Bachelorprojekt

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Harish-Chandra modules over the Lie algebra $\mathfrak{sl}(2, \mathbf{C}) \times \mathfrak{sl}(2, \mathbf{C})$

(Harish-Chandra moduler over Lie algebraen $\mathfrak{sl}(2, \mathbf{C}) \times \mathfrak{sl}(2, \mathbf{C})$)

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Abstract

This paper investigates a special class of modules over the Lie algebra $L = \mathfrak{sl}(2, \mathbf{C}) \times \mathfrak{sl}(2, \mathbf{C})$ called Harish-Chandra modules, which are defined as follows: Given a Lie algebra L and a Lie subalgebra L_k an L-module M is a Harish-Chandra module for the pair (L, L_k) if it can be written as direct sum $M = \bigoplus_i M_i$ of finite dimensional simple L_k-modules, where for each M_{i_0} there are only finitely many M_i equivalent to M_{i_0} in the decomposition. In this paper we consider the pair (L, L_k) with $L = \mathfrak{sl}(2, \mathbf{C}) \times \mathfrak{sl}(2, \mathbf{C})$ and $L_k = \{(u, u) \mid u \in \mathfrak{sl}(2, \mathbf{C})\} \subset L$. We start out by giving a general description of L-modules, and then move on to give a characterization of all simple Harish-Chandra modules for the pair (L, L_k). After this we begin working towards the main goal of this paper: A characterization of the indecomposable Harish-Chandra modules for the pair (L, L_k). We show that overall these can be split into two kinds of modules, singular and non-singular, where the singular further can be split into two different types, open and closed. Here the non-singular modules are completely determined by numbers (ℓ_0, ℓ_1, n), where $\ell_0 \in \frac{1}{2}\mathbf{Z}_{\geq 0}$, $\ell_1 \in \mathbf{C}$, and $n \in \mathbf{N}$, while the singular modules are completely determined by some integers $(s, n_1, m_1, n_2, m_2, \ldots, n_k, m_k)$ (the open type) or by some integers and a complex number μ $(n_1, m_1, \ldots, n_k, m_k, \mu, N)$ (the closed type).

Contents

Chapter 1

Harish-Chandra modules over $\mathfrak{sl}(2, \mathbf{C}) \times \mathfrak{sl}(2, \mathbf{C})$

Let L be a semisimple Lie algebra and let L_k be a Lie subalgebra.

Definition 1.1. *An L-module M is a Harish-Chandra module for the pair (L, L_k) if, regarded as an L_k-module, it can be written as a direct sum*

$$M = \bigoplus_i M_i$$

of finite dimensional simple[1] *L_k-submodules M_i, where for each M_{i_0} only finitely many L_k-submodules equivalent to M_{i_0} occur in the decomposition of M. If L and L_k are clear from the context we will just call M a Harish-Chandra module.*

A Harish-Chandra module M is indecomposable if it cannot be decomposed into the direct sum of non-zero L-submodules.

Our goal is to classify all indecomposable Harish-Chandra modules over (L, L_k) for $L = \mathfrak{sl}(2, \mathbf{C}) \times \mathfrak{sl}(2, \mathbf{C})$ and $L_k = \{(u, u) \mid u \in \mathfrak{sl}(2, \mathbf{C})\}$, where we by $\mathfrak{sl}(2, \mathbf{C}) \times \mathfrak{sl}(2, \mathbf{C})$ mean the following:

For L, L' Lie algebras over F, we consider $L \times L' = L \oplus L'$ as a Lie algebra over F with pointwise addition, multiplication given by $\alpha(a, b) = (\alpha a, \alpha b)$ for $\alpha \in F, a \in L, b \in L'$, and with Lie bracket $[(a_1, b_1), (a_2, b_2)] = ([a_1, a_2], [b_1, b_2])$ for $a_1, a_2 \in L, b_1, b_2 \in L'$.

Remark 1.2. Note that $L \times 0$ and $0 \times L'$ are ideals in $L \times L'$ as given above. Thus we see that $\mathfrak{sl}(2, \mathbf{C}) \times 0$ and $0 \times \mathfrak{sl}(2, \mathbf{C})$ are ideals in $\mathfrak{sl}(2, \mathbf{C}) \times \mathfrak{sl}(2, \mathbf{C})$ with

$$(\mathfrak{sl}(2, \mathbf{C}) \times 0) \oplus (0 \times \mathfrak{sl}(2, \mathbf{C})) = \mathfrak{sl}(2, \mathbf{C}) \times \mathfrak{sl}(2, \mathbf{C}),$$

[1] In [GP67b] the word irreducible is used instead of simple, but we will only use irreducible when talking about representations in this paper.

so $\mathfrak{sl}(2, \mathbf{C}) \times \mathfrak{sl}(2, \mathbf{C})$ is semisimple.

Now if we take $L = \mathfrak{sl}(2, \mathbf{C}) \times \mathfrak{sl}(2, \mathbf{C})$ and $L_k = \{(u, u) \mid u \in \mathfrak{sl}(2, \mathbf{C})\}$ as a Lie subalgebra, it makes sense to talk about Harish-Chandra modules over (L, L_k). Here L_k is clearly a Lie subalgebra since it is a subspace and the Lie bracket on $\mathfrak{sl}(2, \mathbf{C}) \times \mathfrak{sl}(2, \mathbf{C})$ preserves L_k by the definition of the Lie bracket on a product. △

We fix the following as a standard basis for $\mathfrak{sl}(2, \mathbf{C})$:

$$x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{1.1}$$

Giving us the relations:

$$[x, y] = h, \qquad [h, x] = 2x, \qquad [h, y] = -2y, \tag{1.2}$$

cf. [Jan16, p. 35] or [Hum72, p. 6].

We claim now that

$$(x, x), \quad (y, y), \quad \tfrac{1}{2}(h, h), \quad (ix, -ix), \quad (iy, -iy), \quad \tfrac{1}{2}(ih, -ih)$$

is a basis of $\mathfrak{sl}(2, \mathbf{C}) \times \mathfrak{sl}(2, \mathbf{C})$. This is clearly the case since $\dim_{\mathbf{C}} \mathfrak{sl}(2, \mathbf{C}) = 3$, so $\dim_{\mathbf{C}} \mathfrak{sl}(2, \mathbf{C}) \times \mathfrak{sl}(2, \mathbf{C}) = 6$, and we see that the above elements span $\mathfrak{sl}(2, \mathbf{C}) \times \mathfrak{sl}(2, \mathbf{C})$; we have $\frac{1}{2}(x, x) - \frac{i}{2}(ix, -ix) = (x, 0)$ and $\frac{1}{2}(x, x) + \frac{i}{2}(ix, -ix) = (0, x)$ and likewise with h and y.

Putting

$$\begin{aligned} h_+ &= (x, x), & h_- &= (y, y), & h_3 &= \tfrac{1}{2}(h, h), \\ f_+ &= (ix, -ix), & f_- &= (iy, -iy), & f_3 &= \tfrac{1}{2}(ih, -ih) \end{aligned}$$

we get the following commutation relations between these basis elements:

$$\begin{aligned}
[h_+, h_3] &= \tfrac{1}{2}([x, h], [x, h]) = \tfrac{1}{2}(-2x, -2x) = -(x, x) = -h_+, \\
[h_-, h_3] &= \tfrac{1}{2}([y, h], [y, h]) = \tfrac{1}{2}(2y, 2y) = (y, y) = h_-, \\
[h_+, h_-] &= ([x, y], [x, y]) = (h, h) = 2h_3, \\
[h_+, f_+] &= ([x, ix], [x, -ix]) = 0, \\
[h_-, f_-] &= ([y, iy], [y, -iy]) = 0, \\
[h_3, f_3] &= \tfrac{1}{4}([h, ih], [h, -ih]) = 0, \\
[h_+, f_3] &= \tfrac{1}{2}([x, ih], [x, -ih]) = \tfrac{1}{2}(-2ix, 2ix) = -(ix, -ix) = -f_+, \\
[h_-, f_3] &= \tfrac{1}{2}([y, ih], [y, -ih]) = \tfrac{1}{2}(2iy, -2iy) = (iy, -iy) = f_-, \\
[h_+, f_-] &= ([x, iy], [x, -iy]) = (ih, -ih) = 2f_3, \\
[h_3, f_-] &= \tfrac{1}{2}([h, iy], [h, -iy]) = \tfrac{1}{2}(-2iy, 2iy) = -(iy, -iy) = -f_-, \\
[h_-, f_+] &= ([y, ix], [y, -ix]) = (-ih, ih) = -(ih, -ih) = -2f_3, \\
[h_3, f_+] &= \tfrac{1}{2}([h, ix], [h, -ix]) = \tfrac{1}{2}(2ix, -2ix) = (ix, -ix) = f_+, \\
[f_+, f_3] &= \tfrac{1}{2}([ix, ih], [-ix, -ih]) = \tfrac{1}{2}(2x, 2x) = (x, x) = h_+, \\
[f_-, f_3] &= \tfrac{1}{2}([iy, ih], [-iy, -ih]) = \tfrac{1}{2}(-2y, -2y) = -(y, y) = -h_-, \\
[f_+, f_-] &= ([ix, iy], [-ix, -iy]) = (-h, -h) = -(h, h) = -2h_3.
\end{aligned} \tag{1.3}$$

Remark 1.3. Note that these are the same relations as for the complexification of the Lie algebra L of the proper Lorentz group in [GP67b, p. 5], so L is isomorphic to $\mathfrak{sl}(2,\mathbf{C}) \times \mathfrak{sl}(2,\mathbf{C})$. This explains the equivalence of the work in this paper and the work in [GP67a; GP67b; GP67c]. △

Now in the rest of the paper let $L = \mathfrak{sl}(2,\mathbf{C}) \times \mathfrak{sl}(2,\mathbf{C})$ and $L_k = \{(u,u) \mid u \in \mathfrak{sl}(2,\mathbf{C})\}$. Note that L_k is the Lie subalgebra of L with basis h_+, h_-, h_3, and that the above commutation relations gives us that

$$[h_+, h_-] = 2h_3, \qquad [2h_3, h_+] = 2h_+, \qquad [2h_3, h_-] = -2h_-$$

Comparing with eq. (1.2) we see that we have an isomorphism

$$\mathfrak{sl}(2,\mathbf{C}) \to L_k, \qquad u \mapsto (u,u), \tag{1.4}$$

or more explicitly $x \mapsto h_+$, $h_- \mapsto y$, and $h \mapsto 2h_3$, so we can use $\mathfrak{sl}(2,\mathbf{C})$-theory when we want to describe L_k-modules.

1.1 Representations of L_k

Let V be a $\mathbf{C}$ vector space and $\rho\colon L_k \to \mathfrak{gl}(V)$ a representation of L_k. We will use the notation $\rho(a) = A$ for $a \in L_k$ switching to upper case letters when we talk about the representation corresponding to a given element. Note that we will switch freely between the language of representations of L_k and the language of L_k-modules.

We will start out by describing the finite dimensional simple L_k-modules. Recall, cf. [Jan16, p. 36], that we know from $\mathfrak{sl}(2,\mathbf{C})$-theory that for integers $n \geq 0$ there exists a unique simple $\mathfrak{sl}(2,\mathbf{C})$-module $V(n)$ of dimension $n+1$, and $V(n)$ has a basis $(v_0, v_1, \ldots, v_n)$ such that for all i, $0 \leq i \leq n$

$$\begin{aligned}
h.v_i &= (n-2i)v_i, \\
x.v_i &= \begin{cases} (n-i+1)v_{i-1} & \text{if } i > 0, \\ 0 & \text{if } i = 0, \end{cases} \\
y.v_i &= \begin{cases} (i+1)v_{i+1} & \text{if } i < n, \\ 0 & \text{if } i = n. \end{cases}
\end{aligned} \tag{1.5}$$

Now using the isomorphism from eq. (1.4) we see that for integers $n \geq 0$ there exists a unique simple L_k-module $M(n)$ of dimension $n+1$[2], and $M(n)$

[2] We will use the notation $V(n)$ when talking about $\mathfrak{sl}(2,\mathbf{C})$-modules and $M(n)$ when talking about L_k-modules to clarify what kind of module we are talking about, but as vector spaces $V(n)$ and $M(n)$ are isomorphic.

has a basis $(v_0, v_1, \ldots, v_n)$ such that for all i, $0 \le i \le n$

$$
\begin{aligned}
h_3.v_i &= (\tfrac{1}{2}n - i)v_i, \\
h_+.v_i &= \begin{cases} (n-i+1)v_{i-1} & \text{if } i > 0, \\ 0 & \text{if } i = 0, \end{cases} \\
h_-.v_i &= \begin{cases} (i+1)v_{i+1} & \text{if } i < n, \\ 0 & \text{if } i = n. \end{cases}
\end{aligned}
\tag{1.6}
$$

Now consider $M(n)$ as an inner product space over $\mathbf{C}$ with inner product given by

$$\langle v_k, v_j\rangle = \delta_{jk}\binom{n}{k}. \tag{1.7}$$

We will switch to the orthonormal basis $(\overline{v}_0, \overline{v}_1, \ldots, \overline{v}_n)$, where $\overline{v}_i = v_i/\|v_i\|$. Here $\|\cdot\|$ is given by $\|v\| = \sqrt{\langle v, v\rangle}$ as usually, and we note that

$$\overline{v}_i = \frac{1}{\sqrt{\binom{n}{i}}} v_i.$$

Note furthermore that

$$h_3.\overline{v}_i = \frac{1}{\sqrt{\binom{n}{i}}} h_3.v_i = \frac{1}{\sqrt{\binom{n}{i}}}(\tfrac{1}{2}n - i)v_i = (\tfrac{1}{2}n - i)\overline{v}_i$$

for all i, $0 \le i \le n$, and clearly still

$$
\begin{aligned}
h_+.\overline{v}_0 &= 0, \\
h_-.\overline{v}_n &= 0.
\end{aligned}
$$

But for i, $0 < i \le n$

$$
\begin{aligned}
h_+.\overline{v}_i &= \frac{1}{\sqrt{\binom{n}{i}}} h_+.v_i = \frac{1}{\sqrt{\binom{n}{i}}}(n-i+1)v_{i-1} \\
&= \sqrt{\frac{\binom{n}{i-1}}{\binom{n}{i}}}(n-i+1)\frac{1}{\sqrt{\binom{n}{i-1}}}v_{i-1} \\
&= \sqrt{\frac{i}{n-i+1}}(n-i+1)\overline{v}_{i-1} = \sqrt{(n-i+1)i}\,\overline{v}_{i-1},
\end{aligned}
$$

and for i, $0 \leq i < n$

$$\begin{aligned}
h_-.\overline{v}_i &= \frac{1}{\sqrt{\binom{n}{i}}} h_-.v_i = \frac{1}{\sqrt{\binom{n}{i}}}(i+1)v_{i+1} \\
&= \sqrt{\frac{\binom{n}{i+1}}{\binom{n}{i}}}(i+1)\frac{1}{\sqrt{\binom{n}{i+1}}}v_{i+1} \\
&= \sqrt{\frac{n-i}{i+1}}(i+1)\overline{v}_{i+1} = \sqrt{(n-i)(i+1)}\overline{v}_{i+1}.
\end{aligned}$$

Finally write $\ell = \frac{1}{2}n$. We will re-index with $m = \frac{1}{2}(n-2i) = \ell - i$ by setting

$$e_m = \overline{v}_{\ell-m}$$

for $m \in \{-\ell, -\ell+1, \ldots, \ell-1, \ell\}$. Thus we get

$$h_3.e_m = h_3.\overline{v}_{\ell-m} = (\ell-(\ell-m))\overline{v}_{\ell-m} = me_m,$$

and since $e_\ell = \overline{v}_0$ and $e_{-\ell} = \overline{v}_n$ also

$$\begin{aligned}
h_+.e_\ell &= 0, \\
h_-.e_{-\ell} &= 0.
\end{aligned}$$

And for $m \in \{-\ell, -\ell+1, \ldots, \ell-2, \ell-1\}$ we get

$$\begin{aligned}
h_+.e_m = h_+.\overline{v}_{\ell-m} &= \sqrt{(n-(\ell-m)+1)(\ell-m)}\overline{v}_{\ell-m-1} \\
&= \sqrt{(\ell+m+1)(\ell-m)}e_{m+1},
\end{aligned}$$

while for $m \in \{-\ell+1, -\ell+2, \ldots, \ell-1, \ell\}$ we get

$$\begin{aligned}
h_-.e_m = h_-.\overline{v}_{\ell-m} &= \sqrt{(n-(\ell-m))(\ell-m+1)}\overline{v}_{\ell-m+1} \\
&= \sqrt{(\ell+m)(\ell-m+1)}e_{m-1}.
\end{aligned}$$

Thus we get the following Lemma:

Lemma 1.4. *Every simple finite dimensional L_k-module is uniquely given by a number $\ell \in \frac{1}{2}\mathbf{Z}_{\geq 0}$. For such ℓ the unique simple L_k-module $M(2\ell)$ has dimension $2\ell+1$, and $M(2\ell)$ has a basis $(e_{-\ell}, e_{-\ell+1}, \ldots, e_{\ell-1}, e_\ell)$ such that for all $m \in \{-\ell, -\ell+1, \ldots, \ell-1, \ell\}$ we have*

$$\begin{aligned}
h_3.e_m &= me_m, \\
h_+.e_m &= \begin{cases} \sqrt{(\ell+m+1)(\ell-m)}e_{m+1} & \text{if } m \neq \ell, \\ 0 & \text{if } m = \ell, \end{cases} \\
h_-.e_m &= \begin{cases} \sqrt{(\ell+m)(\ell-m+1)}e_{m-1} & \text{if } m \neq -\ell, \\ 0 & \text{if } m = -\ell. \end{cases}
\end{aligned} \tag{1.8}$$

1.1.1 Formulae for the operators $H_+, H_-, H_3, F_+, F_-, F_3$

Let M be a Harish-Chandra L-module. Then we have linear operators $H_+, H_-, H_3, F_+, F_-, F_3 \colon M \to M$ given by the corresponding actions of L satisfying commutation relations as in eq. (1.3), and we want to give expressions for these in terms of other linear operators $E_+, E_-, D_+, D_-, D_0 \colon M \to M$.

We will denote by R_ℓ a finite dimensional L-module which is a (finite) direct sum of L_k-modules $M(2\ell)$ for the same number $\ell \in \frac{1}{2}\mathbf{Z}_{\geq 0}$. Then M is a direct sum of the subspaces R_ℓ since M is Harish-Chandra, and from Lemma 1.4 we know that R_ℓ can be written as the direct sum of subspaces $R_{\ell,m}$, where $R_{\ell,m}$ are eigenspaces for H_3 such that

$$H_3\xi = m\xi \tag{1.9}$$

for $m \in \{-\ell, -\ell+1, \ldots, \ell-1, \ell\}$ and $\xi \in R_{l,m}$. We will use the decomposition

$$M = \bigoplus_{\substack{\ell \in \frac{1}{2}\mathbf{Z}_{\geq 0} \\ m \in \{-\ell, -\ell+1, \ldots, \ell-1, \ell\}}} R_{\ell,m} = \bigoplus_{\ell,m} R_{\ell,m}$$

throughout this paper.

By Lemma 1.4 we also have that H_+ and H_- maps the $R_{\ell,m}$ into each other as follows:

$$\begin{aligned} &H_+ \colon R_{\ell,m} \to R_{\ell,m+1} && \text{if } -\ell \leq m < \ell, && H_+ \colon R_{\ell,\ell} \to 0, \\ &H_- \colon R_{\ell,m} \to R_{\ell,m-1} && \text{if } -\ell < m \leq \ell, && H_- \colon R_{\ell,-\ell} \to 0. \end{aligned}$$

Hence we have linear operators $H_+H_-, H_-H_+ \colon R_{\ell,m} \to R_{\ell,m}$, and by eq. (1.8) we see that

$$\begin{aligned} H_+H_-\xi &= \sqrt{(\ell + (m-1) + 1)(\ell - (m-1))}\sqrt{(\ell+m)(\ell-m+1)}\xi \\ &= (\ell+m)(\ell-m+1)\xi, \\ H_-H_+\xi &= \sqrt{(\ell + (m+1))(\ell - (m+1) + 1)}\sqrt{(\ell+m+1)(\ell-m)}\xi \\ &= (\ell+m+1)(\ell-m)\xi. \end{aligned} \tag{1.10}$$

Note that this also covers the cases $m = \ell$ and $m = -\ell$.

