmosphere (and the off-standard atmospheres) by Equation (3.3) with $g_{sl} = g_0$. This strictly applies, as indicated in Section 3, to a geographic latitude of 45.5425 degrees. The present section defines the relationship between H and Z for all latitudes.

The variation of g with geometric height, Z, at any geographic latitude†, ϕ, is assumed to obey the simple, inverse square law

$$g_{\phi,Z} = g_{\phi,sl}\left[\frac{\text{Earth radius}}{Z + \text{Earth radius}}\right]^2 . \tag{8.1}$$

At sea level the variation of the acceleration due to gravity with latitude, denoted $g_{\phi,sl}$, is given by the Lambert Equation†

$$g_{\phi,sl} = 9.806\ 16(1 - 0.002\ 637\ 3\cos 2\phi + 0.000\ 005\ 9\cos^2 2\phi)\ \ \text{m/s}^2. \tag{8.2}$$

In order to take account of the variation between sea level and geometric height Z of the centripetal component of g (*i.e.* that arising from the Earth's rotation and not from gravitational attraction) a fictitious value of Earth radius, r'_{e_ϕ}, is used which does not correspond to the radius, r_{e_ϕ}, of the International Ellipsoid of Reference 12†. Values of this fictitious Earth radius are given by

$$r'_{e_\phi} = \frac{2g_{\phi,sl}}{3.085\ 462\times 10^{-6} + 2.27\times 10^{-9}\cos 2\phi - 2\times 10^{-12}\cos 4\phi} . \tag{8.3}$$

Integration of Equation (3.3) after substituting $g_{\phi,Z}$ from Equation (8.1) for g gives the following relationship between geopotential and geometric heights

$$H = (r'_{e_\phi})^2\left[\frac{1}{r'_{e_\phi}} - \frac{1}{(r'_{e_\phi} + Z)}\right]\left[\frac{g_{\phi,sl}}{g_0}\right] = Z\left(\frac{r'_{e_\phi}}{r'_{e_\phi} + Z}\right)\left(\frac{g_{\phi,sl}}{g_0}\right) . \tag{8.4}$$

For any geographic latitude ϕ, substitution of the appropriate values of $g_{\phi,sl}$ and r'_{e_ϕ} from Equations (8.2) and (8.3) gives the required numerical relationship between H and Z. In particular, for the ISA,

$$g_{\phi,sl} = g_0 = 9.806\ 65\ \ \text{m/s}^2 (\text{or } 32.174\ 0\ \ \text{ft/s}^2),$$

which corresponds to $\phi = 45.542\ 5$ degrees,

at which $r'_{e_\phi} = 6.356\ 766\times 10^6\ \text{m}\ (\text{or } 20.855\ 53\times 10^6\ \text{ft})$.

† See Section A1 of Appendix A of Item No. 79018 (Reference 12) for a more detailed account of these terms and of the origins of Equations (8.2) and (8.3).

9. DERIVATION AND REFERENCES

Derivation

1.	ISO	*Standard Atmosphere (Identical with the ICAO and WMO Standard Atmospheres from –2 to 32 km*). International Organization for Standardization, ISO 2533, 1975.

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2.	SCORER, R.S.	*Natural aerodynamics*. Pergamon Press, 1958.
3.	–	*US Standard Atmosphere*, *1962*. US Committee on Extension to the Standard Atmosphere, US Government Printing Office, 1962.
4.	–	*Manual of the ICAO Standard Atmosphere*. International Civil Aviation Organization Document 7488/2, 1964.
5.	ESDU	Atmospheric data for performance calculations. Data Item No. 68046, ESDU International, December 1968 (with Amendments A to D, October 1992).
6.	ESDU	Atmospheric data for performance calculations. Addendum: height in feet, data in SI units. Data Item No. 72018, ESDU International, May 1972.
7.	ESDU	Reynolds number, speed of sound, dynamic viscosity, kinetic pressure and total pressure coefficient in air. Data Item No. 73017, ESDU International, August 1973, (with Amendments A and B, December 1977).
8.	–	Conversion factors and tables. British Standards Institution, B.S. 350: Part 1: 1974.
9.	–	*US Standard Atmosphere*, 1976. US Committee on Extension to the Standard Atmosphere, US Government Printing Office, 1976.
10.	ESDU	Properties of a standard atmosphere. Data Item No. 77021, ESDU International, Issued October 1977 (with Amendments A and B, March 2005)
11.	ESDU	Height relationships for non-standard atmospheres. Data Item No. 78012, ESDU International, June 1978 (with Amendment A, February 1986).
12.	ESDU	Example of performance analysis using data obtained concurrently in air-path, body and Earth axes. Data Item No. 79018, ESDU International, November 1979.
13.	ISO	Reference Atmospheres for Aerospace. International Organization for Standardization, ISO 5878, 1982.

