

# Bachelorprojekt

Title

(subtitle)

Daniel Kongsgaard

Vejleder: Jens Carsten Jantzen

Aarhus Universitet



## **Abstract**

Some text

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# Chapter 1

## Harish-Chandra modules over $\mathfrak{sl}(2, \mathbf{C}) \times \mathfrak{sl}(2, \mathbf{C})$

Let  $L$  be a semisimple Lie algebra and let  $L_k$  be a Lie subalgebra.

**Definition 1.1.** An  $L$ -module  $M$  is a Harish-Chandra module for the pair  $(L, L_k)$  if, regarded as an  $L_k$ -module, it can be written as a sum

$$M = \bigoplus_i M_i$$

of finite dimensional simple<sup>1</sup>  $L_k$ -submodules  $M_i$ , where for each  $M_{i_0}$  only finitely many  $L_k$ -submodules equivalent to  $M_{i_0}$  occur in the decomposition of  $M$ . If  $L$  and  $L_k$  are clear from the context we will just call  $M$  a Harish-Chandra module.

A Harish-Chandra module  $M$  is indecomposable if it cannot be decomposed into the direct sum of non-zero  $L$ -submodules.

Our goal is to classify all indecomposable Harish-Chandra modules over  $(L, L_k)$  for  $L = \mathfrak{sl}(2, \mathbf{C}) \times \mathfrak{sl}(2, \mathbf{C})$  and  $L_k = \{(u, u) \mid u \in \mathfrak{sl}(2, \mathbf{C})\}$ , where we by  $\mathfrak{sl}(2, \mathbf{C}) \times \mathfrak{sl}(2, \mathbf{C})$  mean the following:

For  $L, L'$  Lie algebras over  $F$ , we consider  $L \times L' = L \oplus L'$  as a Lie algebra over  $F$  with pointwise addition, multiplication given by  $\alpha(a, b) = (\alpha a, \alpha b)$  for  $\alpha \in F, a \in L, b \in L'$ , and with Lie bracket  $[(a_1, b_1), (a_2, b_2)] = ([a_1, a_2], [b_1, b_2])$  for  $a_1, a_2 \in L, b_1, b_2 \in L'$ .

**Remark 1.2.** Note that  $L \times 0$  and  $0 \times L'$  are ideals in  $L \times L'$  as given above. Thus we see that  $\mathfrak{sl}(2, \mathbf{C}) \times 0$  and  $0 \times \mathfrak{sl}(2, \mathbf{C})$  are ideals in  $\mathfrak{sl}(2, \mathbf{C}) \times \mathfrak{sl}(2, \mathbf{C})$  with

$$(\mathfrak{sl}(2, \mathbf{C}) \times 0) \oplus (0 \times \mathfrak{sl}(2, \mathbf{C})) = \mathfrak{sl}(2, \mathbf{C}) \times \mathfrak{sl}(2, \mathbf{C}),$$

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<sup>1</sup>In [GP67b] the word irreducible is used instead of simple, but we will only use irreducible when talking about representations in this paper.

so  $\mathfrak{sl}(2, \mathbf{C}) \times \mathfrak{sl}(2, \mathbf{C})$  is semisimple.

Now if we take  $L = \mathfrak{sl}(2, \mathbf{C}) \times \mathfrak{sl}(2, \mathbf{C})$  and  $L_k = \{(u, u) \mid u \in \mathfrak{sl}(2, \mathbf{C})\}$  as a Lie subalgebra, it makes sense to talk about Harish-Chandra modules over  $(L, L_k)$ . Here  $L_k$  is clearly a Lie subalgebra since it is a subspace and the Lie bracket on  $\mathfrak{sl}(2, \mathbf{C}) \times \mathfrak{sl}(2, \mathbf{C})$  preserves  $L_k$  by the definition of the Lie bracket on a product.  $\triangle$

We fix the following as a standard basis for  $\mathfrak{sl}(2, F)$ :

$$x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \quad (1.1)$$

Giving us the relations:

$$[x, y] = h, \quad [h, x] = 2x, \quad [h, y] = -2y, \quad (1.2)$$

cf. [Jan16, p. 35] or [Hum72, p. 6].

We claim now that

$$(x, x), \quad (y, y), \quad \frac{1}{2}(h, h), \quad (ix, -ix), \quad (iy, -iy), \quad \frac{1}{2}(ih, -ih)$$

is a basis of  $\mathfrak{sl}(2, \mathbf{C}) \times \mathfrak{sl}(2, \mathbf{C})$ . This is clearly the case since  $\dim_{\mathbf{C}} \mathfrak{sl}(2, \mathbf{C}) = 3$ , so  $\dim_{\mathbf{C}} \mathfrak{sl}(2, \mathbf{C}) \times \mathfrak{sl}(2, \mathbf{C}) = 6$ , and we see that the above elements span  $\mathfrak{sl}(2, \mathbf{C}) \times \mathfrak{sl}(2, \mathbf{C})$ ; we have  $\frac{1}{2}(x, x) - \frac{i}{2}(ix, -ix) = (x, 0)$  and  $\frac{1}{2}(x, x) + \frac{i}{2}(ix, -ix) = (0, x)$  and likewise with  $h$  and  $y$ .

Putting

$$\begin{aligned} h_+ &= (x, x), & h_- &= (y, y), & h_3 &= \frac{1}{2}(h, h), \\ f_+ &= (ix, -ix), & f_- &= (iy, -iy), & f_3 &= \frac{1}{2}(ih, -ih) \end{aligned}$$

we get the following commutation relations between these basis elements:

$$\begin{aligned} [h_+, h_3] &= \frac{1}{2}([x, h], [x, h]) = \frac{1}{2}(-2x, -2x) = -(x, x) = -h_+, \\ [h_-, h_3] &= \frac{1}{2}([y, h], [y, h]) = \frac{1}{2}(2y, 2y) = (y, y) = h_-, \\ [h_+, h_-] &= ([x, y], [x, y]) = (h, h) = 2h_3, \\ [h_+, f_+] &= ([x, ix], [x, -ix]) = 0, \\ [h_-, f_-] &= ([y, iy], [y, -iy]) = 0, \\ [h_3, f_3] &= \frac{1}{4}([h, ih], [h, -ih]) = 0, \\ [h_+, f_3] &= \frac{1}{2}([x, ih], [x, -ih]) = \frac{1}{2}(-2ix, 2ix) = -(ix, -ix) = -f_+, \\ [h_-, f_3] &= \frac{1}{2}([y, ih], [y, -ih]) = \frac{1}{2}(2iy, -2iy) = (iy, -iy) = f_-, \\ [h_+, f_-] &= ([x, iy], [x, -iy]) = (ih, -ih) = 2f_3, \\ [h_3, f_-] &= \frac{1}{2}([h, iy], [h, -iy]) = \frac{1}{2}(-2iy, 2iy) = -(iy, -iy) = -f_-, \\ [h_-, f_+] &= ([y, ix], [y, -ix]) = (-ih, ih) = -(ih, -ih) = -2f_3, \\ [h_3, f_+] &= \frac{1}{2}([h, ix], [h, -ix]) = \frac{1}{2}(2ix, -2ix) = (ix, -ix) = f_+, \\ [f_+, f_3] &= \frac{1}{2}([ix, ih], [-ix, -ih]) = \frac{1}{2}(2x, 2x) = (x, x) = h_+, \\ [f_-, f_3] &= \frac{1}{2}([iy, ih], [-iy, -ih]) = \frac{1}{2}(-2y, -2y) = -(y, y) = -h_-, \\ [f_+, f_-] &= ([ix, iy], [-ix, -iy]) = (-h, -h) = -(h, h) = -2h_3. \end{aligned} \quad (1.3)$$

**Remark 1.3.** Note that these are the same relations as for the complexification of the Lie algebra  $L$  of the proper Lorentz group in [GP67b, p. 5], so  $L$  is isomorphic to  $\mathfrak{sl}(2, \mathbf{C}) \times \mathfrak{sl}(2, \mathbf{C})$ . This explains the equivalence of the work in this paper and the work in [GP67a; GP67b; GP67c].  $\triangle$

Now let  $L = \mathfrak{sl}(2, \mathbf{C}) \times \mathfrak{sl}(2, \mathbf{C})$  and  $L_k = \{(u, u) \mid u \in \mathfrak{sl}(2, \mathbf{C})\}$ . Note that  $L_k$  is the Lie subalgebra of  $L$  with basis  $h_+, h_-, h_3$ , and that the above commutation relations gives us that

$$[h_+, h_-] = 2h_3, \quad [2h_3, h_+] = 2h_+, \quad [2h_3, h_-] = -2h_-$$

Comparing with eq. (1.2) we see that we have an isomorphism

$$\mathfrak{sl}(2, \mathbf{C}) \rightarrow L_k, \quad u \mapsto (u, u), \quad (1.4)$$

or more explicitly  $x \mapsto h_+$ ,  $h_- \mapsto y$ , and  $h \mapsto 2h_3$ , so we can use  $\mathfrak{sl}(2, \mathbf{C})$ -theory when we want to describe  $L_k$ -modules.

## 1.1 Representations of $L_k$

Let  $V$  be a  $\mathbf{C}$  vector space and  $\rho: L_k \rightarrow \mathfrak{gl}(V)$  a representation of  $L_k$ . We will use the notation  $\rho(a) = A$  for  $a \in L_k$  switching to upper case letters when we talk about the representation corresponding to a given element. Note that we will switch freely between the language of representations of  $L_k$  and the language of  $L_k$ -modules.

We will start out by describing the finite dimensional simple  $L_k$ -modules. Recall, cf. [Jan16, p. 36], that we know from  $\mathfrak{sl}(2, \mathbf{C})$ -theory that for integers  $n \geq 0$  there exists a unique simple  $\mathfrak{sl}(2, \mathbf{C})$ -module  $V(n)$  of dimension  $n + 1$ , and  $V(n)$  has a basis  $(v_0, v_1, \dots, v_n)$  such that for all  $i$ ,  $0 \leq i \leq n$

$$\begin{aligned} h.v_i &= (n - 2i)v_i, \\ x.v_i &= \begin{cases} (n - i + 1)v_{i-1} & \text{if } i > 0, \\ 0 & \text{if } i = 0, \end{cases} \\ y.v_i &= \begin{cases} (i + 1)v_{i+1} & \text{if } i < n, \\ 0 & \text{if } i = n. \end{cases} \end{aligned} \quad (1.5)$$

Now using the isomorphism from eq. (1.4) we see that for integers  $n \geq 0$  there exists a unique simple  $L_k$ -module  $M(n)$  of dimension  $n + 1^2$ , and  $M(n)$

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<sup>2</sup>We will use the notation  $V(n)$  when talking about  $\mathfrak{sl}(2, \mathbf{C})$ -modules and  $M(n)$  when talking about  $L_k$ -modules to clarify what kind of module we are talking about, but as vector spaces  $V(n)$  and  $M(n)$  are isomorphic.

has a basis  $(v_0, v_1, \dots, v_n)$  such that for all  $i$ ,  $0 \leq i \leq n$

$$\begin{aligned} h_3.v_i &= (\tfrac{1}{2}n - i)v_i, \\ h_+.v_i &= \begin{cases} (n - i + 1)v_{i-1} & \text{if } i > 0, \\ 0 & \text{if } i = 0, \end{cases} \\ h_-.v_i &= \begin{cases} (i + 1)v_{i+1} & \text{if } i < n, \\ 0 & \text{if } i = n. \end{cases} \end{aligned} \tag{1.6}$$

Now consider  $M(n)$  as an inner product space over  $\mathbf{C}$  with inner product given by

$$\langle v_k, v_j \rangle = \delta_{jk} \binom{n}{k}. \tag{1.7}$$

We will switch to the orthonormal basis  $(\bar{v}_0, \bar{v}_1, \dots, \bar{v}_n)$ , where  $\bar{v}_i = v_i / \|v_i\|$ . Here  $\|\cdot\|$  is given by  $\|v\| = \sqrt{\langle v, v \rangle}$  as usually, and we note that

$$\bar{v}_i = \frac{1}{\sqrt{\binom{n}{i}}} v_i.$$

Note furthermore that

$$h_3.\bar{v}_i = \frac{1}{\sqrt{\binom{n}{i}}} h_3.v_i = \frac{1}{\sqrt{\binom{n}{i}}} (\tfrac{1}{2}n - i)v_i = (\tfrac{1}{2}n - i)\bar{v}_i$$

for all  $i$ ,  $0 \leq i \leq n$ , and clearly still

$$\begin{aligned} h_+.\bar{v}_0 &= 0, \\ h_-. \bar{v}_n &= 0. \end{aligned}$$

But for  $i$ ,  $0 < i \leq n$

$$\begin{aligned} h_+.\bar{v}_i &= \frac{1}{\sqrt{\binom{n}{i}}} h_+.v_i = \frac{1}{\sqrt{\binom{n}{i}}} (n - i + 1)v_{i-1} \\ &= \sqrt{\frac{\binom{n}{i-1}}{\binom{n}{i}}} (n - i + 1) \frac{1}{\sqrt{\binom{n}{i-1}}} v_{i-1} \\ &= \sqrt{\frac{i}{n - i + 1}} (n - i + 1) \bar{v}_{i-1} = \sqrt{(n - i + 1)i} \bar{v}_{i-1}, \end{aligned}$$



and for  $i$ ,  $0 \leq i < n$

$$\begin{aligned} h_{-}.\bar{v}_i &= \frac{1}{\sqrt{\binom{n}{i}}} h_{-}.v_i = \frac{1}{\sqrt{\binom{n}{i}}} (i+1)v_{i+1} \\ &= \sqrt{\frac{\binom{n}{i+1}}{\binom{n}{i}}} (i+1) \frac{1}{\sqrt{\binom{n}{i+1}}} v_{i+1} \\ &= \sqrt{\frac{n-i}{i+1}} (i+1) \bar{v}_{i+1} = \sqrt{(n-i)(i+1)} \bar{v}_{i+1}. \end{aligned}$$

Finally write  $\ell = \frac{1}{2}n$ . We will re-index with  $m = \frac{1}{2}(n - 2i) = \ell - i$  by setting

$$e_m = \bar{v}_{\ell-m}$$

for  $m \in \{-\ell, -\ell+1, \dots, \ell-1, \ell\}$ . Thus we get

$$h_3.e_m = h_3.\bar{v}_{\ell-m} = (\ell - (\ell - m))\bar{v}_{\ell-m} = me_m,$$

and since  $e_\ell = \bar{v}_0$  and  $e_{-\ell} = \bar{v}_n$  also

$$\begin{aligned} h_{+}.e_\ell &= 0, \\ h_{-}.e_{-\ell} &= 0. \end{aligned}$$

And for  $m \in \{-\ell, -\ell+1, \dots, \ell-2, \ell-1\}$  we get

$$\begin{aligned} h_{+}.e_m &= h_{+}.\bar{v}_{\ell-m} = \sqrt{(n - (\ell - m) + 1)(\ell - m)} \bar{v}_{\ell-m-1} \\ &= \sqrt{(\ell + m + 1)(\ell - m)} e_{m+1}, \end{aligned}$$

while for  $m \in \{-\ell+1, -\ell+2, \dots, \ell-1, \ell\}$  we get

$$\begin{aligned} h_{-}.e_m &= h_{-}.\bar{v}_{\ell-m} = \sqrt{(n - (\ell - m))(\ell - m + 1)} \bar{v}_{\ell-m+1} \\ &= \sqrt{(\ell + m)(\ell - m + 1)} e_{m-1}. \end{aligned}$$

Thus we get the following Lemma:

**Lemma 1.4.** *Every simple finite dimensional  $L_k$ -module is uniquely given by a number  $\ell \in \frac{1}{2}\mathbf{Z}_{\geq 0}$ . For such  $\ell$  the unique simple  $L_k$ -module  $M(2\ell)$  has dimension  $2\ell + 1$ , and  $M(2\ell)$  has a basis  $(e_{-\ell}, e_{-\ell+1}, \dots, e_{\ell-1}, e_\ell)$  such that for all  $m \in \{-\ell, -\ell+1, \dots, \ell-1, \ell\}$  we have*

$$\begin{aligned} h_3.e_m &= me_m, \\ h_{+}.e_m &= \begin{cases} \sqrt{(\ell + m + 1)(\ell - m)} e_{m+1} & \text{if } m \neq \ell, \\ 0 & \text{if } m = \ell, \end{cases} \\ h_{-}.e_m &= \begin{cases} \sqrt{(\ell + m)(\ell - m + 1)} e_{m-1} & \text{if } m \neq -\ell, \\ 0 & \text{if } m = -\ell. \end{cases} \end{aligned} \tag{1.8}$$

### 1.1.1 Formulae for the operators $H_+, H_-, H_3, F_+, F_-, F_3$

Let  $M$  be a Harish-Chandra  $L$ -module. Then we have linear operators  $H_+, H_-, H_3, F_+, F_-, F_3: M \rightarrow M$  satisfying commutation relations as in eq. (1.3), and we want to give expressions for these in terms of other linear operators  $E_+, E_-, D_+, D_-, D_0: M \rightarrow M$ .

We will denote by  $R_\ell$  a finite dimensional  $L$ -module which is a (finite) direct sum of  $L_k$ -modules  $M(2\ell)$  for the same number  $\ell \in \frac{1}{2}\mathbf{Z}_{\geq 0}$ . Then  $M$  is a direct sum of the subspaces  $R_\ell$  since  $M$  is Harish-Chandra, and from Lemma 1.4 we know that  $R_\ell$  can be written as the direct sum of subspaces  $R_{\ell,m}$ , where  $R_{\ell,m}$  are eigenspaces for  $H_3$  such that

$$H_3\xi = m\xi \quad (1.9)$$

for  $m \in \{-\ell, -\ell+1, \dots, \ell-1, \ell\}$  and  $\xi \in R_{\ell,m}$ . We will use the decomposition

$$M = \bigoplus_{\substack{\ell \in \frac{1}{2}\mathbf{Z}_{\geq 0} \\ m \in \{-\ell, -\ell+1, \dots, \ell-1, \ell\}}} R_{\ell,m} = \bigoplus_{\ell, m} R_{\ell,m}$$

throughout this paper.

By Lemma 1.4 we also have that  $H_+$  and  $H_-$  maps the  $R_{\ell,m}$  into each other as follows:

$$\begin{aligned} H_+ : R_{\ell,m} &\rightarrow R_{\ell,m+1} & \text{if } -\ell \leq m < \ell, & & H_+ : R_{\ell,\ell} &\rightarrow 0, \\ H_- : R_{\ell,m} &\rightarrow R_{\ell,m-1} & \text{if } -\ell < m \leq \ell, & & H_- : R_{\ell,-\ell} &\rightarrow 0. \end{aligned}$$

Hence we have linear operators  $H_+H_-, H_-H_+ : R_{\ell,m} \rightarrow R_{\ell,m}$ , and by eq. (1.8) we see that

$$\begin{aligned} H_+H_-\xi &= \sqrt{(\ell + (m-1) + 1)(\ell - (m-1))} \sqrt{(\ell + m)(\ell - m + 1)} \xi \\ &= (\ell + m)(\ell - m + 1) \xi, \\ H_-H_+\xi &= \sqrt{(\ell + (m+1))(\ell - (m+1) + 1)} \sqrt{(\ell + m + 1)(\ell - m)} \xi \\ &= (\ell + m + 1)(\ell - m) \xi. \end{aligned} \quad (1.10)$$

Note that this also covers the cases  $m = \ell$  and  $m = -\ell$ .

Now we define  $E_+ : R_{\ell,m} \rightarrow R_{\ell,m+1}$  and  $E_- : R_{\ell,m} \rightarrow R_{\ell,m-1}$  to be the linear maps satisfying

$$\begin{aligned} H_+\xi &= \sqrt{(\ell + m + 1)(\ell - m)} E_+\xi & \text{if } m \neq \ell, \\ E_+\xi &= 0 & \text{if } m = \ell, \\ H_-\xi &= \sqrt{(\ell + m)(\ell - m + 1)} E_-\xi & \text{if } m \neq -\ell, \\ E_-\xi &= 0 & \text{if } m = -\ell, \end{aligned} \quad (1.11)$$

for  $\xi \in R_{\ell,m}$ . Comparing eq. (1.11) and eq. (1.10) we see that

$$\begin{aligned} E_+ E_- \xi &= \xi & \text{if } m \neq -\ell \\ E_- E_+ \xi &= \xi & \text{if } m \neq \ell. \end{aligned}$$

Thus  $E_+ : R_{\ell,m} \rightarrow R_{\ell,m+1}$  and  $E_- : R_{\ell,m+1} \rightarrow R_{\ell,m}$  are isomorphisms for  $m \neq \ell$  and they are each others inverse.

**Remark 1.5.** Note that the above definitions make sense more generally on  $L_k$ -modules  $M$  with a direct sum decomposition  $\bigoplus_{\ell} R_{\ell}$  such that each  $R_{\ell}$  is a finite direct sum of simple  $L_k$ -modules isomorphic to  $M(2\ell)$ , i.e. we do not need the additional structure from  $L$ -modules.

In particular note that on the  $L_k$ -module  $M(2\ell)$  with basis  $(e_{-\ell}, e_{-\ell+1}, \dots, e_{\ell-1}, e_{\ell})$  as in Lemma 1.4, we have that  $E_+ e_m = e_{m+1}$  for  $m \neq \ell$  and  $E_- e_m = e_{m-1}$  for  $m \neq -\ell$ .  $\triangle$

Now note that  $H_+$ ,  $H_-$ , and  $H_3$  are completely determined by eq. (1.9) and eq. (1.11), so we just need to find maps to determine  $F_+$ ,  $F_-$ , and  $F_3$  now, while making sure that we get commutation relations as in eq. (1.3).

We already have that  $L_k = \text{span}_{\mathbb{C}}(h_+, h_-, h_3)$ , but now we will also consider  $L_p = \text{span}_{\mathbb{C}}(f_+, f_-, f_3)$ . We will show shortly that  $u.R_{\ell} \subset R_{\ell-1} \oplus R_{\ell} \oplus R_{\ell+1}$  for all  $u \in L_p$ . This implies that there are maps  $D_-^u : R_{\ell} \rightarrow R_{\ell-1}$ ,  $D_0^u : R_{\ell} \rightarrow R_{\ell}$ , and  $D_+^u : R_{\ell} \rightarrow R_{\ell+1}$  such that  $u.v = D_-^u(v) + D_0^u(v) + D_+^u(v)$  for all  $u \in L_p$  and  $v \in R_{\ell}$ . In the following we will find maps  $D_-$ ,  $D_0$ , and  $D_+$  independent of  $u$  such that we can express  $D_-^u$ ,  $D_0^u$ , and  $D_+^u$  in terms of these and the maps  $E_-$  and  $E_+$  from above, thus we will also be able to express  $F_+$ ,  $F_-$ , and  $F_3$  in terms of  $D_-$ ,  $D_0$ ,  $D_+$ ,  $E_-$ , and  $E_+$ . To be more precise we will find maps  $D_-$ ,  $D_0$ , and  $D_+$  such that we can express  $F_3$  in terms of just these (and multiplication by some constant), and then we can get  $F_+$  and  $F_-$  by the commutation relations.

For reasons that will be clearer later, we want the maps  $D_0$  and  $D_+$  to be defined on  $M = \bigoplus_{\ell,m} R_{\ell,m}$  and  $D_-$  defined on the direct sum without the summands  $R_{\ell,\ell}$  and  $R_{\ell,-\ell}$  to be such that  $D_0 R_{\ell,m} \subset R_{\ell,m}$ ,  $D_+ R_{\ell,m} \subset R_{\ell+1,m}$ , and  $D_- R_{\ell,m} \subset R_{\ell-1,m}$  and the diagrams

Maybe move this to later

$$\begin{array}{ccc} R_{\ell-1,m+1} & \xleftarrow{D_-} & R_{\ell,m+1} \\ E_+ \uparrow & & \uparrow E_+ \\ R_{\ell-1,m} & \xleftarrow{D_-} & R_{\ell,m} \end{array} \quad \begin{array}{ccc} R_{\ell,m+1} & \xrightarrow{D_0} & R_{\ell,m+1} \\ E_+ \uparrow & & \uparrow E_+ \\ R_{\ell,m} & \xrightarrow{D_0} & R_{\ell,m} \end{array} \quad (1.12)$$

$$\begin{array}{ccc} R_{\ell,m+1} & \xrightarrow{D_+} & R_{\ell+1,m+1} \\ E_+ \uparrow & & \uparrow E_+ \\ R_{\ell,m} & \xrightarrow{D_+} & R_{\ell+1,m+1} \end{array}$$

commute, when  $-\ell + 1 \leq m < \ell - 1$  in the top left diagram,  $-\ell \leq m < \ell$  in the other two diagrams. Also similar diagrams with  $E_-$  replacing  $E_+$  commute since  $E_- : R_{\ell, m} \rightarrow R_{\ell, m-1}$  for  $m \neq -\ell$  is inverse to  $E_+ : R_{\ell, m-1} \rightarrow R_{\ell, m}$ . Before we can get to the final description of these maps we need quite a lot of work.

\* \* \* \* \*

Note that eq. (1.3) gives us that  $[L_k, L_p] \subset L_p$ , so by the adjoint representation we can see  $L_p$  as an  $L_k$ -module, and again by eq. (1.3) we see that  $L_p$  is a simple  $L_k$ -module: If  $V$  is an  $L_k$ -submodule and we have a non-zero element  $f = af_+ + bf_- + cf_3 \in V$  for some  $a, b, c \in \mathbf{C}$  not all zero, then

$$\begin{aligned} [h_+, af_+ + bf_- + cf_3] &= 2bf_3 - cf_+, \\ [h_-, af_+ + bf_- + cf_3] &= -2af_3 + cf_-, \\ [h_3, af_+ + bf_- + cf_3] &= af_+ - bf_-. \end{aligned}$$

If  $c \neq 0$ , we get that

$$\begin{aligned} [h_3, [h_+, f]] &= [h_3, 2bf_3 - cf_+] = -cf_+, \\ [h_3, [h_-, f]] &= [h_3, -2af_3 + cf_-] = -cf_-, \end{aligned}$$

so we see that  $f_+, f_- \in V$ , and thus also  $[h_+, \frac{1}{2}f_-] = f_3 \in V$ , so  $V = L_p$ . If on the other hand  $c = 0$ , then

$$\begin{aligned} [h_-, f] &= -2af_3, \\ [h_+, f] &= 2bf_3, \end{aligned}$$

so since either  $a \neq 0$  or  $b \neq 0$ , we see that  $f_3 \in V$ , and thus also  $[h_+, -f_3] = f_+ \in V$  and  $[h_-, f_3] = f_- \in V$ , so  $V = L_p$ . Hence  $L_p$  is indeed a simple  $L_k$ -module. Now since  $L_p$  is a simple finite dimensional  $L_k$ -module of dimension 3, we have that  $L_p \simeq M(2)$  as  $L_k$ -modules.

In general given two  $L$ -modules  $V$  and  $W$ , we consider the tensor product  $V \otimes W$  over  $\mathbf{C}$  of the underlying vector spaces as an  $L$ -module via the action

$$x.(v \otimes w) = x.v \otimes w + v \otimes x.w, \tag{1.13}$$

for  $x \in L$  and  $v \otimes w \in V \otimes W$ , cf. [Hum72, p. 26].