Now we define $E_+ \colon R_{\ell,m} \to R_{\ell,m+1}$ and $E_- \colon R_{\ell,m} \to R_{\ell,m-1}$ to be the linear maps satisfying

$$\begin{aligned} H_+\xi &= \sqrt{(\ell+m+1)(\ell-m)}E_+\xi && \text{if } m \neq \ell, \\ E_+\xi &= 0 && \text{if } m = \ell, \\ H_-\xi &= \sqrt{(\ell+m)(\ell-m+1)}E_-\xi && \text{if } m \neq -\ell, \\ E_-\xi &= 0 && \text{if } m = \ell, \end{aligned} \tag{1.11}$$

for $\xi \in R_{\ell,m}$. Comparing eq. (1.11) and eq. (1.10) we see that

$$\begin{aligned} E_+E_-\xi &= \xi && \text{if } m \neq -\ell \\ E_-E_+\xi &= \xi && \text{if } m \neq \ell. \end{aligned}$$

Thus $E_+ \colon R_{\ell,m} \to R_{\ell,m+1}$ and $E_- \colon R_{\ell,m+1} \to R_{\ell,m}$ are isomorphisms for $m \neq \ell$ and they are each others inverse.

Remark 1.5. Note that the above definitions make sense more generally on L_k-modules M with a direct sum decomposition $\bigoplus_\ell R_\ell$ such that each R_ℓ is a finite direct sum of simple L_k-modules isomorphic to $M(2\ell)$, i.e. we do not need the additional structure from L-modules.

In particular note that on the L_k-module $M(2\ell)$ with basis $(e_{-\ell}, e_{-\ell+1}, \dots, e_{\ell-1}, e_\ell)$ as in Lemma 1.4, we have that $E_+e_m = e_{m+1}$ for $m \neq \ell$ and $E_-e_m = e_{m-1}$ for $m \neq -\ell$. △

Now note that H_+, H_-, and H_3 are completely determined by eq. (1.9) and eq. (1.11), so we just need to find maps to determine F_+, F_-, and F_3 now, while making sure that we get commutation relations as in eq. (1.3).

We already have that $L_k = \operatorname{span}_{\mathbf{C}}(h_+, h_-, h_3)$, but now we will also consider $L_p = \operatorname{span}_{\mathbf{C}}(f_+, f_-, f_3)$. We will show shortly that $u.R_\ell \subset R_{\ell-1} \oplus R_\ell \oplus R_{\ell+1}$ for all $u \in L_p$. This implies that there are maps $D^u_- \colon R_\ell \to R_{\ell-1}$, $D^u_0 \colon R_\ell \to R_\ell$, and $D^u_+ \colon R_\ell \to R_{\ell+1}$ such that $u.v = D^u_-(v) + D^u_0(v) + D^u_+(v)$ for all $u \in L_p$ and $v \in R_\ell$. In the following we will find maps D_-, D_0, and D_+ independent of u such that we can express D^u_-, D^u_0, and D^u_+ in terms of these and the maps E_- and E_+ from above, thus we will also be able to express F_+, F_-, and F_3 in terms of D_-, D_0, D_+, E_-, and E_+. To be more precise we will find maps D_-, D_0, and D_+ such that we can express F_3 in terms of just these (and multiplication by some constant), and then we can get F_+ and F_- by the commutation relations.

For reasons that will be clearer later, we want the maps D_+ to be defined on $M = \bigoplus_{\ell,m} R_{\ell,m}$, D_0 defined on the direct sum without the $R_{0,0}$ summand (if there is one), and D_- defined on the direct sum without the summands $R_{\ell,\ell}$ and $R_{\ell,-\ell}$ to be such that $D_0R_{\ell,m} \subset R_{\ell,m}$, $D_+R_{\ell,m} \subset R_{\ell+1,m}$, and $D_-R_{\ell,m} \subset R_{\ell-1,m}$ and the diagrams

$$
\begin{tikzcd}
R_{\ell-1,m+1} & R_{\ell,m+1} \arrow[l, "D_-"'] \\
R_{\ell-1,m} \arrow[u, "E_+"] & R_{\ell,m} \arrow[l, "D_-"] \arrow[u, "E_+"']
\end{tikzcd}
\qquad
\begin{tikzcd}
R_{\ell,m+1} \arrow[r, "D_0"] & R_{\ell,m+1} \\
R_{\ell,m} \arrow[u, "E_+"] \arrow[r, "D_0"'] & R_{\ell,m} \arrow[u, "E_+"']
\end{tikzcd}
\tag{1.12}
$$

$$
\begin{tikzcd}
R_{\ell,m+1} \arrow[r, "D_+"] & R_{\ell+1,m+1} \\
R_{\ell,m} \arrow[u, "E_+"] \arrow[r, "D_+"'] & R_{\ell+1,m+1} \arrow[u, "E_+"']
\end{tikzcd}
$$

commute, when $-\ell + 1 \leq m < \ell - 1$ in the top left diagram, $-\ell \leq m < \ell$ in the other two diagrams. Also similar diagrams with E_- replacing E_+ commute since $E_- \colon R_{\ell,m} \to R_{\ell,m-1}$ for $m \neq -\ell$ is inverse to $E_+ \colon R_{\ell,m-1} \to R_{\ell,m}$. Before we can get to the final description of these maps we need quite a lot of work.

* * * * *

Note that eq. (1.3) gives us that $[L_k, L_p] \subset L_p$, so by the adjoint representation we can see L_p as an L_k-module, and again by eq. (1.3) we see that L_p is a simple L_k-module: If V is an L_k-submodule and we have a non-zero element $f = af_+ + bf_- + cf_3 \in V$ for some $a, b, c \in \mathbf{C}$ not all zero, then

$$
\begin{aligned}
[h_+, af_+ + bf_- + cf_3] &= 2bf_3 - cf_+, \\
[h_-, af_+ + bf_- + cf_3] &= -2af_3 + cf_-, \\
[h_3, af_+ + bf_- + cf_3] &= af_+ - bf_-.
\end{aligned}
$$

If $c \neq 0$, we get that

$$
\begin{aligned}
[h_3, [h_+, f]] &= [h_3, 2bf_3 - cf_+] = -cf_+, \\
[h_3, [h_-, f]] &= [h_3, -2af_3 + cf_-] = -cf_-,
\end{aligned}
$$

so we see that $f_+, f_- \in V$, and thus also $[h_+, \frac{1}{2}f_-] = f_3 \in V$, so $V = L_p$. If on the other hand $c = 0$, then

$$
\begin{aligned}
[h_-, f] &= -2af_3, \\
[h_+, f] &= 2bf_3,
\end{aligned}
$$

so since either $a \neq 0$ or $b \neq 0$, we see that $f_3 \in V$, and thus also $[h_+, -f_3] = f_+ \in V$ and $[h_-, f_3] = f_- \in V$, so $V = L_p$. Hence L_p is indeed a simple L_k-module. Now since L_p is a simple finite dimensional L_k-module of dimension 3, we have that $L_p \simeq M(2)$ as L_k-modules.

In general given a Lie algebra L_1 and two L_1-modules V and W, we consider the tensor product $V \otimes W$ over $\mathbf{C}$ of the underlying vector spaces as an L_1-module via the action

$$
x.(v \otimes w) = x.v \otimes w + v \otimes x.w, \tag{1.13}
$$

for $x \in L_1$ and $v \otimes w \in V \otimes W$, cf. [Hum72, p. 26].

Now we are interested in the L_k-module $L_p \otimes M$, where M is a Harish-Chandra L-module (and thus an L_k-module) as before. Specifically we will show that the linear map

$$\begin{aligned} \psi \colon L_p \otimes M &\to M \\ x \otimes v &\mapsto x.v \end{aligned} \tag{1.14}$$

is a homomorphism of L_k-modules. For $y \in L_k$ we see that

$$y.(x \otimes v) = y.x \otimes v + x \otimes y.v = [y, x] \otimes v + x \otimes y.v,$$

for $x \otimes v \in L_p \otimes M$, since the action in L_p is by the adjoint representation. So

$$\begin{aligned} \psi(y.(x \otimes v)) &= \psi([y, x] \otimes v) + \psi(x \otimes y.v) = [y, x].v + x.(y.v) \\ &= y.(x.v) - x.(y.v) + x.(y.v) = y.(x.v) = y.\psi(x \otimes v), \end{aligned}$$

i.e. ψ is indeed a homomorphism of L_k-modules.

Now we note that $M = \bigoplus_\ell R_\ell$, so

$$L_p \otimes M = L_p \otimes \big(\bigoplus_\ell R_\ell\big) \simeq \bigoplus_\ell (L_p \otimes R_\ell),$$

as L_k-modules, and since R_ℓ is direct sum of finitely many copies of $M(2\ell)$, we see that

$$\begin{aligned} L_p \otimes R_\ell &\simeq M(2) \otimes \big(M(2\ell)^1 \oplus M(2\ell)^2 \oplus \cdots \oplus M(2\ell)^r\big) \\ &\simeq (M(2) \otimes M(2\ell)^1) \oplus (M(2) \otimes M(2\ell)^2) \oplus \cdots \oplus (M(2) \otimes M(2\ell)^r), \end{aligned}$$

as L_k-modules, since $L_p \simeq M(2)$. Here the superscripts are just indices for the different $M(2\ell)$. Thus we want to describe the L_k-modules $M(2) \otimes M(2\ell)$, which we will do by first describing the $\mathfrak{sl}(2, \mathbf{C})$-modules $V(2) \otimes V(2\ell)$ and then translating back to a solution to our problem.

1.1.2 Describing $V(2) \otimes V(n)$

Let $2\ell = n \in \mathbf{N}$. We want to show that for $\mathfrak{sl}(2, \mathbf{C})$-modules we have that

$$V(2) \otimes V(n) \simeq \begin{cases} V(n-2) \oplus V(n) \oplus V(n+2) & \text{if } n \geq 2, \\ V(3) \oplus V(1) & \text{if } n = 1, \\ V(2) & \text{if } n = 0. \end{cases} \tag{1.15}$$

Note that in all cases there is a summand $V(n+2)$. We can show the above by considerations using formal characters. We will use the notation of [Jan16, Chapter 8], specifically we will do calculations with the functions $e(\lambda)\colon H^* \to \mathbf{Z}$ for $\lambda \in H^*$. Firstly note that in general

$$\mathrm{ch}_V = \sum_{\lambda \in H^*} (\dim V_\lambda) e(\lambda),$$

and use the notation $V(n)_k$ for $V(\lambda)_\mu$ and $e(n)$ for $e(\lambda)$ with $\lambda, \mu \in H^*$ such that $\lambda(h) = n$ and $\mu(h) = k$. We get that

$$\mathrm{ch}_{V(2)} = e(-2) + e(0) + e(2)$$

and

$$\mathrm{ch}_{V(n)} = \sum_{i=0}^{n} e(n-2i),$$

since

$$\dim V(n)_k = \begin{cases} 1 & \text{if } k = n-2i \text{ for some } i \in \{0,1,\dots,n\}, \\ 0 & \text{otherwise.} \end{cases}$$

Now since $e(\lambda) * e(\mu) = e(\lambda+\mu)$ in general cf. [Jan16, p. 93], we see that for $n \geq 2$

$$\begin{aligned}
\mathrm{ch}_{V(2)\otimes V(n)} &= \mathrm{ch}_{V(2)} * \mathrm{ch}_{V(n)} = e(-2) * \mathrm{ch}_{V(n)} + e(0) * \mathrm{ch}_{V(n)} + e(2) * \mathrm{ch}_{V(n)} \\
&= \sum_{i=0}^{n} e(n-2-2i) + \mathrm{ch}_{V(n)} + \sum_{i=0}^{n} e(n+2-2i) \\
&= e(-n-2) + e(-n) + \sum_{i=0}^{n-2} e(n-2-2i) + \mathrm{ch}_{V(n)} \\
&\quad + \sum_{i=0}^{n} e(n+2-2i) \\
&= \mathrm{ch}_{V(n-2)} + \mathrm{ch}_{V(n)} + \sum_{i=0}^{n+2} e(n+2-2i) \\
&= \mathrm{ch}_{V(n-2)} + \mathrm{ch}_{V(n)} + \mathrm{ch}_{V(n+2)} = \mathrm{ch}_{V(n-2)\oplus V(n)\oplus V(n+2)},
\end{aligned}$$

where the first equality follows from the fact that $\mathrm{ch}_{V\otimes W} = \mathrm{ch}_V * \mathrm{ch}_W$ in general, cf. [Hum72, p. 125]. Thus since two $\mathfrak{sl}(2,\mathbf{C})$-modules V and V' are isomorphic if and only if $\mathrm{ch}_V = \mathrm{ch}_{V'}$, cf. [Jan16, p. 90], we see that $V(2)\otimes V(n) \simeq V(n-2) \oplus V(n) \oplus V(n+2)$ if $n \geq 2$.

Likewise we see that

$$\begin{aligned}
\mathrm{ch}_{V(2)\otimes V(1)} &= \mathrm{ch}_{V(2)} * \mathrm{ch}_{V(1)} \\
&= \big(e(-2) + e(0) + e(2)\big) * e(-1) + \big(e(-2) + e(0) + e(2)\big) * e(1) \\
&= e(-3) + e(-1) + e(1) + e(-1) + e(1) + e(3) \\
&= \big(e(-3) + e(-1) + e(1) + e(3)\big) + \big(e(-1) + e(1)\big) \\
&= \mathrm{ch}_{V(3)} + \mathrm{ch}_{V(1)} = \mathrm{ch}_{V(3)\oplus V(1)}
\end{aligned}$$

and

$$\mathrm{ch}_{V(2)\otimes V(0)} = \mathrm{ch}_{V(2)} * \mathrm{ch}_{V(0)} = \mathrm{ch}_{V(2)} * e(0) = \mathrm{ch}_{V(2)},$$

so indeed $V(2) \otimes V(1) \simeq V(3) \oplus V(1)$ and $V(2) \otimes V(0) \simeq V(2)$.

Now consider (w_0, w_1, w_2) a basis for $V(2)$ and $(v_i \mid 0 \leq i \leq n)$ a basis for $V(n)$ such that both satisfies the conditions from eq. (1.5). Then for $w_i \otimes v_j \in V(2) \otimes V(n)$ with $i \in \{0,1,2\}$ and $j \in \{0,1,\dots,n\}$ we see that

$$\begin{aligned} h.(w_i \otimes v_j) &= h.w_i \otimes v_j + w_i \otimes h.v_j = (2-2i)w_i \otimes v_j + (n-2j)w_i \otimes v_j \\ &= (n - 2(i+j-1))w_i \otimes v_j. \end{aligned} \tag{1.16}$$

Hence $v_0 \otimes w_0$ is up to scalar multiple the only vector of weight $n+2$ in $V(2) \otimes V(n)$, so it is necessarily a highest weight vector generating the direct summand isomorphic to $V(n+2)$. Note that by eq. (1.15) we indeed have a direct summand isomorphic to $V(n+2)$ for all $n \in \mathbf{N}$. By $\mathfrak{sl}(2,\mathbf{C})$-theory, cf. [Jan16, p. 36], we know that this summand has a basis $(s_k \mid 0 \leq k \leq n+2)$ satisfying equations as in eq. (1.5), where

$$s_k \coloneqq \frac{1}{k!} y^k.(w_0 \otimes v_0). \tag{1.17}$$

By straightforward calculations, cf. Appendix A.1, we get for $n > 0$ that

$$\begin{aligned} s_0 &= w_0 \otimes v_0, & \\ s_1 &= w_1 \otimes v_0 + w_0 \otimes v_1 & &\text{if } n > 0, \\ s_k &= w_2 \otimes v_{k-2} + w_1 \otimes v_{k-1} + w_0 \otimes v_k & &\text{for } 2 \leq k \leq n, \\ s_{n+1} &= w_2 \otimes v_{n-1} + w_1 \otimes v_n & &\text{if } n > 0, \\ s_{n+2} &= w_2 \otimes v_n. & \end{aligned} \tag{1.18}$$

In case $n = 0$ we likewise see that $s_1 = w_1 \otimes v_0$ and $s_2 = w_2 \otimes v_0$, and we note that (s_0, s_1, s_2) is a basis for $V(2) \otimes V(0) \simeq V(2)$.

Suppose now that $n \geq 1$. Note that by eq. (1.15) we have a direct summand isomorphic to $V(n)$, and by eq. (1.16) the weight space of weight n is spanned by $w_0 \otimes v_1$ and $w_1 \otimes v_0$, so the vector of highest weight n generating the summand corresponding to $V(n)$ must be of the form $aw_0 \otimes v_1 + bw_1 \otimes v_0$ for some $a, b \in \mathbf{C}$. Furthermore we know that for this vector generating the summand corresponding to $V(n)$, we must have that

$$\begin{aligned} 0 &= x.(aw_0 \otimes v_1 + bw_1 \otimes v_0) \\ &= ax.w_0 \otimes v_1 + aw_0 \otimes x.v_1 + bx.w_1 \otimes v_0 + bw_1 \otimes x.v_0 \\ &= 0 + a(n-1+1)w_0 \otimes v_0 + b(2-1+1)w_0 \otimes v_0 + 0 \\ &= (an+2b)w_0 \otimes v_0, \end{aligned}$$

i.e. $an + 2b = 0$ so $b = -\frac{n}{2}a$. This determines the vector generating the summand corresponding to $V(n)$ up to a scalar, so taking $a = 1$, we see that we can take

$$t_0 \coloneqq w_0 \otimes v_1 - \frac{n}{2} w_1 \otimes v_0$$

as our vector generating the summand corresponding to $V(n)$. As before $\mathfrak{sl}(2,\mathbf{C})$-theory now yields that this summand has a basis $(t_k \mid 0 \leq k \leq n)$ satisfying equations as in eq. (1.5), where

$$t_k \coloneqq \frac{1}{k!} y^k.t_0. \tag{1.19}$$

By straightforward calculations, cf. Appendix A.1, we get that

$$\begin{aligned} t_0 &= w_0 \otimes v_1 - \frac{n}{2} w_1 \otimes v_0, \\ t_k &= (k+1) w_0 \otimes v_{k+1} - \frac{n-2k}{2} w_1 \otimes v_k \\ &\quad + (k-1-n) w_2 \otimes v_{k-1} \qquad\qquad \text{for } 1 \leq k \leq n-1, \\ t_n &= \frac{n}{2} w_1 \otimes v_n - w_2 \otimes v_{n-1}. \end{aligned} \tag{1.20}$$

Suppose now that $n \geq 2$. By eq. (1.15) we have a direct summand isomorphic to $V(n-2)$, and by eq. (1.16) the weight space of weight $n-2$ is spanned by $w_0 \otimes v_2$, $w_1 \otimes v_1$, and $w_2 \otimes v_0$, so the vector of highest weight $n-2$ generating the summand corresponding to $V(n)$ must be of the form $aw_0 \otimes v_2 + bw_1 \otimes v_1 + cw_2 \otimes v_0$ for some $a, b, c \in \mathbf{C}$. Furthermore we know that for this vector generating the summand corresponding to $V(n-2)$, we must have

$$\begin{aligned} 0 &= x.(aw_0 \otimes v_2 + bw_1 \otimes v_1 + cw_2 \otimes v_0) \\ &= aw_0 \otimes x.v_2 + bx.w_1 \otimes v_1 + bw_1 \otimes x.v_1 + cx.w_2 \otimes v_0 \\ &= a(n-2+1) w_0 \otimes v_1 + b(2-1+1) w_0 \otimes v_1 + b(n-1+1) w_1 \otimes v_0 \\ &\quad + c(2-2+1) w_1 \otimes v_0 \\ &= \big((n-1)a + 2b\big) w_0 \otimes v_1 + \big(bn + c\big) w_1 \otimes v_0, \end{aligned}$$

i.e. $a(n-1) + 2b = 0$ and $bn + c = 0$. Giving us $c = -bn$ and $b = -\frac{n-1}{2}a$, so

$$c = \frac{n(n-1)}{2} a.$$

This determines the vector generating the summand corresponding to $V(n-2)$ up to a scalar, so taking $a = 1$, we see that we can take

$$u_0 \coloneqq w_0 \otimes v_2 - \frac{n-1}{2} w_1 \otimes v_1 + \frac{n(n-1)}{2} w_2 \otimes v_0$$

as our vector generating the summand corresponding to $V(n-2)$. Again $\mathfrak{sl}(2,\mathbf{C})$-theory now yields that this summand has a basis $(u_k \mid 0 \le k \le n-2)$ satisfying equations as in eq. (1.5), where

$$u_k := \frac{1}{k!} y^k . u_0. \tag{1.21}$$

By straightforward calculations, cf. Appendix A.1, we get that

$$\begin{aligned} u_k = {} & \frac{(k+1)(k+2)}{2} w_0 \otimes v_{k+2} - \frac{(k+1)(n-k-1)}{2} w_1 \otimes v_{k+1} \\ & + \frac{(n-k)(n-k-1)}{2} w_2 \otimes v_k \end{aligned} \tag{1.22}$$

for $0 \le k \le n-2$.