10. EXAMPLES

10.1 Calculation of H_p and ΔT_{H_p}

An aeroplane is flying in conditions where the static pressure is 20 540 N/m^2 and the outside air temperature is 227.5 K. Find the pressure height and the temperature change from ISA.

Pressure height

From Table 11.4 the given static pressure indicates that the relevant expression for pressure height is

$$\begin{aligned} H_p &= 74\,588.142 - 6\,341.615\,6\ \log_e p \\ &= 74\,588.142 - 6\,341.615\,6\ \log_e 20\,540 \\ &= 11\,615\ \mathrm{m}\ . \end{aligned}$$

Temperature change

From Table 11.4 the change from ISA temperature is

$$\begin{aligned} \Delta T_{H_p} &= T - 216.65 \\ &= 227.5 - 216.65 \\ \Delta T_{H_p} &= 10.85\ \mathrm{K}. \end{aligned}$$

10.2 Calculation of H

Find the geopotential height corresponding to a pressure height of 70 000 ft in an ISA + 20 K atmosphere. Assume that sea-level pressure is the Standard value, *i.e.* $H_1 = 0$.

From the third column of Table 11.4 the expression for $H - H_p$ is

$$H - H_p = H_1 + \Delta T_{H_p}\Big(280.243\,69 + 3\,280.839\,9 \times \log_e\big(0.907\,685\,21 + 1.406\,877\,5 \times 10^{-6} H_p\big)\Big).$$

Setting $H_1 = 0$ and substituting for ΔT_{H_p} and H_p gives

$$\begin{aligned} H - H_p &= 20\Big(280.243\,69 + 3\,280.839\,9 \times \log_e\big(0.907\,685\,21 + 1.406\,877\,5 \times 10^{-6} \times 70\,000\big)\Big) \\ H - H_p &= 6\,008\ \mathrm{ft}\ . \end{aligned}$$

Hence the geopotential height for $H_p = 70\,000$, $\Delta T_{H_p} = 20$ is

$$H = 76\,008\ \mathrm{ft}$$

11. TABLES

TABLE 11.1 Constants Used in Calculating Properties of the Standard and Off-Standard Atmospheres

	SI	*British*†
a_0	340.294 m/s	1 116.45 ft/s
‡g_0	**9.806 65** m/s^2	32.174 0 ft/s^2
m_0	28.964 42 kg/kmol	see Section 1.2
p_0	**101 325** N/m^2	2 116.22 lbf/ft^2
$\Re$	8 314.32 N m/kmol K	see Section 1.2
R	287.052 87 N m/kg K	3 089.811 4 ft lbf/slug K
‡r'_{e_ϕ}	$6.356\ 766 \times 10^6$ m	$20.855\ 53 \times 10^6$ ft
S	110.4 K	110.4 K
T_0	**288.15** K	**288.15** K
β_s	1.458×10^{-6} N $s/m^2K^{½}$	$30.450\ 9 \times 10^{-9}$ lbf $s/ft^2K^{½}$
μ_0	17.894×10^{-6} N s/m^2	$0.373\ 72 \times 10^{-6}$ lbf s/ft^2
†ρ_0	**1.225** kg/m^3	$2.376\ 892 \times 10^{-3}$ $slug/ft^3$

† The International Standard Atmosphere of Derivation 1 is defined in terms of SI units only. The values of the constants given in British Units have been obtained by conversion (Reference 8) from the SI values, are correct to the number of figures quoted, and are adequate for most calculation purposes. However, while the value of ρ_0 can be obtained by direct conversion from the SI value, the need for p_0, R, T_0 and ρ_0 to satisfy Equation (3.1) can only be met if an exceptionally large number of significant figures is used, as follows:

$$p_0 = 2116.216\ 624 \quad lbf/ft^2$$
$$R = 3089.811\ 378 \quad ft\ lbf/slug\ K$$
$$T_0 = 288.15 \quad K$$

giving, from Equation (3.1), $\rho_0 = 0.002\ 376\ 892\ 44 \quad slug/ft^3$.

