Quantum Langevin equation

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March 31, 2024

1 General derivation

In this section we derive the quantum Langevin equation. The system of interest is in contact with a bigger system, called bath. The interaction between them is linear, under rotating wave approximation

$$H = H_{sys} + \sum_k \omega_k b_k^\dagger b_k + \sum_k \frac{c_k}{\sqrt{2m_k\omega_k}} \left(b_k S_+ + b_k^\dagger S_- \right), \tag{1}$$

where $S_+,\ S_-$ are positive/negative frequencies operators of the system which realise the bath-system coupling (with the property $S_- = S_+^\dagger$). Considering the self-adjoint operator S, we define $S_+ = \sum_{m>n} \langle n|S|m\rangle |n\rangle\langle m|$, where $|n\rangle, |m\rangle$ are eigenstates of H_{sys} with eigenfrequencies $\omega_n < \omega_m$. Again, we write the equations of motion

$$\begin{aligned} \dot{A} &= -i[A, H_{sys}] - i \sum_k \frac{c_k}{\sqrt{2m_k\omega_k}} \Big([A, S_+] b_k + b_k^\dagger [A, S_-] \Big) \\ \dot{b}_k &= -i\omega_k b_k - i \frac{c_k}{\sqrt{2m_k\omega_k}} S_-. \end{aligned} \tag{2}$$

The formal solution for the bath is

$$b_k = b_k^{homg.}(t) - i \frac{c_k}{\sqrt{2m_k\omega_k}} \int_{t_0}^t e^{-i\omega_k(t-t')} S_-(t') dt', \tag{3}$$

where $b_k^{homg.}(t) = b_k(0) e^{-i\omega_k t}$. We define, as in the previous section, the quantum noise operators as $\xi_- = \sum_k \frac{c_k}{\sqrt{2m_k\omega_k}} b_k^{homg.}$ ($\xi_+ = \xi_-^\dagger$) and, making use of the spectral density $J(\omega) = \pi/2 \sum_k |c_k|^2/(m_k\omega_k)$ we get

$$\begin{aligned} \dot{A} &= -i[A, H_{sys}] - i \left([A, S_+]\xi_- + [A, S_-]\xi_+ \right) \\ &- [A, S_+] \int_{t_0}^t D_+(t-t') S_-(t') dt' + \int_{t_0}^t D_-(t-t') S_+(t') dt' [A, S_-], \end{aligned} \tag{4}$$

where $D_+(t) = \int_0^\infty J(\omega) e^{-i\omega t} d\omega/\pi$ is the positive frequency time-derivative dissipator. We notice that, so far we didn't use the integration by part, as we did before. This is actually a crucial difference, which is introduced by the RWA, and affects the assumption on the shape of the spectral density of the bath. Indeed we see that, in order to recover the usual quantum Langevin equation, we need to assume a flat spectral density,

$$J(\omega) \sim \gamma, \tag{5}$$

which brings the dissipator $D_{\pm}$ to a delta function in time. This gives back the following Langevin equation

$$\dot{A} = -i[A, H_{sys}] - i\left([A, S_+]\xi_- + [A, S_-]\xi_+\right) - \frac{\gamma}{2}\left([A, S_+]S_- - S_+[A, S_-]\right) \tag{6}$$