Now we want to express $w_1 \otimes v_k$ for $0 \le k \le n$ in terms of the bases $(s_k \mid 0 \le k \le n+2)$, $(t_k \mid 0 \le k \le n)$, and $(u_k \mid 0 \le k \le n-2)$. A straightforward but long calculation, cf. Appendix A.2, yields that

$$w_1 \otimes v_k = \frac{2(k+1)(n+1-k)}{(n+1)(n+2)} s_{k+1} - \frac{2(n-2k)}{n(n+2)} t_k - \frac{4}{n(n+1)} u_{k-1} \tag{1.23}$$

for $0 < k < n$, while

$$w_1 \otimes v_0 = \frac{2}{n+2}(s_1 - t_0) \qquad \text{and} \qquad w_1 \otimes v_n = \frac{2}{n+2}(s_{n+1} + t_n) \tag{1.24}$$

if $n \ge 1$. If $n = 0$ we have already seen (just after eq. (1.18)) that $w_1 \otimes v_0 = s_1$. Note that eq. (1.24) is a special case of eq. (1.23) if we set $u_{-1} = u_{n-1} = 0$.

Now consider $V(2)$ and $V(n)$ as inner product spaces over $\mathbf{C}$ with inner products given by

$$\langle w_k, w_j \rangle = \delta_{jk} \binom{2}{k} \qquad \text{and} \qquad \langle v_k, v_j \rangle = \delta_{jk} \binom{n}{k}. \tag{1.25}$$

Then we can consider $V(2) \otimes V(n)$ an inner product space with inner product given by

$$\langle w \otimes v, w' \otimes v' \rangle = \langle w, w' \rangle \cdot \langle v, v' \rangle \tag{1.26}$$

for $w, w' \in V(2)$ and $v, v' \in V(n)$. By straightforward calculations, cf. Appendix A.3, we get that

$$\langle s_0, s_0 \rangle = 1, \qquad \langle t_0, t_0 \rangle = \frac{n(n+2)}{2}, \qquad \langle u_0, u_0 \rangle = \frac{n^2(n+1)(n-1)}{4}. \tag{1.27}$$

Now set $\overline{w}_k = w_k/\|w_k\|$, $\overline{v}_k = v_k/\|v_k\|$, $\overline{s}_k = s_k/\|s_k\|$, $\overline{t}_k = t_k/\|t_k\|$, and $\overline{s}_k = s_k/\|s_k\|$ for all possible k, where $\|\cdot\|$ is given by $\|v\| = \sqrt{\langle v, v \rangle}$ as usually in an inner product space. Note, cf. Appendix A.3, that

$$\begin{aligned}
\langle w_k, w_k\rangle &= \binom{2}{k} \\
\langle v_k, v_k\rangle &= \binom{n}{k} \\
\langle s_k, s_k\rangle &= \langle s_0, s_0\rangle \binom{n+2}{k} = \binom{n+2}{k} \\
\langle t_k, t_k\rangle &= \langle t_0, t_0\rangle \binom{n}{k} = \frac{n(n+2)}{2}\binom{n}{k} \\
\langle u_k, u_k\rangle &= \langle u_0, u_0\rangle \binom{n-2}{k} = \frac{n^2(n+1)(n-1)}{4}\binom{n-2}{k}
\end{aligned} \tag{1.28}$$

for k where it makes sense, so we see that

$$w_k = \sqrt{\binom{2}{k}}\overline{w}_k, \qquad v_k = \sqrt{\binom{n}{k}}\overline{v}_k, \qquad s_k = \sqrt{\binom{n+2}{k}}\overline{s}_k, \tag{1.29}$$

and

$$t_k = \sqrt{\frac{n(n+2)}{2}\binom{n}{k}}\overline{t}_k, \qquad u_k = \sqrt{\frac{n^2(n+1)(n-1)}{4}\binom{n-2}{k}}\overline{u}_k. \tag{1.30}$$

Remark 1.6. Since

$$\overline{v}_k = \frac{1}{\sqrt{\binom{n}{k}}} v_k,$$

we note that we just need to change indices to go to the basis (e_m) from the basis of (v_k) as in the work leading to Lemma 1.4. $\triangle$

By a simple calculation, cf. Appendix A.4, we get that

$$\begin{aligned}
\overline{w}_1 \otimes \overline{v}_k = &\sqrt{\frac{2(k+1)(n+1-k)}{(n+1)(n+2)}}\overline{s}_{k+1} - \frac{n-2k}{\sqrt{n(n+2)}}\overline{t}_k \\
&- \sqrt{\frac{2k(n-k)}{n(n+1)}}\overline{u}_{k-1}.
\end{aligned} \tag{1.31}$$

for $0 \leq k \leq n$. Now changing indices as mentioned in Remark 1.6 to $\ell = \frac{1}{2}n$ and $m = \frac{1}{2}(n-2k) = \ell - k$ as we did to get to Lemma 1.4, i.e. $n = 2\ell$ and

$k = \ell - m$, we get that

$$\begin{aligned}
\overline{w}_1 \otimes e_m &= \overline{w}_1 \otimes \overline{v}_k \\
&= \sqrt{\frac{2(\ell-m+1)(2\ell+1)-(\ell-m)}{(2\ell+1)(2\ell+2)}}\overline{s}_{k+1} - \frac{(2\ell-2(\ell-m))}{\sqrt{2\ell(2\ell+2)}}\overline{t}_k \\
&\quad - \sqrt{\frac{2(\ell-m)(2\ell-(\ell-m))}{2\ell(2\ell+1)}}\overline{u}_{k-1} \\
&= \sqrt{\frac{(\ell-m+1)(\ell+1+m)}{(2\ell+1)(\ell+1)}}\overline{s}_{k+1} - \frac{m}{\sqrt{\ell(\ell+1)}}\overline{t}_k \\
&\quad - \sqrt{\frac{(\ell-m)(\ell+m)}{\ell(2\ell+1)}}\overline{u}_{k-1},
\end{aligned}$$

where e_m is as in the work we did to get Lemma 1.4 except for the fact that we consider $\mathfrak{sl}(2, \mathbf{C})$-modules still. Now setting

$$\widetilde{D}_+(\overline{v}_k) = -\frac{\overline{s}_{k+1}}{\sqrt{(\ell+1)(2\ell+1)}}, \quad \widetilde{D}_0(\overline{v}_k) = \frac{\overline{t}_k}{\sqrt{\ell(\ell+1)}}, \quad \widetilde{D}_-(\overline{v}_k) = -\frac{\overline{u}_{k-1}}{\sqrt{\ell(2\ell+1)}},$$

we see that

$$\begin{aligned}
\overline{w}_1 \otimes e_m &= \overline{w}_1 \otimes \overline{v}_k \\
&= \sqrt{(\ell+1)^2-m^2}\frac{\overline{s}_{k+1}}{\sqrt{(\ell+1)(2\ell+1)}} - m\frac{\overline{t}_k}{\sqrt{\ell(\ell+1)}} \\
&\quad - \sqrt{\ell^2-m^2}\frac{\overline{u}_{k-1}}{\ell(2\ell+1)} \\
&= \sqrt{\ell^2-m^2}\widetilde{D}_-(\overline{v}_k) - m\widetilde{D}_0(\overline{v}_k) - \sqrt{(\ell+1)^2-m^2}\widetilde{D}_+(\overline{v}_k).
\end{aligned} \tag{1.32}$$

Note that for $m \in \{\pm\ell\}$ the $\widetilde{D}_-$ term vanishes, so the formula works here although D_- is not well-defined in these edge cases.

* * * * *

Getting back to the problem at the end of Section 1.1.1, we want to give the maps D_0, D_+, and D_- such that $D_0 R_{\ell,m} \subset R_{\ell,m}$, $D_+ R_{\ell,m} \subset R_{\ell+1,m}$, and $D_- R_{\ell,m} \subset R_{\ell-1,m}$, the diagrams of eq. (1.12) commute, and we can describe F_3, F_+, F_- by the maps D_0, D_+, D_-, E_+, and E_-. Now consider the $\mathfrak{sl}(2, \mathbf{C})$-modules $V(n)$ as L_k-modules $M(n)$ via the isomorphism of eq. (1.4), and note that since

$$R_\ell = M(2\ell)^1 \oplus M(2\ell)^2 \oplus \cdots \oplus M(2\ell)^r$$

and each $M(2\ell)^i$ has a basis $(e^i_{-\ell}, e^i_{-\ell+1}, \ldots, e^i_{\ell-1}, e^i_\ell)$ with $H_3 e^i_m = m e^i_m$ for all m, we have that $R_{\ell,m}$ has basis $(e^1_m, e^2_m, \ldots, e^r_m)$ by definition. So when describing the maps D_0, D_+, and D_-, we just need to describe what the maps should do to each e_m^i. We already know that $E_+ e_m^i = e_{m+1}^i$ and $E_- e_m^i = e_{m-1}^i$ where it makes sense, so if the maps D_0, D_+, and D_- do not depend on m or i, we get the commutative diagrams of eq. (1.12), thus we want to describe what each map does to $M(2\ell)$ in general, so we will stop writing the superscripts.

Since we want to describe the maps F_3, F_+, and F_-, we are actually interested in the actions of L_p, so by using ψ of eq. (1.14) and the considerations at the end of Section 1.1.1, we can start out by describing the L_k-module $M(2) \otimes M(2\ell)$, i.e. we can use the description of $V(2) \otimes V(n)$ from above. Note that we have already seen that $L_p \simeq M(2)$ as L_k-modules, but we would like to better understand how the basis (f_+, f_3, f_-) of L_p corresponds to the basis (w_0, w_1, w_2) of $M(2)$ as in eq. (1.6). In the basis (w_0, w_1, w_2) we have that $h_+.w_0 = 0$ (since this is what corresponds to $x.w_0 = 0$ in $V(2)$ by eq. (1.4)), so by checking eq. (1.3) we see that w_0 must correspond to a multiple of f_3, but the basis is chosen up to scalar, so we can take w_0 to be $-\frac{\sqrt{2}}{2} f_3$. Now we get w_1 by taking $h_-.w_0$ (corresponding to $y.w_0$ in $V(2)$ by eq. (1.4)), thus we get that

$$w_1 = h_-.w_0 = -\tfrac{\sqrt{2}}{2} h_-.f_+ = -\tfrac{\sqrt{2}}{2}[h_-, f_+] = \sqrt{2} f_3.$$

Likewise we get that $w_2 = [h_-, \sqrt{2} f_3] = \sqrt{2} f_-$, so we can take our basis to be $(w_0, w_1, w_2) = (-\frac{\sqrt{2}}{2} f_+, \sqrt{2} f_3, \sqrt{2} f_-)$ when thinking of L_p as the L_k-module $M(2)$. Normalizing as in eq. (1.29), we get that $(\overline{w}_0, \overline{w}_1, \overline{w}_2) = (-\frac{\sqrt{2}}{2} f_+, f_3, \sqrt{2} f_-)$. So by eq. (1.32), we see that in $L_p \otimes M(2\ell)$

$$f_3 \otimes e_m = \sqrt{\ell^2 - m^2}\, \widetilde{D}_-(e_m) - m \widetilde{D}_0(e_m) - \sqrt{(\ell+1)^2 - m^2}\, \widetilde{D}_+(e_m),$$

where $e_m = \overline{v}_k$ for $k = \ell - m$ and $f_3 = \overline{w}_1$.

Remark 1.7. Note that if we have bases $(e_{-\ell-1}^{(2\ell+2)}, e_{-\ell}^{(2\ell+2)}, \dots, e_{\ell}^{(2\ell+2)}, e_{\ell+1}^{(2\ell+2)})$ for $M(2\ell+2)$, $(e_{-\ell}^{(2\ell)}, e_{-\ell+1}^{(2\ell)}, \dots, e_{\ell-1}^{(2\ell)}, e_{\ell}^{(2\ell)})$ for $M(2\ell)$, and $(e_{-\ell+1}^{(2\ell-2)}, e_{-\ell+2}^{(2\ell-2)}, \dots, e_{\ell-2}^{(2\ell-2)}, e_{\ell-1}^{(2\ell-2)})$ for $M(2\ell-2)$ (if $\ell \geq 2$) as in Lemma 1.4, then as above changing indices with $k = \ell + 1 - m$ we see that $e_m^{(2\ell+2)}$ corresponds to $\overline{s}_k$. Likewise changing indices with $k = \ell - m$ we see that $e_m^{(2\ell)}$ corresponds to $\overline{t}_k$, and with $k = \ell - 1 - m$ we see that $e_m^{(2\ell-2)}$ corresponds to $\overline{u}_k$. △

By this remark together with Remark 1.5, we see that $E_+ \overline{v}_k = \overline{v}_{k-1}$ and $E_+ \overline{s}_k = \overline{s}_{k-1}$ where it makes sense, so by the definition of $\widetilde{D}_+$ it commutes with E_+ and E_-. Similarly we can see that $\widetilde{D}_0$ and $\widetilde{D}_-$ commute with E_+ and E_-. Note that it makes sense to talk about using E_+ and E_- here by Remark 1.5.

Now using ψ from eq. (1.14), we see that

$$\begin{aligned} F_3 e_m &= f_3.e_m = \psi(f_3 \otimes e_m) \\ &= \sqrt{\ell^2 - m^2}\, \psi \widetilde{D}_-(e_m) - m \psi \widetilde{D}_0(e_m) - \sqrt{(\ell+1)^2 - m^2}\, \psi \widetilde{D}_+(e_m). \end{aligned} \tag{1.33}$$

So we can take $D_0 = \psi\widetilde{D}_0$, $D_+ = \psi\widetilde{D}_+$, and $D_- = \psi\widetilde{D}_-$ to get three linear maps with which we can describe the map F_3. So far this is just maps on $M(2\ell)$, but we can expand to maps on R_ℓ by using the maps on each summand of $R_\ell = M(2\ell)^1 \oplus \cdots \oplus M(2\ell)^r$, and likewise we can expand further to maps on $M = \bigoplus_\ell R_\ell$ by using the maps on each summand. Also indeed $D_0 R_{\ell,m} \subset R_{\ell,m}$, $D_+ R_{\ell,m} \subset R_{\ell+1,m}$, and $D_- R_{\ell,m} \subset R_{\ell-1,m}$, since for $\xi \in R_{\ell,m}$ we have that

$$\begin{aligned} H_3 D_0(\xi) &= h_3.\psi\widetilde{D}_0(\xi) = \psi h_3.\widetilde{D}_0(\xi) = \psi H_3\widetilde{D}_0(\xi) = m\psi\widetilde{D}_0(\xi) \\ &= mD_0(\xi), \end{aligned}$$

since ψ is an L_k-module homomorphism and by Remark 1.7 we see that $\widetilde{D}_0(e_m)$ is a scalar multiple of $\bar{t}_k = \bar{t}_{\ell-m} = e_m^{(2\ell)}$, and indeed $H_3 e_m^{(2\ell)} = m e_m^{(2\ell)}$. The same reasoning with $\bar{s}_{k+1}$ for D_+ and $\bar{u}_{k-1}$ for D_- yields the other two inclusions. Also note that the diagrams of eq. (1.12) commute by the defintion of D_0, D_+, and D_-, since we have already shown that $\widetilde{D}_0$, $\widetilde{D}_+$, and $\widetilde{D}_-$ commute with E_+ and E_- where it makes sense, and L_k-module homomorphisms in general commute with E_+ and E_-, so also ψ commutes with E_+ and E_-. This is the case since for an L_k-module homomorphism $\varphi\colon M \to M'$ with decompositions $M = \bigoplus_{\ell,m} R_{\ell,m}$ and $M' = \bigoplus_{\ell,m} R'_{\ell,m}$, we have that $\gamma R_{\ell,m} \subset R'_{\ell,m}$, so since γ commutes with H_+ and H_- this implies by eq. (1.11) that γ commutes with E_+ and E_-. To see that $\gamma R_{\ell,m} \subset R'_{\ell,m}$ first note that R_ℓ and R'_ℓ are direct sums of finitely many L_k-modules $M(2\ell)$, where each $M(2\ell)$ is generated by a maximal vector of weight ℓ (weight w.r.t. h_3 in L_k). So since γ takes a maximal vector of weight ℓ to either zero or another maximal vector of weight ℓ, we see that indeed $\gamma R_\ell \subset R'_\ell$. Finally since $\gamma \in \operatorname{Hom}_{L_k}(M, M')$ we have that $\gamma H_3 = H'_3\gamma$, so vectors of weight m go to 0 or vectors of weight m, and thus $\gamma R_{\ell,m} \subset R'_{\ell,m}$.

Remark 1.8. Note that D_0 is not defined at $R_{0,0}$, but D_0 is defined on $R_{\ell,0}$ by using the relation $E_+D_0 = E_+D_0$ repeatedly. Also note that D_- is not defined at $R_{\ell,\pm\ell}$. When we sometimes later use D_0 on $R_{0,0}$, it is taken to be given by multiplication by some constant d_0^0, and also we will use the convention that D_- is 0 on $R_{\ell_0,m}$. △

Now simple calculations, cf. Appendix A.5, gives us that

$$\begin{aligned} F_3\xi &= \sqrt{\ell^2 - m^2}D_-\xi - mD_0\xi - \sqrt{(\ell+1)^2 - m^2}D_+\xi, \\ F_+\xi &= \sqrt{(\ell-m)(\ell-m-1)}D_-E_+\xi - \sqrt{(\ell-m)(\ell+m+1)}D_0E_+\xi \\ &\quad + \sqrt{(\ell+m+1)(\ell+m+2)}E_+D_+\xi, \\ F_-\xi &= -\sqrt{(\ell+m)(\ell+m-1)}D_-E_-\xi - \sqrt{(\ell+m)(\ell-m+1)}D_0E_-\xi \\ &\quad - \sqrt{(\ell-m+1)(\ell-m+2)}E_-D_+\xi \end{aligned} \tag{1.34}$$

for $\xi \in R_{\ell,m}$. Note here that although D_- is not defined on $R_{\ell,\ell}$ and $R_{\ell,-\ell}$ the above still makes sense since in these cases the terms with D_- vanish, either by the coefficient being zero or by E_+ or E_- mapping to zero.