Now we are interested in the  $L_k$ -module  $L_p \otimes M$ , where  $M$  is a Harish-Chandra  $L$ -module (and thus an  $L_k$ -module) as before. Specifically we will show that the linear map

$$\begin{aligned} \psi : L_p \otimes M &\rightarrow M \\ x \otimes v &\mapsto x.v \end{aligned} \tag{1.14}$$

is a homomorphism of  $L_k$ -modules. For  $y \in L_k$  we see that

$$y.(x \otimes v) = y.x \otimes v + x \otimes y.v = [y, x] \otimes v + x \otimes y.v,$$

for  $x \otimes v \in L_p \otimes M$ , since the action in  $L_p$  is by the adjoint representation. So

$$\begin{aligned}\psi(y.(x \otimes v)) &= \psi([y, x] \otimes v) + \psi(x \otimes y.v) = [y, x].v + x.(y.v) \\ &= y.(x.v) - x.(y.v) + x.(y.v) = y.(x.v) = y.\psi(x \otimes v),\end{aligned}$$

i.e.  $\psi$  is indeed a homomorphism of  $L_k$ -modules.

Now we note that  $M = \bigoplus_{\ell} R_{\ell}$ , so

$$L_p \otimes M = L_p \otimes \left( \bigoplus_{\ell} R_{\ell} \right) \simeq \bigoplus_{\ell} (L_p \otimes R_{\ell}),$$

as  $L_k$ -modules, and since  $R_{\ell}$  is direct sum of finitely many copies of  $M(2\ell)$ , we see that

$$\begin{aligned}L_p \otimes R_{\ell} &\simeq M(2) \otimes (M(2\ell)^1 \oplus M(2\ell)^2 \oplus \cdots \oplus M(2\ell)^r) \\ &\simeq (M(2) \otimes M(2\ell)^1) \oplus (M(2) \otimes M(2\ell)^2) \oplus \cdots \oplus (M(2) \otimes M(2\ell)^r),\end{aligned}$$

as  $L_k$ -modules, since  $L_p \simeq M(2)$ . Here the superscripts are just indices for the different  $M(2\ell)$ . Thus we want to describe the  $L_k$ -modules  $M(2) \otimes M(2\ell)$ , which we will do by first describing the  $\mathfrak{sl}(2, \mathbf{C})$ -modules  $V(2) \otimes V(2\ell)$  and then translating back to a solution to our problem.

### 1.1.2 Describing $V(2) \otimes V(n)$

Let  $2\ell = n \in \mathbf{N}$ . We want to show that

$$V(2) \otimes V(n) \simeq \begin{cases} V(n-2) \oplus V(n) \oplus V(n+2) & \text{if } n \geq 2, \\ V(3) \oplus V(1) & \text{if } n = 1, \\ V(2) & \text{if } n = 0. \end{cases} \quad (1.15)$$

Note that in all cases there is a summand  $V(n+2)$ . We can show the above by considerations using formal characters. We will use the notation of [Jan16, Chapter 8], specifically we will do calculations with the functions  $e(\lambda): H^* \rightarrow \mathbf{Z}$  for  $\lambda \in H^*$ . Firstly note that in general

$$\text{ch}_V = \sum_{\lambda \in H^*} (\dim V_{\lambda}) e(\lambda),$$

and use the notation  $V(n)_k$  for  $V(\lambda)_{\mu}$  and  $e(n)$  for  $e(\lambda)$  with  $\lambda, \mu \in H^*$  such that  $\lambda(h) = n$  and  $\mu(h) = k$ . We get that

$$\text{ch}_{V(2)} = e(-2) + e(0) + e(2)$$

and

$$\text{ch}_{V(n)} = \sum_{i=0}^n e(n-2i),$$

since

$$\dim V(n)_k = \begin{cases} 1 & \text{if } k = n - 2i \text{ for some } i \in \{0, 1, \dots, n\}, \\ 0 & \text{otherwise.} \end{cases}$$

Now since  $e(\lambda) * e(\mu) = e(\lambda + \mu)$  in general cf. [Jan16, p. 93], we see that for  $n \geq 2$

$$\begin{aligned} \text{ch}_{V(2) \otimes V(n)} &= \text{ch}_{V(2)} * \text{ch}_{V(n)} = e(-2) * \text{ch}_{V(n)} + e(0) * \text{ch}_{V(n)} + e(2) * \text{ch}_{V(n)} \\ &= \sum_{i=0}^n e(n - 2 - 2i) + \text{ch}_{V(n)} + \sum_{i=0}^n e(n + 2 - 2i) \\ &= e(-n - 2) + e(-n) + \sum_{i=0}^{n-2} e(n - 2 - 2i) + \text{ch}_{V(n)} \\ &\quad + \sum_{i=0}^n e(n + 2 - 2i) \\ &= \text{ch}_{V(n-2)} + \text{ch}_{V(n)} + \sum_{i=0}^{n+2} e(n + 2 - 2i) \\ &= \text{ch}_{V(n-2)} + \text{ch}_{V(n)} + \text{ch}_{V(n+2)} = \text{ch}_{V(n-2) \oplus V(n) \oplus V(n+2)}, \end{aligned}$$

where the first equality follows from the fact that  $\text{ch}_{V \otimes W} = \text{ch}_V * \text{ch}_W$  in general, cf. [Hum72, p. 125]. Thus since two  $L$ -modules  $V$  and  $V'$  are isomorphic if and only if  $\text{ch}_V = \text{ch}_{V'}$ , cf. [Jan16, p. 90], we see that  $V(2) \otimes V(n) \simeq V(n-2) \oplus V(n) \oplus V(n+2)$  if  $n \geq 2$ .

Likewise we see that

$$\begin{aligned} \text{ch}_{V(2) \otimes V(1)} &= \text{ch}_{V(2)} * \text{ch}_{V(1)} \\ &= (e(-2) + e(0) + e(2)) * e(-1) + (e(-2) + e(0) + e(2)) * e(1) \\ &= e(-3) + e(-1) + e(1) + e(-1) + e(1) + e(3) \\ &= (e(-3) + e(-1) + e(1) + e(3)) + (e(-1) + e(1)) \\ &= \text{ch}_{V(3)} + \text{ch}_{V(1)} = \text{ch}_{V(3) \oplus V(1)} \end{aligned}$$

and

$$\text{ch}_{V(2) \otimes V(0)} = \text{ch}_{V(2)} * \text{ch}_{V(0)} = \text{ch}_{V(2)} * e(0) = \text{ch}_{V(2)},$$

so indeed  $V(2) \otimes V(1) \simeq V(3) \oplus V(1)$  and  $V(2) \otimes V(0) \simeq V(2)$ .

Now consider  $(w_0, w_1, w_2)$  a basis for  $V(2)$  and  $(v_i \mid 0 \leq i \leq n)$  a basis for  $V(n)$  such that both satisfies the conditions from eq. (1.5). Then for  $w_i \otimes v_j \in V(2) \otimes V(n)$  with  $i \in \{0, 1, 2\}$  and  $j \in \{0, 1, \dots, n\}$  we see that

$$\begin{aligned} h.(w_i \otimes v_j) &= h.w_i \otimes v_j + w_i \otimes h.v_j = (2 - 2i)w_i \otimes v_j + (n - 2j)w_i \otimes v_j \\ &= (n - 2(i + j - 1))w_i \otimes v_j. \end{aligned} \tag{1.16}$$

Hence  $v_0 \otimes w_0$  is up to scalar multiple the only vector of weight  $n + 2$  in  $V(2) \otimes V(n)$ , so it is necessarily a highest weight vector generating the direct summand isomorphic to  $V(n + 2)$ . Note that by eq. (1.15) we indeed have a direct summand isomorphic to  $V(n + 2)$  for all  $n \in \mathbf{N}$ . By  $\mathfrak{sl}(2, \mathbf{C})$ -theory, cf. [Jan16, p. 36], we know that this summand has a basis  $(s_k \mid 0 \leq k \leq n + 2)$  satisfying equations as in eq. (1.5), where

$$s_k := \frac{1}{k!} y^k \cdot (w_0 \otimes v_0). \quad (1.17)$$

By straightforward calculations, cf. Appendix A.1, we get for  $n > 0$  that

$$\begin{aligned} s_0 &= w_0 \otimes v_0, \\ s_1 &= w_1 \otimes v_0 + w_0 \otimes v_1 && \text{if } n > 0, \\ s_k &= w_2 \otimes v_{k-2} + w_1 \otimes v_{k-1} + w_0 \otimes v_k && \text{for } 2 \leq k \leq n, \\ s_{n+1} &= w_2 \otimes v_{n-1} + w_1 \otimes v_n && \text{if } n > 0, \\ s_{n+2} &= w_2 \otimes v_n. \end{aligned} \quad (1.18)$$

In case  $n = 0$  we likewise see that  $s_1 = w_1 \otimes v_0$  and  $s_2 = w_2 \otimes v_0$ , and we note that  $(s_0, s_1, s_2)$  is a basis for  $V(2) \otimes V(0) \simeq V(2)$ .

Suppose now that  $n \geq 1$ . Note that by eq. (1.15) we have a direct summand isomorphic to  $V(n)$ , and by eq. (1.16) the weight space of weight  $n$  is spanned by  $w_0 \otimes v_1$  and  $w_1 \otimes v_0$ , so the vector of highest weight  $n$  generating the summand corresponding to  $V(n)$  must be of the form  $aw_0 \otimes v_1 + bw_1 \otimes v_0$  for some  $a, b \in \mathbf{C}$ . Furthermore we know that for this vector generating the summand corresponding to  $V(n)$ , we must have that

$$\begin{aligned} 0 &= x \cdot (aw_0 \otimes v_1 + bw_1 \otimes v_0) \\ &= ax \cdot w_0 \otimes v_1 + aw_0 \otimes x \cdot v_1 + bx \cdot w_1 \otimes v_0 + bw_1 \otimes x \cdot v_0 \\ &= 0 + a(n - 1 + 1)w_0 \otimes v_0 + b(2 - 1 + 1)w_0 \otimes v_0 + 0 \\ &= (an + 2b)w_0 \otimes v_0, \end{aligned}$$

i.e.  $an + 2b = 0$  so  $b = -\frac{n}{2}a$ . This determines the vector generating the summand corresponding to  $V(n)$  up to a scalar, so taking  $a = 1$ , we see that we can take

$$t_0 := w_0 \otimes v_1 - \frac{n}{2} w_1 \otimes v_0$$

as our vector generating the summand corresponding to  $V(n)$ . As before  $\mathfrak{sl}(2, \mathbf{C})$ -theory now yields that this summand has a basis  $(t_k \mid 0 \leq k \leq n)$  satisfying equations as in eq. (1.5), where

$$t_k := \frac{1}{k!} y^k \cdot t_0. \quad (1.19)$$

By straightforward calculations, cf. Appendix A.1, we get that

$$\begin{aligned}
 t_0 &= w_0 \otimes v_1 - \frac{n}{2} w_1 \otimes v_0, \\
 t_k &= (k+1)w_0 \otimes v_{k+1} - \frac{n-2k}{2} w_1 \otimes v_k \\
 &\quad + (k-1-n)w_2 \otimes v_{k-1} \quad \text{for } 1 \leq k \leq n-1, \\
 t_n &= \frac{n}{2} w_1 \otimes v_n - w_2 \otimes v_{n-1}.
 \end{aligned} \tag{1.20}$$

Suppose now that  $n \geq 2$ . By eq. (1.15) we have a direct summand isomorphic to  $V(n-2)$ , and by eq. (1.16) the weight space of weight  $n-2$  is spanned by  $w_0 \otimes v_2$ ,  $w_1 \otimes v_1$ , and  $w_2 \otimes v_0$ , so the vector of highest weight  $n-2$  generating the summand corresponding to  $V(n)$  must be of the form  $aw_0 \otimes v_2 + bw_1 \otimes v_1 + cw_2 \otimes v_0$  for some  $a, b, c \in \mathbf{C}$ . Furthermore we know that for this vector generating the summand corresponding to  $V(n-2)$ , we must have

$$\begin{aligned}
 0 &= x.(aw_0 \otimes v_2 + bw_1 \otimes v_1 + cw_2 \otimes v_0) \\
 &= aw_0 \otimes x.v_2 + bx.w_1 \otimes v_1 + bw_1 \otimes x.v_1 + cx.w_2 \otimes v_0 \\
 &= a(n-2+1)w_0 \otimes v_1 + b(2-1+1)w_0 \otimes v_1 + b(n-1+1)w_1 \otimes v_0 \\
 &\quad + c(2-2+1)w_1 \otimes v_0 \\
 &= ((n-1)a + 2b)w_0 \otimes v_1 + (bn + c)w_1 \otimes v_0,
 \end{aligned}$$

i.e.  $a(n-1) + 2b = 0$  and  $bn + c = 0$ . Giving us  $c = -bn$  and  $b = -\frac{n-1}{2}a$ , so

$$c = \frac{n(n-1)}{2}a.$$

This determines the vector generating the summand corresponding to  $V(n-2)$  up to a scalar, so taking  $a = 1$ , we see that we can take

$$u_0 := w_0 \otimes v_2 - \frac{n-1}{2} w_1 \otimes v_1 + \frac{n(n-1)}{2} w_2 \otimes v_0$$

as our vector generating the summand corresponding to  $V(n-2)$ . Again  $\mathfrak{sl}(2, \mathbf{C})$ -theory now yields that this summand has a basis  $(u_k \mid 0 \leq k \leq n-2)$  satisfying equations as in eq. (1.5), where

$$u_k := \frac{1}{k!} y^k . u_0. \tag{1.21}$$

By straightforward calculations, cf. Appendix A.1, we get that

$$\begin{aligned}
 u_k &= \frac{(k+1)(k+2)}{2} w_0 \otimes v_{k+2} - \frac{(k+1)(n-k-1)}{2} w_1 \otimes v_{k+1} \\
 &\quad + \frac{(n-k)(n-k-1)}{2} w_2 \otimes v_k
 \end{aligned} \tag{1.22}$$



for  $0 \leq k \leq n-2$ .

Now we want to express  $w_1 \otimes v_k$  for  $0 \leq k \leq n$  in terms of the bases  $(s_k \mid 0 \leq k \leq n+2)$ ,  $(t_k \mid 0 \leq k \leq n)$ , and  $(u_k \mid 0 \leq k \leq n-2)$ . A straightforward but long calculation, cf. Appendix A.2, yields that

$$w_1 \otimes v_k = \frac{2(k+1)(n+1-k)}{(n+1)(n+2)} s_{k+1} - \frac{2(n-2k)}{n(n+2)} t_k - \frac{4}{n(n+1)} u_{k-1} \quad (1.23)$$

for  $0 < k < n$ , while

$$w_1 \otimes v_0 = \frac{2}{n+2} (s_1 - t_0) \quad \text{and} \quad w_1 \otimes v_n = \frac{2}{n+2} (s_{n+1} + t_n) \quad (1.24)$$

if  $n \geq 1$ . If  $n = 0$  we have already seen (just after eq. (1.18)) that  $w_1 \otimes v_0 = s_1$ . Note that eq. (1.24) is a special case of eq. (1.23) if we set  $u_{-1} = u_{n-1} = 0$ .

Now consider  $V(2)$  and  $V(n)$  as inner product spaces over  $\mathbf{C}$  with inner products given by

$$\langle w_k, w_j \rangle = \delta_{jk} \binom{2}{k} \quad \text{and} \quad \langle v_k, v_j \rangle = \delta_{jk} \binom{n}{k}. \quad (1.25)$$

Then we can clearly also consider  $V(2) \otimes V(n)$  an inner product space with inner product given by

$$\langle w \otimes v, w' \otimes v' \rangle = \langle w, w' \rangle \cdot \langle v, v' \rangle \quad (1.26)$$

for  $w, w' \in V(2)$  and  $v, v' \in V(n)$ . By straightforward calculations, cf. Appendix A.3, we get that

$$\langle s_0, s_0 \rangle = 1, \quad \langle t_0, t_0 \rangle = \frac{n(n+2)}{2}, \quad \langle u_0, u_0 \rangle = \frac{n^2(n+1)(n-1)}{4}. \quad (1.27)$$

Now set  $\bar{w}_k = w_k / \|w_k\|$ ,  $\bar{v}_k = v_k / \|v_k\|$ ,  $\bar{s}_k = s_k / \|s_k\|$ ,  $\bar{t}_k = t_k / \|t_k\|$ , and  $\bar{u}_k = u_k / \|u_k\|$  for all possible  $k$ , where  $\|\cdot\|$  is given by  $\|v\| = \sqrt{\langle v, v \rangle}$  as usually in an inner product space. Note, cf. Appendix A.3, that

$$\begin{aligned} \langle w_k, w_k \rangle &= \binom{2}{k} \\ \langle v_k, v_k \rangle &= \binom{n}{k} \\ \langle s_k, s_k \rangle &= \langle s_0, s_0 \rangle \binom{n+2}{k} = \binom{n+2}{k} \\ \langle t_k, t_k \rangle &= \langle t_0, t_0 \rangle \binom{n}{k} = \frac{n(n+2)}{2} \binom{n}{k} \\ \langle u_k, u_k \rangle &= \langle u_0, u_0 \rangle \binom{n-2}{k} = \frac{n^2(n+1)(n-1)}{4} \binom{n-2}{k} \end{aligned} \quad (1.28)$$

for  $k$  where it makes sense, so we see that

$$w_k = \sqrt{\binom{2}{k}} \bar{w}_k, \quad v_k = \sqrt{\binom{n}{k}} \bar{v}_k, \quad s_k = \sqrt{\binom{n+2}{k}} \bar{s}_k, \quad (1.29)$$

and

$$t_k = \sqrt{\frac{n(n+2)}{2} \binom{n}{k}} \bar{t}_k, \quad u_k = \sqrt{\frac{n^2(n+1)(n-1)}{4} \binom{n-2}{k}} \bar{u}_k. \quad (1.30)$$

**Remark 1.6.** Since

$$\bar{v}_k = \frac{1}{\sqrt{\binom{n}{k}}} v_k,$$

we note that we just need to change indices to go to the basis  $(e_m)$  from the basis of  $(v_k)$  as in the work leading to Lemma 1.4.  $\triangle$

By a simple calculation, cf. Appendix A.4, we get that

$$\begin{aligned} \bar{w}_1 \otimes \bar{v}_k &= \sqrt{\frac{2(k+1)(n+1-k)}{(n+1)(n+2)}} \bar{s}_{k+1} - \frac{(n-2k)}{\sqrt{n(n+2)}} \bar{t}_k \\ &\quad - \sqrt{\frac{2k(n-k)}{n(n+1)}} \bar{u}_{k-1}. \end{aligned} \quad (1.31)$$

for  $0 \leq k \leq n$ . Now changing indices as mentioned in Remark 1.6 to  $\ell = \frac{1}{2}n$  and  $m = \frac{1}{2}(n-2k) = \ell - k$  as we did to get to Lemma 1.4, i.e.  $n = 2\ell$  and  $k = \ell - m$ , we get that

$$\begin{aligned} \bar{w}_1 \otimes e_m &= \bar{w}_1 \otimes \bar{v}_k \\ &= \sqrt{\frac{2(\ell-m+1)(2\ell+1)-(\ell-m)}{(2\ell+1)(2\ell+2)}} \bar{s}_{k+1} - \frac{(2\ell-2(\ell-m))}{\sqrt{2\ell(2\ell+2)}} \bar{t}_k \\ &\quad - \sqrt{\frac{2(\ell-m)(2\ell-(\ell-m))}{2\ell(2\ell+1)}} \bar{u}_{k-1} \\ &= \sqrt{\frac{(\ell-m+1)(\ell+1+m)}{(2\ell+1)(\ell+1)}} \bar{s}_{k+1} - \frac{m}{\sqrt{\ell(\ell+1)}} \bar{t}_k \\ &\quad - \sqrt{\frac{(\ell-m)(\ell+m)}{\ell(2\ell+1)}} \bar{u}_{k-1}, \end{aligned}$$

where  $e_m$  is as in the work we did to get Lemma 1.4 except for the fact that we consider  $\mathfrak{sl}(2, \mathbf{C})$ -modules still. Now setting

$$\tilde{D}_+(\bar{v}_k) = -\frac{\bar{s}_{k+1}}{\sqrt{(\ell+1)(2\ell+1)}}, \quad \tilde{D}_0(\bar{v}_k) = \frac{\bar{t}_k}{\sqrt{\ell(\ell+1)}}, \quad \tilde{D}_-(\bar{v}_k) = -\frac{\bar{u}_{k-1}}{\sqrt{\ell(2\ell+1)}},$$

we see that

$$\begin{aligned}
 \bar{w}_1 \otimes e_m &= \bar{w}_1 \otimes \bar{v}_k \\
 &= \sqrt{(\ell+1)^2 - m^2} \frac{\bar{s}_{k+1}}{\sqrt{(\ell+1)(2\ell+1)}} - m \frac{\bar{t}_k}{\sqrt{\ell(\ell+1)}} \\
 &\quad - \sqrt{\ell^2 - m^2} \frac{\bar{u}_{k-1}}{\ell(2\ell+1)} \\
 &= \sqrt{\ell^2 - m^2} \tilde{D}_-(\bar{v}_k) - m \tilde{D}_0(\bar{v}_k) - \sqrt{(\ell+1)^2 - m^2} \tilde{D}_+(\bar{v}_k).
 \end{aligned} \tag{1.32}$$

Note that for  $m \in \{\pm\ell\}$  the  $\tilde{D}_-$  term vanishes, so the formula works here although  $D_-$  is not well-defined in these edge cases.

\* \* \* \*

Getting back to the problem at the end of Section 1.1.1, we want to give the maps  $D_0$ ,  $D_+$ , and  $D_-$  such that  $D_0 R_{\ell,m} \subset R_{\ell,m}$ ,  $D_+ R_{\ell,m} \subset R_{\ell+1,m}$ , and  $D_- R_{\ell,m} \subset R_{\ell-1,m}$ , the diagrams of eq. (1.12) commute, and we can describe  $F_3$ ,  $F_+$ ,  $F_-$  by the maps  $D_0$ ,  $D_+$ ,  $D_-$ ,  $E_+$ , and  $E_-$ . Now consider the  $\mathfrak{sl}(2, \mathbb{C})$ -modules  $V(n)$  as  $L_k$ -modules  $M(n)$  via the isomorphism of eq. (1.4), and note that since

$$R_\ell = M(2\ell)^1 \oplus M(2\ell)^2 \oplus \cdots \oplus M(2\ell)^r$$

and each  $M(2\ell)^i$  has a basis  $(e_{-\ell}^i, e_{-\ell+1}^i, \dots, e_{\ell-1}^i, e_\ell^i)$  with  $H_3 e_m^i = m e_m^i$  for all  $m$ , we have that  $R_{\ell,m}$  has basis  $(e_m^1, e_m^2, \dots, e_m^r)$  by definition. So when describing the maps  $D_0$ ,  $D_+$ , and  $D_-$ , we just need to describe what the maps should do to each  $e_m^i$ . We already know that  $E_+ e_m^i = e_{m+1}^i$  and  $E_- e_m^i = e_{m-1}^i$  where it makes sense, so if the maps  $D_0$ ,  $D_+$ , and  $D_-$  do not depend on  $m$  or  $i$ , we get the commutative diagrams of eq. (1.12), thus we want to describe what each map does to  $M(2\ell)$  in general, so we will stop writing the superscripts.

Since we want to describe the maps  $F_3$ ,  $F_+$ , and  $F_-$ , we are actually interested in the actions of  $L_p$ , so by using  $\psi$  of eq. (1.14) and the considerations at the end of Section 1.1.1, we can start out by describing the  $L_k$ -module  $M(2) \otimes M(2\ell)$ , i.e. we can use the description of  $V(2) \otimes V(n)$  from above. Note that we have already seen that  $L_p \simeq M(2)$  as  $L_k$ -modules, but we would like to better understand how the basis  $(f_+, f_3, f_-)$  of  $L_p$  corresponds to the basis  $(w_0, w_1, w_2)$  of  $M(2)$  as in eq. (1.6). In the basis  $(w_0, w_1, w_2)$  we have that  $h_+.w_0 = 0$  (since this is what corresponds to  $x.w_0 = 0$  in  $V(2)$  by eq. (1.4)), so by checking eq. (1.3) we see that  $w_0$  must correspond to a multiple of  $f_3$ , but the basis is chosen up to scalar, so we can take  $w_0$  to be  $-\frac{\sqrt{2}}{2} f_3$ . Now we get  $w_1$  by taking  $h_-.w_0$  (corresponding to  $y.w_0$  in  $V(2)$  by eq. (1.4)), thus we get that

$$w_1 = h_-.w_0 = -\frac{\sqrt{2}}{2} h_-.f_+ = -\frac{\sqrt{2}}{2} [h_-, f_+] = \sqrt{2} f_3.$$

Likewise we get that  $w_2 = [h_-, \sqrt{2} f_3] = \sqrt{2} f_-$ , so we can take our basis to be  $(w_0, w_1, w_2) = (-\frac{\sqrt{2}}{2} f_+, \sqrt{2} f_3, \sqrt{2} f_-)$  when thinking of  $L_p$  as the  $L_k$ -module  $M(2)$ . Normalizing as in eq. (1.29), we get that  $(\bar{w}_0, \bar{w}_1, \bar{w}_2) =$

$(-\frac{\sqrt{2}}{2}f_+, f_3, \sqrt{2}f_-)$ . So by eq. (1.32), we see that in  $L_p \otimes M(2\ell)$

$$f_3 \otimes e_m = \sqrt{\ell^2 - m^2} \tilde{D}_-(e_m) - m \tilde{D}_0(e_m) - \sqrt{(\ell+1)^2 - m^2} \tilde{D}_+(e_m),$$

where  $e_m = \bar{v}_k$  for  $k = \ell - m$  and  $f_3 = \bar{w}_1$ .

**Remark 1.7.** Note that if we have bases  $(e_{-\ell-1}^{(2\ell+2)}, e_{-\ell}^{(2\ell+2)}, \dots, e_{\ell}^{(2\ell+2)}, e_{\ell+1}^{(2\ell+2)})$  for  $M(2\ell+2)$ ,  $(e_{-\ell}^{(2\ell)}, e_{-\ell+1}^{(2\ell)}, \dots, e_{\ell-1}^{(2\ell)}, e_{\ell}^{(2\ell)})$  for  $M(2\ell)$ , and  $(e_{-\ell+1}^{(2\ell-2)}, e_{-\ell+2}^{(2\ell-2)}, \dots, e_{\ell-2}^{(2\ell-2)}, e_{\ell-1}^{(2\ell-2)})$  for  $M(2\ell-2)$  (if  $\ell \geq 1$ ) as in Lemma 1.4, then as above changing indices with  $k = \ell + 1 - m$  we see that  $e_m^{(2\ell+2)}$  corresponds to  $\bar{s}_k$ . Likewise changing indices with  $k = \ell - m$  we see that  $e_m^{(2\ell)}$  corresponds to  $\bar{t}_k$ , and with  $k = \ell - 1 - m$  we see that  $e_m^{(2\ell-2)}$  corresponds to  $\bar{u}_k$ .  $\triangle$

By this remark together with Remark 1.5, we see that  $E_+ \bar{v}_k = \bar{v}_{k-1}$  and  $E_+ \bar{s}_k = \bar{s}_{k-1}$  where it makes sense, so by the definition of  $\tilde{D}_+$  it commutes with  $E_+$  and  $E_-$ . Similarly we can see that  $\tilde{D}_0$  and  $\tilde{D}_-$  commute with  $E_+$  and  $E_-$ .