‡ The standard sea-level value, g_0, of acceleration due to gravity, used in defining the International Standard Atmosphere, corresponds to a geographic latitude of $\phi = 45.5425$ degrees in the Lambert equation (Equation (8.2)). The value of the fictitious Earth radius, r'_{e_ϕ}, at this latitude is deduced from Equation (8.3).

TABLE 11.2 Standard Atmosphere Temperature Gradients and Temperatures and Pressures at Break Points

SI UNITS

Geopotential height, $H_{b_{std}}$ m	*Geometric height*, $Z_{b_{std}}$ m	*Temperature gradient*, L_{std} K/m	*Temperature*, $T_{b_{std}}$ K	*Pressure*, p_b N/m^2
0	0		288.15	101 325
		-6.5×10^{-3}		
11 000	11 019.1		216.65	22 632.0
		0		
20 000	20 063.1		216.65	5 474.87
		1.0×10^{-3}		
32 000	32 161.9		228.65	868.014
		2.8×10^{-3}		
47 000	47 350.1		270.65	110.906
		0		
50 000	50 396.4		270.65	75.944 3

BRITISH UNITS

Geopotential height, $H_{b_{std}}$ ft	*Geometric height*, $Z_{b_{std}}$ ft	*Temperature gradient*, L_{std} K/ft	*Temperature*, $T_{b_{std}}$ K	†*Pressure*, p_b lbf/ft^2
0	0		288.15	2 116.22
		$-1.981\ 2 \times 10^{-3}$		
36 089.2	36 151.8		216.65	472.680
		0		
65 616.8	65 823.9		216.65	114.345
		$0.304\ 8 \times 10^{-3}$		
104 987	105 518		228.65	18.128 8
		$0.853\ 44 \times 10^{-3}$		
154 199	155 348		270.65	2.316 32
		0		
164 042	165 343		270.65	1.586 13

† These values are direct conversions of those given in SI units.

TABLE 11.3 Numerical Expressions for Calculating Properties of Standard and Off-Standard Atmospheres for Given Values of Pressure Height

(SI UNITS)

Quantity	*Format of numerical expressions*	*Constants in numerical expressions*	*Range of pressure height,* H_p m: *up to* 11 000	11 000 *to* 20 000	20 000 *to* 32 000	32 000 *to* 47 000	47 000 *to* 50 000
Temperature in International Standard Atmosphere, T_{std} K	$\mathbf{A} + \mathbf{B}H_p$	**A** = **B** =	288.15 -6.5×10^{-3}	216.65 0	196.65 10^{-3}	139.05 2.8×10^{-3}	270.65 0
Pressure in International Standard and off-standard atmospheres, p N/m²	(i) $\left(\mathbf{C} + \mathbf{D}H_p\right)^{\mathbf{E}}$ (ii) $\mathbf{F} \exp\left(\mathbf{G}H_p\right)$	**C** = **D** = **E** = **F** = **G** =	8.961 963 8 $-0.202\,161\,25 \times 10^{-3}$ 5.255 879 7	128 244.5 $-0.157\,688\,52 \times 10^{-3}$	0.705 518 48 $3.587\,686\,1 \times 10^{-6}$ −34.163 218	0.349 268 67 $7.033\,098\,0 \times 10^{-6}$ −12.201 149	41 828.420 $-0.126\,226\,56 \times 10^{-3}$
Density in International Standard Atmosphere, ρ_{std} kg/m³	(i) $\left(\mathbf{I} + \mathbf{J}H_p\right)^{\mathbf{L}}$ (ii) $\mathbf{M} \exp\left(\mathbf{N}H_p\right)$	**I** = **J** = **L** = **M** = **N** =	1.048 840 $-23.659\,414 \times 10^{-6}$ 4.255 879 7	2.062 140 0 $-0.157\,688\,52 \times 10^{-3}$	0.972 630 9 $4.946\,00 \times 10^{-6}$ −35.163 218	0.843 929 29 $16.993\,902 \times 10^{-6}$ −13.201 149	0.538 395 63 $-0.126\,226\,56 \times 10^{-3}$
Density in off-standard atmospheres, ρ kg/m³	$\dfrac{\rho_{std}}{1 + \dfrac{\Delta T_{H_p}}{T_{std}}}$						