We claim now that the formulae eq. (1.34) for the linear operators F_+, F_-, and F_3 together with the formulae eqs. (1.9) and (1.11) for H_+, H_-, and H_3 define a representation of L, i.e. they satisfy the commutation relations of eq. (1.3), if and only if D_0, D_+, and D_- satisfy

$$\begin{aligned} \ell D_+D_0\xi &= (\ell+2)D_0D_+\xi, \\ (\ell+1)D_-D_0\xi &= (\ell-1)D_0D_-\xi, \\ \xi &= (2\ell-1)D_+D_-\xi - (2\ell+3)D_-D_+\xi - D_0^2\xi \end{aligned} \tag{1.35}$$

for $\xi \in R_{\ell,m}$, $\ell \neq 0$ except in the first equation. Here the term D_+D_- should be replaced with $E_+D_+D_-E_-$ for $\xi \in R_{\ell,\ell}$ and $E_-D_+D_-E_+$ for $\xi \in R_{\ell,-\ell}$.

1.1.3 Simple Harish-Chandra modules for the pair (L, L_k)

We are now almost ready to classify the simple Harish-Chandra modules for the pair (L, L_k), but before most of the work we need some basic results.

Let M be an simple Harish-Chandra module over L and suppose that each non-trivial subspace $R_{\ell,m}$ in $M = \bigoplus_{\ell,m} R_{\ell,m}$ is one dimensional. In this case each L_k-module R_ℓ is either isomorphic to the simple module $M(2\ell)$ or 0. We claim that actually all simple Harish-Chandra modules are of this kind, so we indeed get a classification of the simple Harish-Chandra modules in the following.

To see this assume that $V \subset R_\ell$ is a simple L_k-submodule in M, i.e. $V \simeq M(2\ell)$, and put

$$\overline{D_+V} = D_+(V) + E_+(D_+(V)) + E_-(D_+(V)).$$

We claim that $\overline{D_+V}$ is either a simple L_k-submodule or 0. This is clear since by construction if $\overline{D_+V}$ is non-zero, it sits inside an L_k-submodule isomorphic to $M(2\ell+2)$, and here we have the weight m elements for $m \in \{-\ell, -\ell+1, \ldots, \ell-1, \ell\}$ sitting in $D_+(V)$, the $-\ell-1$ weight elements in $E_-(D_+(V))$, and the $\ell+1$ elements in $E_+(D_+(V))$, so $\overline{D_+V} \simeq M(2\ell+2)$. Now if $D_0(V) \subset V$ then eq. (1.35) imply that $D_0(\overline{D_+V}) \subset \overline{D_+V}$ since E_+ and E_- commute with D_0. Likewise if $D_+D_-(V) \subset V$ we get that $D_-(\overline{D_+V}) \subset V$ and $D_+D_-(\overline{D_+V}) \subset \overline{D_+V}$.

Now consider an eigenvector $v \in R_{\ell_0,\ell_0}$ of D_0 (or of Δ_2 in case $\ell_0 = 0$), and let V_{ℓ_0} be the subspace spanned by all $(E_-)^r v$. This is clearly a simple L_k-module isomorphic to $M(2\ell_0)$ (up to constants on the terms it is exactly the construction of $M(2\ell_0)$). Inductively define $V_\ell = \overline{D_+V_{\ell-1}}$ for all $\ell > \ell_0$. Then $\bigoplus_{\ell \geq \ell_0} V_\ell$ is an L-submodule of M since by definition it is invariant under D_+, E_+, and E_-, and by eq. (1.35) $D_0(V_{\ell_0}) \subset V_{\ell_0}$ and $D_+D_-(V_{\ell_0}) \subset V_{\ell_0}$ so $\bigoplus_{\ell \geq \ell_0} V_\ell$ is also invariant under D_0 and D_- by the above. Hence if M is simple, we get that $M = \bigoplus_{\ell \geq \ell_0} V_\ell$, and thus indeed each non-trivial subspace $R_{\ell,m}$ in M is one dimensional.

Denote by ℓ_0 the minimal index ℓ in the decomposition $M = \bigoplus_\ell R_\ell$. Note that

$$M' = \bigoplus_{\ell' \in \{\ell_0, \ell_0+1, \dots\}} R_{\ell'}$$

is invariant under E_+, E_-, D_0, D_+, and D_-, so by the formulae eq. (1.34) for F_+, F_-, and F_3, we see that M' is a submodule since we already know that it is an L_k-submodule because $R_{\ell'}$ all are L_k-submodules. Thus $M' = M$ since M is simple and hence the index ℓ in $M = \bigoplus_\ell R_\ell$ range over only integral values or only half-integral values.

Additionally we want to show that the kernel of the map $D_-\colon M \to M$ is R_{ℓ_0}. To do this assume for contradiction that $D_- R_{\ell', m_0} = 0$ for some index $\ell' > \ell_0$ and $m_0 \in \{-\ell_0, -\ell_0 + 1, \dots, \ell_0 - 1, \ell_0\}$. Then by the commutative diagram in eq. (1.12) with D_-, i.e. $D_- E_+ = E_+ D_-$, and the fact that $E_+\colon R_{\ell',m} \to R_{\ell',m+1}$ is an isomorphism for $m < \ell'$, we see that $D_- R_{\ell',m} = 0$ for all $m \in \{-\ell', -\ell' + 1, \dots, \ell' - 1, \ell'\}$. But then

$$M'' = \bigoplus_{\ell'' \in \{\ell', \ell'+1, \dots\}} R_{\ell''}$$

is a proper L-submodule of M, which contradicts the simplicity of M. Thus indeed $\ker D_- = R_{\ell_0}$.

Likewise we see that if M is infinite dimensional, then $D_+\colon M \to M$ has trivial kernel since if $D_+ R_{\ell'} = 0$, then $M = \bigoplus_{\ell \in \{\ell_0, \ell_0+1, \dots\}} R_\ell$ is finite dimensional. This is the case since all terms with $\ell > \ell'$ must be trivial since otherwise

$$M'' = \bigoplus_{\ell'' \in \{\ell_0, \ell_0+1, \dots, \ell'\}} R_{\ell''}$$

is a proper L-submodule of M, which contradicts the simplicity of M.

Infinite dimensional simple modules

Assume that M is a Harish-Chandra module of the above kind and is infinite dimensional. Because all $R_{\ell,m}$ are one dimensional, the diagram with E_+ and D_+ in eq. (1.12) commute, i.e. $D_+ E_+ = E_+ D_+$, and D_+ has trivial kernel, while E_+ is an isomorphism for $m \neq \ell$, we see that we can choose a basis $\{\xi_{\ell,m}\}$ of M such that $\xi_{\ell,m} \in R_{\ell,m}$ and

$$\begin{aligned} E_+ \xi_{\ell,m} &= \xi_{\ell,m+1} && \text{for } -\ell \leq m < \ell, \\ D_+ \xi_{\ell,m} &= \xi_{\ell+1,m} && \text{for } \ell \in \{\ell_0, \ell_0 + 1, \dots\}. \end{aligned}$$

In this basis we get that

$$\begin{aligned} E_-\xi_{\ell,m} &= \xi_{\ell,m-1} && \text{for } -\ell < m \le \ell, \\ D_0\xi_{\ell,m} &= d^0_\ell \xi_{\ell,m} && \text{for } \ell \in \{\ell_0, \ell_0+1, \dots\}, \\ D_-\xi_{\ell,m} &= d^-_\ell \xi_{\ell-1,m} && \text{for } \ell \in \{\ell_0+1, \ell_0+2, \dots\}, \\ D_-\xi_{\ell_0,m} &= 0, \end{aligned} \tag{1.36}$$

where the first equation comes from the fact that $E_-\colon R_{\ell,m} \to R_{\ell,m-1}$ for $m \neq -\ell$ is the inverse of $E_+\colon R_{\ell,m-1} \to R_{\ell,m}$, while the independence of m in the other equations comes from the commutativity of the diagrams of eq. (1.12). Note that we here unlike normally define D_0 on $R_{0,0}$ by multiplication by some constant d^0_0.

Now eqs. (1.35) and (1.36) implies that

$$\begin{aligned} \ell d^0_\ell &= (\ell+2) d^0_{\ell+1}, \\ (\ell+1) d^-_\ell d^0_\ell &= (\ell-1) d^0_{\ell-1} d^-_\ell, \\ 1 &= (2\ell-1) d^-_\ell - (2\ell+3) d^-_{\ell+1} - (d^0_\ell)^2, \\ d^-_{\ell_0} &= 0, \end{aligned} \tag{1.37}$$

for $\ell \in \{\ell_0, \ell_0+1, \dots\}$, $\ell \neq 0$ except in the first equation, and in the second equation we also demand that $\ell > \ell_0$. We see that

$$d^0_{\ell+1} = \frac{\ell}{\ell+2} d^0_\ell.$$

So if $\ell_0 \neq 0$, then for some constant c

$$d^0_{\ell_0} = \frac{c}{\ell_0(\ell_0+1)},$$

so we see inductively that if

$$d^0_\ell = \frac{c}{\ell(\ell+1)}, \tag{1.38}$$

then

$$\begin{aligned} d^0_{\ell+1} &= \frac{\ell}{\ell+2} d^0_\ell = \frac{\ell}{\ell+2}\frac{c}{\ell(\ell+1)} \\ &= \frac{c}{(\ell+1)(\ell+2)}. \end{aligned}$$

Thus if $\ell_0 \neq 0$ eq. (1.38) holds true in general for some constant c.

If on the other hand $\ell_0 = 0$, then we see that

$$2d^0_{\ell_0+1} = 0,$$

so $d^0_{\ell_0+1} = 0$, and thus

$$d^0_\ell = \frac{\ell-1}{\ell+1} d^0_{\ell-1} = 0$$

for all $\ell \in \{1, 2, \ldots\}$. Also in this case have $d^0_0 = c_1$, where c_1 is some constant.

To unify these two cases we set $c = i\ell_0\ell_1$ and $c_1 = i\ell_1$ for some complex constant ℓ_1 such that

$$d^0_\ell = \frac{i\ell_0\ell_1}{\ell(\ell+1)} \tag{1.39}$$

for $\ell \in \{\ell_0, \ell_0+1, \ldots\}$. Substituting this expression with d^0_ℓ in the third equation of eq. (1.37) we get that

$$(2\ell-1)d^-_\ell - (2\ell+3)d^-_{\ell+1} = 1 - \frac{\ell_0^2\ell_1^2}{\ell^2(\ell+1)^2},$$

and a simple calculation, cf. Appendix A.7, yields that

$$d^-_\ell = -\frac{\left(\ell^2-\ell_1^2\right)\left(\ell^2-\ell_0^2\right)}{\ell^2(4\ell^2-1)}, \tag{1.40}$$

for $\ell > \ell_0$.

Since we showed in the beginning of this subsection that the kernel of D_- is R_{ℓ_0}, we must have that $d^-_\ell \neq 0$ for all $\ell > \ell_0$. Thus $\ell^2 - \ell_1^2 \neq 0$ for $\ell > \ell_0$, so $|\ell_1| - \ell_0$ cannot be a positive integer if ℓ_1 is real, because if that was the case then $|\ell_1| = \ell_0 + (|\ell_1| - \ell_0) \in \{\ell_0+1, \ell_0+2, \ldots\}$, but $|\ell_1|^2 - \ell_1^2 = 0$ since $\ell_1 \in \mathbf{R}$.

Hence altogether by eqs. (1.11) and (1.34) in the basis $\{\xi_{\ell,m}\}$ the operators H_+, H_-, H_3, F_+, F_-, and F_3 are given by the formulae

$$\begin{aligned}
H_3\xi_{\ell,m} &= m\xi_{\ell,m},\\
H_+\xi_{\ell,m} &= \sqrt{(\ell+m+1)(\ell-m)}\xi_{\ell,m+1},\\
H_-\xi_{\ell,m} &= \sqrt{(\ell+m)(\ell-m+1)}\xi_{\ell,m-1},\\
F_3\xi_{\ell,m} &= \sqrt{\ell^2-m^2}d^-_\ell\xi_{\ell-1,m} - md^0_\ell\xi_{\ell,m} - \sqrt{(\ell+1)^2-m^2}d^+_\ell\xi_{\ell+1,m},\\
F_+\xi_{\ell,m} &= \sqrt{(\ell-m)(\ell-m-1)}d^-_\ell\xi_{\ell-1,m+1} - \sqrt{(\ell-m)(\ell+m+1)}d^0_\ell\xi_{\ell,m+1}\\
&\quad + \sqrt{(\ell+m+1)(\ell+m+2)}d^+_\ell\xi_{\ell+1,m+1},\\
F_-\xi_{\ell,m} &= -\sqrt{(\ell+m)(\ell+m-1)}\xi_{\ell-1,m-1} - \sqrt{(\ell+m)(\ell-m+1)}\xi_{\ell,m-1}\\
&\quad - \sqrt{(\ell-m+1)(\ell-m+2)}\xi_{\ell+1,m-1},
\end{aligned} \tag{1.41}$$

where

$$d^0_\ell = \frac{i\ell_0\ell_1}{\ell(\ell+1)}, \qquad d^-_\ell = -\frac{\left(\ell^2-\ell_1^2\right)\left(\ell^2-\ell_0^2\right)}{\ell^2(4\ell^2-1)}, \qquad d^+_\ell = 1, \tag{1.42}$$

for $\ell \in \{\ell_0, \ell_0+1, \ldots\}$, and where ℓ_1 is a complex number such that $|\ell_1| - \ell_0$ is not a positive integer if ℓ_1 is real. Here we use the convention that $\xi_{\ell',m'} = 0$ for pairs ℓ', m' where there is no such basis element. Also since the formuale of eq. (1.35) hold by definition, we see that given such maps we indeed get a simple L-module.

Finite dimensional simple modules

Assume that M is a Harish-Chandra module of the above kind and that M is finite dimensional, i.e. $M = \bigoplus_{\ell,m} R_{\ell,m}$ where $R_{\ell,m}$ are one dimensional subspaces for $\ell_0 \leq \ell < |\ell_1|$. Here ℓ_1 is some real number such that $|\ell_1| \geq \ell_0$ and $|\ell_1| - \ell_0$ is integral. We can choose a basis $\{\xi_{\ell,m}\}$ as in the infinite dimensional case and by similar considerations we still get the formulae eqs. (1.41) and (1.42) describing the actions of H_+, H_-, H_3, F_+, F_-, and F_3, though now in this basis we only consider $\ell \in \{\ell_0, \ell_0+1, \ldots, |\ell_1|-1\}$.

Here it is worth noting that we can actually describe the modules accurately using some algebraic results which we will not show in this paper. Firstly note that we know that all simple finite dimensional $\mathfrak{sl}(2,\mathbf{C})$-modules are the $V(n)$ we have described earlier, and from this it can be shown that all simple finite dimensional $\mathfrak{sl}(2,\mathbf{C}) \times \mathfrak{sl}(2,\mathbf{C})$-modules are of the form $V(n) \otimes V(m)$, where the action of $\mathfrak{sl}(2,\mathbf{C}) \times \mathfrak{sl}(2,\mathbf{C})$ on $V(n) \otimes V(m)$ is given by

$$(x,y).v_1 \otimes v_2 = (x.v_1) \otimes v_2 + v_1 \otimes (y.v_2).$$

Note that considering $V(n) \otimes V(m)$ as $\mathfrak{sl}(2,\mathbf{C})$-modules via the diagonal map $\mathfrak{sl}(2,\mathbf{C}) \to \mathfrak{sl}(2,\mathbf{C}) \times \mathfrak{sl}(2,\mathbf{C})$, $u \mapsto (u,u)$, we get the standard action of $\mathfrak{sl}(2,\mathbf{C})$ on $V(n) \otimes V(m)$ as described in eq. (1.13), and now Clebsch-Gordan's formula (which we proved in the special case of $V(2) \otimes V(n)$ earlier) implies that

$$V(n) \otimes V(m) \simeq \bigoplus_{i=0}^{\min(n,m)} V(n+m-2i).$$

Note that the last term of the direct sum if $V(|n-m|)$. Thus translating to the language of L and L_k-modules, we have that all simple finite dimensional L-modules are of the form $\bigoplus_{i=0}^{\min(2r,2s)} M(2r+2s-2i)$ where $r, s \in \frac{1}{2}\mathbf{Z}_{\geq 0}$. Here we see that $\ell_0 = |r-s|$ and $|\ell_1| - 1 = r+s$.

1.2 Decomposition of modules into indecomposables

Now we want to continue our work using our knowledge of the classification of simple Harish-Chandra modules for the pair (L, L_k) to begin our classification of indecomposable Harish-Chandra modules for the pair (L, L_k). To do this we will first need to do some work with Laplace operators.

1.2.1 Laplace operators

Let $U(L)$ be the universal enveloping algebra of L, cf. [Jan16, Appendix E]. We know, cf. [Jan16, p. E-9], that M is an L-module if and only if it is an $U(L)$-module, so we can describe L-modules by describing $U(L)$-modules. To do this we will first need to have an explicit description of the center $Z(U(L))$ of $U(L)$. We will begin this description by first noting that $Z(U(\mathfrak{sl}(2,\mathbf{C}) \times \mathfrak{sl}(2,\mathbf{C}))) \simeq Z(U(\mathfrak{sl}(2,\mathbf{C}))) \otimes Z(U(\mathfrak{sl}(2,\mathbf{C})))$, which follows from the fact that $Z(U(L_1 \times L_2)) \simeq Z(U(L_1)) \otimes Z(U(L_2))$ for Lie algebras L_1 and L_2 in general cf. Appendix B.1.

It is a result in Lie algebra that $Z(U(\mathfrak{sl}(2,\mathbf{C})))$ is the algebra of polynomials in $C = h^2 + 2h + 4yx$, i.e. $Z(U(\mathfrak{sl}(2,\mathbf{C}))) = \mathbf{C}[C]$ (we saw this in Exercise 11 in the Lie algebra course). Thus we see that $Z(U(\mathfrak{sl}(2,\mathbf{C}))) \otimes Z(U(\mathfrak{sl}(2,\mathbf{C})))$ is the algebra of polynomials in $C \otimes 1$ and $1 \otimes C$, or equivalently the algebra of polynomials in $C \otimes 1 - 1 \otimes C$ and $C \otimes 1 + 1 \otimes C$. So we want to describe $C \otimes 1 - 1 \otimes C$ and $C \otimes 1 + 1 \otimes C$ in terms of our basis $h_+, h_-, h_3, f_+, f_-, f_3$. We note that $(u, u') \in L = \mathfrak{sl}(2,\mathbf{C}) \times \mathfrak{sl}(2,\mathbf{C})$ in $U(L) = U(\mathfrak{sl}(2,\mathbf{C}) \times \mathfrak{sl}(2,\mathbf{C}))$ is identified with $u \otimes 1 + 1 \otimes u'$ in $U(\mathfrak{sl}(2,\mathbf{C})) \otimes U(\mathfrak{sl}(2,\mathbf{C}))$, so

$$
\begin{aligned}
&\tfrac{1}{2}(h_-f_+ + f_-h_+) + h_3f_3 + f_3 \\
&= \tfrac{1}{2}\big((y \otimes 1 + 1 \otimes y)(ix \otimes 1 - 1 \otimes ix) + (iy \otimes 1 - 1 \otimes iy)(x \otimes 1 + 1 \otimes x)\big) \\
&\quad + \tfrac{1}{4}(h \otimes 1 + 1 \otimes h)(ih \otimes 1 - 1 \otimes ih) + \tfrac{1}{2}(ih \otimes 1 - 1 \otimes ih) \\
&= \tfrac{1}{2}\big(iyx \otimes 1 - iy \otimes x + ix \otimes y - i \otimes yx + iyx \otimes 1 + iy \otimes x - ix \otimes y - i \otimes yx\big) \\
&\quad + \tfrac{1}{4}(ih^2 \otimes 1 - ih \otimes h + ih \otimes h - i \otimes h^2) + \tfrac{1}{2}(ih \otimes 1 - 1 \otimes ih) \\
&= \tfrac{1}{2}\big(2iyx \otimes 1 - 2i \otimes yx\big) + \tfrac{1}{4}(ih^2 \otimes 1 - i \otimes h^2) + \tfrac{1}{2}(ih \otimes 1 - i \otimes h) \\
&= iyx \otimes 1 + \tfrac{1}{4}ih^2 \otimes 1 + \tfrac{1}{2}ih \otimes 1 - i \otimes yx - \tfrac{1}{4}i \otimes h^2 - \tfrac{1}{2}i \otimes h \\
&= \tfrac{i}{4}(h^2 + 2h + yx) \otimes 1 - \tfrac{i}{4}1 \otimes (h^2 + 2h + yx) \\
&= \tfrac{i}{4}(C \otimes 1 - 1 \otimes C),
\end{aligned}
$$

and likewise $h_-h_+ - f_-f_+ + h_3^2 - f_3^2 + 2h_3 = \frac{1}{2}(C \otimes 1 + 1 \otimes C)$.