Now using  $\psi$  from eq. (1.14), we see that

$$\begin{aligned} F_3 e_m &= f_3 \cdot e_m = \psi(f_3 \otimes e_m) \\ &= \sqrt{\ell^2 - m^2} \psi \tilde{D}_-(e_m) - m \psi \tilde{D}_0(e_m) - \sqrt{(\ell+1)^2 - m^2} \psi \tilde{D}_+(e_m). \end{aligned} \tag{1.33}$$

So we can take  $D_0 = \psi \tilde{D}_0$ ,  $D_+ = \psi \tilde{D}_+$ , and  $D_- = \psi \tilde{D}_-$  to get three linear maps with which we can describe the map  $F_3$ . So far this is just maps on  $M(2\ell)$ , but we can expand to maps on  $R_\ell$  by using the maps on each summand of  $R_\ell = M(2\ell)^1 \oplus \dots \oplus M(2\ell)^r$ , and likewise we can expand further to maps on  $M = \bigoplus_\ell R_\ell$  by using the maps on each summand. Also indeed  $D_0 R_{\ell,m} \subset R_{\ell,m}$ ,  $D_+ R_{\ell,m} \subset R_{\ell+1,m}$ , and  $D_- R_{\ell,m} \subset R_{\ell-1,m}$ , since for  $\xi \in R_{\ell,m}$  we have that

$$\begin{aligned} H_3 D_0(\xi) &= h_3 \cdot \psi \tilde{D}_0(\xi) = \psi h_3 \cdot \tilde{D}_0(\xi) = \psi H_3 \tilde{D}_0(\xi) = m \psi \tilde{D}_0(\xi) \\ &= m D_0(\xi), \end{aligned}$$

since  $\psi$  is an  $L_k$ -module homomorphism and by Remark 1.7 we see that  $\tilde{D}_0(e_m)$  is a scalar multiple of  $\bar{t}_k = \bar{t}_{\ell-m} = e_m^{(2\ell)}$ , and indeed  $H_3 e_m^{(2\ell)} = m e_m^{(2\ell)}$ . The same reasoning with  $\bar{s}_{k+1}$  for  $D_+$  and  $\bar{u}_{k-1}$  for  $D_-$  yields the other two inclusions. Also note that the diagrams of eq. (1.12) commute by the definition of  $D_0$ ,  $D_+$ , and  $D_-$ , since we have already shown that  $\tilde{D}_0$ ,  $\tilde{D}_+$ , and  $\tilde{D}_-$  commute with  $E_+$  and  $E_-$  where it makes sense, and  $L_k$ -module homomorphisms in general commute with  $E_+$  and  $E_-$ , so also  $\psi$  commutes with  $E_+$  and  $E_-$ . This is the case since for a  $L_k$ -module homomorphism  $\varphi: M \rightarrow M'$  with decompositions  $M = \bigoplus_{\ell,m} R_{\ell,m}$  and  $M' = \bigoplus_{\ell,m} R'_{\ell,m}$ , we have first that  $\varphi(R_\ell) \subset R'_\ell$  and then  $\varphi(R_{\ell,m}) \subset R'_{\ell,m}$  — for more details see .

Add reference

**Remark 1.8.** Note that  $D_0$  is not defined at  $R_{0,0}$ , but  $D_0$  is defined on  $R_{\ell,0}$  by using the relation  $E_+D_0 = E_+D_0$  repeatedly. Also note that  $D_-$  is not defined at  $R_{\ell,\pm\ell}$ .  $\triangle$

Now simple calculations, cf. Appendix A.5, gives us that

$$\begin{aligned}
 F_3\xi &= \sqrt{\ell^2 - m^2}D_-\xi - mD_0\xi - \sqrt{(\ell+1)^2 - m^2}D_+\xi, \\
 F_+\xi &= \sqrt{(\ell-m)(\ell-m-1)}D_-E_+\xi - \sqrt{(\ell-m)(\ell+m+1)}D_0E_+\xi \\
 &\quad + \sqrt{(\ell+m+1)(\ell+m+2)}E_+D_+\xi, \\
 F_-\xi &= -\sqrt{(\ell+m)(\ell+m-1)}D_-E_-\xi - \sqrt{(\ell+m)(\ell-m+1)}D_0E_-\xi \\
 &\quad - \sqrt{(\ell-m+1)(\ell-m+2)}E_-D_+\xi
 \end{aligned} \tag{1.34}$$

for  $\xi \in R_{\ell,m}$ . Note here that although  $D_-$  is not defined on  $R_{\ell,\ell}$  and  $R_{\ell,-\ell}$  the above still makes sense since in these cases the terms with  $D_-$  vanish, either by the coefficient being zero or by  $E_+$  or  $E_-$  mapping to zero.

We claim now that the formulae eq. (1.34) for the linear operators  $F_+$ ,  $F_-$ , and  $F_3$  together with the formulae eqs. (1.9) and (1.11) for  $H_+$ ,  $H_-$ , and  $H_3$  define a representation of  $L$ , i.e. they satisfy the commutation relations of eq. (1.3), if and only if  $D_0$ ,  $D_+$ , and  $D_-$  satisfy

$$\begin{aligned}
 \ell D_+D_0\xi &= (\ell+2)D_0D_+\xi, \\
 (\ell+1)D_-D_0\xi &= (\ell-1)D_0D_-\xi, \\
 \xi &= (2\ell-1)D_+D_-\xi - (2\ell+3)D_-D_+\xi - D_0^2\xi
 \end{aligned} \tag{1.35}$$

for  $\xi \in R_{\ell,m}$ ,  $\ell \neq 0$  except in the first line where  $\ell = 0$  is allowed. Here the term  $D_+D_-$  should be replaced with  $E_+D_+D_-E_-$  for  $\xi \in R_{\ell,\ell}$  and  $E_-D_+D_-E_+$  for  $\xi \in R_{\ell,-\ell}$ .

### 1.1.3 Simple Harish-Chandra modules for the pair $(L, L_k)$

We want to classify the simple Harish-Chandra modules for the pair  $(L, L_k)$  for later use. Before most of the work we need some basic results.

Let  $M$  be an simple Harish-Chandra module over  $L$  and suppose that each non-trivial subspace  $R_{\ell,m}$  in  $M = \bigoplus_{\ell,m} R_{\ell,m}$  is one dimensional. In this case each  $L_k$ -module  $R_\ell$  is either isomorphic to the simple module  $M(2\ell)$  or 0. We claim that actually all simple Harish-Chandra modules are of this kind, so we indeed get a classification of the simple Harish-Chandra modules in the following.

To see this assume that  $V \subset R_\ell$  is a simple  $L_k$ -submodule in  $M$ , i.e.  $V \simeq M(2\ell)$ , and put

$$\overline{D_+V} = D_+(V) + E_+(D_+(V)) + E_-(D_+(V)).$$

We claim that  $\overline{D_+V}$  is either a simple  $L_k$ -submodule or 0. This is clear since by construction if  $\overline{D_+V}$  is non-zero, it sits inside an  $L_k$ -submodule isomorphic to  $M(2\ell + 2)$ , and here we have the weight  $m$  elements for  $m \in \{-\ell, -\ell + 1, \dots, \ell - 1, \ell\}$  sitting in  $D_+(V)$ , the  $-\ell - 1$  weight elements in  $E_-(D_+(V))$ , and the  $\ell + 1$  elements in  $E_+(D_+(V))$ , so  $\overline{D_+V} \simeq M(2\ell + 2)$ . Now if  $D_0(V) \subset V$  then eq. (1.35) imply that  $D_0(\overline{D_+V}) \subset \overline{D_+V}$  since  $E_+$  and  $E_-$  commute with  $D_0$ . Likewise if  $D_+D_-(V) \subset V$  we get that  $D_-(\overline{D_+V}) \subset V$  and  $D_+D_-(\overline{D_+V}) \subset \overline{D_+V}$ .

Now consider an eigenvector  $v \in R_{\ell_0, \ell_0}$  of  $D_0$  (or of  $\Delta_2$  in case  $\ell_0 = 0$ ), and let  $V_{\ell_0}$  be the subspace spanned by all  $(E_-)^r v$ . This is clearly a simple  $L_k$ -module isomorphic to  $M(2\ell_0)$  (up to constants on the terms it is exactly the construction of  $M(2\ell_0)$ ). Inductively define  $V_\ell = \overline{D_+V_{\ell-1}}$  for all  $\ell > \ell_0$ . Then  $\bigoplus_{\ell \geq \ell_0} V_\ell$  is an  $L$ -submodule of  $M$  since by definition it is invariant under  $D_+$ ,  $E_+$ , and  $E_-$ , and by eq. (1.35)  $D_0(V_{\ell_0}) \subset V_{\ell_0}$  and  $D_+D_-(V_{\ell_0}) \subset V_{\ell_0}$  so  $\bigoplus_{\ell \geq \ell_0} V_\ell$  is also invariant under  $D_0$  and  $D_-$  by the above. Hence if  $M$  is simple, we get that  $M = \bigoplus_{\ell \geq \ell_0} V_\ell$ , and thus indeed each non-trivial subspace  $R_{\ell, m}$  in  $M$  is one dimensional.

Denote by  $\ell_0$  the minimal index  $\ell$  in the decomposition  $M = \bigoplus_\ell R_\ell$ . Note that

$$M' = \bigoplus_{\ell' \in \{\ell_0, \ell_0+1, \dots\}} R_{\ell'}$$

is invariant under  $E_+$ ,  $E_-$ ,  $D_0$ ,  $D_+$ , and  $D_-$ , so by the formulae eq. (1.34) for  $F_+$ ,  $F_-$ , and  $F_3$ , we see that  $M'$  is a submodule since we already know that it is an  $L_k$ -submodule because  $R_{\ell'}$  all are  $L_k$ -submodules. Thus  $M' = M$  since  $M$  is simple and hence the index  $\ell$  in  $M = \bigoplus_\ell R_\ell$  range over only integral values or only half-integral values.

Additionally we want to show that the kernel of the map  $D_- : M \rightarrow M$  is  $R_{\ell_0}$ . To do this assume for contradiction that  $D_- R_{\ell', m_0} = 0$  for some index  $\ell' > \ell_0$  and  $m_0 \in \{-\ell_0, -\ell_0 + 1, \dots, \ell_0 - 1, \ell_0\}$ . Then by the commutative diagram in eq. (1.12) with  $D_-$ , i.e.  $D_- E_+ = E_+ D_-$ , and the fact that  $E_+ : R_{\ell', m} \rightarrow R_{\ell', m+1}$  is an isomorphism for  $m < \ell'$ , we see that  $D_- R_{\ell', m} = 0$  for all  $m \in \{-\ell', -\ell' + 1, \dots, \ell' - 1, \ell'\}$ . But then

$$M'' = \bigoplus_{\ell'' \in \{\ell', \ell'+1, \dots\}} R_{\ell''}$$

is a proper  $L$ -submodule of  $M$ , which contradicts the simplicity of  $M$ . Thus indeed  $\ker D_- = R_{\ell_0}$ .

Likewise we see that if  $M$  is infinite dimensional, then  $D_+ : M \rightarrow M$  has trivial kernel since if  $D_+ R_{\ell'} = 0$ , then  $M = \bigoplus_{\ell \in \{\ell_0, \ell_0+1, \dots\}} R_\ell$  is finite dimensional. This is the case since all terms with  $\ell > \ell'$  must be trivial since

otherwise

$$M'' = \bigoplus_{\ell'' \in \{\ell_0, \ell_0+1, \dots, \ell'\}} R_{\ell''}$$

is a proper  $L$ -submodule of  $M$ , which contradicts the simplicity of  $M$ .

### Infinite dimensional simple modules

Assume that  $M$  is a Harish-Chandra module of the above kind and is infinite dimensional. Because all  $R_{\ell,m}$  are one dimensional, the diagram with  $E_+$  and  $D_+$  in eq. (1.12) commute, i.e.  $D_+E_+ = E_+D_+$ , and  $D_+$  has trivial kernel, while  $E_+$  is an isomorphism for  $m \neq \ell$ , we see that we can choose a basis  $\{\xi_{\ell,m}\}$  of  $M$  such that  $\xi_{\ell,m} \in R_{\ell,m}$  and

$$\begin{aligned} E_+\xi_{\ell,m} &= \xi_{\ell,m+1} & \text{for } -\ell \leq m < \ell, \\ D_+\xi_{\ell,m} &= \xi_{\ell+1,m} & \text{for } \ell \in \{\ell_0, \ell_0+1, \dots\}. \end{aligned}$$

In this basis we get that

$$\begin{aligned} E_-\xi_{\ell,m} &= \xi_{\ell,m-1} & \text{for } -\ell < m \leq \ell, \\ D_0\xi_{\ell,m} &= d_\ell^0 \xi_{\ell,m} & \text{for } \ell \in \{\ell_0, \ell_0+1, \dots\}, \\ D_-\xi_{\ell,m} &= d_\ell^- \xi_{\ell-1,m} & \text{for } \ell \in \{\ell_0+1, \ell_0+2, \dots\}, \\ D_-\xi_{\ell_0,m} &= 0, \end{aligned} \tag{1.36}$$

where the first equation comes from the fact that  $E_-: R_{\ell,m} \rightarrow R_{\ell,m-1}$  for  $m \neq -\ell$  is the inverse of  $E_+: R_{\ell,m-1} \rightarrow R_{\ell,m}$ , while the independence of  $m$  in the other equations comes from the commutativity of the diagrams of eq. (1.12).

Now eqs. (1.35) and (1.36) implies that

$$\begin{aligned} \ell d_\ell^0 &= (\ell+2)d_{\ell+1}^0, \\ (\ell+1)d_\ell^- d_\ell^0 &= (\ell-1)d_{\ell-1}^0 d_\ell^-, \\ 1 &= (2\ell-1)d_\ell^- - (2\ell+3)d_{\ell+1}^- - (d_\ell^0)^2, \\ d_{\ell_0}^- &= 0, \end{aligned} \tag{1.37}$$

for  $\ell \in \{\ell_0, \ell_0+1, \dots\}$  except in the second equation where we also demand that  $\ell > \ell_0$ . We see that

$$d_{\ell+1}^0 = \frac{\ell}{\ell+2} d_\ell^0.$$

So if  $\ell_0 \neq 0$ , then for some constant  $c$

$$d_{\ell_0}^0 = \frac{c}{\ell_0(\ell_0+1)},$$

so we see inductively that if

$$d_\ell^0 = \frac{c}{\ell(\ell+1)}, \quad (1.38)$$

then

$$\begin{aligned} d_{\ell+1}^0 &= \frac{\ell}{\ell+2} d_\ell^0 = \frac{\ell}{\ell+2} \frac{c}{\ell(\ell+1)} \\ &= \frac{c}{(\ell+1)(\ell+2)}. \end{aligned}$$

Thus if  $\ell_0 \neq 0$  eq. (1.38) holds true in general for some constant  $c$ .

If on the other hand  $\ell_0 = 0$ , then we see that

$$2d_{\ell_0+1}^0 = 0,$$

so  $d_{\ell_0+1}^0 = 0$ , and thus

$$d_\ell^0 = \frac{\ell-1}{\ell+1} d_{\ell-1}^0 = 0$$

for all  $\ell \in \{1, 2, \dots\}$ . Also in this case have  $d_0^0 = c_1$ , where  $c_1$  is some constant.

To unify these two cases we set  $c = i\ell_0\ell_1$  and  $c_1 = i\ell_1$  for some complex constant  $\ell_1$  such that

$$d_\ell^0 = \frac{i\ell_0\ell_1}{\ell(\ell+1)} \quad (1.39)$$

for  $\ell \in \{\ell_0, \ell_0 + 1, \dots\}$ . Substituting this expression with  $d_\ell^0$  in the third equation of eq. (1.37) we get that

$$(2\ell-1)d_\ell^- - (2\ell+3)d_{\ell+1}^- = 1 - \frac{\ell_0^2\ell_1^2}{\ell^2(\ell+1)^2},$$

and a simple calculation, cf. Appendix A.7, yields that

$$d_\ell^- = -\frac{(\ell^2 - \ell_1^2)(\ell^2 - \ell_0^2)}{\ell^2(4\ell^2 - 1)}, \quad (1.40)$$

for  $\ell > \ell_0$ .

Since we showed in the beginning of this subsection that the kernel of  $D_-$  is  $R_{\ell_0}$ , we must have that  $d_\ell^- \neq 0$  for all  $\ell > \ell_0$ . Thus  $\ell^2 - \ell_1^2 \neq 0$  for  $\ell > \ell_0$ , so  $|\ell_1| - \ell_0$  cannot be a positive integer if  $\ell_1$  is real, because if that was the case then  $|\ell_1| = \ell_0 + (|\ell_1| - \ell_0) \in \{\ell_0 + 1, \ell_0 + 2, \dots\}$ , but  $|\ell_1|^2 - \ell_1^2 = 0$  since  $\ell_1 \in \mathbf{R}$ .



Hence altogether by eqs. (1.11) and (1.34) in the basis  $\{\xi_{\ell,m}\}$  the operators  $H_+$ ,  $H_-$ ,  $H_3$ ,  $F_+$ ,  $F_-$ , and  $F_3$  are given by the formulae

$$\begin{aligned}
 H_3 \xi_{\ell,m} &= m \xi_{\ell,m}, \\
 H_+ \xi_{\ell,m} &= \sqrt{(\ell+m+1)(\ell-m)} \xi_{\ell,m+1}, \\
 H_- \xi_{\ell,m} &= \sqrt{(\ell+m)(\ell-m+1)} \xi_{\ell,m-1}, \\
 F_3 \xi_{\ell,m} &= \sqrt{\ell^2 - m^2} d_\ell^- \xi_{\ell-1,m} - m d_\ell^0 \xi_{\ell,m} - \sqrt{(\ell+1)^2 - m^2} d_\ell^+ \xi_{\ell+1,m}, \\
 F_+ \xi_{\ell,m} &= \sqrt{(\ell-m)(\ell-m-1)} d_\ell^- \xi_{\ell-1,m+1} - \sqrt{(\ell-m)(\ell+m+1)} d_\ell^0 \xi_{\ell,m+1} \\
 &\quad + \sqrt{(\ell+m+1)(\ell+m+2)} d_\ell^+ \xi_{\ell+1,m+1}, \\
 F_- \xi_{\ell,m} &= -\sqrt{(\ell+m)(\ell+m-1)} \xi_{\ell-1,m-1} - \sqrt{(\ell+m)(\ell-m+1)} \xi_{\ell,m-1} \\
 &\quad - \sqrt{(\ell-m+1)(\ell-m+2)} \xi_{\ell+1,m-1},
 \end{aligned} \tag{1.41}$$

where

$$d_\ell^0 = \frac{i\ell_0\ell_1}{\ell(\ell+1)}, \quad d_\ell^- = -\frac{(\ell^2 - \ell_1^2)(\ell^2 - \ell_0^2)}{\ell^2(4\ell^2 - 1)}, \quad d_\ell^+ = 1, \tag{1.42}$$

for  $\ell \in \{\ell_0, \ell_0 + 1, \dots\}$ , and where  $\ell_1$  is a complex number such that  $|\ell_1| - \ell_0$  is not a positive integer if  $\ell_1$  is real. Here we use the convention that  $\xi_{\ell',m'} = 0$  for pairs  $\ell', m'$  where there is no such basis element.

### Finite dimensional simple modules

Assume that  $M$  is a Harish-Chandra module of the above kind and that  $M$  is finite dimensional, i.e.  $M = \bigoplus_{\ell,m} R_{\ell,m}$  where  $R_{\ell,m}$  are one dimensional subspaces for  $\ell_0 \leq \ell < |\ell_1|$ . Here  $\ell_1$  is some real number such that  $|\ell_1| \geq \ell_0$  and  $|\ell_1| - \ell_0$  is integral. We can choose a basis  $\{\xi_{\ell,m}\}$  as in the infinite dimensional case and by similar considerations we still get the formulae eqs. (1.41) and (1.42) describing the actions of  $H_+$ ,  $H_-$ ,  $H_3$ ,  $F_+$ ,  $F_-$ , and  $F_3$ , though now in this basis we only consider  $\ell \in \{\ell_0, \ell_0 + 1, \dots, |\ell_1| - 1\}$ .

Here it is worth noting that we can actually describe the modules accurately using some algebraic results which we will not show in this paper. Firstly note that we know that all finite dimensional  $\mathfrak{sl}(2, \mathbf{C})$ -modules are the  $V(n)$  we have described earlier, and from this it can be shown that all finite dimensional  $\mathfrak{sl}(2, \mathbf{C}) \times \mathfrak{sl}(2, \mathbf{C})$ -modules are of the form  $V(n) \otimes V(m)$ , where the action of  $\mathfrak{sl}(2, \mathbf{C}) \times \mathfrak{sl}(2, \mathbf{C})$  on  $V(n) \otimes V(m)$  is given by

$$(x, y).v_1 \otimes v_2 = (x.v_1) \otimes v_2 + v_1 \otimes (y.v_2).$$

Note that considering  $V(n) \otimes V(m)$  as  $\mathfrak{sl}(2, \mathbf{C})$ -modules via the diagonal map  $\mathfrak{sl}(2, \mathbf{C}) \rightarrow \mathfrak{sl}(2, \mathbf{C}) \times \mathfrak{sl}(2, \mathbf{C})$ ,  $u \mapsto (u, u)$ , we get the standard action of  $\mathfrak{sl}(2, \mathbf{C})$

on  $V(n) \otimes V(m)$  as described in eq. (1.13), and now Clebsch-Gordan's formula implies that

$$V(n) \otimes V(m) \simeq \bigoplus_{i=0}^{\min(n,m)} V(n+m-2i).$$

Note that the last term of the direct sum is  $V(|n-m|)$ . Thus translating to the language of  $L$  and  $L_k$ -modules, we have that all finite dimensional  $L$ -modules are of the form  $\bigoplus_{i=0}^{\min(2r,2s)} M(2r+2s-2i)$  where  $r, s \in \frac{1}{2}\mathbf{Z}_{\geq 0}$ . Here we see that  $\ell_0 = |r-s|$  and  $|\ell_1| - 1 = r+s$ .

## 1.2 Decomposition of modules into indecomposables

Now we want to continue our work using our knowledge of the classification of simple Harish-Chandra modules for the pair  $(L, L_k)$  to begin our classification of indecomposable Harish-Chandra modules for the pair  $(L, L_k)$ . To do this we will first need to some work with Laplace operators.

### 1.2.1 Laplace operators

Let  $U(L)$  be the universal enveloping algebra of  $L$ , cf. [Jan16, Appendix E]. We know, cf. [Jan16, p. E-9], that  $M$  is an  $L$ -module if and only if it is an  $U(L)$ -module, so we can describe  $L$ -modules by describing  $U(L)$ -modules. To do this we will first need to have an explicit description of the center  $Z(U(L))$  of  $U(L)$ . We will begin this description by first noting that  $Z(U(\mathfrak{sl}(2, \mathbf{C}) \times \mathfrak{sl}(2, \mathbf{C}))) \simeq Z(U(\mathfrak{sl}(2, \mathbf{C}))) \otimes Z(U(\mathfrak{sl}(2, \mathbf{C})))$ , which follows from the fact that  $Z(U(L_1 \times L_2)) \simeq Z(U(L_1)) \otimes Z(U(L_2))$  for Lie algebras  $L_1$  and  $L_2$  in general cf. Appendix B.1.

We have seen in Exercise 11 in the Lie algebra course that  $Z(U(\mathfrak{sl}(2, \mathbf{C})))$  is the algebra of polynomials in  $C = h^2 + 2h + 4yx$ , i.e.  $Z(U(\mathfrak{sl}(2, \mathbf{C}))) = \mathbf{C}[C]$ . Thus we see that  $Z(U(\mathfrak{sl}(2, \mathbf{C}))) \otimes Z(U(\mathfrak{sl}(2, \mathbf{C})))$  is the algebra of polynomials in  $C \otimes 1$  and  $1 \otimes C$ , or equivalently the algebra of polynomials in  $C \otimes 1 - 1 \otimes C$  and  $C \otimes 1 + 1 \otimes C$ . So we want to describe  $C \otimes 1 - 1 \otimes C$  and  $C \otimes 1 + 1 \otimes C$  in terms of our basis  $h_+, h_-, h_3, f_+, f_-, f_3$ . We note that  $(u, u') \in L = \mathfrak{sl}(2, \mathbf{C}) \times \mathfrak{sl}(2, \mathbf{C})$  in  $U(L) = U(\mathfrak{sl}(2, \mathbf{C}) \times \mathfrak{sl}(2, \mathbf{C}))$  is identified with  $u \otimes 1 + 1 \otimes u'$  in  $U(\mathfrak{sl}(2, \mathbf{C})) \otimes U(\mathfrak{sl}(2, \mathbf{C}))$ , so

$$\begin{aligned} & \frac{1}{2}(h_-f_+ + f_-h_+) + h_3f_3 + f_3 \\ &= \frac{1}{2}((y \otimes 1 + 1 \otimes y)(ix \otimes 1 - 1 \otimes ix) + (iy \otimes 1 - 1 \otimes iy)(x \otimes 1 + 1 \otimes x)) \\ & \quad + \frac{1}{4}(h \otimes 1 + 1 \otimes h)(ih \otimes 1 - 1 \otimes ih) + \frac{1}{2}(ih \otimes 1 - 1 \otimes ih) \\ &= \frac{1}{2}(iyx \otimes 1 - iy \otimes x + ix \otimes y - i \otimes yx + iyx \otimes 1 + iy \otimes x - ix \otimes y - i \otimes yx) \\ & \quad + \frac{1}{4}(ih^2 \otimes 1 - ih \otimes h + ih \otimes h - i \otimes h^2) + \frac{1}{2}(ih \otimes 1 - 1 \otimes ih) \\ &= \frac{1}{2}(2iyx \otimes 1 - 2i \otimes yx) + \frac{1}{4}(ih^2 \otimes 1 - i \otimes h^2) + \frac{1}{2}(ih \otimes 1 - i \otimes h) \end{aligned}$$

Maybe write the argument in an appendix or find better reference

## 1.2. Decomposition of modules into indecomposables

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$$\begin{aligned}
&= iyx \otimes 1 + \frac{1}{4}ih^2 \otimes 1 + \frac{1}{2}ih \otimes 1 - i \otimes yx - \frac{1}{4}i \otimes h^2 - \frac{1}{2}i \otimes h \\
&= \frac{i}{4}(h^2 + 2h + yx) \otimes 1 - \frac{i}{4}1 \otimes (h^2 + 2h + yx) \\
&= \frac{i}{4}(C \otimes 1 - 1 \otimes C),
\end{aligned}$$

and likewise  $h_-h_+ - f_-f_+ + h_3^2 - f_3^2 + 2h_3 = \frac{1}{2}(C \otimes 1 + 1 \otimes C)$ .

Thus since the constants don't matter when we look at the algebra of polynomials in  $C \otimes 1 - 1 \otimes C$  and  $C \otimes 1 + 1 \otimes C$ , we see that setting

$$\begin{aligned}
\Delta_1 &= \frac{1}{2}(h_-f_+ + f_-h_+) + h_3f_3 + f_3, \\
\Delta_2 &= h_-h_+ - f_-f_+ + h_3^2 - f_3^2 + 2h_3,
\end{aligned} \tag{1.43}$$

we have that  $Z(U(L))$  is the algebra of polynomials in  $\Delta_1$  and  $\Delta_2$ . Thus in term of the corresponding linear operators on a Harish-Chandra module  $M$  for the pair  $(L, L_k)$ , we define linear operators

$$\begin{aligned}
\Delta_1 &:= \frac{1}{2}(H_-F_+ + F_-H_+) + H_3F_3 + F_3 \\
\Delta_2 &:= H_-H_+ - F_-F_+ + H_3^2 - F_3^2 + 2H_3,
\end{aligned} \tag{1.44}$$

which are called Laplace operators. Note that by eqs. (1.9), (1.10) and (1.34), cf. Appendix A.8, we get that

$$\begin{aligned}
\Delta_1\xi &= -\ell(\ell+1)D_0\xi \\
\Delta_2\xi &= (\ell^2 - 1)\xi - (\ell+1)^2D_0^2\xi + (4\ell^2 - 1)D_+D_-\xi
\end{aligned} \tag{1.45}$$

for  $\xi \in R_\ell$ . Here as in the relations for the  $D$ 's earlier, we replace  $D_+D_-$  with  $E_+D_+D_-E_-$  for  $\xi \in R_{\ell,\ell}$  and  $E_-D_+D_-E_+$  for  $\xi \in R_{\ell,-\ell}$ . Alternatively by eq. (1.35), cf. Appendix A.8, we also get that

$$\Delta_2\xi = ((\ell+1)^2 - 1)\xi + \ell^2D_0^2\xi + (4(\ell+1)^2 - 1)D_-D_+\xi \tag{1.46}$$

for  $\xi \in R_\ell$ , which will sometimes be more useful.

