FASIT OPPGAVER UKE 7-8, STA-1001 2024

4.23

a) $E(XY^2) = \sum_x \sum_y x \cdot y^2 \cdot f(x,y)$
$= 2 \cdot 1^2 \cdot 0.15 + 2 \cdot 3^2 \cdot 0.25 + 2 \cdot 5^2 \cdot 0.15 + 4 \cdot 1^2 \cdot 0.10 + 4 \cdot 3^2 \cdot 0.25 + 4 \cdot 5^2 \cdot 0.10 = 31.7$

b)

x	2	4
$f(x)$	0.55	0.45

y	1	3	5
$f(y)$	0.25	0.50	0.25

$E(X) = \sum_x xf(x) = 2.90$ $\qquad E(Y) = \sum_y yf(y) = 3.00$

4.35

$\mu = E(X) = \sum_x xf(x) = 4.11$
$E(X^2) = \sum_x x^2 f(x) = 2^2 \cdot 0.01 + 3^2 \cdot 0.25 + 4^2 \cdot 0.4 + 5^2 \cdot 0.3 + 6^2 \cdot 0.04 = 17.63$
$\sigma^2 = Var(X) = E(X^2) - \mu^2 = 17.63 - 4.11^2 = 0.7379$

4.41

$\mu_{h(X)} = E[h(X)] = \sum_x h(x))^2 f(x) = \sum_x (3x+1)^2 f(x)$
$= (3 \cdot (-3)x + 1)^2 \cdot \frac{1}{6} + (3 \cdot 6 + 1)^2 \cdot \frac{1}{2} + (3 \cdot 9 + 1)^2 \cdot \frac{1}{3} = 452.5$
$\sigma^2_{h(X)} = Var[h(X)] = \sum_x (h(x) - \mu_{h(X)})^2 f(x) = \sum_x [(3x+1)^2 - 452,5]^2 f(x) = 65972.25$
$\sigma_{h(X)} = \sqrt{\sigma^2_{h(X)}} = 256.85$

4.53

Fra 4.35: $E(X) = 4.11$ og $Var(X) = 0.74$
$E(Z) = E(3X - 2) = 2 \cdot (E(X) - 2 = 10.33$
$Var(Z) = Var(3X - 2) = 3^2 \cdot Var(X) = 6.66$

4.60

a) $E(X) = 2.9 \qquad E(Y) = 3 \qquad E(2X - 3Y) = 2E(X) - 3E(Y) = -3.20$
Alternativt: $E(2X - 3Y) = (2 \cdot 2 + 3 \cdot 1) \cdot 0.15 + \cdots + (2 \cdot 4 - 3 \cdot 5) \cdot 0.10 = -3.20$

b) $E(XY) = 2 \cdot 1 \cdot 0.15 + \cdots + 4 \cdot 5 \cdot 0.10 = 8.70$
Merk at det er en feil i oppgaveteksten. Disse variablene er faktisk ikke uavhengige, men de har kovarians 0! Dette fordi $E(XY) = E(X)E(Y)$.

4.62

$Var(Z) = Var(-2X + 4Y - 3) = (-2)^2 Var(X) + 4^2 Var(Y) = 4 \cdot 5 + 16 \cdot 3 = 68$

4.63

$Var(Z) = Var(-2X + 4Y - 3) = (-2)^2 Var(X) + 4^2 Var(Y) + 2 \cdot (-2) \cdot 4 \cdot Cov(X,Y) = 4 \cdot 5 + 16 \cdot 3 - 16 \cdot 1 = 52$

4.78

$E(X) = \frac{1}{2} \qquad E(X^2) = \frac{2}{7} \qquad Var(X) = \frac{1}{28} \qquad SD(X) = 0.1890$

$$P(\mu - 2\sigma < X < \mu + 2\sigma) \quad = \quad \int\limits_{0.1220}^{0.8780} 30x^2(1-x)^2 dx = 0.9700$$

Tsjebytsjev: $P(\mu - 2\sigma < X < \mu + 2\sigma) \geq 1 - \frac{1}{2^2} = 0.75$

A2

a)

$$\begin{aligned} E(X) &= \int\limits_2^\infty x \cdot \tfrac{160}{x^6} dx = \tfrac{5}{2} \\ E(X^2) &= \int\limits_2^\infty x^2 \cdot \tfrac{160}{x^6} dx = \tfrac{20}{3} \\ Var(X) &= E(X^2) - [E(X)]^2 = \tfrac{5}{12} \end{aligned}$$

b)

$$\begin{aligned} \int\limits_2^m f(x)\, dx &= 0.5 \\ \int\limits_2^m \tfrac{160}{x^6}\, dx &= 0.5 \\ -32 \cdot m^{-5} &= 0.5 \\ m &= \left(\tfrac{1}{64}\right)^{-1/5} = 2.297 \end{aligned}$$

c)

$$g(x) = \begin{cases} 100, & x \leq 2.5 \\ 100 - 30 \cdot (x - 2.5), & x > 2.5 \end{cases}$$

$$\begin{aligned} E[g(X)] &= \int\limits_2^\infty g(x) \cdot \tfrac{160}{x^6} dx = \int\limits_2^{2.5} 100 \cdot \tfrac{160}{x^6} dx + \int\limits_{2.5}^\infty [100 - 30 \cdot (x - 2.5)] \cdot \tfrac{160}{x^6} dx \\ &= 67.232 + 26.624 = 93.856 \end{aligned}$$