Thus since the constants don't matter when we look at the algebra of polynomials in $C \otimes 1 - 1 \otimes C$ and $C \otimes 1 + 1 \otimes C$, we see that setting

$$
\begin{aligned}
\Delta_1 &= \tfrac{1}{2}(h_-f_+ + f_-h_+) + h_3f_3 + f_3, \\
\Delta_2 &= h_-h_+ - f_-f_+ + h_3^2 - f_3^2 + 2h_3,
\end{aligned}
\tag{1.43}
$$

we have that $Z(U(L))$ is the algebra of polynomials in Δ_1 and Δ_2. Thus in term of the corresponding linear operators on a Harish-Chandra module M for the pair (L, L_k), we define linear operators

$$
\begin{aligned}
\Delta_1 &:= \tfrac{1}{2}(H_-F_+ + F_-H_+) + H_3F_3 + F_3 \\
\Delta_2 &:= H_-H_+ - F_-F_+ + H_3^2 - F_3^2 + 2H_3,
\end{aligned}
\tag{1.44}
$$

which are called Laplace operators. Note that by eqs. (1.9), (1.10) and (1.34), cf. Appendix A.8, we get that

$$\begin{aligned} \Delta_1\xi &= -\ell(\ell+1)D_0\xi \\ \Delta_2\xi &= (\ell^2-1)\xi - (\ell+1)^2 D_0^2\xi + (4\ell^2-1)D_+D_-\xi \end{aligned} \tag{1.45}$$

for $\xi \in R_\ell$, $\ell \neq 0$. Here as in the relations for the D's earlier, we replace D_+D_- with $E_+D_+D_-E_-$ for $\xi \in R_{\ell,\ell}$ and $E_-D_+D_-E_+$ for $\xi \in R_{\ell,-\ell}$. Alternatively by eq. (1.35), cf. Appendix A.8, we also get that

$$\Delta_2\xi = \big((\ell+1)^2-1\big)\xi + \ell^2 D_0^2\xi + \big(4(\ell+1)^2-1\big)D_-D_+\xi \tag{1.46}$$

for $\xi \in R_\ell$, which will sometimes be more useful.

Now by noting that D_0, D_+D_-, and D_0^2 all preserve $R_{\ell,m}$ eq. (1.45) gives us the following Lemma:

Lemma 1.9. *Each subspace $R_{\ell,m}$ is invariant under the Laplace operators Δ_1 and Δ_2.*

Additionally we are ready to prove the Lemma:

Lemma 1.10. *The linear operators D_+, D_-, D_0, E_+, and E_- commute with the Laplace operators Δ_1 and Δ_2.*

Proof. Denote by $(\Delta_i)_{\ell,m}$ the restriction of Δ_i to $R_{\ell,m}$ for $i = 1,2$. Lemma 1.9 implies that $\Delta_i = \bigoplus_{\ell,m}(\Delta_i)_{\ell,m}$ for $i = 1,2$, so it is enough to check that $(\Delta)_{\ell,m}$ commutes with the operators for all ℓ and m. Therefore eqs. (1.45) and (1.46) implies that Δ_i commute with E_+ and E_- since D_+, D_-, and D_0 commute with E_+ and E_- where it makes sense and using eq. (1.46) for Δ_2 it makes sense for all $R_{\ell,m}$.

Now multiplying the first equation of eq. (1.35) with $\ell+1$, we see that

$$\ell(\ell+1)D_+D_0\xi = (\ell+1)(\ell+2)D_0D_+\xi$$

for $\xi \in R_{\ell,m}$, so by eq. (1.45), we see that

$$D_+\Delta_1\xi = -\ell(\ell+1)D_+D_0\xi = -(\ell+1)(\ell+2)D_0D_+\xi = \Delta_1 D_+\xi$$

for $\xi \in R_{\ell,m}$. Thus Δ_1 indeed commutes with D_+. Similarly the second equation of eq. (1.35) imply that Δ_1 commutes with D_-, and also it is obvious from eq. (1.45) that Δ_1 commutes with D_0.

Likewise the first equation of eq. (1.35) together with eqs. (1.45) and (1.46) implies that

$$\begin{aligned} \Delta_2 D_+\xi &= \big((\ell+1)^2-1\big)D_+\xi - (\ell+2)^2 D_0^2 D_+\xi + \big(4(\ell+1)^2-1\big)D_+D_-D_+\xi \\ &= \big((\ell+1)^2-1\big)D_+\xi - \ell^2 D_+D_0^2\xi + \big(4(\ell+1)^2-1\big)D_+D_-D_+\xi \\ &= D_+\Delta_2\xi \end{aligned}$$

for $\xi \in R_{\ell,m}$. Thus Δ_2 commutes with D_+, and similarly using the second equation of eq. (1.35) we get that Δ_2 commutes with D_-. Finally it is clear that D_0 commutes with the first two terms of Δ_2, so we just need to show that $D_0(D_+D_-)\xi = (D_+D_-)D_0\xi$ for $\xi \in R_{\ell,m}$ where it makes sense. But now the first and second equation of eq. (1.35) imply that

$$(\ell+1)D_0D_+D_-\xi = (\ell-1)D_+D_0D_-\xi = (\ell+1)D_+D_-D_0\xi$$

for $\xi \in R_{\ell,m}$, so since $\ell \geq 0$ and thus $\ell \neq -1$, we get that $D_0(D_+D_-)\xi = (D_+D_-)D_0\xi$. Hence Δ_2 indeed commutes with D_0 also. □

1.2.2 Properties of the Laplace operators in indecomposable modules

Now we are finally ready to begin considering the properties of Δ_1 and Δ_2 in indecomposable Harish-Chandra modules, which will end up being an important part of our characterization of indecomposable Harish-Chandra modules for the pair (L, L_k).

Proposition 1.11. *A Harish-Chandra module M for the pair (L, L_k) is decomposable into the direct sum of a countable number of indecomposable modules such that on each indecomposable module the Laplace operators Δ_1 and Δ_2 have each one eigenvalue, λ_1 and λ_2 respectively.*

Proof. Since each of the subspaces $R_{\ell,m}$ is invariant under Δ_1 and Δ_2 by Lemma 1.9 and since these operators commute with each other, we get that $R_{\ell,m}$ can be written as a direct sum of subspaces $R_{\ell,m}(\lambda_1^i, \lambda_2^i)$ on each of which each of the operators Δ_1 and Δ_2 has one eigenvalue λ_1^i and λ_2^i respectively. Note that here the index set of i is finite since $R_{\ell,m}$ is finite dimensional.

Consider now fixed λ_1 and λ_2 and the set S of those (ℓ, m) for which there exists subspaces $R_{\ell,m}(\lambda_1^i, \lambda_2^i)$ with $\lambda_1 = \lambda_1^i$ and $\lambda_2 = \lambda_2^i$. Denote by $M(\lambda_1, \lambda_2)$ the subspace of M with $M(\lambda_1, \lambda_2) = \bigoplus_{(\ell,m)\in S} R_{\ell,m}(\lambda_1, \lambda_2)$ such that in $M(\lambda_1, \lambda_2)$ each of the operators Δ_1 and Δ_2 has one eigenvalue, λ_1 and λ_2 respectively. We want to show that $M(\lambda_1, \lambda_2)$ is a submodule of M, i.e. that it is invariant under H_+, H_-, H_3, F_+, F_-, and F_3, but this is clearly the case since $M(\lambda_1, \lambda_2)$ is invariant under E_+, E_-, D_+, D_-, and D_0 because Δ_1 and Δ_2 commute with these operators by Lemma 1.10. Finally note that the number of $M(\lambda_1, \lambda_2)$ in the decomposition of M is countable since the number of $R_{\ell,m}$ is countable and the number of $R_{\ell,m}(\lambda_1^i, \lambda_2^i)$ in a given $R_{\ell,m}$ is finite, so decomposing each $M(\lambda_1, \lambda_2)$ we get the result. □

Proposition 1.12. *Let M be a Harish-Chandra module in which each of the Laplace operators Δ_1 and Δ_2 has one eigenvalue. Then there exists an integral or half-integral number $\ell_0 \geq 0$ and a complex number ℓ_1 such that the eigenvalues λ_1 and λ_2 have the form*

$$\lambda_1 = -i\ell_0\ell_1, \qquad \lambda_2 = \ell_0^2 + \ell_1^2 - 1. \tag{1.47}$$

Proof. Denote by ℓ_0 the minimal index in the decomposition $M = \bigoplus_\ell R_\ell$ of M into L_k-submodules of R_ℓ. By the definition of D_- it maps R_{ℓ_0} to zero, so by eq. (1.45) we get that

$$\begin{aligned} \Delta_1 \xi &= -\ell_0(\ell_0+1)D_0\xi \\ \Delta_2 \xi &= (\ell_0^2-1)\xi - (\ell_0+1)D_0^2\xi \end{aligned} \tag{1.48}$$

for $\xi \in R_{\ell_0}$. Now if $\ell_0 \neq 0$ the subspace R_{ℓ_0} is invariant under D_0, so we can find an eigenvector ξ_0 for D_0 such that $D_0\xi_0 = \mu\xi_0$ for some $\mu \in \mathbf{C}$. Thus we see that

$$\begin{aligned} \Delta_1 \xi_0 &= -\ell_0(\ell_0+1)\mu\xi_0 \\ \Delta_2 \xi_0 &= (\ell_0^2-1)\xi_0 - (\ell_0+1)\mu^2\xi_0, \end{aligned}$$

so we get eigenvalues λ_1 and λ_2 of Δ_1 and Δ_2 with

$$\lambda_1 = -\ell_0(\ell_0+1)\mu, \qquad \lambda_2 = (\ell_0^2-1) - (\ell_0+1)\mu^2.$$

Hence putting $(\ell_0+1)\mu = i\ell_1$, we get that

$$\lambda_1 = -i\ell_0\ell_1, \qquad \lambda_2 = \ell_0^2 + \ell_1^2 - 1.$$

Now by assumption each of Δ_1 and Δ_2 has only one eigenvalue on M, and thus these eigenvalues are expressed in terms of the ℓ_0 and ℓ_1 as in eq. (1.47). Also if $\ell_0 = 0$ it is clear that we can still find a complex number ℓ_1 such that the above equations are satisfied. $\square$

Note that all such eigenvalues λ_1 and λ_2 for Δ_1 and Δ_2 are possible, since in the case of finite dimensional simple modules N as in Section 1.1.3 we have by eqs. (1.42) and (1.45) that

$$\begin{aligned} \Delta_1\xi_{\ell,m} &= -\ell(\ell+1)d_\ell^0\xi_{\ell,m} = -i\ell_0\ell_1, \\ \Delta_2\xi_{\ell,m} &= (\ell^2-1)\xi_{\ell,m} - (\ell+1)^2(d_\ell^0)^2\xi_{\ell,m} + (4\ell^2-1)d_{\ell+1}^+d_\ell^-\xi_{\ell,m} \\ &= (\ell^2-1)\xi_{\ell,m} + \frac{\ell_0^2\ell_1^2}{\ell^2}\xi_{\ell,m} - \frac{(\ell^2-\ell_1^2)(\ell^2-\ell_0^2)}{\ell^2}\xi_{\ell,m} \\ &= (\ell^2-1)\xi_{\ell,m} - (\ell^2-\ell_0^2-\ell_1^2)\xi_{\ell,m} \\ &= (\ell_0^2+\ell_1^2-1)\xi_{\ell,m}, \end{aligned}$$

for $\xi_{\ell,m} \in R_{\ell,m}$, where $\ell_0 \geq 0$ is an integral or half-integral number and ℓ_1 is a complex number. Here we can construct N such that ℓ_0 and ℓ_1 are as we want.

Proposition 1.13. *Let M be a Harish-Chandra module in which the Laplace operators Δ_1 and Δ_2 have only one eigenvalue λ_1 and λ_2 respectively. Then on each subspace R_ℓ the operators D_+D_-, D_-D_+, and D_0 have only one eigenvalue*

d_ℓ^-, d_ℓ^+, and d_ℓ^0 respectively. Here the numbers d_ℓ^-, d_ℓ^+, and d_ℓ^0 are expressed in terms of ℓ_0 and ℓ_1 in the following way:

$$\begin{aligned}
d_0^- &= d_{1/2}^- = 0, \\
d_\ell^- &= \frac{(\ell^2-\ell_0^2)(\ell_1^2-\ell^2)}{(4\ell^2-1)\ell^2} && \textit{if } \ell \neq 0, \tfrac{1}{2}, \\
d_\ell^+ &= \frac{((\ell+1)^2-\ell_0^2)(\ell_1^2-(\ell+1)^2)}{(4(\ell+1)^2-1)(\ell+1)^2} \\
d_0^0 &= i\ell_1, \\
d_\ell^0 &= \frac{i\ell_0\ell_1}{\ell(\ell+1)} && \textit{if } \ell \neq 0.
\end{aligned} \tag{1.49}$$

Proof. By eq. (1.45) we see that

$$\begin{aligned}
(4\ell^2-1)D_+D_-\xi &= \Delta_2\xi - (\ell^2-1)\xi + (\ell+1)^2D_0^2\xi, \\
(\ell+1)D_0\xi &= -\frac{\Delta_1}{\ell}\xi
\end{aligned}$$

for $\xi \in R_\ell$ with $\ell > \ell_0$ such that $\ell \neq 0$. Thus

$$(4\ell^2-1)D_+D_-\xi = \Delta_2\xi - (\ell^2-1)\xi + \frac{\Delta_1^2}{\ell^2}\xi$$

for $\xi \in R_\ell$ with $\ell > \ell_0$. Hence since Δ_1 and Δ_2 each only have one eigenvalue on R_ℓ so thus D_+D_-, and we see by eq. (1.47) that

$$\begin{aligned}
d_\ell^- &= \frac{1}{(4\ell^2-1)}\Big(\lambda_2 - (\ell-1)^2 + \frac{\lambda_1^2}{\ell^2}\Big) \\
&= \frac{(\ell_0^2+\ell_1^2-1)\ell^2 - (\ell-1)^2\ell^2 - \ell_0^2\ell_1^2}{(4\ell^2-1)\ell^2} \\
&= \frac{(\ell_0^2+\ell_1^2-\ell^2)\ell^2 - \ell_0^2\ell_1^2}{(4\ell^2-1)\ell^2} \\
&= \frac{(\ell^2-\ell_0^2)(\ell_1^2-\ell^2)}{(4\ell^2-1)\ell^2}
\end{aligned}$$

for $\ell \neq 0, \frac{1}{2}$. Now since D_- by definition maps R_0 and $R_{1/2}$ to zero if they occur in the decomposition of M, we see that $d_0^- = 0$ and $d_{1/2}^- = 0$, and likewise we know that D_- maps R_{ℓ_0} to zero so $d_{\ell_0}^- = 0$, which also holds true with the formula above. Thus we have proven the formulae for d_ℓ^-, and the other formulae are proven similarly. □

Now denote by $C_s(\lambda_1, \lambda_2)$ for $s = 1$ or $s = \frac{1}{2}$ the set of all Harish-Chandra modules for the pair (L, L_k) in which the Laplace operators have the eigenvalues

λ_1 and λ_2, and in which if $s=1$ every $M\in C_1(\lambda_1,\lambda_2)$ has only integral numbers as indices in the decomposition $M=\bigoplus_\ell R_\ell$, and if $s=\frac{1}{2}$ every $M\in C_{1/2}(\lambda_1,\lambda_2)$ has only half-integral numbers as indices in the decomposition $M=\bigoplus_\ell R_\ell$.

Proposition 1.14. *Let $M\in C_s(\lambda_1,\lambda_2)$, $M'\in C_{s'}(\lambda_1',\lambda_2')$, where $(s,\lambda_1,\lambda_2)\neq(s',\lambda_1',\lambda_2')$. Then $\mathrm{Hom}_L(M,M')=0$.*

Proof. Let $\gamma\in\mathrm{Hom}_L(M,M')$ and assume that $\gamma\neq 0$. First we will show that $\gamma R_\ell\subset R_\ell'$, where R_ℓ comes from the decomposition of M and R_ℓ' from the decomposition of M'. To see this note that R_ℓ and R_ℓ' are direct sums of finitely many L_k-modules $M(2\ell)$, where each $M(2\ell)$ is generated by a maximal vector of weight ℓ (weight w.r.t. h_3 in L_k). So since γ takes a maximal vector of weight ℓ to either zero or another maximal vector of weight ℓ, we see that indeed $\gamma R_\ell\subset R_\ell'$.

Denoting by Δ_i the Laplace operators in M and by Δ_i' the Laplace operators in M', we also have that $\Delta_i'\gamma=\gamma\Delta_i$ for $i=1,2$, since in both cases Δ_i and Δ_i' correspond to the actions in $U(L)$ of eq. (1.43) and $\gamma\in\mathrm{Hom}_L(M,M')=\mathrm{Hom}_{U(L)}(M,M')$. Now since $\gamma\neq 0$ we get that $\gamma R_\ell\subset R_\ell'$ implies that $s=s'$ and $\Delta_i'\gamma=\gamma\Delta_i$ implies that $\lambda_i=\lambda_i'$ for $i=1,2$, but this is a contradiction since $(s,\lambda_1,\lambda_2)\neq(s',\lambda_1',\lambda_2')$. Hence we must have that $\gamma=0$, and thus indeed $\mathrm{Hom}_L(M,M')=0$. $\square$

This implies that the study of the category of Harish-Chandra modules for the pair (L,L_k) can be reduced to the study of the category $C_s(\lambda_1,\lambda_2)$ of modules in which the Laplace operators have exactly one eigenvalue.

Remark 1.15. From now on in places where the index s is not important, we will simply denote $C_s(\lambda_1,\lambda_2)$ by $C(\lambda_1,\lambda_2)$. $\triangle$

Definition 1.16. *The category of modules $C(\lambda_1,\lambda_2)$ is called singular if the numbers ℓ_0 and ℓ_1 constructed from λ_1 and λ_2 as in eq.* (1.47) *are such that $\ell_1-\ell_0$ is an integer. Otherwise it is called non-singular.*

We will see that the study of the non-singular categories $C(\lambda_1,\lambda_2)$ is simpler than that of the singular ones, and we are now ready to begin our description of the category of the singular categories $C(\lambda_1,\lambda_2)$.

1.3 The non-singular category $C(\lambda_1,\lambda_2)$

Let $M\in C(\lambda_1,\lambda_2)$ be an L-module, where (λ_1,λ_2) is a non-singular pair, i.e. $\ell_1-\ell_0$ is not an integer. We now want to that this module M is completely determined by a finite dimensional vector space and a nilpotent map a on this vector space, where an isomorphism of the modules is equivalent to similarity of the linear map a.

Define on the finite dimensional linear subspace R_{ℓ_0,m_0} for some $m_0 \in \{-\ell_0, -\ell_0+1, \ldots, \ell_0-1, \ell_0\}$ a linear map $a\colon R_{\ell_0,m_0} \to R_{\ell_0,m_0}$ by

$$a\xi = D_0\xi - \frac{i\ell_1}{\ell_0+1}\xi \tag{1.50}$$

for $\xi \in R_{\ell_0,m_0}$. This map is nilpotent since by Proposition 1.13 the only eigenvalue of D_0 on R_{ℓ_0} is

$$d^0_{\ell_0} = \frac{i\ell_1}{\ell_0+1},$$

and

$$\det(a - t\,\mathrm{id}) = \det\Big(D_0 - \Big(t + \frac{i\ell_1}{\ell_0+1}\Big)\,\mathrm{id}\Big),$$

so the only eigenvalue of a on R_ℓ is zero, and thus a is clearly nilpotent by Cayley-Hamilton Theorem.