Now by noting that  $D_0$ ,  $D_+D_-$ , and  $D_0^2$  all preserve  $R_{\ell,m}$  eq. (1.45) gives us the following Lemma:

**Lemma 1.9.** *Each subspace  $R_{\ell,m}$  is invariant under the Laplace operators  $\Delta_1$  and  $\Delta_2$ .*

Additionally we are ready to prove the Lemma:

**Lemma 1.10.** *The linear operators  $D_+$ ,  $D_-$ ,  $D_0$ ,  $E_+$ , and  $E_-$  commute with the Laplace operators  $\Delta_1$  and  $\Delta_2$ .*

*Proof.* Denote by  $(\Delta_i)_{\ell,m}$  the restriction of  $\Delta_i$  to  $R_{\ell,m}$  for  $i = 1, 2$ . Lemma 1.9 implies that  $\Delta_i = \bigoplus_{\ell,m} (\Delta_i)_{\ell,m}$  for  $i = 1, 2$ , so it is enough to check that  $(\Delta)_{\ell,m}$  commutes with the operators for all  $\ell$  and  $m$ . Therefore eqs. (1.45) and (1.46) implies that  $\Delta_i$  commute with  $E_+$  and  $E_-$  since  $D_+$ ,  $D_-$ , and  $D_0$  commute

with  $E_+$  and  $E_-$  where it makes sense and using eq. (1.46) for  $\Delta_2$  it makes sense for all  $R_{\ell,m}$ .

Now multiplying the first equation of eq. (1.35) with  $\ell + 1$ , we see that

$$\ell(\ell + 1)D_+D_0\xi = (\ell + 1)(\ell + 2)D_0D_+\xi$$

for  $\xi \in R_{\ell,m}$ , so by eq. (1.45), we see that

$$D_+\Delta_1\xi = -\ell(\ell + 1)D_+D_0\xi = -(\ell + 1)(\ell + 2)D_0D_+\xi = \Delta_1D_+\xi$$

for  $\xi \in R_{\ell,m}$ . Thus  $\Delta_1$  indeed commutes with  $D_+$ . Similarly the second equation of eq. (1.35) imply that  $\Delta_1$  commutes with  $D_-$ , and also it is obvious from eq. (1.45) that  $\Delta_1$  commutes with  $D_0$ .

Likewise the first equation of eq. (1.35) together with eqs. (1.45) and (1.46) implies that

$$\begin{aligned}\Delta_2D_+\xi &= ((\ell + 1)^2 - 1)D_+\xi - (\ell + 2)^2D_0^2D_+\xi + (4(\ell + 1)^2 - 1)D_+D_-D_+\xi \\ &= ((\ell + 1)^2 - 1)D_+\xi - \ell^2D_+D_0^2\xi + (4(\ell + 1)^2 - 1)D_+D_-D_+\xi \\ &= D_+\Delta_2\xi\end{aligned}$$

for  $\xi \in R_{\ell,m}$ . Thus  $\Delta_2$  commutes with  $D_+$ , and similarly using the second equation of eq. (1.35) we get that  $\Delta_2$  commutes with  $D_-$ . Finally it is clear that  $D_0$  commutes with the first two terms of  $\Delta_2$ , so we just need to show that  $D_0(D_+D_-)\xi = (D_+D_-)D_0\xi$  for  $\xi \in R_{\ell,m}$  where it makes sense. But now the first and second equation of eq. (1.35) imply that

$$(\ell + 1)D_0D_+D_-\xi = (\ell - 1)D_+D_0D_-\xi = (\ell + 1)D_+D_-D_0\xi$$

for  $\xi \in R_{\ell,m}$ , so since  $\ell \geq 0$  and thus  $\ell \neq -1$ , we get that  $D_0(D_+D_-)\xi = (D_+D_-)D_0\xi$ . Hence  $\Delta_2$  indeed commutes with  $D_0$  also.  $\square$

### 1.2.2 Properties of the Laplace operators in indecomposable modules

Now we are finally ready to begin considering the properties of  $\Delta_1$  and  $\Delta_2$  in indecomposable Harish-Chandra modules, which will end up being an important part of our characterization of indecomposable Harish-Chandra modules for the pair  $(L, L_k)$ .

**Proposition 1.11.** *A Harish-Chandra module  $M$  for the pair  $(L, L_k)$  is decomposable into the direct sum of a countable number of indecomposable modules such that on each indecomposable module the Laplace operators  $\Delta_1$  and  $\Delta_2$  have each one eigenvalue,  $\lambda_1$  and  $\lambda_2$  respectively.*

*Proof.* Since each of the subspaces  $R_{\ell,m}$  is invariant under  $\Delta_1$  and  $\Delta_2$  by Lemma 1.9 and since these operators commute with each other, we get that

$R_{\ell,m}$  can be written as a direct sum of subspaces  $R_{\ell,m}(\lambda_1^i, \lambda_2^i)$  on each of which each of the operators  $\Delta_1$  and  $\Delta_2$  has one eigenvalue  $\lambda_1^i$  and  $\lambda_2^i$  respectively. Note that here the index set of  $i$  is finite since  $R_{\ell,m}$  is finite dimensional.

Consider now fixed  $\lambda_1$  and  $\lambda_2$  and the set  $S$  of those  $(\ell, m)$  for which there exists subspaces  $R_{\ell,m}(\lambda_1^i, \lambda_2^i)$  with  $\lambda_1 = \lambda_1^i$  and  $\lambda_2 = \lambda_2^i$ . Denote by  $M(\lambda_1, \lambda_2)$  the subspace of  $M$  with  $M(\lambda_1, \lambda_2) = \bigoplus_{(\ell,m) \in S} R_{\ell,m}(\lambda_1, \lambda_2)$  such that in  $M(\lambda_1, \lambda_2)$  each of the operators  $\Delta_1$  and  $\Delta_2$  has one eigenvalue,  $\lambda_1$  and  $\lambda_2$  respectively. We want to show that  $M(\lambda_1, \lambda_2)$  is a submodule of  $M$ , i.e. that it is invariant under  $H_+$ ,  $H_-$ ,  $H_3$ ,  $F_+$ ,  $F_-$ , and  $F_3$ , but this is clearly the case since  $M(\lambda_1, \lambda_2)$  is invariant under  $E_+$ ,  $E_-$ ,  $D_+$ ,  $D_-$ , and  $D_0$  because  $\Delta_1$  and  $\Delta_2$  commute with these operators by Lemma 1.10. Finally note that the number of  $M(\lambda_1, \lambda_2)$  in the decomposition of  $M$  is countable since the number of  $R_{\ell,m}$  is countable and the number of  $R_{\ell,m}(\lambda_1^i, \lambda_2^i)$  in a given  $R_{\ell,m}$  is finite, so decomposing each  $M(\lambda_1, \lambda_2)$  we get the result.  $\square$

**Proposition 1.12.** *Let  $M$  be a Harish-Chandra module in which each of the Laplace operators  $\Delta_1$  and  $\Delta_2$  has one eigenvalue. Then there exists an integral or half-integral number  $\ell_0 \geq 0$  and a complex number  $\ell_1$  such that the eigenvalues  $\lambda_1$  and  $\lambda_2$  have the form*

$$\lambda_1 = -i\ell_0\ell_1, \quad \lambda_2 = \ell_0^2 + \ell_1^2 - 1. \quad (1.47)$$

*Proof.* Denote by  $\ell_0$  the minimal index in the decomposition  $M = \bigoplus_{\ell} R_{\ell}$  of  $M$  into  $L_k$ -submodules of  $R_{\ell}$ . By the definition of  $D_-$  it maps  $R_{\ell_0}$  to zero, so by eq. (1.45) we get that

$$\begin{aligned} \Delta_1 \xi &= -\ell_0(\ell_0 + 1)D_0 \xi \\ \Delta_2 \xi &= (\ell_0^2 - 1)\xi - (\ell_0 + 1)D_0^2 \xi \end{aligned} \quad (1.48)$$

for  $\xi \in R_{\ell_0}$ . Now the subspace  $R_{\ell_0}$  is invariant under  $D_0$ , so we can find an eigenvector  $\xi_0$  for  $D_0$  such that  $D_0 \xi_0 = \mu \xi_0$  for some  $\mu \in \mathbf{C}$ . Thus we see that

$$\begin{aligned} \Delta_1 \xi_0 &= -\ell_0(\ell_0 + 1)\mu \xi_0 \\ \Delta_2 \xi_0 &= (\ell_0^2 - 1)\xi_0 - (\ell_0 + 1)\mu^2 \xi_0, \end{aligned}$$

so we get eigenvalues  $\lambda_1$  and  $\lambda_2$  of  $\Delta_1$  and  $\Delta_2$  with

$$\lambda_1 = -\ell_0(\ell_0 + 1)\mu, \quad \lambda_2 = (\ell_0^2 - 1) - (\ell_0 + 1)\mu^2.$$

Hence putting  $(\ell_0 + 1)\mu = i\ell_1$ , we get that

$$\lambda_1 = -i\ell_0\ell_1, \quad \lambda_2 = \ell_0^2 + \ell_1^2 - 1.$$

Now by assumption each of  $\Delta_1$  and  $\Delta_2$  has only one eigenvalue on  $M$ , and thus these eigenvalues are expressed in terms of the  $\ell_0$  and  $\ell_1$  as in eq. (1.47).  $\square$

Note that all such eigenvalues  $\lambda_1$  and  $\lambda_2$  for  $\Delta_1$  and  $\Delta_2$  are possible, since in the case of finite dimensional simple modules  $N$  as in Section 1.1.3 we have by eqs. (1.42) and (1.45) that

$$\begin{aligned}\Delta_1 \xi_{\ell,m} &= -\ell(\ell+1)d_\ell^0 \xi_{\ell,m} = -i\ell_0 \ell_1, \\ \Delta_2 \xi_{\ell,m} &= (\ell^2 - 1)\xi_{\ell,m} - (\ell+1)^2(d_\ell^0)^2 \xi_{\ell,m} + (4\ell^2 - 1)d_{\ell+1}^+ d_\ell^- \xi_{\ell,m} \\ &= (\ell^2 - 1)\xi_{\ell,m} + \frac{\ell_0^2 \ell_1^2}{\ell^2} \xi_{\ell,m} - \frac{(\ell^2 - \ell_1^2)(\ell^2 - \ell_0^2)}{\ell^2} \xi_{\ell,m} \\ &= (\ell^2 - 1)\xi_{\ell,m} - (\ell^2 - \ell_0^2 - \ell_1^2)\xi_{\ell,m} \\ &= (\ell_0^2 + \ell_1^2 - 1)\xi_{\ell,m},\end{aligned}$$

for  $\xi_{\ell,m} \in R_{\ell,m}$ , where  $\ell_0 \geq 0$  is an integral or half-integral number and  $\ell_1$  is a complex number such that we can construct  $N$  such that  $\ell_0$  and  $\ell_1$  are as we want.

**Proposition 1.13.** *Let  $M$  be a Harish-Chandra module in which the Laplace operators  $\Delta_1$  and  $\Delta_2$  have only one eigenvalue  $\lambda_1$  and  $\lambda_2$  respectively. Then on each subspace  $R_\ell$  the operators  $D_+D_-$ ,  $D_-D_+$ , and  $D_0$  have only one eigenvalue  $d_\ell^-$ ,  $d_\ell^+$ , and  $d_\ell^0$  respectively. Here the numbers  $d_\ell^-$ ,  $d_\ell^+$ , and  $d_\ell^0$  are expressed in terms of  $\ell_0$  and  $\ell_1$  in the following way:*

$$\begin{aligned}d_0^- &= d_{1/2}^- = 0, \\ d_\ell^- &= \frac{(\ell^2 - \ell_0^2)(\ell_1^2 - \ell^2)}{(4\ell^2 - 1)\ell^2} && \text{if } \ell \neq 0, \frac{1}{2}, \\ d_\ell^+ &= \frac{((\ell+1)^2 - \ell_0^2)(\ell_1^2 - (\ell+1)^2)}{(4(\ell+1)^2 - 1)(\ell+1)^2} \\ d_0^0 &= i\ell_1, \\ d_\ell^0 &= \frac{i\ell_0 \ell_1}{\ell(\ell+1)} && \text{if } \ell \neq 0.\end{aligned} \tag{1.49}$$

I might want to change the result slightly. I need to say that we if have  $R_{\ell+1} = 0$  so that  $d_\ell^+ = 0$  then  $\ell+1 = |\ell_1|$  and  $\ell_1$  is real — I either need to change the result or go back through the paper and correct some things and argue that this will not happen/will not matter

*Proof.* By eq. (1.45) we see that

$$\begin{aligned}(4\ell^2 - 1)D_+D_- \xi &= \Delta_2 \xi - (\ell^2 - 1)\xi + (\ell+1)^2 D_0^2 \xi, \\ (\ell+1)D_0 \xi &= -\frac{\Delta_1}{\ell} \xi\end{aligned}$$

for  $\xi \in R_\ell$  with  $\ell > \ell_0$  such that  $\ell \neq 0$ . Thus

$$(4\ell^2 - 1)D_+D_- \xi = \Delta_2 \xi - (\ell^2 - 1)\xi + \frac{\Delta_1^2}{\ell^2} \xi$$

for  $\xi \in R_\ell$  with  $\ell > \ell_0$ . Hence since  $\Delta_1$  and  $\Delta_2$  each only have one eigenvalue on  $R_\ell$  so thus  $D_+D_-$ , and we see by eq. (1.47) that

$$\begin{aligned} d_\ell^- &= \frac{1}{(4\ell^2 - 1)} \left( \lambda_2 - (\ell - 1)^2 + \frac{\lambda_1^2}{\ell^2} \right) \\ &= \frac{(\ell_0^2 + \ell_1^2 - 1)\ell^2 - (\ell - 1)^2\ell^2 - \ell_0^2\ell_1^2}{(4\ell^2 - 1)\ell^2} \\ &= \frac{(\ell_0^2 + \ell_1^2 - \ell^2)\ell^2 - \ell_0^2\ell_1^2}{(4\ell^2 - 1)\ell^2} \\ &= \frac{(\ell^2 - \ell_0^2)(\ell_1^2 - \ell^2)}{(4\ell^2 - 1)\ell^2} \end{aligned}$$

for  $\ell \neq 0, \frac{1}{2}$ . Now since  $D_-$  by definition maps  $R_0$  and  $R_{1/2}$  to zero if they occur in the decomposition of  $M$ , we see that  $d_0^- = 0$  and  $d_{1/2}^- = 0$ , and likewise we know that  $D_-$  maps  $R_{\ell_0}$  to zero so  $d_{\ell_0}^- = 0$ , which also holds true with the formula above. Thus we have proven the formulae for  $d_\ell^-$ , and the other formulae are proven similarly.  $\square$

Now denote by  $C_s(\lambda_1, \lambda_2)$  for  $s = 1$  or  $s = \frac{1}{2}$  the set of all Harish-Chandra modules for the pair  $(L, L_k)$  in which the Laplace operators have the eigenvalues  $\lambda_1$  and  $\lambda_2$ , and in which if  $s = 1$  every  $M \in C_1(\lambda_1, \lambda_2)$  has only integral numbers as indices in the decomposition  $M = \bigoplus_\ell R_\ell$ , and if  $s = \frac{1}{2}$  every  $M \in C_{1/2}(\lambda_1, \lambda_2)$  has only half-integral numbers as indices in the decomposition  $M = \bigoplus_\ell R_\ell$ .

**Proposition 1.14.** *Let  $M \in C_s(\lambda_1, \lambda_2)$ ,  $M' \in C_{s'}(\lambda'_1, \lambda'_2)$ , where  $(s, \lambda_1, \lambda_2) \neq (s', \lambda'_1, \lambda'_2)$ . Then  $\text{Hom}_L(M, M') = 0$ .*

*Proof.* Let  $\gamma \in \text{Hom}_L(M, M')$  and assume that  $\gamma \neq 0$ . First we will show that  $\gamma R_\ell \subset R'_\ell$ , where  $R_\ell$  comes from the decomposition of  $M$  and  $R'_\ell$  from the decomposition of  $M'$ . To see this note that  $R_\ell$  and  $R'_\ell$  are direct sums of finitely many  $L_k$ -modules  $M(2\ell)$ , where each  $M(2\ell)$  is generated by a maximal vector of weight  $\ell$  (weight w.r.t.  $h_3$  in  $L_k$ ). So since  $\gamma$  takes a maximal vector of weight  $\ell$  to either zero or another maximal vector of weight  $\ell$ , we see that indeed  $\gamma R_\ell \subset R'_\ell$ .

Denoting by  $\Delta_i$  the Laplace operators in  $M$  and by  $\Delta'_i$  the Laplace operators in  $M'$ , we also have that  $\Delta'_i \gamma = \gamma \Delta_i$  for  $i = 1, 2$ , since in both cases  $\Delta_i$  and  $\Delta'_i$  correspond to the actions in  $U(L)$  of eq. (1.43) and  $\gamma \in \text{Hom}_L(M, M') = \text{Hom}_{U(L)}(M, M')$ . Now since  $\gamma \neq 0$  we get that  $\gamma R_\ell \subset R'_\ell$  implies that  $s = s'$  and  $\Delta'_i \gamma = \gamma \Delta_i$  implies that  $\lambda_i = \lambda'_i$  for  $i = 1, 2$ , but this is a contradiction since  $(s, \lambda_1, \lambda_2) \neq (s', \lambda'_1, \lambda'_2)$ . Hence we must have that  $\gamma = 0$ , and thus indeed  $\text{Hom}_L(M, M') = 0$ .  $\square$

**Remark 1.15.** From now on in places where the index  $s$  is not important, we will simply denote  $C_s(\lambda_1, \lambda_2)$  by  $C(\lambda_1, \lambda_2)$ .  $\triangle$

**Definition 1.16.** The category of modules  $C(\lambda_1, \lambda_2)$  is called *singular* if the numbers  $\ell_0$  and  $\ell_1$  constructed from  $\lambda_1$  and  $\lambda_2$  as in eq. (1.47) are such that  $\ell_1 - \ell_0$  is an integer. Otherwise it is called *non-singular*.

We will see that the study of the non-singular categories  $C(\lambda_1, \lambda_2)$  is simpler than that of the singular ones, and we are now ready to begin our description of the category of the singular categories  $C(\lambda_1, \lambda_2)$  which will be important to our classification of indecomposable Harish-Chandra modules.

### 1.3 The non-singular category $C(\lambda_1, \lambda_2)$

Let  $M \in C(\lambda_1, \lambda_2)$  be an  $L$ -module, where  $(\lambda_1, \lambda_2)$  is a non-singular pair, i.e.  $\ell_1 - \ell_0$  is not an integer. We now want to show that this module  $M$  is completely determined by a finite dimensional vector space and a nilpotent map  $a$  on this vector space, where an isomorphism of the modules is equivalent to similarity of the linear maps  $a$ .

Define on the finite dimensional linear subspace  $R_{\ell_0, m_0}$  for some  $m_0 \in \{-\ell_0, -\ell_0 + 1, \dots, \ell_0 - 1, \ell_0\}$  a linear map  $a: R_{\ell_0, m_0} \rightarrow R_{\ell_0, m_0}$  by

$$a\xi = D_0\xi - \frac{i\ell_1}{\ell_0 + 1}\xi \quad (1.50)$$

for  $\xi \in R_{\ell_0, m_0}$ . This map is nilpotent since by Proposition 1.13 the only eigenvalue of  $D_0$  on  $R_{\ell_0}$  is

$$d_{\ell_0}^0 = \frac{i\ell_1}{\ell_0 + 1},$$

and

$$\det(a - t \text{id}) = \det\left(D_0 - \left(t + \frac{i\ell_1}{\ell_0 + 1} \text{id}\right)\right),$$

so the only eigenvalue of  $a$  on  $R_{\ell}$  is zero, and thus  $a$  is clearly nilpotent by Cayley-Hamilton Theorem.

We want to show that the module  $M$  is completely determined by the finite dimensional vector space  $R_{\ell_0, m_0}$  and the linear map  $a: R_{\ell_0, m_0} \rightarrow R_{\ell_0, m_0}$  when  $C(\lambda_1, \lambda_2)$  is non-singular. To do this we first need some lemmas.

**Lemma 1.17.** In a non-singular module  $M \in C(\lambda_1, \lambda_2)$ , the maps

$$\begin{aligned} D_+ &: R_{\ell, m} \rightarrow R_{\ell+1, m} \\ D_- &: R_{\ell+1, m} \rightarrow R_{\ell, m} \end{aligned}$$

for  $\ell \geq \ell_0$  are isomorphisms.



*Proof.* By Proposition 1.13 the eigenvalues of  $D_+D_-$  and  $D_-D_+$  on  $R_\ell$  for  $\ell \neq 0, \frac{1}{2}$  are

$$\begin{aligned} d_\ell^- &= \frac{(\ell^2 - \ell_0^2)(\ell_1^2 - \ell^2)}{(4\ell^2 - 1)\ell^2} && \text{for } \ell > \ell_0, \\ d_\ell^+ &= \frac{((\ell + 1)^2 - \ell_0^2)(\ell_1^2 - (\ell + 1)^2)}{(4(\ell + 1)^2 - 1)(\ell + 1)^2}. \end{aligned}$$

Now by assumption  $M$  is non-singular, i.e.  $\ell_1 - \ell_0$  is not an integer, and we want to show that  $\ell_1^2 - \ell^2 \neq 0$  for all  $\ell \in \{\ell_0, \ell_0 + 1, \dots\}$ .

Assume that  $\ell_1^2 - \ell^2 = 0$  for some  $\ell = \ell_0 + k$ , where  $k$  is a non-negative integer. Since  $\ell_1 - \ell_0$  is not an integer, we also have that  $\ell_1 - (\ell_0 + k)$  is not an integer and hence not equal to zero, so we must have that  $\ell_1 + \ell_0 + k = 0$  since  $\ell_1^2 - \ell^2 = (\ell_1 - \ell)(\ell_1 + \ell)$ . But this would imply that  $\ell_1 = -\ell_0 - k$  and hence  $\ell_1 - \ell_0 = -2\ell_0 - k$  is an integer, which is a contradiction with the non-singularity of  $M$ .

Thus we see that  $\ell_1^2 - \ell^2$  is non-zero for all  $\ell \in \{\ell_0, \ell_0 + 1, \dots\}$ , and therefore the eigenvalues  $d_\ell^+$  and  $d_\ell^-$  are different from zero for all  $\ell$  except  $\ell_0$  in the case of  $d_\ell^-$ . Hence the maps  $D_+D_-: R_{\ell,m} \rightarrow R_{\ell,m}$  for  $\ell \neq \ell_0$  and  $D_-D_+: R_{\ell,m} \rightarrow R_{\ell,m}$  have diagonals without zeros in the Schur decomposition, so they are invertible. Therefore  $D_-: R_{\ell+1,m} \rightarrow R_{\ell,m}$  and  $D_+: R_{\ell,m} \rightarrow R_{\ell+1,m}$  are injective, and thus  $\dim R_{\ell,m} = \dim R_{\ell+1,m}$ , which again implies that  $D_-$  and  $D_+$  as above are actually isomorphisms.  $\square$

**Lemma 1.18.** *In a non-singular module  $M \in C(\lambda_1, \lambda_2)$  the Laplace operators  $\Delta_1$  and  $\Delta_2$  are such that each operator  $(\Delta_i)_{\ell,m}$  is similar to  $(\Delta_i)_{\ell_0,m_0}$  via the same matrix for  $i = 1, 2$ .*

*Proof.* Recall that the maps  $E_+: R_{\ell_0,m} \rightarrow R_{\ell_0,m+1}$  for  $-\ell_0 \leq m < \ell_0$  and  $E_-: R_{\ell_0,m} \rightarrow R_{\ell_0,m-1}$  for  $-\ell_0 < m \leq \ell_0$  are isomorphisms, and the Laplace operators  $\Delta_1$  and  $\Delta_2$  commute with these maps by Lemma 1.10, so  $(\Delta_i)_{\ell_0,m}$  is similar to  $(\Delta_i)_{\ell_0,m_0}$  for each  $m$  and  $i = 1, 2$ .

Likewise the  $D_+: R_{\ell,m} \rightarrow R_{\ell+1,m}$  are also isomorphisms for all  $\ell$  and commute with both  $\Delta_1$  and  $\Delta_2$ , so the map  $(\Delta_i)_{\ell_0+1,m}$  is similar to  $(\Delta_i)_{\ell_0,m}$ , and inductively  $(\Delta_i)_{\ell,m}$  is similar to  $(\Delta_i)_{\ell_0,m}$  for all  $\ell \in \{\ell_0, \ell_0 + 1, \dots\}$ . Hence indeed  $(\Delta_i)_{\ell,m}$  is similar to  $(\Delta_i)_{\ell_0,m_0}$ .  $\square$

**Lemma 1.19.** *If  $M \in C(\lambda_1, \lambda_2)$  is a non-singular module, then the Laplace operators  $\Delta_1$  and  $\Delta_2$  are connected on the whole of  $M$  by the relation*

$$\Delta_1^2 + \ell_0^2 \Delta_2 - \ell_0^2 (\ell_0^2 - 1) \text{id} = 0. \quad (1.51)$$

*Proof.* Suppose that  $\ell_0 \neq 0$ . By eq. (1.48) we get that

$$\begin{aligned} \Delta_2 \xi + \frac{\Delta_1^2}{\ell_0^2} \xi - (\ell_0^2 - 1) \text{id} \xi &= (\ell_0^2 - 1) \xi - (\ell_0 + 1)^2 D_0^2 \xi \\ &\quad + (\ell_0 + 1)^2 D_0^2 \xi - (\ell_0^2 - 1) \xi = 0, \end{aligned} \quad (1.52)$$

for  $\xi \in R_{\ell_0, m_0}$ , so multiplying by  $\ell_0^2$  we get eq. (1.51) on  $R_{\ell_0, m_0}$ . By Lemma 1.18  $(\Delta_i)_{\ell, m}$  is similar to  $(\Delta_i)_{\ell_0, m_0}$  via the same matrix for  $i = 1, 2$ , so the relation holds true for any  $\xi \in R_{\ell, m}$ , and thus on all of  $M$ .

