Algorithm Design

Exercise Sheet 2

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General Terms: You can work on the exercises in teams of two people, and hand them in accordingly. However, make sure that each one of you has fully understood every solution you present. Do not copy any work from other students, the internet or other sources, and do not share your work with others outside your team. If at any point, any part of the exercises you hand in is apparent to be a copy of other work, this will result in the following consequences: All of your exercises, previous as well as upcoming ones, will be treated as if you did not hand them in at all, and you will have to participate in the written exam to make up for this. Please note that there will be no exception made, even if you are the original author of work someone else copied, or if your exercise partner is the one responsible. Therefore, please make sure to only choose a partner that you trust, and do not hand out your exercise solutions to others.

Late Policy: If you hand in your exercises after the due date, each day that you are late will result in a discount to your score, i.e. you will only receive 90% of the points if you are one day late, 80% on the second day after the due date, and 70% on the third. If you are late by more than three days, the assignment will get zero points in total.

Formal Requirements: Please typeset your solutions (11 pt at least), and state the names of both team members at the beginning of each sheet you hand in. Make sure to name the exact exercise each part of the solution refers to, otherwise it will not be graded. Please start a new page for each main exercise in the assignment (i.e. exercise 1, 2, etc., but not for each subquestion in them). Make sure your solution takes no more than one page for each main exercise (plus at most one extra page for the code, if an implementation is required). Everything after the first page will not be taken into account. So, for example, when the assignment has four exercises, please hand in four pages, one for each exercise. All solutions have to be sent via email to

lazos@diag.uniroma1.it **or** rebeccar@diag.uniroma1.it

Office Hours: There will be special hours for questions about the exercises announced via the piazza site. Presumably, this will take place in form of a zoom meeting due to the Covid19-measures.

Exercise 1

Santa is worried about his employee relations, since christmas preparations have led to a lot of overtime. To make sure all the elves are happy, he wants to recruit some of them as *complaint officers*, with weekly meetings to report any complaints or worries to him. His worker elves W are pretty busy already, so Santa wants to task no more than k elves with this additional workload. Still, Santa wants to make sure that for as many elves $e \in W$ as possible, at least one of his friends (whose identities he knows) is a complaint officer.

1. Give an intuitive greedy algorithm that outputs k elves that will serve as compliant officers.
2. Prove that for large numbers of k the algorithm approximates a solution with ratio no more than $(1 - \frac{1}{e})$.

Exercise 2

You are tasked with shipping a number n of goods $g_i \in \{1, \ldots, n\}$ to a target location t_i. In your very rural area, the roads are in bad shape and often blocked by trees and the like, and also, there are only three cargo companies you can use for the transport of any good g_i. You have to order one truck for each good, and know the companies' routes $P_{i,1}$, $P_{i,2}$ and $P_{i,3}$ they would take to all specific locations, where each route consists of a sequence of road segments that it uses. Now since you are aware of the frequent road blockages, and you want not too many of your transports to be obstructed, your aim is to pick the paths/companies for the goods in such a way that a single blocked road can intercept as few shipments as possible. I.e. choose a path $P_{i,j}$ for every good g_i such that $\max_{e \in E} \{|\{P_{i,j} | e \in P_{i,j}\}|\}$ is as small as possible, where E is the set of all road segments.

(a) Formulate the problem as an ILP, and relax to an according LP.

(b) Give a rounding algorithm to compute a feasible solution starting from the LP optimum that guarantees a 3-approximation to the optimal solution, and prove its approximation guarantee.

Exercise 3

Consider the following variation of the min-cut algorithm presented in class. We start with a graph G with n vertices, and we use the randomized min-cut algorithm to contract the graph down to a graph G_k with $k = \sqrt{n}$ vertices. Next, we make $l = \sqrt{n}$ copies of the graph G_k, and run the randomized algorithm independently on each copy of the reduced graph. We output the smallest min-cut set found in all the executions.

(a) What is the probability that the reduced graph G_k has the same cut-set value as the original graph G?

(b) What is the probability that the algorithm outputs a correct min-cut set?

Hint: For $a << b$ you can use $(1 - 1/b)^a \leq (1 - a/b)$.

(c) Compare the number of contractions and the resulting error probability to that when running the original algorithm twice and taking the minimum cut-set value.

Exercise 4

You are organizing a conference that has received n submitted papers. Your goal is to get people to review as many of them as possible. To do this, you have enlisted the help of k reviewers. Each reviewer i has a cost s_{ij} for writing a review for paper j. The strategy of each reviewer i is to select a subset of papers to write a review for. They can select any subset $S_i \subset \{1, 2, \ldots, n\}$, as long as the total cost to write all reviews is less than T (the time before the deadline):

$$\sum_{j \in S_i} s_{ij} \leq T.$$

Reviews are costly, so you want to reward them for their efforts. However, each paper and review has to be treated equally: specifically, there is a budget of 1 for each paper, that will be evenly shared across all reviewers who reviewed that paper. For example, if 4 reviewers reviewed a paper they will receive $1/4$ each. If only one reviewer reviews the paper they will receive all the reward. Of course, each reviewer i wants to maximize their utility u_i, which is the reward R_i over all papers they receive minus the effort they put into writting reviews:

$$u_i = R_i - \sum_{j \in S_i} s_{ij}.$$

You can assume that for the given s_{ij}'s, there is a combination of strategies S_i where every reviewer has positive utility and all papers get at least one review. However, this outcome might not be a pure Nash equilibrium. As a designer, your goal is to find the *fraction* of papers that receive at least 1 review at a pure Nash equilibrium, for the worst possible combination of s_{ij}'s satisfying the assumption.

(a) Show that there exists a set of s_{ij}'s such that the fraction of papers that receive reviews is close to $1/n$. Given the previous *negative* result, you think about increasing the reward of each paper from 1 to $B > 1$.

(b) Show that for $B = 2$ this fraction is close to $1/3$. [Hint: You can consider an instance with $3n + 1$ papers and only n will be reviewed.]