Kan her innvende at funksjonen $g(x)$ aldri burde gå lavere enn 0 for å være meir realistisk. Det kan vi gjøre ved å sette $g(x) = 0$ for $x > \frac{35}{6}$. Forventninga blir da omtrent 93.4 millioner. Grunnen til at den ikke endres mye er fordi det er veldig lav sannsynlighet for $x > \frac{35}{6}$.

A3

OBS: For å forenkle utregningene er ofte gamma-funksjonen brukt:

$\Gamma(n) = \int\limits_0^\infty x^{n-1} e^{-x} dx = (n-1)!, \quad n = 1, 2, \ldots$

Se ellers "Tabeller og formler i statistikk". Etter substitusjon kan en òg se at:

$\int\limits_0^\infty x^{n-1} e^{-ax} dx = \frac{\Gamma(n)}{a^n} = \frac{(n-1)!}{a^n}, \quad n = 1, 2, \ldots, a > 0$

a)

$$
\begin{aligned}
E(X+Y) &= \int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}(x+y)f(x,y)dxdy = \int\limits_{0}^{\infty}\int\limits_{0}^{\infty}(x+y)2e^{-(2x+y)}dxdy \\
&= \int\limits_{0}^{\infty}e^{-y}\int\limits_{0}^{\infty}2xe^{-2x}dxdy + \int\limits_{0}^{\infty}ye^{-y}\int\limits_{0}^{\infty}2e^{-2x}dxdy \\
&= \int\limits_{0}^{\infty}\frac{e^{-y}}{2}dy + \int\limits_{0}^{\infty}ye^{-y}dy = \frac{\Gamma(1}{2}) + \Gamma(2) = \frac{1}{2} + 1 = \frac{3}{2}
\end{aligned}
$$

b)

$$
\begin{aligned}
E(X) &= \int\limits_{-\infty}^{\infty} xg(x)\,dx = \int\limits_{0}^{\infty} x2e^{-2x}\,dx = \frac{\Gamma(1)}{2} = \frac{1}{2} \\
E(X^2) &= \int\limits_{-\infty}^{\infty} x^2g(x)\,dx = \int\limits_{0}^{\infty} x^2 2e^{-2x}\,dx = \frac{\Gamma(3)}{4} = \frac{1}{2} \\
Var(X) &= E(X^2) - \mu_X^2 = \frac{1}{2} - \left(\frac{1}{2}\right)^2 = \frac{1}{4} \\
E(Y) &= \int\limits_{-\infty}^{\infty} yh(y)\,dy = \int\limits_{0}^{\infty} y\cdot e^{-y}\,dy = \Gamma(2) = 1 \\
E(Y^2) &= \int\limits_{-\infty}^{\infty} y^2h(y)\,dy = \int\limits_{0}^{\infty} y^2\cdot e^{-y}\,dy = \Gamma(3) = 2 \\
Var(Y) &= E(Y^2) - \mu_Y^2 = 2 - 1^2 = 1
\end{aligned}
$$

Merk at $E(X+Y) = E(X) + E(Y)$ som det skal være.

c)

$$
\begin{aligned}
Cov(X,Y) &= E(XY) - \mu_X\mu_Y = \int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} xyf(x,y)dxdy - \frac{1}{2}\cdot 1 \\
&= \int\limits_{0}^{\infty}\int\limits_{0}^{\infty} xy2e^{-(2x+y)}dxdy - \frac{1}{2} = \int\limits_{0}^{\infty} ye^{-y}\int\limits_{0}^{\infty} 2xe^{-2x}dxdy - \frac{1}{2} \\
&= \int\limits_{0}^{\infty}\frac{ye^{-y}}{2}dy - \frac{1}{2} = \frac{\Gamma(2)}{2} - \frac{1}{2} = 0
\end{aligned}
$$

Dette kunne vi forøvrig se fra oppgave a), da $f(x,y) = g(x)\cdot h(y)$. Da er X og Y uavhengige og kovariansen vil alltid bli 0.

$Var(X+Y) = Var(X) + Var(Y) + 2\,Cov(X,Y) = Var(X) + Var(Y) = \frac{1}{4} + 1 = \frac{5}{4} = 1.25$