Suppose otherwise that  $\ell_0 = 0$ . Then eq. (1.48) implies that  $(\Delta_1)_{0,0}$  is zero, and thus the relation follows easily on  $R_{0,0}$ , and we can expand to all of  $M$  as above.  $\square$

**Remark 1.20.** Note that Lemmas 1.17 to 1.19 are not true in the singular case.  $\triangle$

We have seen above that to each non-singular module  $M \in C(\lambda_1, \lambda_2)$  there corresponds a finite dimensional vector space  $P = R_{\ell_0, m_0}$  and a nilpotent linear map  $a: P \rightarrow P$  given by

$$a\xi = \left(D_0 - \frac{i\ell_1}{\ell_0 + 1} \text{id}\right)\xi$$

for  $\xi \in R_{\ell_0, m_0} = P$ . Denote now by  $\tilde{A}$  the pair  $(P, a)$  consisting of a finite dimensional vector space  $P$  and a nilpotent mapping  $a: P \rightarrow P$ .

**Theorem 1.21.** *To each pair  $\tilde{A}$  and non-singular pair  $(\lambda_1, \lambda_2)$  of numbers there is a corresponding  $L$ -module  $M \in C(\lambda_1, \lambda_2)$  such that  $P = R_{\ell_0, m_0}$  and  $a$  is related to  $D_0$  by eq. (1.50).*

*Proof.* Denote by  $R_{\ell_0, m_0}$  the space  $P$  and consider the linear transformation

$$D_0\xi = a\xi + \frac{i\ell_1}{\ell_0 + 1}\xi$$

for  $\xi \in R_{\ell_0, m_0}$ . Consider the space

$$M = \bigoplus_{\substack{\ell \in \{\ell_0, \ell_0+1, \dots\} \\ m \in \{-\ell, -\ell+1, \dots, \ell-1, \ell\}}} R_{\ell, m},$$

which is a direct sum of vector spaces with  $\dim R_{\ell, m} = \dim P$  for all  $\ell$  and  $m$ .

Now take an isomorphism  $E_+: R_{\ell, m} \rightarrow R_{\ell, m+1}$  for  $m \neq \ell$  and put  $E_+: R_{\ell, \ell} \rightarrow 0$ , which we can do since  $\dim R_{\ell, m} = \dim R_{\ell, m+1}$ . Define an isomorphism  $E_-: R_{\ell, m+1} \rightarrow R_{\ell, m}$  such that it is inverse to  $E_+: R_{\ell, m} \rightarrow R_{\ell, m+1}$  and put  $E_-: R_{\ell, -\ell} \rightarrow 0$ . Take now isomorphisms  $D_+: R_{\ell, m_0} \rightarrow R_{\ell+1, m_0}$  for some fixed  $m_0$ , and define on all the remaining  $R_{\ell, m}$  linear maps  $D_+: R_{\ell, m} \rightarrow R_{\ell+1, m}$  such that the diagram

$$\begin{array}{ccc} R_{\ell, m+1} & \xrightarrow{D_+} & R_{\ell+1, m+1} \\ E_+ \uparrow & & \uparrow E_+ \\ R_{\ell, m} & \xrightarrow{D_+} & R_{\ell+1, m} \end{array}$$

commutes for  $-\ell \leq m < \ell$ , i.e.  $(D_+)_{\ell, m+1} = (E_+)_{\ell, m+1}^{-1} (D_+)_{\ell, m} (E_+)_{\ell+1, m}$ . Now we only need to construct linear maps  $D_0$  and  $D_-$  on  $M$  satisfying properties as we have seen earlier, but to do this we will first define linear maps  $\Delta_1$  and  $\Delta_2$  corresponding to the Laplace operators.

On  $R_{\ell_0, m_0}$  set

$$\begin{aligned} \Delta_1 \xi &= -\ell_0(\ell_0 + 1)D_0 \xi \\ &= -\ell_0(\ell_0 + 1)a\xi - i\ell_1\ell_0\xi, \\ \Delta_2 \xi &= (\ell_0^2 - 1)\xi - (\ell_0 + 1)^2 D_0^2 \xi \\ &= (\ell_0^2 - 1)\xi + \ell_1^2 \xi - (\ell_0 + 1)i\ell_1 a\xi - (\ell_0 + 1)^2 a^2 \xi \\ &= (\ell_0^2 + \ell_1^2 - 1)\xi - (\ell_0 + 1)^2 \left( a^2 \xi + 2 \frac{i\ell_1}{\ell_0 + 1} a\xi \right) \end{aligned} \quad (1.53)$$

for  $\xi \in R_{\ell_0, m_0}$ . Now note that for arbitrary  $R_{\ell, m}$  the linear map given by  $J_{\ell, m} = (E_+)^{m-m_0} (D_+)^{\ell-\ell_0}$ , where  $(E_+)^- = E_-$ , is a composition of isomorphisms and hence an isomorphism, so we can define

$$(\Delta_i)_{\ell, m} \xi = J_{\ell, m} (\Delta_i)_{\ell_0, m_0} (J_{\ell, m})^{-1}$$

for  $\xi \in R_{\ell, m}$  and  $i = 1, 2$ . Thus we have defined  $\Delta_1$  and  $\Delta_2$  on all of  $M$ .

Now define  $D_0: R_{\ell, m} \rightarrow R_{\ell, m}$  by

$$D_0 \xi = -\frac{1}{\ell(\ell + 1)} \Delta_1 \xi \quad (1.54)$$

for  $\xi \in R_{\ell, m}$ ,  $\ell \neq 0$ , and  $D_+ D_-: R_{\ell, m} \rightarrow R_{\ell, m}$  by

$$\begin{aligned} D_+ D_- \xi &= \frac{1}{4\ell^2 - 1} (\Delta_2 \xi - (\ell^2 - 1)\xi + (\ell + 1)^2 D_0^2 \xi) \\ &= \frac{1}{4\ell^2 - 1} \left( \Delta_2 \xi - (\ell^2 - 1)\xi + \frac{\Delta_1^2}{\ell^2} \xi \right) \end{aligned} \quad (1.55)$$

for  $\xi \in R_{\ell, m}$ ,  $\ell \neq \ell_0$ , which we can do since  $D_+$  is an isomorphism. Using this we define  $D_-: R_{\ell, m} \rightarrow R_{\ell-1, m}$  to be the map  $(D_+)^{-1} (D_+ D_-)$  for  $\ell \neq \ell_0$ , and equal to zero for  $\ell = \ell_0$ .

Now the maps  $E_+$ ,  $E_-$ ,  $D_0$ ,  $D_+$ , and  $D_-$  constructed above satisfy the relations of Section 1.1.1, so the operators  $F_+$ ,  $F_-$ ,  $F_3$ ,  $H_+$ ,  $H_-$ , and  $H_3$  constructed from these maps as in eqs. (1.9), (1.11) and (1.35) gives  $M$  an  $L$ -module structure. Finally we get  $P = R_{\ell_0, m_0}$  and eq. (1.50) by construction, and we note that  $M$  is non-singular since the pair  $(\lambda_1, \lambda_2)$  with the corresponding  $\ell_0$  and  $\ell_1$  is non-singular by assumption.  $\square$

Write this more explicitly

**Corollary 1.22.** *For modules  $M$  and  $M'$  from the non-singular category  $C(\lambda_1, \lambda_2)$  to be equivalent it is necessary and sufficient that the subspaces  $R_{\ell_0, m_0}$  and  $R'_{\ell_0, m_0}$  in these modules have the same dimension, and that their maps  $D_0: R_{\ell_0, m_0} \rightarrow R_{\ell_0, m_0}$  and  $D'_0: R'_{\ell_0, m_0} \rightarrow R'_{\ell_0, m_0}$  are similar.*

Write a little before this

Now let  $S$  be the category with objects pairs  $A = (P, a)$ , where  $P$  is a finite dimensional vector space (over  $\mathbf{C}$ ) and  $a: P \rightarrow P$  is a nilpotent linear transformation, and with morphisms  $\gamma: A \rightarrow A'$  given by linear maps  $\gamma: P \rightarrow P'$  such that

$$\begin{array}{ccc} P & \xrightarrow{a} & P \\ \gamma \downarrow & & \downarrow \gamma \\ P' & \xrightarrow{a'} & P' \end{array}$$

commutes. We have already shown that there is a correspondence between non-singular modules  $M \in C(\lambda_1, \lambda_2)$  and pairs  $A = (P, a) \in S$ , and now we want to show that this correspondence is functorial.

**Theorem 1.23.** *The non-singular category  $C(\lambda_1, \lambda_2)$  is equivalent to the category  $S$ .*

*Proof.* By Theorem 1.21 we have a correspondence between objects  $M \in C(\lambda_1, \lambda_2)$  and the objects  $A \in S$ , so we just need to establish a correspondence between the morphisms. Let  $\Gamma: M \rightarrow M'$  be a morphism between the two modules  $M, M' \in C(\lambda_1, \lambda_2)$ . We have already seen in the proof of Proposition 1.14 that  $\Gamma R_\ell \subset R'_\ell$  since  $\Gamma$  commutes with  $H_3$ , and since  $\Gamma$  also commutes with  $H_+$  and  $H_-$ , we get that  $\Gamma R_{\ell, m} \subset R'_{\ell, m}$ . Thus  $\Gamma$  is the direct sum of morphisms  $\gamma_{\ell, m}: R_{\ell, m} \rightarrow R'_{\ell, m}$ , so choosing indices  $\ell_0$  and  $m_0$  we get a morphism  $\gamma := \gamma_{\ell_0, m_0}: R_{\ell_0, m_0} \rightarrow R'_{\ell_0, m_0}$ . Here  $\gamma$  gives a morphism in  $S$  from  $\tilde{A} = (R_{\ell_0, m_0}, a)$  to  $\tilde{A}' = (R'_{\ell_0, m_0}, a')$ . To see this it is enough to show that  $D'_0 \gamma = \gamma D_0$ , which is true for  $\ell_0 \neq 0^3$  since by the proof of Proposition 1.14  $\Delta'_1 \Gamma = \Gamma \Delta_1$ , so for  $\ell \neq 0$  also  $D'_0 \Gamma = \Gamma D_0$  on  $R_\ell$  because  $\Delta_1 = -\ell(\ell+1)D_0$  on  $R_\ell$  and  $\Delta'_1 = -\ell(\ell+1)D'_0$  on  $R'_\ell$ , and therefore specifically  $D'_0 \gamma = \gamma D_0$ . Hence  $\gamma: \tilde{A} \rightarrow \tilde{A}'$  is indeed a morphism in  $S$ .

Now suppose conversely that we are given a morphism  $\gamma: A \rightarrow A'$  in  $S$ , i.e. a linear map  $\gamma: P \rightarrow P'$  such that  $\gamma a = a' \gamma$ . Construct modules  $M$  and  $M'$  with  $R_{\ell_0, m_0} = 0$  and  $R'_{\ell_0, m_0} = P'$  as in the proof of Theorem 1.21, then we have linear map  $\gamma: R_{\ell_0, m_0} \rightarrow R'_{\ell_0, m_0}$  such that  $\gamma a = a' \gamma$ , which implies that  $\gamma D_0 = D'_0 \gamma$  since  $D_0 = a + \frac{i\ell_1}{\ell_0+1} \text{id}$ . From this we will construct linear maps  $\gamma_{\ell, m}: R_{\ell, m} \rightarrow R'_{\ell, m}$  by noting that  $J_{\ell, m} = E_+^{m-m_0} D_+^{\ell-\ell_0}: R_{\ell_0, m_0} \rightarrow R_{\ell, m}$  is an isomorphism for  $\ell$  and  $m$  where it makes sense, so we get linear maps

$$\gamma_{\ell, m} = J'_{\ell, m} \gamma J_{\ell, m}^{-1}, \quad (1.56)$$

where  $J'_{\ell, m}: R'_{\ell, m} \rightarrow R'_{\ell_0, m_0}$  is as above also. We want to show that the direct sum  $\Gamma$  of the  $\gamma_{\ell, m}$  gives a morphism of  $L$ -modules from  $M = \bigoplus_{\ell, m} R_{\ell, m}$  to  $M' = \bigoplus_{\ell, m} R'_{\ell, m}$ .

<sup>3</sup>Recall that  $D_0$  is not defined on  $R_{0,0}$ , so this is what we want to check.

Recall, cf. Lemma 1.18, that  $(\Delta_i)_{\ell,m}$  is similar to  $(\Delta_i)_{\ell_0,m_0}$  for all  $\ell$  and  $m$ ,  $i = 1, 2$ , with  $(\Delta_i)_{\ell,m} = J_{\ell,m}(\Delta_i)_{\ell_0,m_0}J_{\ell,m}^{-1}$ . Hence since  $\gamma(\Delta_i)_{\ell_0,m_0} = (\Delta'_i)_{\ell_0,m_0}\gamma$  by eq. (1.48) since  $\gamma D_0 = D'_0\gamma$ , we see that

$$\begin{aligned}\gamma_{\ell,m}(\Delta_i)_{\ell,m} &= J'_{\ell,m}\gamma J_{\ell,m}^{-1}J_{\ell,m}(\Delta_i)_{\ell_0,m_0}J_{\ell,m}^{-1} = J'_{\ell,m}\gamma(\Delta_i)_{\ell_0,m_0}J_{\ell,m}^{-1} \\ &= J'_{\ell,m}(\Delta'_i)_{\ell_0,m_0}\gamma J_{\ell,m}^{-1} = (\Delta'_i)_{\ell,m}J'_{\ell,m}\gamma J_{\ell,m}^{-1} = (\Delta'_i)_{\ell,m}\gamma_{\ell,m}.\end{aligned}$$

Now since

$$\begin{aligned}D_0\xi &= -\frac{1}{\ell(\ell+1)}\Delta_1\xi, \\ D_+D_-\xi &= \frac{1}{4\ell^2-1}\left(\Delta_2\xi - (\ell^2-1)\xi + \frac{\Delta_1^2}{\ell^2}\xi\right)\end{aligned}$$

for  $\xi \in R_{\ell,m}$  with  $\ell \neq 0$ , cf. the proof of Theorem 1.21, we get that

$$\begin{aligned}(D'_0)_{\ell,m}\gamma_{\ell,m} &= \gamma_{\ell,m}(D_0)_{\ell,m}, \\ (D'_+D'_-)_{\ell,m}\gamma_{\ell,m} &= \gamma_{\ell,m}(D_+D_-)_{\ell,m}.\end{aligned}$$

Also noting that since  $E_+$  and  $D_+$  commute we have that  $E_+J_{\ell,m} = J_{\ell,m}E_+ = J_{\ell,m+1}$  and  $D_+J_{\ell,m} = J_{\ell,m}D_+ = J_{\ell+1,m}$ , so

$$\gamma_{\ell,m}E_+ = J'_{\ell,m}\gamma J_{\ell,m}^{-1}E_+ = E'_+J'_{\ell,m-1}\gamma J_{\ell,m-1}^{-1} = E'_+\gamma_{\ell,m-1}, \quad \square$$

and likewise  $\gamma_{\ell,m}D_+ = D'_+\gamma_{\ell-1,m}$  where it makes sense. Thus  $\Gamma$  commutes with  $D_+$  and  $D_+D_-$ , so by Lemma 1.17  $\Gamma$  also commutes with  $D_-$ , and likewise since  $E_-$  is the inverse of  $E_+$  or zero, we get that  $\Gamma$  commutes with  $E_+$ . Hence  $\Gamma$  commutes with  $E_+$ ,  $E_-$ ,  $D_0$ ,  $D_+$ , and  $D_-$ , so it commutes with  $H_+$ ,  $H_-$ ,  $H_3$ ,  $F_+$ ,  $F_-$ , and  $F_3$ , i.e. it is a morphism of  $L$ -modules.

**Corollary 1.24.** *An indecomposable module  $M$  in the non-singular category  $C(\lambda_1, \lambda_2)$  corresponds to an indecomposable object  $A$  in the category  $S$ .*

Here it follows from linear algebra that the indecomposable objects  $A \in S$  are finite dimensional vector spaces  $P$  with nilpotent linear maps  $a: P \rightarrow P$  whose matrices in a suitable basis have the form of a single Jordan block.

**Remark 1.25.** Note that by the above an indecomposable non-singular Harish-Chandra module for the pair  $(L, L_k)$  has one additional invariant when compared to the simple case. In the simple case we just need to numbers  $\ell_0$  and  $\ell_1$ , but in the indecomposable case we additionally need a number  $n$  giving the dimension of the Jordan block.  $\triangle$

Finally to end our description of the non-singular Harish-Chandra modules, we will give the explicit form of  $E_+$ ,  $E_-$ ,  $D_+$ ,  $D_-$ , and  $D_0$  in a non-singular Harish-Chandra module  $M \in C(\lambda_1, \lambda_2)$ . Here we denote by  $[E_+]_{\ell,m}$  the matrix representation of the map  $E_+: R_{\ell,m} \rightarrow R_{\ell,m+1}$  in a given basis, by  $[E_-]_{\ell,m}$  the matrix representation of the map  $E_-: R_{\ell,m} \rightarrow R_{\ell,m-1}$  in a given basis, and so on.

Maybe write a little more clearly why this is an equivalence of categories

**Theorem 1.26.** *Let  $M \in C(\lambda_1, \lambda_2)$  be a non-singular indecomposable Harish-Chandra module for the pair  $(L, L_k)$ . Then all the subspaces  $R_{\ell, m}$ ,  $\ell = \ell_0, \ell_0 + 1, \dots$ , have the same dimension, and bases can be chosen in them such that  $[E_+]_{\ell, m}$  for  $m \neq \ell$ ,  $[E_-]_{\ell, m}$  for  $m \neq -\ell$ , and  $[D_+]_{\ell, m}$  are identity matrices. Furthermore the matrices  $[D_0]_{\ell, m}$  and  $[D_-]_{\ell, m}$  can be expressed in terms of the matrix*

$$[a_0] = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

by the formulae

$$\begin{aligned} [D_0]_{\ell, m} &= i \frac{\ell_0 \ell_1}{\ell(\ell+1)} [\text{id}] + \frac{\ell_0(\ell_0+1)}{\ell(\ell+1)} [a_0] \\ [D_-]_{\ell, m} &= \frac{\ell_0^2 - \ell^2}{\ell^2(4\ell^2 - 1)} \left( (\ell^2 - \ell_1^2) [\text{id}] + 2i\ell_1(\ell_0+1)[a_0] + (\ell_0+1)^2 [a_0]^2 \right), \end{aligned}$$

where  $[\text{id}]$  is the identity matrix of the same dimension as  $[a_0]$ .

*Proof.* We follow the proof of Theorem 1.21. First take  $[D_0]$  to be the linear transformation

$$[D_0]\xi = [a_0]\xi + \frac{i\ell_1}{\ell_0+1} [\text{id}]\xi$$

for  $\xi \in R_{\ell_0, m_0}$ . Now following the construction in Theorem 1.21, we get that all the subspaces  $R_{\ell, m}$  have the same dimension, and we can find bases of  $R_{\ell, m}$  such that  $[E_+]_{\ell, m}$  for  $m \neq \ell$ ,  $[E_-]_{\ell, m}$  for  $m \neq -\ell$ , and  $[D_+]_{\ell, m}$  are identity matrices. By Lemma 1.10  $E_+$ ,  $E_-$ , and  $D_+$  commute with  $\Delta_i$ ,  $i = 1, 2$ , so  $[\Delta_i]_{\ell, m}$  is independent of  $\ell$  and  $m$ , so we can focus on  $R_{\ell_0, m_0}$ , where eq. (1.53) gives us that

$$\begin{aligned} [\Delta_1]_{\ell_0, m_0} \xi &= -\ell_0(\ell_0+1)[a_0]\xi - i\ell_1\ell_0[\text{id}]\xi, \\ [\Delta_2]_{\ell_0, m_0} \xi &= (\ell_0^2 + \ell_1^2 - 1)[\text{id}]\xi - (\ell_0+1)^2 \left( [a_0]^2 \xi + 2 \frac{i\ell_1}{\ell_0+1} [a_0]\xi \right) \\ &= (\ell_0^2 + \ell_1^2 - 1)[\text{id}]\xi - 2i\ell_1(\ell_0+1)[a_0]\xi - (\ell_0+1)^2 [a_0]^2 \xi, \end{aligned}$$

for  $\xi \in R_{\ell_0, m_0}$ . Thus we get that

$$\begin{aligned} [\Delta_1]_{\ell, m} &= -\ell_0(\ell_0+1)[a_0] - i\ell_1\ell_0[\text{id}], \\ [\Delta_2]_{\ell, m} &= (\ell_0^2 + \ell_1^2 - 1)[\text{id}] - 2i\ell_1(\ell_0+1)[a_0] - (\ell_0+1)^2 [a_0]^2, \end{aligned}$$

since  $[\Delta_i]_{\ell, m}$  is independent of  $\ell$  and  $m$ . Continuing as in Theorem 1.21, we get by eqs. (1.54) and (1.55) that

$$\begin{aligned}
 [D_0]_{\ell, m} &= -\frac{1}{\ell(\ell+1)}[\Delta_1]_{\ell, m} \\
 &= \frac{\ell_0(\ell_0+1)}{\ell(\ell+1)}[a_0] + \frac{i\ell_1\ell_0}{\ell(\ell+1)}[\text{id}], \\
 [D_+D_-]_{\ell, m} &= \frac{1}{4\ell^2-1} \left( [\Delta_2]_{\ell, m} - (\ell^2-1)[\text{id}] + \frac{[\Delta_1]_{\ell, m}^2}{\ell^2} \right) \\
 &= \frac{1}{(4\ell^2-1)\ell^2} \left( (\ell_0^2 + \ell_1^2 - 1)\ell^2[\text{id}] - 2i\ell_1(\ell_0+1)\ell^2[a_0] \right. \\
 &\quad \left. - (\ell_0+1)^2\ell^2[a_0]^2 - (\ell^2-1)\ell^2[\text{id}] + \ell_0^2(\ell_0+1)^2[a_0]^2 \right. \\
 &\quad \left. - \ell_1^2\ell_0^2[\text{id}] + 2i\ell_0^2(\ell_0+1)\ell_1[a_0] \right) \\
 &= \frac{\ell_0^2 - \ell^2}{(4\ell^2-1)\ell^2} \left( (\ell^2 - \ell_1^2)[\text{id}] + 2i\ell_1(\ell_0+1)[a_0] + (\ell_0+1)^2[a_0]^2 \right)
 \end{aligned}$$

since

$$\begin{aligned}
 (\ell_0^2 + \ell_1^2 - 1)\ell^2 - (\ell^2 - 1)\ell^2 - \ell_1^2\ell_0^2 &= -\ell^4 + \ell_0^2\ell^2 + \ell_1^2\ell^2 - \ell_1^2\ell_0^2 \\
 &= (\ell_0^2 - \ell^2)(\ell^2 - \ell_1^2), \\
 -2i\ell_1(\ell_0+1)\ell^2 + 2i\ell_0^2(\ell_0+1)\ell_1 &= (\ell_0^2 - \ell^2)2i\ell_1(\ell_0+1), \\
 -(\ell_0+1)^2\ell^2 + \ell_0^2(\ell_0+1)^2 &= (\ell_0^2 - \ell^2)(\ell_0+1)^2.
 \end{aligned}$$

Now since  $[D_+]_{\ell, m}$  was just the identity matrix, we see that  $[D_-]_{\ell, m}$  has the same expression as  $[D_+D_-]_{\ell, m}$  and thus we get the theorem.  $\square$

## 1.4 The singular category $C(\lambda_1, \lambda_2)$

Now we want to describe the singular category  $C(\lambda_1, \lambda_2)$ , i.e. Harish-Chandra modules for the pair  $(L, L_k)$  with  $\ell_1 - \ell_0$  an integer. The description of such modules turns out to be quite a bit more complicated than in the non-singular case, where a finite dimensional vector space  $P$  and a nilpotent linear map  $a: P \rightarrow P$  describes the module.

We define at first a category  $S_0$  as follows. The objects  $\tilde{A}$  of  $S_0$  are finite dimensional vector spaces  $P_1$  and  $P_2$  with four linear maps

$$d_+: P_1 \rightarrow P_2, \quad d_-: P_2 \rightarrow P_1, \quad \delta_1: P_1 \rightarrow 0 \quad \delta_2: P_2 \rightarrow P_2 \quad (1.57)$$

that satisfy the conditions

$$\begin{aligned}
 d_- \delta_2 &= \delta_2 d_+ = 0, \\
 \delta_2 \text{ and } d_+ d_- &\text{ are nilpotent.}
 \end{aligned} \quad (1.58)$$

The morphisms  $\gamma: \tilde{A} \rightarrow \tilde{A}'$  of  $S_0$  are pairs of linear maps  $\gamma = (\gamma_1, \gamma_2)$  with  $\gamma_1: P_1 \rightarrow P'_1$  and  $\gamma_2: P_2 \rightarrow P'_2$  such that the diagram

$$\begin{array}{ccccccc}
 P_1 & \xrightarrow{d_+} & P_2 & \xrightarrow{\delta_2} & P_2 & \xrightarrow{d_-} & P_1 \\
 \downarrow \gamma_1 & & \downarrow \gamma_2 & & \downarrow \gamma_2 & & \downarrow \gamma_1 \\
 P'_1 & \xrightarrow{d'_+} & P'_2 & \xrightarrow{\delta'_2} & P'_2 & \xrightarrow{d'_-} & P'_1
 \end{array} \tag{1.59}$$

commutes. Similarly to the non-singular case we now want to prove that the singular category  $C(\lambda_1, \lambda_2)$  is equivalent to the category  $S_0$ , but before we can do that we need some lemmas.

**Lemma 1.27.** *In a singular module  $M \in C(\lambda_1, \lambda_2)$  all the subspaces  $R_{\ell, m}$  for  $\ell_0 \leq \ell \leq |\ell_1| - 1$  and all  $m$  where it makes sense have the same dimension. The subspaces  $R_{\ell, m}$  for  $\ell \geq |\ell_1|$  and all  $m$  where it makes sense also have the same dimension. Furthermore the linear maps*

$$\begin{aligned}
 D_+ : R_{\ell, m} &\rightarrow R_{\ell+1, m} && \text{for } \ell \neq |\ell_1| - 1, \\
 D_- : R_{\ell, m} &\rightarrow R_{\ell-1, m} && \text{for } \ell \neq \ell_0, |\ell_1|
 \end{aligned}$$

are isomorphisms.