TABLE 11.3 Numerical Expressions for Calculating Properties of Standard and Off-Standard Atmospheres for Given Values of Pressure Height (continued)

(BRITISH UNITS)

	Range of pressure height, H_p ft		*up to* 36 089.2	36 089.2 *to* 65 616.8	65 616.8 *to* 104 987	104 987 *to* 154 199	154 199 *to* 164 042
Quantity	*Format of numerical expressions*	*Constants in numerical expressions*					
Temperature in International Standard Atmosphere, T_{std} K	$\mathbf{A} + \mathbf{B}H_p$	**A** = **B** =	288.15s $-1.981\ 2 \times 10^{-3}$	216.65 0	196.65 $0.304\ 8 \times 10^{-3}$	139.05 $0.853\ 44 \times 10^{-3}$	270.65 0
Pressure in International Standard and off-standard atmospheres, p lbf/ft^2	(i) $\left(\mathbf{C} + \mathbf{D}H_p\right)^{\mathbf{E}}$ (ii) $\mathbf{F}\exp\left(\mathbf{G}H_p\right)$	**C** = **D** = **E** = **F** = **G** =	4.292 708 5 $-29.514\ 885 \times 10^{-6}$ 5.255 879 7	 2 678.442 0 $-48.063\ 462 \times 10^{-6}$	0.790 112 02 $1.224\ 643\ 5 \times 10^{-6}$ $-34.163\ 218$	0.479 583 69 $2.943\ 516\ 0 \times 10^{-6}$ $-12.201\ 149$	 873.604 72 $-38.473\ 855 \times 10^{-6}$
Density in International Standard Atmosphere, ρ_{std} slug/ft^3	(i) $\left(\mathbf{I} + \mathbf{J}H_p\right)^{\mathbf{L}}$ (ii) $\mathbf{M}\exp\left(\mathbf{N}H_p\right)$	**I** = **J** = **L** = **M** = **N** =	0.241 792 85 $-1.662\ 467\ 5 \times 10^{-6}$ 4.255 879 7	 $4.001\ 212\ 2 \times 10^{-3}$ $-48.063\ 462 \times 10^{-6}$	1.161 656 4 $1.800\ 523\ 2 \times 10^{-6}$ $-35.163\ 218$	1.354 416 7 $8.312\ 933\ 5 \times 10^{-6}$ $-13.201\ 149$	 $1.044\ 660\ 0 \times 10^{-3}$ $-38.473\ 855 \times 10^{-6}$
Density in off-standard atmospheres, ρ slug/ft^3	$\dfrac{\rho_{std}}{1 + \dfrac{\Delta T_{H_p}}{T_{std}}}$						

TABLE 11.4 Numerical Expressions For Calculating Properties of Standard and Off-Standard Atmospheres for Given Values of Temperature and Pressure

(SI UNITS)

Quantity	Format of numerical expressions	Constants in numerical expressions					
	Range of pressure, p N/m^2		*greater than* 22 632.0	22 632.0 *to* 5 474.87	5 474.87 *to* 868.014	868.014 *to* 110.906	110.906 *to* 75.944 3
	Range of relative pressure δ		*greater than* 0.223 360	0.223 360 *to* 0.054 032 8	0.054 032 8 *to* 0.008 566 63	0.008 566 63 *to* 0.001 094 56	0.001 094 56 *to* 0.000 749 512
	Range of pressure height, H_p m		*up to* 11 000	11 000 *to* 20 000	20 000 *to* 32 000	32 000 *to* 47 000	47 000 *to* 50 000
Pressure height, H_p m	(i) $\mathbf{A}+\mathbf{B}p^{\mathbf{C}}$	**A** =	44 330.769		–196 650	–49 660.714	
		B =	–4 946.546 3		278 731.18	142 184.85	
		C =	0.190 263 11		–0.029 271 247	–0.081 959 491	
	(ii) $\mathbf{D}+\mathbf{E}\log_e p$	**D** =		74 588.142			84 303.425
		E =		–6 341.615 6			–7 922.263 0
Temperature difference from International Standard Atmosphere, ΔT_{H_p} K	(i) $T+\mathbf{F}p^{\mathbf{G}}$	**F** =	–32.152 551		–278.731 18	–398.117 59	
		G =	0.190 263 11		–0.029 271 247	–0.081 959 491	
	(ii) $T+\mathbf{I}$	**I** =		–216.65			–270.65
Difference between geopotential and pressure heights in off-standard atmospheres, $(H-H_p)$ m	(i)† $H_1+\Delta T_{H_p}\left(\mathbf{J}+\mathbf{L}\log_e\left(\mathbf{M}+\mathbf{N}H_p\right)\right)$	**J** =	0		85.418 277	139.327 58	
		L =	–153.846 15		1 000	357.142 86	
		M =	1.0		0.907 685 2	0.608 134 7	
		N =	–22.557 696 × 10^{-6}		4.615 739 7 × 10^{-6}	12.245 791 × 10^{-6}	
	(ii)† $H_1+\Delta T_{H_p}\left(\mathbf{Q}+\mathbf{R}H_p\right)$	**Q** =		–6.896 516 5			25.898 003
		R =		4.615 739 7 × 10^{-3}			3.694 808 8 × 10^{-3}