We want to show that the module M is completely determined by the finite dimensional vector space R_{ℓ_0,m_0} and the linear map $a\colon R_{\ell_0,m_0} \to R_{\ell_0,m_0}$ when $C(\lambda_1, \lambda_2)$ is non-singular. To do this we first need some lemmas.

Lemma 1.17. *In a non-singular module $M \in C(\lambda_1, \lambda_2)$, the maps*

$$\begin{aligned} D_+ &: R_{\ell,m} \to R_{\ell+1,m} \\ D_- &: R_{\ell+1,m} \to R_{\ell,m} \end{aligned}$$

for $\ell \geq \ell_0$ are isomorphisms.

Proof. By Proposition 1.13 the eigenvalues of D_+D_- and D_-D_+ on R_ℓ for $\ell \neq 0, \frac{1}{2}$ are

$$\begin{aligned} d^-_\ell &= \frac{(\ell^2-\ell_0^2)(\ell_1^2-\ell^2)}{(4\ell^2-1)\ell^2} \qquad \text{for } \ell > \ell_0, \\ d^+_\ell &= \frac{((\ell+1)^2-\ell_0^2)(\ell_1^2-(\ell+1)^2)}{(4(\ell+1)^2-1)(\ell+1)^2}. \end{aligned}$$

Now by assumption M is non-singular, i.e. $\ell_1 - \ell_0$ is not an integer, and we want to show that $\ell_1^2 - \ell^2 \neq 0$ for all $\ell \in \{\ell_0, \ell_0+1, \ldots\}$.

Assume that $\ell_1^2 - \ell^2 = 0$ for some $\ell = \ell_0 + k$, where k is a non-negative integer. Since $\ell_1 - \ell_0$ is not an integer, we also have that $\ell_1 - (\ell_0 + k)$ is not an integer and hence not equal to zero, so we must have that $\ell_1 + \ell_0 + k = 0$ since $\ell_1^2 - \ell^2 = (\ell_1 - \ell)(\ell_1 + \ell)$. But this would imply that $\ell_1 = -\ell_0 - k$ and hence $\ell_1 - \ell_0 = -2\ell_0 - k$ is an integer, which is a contradiction with the non-singularity of M.

Thus we see that $\ell_1^2 - \ell^2$ is non-zero for all $\ell \in \{\ell_0, \ell_0+1, \ldots\}$, and therefore the eigenvalues d^+_ℓ and d^-_ℓ are different from zero for all ℓ except

ℓ_0 in the case of d_ℓ^-. Hence the maps $D_+D_-\colon R_{\ell,m} \to R_{\ell,m}$ for $\ell \neq \ell_0$ and $D_-D_+\colon R_{\ell,m} \to R_{\ell,m}$ have diagonals without zeros in the Schur decomposition, so they are invertible. Therefore $D_-\colon R_{\ell+1,m} \to R_{\ell,m}$ and $D_+\colon R_{\ell,m} \to R_{\ell+1,m}$ are injective, and thus $\dim R_{\ell,m} = \dim R_{\ell+1,m}$, which again implies that D_- and D_+ as above are actually isomorphisms. □

Lemma 1.18. *In a non-singular module $M \in C(\lambda_1, \lambda_2)$ the Laplace operators Δ_1 and Δ_2 are such that each operator $(\Delta_i)_{\ell,m}$ is similar to $(\Delta_i)_{\ell_0,m_0}$ via the same matrix for $i = 1, 2$.*

Proof. Recall that the maps $E_+\colon R_{\ell_0,m} \to R_{\ell_0,m+1}$ for $-\ell_0 \leq m < \ell_0$ and $E_-\colon R_{\ell_0,m} \to R_{\ell_0,m-1}$ for $-\ell_0 < m \leq \ell_0$ are isomorphisms, and the Laplace operators Δ_1 and Δ_2 commute with these maps by Lemma 1.10, so $(\Delta_i)_{\ell_0,m}$ is similar to $(\Delta_i)_{\ell_0,m_0}$ for each m and $i = 1, 2$.

Likewise the $D_+\colon R_{\ell,m} \to R_{\ell+1,m}$ are also isomorphisms for all ℓ and commute with both Δ_1 and Δ_2, so the map $(\Delta_i)_{\ell_0+1,m}$ is similar to $(\Delta_i)_{\ell_0,m}$, and inductively $(\Delta_i)_{\ell,m}$ is similar to $(\Delta_i)_{\ell_0,m}$ for all $\ell \in \{\ell_0, \ell_0+1, \ldots\}$. Hence indeed $(\Delta_i)_{\ell,m}$ is similar to $(\Delta_i)_{\ell_0,m_0}$, $i = 1, 2$, and by construction it is clearly by the same matrix for $i = 1, 2$. □

Lemma 1.19. *If $M \in C(\lambda_1, \lambda_2)$ is a non-singular module, then the Laplace operators Δ_1 and Δ_2 are connected on the whole of M by the relation*

$$\Delta_1^2 + \ell_0^2 \Delta_2 - \ell_0^2(\ell_0^2 - 1)\,\mathrm{id} = 0. \tag{1.51}$$

Proof. Suppose that $\ell_0 \neq 0$. By eq. (1.48) we get that

$$\begin{aligned} \Delta_2\xi + \frac{\Delta_1^2}{\ell_0^2}\xi - (\ell_0^2 - 1)\,\mathrm{id}\,\xi &= (\ell_0^2 - 1)\xi - (\ell_0+1)^2 D_0^2 \xi \\ &\quad + (\ell_0+1)^2 D_0^2\xi - (\ell_0^2 - 1)\xi = 0, \end{aligned} \tag{1.52}$$

for $\xi \in R_{\ell_0,m_0}$, so multiplying by ℓ_0^2 we get eq. (1.51) on R_{ℓ_0,m_0}. By Lemma 1.18 $(\Delta_i)_{\ell,m}$ is similar to $(\Delta_i)_{\ell_0,m_0}$ via the same matrix for $i = 1, 2$, so the relation holds true for any $\xi \in R_{\ell,m}$, and thus on all of M.

Suppose otherwise that $\ell_0 = 0$. Then eq. (1.48) implies that $(\Delta_1)_{0,0}$ is zero, and thus the relation follows easily on $R_{0,0}$, and we can expand to all of M as above. □

Remark 1.20. Note that Lemmas 1.17 to 1.19 are not true in the singular case. △

We have seen above that to each non-singular module $M \in C(\lambda_1, \lambda_2)$ there is a corresponding finite dimensional vector space $P = R_{\ell_0,m_0}$ and a nilpotent linear map $a\colon P \to P$ given by

$$a\xi = \Big(D_0 - \frac{i\ell_1}{\ell_0+1}\,\mathrm{id}\Big)\xi$$

for $\xi \in R_{\ell_0,m_0} = P$. Denote now by $\widetilde{A}$ the pair (P, a) consisting of a finite dimensional vector space P and a nilpotent mapping $a\colon P \to P$.

Theorem 1.21. *To each pair $\widetilde{A}$ and non-singular pair (λ_1, λ_2) of numbers there is a corresponding L-module $M \in C(\lambda_1, \lambda_2)$ such that $P = R_{\ell_0,m_0}$ and a is related to D_0 by eq.* (1.50).

Proof. Denote by R_{ℓ_0,m_0} the space P and consider the linear transformation

$$D_0\xi = a\xi + \frac{i\ell_1}{\ell_0+1}\xi$$

for $\xi \in R_{\ell_0,m_0}$. Consider the space

$$M = \bigoplus_{\substack{\ell\in\{\ell_0,\ell_0+1,\ldots\}\\ m\in\{-\ell,-\ell+1,\ldots,\ell-1,\ell\}}} R_{\ell,m},$$

which is a direct sum of vector spaces with $\dim R_{\ell,m} = \dim P$ for all ℓ and m.

Now take an isomorphism $E_+\colon R_{\ell,m} \to R_{\ell,m+1}$ for $m \neq \ell$ and put $E_+\colon R_{\ell,\ell} \to 0$, which we can do since $\dim R_{\ell,m} = \dim R_{\ell,m+1}$. Define an isomorphism $E_-\colon R_{\ell,m+1} \to R_{\ell,m}$ such that it is inverse to $E_+\colon R_{\ell,m} \to R_{\ell,m+1}$ and put $E_-\colon R_{\ell,-\ell} \to 0$. Take now isomorphisms $D_+\colon R_{\ell,m_0} \to R_{\ell+1,m_0}$ for some fixed m_0, and define on all the remaining $R_{\ell,m}$ linear maps $D_+\colon R_{\ell,m} \to R_{\ell+1,m}$ such that the diagram

$$\begin{array}{ccc} R_{\ell,m+1} & \xrightarrow{D_+} & R_{\ell+1,m+1} \\ {\scriptstyle E_+}\uparrow & & \uparrow{\scriptstyle E_+} \\ R_{\ell,m} & \xrightarrow[D_+]{} & R_{\ell+1,m} \end{array}$$

commutes for $-\ell \leq m < \ell$, i.e. $(D_+)_{\ell,m+1} = (E_+)_{\ell+1,m}(D_+)_{\ell,m}(E_+)^{-1}_{\ell,m}$, where $(E_+)^{-1}_{\ell,m} = (E_-)_{\ell,m+1}$. Now we only need to construct linear maps D_0 and D_- on M satisfying properties as we have seen earlier, but to do this we will first define linear maps Δ_1 and Δ_2 corresponding to the Laplace operators.

On R_{ℓ_0,m_0} set

$$\begin{aligned} \Delta_1\xi &= -\ell_0(\ell_0+1)D_0\xi \\ &= -\ell_0(\ell_0+1)a\xi - i\ell_1\ell_0\xi, \\ \Delta_2\xi &= (\ell_0^2-1)\xi - (\ell_0+1)^2D_0^2\xi \\ &= (\ell_0^2-1)\xi + \ell_1^2\xi - (\ell_0+1)i\ell_1 a\xi - (\ell_0+1)^2a^2\xi \\ &= (\ell_0^2+\ell_1^2-1)\xi - (\ell_0+1)^2\Big(a^2\xi + 2\frac{i\ell_1}{\ell_0+1}a\xi\Big) \end{aligned} \tag{1.53}$$

for $\xi \in R_{\ell_0,m_0}$. Now note that for arbitrary $R_{\ell,m}$ the linear map given by $J_{\ell,m} = (E_+)^{m-m_0}(D_+)^{\ell-\ell_0}\colon R_{\ell_0,m_0} \to R_{\ell,m}$, where $(E_+)^{-1} = E_-$, is a composition of isomorphisms and hence an isomorphism, so we can define

$$(\Delta_i)_{\ell,m}\xi = J_{\ell,m}(\Delta_i)_{\ell_0,m_0}(J_{\ell,m})^{-1}\xi$$

for $\xi \in R_{\ell,m}$ and $i = 1, 2$. Thus we have defined Δ_1 and Δ_2 on all of M.

Now define $D_0 \colon R_{\ell,m} \to R_{\ell,m}$ by

$$D_0\xi = -\frac{1}{\ell(\ell+1)}\Delta_1\xi \tag{1.54}$$

for $\xi \in R_{\ell,m}$, $\ell \neq 0$, and $D_+D_- \colon R_{\ell,m} \to R_{\ell,m}$ by

$$\begin{aligned} D_+D_-\xi &= \frac{1}{4\ell^2-1}(\Delta_2\xi - (\ell^2-1)\xi + (\ell+1)^2 D_0^2\xi) \\ &= \frac{1}{4\ell^2-1}\Big(\Delta_2\xi - (\ell^2-1)\xi + \frac{\Delta_1^2}{\ell^2}\xi\Big) \end{aligned} \tag{1.55}$$

for $\xi \in R_{\ell,m}$, $\ell \neq \ell_0$, which we can do since D_+ is an isomorphism. Using this we define $D_- \colon R_{\ell,m} \to R_{\ell-1,m}$ to be the map $(D_+)^{-1}(D_+D_-)$ for $\ell \neq \ell_0$, and equal to zero for $\ell = \ell_0$.

Now the maps E_+, E_-, D_0, D_+, and D_- constructed above are maps as in Section 1.1.1 (by construction the E's and D's commute and satisfy the other properties we want), so the operators F_+, F_-, F_3, H_+, H_-, and H_3 constructed from these maps as in eqs. (1.9), (1.11) and (1.34) gives M an L-module structure if the equations of eq. (1.35) hold. We see that for $\ell \neq 0$

$$\begin{aligned} \ell(D_+)_{\ell,m}(D_0)_{\ell,m} &= -\frac{\ell}{\ell(\ell+1)}(D_+)_{\ell,m}(\Delta_1)_{\ell,m} \\ &= -\frac{1}{\ell+1}(\Delta_1)_{\ell+1,m}(D_+)_{\ell,m} \\ &= -(\ell+2)\frac{1}{(\ell+1)(\ell+2)}(\Delta_1)_{\ell+1,m}(D_+)_{\ell,m} \\ &= (\ell+2)(D_0)_{\ell+1,m}(D_+)_{\ell,m}, \end{aligned}$$

which gives us the first equation, and the rest can be checked similarly, so we get an L-module structure on M.

Finally we get $P = R_{\ell_0,m_0}$ and eq. (1.50) by construction, and we note that M is non-singular since the pair (λ_1, λ_2) with the corresponding ℓ_0 and ℓ_1 is non-singular by assumption. □

Looking at the construction of the L-module M above, we get the following corollary:

Corollary 1.22. *For modules M and M' from the non-singular category $C(\lambda_1, \lambda_2)$ to be equivalent it is necessary and sufficient that the subspaces R_{ℓ_0,m_0} and R'_{ℓ_0,m_0} in these modules have the same dimension, and that their maps $D_0 \colon R_{\ell_0,m_0} \to R_{\ell_0,m_0}$ and $D'_0 \colon R'_{\ell_0,m_0} \to R'_{\ell_0,m_0}$ are similar.*

Now let S be the category with objects pairs $A = (P, a)$, where P is a finite dimensional vector space (over $\mathbf{C}$) and $a\colon P \to P$ is a nilpotent linear transformation, and with morphisms $\gamma\colon A \to A'$ given by linear maps $\gamma\colon P \to P'$ such that

$$\begin{array}{ccc} P & \xrightarrow{a} & P \\ {\scriptstyle\gamma}\big\downarrow & & \big\downarrow{\scriptstyle\gamma} \\ P' & \xrightarrow[a']{} & P' \end{array}$$

commutes. We have already shown that there is a correspondence between non-singular modules $M \in C(\lambda_1, \lambda_2)$ and pairs $A = (P, a) \in S$, and now we want to show that this correspondence is functorial and actually gives an equivalence of categories.

Theorem 1.23. *The non-singular category $C(\lambda_1, \lambda_2)$ is equivalent to the category S.*

Proof. By Theorem 1.21 we have a correspondence between objects $M \in C(\lambda_1, \lambda_2)$ and the objects $A \in S$, so we just need to establish a correspondence between the morphisms. Let $\Gamma\colon M \to M'$ be a morphism between the two modules $M, M' \in C(\lambda_1, \lambda_2)$. We have already seen in the proof of Proposition 1.14 that $\Gamma R_\ell \subset R'_\ell$ since Γ commutes with H_3, and this furthermore implies that $\Gamma R_{\ell,m} \subset R'_{\ell,m}$ since Γ must map vectors of weight m to either 0 or vectors of weight m. Thus Γ is the direct sum of morphisms $\gamma_{\ell,m}\colon R_{\ell,m} \to R'_{\ell,m}$, so choosing indices ℓ_0 and m_0 we get a morphism $\gamma := \gamma_{\ell_0,m_0}\colon R_{\ell_0,m_0} \to R_{\ell'_0,m'_0}$. Here γ gives a morphism in S from $\widetilde{A} = (R_{\ell_0,m_0}, a)$ to $\widetilde{A}' = (R'_{\ell_0,m_0}, a')$. To see this by eq. (1.50) it is enough to show that $D'_0\gamma = \gamma D_0$, which is true for $\ell_0 \neq 0$[3] since by the proof of Proposition 1.14 $\Delta'_1\Gamma = \Gamma\Delta_1$, so for $\ell \neq 0$ also $D'_0\Gamma = \Gamma D_0$ on R_ℓ because $\Delta_1 = -\ell(\ell+1)D_0$ on R_ℓ and $\Delta'_1 = -\ell(\ell+1)D'_0$ on R'_ℓ, and therefore specifically $D'_0\gamma = \gamma D_0$. Hence $\gamma\colon \widetilde{A} \to \widetilde{A}'$ is indeed a morphism in S.

Now suppose conversely that we are given a morphism $\gamma\colon A \to A'$ in S, i.e. a linear map $\gamma\colon P \to P'$ such that $\gamma a = a'\gamma$. Construct modules M and M' with $R_{\ell_0,m_0} = P$ and $R'_{\ell_0,m_0} = P'$ as in the proof of Theorem 1.21, then we have a linear map $\gamma\colon R_{\ell_0,m_0} \to R'_{\ell_0,m_0}$ such that $\gamma a = a'\gamma$, which implies that $\gamma D_0 = D'_0\gamma$ since $D_0 = a + \frac{i\ell_1}{\ell_0+1}\,\mathrm{id}$. From this we will construct linear maps $\gamma_{\ell,m}\colon R_{\ell,m} \to R'_{\ell,m}$ by noting that $J_{\ell,m} = E_+^{m-m_0}D_+^{\ell-\ell_0}\colon R_{\ell_0,m_0} \to R_{\ell,m}$ is an isomorphism for ℓ and m where it makes sense, so we get linear maps

$$\gamma_{\ell,m} = J'_{\ell,m}\gamma J_{\ell,m}^{-1}, \tag{1.56}$$

where $J'_{\ell,m}\colon R'_{\ell,m} \to R'_{\ell,m}$ is as above also. We want to show that the direct sum Γ of the $\gamma_{\ell,m}$ gives a morphism of L-modules from $M = \bigoplus_{\ell,m} R_{\ell,m}$ to $M' = \bigoplus_{\ell,m} R'_{\ell,m}$.

[3]Recall that D_0 is not defined on $R_{0,0}$, so this is what we want to check.