*Proof.* By eq. (1.49), we have that the eigenvalues  $d_\ell^-$  and  $d_\ell^+$  for  $D_+ D_-$  and  $D_- D_+$  on  $R_{\ell, m}$  are

$$d_\ell^+ = \frac{((\ell+1)^2 - \ell_0^2)(\ell_1^2 - (\ell+1)^2)}{(4(\ell+1)^2 - 1)(\ell+1)^2}, \quad d_\ell^- = \frac{(\ell^2 - \ell_0^2)(\ell_1^2 - \ell^2)}{(4\ell^2 - 1)\ell^2}$$

for  $\ell \neq \ell_0$  in the case of  $d_\ell^-$ , where  $d_{\ell_0}^- = 0$ . Now  $M$  is singular, so  $\ell_1$  is real because  $\ell_1 - \ell_0$  is an integer and  $\ell_0$  is real, and  $|\ell_1| - \ell_0$  is a positive integer by assumption, so we get that  $d_\ell^+ = 0$  only for  $\ell = |\ell_1| - 1 = \ell_0 + (|\ell_1| - \ell_0) - 1 \in \{\ell_0, \ell_0 + 1, \dots\}$  and  $d_\ell^- = 0$  only for  $\ell = \ell_0$  and  $\ell = |\ell_1| = \ell_0 + (|\ell_1| - \ell_0) \in \{\ell_0 + 1, \ell_0 + 2, \dots\}$ . Hence the maps

$$\begin{aligned}
 D_- D_+ : R_{\ell, m} &\rightarrow R_{\ell, m} && \text{for } \ell \neq |\ell_1| - 1, \\
 D_+ D_- : R_{\ell, m} &\rightarrow R_{\ell, m} && \text{for } \ell \neq \ell_0, |\ell_1| \text{ and } m \neq \pm \ell
 \end{aligned}$$

have diagonals without zeros in their Schur decomposition, so they are invertible, and thus the maps

$$\begin{aligned}
 D_+ : R_{\ell, m} &\rightarrow R_{\ell+1, m} && \text{for } \ell \neq |\ell_1| - 1, \\
 D_- : R_{\ell, m} &\rightarrow R_{\ell-1, m} && \text{for } \ell \neq \ell_0, |\ell_1| \text{ and } m \neq \pm \ell
 \end{aligned}$$

are injective. Since  $E_+$  and  $E_-$  are isomorphisms we already have that  $R_{\ell, m}$  and  $R_{\ell, m'}$  have the same dimension as we have already seen a few times, and therefore the above implies that the subspaces  $R_{\ell, m}$  for  $\ell = \ell_0, \dots, |\ell_1| - 1$  and  $m$  where it makes sense all have the same dimension, and that the subspaces  $R_{\ell, m}$  for  $\ell \geq |\ell_1|$  and  $m$  where it makes sense all have the same dimension.  $\square$



Since  $E_+$ ,  $E_-$ ,  $D_+$ , and  $D_-$  commute with  $\Delta_1$  and  $\Delta_2$  by Lemma 1.10, we get by the same method as in the proof of Lemma 1.18 that:

**Lemma 1.28.** *In a singular module  $M \in C(\lambda_1, \lambda_2)$  all the operators  $(\Delta_i)_{\ell,m}$  with  $\ell_0 \leq \ell \leq |\ell_1| - 1$  are similar to  $(\Delta_i)_{\ell_0, m_0}$  via the same matrix for  $i = 1, 2$ , and the corresponding operators  $(\Delta_i)_{\ell,m}$  with  $\ell \geq |\ell_1|$  are similar to  $(\Delta_i)_{|\ell_1|, m_0}$  via the same matrix for  $i = 1, 2$ .*

**Lemma 1.29.** *Define in the singular module  $M \in C(\lambda_1, \lambda_2)$  a linear operator  $\delta$  by*

$$\delta = \frac{1}{\ell_0^2 - \ell_1^2} \left( \Delta_2 + \frac{\Delta_1^2}{\ell_0^2} - (\ell_0^2 - 1) \text{id} \right) \quad (1.60)$$

*for  $\ell_0 \neq 0$ . Then on the subspaces  $R_{\ell,m}$  for which  $\ell_0 \leq \ell \leq |\ell_1| - 1$  the operator  $\delta$  is zero, and on the remaining subspaces  $R_{\ell,m}$   $\delta$  is nilpotent.*

*Proof.* That  $\delta = 0$  on the subspaces  $R_{\ell,m}$  for which  $\ell_0 \leq \ell \leq |\ell_1| - 1$  follows by Lemma 1.28 and the same argument as in the proof of Lemma 1.19. And since  $\Delta_1$  and  $\Delta_2$  only have one eigenvalues  $\lambda_1 = -i\ell_0\ell_1$  and  $\lambda_2 = \ell_0^2 + \ell_1^2 - 1$  respectively we see that  $\delta$  only has the eigenvalues

$$\begin{aligned} \frac{1}{\ell_0^2 - \ell_1^2} \left( \lambda_2 + \frac{\lambda_1^2}{\ell_0^2} - (\ell_0^2 - 1) \text{id} \right) &= \frac{1}{\ell_0^2 - \ell_1^2} \left( \ell_0^2 + \ell_1^2 - 1 - \ell_1^2 - (\ell_0^2 - 1) \text{id} \right) \\ &= 0, \end{aligned}$$

so by Cayley-Hamilton Theorem  $\delta$  is nilpotent in general which gives the result.  $\square$

**Remark 1.30.** Using eq. (1.49) the argument showing that  $\delta$  is nilpotent can also be used to show that

$$\begin{aligned} D_- D_+ : R_{|\ell_1|-1, m} &\rightarrow R_{|\ell_1|-1, m} \\ D_+ D_- : R_{|\ell_1|, m} &\rightarrow R_{|\ell_1|, m} \end{aligned}$$

for  $m \neq \pm \ell_1$  in the latter case, are both nilpotent.  $\triangle$

**Remark 1.31.** Comparing Lemma 1.29 with Lemma 1.19, we see that this property of  $\delta$  is one of the key differences between the singular and non-singular cases.  $\triangle$

We thus want to understand the map  $\delta$  better for which we will need the lemma:

**Lemma 1.32.** *We have that*

$$\begin{aligned} \delta D_+ \xi &= 0 \quad \text{for } \xi \in R_{|\ell_1|-1, m}, \\ D_- \delta \xi &= 0 \quad \text{for } \xi \in R_{|\ell_1|, m}. \end{aligned} \quad (1.61)$$

*Proof.* The operator  $\delta$  is a linear combination of  $\Delta_1^2$ ,  $\Delta_2$ , and  $\text{id}$ , so it commutes with  $D_+$  and  $D_-$ . Hence since  $\delta = 0$  on  $R_{|\ell_1|-1,m}$  by Lemma 1.29  $\delta D_+ \xi = D_+ \delta \xi = 0$  for  $\xi \in R_{|\ell_1|-1,m}$  and  $D_- \delta \xi = \delta D_- \xi = 0$  for  $\xi \in R_{|\ell_1|,m}$  since then  $D_- \xi \in R_{|\ell_1|-1,m}$ .  $\square$

Now we are ready to begin showing the equivalence of the singular category  $C(\lambda_1, \lambda_2)$  and the category  $S_0$ . First we will show that we from objects  $M \in C(\lambda_1, \lambda_2)$  can construct objects  $\tilde{A} \in S_0$ .

Let  $M \in C(\lambda_1, \lambda_2)$  be a singular module, and consider the maps

$$\begin{aligned} D_+ : R_{|\ell_1|-1,m_0} &\rightarrow R_{|\ell_1|,m_0}, & D_- : R_{|\ell_1|,m_0} &\rightarrow R_{|\ell_1|-1,m_0}, \\ \delta : R_{|\ell_1|-1,m_0} &\rightarrow 0, & \delta : R_{|\ell_1|,m_0} &\rightarrow R_{|\ell_1|,m_0} \end{aligned}$$

for some  $m_0$  where it makes sense. Writing  $P_1 = R_{|\ell_1|-1,m_0}$  and  $P_2 = R_{|\ell_1|,m_0}$  for the finite dimensional vector spaces and  $d_+$ ,  $d_-$ ,  $\delta_1$ , and  $\delta_2$  for the maps above, we have that  $(P_1, P_2, d_+, d_-, \delta_1, \delta_2)$  is an object of the category  $S_0$ , since  $d_- \delta_2 = \delta_2 d_+ = 0$  by Lemma 1.32,  $\delta_2$  is nilpotent by Lemma 1.29, and  $d_+ d_-$  is nilpotent by Remark 1.30.

Likewise from a morphism  $\Gamma : M \rightarrow M'$  for  $M, M' \in C(\lambda_1, \lambda_2)$  we get a corresponding morphism  $\gamma = (\gamma_1, \gamma_2)$  from  $\tilde{A}$  to  $\tilde{A}'$ . As in the proof of Theorem 1.23 we have that  $\Gamma R_{\ell,m} \subset R'_{\ell,m}$  for all  $\ell$  and  $m$  where it makes sense, so setting  $\gamma_1 = (\Gamma)_{|\ell_1|-1,m_0} : P_1 \rightarrow P'_1$  and  $\gamma_2 = (\Gamma)_{|\ell_1|,m_0} : P_2 \rightarrow P'_2$ , we want to show that  $\gamma = (\gamma_1, \gamma_2)$  gives a morphism in  $S_0$ .

Show that the diagram eq. (1.59) commutes

Now we want to show that to each object  $\tilde{A} = (P_1, P_2, d_+, d_-, \delta_1, \delta_2) \in S_0$  there is a corresponding module singular module  $M \in C(\lambda_1, \lambda_2)$  with  $\lambda_1 = -i\ell_0\ell_1$  and  $\lambda_2 = \ell_0^2 + \ell_1^2 - 1$ . Let such an  $\tilde{A}$  be given, then we want to construct  $M$ . First choose an  $m_0$  such that it makes sense to write  $R_{|\ell_1|-1,m_0}$  and put  $R_{|\ell_1|-1,m} = P_1$  and  $R_{|\ell_1|,m_0} = P_2$ , and define

$$D_+ := d_+ : R_{|\ell_1|-1,m} \rightarrow R_{|\ell_1|,m}, \quad D_- := d_- : R_{|\ell_1|,m_0} \rightarrow R_{|\ell_1|-1,m_0}.$$

Consider the space  $M = \bigoplus_{\ell,m} R_{\ell,m}$  for  $\ell = \ell_0, \ell_0 + 1, \dots, |\ell_1| + 1, |\ell_1|, \dots$  and  $m$  where it makes sense, where  $\dim R_{\ell,m} = \dim R_{|\ell_1|-1,m} = \dim P_1$  for all  $\ell \leq |\ell_1| - 1$ , and  $\dim R_{\ell,m} = \dim R_{|\ell_1|,m} = \dim P_2$  for all  $\ell \geq |\ell_1|$ . We want to construct the maps  $E_+$ ,  $E_-$ ,  $D_+$ ,  $D_-$ , and  $D_0$ . First take  $E_+ : R_{\ell,m} \rightarrow R_{\ell,m+1}$  to be any isomorphism and  $E_+ : R_{\ell,\ell} \rightarrow 0$ , and then take  $E_- : R_{\ell,m} \rightarrow R_{\ell,m-1}$  for  $m \neq -\ell$  to be the inverse isomorphism of the corresponding  $E_+$  and take  $E_- : R_{\ell,-\ell} \rightarrow 0$ . Likewise take  $D_+ : R_{\ell,m_0} \rightarrow R_{\ell+1,m_0}$  for  $\ell \neq |\ell_1| - 1$  to be any isomorphism, and for  $\ell = |\ell_1| - 1$  we already have defined  $D_+$  on  $R_{|\ell_1|-1,m_0}$ . We expand to all  $R_{\ell,m}$  and  $R_{|\ell_1|-1,m}$  by using  $E_+$  either on the left or on the right such that

$$\begin{array}{ccc} R_{\ell,m+1} & \xrightarrow{D_+} & R_{\ell+1,m+1} \\ E_+ \uparrow & & \uparrow E_+ \\ R_{\ell,m} & \xrightarrow{D_+} & R_{\ell+1,m} \end{array}$$

commutes for  $-\ell \leq m < \ell$ , i.e.  $(D_+)_{\ell, m+1} = (E_+)_{\ell, m+1}^{-1} (D_+)_{\ell, m} (E_+)_{\ell+1, m}$ .

Now before constructing  $D_0$  and  $D_-$  we want to construct  $\Delta_1$  and  $\Delta_2$ . Since  $D_-$  and  $\delta = \delta_2$  are defined on  $R_{|\ell_1|, m_0}$ , we can define

$$\Delta_2 \xi = (\ell_1^2 + \ell_0^2 - 1)\xi + \ell_1^2 \frac{4\ell_1^2 - 1}{\ell_1^2 - \ell_0^2} D_+ D_- \xi + \ell_0^2 \delta \xi$$

for  $\xi \in R_{|\ell_1|, m_0}$ . Since  $D_+ D_- = d_+ d_-$  and  $\delta = \delta_2$  on  $R_{|\ell_1|, m_0}$  are nilpotent by assumption, we see that  $\Delta_2$  has only one eigenvalue and that is  $\ell_1^2 + \ell_0^2 - 1$ . Now since we still want the equation

$$\delta = \frac{1}{\ell_0^2 - \ell_1^2} \left( \Delta_2 + \frac{\Delta_1^2}{\ell_0^2} - (\ell_0^2 - 1) \text{id} \right)$$

to hold true, we define

$$\begin{aligned} \Delta_1^2 \xi &= \ell_0^2 \left( (\ell_0^2 - 1)\xi - \Delta_2 \xi + (\ell_0^2 - \ell_1^2) \delta \xi \right) \\ &= \ell_0^2 \left( (\ell_0^2 - 1)\xi - (\ell_1^2 + \ell_0^2 - 1)\xi - \ell_1^2 \frac{4\ell_1^2 - 1}{\ell_1^2 - \ell_0^2} D_+ D_- \xi - \ell_0^2 \delta \xi \right. \\ &\quad \left. + (\ell_0^2 - \ell_1^2) \delta \xi \right) \\ &= \ell_0^2 \left( -\ell_1^2 \xi - \ell_1^2 \frac{4\ell_1^2 - 1}{\ell_1^2 - \ell_0^2} D_+ D_- \xi - \ell_1^2 \delta \xi \right) \\ &= -\ell_0^2 \ell_1^2 \left( \xi + \frac{4\ell_1^2 - 1}{\ell_1^2 - \ell_0^2} D_+ D_- \xi + \delta \xi \right) \end{aligned}$$

for  $\xi \in R_{|\ell_1|, m_0}$ . Again since  $D_+ D_-$  and  $\delta$  are nilpotent on  $R_{|\ell_1|, m_0}$  by assumption, we see that  $\Delta_1^2$  only has the eigenvalue  $-\ell_0^2 \ell_1^2$ , which is non-zero, and thus  $\Delta_1$ 's only eigenvalue is non-zero. Therefore Appendix B.2 implies that the above determines  $\Delta_1$  uniquely on  $R_{|\ell_1|, m_0}$ .

For  $\ell \geq |\ell_1|$  consider the map  $J_{r,s} = (E_+)^s (D_+)^r$  as earlier, where  $(E_+)^{-1} = E_-$ , and put

$$(\Delta_i)_{\ell, m} = J_{\ell-|\ell_1|, m-m_0} (\Delta_i)_{|\ell_1|, m_0} J_{\ell-|\ell_1|, m-m_0}^{-1}$$

for  $i = 1, 2$ , such that we have expanded  $\Delta_1$  and  $\Delta_2$  to all  $R_{\ell, m}$ . Now following the formulae of eqs. (1.54) and (1.55) we put

$$\begin{aligned} (D_0)_{\ell, m} &= -\frac{1}{\ell(\ell+1)} (\Delta_1)_{\ell, m}, \\ (D_+ D_-)_{\ell, m} &= \frac{1}{4\ell^2 - 1} \left( (\Delta_2)_{\ell, m} - (\ell^2 - 1) (\text{id})_{\ell, m} + \frac{(\Delta_1)_{\ell, m}^2}{\ell^2} \right). \end{aligned}$$

Since  $(D_+)_{\ell, m}$  is an isomorphism for all  $\ell \geq |\ell_1|$ , we get that this determines  $(D_-)_{\ell, m}$  for  $\ell \geq |\ell_1| + 1$ , and we already have  $(D_-)_{|\ell_1|, m}$  given.

Similarly we can find  $D_0$  and  $D_-$  for  $\ell_0 \leq \ell < |\ell_1|$ , and thus we get the linear maps  $E_+$ ,  $E_-$ ,  $D_+$ ,  $D_-$ , and  $D_0$  defined everywhere where it makes sense, and actually we get by our standard construction operators  $H_+$ ,  $H_-$ ,  $H_3$ ,  $F_+$ ,  $F_-$ , and  $F_3$ , which gives a representation, i.e. we indeed get a singular module  $M \in C(\lambda_1, \lambda_2)$  for each object  $\tilde{A} = (P_1, P_2, d_+, d_-, \delta_1, \delta_2) \in S_0$ . To see this we need only check that the equations of eq. (1.35) hold. We see that for  $\ell \neq 0$

$$\begin{aligned} \ell(D_+)_{\ell,m}(D_0)_{\ell,m} &= -\frac{\ell}{\ell(\ell+1)}(D_+)_{\ell,m}(\Delta_1)_{\ell,m} \\ &= -\frac{1}{\ell+1}(\Delta_1)_{\ell+1,m}(D_+)_{\ell,m} \\ &= -(\ell+2)\frac{1}{(\ell+1)(\ell+2)}(\Delta_1)_{\ell+1,m}(D_+)_{\ell,m} \\ &= (\ell+2)(D_0)_{\ell+1,m}(D_+)_{\ell,m}, \end{aligned}$$

which gives us the first equation, and the rest can be checked similarly. Additionally it can be checked like in Theorem 1.23 that there is a correspondence between the morphisms, and thus altogether we get the theorem:

**Theorem 1.33.** *The singular category  $C(\lambda_1, \lambda_2)$  is equivalent to the category  $S_0$ .*

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# Appendix A

## Calculations

Throughout the paper there are situations where we need to do some straightforward but rather long calculations, so to clean up the exposition somewhat we will collect most of these calculations in this appendix and then just use the results in the paper.

### A.1 Bases of $V(2) \otimes V(n)$

We want to describe the  $s_k$ 's of eq. (1.17) more explicitly. We have that  $s_0 = w_0 \otimes v_0$  and  $s_k = \frac{1}{k!} y^k . s_0$ , and we note that if  $n > 0$  then

$$\begin{aligned} s_1 &= y.(w_0 \otimes v_0) = y.w_0 \otimes v_0 + w_0 \otimes y.v_0 \\ &= w_1 \otimes v_0 + w_0 \otimes v_1 \end{aligned}$$

and

$$\begin{aligned} s_2 &= \frac{1}{2} y . s_1 \\ &= \frac{1}{2} y . w_1 \otimes v_0 + \frac{1}{2} w_1 \otimes y . v_0 + \frac{1}{2} y . w_0 \otimes v_1 + w_0 \otimes \frac{1}{2} y . v_1 \\ &= w_2 \otimes v_0 + \frac{1}{2} w_1 \otimes v_1 + \frac{1}{2} w_1 \otimes v_1 + w_0 \otimes v_2 \\ &= w_2 \otimes v_0 + w_1 \otimes v_1 + w_0 \otimes v_2. \end{aligned}$$

Inductively we see that

$$s_k = w_2 \otimes v_{k-2} + w_1 \otimes v_{k-1} + w_0 \otimes v_k$$

for  $k \leq n$ , since the base case holds and given the equality for  $k < n$  we get

$$\begin{aligned} s_{k+1} &= \frac{1}{k+1} y . s_k \\ &= w_2 \otimes \frac{1}{k+1} y . v_{k-2} + \frac{1}{k+1} y . w_1 \otimes v_{k-1} + w_1 \otimes \frac{1}{k+1} y . v_{k-1} \\ &\quad + \frac{1}{k+1} y . w_0 \otimes v_k + w_0 \otimes \frac{1}{k+1} y . v_k \end{aligned}$$

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$$\begin{aligned}
&= \frac{k-1}{k+1}w_2 \otimes v_{k-1} + \frac{2}{k+1}w_2 \otimes v_{k-1} + \frac{k}{k+1}w_1 \otimes v_k + \frac{1}{k+1}w_1 \otimes v_k \\
&\quad + w_0 \otimes v_{k+1} \\
&= w_2 \otimes v_{k-1} + w_1 \otimes v_k + w_0 \otimes v_{k+1}.
\end{aligned}$$

We likewise see that for  $k = n+1$  the last term vanishes, so we have  $s_{k+1} = w_2 \otimes v_{n-1} + w_1 \otimes v_n$ , and for  $k = n+2$  the two last terms vanish, so we get  $s_{k+2} = w_2 \otimes v_n$ . Thus altogether we get the description in eq. (1.18).

Suppose now that  $n \geq 1$ . We want to describe the  $t_k$ 's of eq. (1.19) more explicitly. We have that  $t_0 = w_0 \otimes v_1 - \frac{n}{2}w_1 \otimes v_0$  and  $t_k = \frac{1}{k!}y^k.t_0$ . We see that

$$\begin{aligned}
t_1 &= y.(w_0 \otimes v_1 - \frac{n}{2}w_1 \otimes v_0) \\
&= y.w_0 \otimes v_1 + w_0 \otimes y.v_1 - \frac{n}{2}y.w_1 \otimes v_0 + \frac{n}{2}w_1 \otimes y.v_0 \\
&= w_1 \otimes v_1 + 2w_0 \otimes v_2 - nw_2 \otimes v_0 - \frac{n}{2}w_1 \otimes v_1 \\
&= 2w_0 \otimes v_2 - \frac{n-2}{2}w_1 \otimes v_1 - nw_2 \otimes v_0,
\end{aligned}$$

and inductively we get that

$$t_k = (k+1)w_0 \otimes v_{k+1} - \frac{n-2k}{2}w_1 \otimes v_k + (k-1-n)w_2 \otimes v_{k-1}$$

for  $1 \leq k \leq n-1$ , since the base case holds and given the equality for  $k < n-1$  we get

$$\begin{aligned}
t_{k+1} &= \frac{1}{k+1}y.t_k \\
&= y.w_0 \otimes v_{k+1} + w_0 \otimes y.v_{k+1} - \frac{n-2k}{2(k+1)}y.w_1 \otimes v_k \\
&\quad - \frac{n-2k}{2(k+1)}w_1 \otimes y.v_k + \frac{k-1-n}{k+1}w_2 \otimes y.v_{k-1} \\
&= w_1 \otimes v_{k+1} + (k+2)w_0 \otimes v_{k+2} - \frac{n-2k}{k+1}w_2 \otimes v_k \\
&\quad - \frac{n-2k}{2}w_1 \otimes v_{k+1} + \frac{(k-1-n)k}{k+1}w_2 \otimes v_k \\
&= (k+2)w_0 \otimes v_{k+2} - \frac{n-2(k+1)}{2}w_1 \otimes v_{k+1} \\
&\quad + \left( \frac{k^2 - k - nk - n + 2k}{k+1} \right) w_2 \otimes v_k \\
&= (k+2)w_0 \otimes v_{k+2} - \frac{n-2(k+1)}{2}w_1 \otimes v_{k+1} + (k-n)w_2 \otimes v_k,
\end{aligned}$$

where we in the last equality use that  $(k+1)(k-n) = k^2 - nk + k - n = k^2 - k - nk - n + 2k$ . We likewise see that for  $k = n$  the first term vanishes so

$$t_n = \frac{n}{2}w_1 \otimes v_n - w_2 \otimes v_{n-1}.$$



Thus we altogether get the description in eq. (1.20).

Suppose now that  $n \geq 2$ . We want to describe the  $u_k$ 's of eq. (1.21) more explicitly. We have that

$$u_0 := w_0 \otimes v_2 - \frac{n-1}{2} w_1 \otimes v_1 + \frac{n(n-1)}{2} w_2 \otimes v_0$$

and  $u_k = \frac{1}{k!} y^k \cdot u_0$ . We see inductively that

$$\begin{aligned} u_k = & \frac{(k+1)(k+2)}{2} w_0 \otimes v_{k+2} - \frac{(k+1)(n-k-1)}{2} w_1 \otimes v_{k+1} \\ & + \frac{(n-k)(n-k-1)}{2} w_2 \otimes v_k \end{aligned}$$

for  $0 \leq k \leq n-2$ , since the base case holds and given the equality for  $k < n-2$  we get

$$\begin{aligned} u_{k+1} &= \frac{1}{k+1} y \cdot u_k \\ &= \frac{k+2}{2} y \cdot w_0 \otimes v_{k+2} + \frac{k+2}{2} w_0 \otimes y \cdot v_{k+2} \\ &\quad - \frac{n-k-1}{2} y \cdot w_1 \otimes v_{k+1} - \frac{n-k-1}{2} w_1 \otimes y \cdot v_{k+1} \\ &\quad + \frac{(n-k)(n-k-1)}{2(k+1)} w_2 \otimes y \cdot v_k \\ &= \frac{k+2}{2} w_1 \otimes v_{k+2} + \frac{(k+2)(k+3)}{2} w_0 \otimes v_{k+3} \\ &\quad - (n-k-1) w_2 \otimes v_{k+1} - \frac{(n-k-1)(k+2)}{2} w_1 \otimes v_{k+2} \\ &\quad + \frac{(n-k)(n-k-1)}{2} w_2 \otimes v_{k+1} \\ &= \frac{(k+2)(k+3)}{2} w_0 \otimes v_{k+3} \\ &\quad - \frac{(n-k-1)(k+2) - (k+2)}{2} w_1 \otimes v_{k+2} \\ &\quad + \frac{(n-k)(n-k-1) - 2(n-k-1)}{2} w_2 \otimes v_{k+1} \\ &= \frac{(k+2)(k+3)}{2} w_0 \otimes v_{k+3} \\ &\quad - \frac{(k+2)(n-k-2)}{2} w_1 \otimes v_{k+2} \\ &\quad + \frac{(n-k-1)(n-k-2)}{2} w_2 \otimes v_{k+1}. \end{aligned}$$

Thus we altogether get the description in eq. (1.22).

## A.2 Finding $w_1 \otimes v_k$

Using the bases  $(s_k \mid 0 \leq k \leq n+2)$  of eq. (1.18),  $(t_k \mid 0 \leq k \leq n)$  of eq. (1.20), and  $(u_k \mid 0 \leq k \leq n-2)$  of eq. (1.22), we see that

$$\begin{aligned}
 & \frac{2(k+1)(n+1-k)}{(n+1)(n+2)} s_{k+1} - \frac{2(n-2k)}{n(n+2)} t_k - \frac{4}{n(n+1)} u_{k-1} \\
 &= \frac{2(k+1)(n+1-k)}{(n+1)(n+2)} \left( w_0 \otimes v_{k+1} + w_1 \otimes v_k + w_2 \otimes v_{k-1} \right) \\
 & \quad - \frac{2(n-2k)}{n(n+2)} \left( (k+1)w_0 \otimes v_{k+1} - \frac{n-2k}{2} w_1 \otimes v_k \right. \\
 & \quad \left. + (k-1-n)w_2 \otimes v_{k-1} \right) \\
 & \quad - \frac{4}{n(n+1)} \left( \frac{k(k+1)}{2} w_0 \otimes v_{k+1} - \frac{k(n-k)}{2} w_1 \otimes v_k \right. \\
 & \quad \left. + \frac{(n-k+1)(n-k)}{2} w_2 \otimes v_{k-1} \right) \\
 &= \frac{\left( 2(k+1)(n+1-k)n - 2(n-2k)(k+1)(n+1) - 2k(k+1)(n+2) \right)}{n(n+1)(n+2)} w_0 \otimes v_{k+1} \\
 & \quad + \frac{\left( 2(k+1)(n+1-k)n + (n-2k)(n-2k)(n+1) + 2k(n-k)(n+2) \right)}{n(n+1)(n+2)} w_1 \otimes v_k \\
 & \quad + \frac{\left( 2(k+1)(n+1-k)n - 2(n-2k)(k-1-n)(n+1) - 2(n-k+1)(n-k)(n+2) \right)}{n(n+1)(n+2)} w_2 \otimes v_{k-1} \\
 &= 2(k+1) \frac{(n+1-k)n - (n-2k)(n+1) - k(n+2)}{n(n+1)(n+2)} w_0 \otimes v_{k+1} \\
 & \quad + \frac{\left( 2(k+1)(n+1-k)n + (n-2k)(n-2k)(n+1) + 2k(n-k)(n+2) \right)}{n(n+1)(n+2)} w_1 \otimes v_k \\
 & \quad + 2(n+1-k) \frac{(k+1)n + (n-2k)(n+1) - (n-k)(n+2)}{n(n+1)(n+2)} w_2 \otimes v_{k-1}.
 \end{aligned}$$

Now we note that

$$\begin{aligned}
 & (n+1-k)n - (n-2k)(n+1) - k(n+2) \\
 &= n \left( (n+1-k) - (n-2k) - k \right) - (n-2k) - 2k \\
 &= n - (n-2k) - 2k = 0,
 \end{aligned}$$

and

$$\begin{aligned} & (k+1)n + (n-2k)(n+1) - (n-k)(n+2) \\ &= n\left((k+1) + (n-2k) - (n-k)\right) + (n-2k) - 2(n-k) \\ &= n + n - 2k - 2n + 2k = 0, \end{aligned}$$

while

$$\begin{aligned} & 2(k+1)(n+1-k)n + (n-2k)(n-2k)(n+1) + 2k(n-k)(n+2) \\ &= n\left(2(k+1)(n+1-k) + (n-2k)(n+1) + 2k(n-k)\right) \\ &\quad - 2k(n-2k)(n+1) + 4k(n-k) \\ &= n\left(2(k+1)(n+1-k) + (n-2k)(n+1) + 2k(n-k)\right) \\ &\quad - 2kn(n-2k) - 2k(n-2k) + 4k(n-k) \\ &= n\left(2(k+1)(n+1-k) + (n-2k)(n+1) + 2k(n-k)\right) \\ &\quad - 2kn(n-2k) + 2kn \\ &= n\left(2(k+1)(n+1-k) + (n-2k)(n+1) + 2k(n-k) - 2k(n-2k) \right. \\ &\quad \left. + 2k\right), \end{aligned}$$

where

$$\begin{aligned} & 2(k+1)(n+1-k) + (n-2k)(n+1) + 2k(n-k) - 2k(n-2k) + 2k \\ &= (n+1)\left(2(k+1) + (n-2k)\right) - 2k(k+1) \\ &\quad + 2k\left((n-k) - (n-2k) + 1\right) \\ &= (n+1)(n+2) - 2k(k+1) + 2k(k+1) \\ &= (n+1)(n+2), \end{aligned}$$

so

$$\begin{aligned} & 2(k+1)(n+1-k)n + (n-2k)(n-2k)(n+1) + 2k(n-k)(n+2) \\ &= n(n+1)(n+2). \end{aligned}$$

Thus we see that

$$\begin{aligned} & \frac{2(k+1)(n+1-k)}{(n+1)(n+2)}s_{k+1} - \frac{2(n-2k)}{n(n+2)}t_k - \frac{4}{n(n+1)}u_{k-1} \\ &= 0 + \frac{n(n+1)(n+2)}{n(n+1)(n+2)}w_1 \otimes v_k + 0 \\ &= w_1 \otimes v_k \end{aligned}$$

I will probably remove some of this and just say that algebraic manipulation shows that ...

giving us eq. (1.23).