† NOTE (i) The quantity H_1 is the geopotential height at which $p = p_0 = 101\,325$ N/m^2.

(ii) In all layers of the atmosphere, $H - H_p = H_1 - 29.271\,247\Delta T_{H_p}\log_e\delta$ m.

TABLE 11.4 Numerical Expressions For Calculating Properties of Standard and Off-Standard Atmospheres for Given Values of Temperature and Pressure (continued)

(BRITISH UNITS)

	Range of pressure, p lbf/ft^2		*greater than* 472.680	472.680 *to* 114.345	114.345 *to* 18.128 8	18.128 8 *to* 2.316 32	2.316 32 *to* 1.586 13
	Range of relative pressure, δ		*greater than* 0.223 360	0.223 360 *to* 0.054 032 8	0.054 032 8 *to* 0.008 566 63	0.008 566 63 *to* 0.001 094 56	0.001 094 56 *to* 0.000 749 512
	Range of pressure height, H_p ft		up to 36 089.2	36 089.2 to 65 616.8	65 616.8 to 104 987	104 987 to 154 199	154 199 to 164 042
Quantity	*Format of numerical expressions*	*Constants in numerical expressions*					
Pressure height, H_p ft	(i) $\mathbf{A}+\mathbf{B}p^{\mathbf{C}}$	**A** = **B** = **C** =	145 442.16 –33 881.210 0.190 263 11		–645 177.17 816 564.17 –0.029 271 247	–162 928.85 339 729.76 –0.081 959 491	
	(ii) $\mathbf{D}+\mathbf{E}\,\log_e p$	**D** = **E** =		164 220.19 –20 805.826			176 031.96 –25 991.676
Temperature difference from International Standard Atmosphere, ΔT_{H_p} K	(i) $T+\mathbf{F}p^{\mathbf{G}}$	**F** = **G** =	–67.125 453 0.190 263 11		–248.888 76 –0.029 271 247	–289.938 97 –0.081 959 491	
	(ii) $T+\mathbf{I}$	**I** =		–216.65			–270.65
Difference between geopotential and pressure heights in off-standard atmospheres, $(H-H_p)$ ft	(i)† $H_1+\Delta T_{H_p}\left(\mathbf{J}+\mathbf{L}\log_e\left(\mathbf{M}+\mathbf{N}H_p\right)\right)$	**J** = **L** = **M** = **N** =	0 –504.744 60 1.0 $-6.875\,585\,6\times10^{-6}$		280.243 69 3 280.839 9 0.907 685 21 $1.406\,877\,5\times10^{-6}$	457.111 48 1 171.728 5 0.608 134 70 $3.732\,516\,9\times10^{-6}$	
	(ii)† $H_1+\Delta T_{H_p}\left(\mathbf{Q}+\mathbf{R}H_p\right)$	**Q** = **R** =		–22.626 366 $4.615\,739\,7\times10^{-3}$			84.967 20 $3.694\,808\,8\times10^{-3}$

† NOTE (i) The quantity H_1 is the geopotential height at which $p = p_0 = 2\ 116.22$ lbf/ft^2.

(ii) In all layers of the atmosphere, $H-H_p = H_1 - 96.034\ 275\ \Delta T_{H_p} \log_e \delta$ ft.

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