Recall, cf. Lemma 1.18, that $(\Delta_i)_{\ell,m}$ is similar to $(\Delta_i)_{\ell_0,m_0}$ for all ℓ and m, $i = 1,2$, with $(\Delta_i)_{\ell,m} = J_{\ell,m}(\Delta_i)_{\ell_0,m_0}J_{\ell,m}^{-1}$. Hence since $\gamma(\Delta_i)_{\ell_0,m_0} = (\Delta_i')_{\ell_0,m_0}\gamma$ by eq. (1.48) since $\gamma D_0 = D_0'\gamma$, we see that

$$\begin{aligned}\gamma_{\ell,m}(\Delta_i)_{\ell,m} &= J'_{\ell,m}\gamma J_{\ell,m}^{-1}J_{\ell,m}(\Delta_i)_{\ell_0,m_0}J_{\ell,m}^{-1} = J'_{\ell,m}\gamma(\Delta_i)_{\ell_0,m_0}J_{\ell,m}^{-1}\\ &= J'_{\ell,m}(\Delta_i')_{\ell_0,m_0}\gamma J_{\ell,m}^{-1} = (\Delta_i')_{\ell,m}J'_{\ell,m}\gamma J_{\ell,m}^{-1} = (\Delta_i')_{\ell,m}\gamma_{\ell,m}.\end{aligned}$$

Now since

$$\begin{aligned}D_0\xi &= -\frac{1}{\ell(\ell+1)}\Delta_1\xi,\\ D_+D_-\xi &= \frac{1}{4\ell^2-1}\Big(\Delta_2\xi - (\ell^2-1)\xi + \frac{\Delta_1^2}{\ell^2}\xi\Big)\end{aligned}$$

for $\xi \in R_{\ell,m}$ with $\ell \neq 0$, cf. the proof of Theorem 1.21, we get that

$$\begin{aligned}(D_0')_{\ell,m}\gamma_{\ell,m} &= \gamma_{\ell,m}(D_0)_{\ell,m},\\ (D_+'D_-')_{\ell,m}\gamma_{\ell,m} &= \gamma_{\ell,m}(D_+D_-)_{\ell,m}.\end{aligned}$$

Also noting that since E_+ and D_+ commute where it makes sense, we have that $E_+J_{\ell,m} = J_{\ell,m}E_+ = J_{\ell,m+1}$ and $D_+J_{\ell,m} = J_{\ell,m}D_+ = J_{\ell+1,m}$, so

$$\gamma_{\ell,m}E_+ = J'_{\ell,m}\gamma J_{\ell,m}^{-1}E_+ = E_+'J'_{\ell,m-1}\gamma J_{\ell,m-1}^{-1} = E_+'\gamma_{\ell,m-1},$$

and likewise $\gamma_{\ell,m}D_+ = D_+'\gamma_{\ell-1,m}$ where it makes sense. Thus Γ commutes with D_+ and D_+D_-, so by Lemma 1.17 Γ also commutes with D_-, and likewise since E_- is the inverse of E_+ or zero, we get that Γ commutes with E_-. Hence Γ commutes with E_+, E_-, D_0, D_+, and D_-, so it commutes with H_+, H_-, H_3, F_+, F_-, and F_3, i.e. it is a morphism of L-modules. □

Corollary 1.24. *An indecomposable module M in the non-singular category $C(\lambda_1,\lambda_2)$ corresponds to an indecomposable object A in the category S.*

Here it follows from linear algebra that the indecomposable objects $A \in S$ are finite dimensional vector spaces P with nilpotent linear maps $a\colon P \to P$ whose matrices in a suitable basis have the form of a single Jordan block.

Remark 1.25. Note that by the above an indecomposable non-singular Harish-Chandra module for the pair (L, L_k) has one additional invariant when compared to the simple case. In the simple case we just need to numbers ℓ_0 and ℓ_1, but in the indecomposable case we additionally need a number n giving the dimension of the Jordan block. △

Finally to end our description of the non-singular Harish-Chandra modules, we will give the explicit form of E_+, E_-, D_+, D_-, and D_0 in a non-singular Harish-Chandra module $M \in C(\lambda_1,\lambda_2)$. Here we denote by $[E_+]_{\ell,m}$ the matrix representation of the map $E_+\colon R_{\ell,m} \to R_{\ell,m+1}$ in a given basis, by $[E_-]_{\ell,m}$ the matrix representation of the map $E_-\colon R_{\ell,m} \to R_{\ell,m-1}$ in a given basis, and so on.

Theorem 1.26. *Let $M \in C(\lambda_1, \lambda_2)$ be a non-singular indecomposable Harish-Chandra module for the pair (L, L_k). Then all the subspaces $R_{\ell,m}$, $\ell = \ell_0, \ell_0 + 1, \dots$, have the same dimension, and bases can be chosen in them such that $[E_+]_{\ell,m}$ for $m \neq \ell$, $[E_-]_{\ell,m}$ for $m \neq -\ell$, and $[D_+]_{\ell,m}$ are identity matrices. Furthermore the matrices $[D_0]_{\ell,m}$ and $[D_-]_{\ell,m}$ can be expressed in terms of the matrix*

$$[a_0] = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

by the formulae

$$\begin{aligned} [D_0]_{\ell,m} &= i\frac{\ell_0\ell_1}{\ell(\ell+1)}[\mathrm{id}] + \frac{\ell_0(\ell_0+1)}{\ell(\ell+1)}[a_0] \\ [D_-]_{\ell,m} &= \frac{\ell_0^2 - \ell^2}{\ell^2(4\ell^2-1)}\Big((\ell^2 - \ell_1^2)[\mathrm{id}] + 2i\ell_1(\ell_0+1)[a_0] + (\ell_0+1)^2[a_0]^2\Big), \end{aligned}$$

where $[\mathrm{id}]$ *is the identity matrix of the same dimension as* $[a_0]$.

Proof. We follow the proof of Theorem 1.21. First take $[D_0]$ to be the linear transformation

$$[D_0]\xi = [a_0]\xi + \frac{i\ell_1}{\ell_0+1}[\mathrm{id}]\xi$$

for $\xi \in R_{\ell_0,m_0}$. Now following the construction in Theorem 1.21, we get that all the subspaces $R_{\ell,m}$ have the same dimension, and we can find bases of $R_{\ell,m}$ such that $[E_+]_{\ell,m}$ for $m \neq \ell$, $[E_-]_{\ell,m}$ for $m \neq -\ell$, and $[D_+]_{\ell,m}$ are identity matrices. By Lemma 1.10 E_+, E_-, and D_+ commute with Δ_i, $i = 1, 2$, so $[\Delta_i]_{\ell,m}$ is independent of ℓ and m, so we can focus on R_{ℓ_0,m_0}, where eq. (1.53) gives us that

$$\begin{aligned} [\Delta_1]_{\ell_0,m_0}\xi &= -\ell_0(\ell_0+1)[a_0]\xi - i\ell_1\ell_0[\mathrm{id}]\xi, \\ [\Delta_2]_{\ell_0,m_0}\xi &= (\ell_0^2 + \ell_1^2 - 1)[\mathrm{id}]\xi - (\ell_0+1)^2\Big([a_0]^2\xi + 2\frac{i\ell_1}{\ell_0+1}[a_0]\xi\Big) \\ &= (\ell_0^2 + \ell_1^2 - 1)[\mathrm{id}]\xi - 2i\ell_1(\ell_0+1)[a_0]\xi - (\ell_0+1)^2[a_0]^2\xi, \end{aligned}$$

for $\xi \in R_{\ell_0,m_0}$. Thus we get that

$$\begin{aligned} [\Delta_1]_{\ell,m} &= -\ell_0(\ell_0+1)[a_0] - i\ell_1\ell_0[\mathrm{id}], \\ [\Delta_2]_{\ell,m} &= (\ell_0^2 + \ell_1^2 - 1)[\mathrm{id}] - 2i\ell_1(\ell_0+1)[a_0] - (\ell_0+1)^2[a_0]^2, \end{aligned}$$

since $[\Delta_i]_{\ell,m}$ is independent of ℓ and m. Continuing as in Theorem 1.21, we get by eqs. (1.54) and (1.55) that

$$\begin{aligned}
[D_0]_{\ell,m} &= -\frac{1}{\ell(\ell+1)}[\Delta_1]_{\ell,m} \\
&= \frac{\ell_0(\ell_0+1)}{\ell(\ell+1)}[a_0] + \frac{i\ell_1\ell_0}{\ell(\ell+1)}[\mathrm{id}], \\
[D_+D_-]_{\ell,m} &= \frac{1}{4\ell^2-1}\Big([\Delta_2]_{\ell,m} - (\ell^2-1)[\mathrm{id}] + \frac{[\Delta_1]^2_{\ell,m}}{\ell^2}\Big) \\
&= \frac{1}{(4\ell^2-1)\ell^2}\Big((\ell_0^2+\ell_1^2-1)\ell^2[\mathrm{id}] - 2i\ell_1(\ell_0+1)\ell^2[a_0] \\
&\quad - (\ell_0+1)^2\ell^2[a_0]^2 - (\ell^2-1)\ell^2[\mathrm{id}] + \ell_0^2(\ell_0+1)^2[a_0]^2 \\
&\quad - \ell_1^2\ell_0^2[\mathrm{id}] + 2i\ell_0^2(\ell_0+1)\ell_1[a_0]\Big) \\
&= \frac{\ell_0^2-\ell^2}{(4\ell^2-1)\ell^2}\Big((\ell^2-\ell_1^2)[\mathrm{id}] + 2i\ell_1(\ell_0+1)[a_0] + (\ell_0+1)^2[a_0]^2\Big)
\end{aligned}$$

since

$$\begin{aligned}
(\ell_0^2+\ell_1^2-1)\ell^2 - (\ell^2-1)\ell^2 - \ell_1^2\ell_0^2 &= -\ell^4 + \ell_0^2\ell^2 + \ell_1^2\ell^2 - \ell_1^2\ell_0^2 \\
&= (\ell_0^2-\ell^2)(\ell^2-\ell_1^2), \\
-2i\ell_1(\ell_0+1)\ell^2 + 2i\ell_0^2(\ell_0+1)\ell_1 &= (\ell_0^2-\ell^2)2i\ell_1(\ell_0+1), \\
-(\ell_0+1)^2\ell^2 + \ell_0^2(\ell_0+1)^2 &= (\ell_0^2-\ell^2)(\ell_0+1)^2.
\end{aligned}$$

Now since $[D_+]_{\ell,m}$ was just the identity matrix, we see that $[D_-]_{\ell,m}$ has the same expression as $[D_+D_-]_{\ell,m}$ and thus we get the theorem. $\square$

1.4 The singular category $C(\lambda_1,\lambda_2)$

Now we want to describe the singular category $C(\lambda_1,\lambda_2)$, i.e. Harish-Chandra modules for the pair (L, L_k) with $\ell_1-\ell_0$ an integer. The description of such modules turns out to be quite a bit more complicated than in the non-singular case, where a finite dimensional vector space P and a nilpotent linear map $a\colon P\to P$ describes the module.

We define at first a category S_0 as follows. The objects $\widetilde{A}$ of S_0 are finite dimensional vector spaces P_1 and P_2 with four linear maps

$$d_+\colon P_1\to P_2,\qquad d_-\colon P_2\to P_1,\qquad \delta_1\colon P_1\to 0\qquad \delta_2\colon P_2\to P_2 \tag{1.57}$$

that satisfy the conditions

$$\begin{aligned}
&d_-\delta_2 = \delta_2 d_+ = 0,\\
&\delta_2 \text{ and } d_+d_- \text{ are nilpotent.}
\end{aligned}\tag{1.58}$$

The morphisms $\gamma\colon \widetilde{A} \to \widetilde{A}'$ in S_0 are pairs of linear maps $\gamma = (\gamma_1, \gamma_2)$ with $\gamma_1\colon P_1 \to P_1'$ and $\gamma_2\colon P_2 \to P_2'$ such that the diagram

$$\begin{tikzcd}
P_1 \arrow[r, "d_+"] \arrow[d, "\gamma_1"] & P_2 \arrow[r, "\delta_2"] \arrow[d, "\gamma_2"] & P_2 \arrow[r, "d_-"] \arrow[d, "\gamma_2"] & P_1 \arrow[d, "\gamma_1"] \\
P_1' \arrow[r, "d_+'"'] & P_2' \arrow[r, "\delta_2'"'] & P_2' \arrow[r, "d_-'"'] & P_1'
\end{tikzcd} \tag{1.59}$$

commutes. Similarly to the non-singular case we now want to prove that the singular category $C(\lambda_1, \lambda_2)$ is equivalent to the category S_0, but before we can do that we need some lemmas.

Lemma 1.27. *In a singular module $M \in C(\lambda_1, \lambda_2)$ all the subspaces $R_{\ell,m}$ for $\ell_0 \leq \ell \leq |\ell_1| - 1$ and all m where it makes sense have the same dimension. The subspaces $R_{\ell,m}$ for $\ell \geq |\ell_1|$ and all m where it makes sense also have the same dimension. Furthermore the linear maps*

$$\begin{aligned}
&D_+\colon R_{\ell,m} \to R_{\ell+1,m} && \textit{for } \ell \neq |\ell_1| - 1,\\
&D_-\colon R_{\ell,m} \to R_{\ell-1,m} && \textit{for } \ell \neq \ell_0, |\ell_1|
\end{aligned}$$

are isomorphisms.

Proof. By eq. (1.49), we have that the eigenvalues d_ℓ^- and d_ℓ^+ for D_+D_- and D_-D_+ on $R_{\ell,m}$ are

$$d_\ell^+ = \frac{((\ell+1)^2 - \ell_0^2)(\ell_1^2 - (\ell+1)^2)}{(4(\ell+1)^2 - 1)(\ell+1)^2}, \qquad d_\ell^- = \frac{(\ell^2 - \ell_0^2)(\ell_1^2 - \ell^2)}{(4\ell^2 - 1)\ell^2}$$

for $\ell \neq \ell_0$ in the case of d_ℓ^-, where $d_{\ell_0}^- = 0$. Now M is singular, so ℓ_1 is real because $\ell_1 - \ell_0$ is an integer and ℓ_0 is real, and $|\ell_1| - \ell_0$ is a positive integer by assumption, so we get that $d_\ell^+ = 0$ only for $\ell = |\ell_1| - 1 = \ell_0 + (|\ell_1| - \ell_0) - 1 \in \{\ell_0, \ell_0 + 1, \ldots\}$ and $d_\ell^- = 0$ only for $\ell = \ell_0$ and $\ell = |\ell_1| = \ell_0 + (|\ell_1| - \ell_0) \in \{\ell_0 + 1, \ell_0 + 2, \ldots\}$. Hence the maps

$$\begin{aligned}
&D_-D_+\colon R_{\ell,m} \to R_{\ell,m} && \text{for } \ell \neq |\ell_1| - 1,\\
&D_+D_-\colon R_{\ell,m} \to R_{\ell,m} && \text{for } \ell \neq \ell_0, |\ell_1| \text{ and } m \neq \pm\ell
\end{aligned}$$

have diagonals without zeros in their Schur decomposition, so they are invertible, and thus the maps

$$\begin{aligned}
&D_+\colon R_{\ell,m} \to R_{\ell+1,m} && \text{for } \ell \neq |\ell_1| - 1,\\
&D_-\colon R_{\ell,m} \to R_{\ell-1,m} && \text{for } \ell \neq \ell_0, |\ell_1| \text{ and } m \neq \pm\ell
\end{aligned}$$

are injective. Since E_+ and E_- are isomorphisms we already have that $R_{\ell,m}$ and $R_{\ell,m'}$ have the same dimension as we have already seen a few times, and therefore the above implies that the subspaces $R_{\ell,m}$ for $\ell = \ell_0, \ldots, |\ell_1| - 1$ and m where it makes sense all have the same dimension, and that the subspaces $R_{\ell,m}$ for $\ell \geq |\ell_1|$ and m where it makes sense all have the same dimension. $\square$

Since E_+, E_-, D_+, and D_- commute with Δ_1 and Δ_2 by Lemma 1.10, we get by the same method as in the proof of Lemma 1.18 that:

Lemma 1.28. *In a singular module $M \in C(\lambda_1, \lambda_2)$ all the operators $(\Delta_i)_{\ell,m}$ with $\ell_0 \leq \ell \leq |\ell_1| - 1$ are similar to $(\Delta_i)_{\ell_0,m_0}$ via the same matrix for $i = 1, 2$, and the operators $(\Delta_i)_{\ell,m}$ with $\ell \geq |\ell_1|$ are similar to $(\Delta_i)_{|\ell_1|,m_0}$ via the same matrix for $i = 1, 2$.*

Lemma 1.29. *Define in the singular module $M \in C(\lambda_1, \lambda_2)$ a linear operator δ by*

$$\delta = \frac{1}{\ell_0^2 - \ell_1^2}\Big(\Delta_2 + \frac{\Delta_1^2}{\ell_0^2} - (\ell_0^2 - 1)\,\mathrm{id}\Big) \tag{1.60}$$

for $\ell_0 \neq 0$. Then on the subspaces $R_{\ell,m}$ for which $\ell_0 \leq \ell \leq |\ell_1| - 1$ the operator δ is zero, and on the remaining subspaces $R_{\ell,m}$ δ is nilpotent.

Proof. That $\delta = 0$ on the subspaces $R_{\ell,m}$ for which $\ell_0 \leq \ell \leq |\ell_1| - 1$ follows by Lemma 1.28 and the same argument as in the proof of Lemma 1.19. And since Δ_1 and Δ_2 only have one eigenvalues $\lambda_1 = -i\ell_0\ell 1$ and $\lambda_2 = \ell_0^2 + \ell_1^2 - 1$ respectively we see that δ only has the eigenvalues

$$\begin{aligned}\frac{1}{\ell_0^2 - \ell_1^2}\Big(\lambda_2 + \frac{\lambda_1^2}{\ell_0^2} - (\ell_0^2 - 1)\Big) &= \frac{1}{\ell_0^2 - \ell_1^2}\Big(\ell_0^2 + \ell_1^2 - 1 - \ell_1^2 - (\ell_0^2 - 1)\Big) \\ &= 0,\end{aligned}$$

so by Cayley-Hamilton Theorem δ is nilpotent in general which gives the result. ☐

Remark 1.30. Using eq. (1.49) the argument showing that δ is nilpotent can also be used to show that

$$\begin{aligned} D_-D_+ \colon R_{|\ell_1|-1,m} &\to R_{|\ell_1|-1,m} \\ D_+D_- \colon R_{|\ell_1|,m} &\to R_{|\ell_1|,m} \end{aligned}$$

for $m \neq \pm\ell_1$ in the latter case, are both nilpotent. △

Remark 1.31. Comparing Lemma 1.29 with Lemma 1.19, we see that this property of δ is one of the key differences between the singular and non-singular cases. △

We thus want to understand the map δ better for which we will need the lemma:

Lemma 1.32. *We have that*

$$\begin{aligned} \delta D_+\xi &= 0 \quad \textit{for } \xi \in R_{|\ell_1|-1,m}, \\ D_-\delta\xi &= 0 \quad \textit{for } \xi \in R_{|\ell_1|,m}. \end{aligned} \tag{1.61}$$

Proof. The operator δ is a linear combination of Δ_1^2, Δ_2, and id, so it commutes with D_+ and D_-. Hence since $\delta = 0$ on $R_{|\ell_1|-1,m}$ by Lemma 1.29 $\delta D_+\xi = D_+\delta\xi = 0$ for $\xi \in R_{|\ell_1|-1,m}$ and $D_-\delta\xi = \delta D_-\xi = 0$ for $\xi \in R_{|\ell_1|,m}$ since then $D_-\xi \in R_{|\ell_1|-1,m}$. □

Now we are ready to begin showing the equivalence of the singular category $C(\lambda_1, \lambda_2)$ and the category S_0. First we will show that we from objects $M \in C(\lambda_1, \lambda_2)$ can construct objects $\widetilde{A} \in S_0$.

Let $M \in C(\lambda_1, \lambda_2)$ be a singular module, and consider the maps

$$\begin{aligned} D_+ &\colon R_{|\ell_1|-1,m_0} \to R_{|\ell_1|,m_0}, & D_- &\colon R_{|\ell_1|,m_0} \to R_{|\ell_1|-1,m_0}, \\ \delta &\colon R_{|\ell_1|-1,m_0} \to 0, & \delta &\colon R_{|\ell_1|,m_0} \to R_{|\ell_1|,m_0} \end{aligned}$$

for some m_0 where it makes sense. Writing $P_1 = R_{|\ell_1|-1,m_0}$ and $P_2 = R_{|\ell_1|,m_0}$ for the finite dimensional vector spaces and d_+, d_-, δ_1, and δ_2 for the maps above, we have that $(P_1, P_2, d_+, d_-, \delta_1, \delta_2)$ is an object of the category S_0, since $d_-\delta_2 = \delta_2 d_+ = 0$ by Lemma 1.32, δ_2 is nilpotent by Lemma 1.29, and d_+d_- is nilpotent by Remark 1.30.