Likewise for  $n \geq 1$ , we get that

$$\begin{aligned} \frac{2}{n+2}(s_1 - t_0) &= \frac{2}{n+2} \left( w_0 \otimes v_1 + w_1 \otimes v_0 - w_0 \otimes v_1 + \frac{n}{2} w_1 \otimes v_0 \right) \\ &= \frac{2}{n+2} \frac{n+2}{2} w_1 \otimes v_0 \\ &= w_1 \otimes v_0 \end{aligned}$$

and

$$\begin{aligned} \frac{2}{n+2}(s_{n+1} + t_n) &= \frac{2}{n+2} \left( w_2 \otimes v_{n+1} + w_1 \otimes v_n + \frac{n}{2} w_1 \otimes v_n - w_2 \otimes v_{n-1} \right) \\ &= \frac{2}{n+2} \frac{n+2}{2} w_1 \otimes v_n \\ &= w_1 \otimes v_n \end{aligned}$$

giving us eq. (1.24).

### A.3 Inner products in $V(2) \otimes V(n)$

Given  $s_0 = w_0 \otimes v_0$ ,  $t_0 = w_0 \otimes v_1 - \frac{n}{2} w_1 \otimes v_0$ , and  $u_0 = w_0 \otimes v_2 - \frac{n-1}{2} w_1 \otimes v_1 + \frac{n(n-1)}{2} w_2 \otimes v_0$  from eq. (1.18), eq. (1.20), and eq. (1.22), we want to find  $\langle s_0, s_0 \rangle$ ,  $\langle t_0, t_0 \rangle$ , and  $\langle u_0, u_0 \rangle$  using the inner products of eq. (1.25) and eq. (1.26). Noting that all terms with  $\langle w_i \otimes v_j, w_k \otimes v_\ell \rangle$  with  $i \neq k$  or  $j \neq \ell$  vanish since then either  $\langle w_i, w_k \rangle = 0$  or  $\langle v_j, v_\ell \rangle = 0$ , we see that

$$\begin{aligned} \langle u_0, u_0 \rangle &= \left\langle w_0 \otimes v_2 - \frac{n-1}{2} w_1 \otimes v_1 + \frac{n(n-1)}{2} w_2 \otimes v_0, \right. \\ &\quad \left. w_0 \otimes v_2 - \frac{n-1}{2} w_1 \otimes v_1 + \frac{n(n-1)}{2} w_2 \otimes v_0 \right\rangle \\ &= \langle w_0 \otimes v_2, w_0 \otimes v_2 \rangle + \frac{(n-1)^2}{4} \langle w_1 \otimes v_1, w_1 \otimes v_1 \rangle \\ &\quad + \frac{n^2(n-1)^2}{4} \langle w_2 \otimes v_0, w_2 \otimes v_0 \rangle \\ &= \langle w_0, w_0 \rangle \cdot \langle v_2, v_2 \rangle + \frac{(n-1)^2}{4} \langle w_1, w_1 \rangle \cdot \langle v_1, v_1 \rangle \\ &\quad + \frac{n^2(n-1)^2}{4} \langle w_2, w_2 \rangle \cdot \langle v_0, v_0 \rangle \\ &= \binom{2}{0} \cdot \binom{n}{2} + \frac{(n-1)^2}{4} \binom{2}{1} \binom{n}{1} + \frac{n^2(n-1)^2}{4} \binom{2}{2} \cdot \binom{n}{0} \\ &= \frac{n(n-1)}{2} + \frac{n(n-1)^2}{2} + \frac{n^2(n-1)^2}{4} \\ &= n(n-1) \frac{2 + 2(n-1) + n(n-1)}{4} \end{aligned}$$

$$= n(n-1) \frac{n^2 + n}{4} = \frac{n^2(n+1)(n-1)}{4}.$$

Similarly we get that

$$\langle s_0, s_0 \rangle = 1$$

and

$$\langle t_0, t_0 \rangle = \frac{n(n+2)}{2}.$$

Thus we indeed get eq. (1.27).

Now we want to show that we also have eq. (1.28). Working with the basis  $(v_0, \dots, v_n)$  of  $V(n)$  from eq. (1.5) first note that by eq. (1.25)  $\langle h.v_j, v_k \rangle = (n-2j)\delta_{jk}\binom{n}{k} = \langle v_j, h.v_k \rangle$ . Also for  $j \neq k+1$ , we have  $\langle x.v_j, v_k \rangle = 0 = \langle v_j, y.v_k \rangle$ , while  $\langle x.v_{k+1}, v_k \rangle = (n-k)\binom{n}{k} = (k+1)\binom{n}{k+1} = \langle v_{k+1}, y.v_k \rangle$ , so  $\langle x.v_j, v_k \rangle = \langle v_j, y.v_k \rangle$  for all  $j$ . By symmetry also  $\langle y.v_j, v_k \rangle = \langle v_j, x.v_k \rangle$ , and so for all  $v, w \in V(n)$

$$\langle X.v, w \rangle = \langle v, X^H.w \rangle \quad \text{for all } X \in \mathfrak{sl}(2, \mathbf{C}), \quad (\text{A.1})$$

since  $h^H = h$ ,  $x^H = y$ , and  $y^H = x$  by the definitions in eq. (1.1). The property of eq. (A.1) determines the inner product up to a positive factor. To see this note that  $h^H = h$ , so by linear algebra its eigenvalues are in  $\mathbf{R}$  and distinct eigenspaces of  $h$  are orthogonal. Therefore since  $(v_0, \dots, v_n)$  is a basis with eigenvectors of  $h$  from distinct eigenspaces the property eq. (A.1) implies that  $\langle v_k, v_j \rangle = 0$  for  $j \neq k$ . Also the property implies that  $\langle y.v_k, v_{k+1} \rangle = \langle v_k, x.v_{k+1} \rangle$ , so

$$(k+1)\langle v_{k+1}, v_{k+1} \rangle = \langle y.v_k, v_{k+1} \rangle = \langle v_k, x.v_{k+1} \rangle = (n-k)\langle v_k, v_k \rangle,$$

and inductively we see that  $\langle v_k, v_k \rangle$  is determined by  $\langle v_0, v_0 \rangle$ , so by the above the inner product is determined by  $\langle v_0, v_0 \rangle$ . More precisely since

$$\frac{n-k}{k+1} \binom{n}{k} = \binom{n}{k+1}$$

we get inductively that

$$\langle v_k, v_k \rangle = \langle v_0, v_0 \rangle \binom{n}{k}.$$

Since the inner products of  $V(2)$  and  $V(n)$  satisfy eq. (A.1), it is clear by eq. (1.26) that also the inner product of  $V(2) \otimes V(n)$  satisfy eq. (A.1), and so clearly also the restrictions to the submodules corresponding to either  $V(n-2)$ ,  $V(n)$ , or  $V(n+2)$  also satisfy this property. Therefore as above we get e.g.

$$\langle s_k, s_k \rangle = \langle s_0, s_0 \rangle \binom{n+2}{k}$$

by working in  $V(n+2)$  instead of  $V(n)$ . Similarly we get the analogous results for  $t_k$  and  $u_k$  giving us eq. (1.28).

#### A.4 Finding $\bar{w}_1 \otimes \bar{v}_k$

We want to find  $\bar{w}_1 \otimes \bar{v}_k$  in terms of  $\bar{s}_k$ ,  $\bar{t}_k$ , and  $\bar{u}_k$  from eqs. (1.29) and (1.30). First we note that for  $0 < k < n$

$$\begin{aligned}
 \sqrt{2 \binom{n}{k}} \bar{w}_1 \otimes \bar{v}_k &= \sqrt{\binom{2}{1}} \bar{w}_1 \otimes \sqrt{\binom{n}{k}} \bar{v}_k \\
 &= w_1 \otimes v_k \\
 &= \frac{2(k+1)(n+1-k)}{(n+1)(n+2)} s_{k+1} - \frac{2(n-2k)}{n(n+2)} t_k - \frac{4}{n(n+1)} u_{k-1} \\
 &= \frac{2(k+1)(n+1-k)}{(n+1)(n+2)} \sqrt{\binom{n+2}{k+1}} \bar{s}_{k+1} \\
 &\quad - \frac{2(n-2k)}{n(n+2)} \sqrt{\frac{n(n+2)}{2} \binom{n}{k}} \bar{t}_k \\
 &\quad - \frac{4}{n(n+1)} \sqrt{\frac{n^2(n+1)(n-1)}{4} \binom{n-2}{k-1}} \bar{u}_{k-1} \\
 &= \frac{2(k+1)(n+1-k)}{(n+1)(n+2)} \sqrt{\binom{n+2}{k+1}} \bar{s}_{k+1} \\
 &\quad - \frac{\sqrt{2}(n-2k)}{\sqrt{n(n+2)}} \sqrt{\binom{n}{k}} \bar{t}_k \\
 &\quad - \frac{2\sqrt{(n-1)}}{\sqrt{(n+1)}} \sqrt{\binom{n-2}{k-1}} \bar{u}_{k-1}.
 \end{aligned}$$

Now since

$$\frac{\binom{n+2}{k+1}}{\binom{n}{k}} = \frac{(n+2)(n+1)}{(k+1)(n+1-k)}, \quad \frac{\binom{n-2}{k-1}}{\binom{n}{k}} = \frac{k(n-k)}{n(n-1)},$$

we see that

$$\begin{aligned}
 \bar{w}_1 \otimes \bar{v}_k &= \frac{\sqrt{2}(k+1)(n+1-k)}{(n+1)(n+2)} \sqrt{\frac{(n+2)(n+1)}{(k+1)(n+1-k)}} \bar{s}_{k+1} \\
 &\quad - \frac{(n-2k)}{\sqrt{n(n+2)}} \bar{t}_k \\
 &\quad - \frac{\sqrt{2}(n-1)}{\sqrt{(n+1)}} \sqrt{\frac{k(n-k)}{n(n-1)}} \bar{u}_{k-1}
 \end{aligned}$$

$$\begin{aligned}
&= \sqrt{\frac{2(k+1)(n+1-k)}{(n+1)(n+2)}} \bar{s}_{k+1} - \frac{(n-2k)}{\sqrt{n(n+2)}} \bar{t}_k \\
&\quad - \sqrt{\frac{2k(n-k)}{n(n+1)}} \bar{u}_{k-1}.
\end{aligned}$$

Also since eq. (1.24) is a special case of eq. (1.23) the above formula also holds for  $k \in \{0, n\}$  if we take the coefficient in front of  $\bar{u}_{k-1}$  to be 0. Thus we indeed get eq. (1.31)

### A.5 $F_3, F_+, F_-$ in terms of $E_+, E_-, D_0, D_+, D_-$

We have already seen that

$$F_3\xi = \sqrt{\ell^2 - m^2}D_-\xi - mD_0\xi - \sqrt{(\ell+1)^2 - m^2}D_+\xi$$

for  $\xi \in R_{\ell,m}$  by using eq. (1.33) and the definition of how we expanded  $D_0, D_+,$  and  $D_-$  to maps on all of  $M$ . Now we get by eqs. (1.3) and (1.11) and the commutative diagrams in eq. (1.12) that

$$\begin{aligned}
F_+\xi &= [F_3, H_+]\xi = F_3H_+\xi - H_+F_3\xi \\
&= \sqrt{(\ell+m+1)(\ell-m)}F_3E_+\xi - \sqrt{\ell^2 - m^2}H_+D_-\xi + mH_+D_0\xi \\
&\quad + \sqrt{(\ell+1)^2 - m^2}H_+D_+\xi \\
&= \sqrt{(\ell+m+1)(\ell-m)}\left(\sqrt{\ell^2 - (m+1)^2}D_-E_+\xi - (m+1)D_0E_+\xi \right. \\
&\quad \left. - \sqrt{(\ell+1)^2 - (m+1)^2}D_+E_+\xi\right) \\
&\quad - \sqrt{\ell^2 - m^2}\sqrt{((\ell-1)+m+1)((\ell-1)-m)}E_+D_-\xi \\
&\quad + m\sqrt{(\ell+m+1)(\ell-m)}E_+D_0\xi \\
&\quad + \sqrt{(\ell+1)^2 - m^2}\sqrt{((\ell+1)+m+1)((\ell+1)-m)}E_+D_+\xi \\
&= \sqrt{(\ell+m+1)(\ell-m)}\left(\sqrt{\ell^2 - (m+1)^2}D_-E_+\xi - (m+1)D_0E_+\xi \right. \\
&\quad \left. - \sqrt{(\ell+1)^2 - (m+1)^2}D_+E_+\xi\right) \\
&\quad - \sqrt{\ell^2 - m^2}\sqrt{(\ell+m)(\ell-m-1)}D_-E_+\xi \\
&\quad + m\sqrt{(\ell+m+1)(\ell-m)}D_0E_+\xi \\
&\quad + \sqrt{(\ell+1)^2 - m^2}\sqrt{(\ell+m+2)(\ell-m+1)}D_+E_+\xi
\end{aligned}$$

$$\begin{aligned}
&= \left( \sqrt{(\ell+m+1)(\ell-m)(\ell^2-(m+1)^2)} \right. \\
&\quad \left. - \sqrt{(\ell^2-m^2)(\ell+m)(\ell-m-1)} \right) D_- E_+ \xi \\
&\quad - \sqrt{(\ell+m+1)(\ell-m)} D_0 E_+ \xi \\
&\quad + \left( \sqrt{((\ell+1)^2-m^2)(\ell+m+2)(\ell-m+1)} \right. \\
&\quad \left. - \sqrt{(\ell+m+1)(\ell-m)((\ell+1)^2-(m+1)^2)} \right) D_+ E_+ \xi
\end{aligned}$$

for  $\xi \in R_{\ell,m}$  and  $-\ell+1 \leq m < \ell-1$ . In the case where  $m = -\ell$  the only problem is at the term with  $E_+ D_-$ , but this is not a problem because the term vanishes since there is  $\ell+m$  as part of the coefficient, so the formula also holds true in this case. In case  $m = \ell-1$  the only problem is at the term with  $D_- E_+$ , but here we have  $\ell^2 - (m+1)^2$  as part of the coefficient, so this term also vanishes, and the formula also holds true in this case. Finally in case  $m = \ell$  the terms with  $D_- E_+$ ,  $D_0 E_+$ ,  $D_+ E_+$ ,  $E_+ D_-$ , and  $E_+ D_0$  all cause problems, but again all of these terms vanish, so the formula still holds true in this case. Now by noting that  $\ell^2 - m^2 = (\ell+m)(\ell-m)$  and  $\ell^2 - (m+1)^2 = (\ell+m+1)(\ell-m-1)$ , we see that

$$\begin{aligned}
&\sqrt{(\ell+m+1)(\ell-m)(\ell^2-(m+1)^2)} - \sqrt{(\ell^2-m^2)(\ell+m)(\ell-m-1)} \\
&= \sqrt{(\ell+m+1)(\ell-m)(\ell+m+1)(\ell-m-1)} \\
&\quad - \sqrt{(\ell+m)(\ell-m)(\ell+m)(\ell-m-1)} \\
&= (\ell+m+1)\sqrt{(\ell-m)(\ell-m-1)} - (\ell+m)\sqrt{(\ell-m)(\ell-m-1)} \\
&= \sqrt{(\ell-m)(\ell-m-1)}
\end{aligned}$$

and similarly

$$\begin{aligned}
&\sqrt{((\ell+1)^2-m^2)(\ell+m+2)(\ell-m+1)} \\
&\quad - \sqrt{(\ell+m+1)(\ell-m)((\ell+1)^2-(m+1)^2)} \\
&= \sqrt{(\ell+m+1)(\ell+m+2)}.
\end{aligned}$$

So we get that

$$\begin{aligned}
F_+ \xi &= \sqrt{(\ell-m)(\ell-m-1)} D_- E_+ \xi - \sqrt{(\ell+m+1)(\ell-m)} D_0 E_+ \xi \\
&\quad - \sqrt{(\ell+m+1)(\ell+m+2)} D_+ E_+ \xi
\end{aligned}$$

for  $\xi \in R_{\ell,m}$  and  $-\ell \leq m \leq \ell$ .

Similarly we get that

$$\begin{aligned}
F_- \xi &= -\sqrt{(\ell+m)(\ell+m-1)} D_- E_- \xi - \sqrt{(\ell+m)(\ell-m+1)} D_0 E_- \xi \\
&\quad - \sqrt{(\ell-m+1)(\ell-m+2)} D_+ E_- \xi
\end{aligned}$$

for  $\xi \in R_{\ell,m}$ , and thus indeed we get eq. (1.34).



## A.6 Relations for $D_0, D_+, D_-$

We want to show that the formulae eq. (1.34) for the linear operators  $F_+, F_-$ , and  $F_3$  together with the formulae eqs. (1.9) and (1.11) for  $H_+, H_-$ , and  $H_3$  define a representation of  $L$ , i.e. they satisfy the commutation relations of eq. (1.3), if and only if  $D_0, D_+$ , and  $D_-$  satisfy eq. (1.35).

We claim that all commutators containing  $H$ 's already satisfy the relations. We already have by construction that all relations with only  $H$ 's satisfy the relations, so we just need to check the relations for commutators with one  $F$  and one  $H$ . We will here check that  $[H_-, F_+] = -2F_3$  as we want, and then we just note that the rest follow by similar considerations.

By eqs. (1.10) and (1.34) first note that for  $\xi \in R_{\ell, \ell}$  with  $\ell \neq 0$  we have that  $F_+\xi = \sqrt{(2\ell+1)(2\ell+2)}E_+D_+\xi \in R_{\ell+1, \ell+1}$ , so

$$\begin{aligned} H_-F_+\xi &= \sqrt{(2\ell+2) \cdot 1} \sqrt{(2\ell+1)(2\ell+2)} E_-E_+D_+\xi \\ &= 2(\ell+1)\sqrt{2\ell+1}D_+\xi \\ &= 2\ell\sqrt{2\ell+1}D_+\xi + 2\sqrt{2\ell+1}D_+\xi, \end{aligned}$$

while  $H_-\xi = \sqrt{2\ell}E_-\xi \in R_{\ell, \ell-1}$ , so

$$\begin{aligned} F_+H_-\xi &= -\sqrt{1 \cdot 2\ell}\sqrt{2\ell}D_0E_+E_-\xi + \sqrt{2\ell(2\ell+1)}\sqrt{2\ell}E_+D_+E_-\xi \\ &= -2\ell D_0\xi + 2\ell\sqrt{2\ell+1}E_+E_-\xi \\ &= -2\ell D_0\xi + 2\ell\sqrt{2\ell+1}D_+\xi. \end{aligned}$$

Therefore we get that

$$\begin{aligned} [H_-, F_+]\xi &= H_-F_+\xi - F_+H_-\xi \\ &= 2\ell D_0\xi + 2\sqrt{2\ell+1}D_+\xi. \end{aligned}$$

Also

$$\begin{aligned} -2F_3\xi &= 2\ell D_0\xi + 2\sqrt{(\ell+1)^2 - \ell^2}D_+\xi \\ &= 2\ell D_0\xi + 2\sqrt{2\ell+1}D_+\xi, \end{aligned}$$

and thus indeed  $[H_-, F_+]\xi = -F_3\xi$  for  $\xi \in R_{\ell, \ell}$ . Since  $D_0$  and  $D_+$  commute with  $E_-$  for  $m \neq -\ell$  this also gives the result for all  $\xi \in R_{\ell, m}$ ,  $m \in \{-\ell, -\ell+1, \dots, \ell-1, \ell\}$ , by applying the isomorphism  $E_-^{\ell-m}: R_{\ell, \ell} \rightarrow R_{\ell, m}$ .

Now we claim that  $\ell D_+D_0\xi = (\ell+2)D_0D_+\xi$  corresponds to the relation  $[F_+, F_3] = F_+$ , while  $(\ell+1)D_-D_0\xi = (\ell-1)D_0D_-\xi$  corresponds to the relation  $[F_-, F_3] = -H_-$ , and  $\xi = (2\ell-1)D_+D_-\xi - (2\ell+3)D_-D_+\xi - D_0^2\xi$  corresponds to the relation  $[F_+, F_-] = -2H_3$ . Here will only show the last claim and note that the others are similar (although a little simpler), and that the first and second claim can be shown first without assuming the third.

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Note that for  $\xi \in R_{\ell,\ell}$  with  $\ell \neq 0$  we have that  $F_+\xi = \sqrt{(2\ell+1)(2\ell+2)}E_+D_+\xi \in R_{\ell+1,\ell+1}$ , so

$$\begin{aligned} F_-F_+\xi &= -\sqrt{(2\ell+2)(2\ell+1)}\sqrt{(2\ell+1)(2\ell+2)}D_-E_-E_+D_+\xi \\ &\quad - \sqrt{2\ell+2}\sqrt{(2\ell+1)(2\ell+2)}D_0E_-E_+D_+\xi \\ &\quad - \sqrt{2}\sqrt{(2\ell+1)(2\ell+2)}E_-D_+E_+D_+\xi \\ &= -(2\ell+1)(2\ell+2)D_-D_+\xi - (2\ell+2)\sqrt{2\ell+1}D_0D_+\xi \\ &\quad - 2\sqrt{(2\ell+1)(\ell+1)}E_+E_-D_+D_+\xi \\ &= -(2\ell+1)(2\ell+2)D_-D_+\xi - (2\ell+2)\sqrt{2\ell+1}D_0D_+\xi \\ &\quad - 2\sqrt{(2\ell+1)(\ell+1)}D_+D_+\xi. \end{aligned}$$

Likewise we have that

$$F_-\xi = -\sqrt{2\ell(2\ell-1)}D_-E_-\xi - \sqrt{2\ell}D_0E_-\xi - \sqrt{2}E_-D_+\xi,$$

where  $D_-E_-\xi \in R_{\ell-1,\ell-1}$ ,  $D_0E_-\xi \in R_{\ell,\ell-1}$ , and  $E_-D_+\xi \in R_{\ell+1,\ell-1}$ , so

$$\begin{aligned} F_+F_-\xi &= -\sqrt{(2\ell-1)2\ell}\sqrt{2\ell(2\ell-1)}E_+D_+D_-E_-\xi \\ &\quad + \sqrt{2\ell}\sqrt{2\ell}D_0E_+D_0E_-\xi - \sqrt{2\ell(2\ell+1)}\sqrt{2\ell}E_+D_+D_0E_-\xi \\ &\quad - \sqrt{2}\sqrt{2}D_-E_+E_-D_+\xi + \sqrt{2(2\ell+1)}\sqrt{2}D_0E_+E_-D_+\xi \\ &\quad - \sqrt{(2\ell+1)(2\ell+2)}\sqrt{2}E_+D_+E_-D_+\xi \\ &= -2\ell(2\ell-1)E_+D_+D_-E_-\xi + 2\ell D_0^2\xi - 2\ell\sqrt{2\ell+1}D_+D_0\xi \\ &\quad - 2D_-D_+\xi + 2\sqrt{2\ell+1}D_0D_+ - 2\sqrt{(2\ell+1)(\ell+1)}D_+^2\xi. \end{aligned}$$

Hence we get that

$$\begin{aligned} [F_+, F_-]\xi &= F_+F_-\xi - F_-F_+\xi \\ &= -2\ell(2\ell-1)E_+D_+D_-E_-\xi + 2\ell D_0^2\xi - 2\ell\sqrt{2\ell+1}D_+D_0\xi \\ &\quad - (2 - (2\ell+1)(2\ell+2))D_-D_+\xi + (2 + (2\ell+2))\sqrt{2\ell+1}D_0D_+ \\ &= -2\ell(2\ell-1)E_+D_+D_-E_-\xi + 2\ell D_0^2\xi - 2\ell\sqrt{2\ell+1}D_+D_0\xi \\ &\quad + 2\ell(2\ell+3)D_-D_+\xi + 2(\ell+2)\sqrt{2\ell+1}D_0D_+\xi \\ &= -2\ell(2\ell-1)E_+D_+D_-E_-\xi + 2\ell D_0^2\xi \\ &\quad + 2\sqrt{2\ell+1}((\ell+2)D_0D_+\xi - \ell D_+D_0\xi) \\ &\quad + 2\ell(2\ell+3)D_-D_+\xi + 2(\ell+2)\sqrt{2\ell+1}D_0D_+\xi \\ &= -2\ell(2\ell-1)E_+D_+D_-E_-\xi + 2\ell D_0^2\xi + 2\ell(2\ell+3)D_-D_+\xi, \end{aligned}$$

since  $2 - (2\ell+1)(2\ell+2) = 2 - 4\ell^2 - 6\ell - 2 = -4\ell^2 - 6\ell = -2\ell(2\ell+3)$  and  $(\ell+2)D_0D_+\xi = \ell D_+D_0\xi$ . Also

$$-2H_3\xi = -2\ell\xi,$$

so dividing by  $-2\ell$  (noting that  $\ell \neq 0$ ) we see that  $[F_+, F_-]\xi = -2H_3\xi$  for  $\xi \in R_{\ell,\ell}$  if and only if

$$\xi = (2\ell - 1)E_+D_+D_-E_-\xi - (2\ell + 3)D_-D_+\xi - D_0^2\xi.$$

Here we note that for  $\xi \in R_{\ell,m}$ ,  $m \neq \pm\ell$ , the term  $E_+D_+D_-E_-\xi$  simplifies to  $D_+D_-\xi$ , while for  $m = -\ell$  it becomes  $E_-D_+D_-E_+\xi$ .

How to expand —  
also further up.

## A.7 Finding $d_\ell^-$

We want to find  $d_\ell^-$  in general given that we already know that  $d_{\ell_0}^- = 0$  and

$$(2\ell - 1)d_\ell^- - (2\ell + 3)d_{\ell+1}^- = 1 - \frac{\ell_0^2\ell_1^2}{\ell^2(\ell+1)^2}.$$

Multiplying the left side of the above equation by  $2\ell + 1$  we get

$$(4\ell^2 - 1)d_\ell^- - (4\ell^1 + 2\ell - 3)d_{\ell+1}^- = (4\ell^2 - 1)d_\ell^- - (4(\ell + 1)^2 - 1)d_{\ell+1}^-$$

and multiplying the right side by  $2\ell + 1$  we get

$$2\ell + 1 - \ell_0^2\ell_1^2 \frac{2\ell + 1}{\ell^2(\ell + 1)^2} = 2\ell + 1 - \ell_0^2\ell_1^2 \left( \frac{1}{\ell^2} - \frac{1}{(\ell + 1)^2} \right),$$

so we see that

$$(4\ell^2 - 1)d_\ell^- - (4(\ell + 1)^2 - 1)d_{\ell+1}^- = 2\ell + 1 - \ell_0^2\ell_1^2 \left( \frac{1}{\ell^2} - \frac{1}{(\ell + 1)^2} \right). \quad (\text{A.2})$$

Now we know that  $d_{\ell_0}^- = 0$ , so

$$\begin{aligned} -(4(\ell_0 + 1)^2 - 1)d_{\ell_0+1}^- &= 2\ell_0 + 1 - \ell_1^2 \left( 1 - \frac{\ell_0^2}{(\ell_0 + 1)^2} \right) \\ &= (\ell_0 + 1)^2 - \ell_0^2 - \ell_1^2 \frac{(\ell_0 + 1)^2 - \ell_0^2}{(\ell_0 + 1)^2} \\ &= \frac{((\ell_0 + 1)^2 - \ell_1^2)((\ell_0 + 1)^2 - \ell_0^2)}{(\ell_0 + 1)^2}, \end{aligned}$$

and thus

$$d_{\ell_0+1}^- = -\frac{((\ell_0 + 1)^2 - \ell_1^2)((\ell_0 + 1)^2 - \ell_0^2)}{(\ell_0 + 1)^2(4(\ell_0 + 1)^2 - 1)}.$$

We get inductively that

$$d_\ell^- = -\frac{(\ell^2 - \ell_1^2)(\ell^2 - \ell_0^2)}{\ell^2(4\ell^2 - 1)},$$

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for  $\ell > \ell_0$ , since we already have the base case, and assuming the equality for  $\ell > \ell_0$  we get by eq. (A.2) that

$$\begin{aligned} & -(4(\ell+1)^2 - 1)d_{\ell+1}^- \\ &= \frac{(\ell^2 - \ell_1^2)(\ell^2 - \ell_0^2)}{\ell^2} + 2\ell + 1 - \ell_0^2 \ell_1^2 \left( \frac{1}{\ell^2} - \frac{1}{(\ell+1)^2} \right) \\ &= \frac{(\ell+1)^2(\ell^2 - \ell_1^2)(\ell^2 - \ell_0^2) + \ell^2(\ell+1)^2(2\ell+1) - \ell_0^2 \ell_1^2(2\ell+1)}{\ell^2(\ell+1)^2}. \end{aligned}$$

So since

$$\begin{aligned} & (\ell+1)^2(\ell^2 - \ell_1^2)(\ell^2 - \ell_0^2) + \ell^2(\ell+1)^2(2\ell+1) - \ell_0^2 \ell_1^2(2\ell+1) \\ &= \ell^2(\ell^2 - \ell_1^2)(\ell^2 - \ell_0^2) \\ &\quad + (2\ell+1)((\ell^2 - \ell_1^2)(\ell^2 - \ell_0^2) + \ell^2(\ell+1)^2 - \ell_0^2 \ell_1^2) \\ &= \ell^2(\ell^2 - \ell_1^2)(\ell^2 - \ell_0^2) \\ &\quad + (2\ell+1)(\ell^4 - \ell^2 \ell_0^2 - \ell^2 \ell_1^2 + \ell^2(\ell+1)^2) \\ &= \ell^2 \left( (\ell^2 - \ell_1^2)(\ell^2 - \ell_0^2) \right. \\ &\quad \left. + (2\ell+1)(\ell^2 - \ell_0^2 - \ell_1^2 + (\ell+1)^2) \right) \end{aligned}$$

and

$$\begin{aligned} & ((\ell+1)^2 - \ell_1^2)((\ell+1)^2 - \ell_0^2) = (\ell^2 - \ell_1^2 + 2\ell+1)(\ell^2 - \ell_0^2 + 2\ell+1) \\ &= (\ell^2 - \ell_1^2)(\ell^2 - \ell_0^2) \\ &\quad + (2\ell+1)((\ell^2 - \ell_0^2 + 2\ell+1) + (\ell^2 - \ell_1^2)) \\ &= (\ell^2 - \ell_1^2)(\ell^2 - \ell_0^2) \\ &\quad + (2\ell+1)((\ell+1)^2 - \ell_0^2 + \ell^2 - \ell_1^2), \end{aligned}$$

we see that

$$-(4(\ell+1)^2 - 1)d_{\ell+1}^- = \frac{((\ell+1)^2 - \ell_1^2)((\ell+1)^2 - \ell_0^2)}{(\ell+1)^2},$$

and thus indeed

$$d_{\ell+1}^- = -\frac{((\ell+1)^2 - \ell_1^2)((\ell+1)^2 - \ell_0^2)}{(\ell+1)^2(4(\ell+1)^2 - 1)}.$$

## A.8 Finding $\Delta_1 \xi$ and $\Delta_2 \xi$

Rewrite - remember Appendix A.6

We have

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$$\begin{aligned}\Delta_1 &:= \frac{1}{2}(H_-F_+ + F_-H_+) + H_3F_3 + F_3 \\ \Delta_2 &:= H_-H_+ - F_-F_+ + H_3^2 - F_3^2 + 2H_3\end{aligned}$$

as in eq. (1.44), and we want to find  $\Delta_1\xi$  and  $\Delta_2\xi$  for  $\xi \in R_{\ell,m}$ . By eqs. (1.9), (1.10) and (1.34) we see that

$$\begin{aligned}\Delta_1\xi &= \frac{1}{2}H_-F_+\xi + \frac{1}{2}F_-H_+\xi + H_3F_3\xi + F_3\xi \\ &= \frac{1}{2}\sqrt{(\ell-m)(\ell-m-1)}H_-D_-E_+\xi \\ &\quad - \frac{1}{2}\sqrt{(\ell-m)((\ell+m+1))}H_-D_0E_+\xi \\ &\quad + \frac{1}{2}\sqrt{(\ell+m+1)(\ell+m+2)}H_-E_+D_+\xi \\ &\quad + \frac{1}{2}\sqrt{(\ell+m+1)(\ell-m)}F_-E_+\xi \\ &\quad + \sqrt{\ell^2-m^2}H_3D_-\xi - mH_3D_0\xi - \sqrt{(\ell+1)^2-m^2}H_3D_+\xi \\ &\quad + \sqrt{\ell^2-m^2}D_-\xi - mD_0\xi - \sqrt{(\ell+1)^2-m^2}D_+\xi \\ &= \frac{1}{2}\sqrt{(\ell-m)(\ell-m-1)} \\ &\quad \cdot \sqrt{((\ell-1)+(m+1))((\ell-1)-(m+1)+1)}E_-D_-E_+\xi \\ &\quad - \frac{1}{2}\sqrt{(\ell-m)((\ell+m+1))}\sqrt{(\ell+(m+1))(\ell-(m+1)+1)}E_-D_0E_+\xi \\ &\quad + \frac{1}{2}\sqrt{(\ell+m+1)(\ell+m+2)} \\ &\quad \cdot \sqrt{((\ell+1)+(m+1))((\ell+1)-(m+1)+1)}E_-E_+D_+\xi \\ &\quad + \frac{1}{2}\sqrt{(\ell+m+1)(\ell-m)}\left(-\sqrt{(\ell+(m+1))(\ell+(m+1)-1)}D_-E_-E_+\xi\right. \\ &\quad \left.- \sqrt{(\ell+(m+1))(\ell-(m+1)+1)}D_0E_-E_+\xi\right. \\ &\quad \left.- \sqrt{(\ell-(m+1)+1)(\ell-(m+1)+2)}E_-D_+E_+\xi\right) \\ &\quad + \sqrt{\ell^2-m^2}mD_-\xi - m \cdot mD_0\xi - \sqrt{(\ell+1)^2-m^2}mD_+\xi \\ &\quad + \sqrt{\ell^2-m^2}D_-\xi - mD_0\xi - \sqrt{(\ell+1)^2-m^2}D_+\xi \\ &= \frac{1}{2}(\ell-m-1)\sqrt{\ell^2-m^2}D_-\xi - \frac{1}{2}(\ell-m)((\ell+m+1))D_0\xi \\ &\quad + \frac{1}{2}(\ell+m+2)\sqrt{(\ell+1)^2-m^2}D_+\xi \\ &\quad + \frac{1}{2}\sqrt{(\ell+m+1)(\ell-m)}\left(-\sqrt{(\ell+m+1)(\ell+m)}D_-\xi\right. \\ &\quad \left.- \sqrt{(\ell+m+1)(\ell-m)}D_0\xi - \sqrt{(\ell-m)(\ell-m+1)}D_+\xi\right) \\ &\quad + \sqrt{\ell^2-m^2}mD_-\xi - m^2D_0\xi - \sqrt{(\ell+1)^2-m^2}mD_+\xi \\ &\quad + \sqrt{\ell^2-m^2}D_-\xi - mD_0\xi - \sqrt{(\ell+1)^2-m^2}D_+\xi\end{aligned}$$

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$$\begin{aligned}
&= \frac{1}{2}(\ell - m - 1)\sqrt{\ell^2 - m^2}D_- \xi - \frac{1}{2}(\ell - m)(\ell + m + 1)D_0 \xi \\
&\quad + \frac{1}{2}(\ell + m + 2)\sqrt{(\ell + 1)^2 - m^2}D_+ \xi - \frac{1}{2}(\ell + m + 1)\sqrt{\ell^2 - m^2}D_- \xi \\
&\quad - \frac{1}{2}(\ell + m + 1)(\ell - m)D_0 \xi - \frac{1}{2}(\ell - m)\sqrt{(\ell + 1)^2 - m^2}D_+ \xi \\
&\quad + \sqrt{\ell^2 - m^2}mD_- \xi - m^2D_0 \xi - \sqrt{(\ell + 1)^2 - m^2}mD_+ \xi \\
&\quad + \sqrt{\ell^2 - m^2}D_- \xi - mD_0 \xi - \sqrt{(\ell + 1)^2 - m^2}D_+ \xi \\
&= \left( \frac{1}{2}(\ell - m - 1) - \frac{1}{2}(\ell + m + 1) + m + 1 \right) \sqrt{\ell^2 - m^2}D_- \xi \\
&\quad + \left( -\frac{1}{2}(\ell - m)(\ell + m + 1) - \frac{1}{2}(\ell + m + 1)(\ell - m) - m^2 - m \right) D_0 \xi \\
&\quad + \left( \frac{1}{2}(\ell + m + 2) - \frac{1}{2}(\ell - m) - m - 1 \right) \sqrt{(\ell + 1)^2 - m^2}D_+ \xi \\
&= 0 + (-\ell^2 - \ell + m^2 + m - m^2 - m)D_0 \xi + 0 \\
&= -\ell(\ell + 1)D_0 \xi
\end{aligned}$$

for  $\xi \in R_{\ell, m}$ , where  $-\ell + 1 \leq m \leq \ell - 1$ . Now as in Appendix A.5, we note that the coefficients causing problems in the edge cases vanish, so we get the above equality for all  $m$ , and the formula is independent of  $m$ , we see that we actually have

$$\Delta_1 \xi = -\ell(\ell + 1)D_0 \xi$$

for all  $\xi \in R_\ell$ .

Similar calculations show that

$$\Delta_2 \xi = (\ell^2 - 1)\xi - (\ell + 1)^2 D_0^2 \xi + (4\ell^2 - 1)D_+ D_- \xi$$

for all  $\xi \in R_\ell$ .

Additionally by eq. (1.35) we have that  $\xi = (2\ell - 1)D_+ D_- \xi - (2\ell + 3)D_- D_+ \xi - D_0^2 \xi$ , so we get that

$$\begin{aligned}
(4\ell^2 - 1)D_+ D_- \xi &= (2\ell + 1)(2\ell - 1)D_+ D_- \xi \\
&= (2\ell + 1)\xi + (2\ell + 1)(2\ell + 3)D_- D_+ \xi + (2\ell + 1)D_0^2 \xi \\
&= (2\ell + 1)\xi + (4(\ell + 1)^2 - 1)D_- D_+ \xi + (2\ell + 1)D_0^2 \xi
\end{aligned}$$

for  $\xi \in R_\ell$  since  $(2\ell + 1)(2\ell + 3) = (2(\ell + 1) - 1)(2(\ell + 1) + 1) = 4(\ell + 1)^2 - 1$ , and therefore also

$$\begin{aligned}
\Delta_2 \xi &= (\ell^2 - 1)\xi - (\ell + 1)^2 D_0^2 \xi + (2\ell + 1)\xi + (4(\ell + 1)^2 - 1)D_- D_+ \xi + (2\ell + 1)D_0^2 \xi \\
&= ((\ell + 1)^2 - 1)\xi + \ell^2 D_0^2 \xi + (4(\ell + 1)^2 - 1)D_- D_+ \xi
\end{aligned}$$

for  $\xi \in R_\ell$ .

## Appendix B

### Auxiliary results

In this appendix we will collect the proofs of some auxiliary results that we will need in the paper.

#### B.1 $Z(U(L_1 \times L_2)) \simeq Z(U(L_1)) \otimes Z(U(L_2))$

Let  $L = L_1 \times L_2$  be a product of two Lie algebras, and let  $\iota_1: L_1 \rightarrow U(L_1)$ ,  $\iota_2: L_2 \rightarrow U(L_2)$ , and  $\iota: L \rightarrow U(L)$  be the canonical homomorphisms of Lie algebras, we get from the universal property of universal enveloping algebras. We want to show first that  $U(L) \simeq U(L_1) \otimes U(L_2)$ .

Consider the map

$$\rho: L \rightarrow U(L_1) \otimes U(L_2), \quad (u_1, u_2) \mapsto \iota_1(u_1) \otimes 1 + 1 \otimes \iota_2(u_2),$$

which is a homomorphisms of Lie algebras since it is clearly linear and

$$\begin{aligned} [\rho(u_1, u_2), \rho(v_1, v_2)] &= [u_1 \otimes 1 + 1 \otimes u_2, v_1 \otimes 1 + 1 \otimes v_2] \\ &= (u_1 \otimes 1 + 1 \otimes u_2)(v_1 \otimes 1 + 1 \otimes v_2) \\ &\quad - (v_1 \otimes 1 + 1 \otimes v_2)(u_1 \otimes 1 + 1 \otimes u_2) \\ &= u_1 v_1 \otimes 1 + u_1 \otimes v_2 + v_1 \otimes u_2 + 1 \otimes u_2 v_2 \\ &\quad - v_1 u_1 \otimes 1 - v_1 \otimes u_2 - u_1 \otimes v_2 - 1 \otimes v_2 u_2 \\ &= (u_1 v_1 - v_1 u_1) \otimes 1 + 1 \otimes (u_2 v_2 - v_2 u_2) \\ &= [u_1, v_1] \otimes 1 + 1 \otimes [u_2, v_2] \\ &= \rho([u_1, v_1], [u_2, v_2]) \\ &= \rho([(u_1, u_2), (v_1, v_2)]) \end{aligned}$$

for  $(u_1, u_2), (v_1, v_2) \in L$  by the definition of the tensor product of an algebra. Thus by the universal property of  $(U(L), \iota)$  we get a unique homomorphisms of associative algebras  $\varphi: U(L) \rightarrow U(L_1) \otimes U(L_2)$  such that the following diagram

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commutes:

$$\begin{array}{ccc} L & \xrightarrow{\iota} & U(L) \\ & \searrow \rho & \downarrow \varphi \\ & & U(L_1) \otimes U(L_2) \end{array}$$

Now let  $i_1: L_1 \rightarrow L$  be the inclusion of  $L_1$  into  $L$  given by  $u \mapsto (u, 0)$  for  $u \in L_1$ . By the definition of the bracket on  $L = L_1 \times L_2$  it is easy to see that  $i_1$  is a Lie algebra homomorphism, and thus the map  $\iota \circ i_1: L_1 \rightarrow L \rightarrow U(L)$  is also a Lie algebra homomorphism. Hence by the universal property of  $(U(L_1), \iota_1)$  we get a unique homomorphism of associative algebras  $\psi_1: U(L_1) \rightarrow U(L)$  such that the following diagram commutes:

$$\begin{array}{ccc} L_1 & \xrightarrow{\iota_1} & U(L_1) \\ & \searrow \iota \circ i_1 & \downarrow \psi_1 \\ & & U(L) \end{array}$$

Likewise we get a unique homomorphism of associative algebras  $\psi_2: U(L_2) \rightarrow U(L)$  such that  $\iota \circ i_2 = \psi_1 \circ \iota_2$ . Now since  $[(u_1, 0), (0, u_2)] = ([u_1, 0], [0, u_2]) = 0$  for  $u_1 \in L_1$  and  $u_2 \in L_2$ , we see that

$$\begin{aligned} 0 &= \iota([(u_1, 0), (0, u_2)]) = [\iota i_1(u_1), \iota i_2(u_2)] = [\psi_1 \iota_1(u_1), \psi_2 \iota_2(u_2)] \\ &= \psi_1 \iota_1(u_1) \psi_2 \iota_2(u_2) - \psi_2 \iota_2(u_2) \psi_1 \iota_1(u_1). \end{aligned}$$

Thus  $\psi_1 \iota_1(u_1) \psi_2 \iota_2(u_2) = \psi_2 \iota_2(u_2) \psi_1 \iota_1(u_1)$  for all  $u_1 \in L_1$  and  $u_2 \in L_2$ . Hence since the  $\iota_j(u_j)$  for  $u_j \in L_j$  generate  $U(L_j)$  by the PBW theorem for  $j = 1, 2$ , cf. [Jan16, p. E-7], we get that  $\psi_1(u_1) \psi_2(u_2) = \psi_2(u_2) \psi_1(u_1)$  for all  $u_1 \in U(L_1)$  and  $u_2 \in U(L_2)$ . Therefore the map

$$\psi: U(L_1) \otimes U(L_2) \rightarrow U(L), \quad u_1 \otimes u_2 \mapsto \psi_1(u_1) \psi_2(u_2), \quad (\text{B.1})$$

is a homomorphism of associative algebras since

$$\begin{aligned} \psi((u_1 \otimes u_2)(v_1 \otimes v_2)) &= \psi(u_1 v_1 \otimes v_1 v_2) = \psi_1(u_1 v_1) \psi_2(v_1 v_2) \\ &= \psi_1(u_1) \psi_1(v_1) \psi_2(u_2) \psi_2(v_2) \\ &= \psi_1(u_1) \psi_2(u_2) \psi_1(v_1) \psi_2(v_2) \\ &= \psi(u_1 \otimes u_2) \psi(v_1 \otimes v_2). \end{aligned}$$

Note now that

$$\begin{aligned} \psi \varphi(u_1, u_2) &= \psi \rho(u_1, u_2) = \psi(\iota_1(u_1) \otimes 1 + 1 \otimes \iota_2(u_2)) \\ &= \psi_1 \iota_1(u_1) \psi_2(1) + \psi_1(1) \psi_2 \iota_2(u_2) \\ &= \iota(u_1, 0) + \iota(0, u_2) = \iota(u_1, u_2) \end{aligned}$$



for all  $(u_1, u_2) \in L$ , so by the PBW theorem as above we get that  $\psi\varphi = \text{id}_{U(L)}$ . Likewise

$$\begin{aligned}\varphi\psi(\iota_1(u_1) \otimes 1 + 1 \otimes \iota_2(u_2)) &= \varphi(\psi_1\iota_1(u_1)\psi_2(1) + \psi_1(1)\psi_2\iota_2(u_2)) \\ &= \varphi(\iota(u_1, 0) + \iota(0, u_2)) = \varphi\iota(u_1, u_2) \\ &= \rho(u_1, u_2) = \iota(u_1) \otimes 1 + 1 \otimes \iota_2(u_2)\end{aligned}$$

for all  $u_1 \in L_1$  and  $u_2 \in L_2$ . Now by the PBW theorem the  $\iota_1(u_1)$  for  $u_1 \in L_1$  generate  $U(L_1)$  and the  $\iota_2(u_2)$  for  $u_2 \in L_2$  generate  $U(L_2)$ , so we see that the  $\iota_1(u_1) \otimes 1 + 1 \otimes \iota_2(u_2)$  for  $u_1 \in L_1$  and  $u_2 \in L_2$  generate  $U(L_1) \otimes U(L_2)$  and thus  $\varphi\psi = \text{id}_{U(L_1) \otimes U(L_2)}$ . Hence we see that  $\varphi$  and  $\psi$  are isomorphisms between  $U(L)$  and  $U(L_1) \otimes U(L_2)$ , so indeed  $U(L) \simeq U(L_1) \otimes U(L_2)$ .

Note that the above also gives us an isomorphism  $Z(U(L)) \simeq Z(U(L_1) \otimes U(L_2))$ . Now we want to show that we also have that  $Z(U(L_1) \otimes U(L_2)) = Z(U(L_1)) \otimes Z(U(L_2))$  such that when describing  $Z(U(L))$  we can instead describe  $Z(U(L_1)) \otimes Z(U(L_2))$ . For  $z_1 \otimes z_2 \in Z(U(L_1)) \otimes Z(U(L_2))$  we get that

$$(z_1 \otimes z_2)(u_1 \otimes u_2) = z_1 u_1 \otimes z_2 u_2 = u_1 z_1 \otimes u_2 z_2 = (u_1 \otimes u_2)(z_1 \otimes z_2)$$

for all  $u_1 \otimes u_2 \in U(L_1) \otimes U(L_2)$ , so we have the inclusion  $Z(U(L_1) \otimes U(L_2)) \subseteq Z(U(L_1)) \otimes Z(U(L_2))$ .

To get the other inclusion let  $z = \sum_i u_i \otimes v_i \in Z(U(L_1) \otimes U(L_2))$ . By combining terms with linearly dependent  $v_i$ 's, we can assume that the  $v_i$ 's in the sum are linearly independent. Now for  $u \otimes 1 \in U(L_1) \otimes U(L_2)$  we have that  $z(u \otimes 1) = (u \otimes 1)z$ , so

$$0 = z(u \otimes 1) - (u \otimes 1)z = \sum_i (u_i u - u u_i) \otimes v_i.$$

Thus since the  $v_i$ 's are linearly independent, we must have that  $u_i u - u u_i = 0$  for all  $i$ , i.e.  $u_i \in Z(U(L_1))$  for all  $i$ . Likewise we get that  $v_i \in Z(U(L_2))$  for all  $i$ , and hence  $z = \sum_i u_i \otimes v_i \in Z(U(L_1)) \otimes Z(U(L_2))$ . Therefore we get the inclusion  $Z(U(L_1) \otimes U(L_2)) \subseteq Z(U(L_1)) \otimes Z(U(L_2))$ , and thus indeed we have the equality  $Z(U(L_1) \otimes U(L_2)) = Z(U(L_1)) \otimes Z(U(L_2))$ . So altogether we have an isomorphism  $Z(U(L)) \simeq Z(U(L_1)) \otimes Z(U(L_2))$ .

## B.2 Determining a linear map from its square and eigenvalue

Let  $A: V \rightarrow V$  be a linear operator on a finite dimensional vector space  $V$  over  $\mathbf{C}$ , and assume that  $A$  only has only one eigenvalue,  $\lambda \neq 0$ . We claim then that  $A$  is uniquely determined by  $A^2$  and  $\lambda$ . To see this first note that

$$\ker(A - \lambda \text{id}_V)^r = \ker(A^2 - \lambda^2 \text{id}_V)^r$$

## B. AUXILIARY RESULTS

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for all integers  $r > 0$ . This is the case since  $(A^2 - \lambda^2 \text{id}_V)^r = (A + \lambda \text{id}_V)^r (A - \lambda \text{id}_V)^r$ , and since  $A + \lambda \text{id}_V$  is bijective for  $\lambda \neq 0$  because  $\det(A + \lambda \text{id}_V)$  cannot be 0 since  $-\lambda$  is not an eigenvalue of  $A$ .

Now choose a basis of  $\ker(A^2 - \lambda^2 \text{id}_V)$ , expand to a basis of  $\ker(A^2 - \lambda^2 \text{id}_V)^2$ , expand further to a basis of  $\ker(A^2 - \lambda^2 \text{id}_V)^3$ , and so on. Since  $A - \lambda \text{id}_V$  takes  $\ker(A^2 - \lambda^2 \text{id}_V)^r = \ker(A - \lambda \text{id}_V)^r$  to  $\ker(A^2 - \lambda^2 \text{id}_V)^{r-1} = \ker(A - \lambda \text{id}_V)^{r-1}$ , we see that the matrix of  $A$  with respect to this basis is upper triangular with all diagonal entries equal to  $\lambda$ . To see this more clearly suppose that  $(v_1, \dots, v_s)$  is the basis of  $\ker(A^2 - \lambda^2 \text{id}_V)^{r-1} = \ker(A - \lambda \text{id}_V)^{r-1}$  and that  $(v_1, \dots, v_s, \dots, v_n)$  is the basis of  $\ker(A^2 - \lambda^2 \text{id}_V)^r = \ker(A - \lambda \text{id}_V)^r$ . Then we have that  $(A - \lambda \text{id}_V)v_\ell \in \ker(A - \lambda \text{id}_V)^{r-1}$  for all  $\ell \in \{1, \dots, n\}$ , i.e.  $Av_\ell - \lambda v_\ell = \sum_{k=1}^s \beta_k v_k$  for some  $\beta_k \in \mathbf{C}$ , so for  $\ell > s$  we have that  $Av_\ell = \sum_{k=1}^s \beta_k v_k + \lambda v_\ell$ , so by induction in  $r$  we have the claim since for  $v \in \ker(A - \lambda \text{id}_V)$  we have that  $Av = \lambda v$  which gives the base case. Write  $A = (a_{ij})$  in this basis, where we now know that  $a_{ij} = 0$  if  $i > j$ , and note that writing  $m = \dim V$  we have that

$$(A^2)_{ij} = \sum_{k=1}^m a_{ik} a_{kj} = \sum_{k=i}^j a_{ik} a_{kj} = \lambda a_{ij} + \sum_{k=i+1}^{j-1} a_{ik} a_{kj},$$

since for  $k < i$   $a_{ik} = 0$  and for  $k > j$   $a_{kj} = 0$ , and since  $a_{ii} = a_{jj} = \lambda$ . Hence by induction in  $i - j$  we get that  $A$  is determined by  $A^2$  and  $\lambda$ , since clearly  $a_{ii} = \lambda$  satisfies this and inductively we can find  $a_{ij}$  from the above formula knowing  $A^2$ ,  $\lambda$ , and  $a_{k\ell}$  with  $k - \ell < i - j$ .