Likewise from a morphism $\Gamma\colon M \to M'$ for $M, M' \in C(\lambda_1, \lambda_2)$ we get a corresponding morphism $\gamma = (\gamma_1, \gamma_2)$ from $\widetilde{A}$ to $\widetilde{A}'$. As in the proof of Theorem 1.23 we have that $\Gamma R_{\ell,m} \subset R'_{\ell,m}$ for all ℓ and m where it makes sense, so setting $\gamma_1 = (\Gamma)_{|\ell_1|-1,m_0}\colon P_1 \to P_1'$ and $\gamma_2 = (\Gamma)_{|\ell_1|,m_0}\colon P_2 \to P_2'$, we want to show that $\gamma = (\gamma_1, \gamma_2)$ gives a morphism in S_0, i.e. we want to show that eq. (1.59) commutes. First note that the middle square commutes since δ is defined via the identity map and Δ_1 and Δ_2, and we have already seen that Γ commutes with Δ_1 and Δ_2 in the proof of Proposition 1.14. Now to see that the left square commute note that for $\xi \in P_1 = R_{|\ell_1|-1,|\ell_1|-1}$ the equation $F'_+\Gamma\xi = \Gamma F_+\xi$ implies by eq. (1.34) that $E'_+D'_+\Gamma\xi = \Gamma E_+D_+\xi$ for $\xi \in R_{|\ell_1|-1,|\ell_1|-1}$. Thus since we saw just before Remark 1.8 that Γ and E_+ commute, we get that $E'_+D'_+\Gamma\xi = E'_+\Gamma D_+\xi$, and therefore $D'_+\Gamma\xi = \Gamma D_+\xi$ for $\xi \in R_{|\ell_1|-1,|\ell_1|-1}$ because E'_+ is an isomorphism on $R_{|\ell_1|,|\ell_1|-1}$. Finally using that E_- commutes with D_+ and Γ, we can expand the equality all the way to $D'_+\Gamma = \Gamma D_+$ on $R_{|\ell_1|-1,m_0}$, which gives the commutativity of the first square in eq. (1.59), and the commutativity of the rest can be shown similarly. Hence a morphism in $C(\lambda_1, \lambda_2)$ indeed gives a morphism in S_0.

Now we want to show that to each object $\widetilde{A} = (P_1, P_2, d_+, d_-, \delta_1, \delta_2) \in S_0$ there is a corresponding singular module $M \in C(\lambda_1, \lambda_2)$ with $\lambda_1 = -i\ell_0\ell_1$ and $\lambda_2 = \ell_0^2 + \ell_1^2 - 1$. Let such an $\widetilde{A}$ be given, then we want to construct M. First choose an m_0 such that it makes sense to write $R_{|\ell_1|-1,m_0}$ and put $R_{|\ell_1|-1,m_0} = P_1$ and $R_{|\ell_1|,m_0} = P_2$, and define

$$D_+ \coloneqq d_+\colon R_{|\ell_1|-1,m_0} \to R_{|\ell_1|,m_0}, \qquad D_- \coloneqq d_-\colon R_{|\ell_1|,m_0} \to R_{|\ell_1|-1,m_0}.$$

Consider the space $M = \bigoplus_{\ell,m} R_{\ell,m}$ for $\ell = \ell_0, \ell_0+1, \ldots, |\ell_1|-1, |\ell_1|, \ldots$ and m where it makes sense, where $\dim R_{\ell,m} = \dim R_{|\ell_1|-1,m} = \dim P_1$ for all

$\ell \leq |\ell_1| - 1$, and $\dim R_{\ell,m} = \dim R_{|\ell_1|,m} = \dim P_2$ for all $\ell \geq |\ell_1|$. We want to construct the maps E_+, E_-, D_+, D_-, and D_0. First take $E_+ \colon R_{\ell,m} \to R_{\ell,m+1}$ to be any isomorphism for $m \neq \ell$ and $E_+ \colon R_{\ell,\ell} \to 0$, and then take $E_- \colon R_{\ell,m} \to R_{\ell,m-1}$ for $m \neq -\ell$ to be the inverse isomorphism of the corresponding E_+ and take $E_- \colon R_{\ell,-\ell} \to 0$. Likewise take $D_+ \colon R_{\ell,m_0} \to R_{\ell+1,m_0}$ for $\ell \neq |\ell_1| - 1$ to be any isomorphism, and for $\ell = |\ell_1| - 1$ we already have defined D_+ on $R_{|\ell_1|-1,m_0}$. We expand to all $R_{\ell,m}$ and $R_{|\ell_1|-1,m}$ by using E_+ either on the left or on the right such that

$$\begin{tikzcd} R_{\ell,m+1} \arrow[r, "D_+"] & R_{\ell+1,m+1} \\ R_{\ell,m} \arrow[u, "E_+"] \arrow[r, "D_+"'] & R_{\ell+1,m} \arrow[u, "E_+"'] \end{tikzcd}$$

commutes for $-\ell \leq m < \ell$, i.e. $(D_+)_{\ell,m+1} = (E_+)_{\ell+1,m}(D_+)_{\ell,m}(E_+)^{-1}_{\ell,m}$.

Now before constructing D_0 and D_- we want to construct Δ_1 and Δ_2. Since D_- and $\delta = \delta_2$ are defined on $R_{|\ell_1|,m_0}$, we can define

$$\Delta_2\xi = (\ell_1^2 + \ell_0^2 - 1)\xi + \ell_1^2 \frac{4\ell_1^2 - 1}{\ell_1^2 - \ell_0^2} D_+D_-\xi + \ell_0^2\delta\xi$$

for $\xi \in R_{|\ell_1|,m_0}$. Since $D_+D_- = d_+d_-$ and $\delta = \delta_2$ on $R_{|\ell_1|,m_0}$ are nilpotent by assumption, we see that Δ_2 has only one eigenvalue and that is $\ell_1^2 + \ell_0^2 - 1$. Now since we still want the equation

$$\delta = \frac{1}{\ell_0^2 - \ell_1^2}\Big(\Delta_2 + \frac{\Delta_1^2}{\ell_0^2} - (\ell_0^2 - 1)\,\mathrm{id}\Big)$$

to hold true, we define

$$\begin{aligned}
\Delta_1^2\xi &= \ell_0^2\Big((\ell_0^2 - 1)\xi - \Delta_2\xi + (\ell_0^2 - \ell_1^2)\delta\xi\Big) \\
&= \ell_0^2\Big((\ell_0^2 - 1)\xi - (\ell_1^2 + \ell_0^2 - 1)\xi - \ell_1^2\frac{4\ell_1^2 - 1}{\ell_1^2 - \ell_0^2}D_+D_-\xi - \ell_0^2\delta\xi \\
&\quad + (\ell_0^2 - \ell_1^2)\delta\xi\Big) \\
&= \ell_0^2\Big(-\ell_1^2\xi - \ell_1^2\frac{4\ell_1^2 - 1}{\ell_1^2 - \ell_0^2}D_+D_-\xi - \ell_1^2\delta\xi\Big) \\
&= -\ell_0^2\ell_1^2\Big(\xi + \frac{4\ell_1^2 - 1}{\ell_1^2 - \ell_0^2}D_+D_-\xi + \delta\xi\Big)
\end{aligned}$$

for $\xi \in R_{|\ell_1|,m_0}$. Again since D_+D_- and δ are nilpotent on $R_{|\ell_1|,m_0}$ by assumption, we see that Δ_1^2 only has the eigenvalue $-\ell_0^2\ell_1^2$, which is non-zero, and thus Δ_1's only eigenvalue is non-zero. Therefore Appendix B.2 implies that the above determines Δ_1 uniquely on $R_{|\ell_1|,m_0}$.

For $\ell \geq |\ell_1|$ consider the map $J_{r,s} = (E_+)^s (D_+)^r$ as earlier, where $(E_+)^{-1} = E_-$, and put

$$(\Delta_i)_{\ell,m} = J_{\ell-|\ell_1|,m-m_0} (\Delta_i)_{|\ell_1|,m_0} J^{-1}_{\ell-|\ell_1|,m-m_0}$$

for $i = 1, 2$, such that we have expanded Δ_1 and Δ_2 to all $R_{\ell,m}$. Now following the formulae of eqs. (1.54) and (1.55) we put

$$\begin{aligned}
(D_0)_{\ell,m} &= -\frac{1}{\ell(\ell+1)} (\Delta_1)_{\ell,m}, \\
(D_+ D_-)_{\ell,m} &= \frac{1}{4\ell^2 - 1} \Big((\Delta_2)_{\ell,m} - (\ell^2 - 1)(\mathrm{id})_{\ell,m} + \frac{(\Delta_1)^2_{\ell,m}}{\ell^2} \Big).
\end{aligned}$$

Since $(D_+)_{\ell,m}$ is an isomorphism for all $\ell \geq |\ell_1|$, we get that this determines $(D_-)_{\ell,m}$ for $\ell \geq |\ell_1| + 1$, and we already have $(D_-)_{|\ell_1|,m}$ given.

Similarly we can find D_0 and D_- for $\ell_0 \leq \ell < |\ell_1|$, and thus we get the linear maps E_+, E_-, D_+, D_-, and D_0 defined everywhere where it makes sense, and actually we get by our standard construction operators H_+, H_-, H_3, F_+, F_-, and F_3, which gives a representation, i.e. we indeed get a singular module $M \in C(\lambda_1, \lambda_2)$ for each object $\widetilde{A} = (P_1, P_2, d_+, d_-, \delta_1, \delta_2) \in S_0$. To see this we need only check that the equations of eq. (1.35) hold. As in Theorem 1.21 we see that for $\ell \neq 0$

$$\begin{aligned}
\ell (D_+)_{\ell,m} (D_0)_{\ell,m} &= -\frac{\ell}{\ell(\ell+1)} (D_+)_{\ell,m} (\Delta_1)_{\ell,m} \\
&= -\frac{1}{\ell+1} (\Delta_1)_{\ell+1,m} (D_+)_{\ell,m} \\
&= -(\ell+2) \frac{1}{(\ell+1)(\ell+2)} (\Delta_1)_{\ell+1,m} (D_+)_{\ell,m} \\
&= (\ell+2)(D_0)_{\ell+1,m} (D_+)_{\ell,m},
\end{aligned}$$

which gives us the first equation, and the rest can be checked similarly. Additionally it can be checked like in Theorem 1.23 that there is a correspondence between the morphisms, and thus altogether we get the theorem:

Theorem 1.33. *The singular category $C(\lambda_1, \lambda_2)$ is equivalent to the category S_0.*

Corollary 1.34. *An indecomposable module M in the singular category $C(\lambda_1, \lambda_2)$ corresponds to an indecomposable object A in the category S_0.*

Now we have reduced the problem of characterizing the singular indecomposable modules to a problem of linear algebra, or more precisely to working with linear relations, which is beyond the scope of this paper — for the details of this see [GP67b] Chapter II. However we will now give the resulting description:

We divide the indecomposable objects of S_0 into two types that we call open or closed. A simple type of open object A from which we build general open objects is called a strand, and it has the following properties: P_1 has a basis $(e_1, \ldots, e_n)$ and P_2 has a basis $(f_0, f_1, \ldots, f_n, f'_1, f'_2, \ldots, f'_m)$ such that the operators d_+, d_-, and δ satisfy

$$\begin{aligned} d_- f_i &= e_{i+1} \quad (i < n), & d_- f_n &= 0, & d_- f'_i &= 0, & d_+ e_i &= f_i, \\ \delta f_i &= 0 \quad (i > 0), & \delta f_0 &= f'_1, & \delta f'_i &= f'_{i+1} \quad (i < m), & \delta f'_m &= 0. \end{aligned}$$

Here f_n and f'_m are called tail vectors in the strand, and the two numbers (n, m) are invariants of a strand.

An open object is composed of strands and is given by the set of numbers $(s, n_1, m_1, n_2, m_2, \ldots, n_k, m_k)$, where $s \in \{0, 1\}$, $n_1 \geq 0$, $n_i > 0$ for $i \neq 1$, $m_i > 0$ for $i \neq k$, $m_k \geq -1$. Here the i'th pair (n_i, m_i) corresponds to the i'th strand, and the tail vectors of these strands must satisfy

$$f'_{m_1} = f_{n_2}, \quad f'_{m_2} = f_{n_3}, \quad \ldots \quad f'_{m_{k-1}} = f_{n_k}. \tag{1.62}$$

Given this one can show the following proposition:

Proposition 1.35. *Let A and A' be two open indecomposable objects in S_0 which are given respectively by $(s, n_1, m_1, \ldots, n_k, m_k)$ and $(s', n'_1, m'_1 \ldots, n'_k, m'_k)$. Then A and A' are equivalent if and only if $s = s'$, $n_i = n'_i$, and $m_i = m'_i$ for all i.*

A simple closed object is obtained from one open object with $s = 0$, $n_1 > 0$, $m_k > 0$, and f_{n_1} and f'_{m_k} non-zero. Here f_{n_1} is called the starting vector and f'_{m_k} the terminating vector of the open object. The closed simple object is defined simply by adding the relation $f'_{m_k} = \mu f_{n_1}$ to the relations of eq. (1.62), where μ is some complex number. Thus in the simple case a closed object is given by numbers $(n_1, m_1, \ldots, n_k, m_k, \mu)$, where $n_i, m_i > 0$ and $\mu \in \mathbf{C}$.

In general a closed object is given by numbers $(n_1, m_1, \ldots, n_k, m_k, \mu, N)$, where $n_i, m_i, N > 0$ are integers and $\mu \in \mathbf{C}$, and it is made from N open objects given by $(0, n_1, m_1, \ldots, n_k, m_k)$ for the same n_i and m_i as above. If we denote for $j = 1, \ldots, N$ the starting object of the j'th open object by f_j and the terminating object by f'_j, then the closed object is given by adding the relations

$$f'_1 = \mu f_1, \qquad f'_i = \mu f_i + f_{i-1} \text{ for } i = 2, 3, \ldots, N.$$

Given this one can as in the open case show the following proposition:

Proposition 1.36. *Let A and A' be two closed indecomposable objects in S_0 which are given respectively by $(n_1, m_1, \ldots, n_k, m_k, \mu, N)$ and $(n'_1, m'_1, \ldots, n'_k, m'_k, \mu', N')$. Then A and A' are equivalent if and only if $\mu = \mu'$, $N = N'$, and the sequence $\{n'_i, m'_i\}$ is a cyclic permutation of the sequence $\{n_i, m_i\}$.*

Now since all indecomposable objects in S_0 are either open or closed, this gives a complete description of the indecomposables of S_0, and thus by Corollary 1.34 a description of the indecomposables of singular Harish-Chandra modules in $C(\lambda_1, \lambda_2)$. Altogether we end up with the theorem:

Theorem 1.37. *Let M be an indecomposable Harish-Chandra module for the pair (L, L_k) belonging to the singular category $C(\lambda_1, \lambda_2)$. Then either M can be described by some integers $(s, n_1, m_1, n_2, m_2, \ldots, n_k, m_k)$ or by some integers and a complex number μ $(n_1, m_1, \ldots, n_k, m_k, \mu, N)$, and two such modules M and M' are equivalent if and only if $\lambda_1 = \lambda_1'$, $\lambda_2 = \lambda_2'$, $\mu = \mu'$ (in the case with μ in the description), and all the corresponding integers agree.*

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Appendix A

Calculations

Throughout the paper there are situations where we need to do some straightforward but rather long calculations, so to clean up the exposition somewhat we will collect most of these calculations in this appendix and then just use the results in the paper.

A.1 Bases of $V(2) \otimes V(n)$

We want to describe the s_k's of eq. (1.17) more explicitly. We have that $s_0 = w_0 \otimes v_0$ and $s_k = \frac{1}{k!} y^k.s_0$, and we note that if $n > 0$ then

$$\begin{aligned} s_1 &= y.(w_0 \otimes v_0) = y.w_0 \otimes v_0 + w_0 \otimes y.v_0 \\ &= w_1 \otimes v_0 + w_0 \otimes v_1 \end{aligned}$$

and

$$\begin{aligned} s_2 &= \tfrac{1}{2} y.s_1 \\ &= \tfrac{1}{2} y.w_1 \otimes v_0 + \tfrac{1}{2} w_1 \otimes y.v_0 + \tfrac{1}{2} y.w_0 \otimes v_1 + w_0 \otimes \tfrac{1}{2} y.v_1 \\ &= w_2 \otimes v_0 + \tfrac{1}{2} w_1 \otimes v_1 + \tfrac{1}{2} w_1 \otimes v_1 + w_0 \otimes v_2 \\ &= w_2 \otimes v_0 + w_1 \otimes v_1 + w_0 \otimes v_2. \end{aligned}$$

Inductively we see that

$$s_k = w_2 \otimes v_{k-2} + w_1 \otimes v_{k-1} + w_0 \otimes v_k$$

for $k \leq n$, since the base case holds and given the equality for $k < n$ we get

$$\begin{aligned} s_{k+1} &= \frac{1}{k+1} y.s_k \\ &= w_2 \otimes \frac{1}{k+1} y.v_{k-2} + \frac{1}{k+1} y.w_1 \otimes v_{k-1} + w_1 \otimes \frac{1}{k+1} y.v_{k-1} \\ &\quad + \frac{1}{k+1} y.w_0 \otimes v_k + w_0 \otimes \frac{1}{k+1} y.v_k \end{aligned}$$

$$
\begin{aligned}
&= \frac{k-1}{k+1} w_2 \otimes v_{k-1} + \frac{2}{k+1} w_2 \otimes v_{k-1} + \frac{k}{k+1} w_1 \otimes v_k + \frac{1}{k+1} w_1 \otimes v_k \\
&\quad + w_0 \otimes v_{k+1} \\
&= w_2 \otimes v_{k-1} + w_1 \otimes v_k + w_0 \otimes v_{k+1}.
\end{aligned}
$$

We likewise see that for $k = n+1$ the last term vanishes, so we have $s_{k+1} = w_2 \otimes v_{n-1} + w_1 \otimes v_n$, and for $k = n+2$ the two last terms vanish, so we get $s_{k+2} = w_2 \otimes v_n$. Thus altogether we get the description in eq. (1.18).

Suppose now that $n \geq 1$. We want to describe the t_k's of eq. (1.19) more explicitly. We have that $t_0 = w_0 \otimes v_1 - \frac{n}{2} w_1 \otimes v_0$ and $t_k = \frac{1}{k!} y^k . t_0$. We see that

$$
\begin{aligned}
t_1 &= y.\big(w_0 \otimes v_1 - \frac{n}{2} w_1 \otimes v_0\big) \\
&= y.w_0 \otimes v_1 + w_0 \otimes y.v_1 - \frac{n}{2} y.w_1 \otimes v_0 + \frac{n}{2} w_1 \otimes y.v_0 \\
&= w_1 \otimes v_1 + 2w_0 \otimes v_2 - n w_2 \otimes v_0 - \frac{n}{2} w_1 \otimes v_1 \\
&= 2w_0 \otimes v_2 - \frac{n-2}{2} w_1 \otimes v_1 - n w_2 \otimes v_0,
\end{aligned}
$$

and inductively we get that

$$
t_k = (k+1) w_0 \otimes v_{k+1} - \frac{n-2k}{2} w_1 \otimes v_k + (k-1-n) w_2 \otimes v_{k-1}
$$

for $1 \leq k \leq n-1$, since the base case holds and given the equality for $k < n-1$ we get

$$
\begin{aligned}
t_{k+1} &= \frac{1}{k+1} y.t_k \\
&= y.w_0 \otimes v_{k+1} + w_0 \otimes y.v_{k+1} - \frac{n-2k}{2(k+1)} y.w_1 \otimes v_k \\
&\quad - \frac{n-2k}{2(k+1)} w_1 \otimes y.v_k + \frac{k-1-n}{k+1} w_2 \otimes y.v_{k-1} \\
&= w_1 \otimes v_{k+1} + (k+2) w_0 \otimes v_{k+2} - \frac{n-2k}{k+1} w_2 \otimes v_k \\
&\quad - \frac{n-2k}{2} w_1 \otimes v_{k+1} + \frac{(k-1-n)k}{k+1} w_2 \otimes v_k \\
&= (k+2) w_0 \otimes v_{k+2} - \frac{n-2(k+1)}{2} w_1 \otimes v_{k+1} \\
&\quad + \left(\frac{k^2 - k - nk - n + 2k}{k+1} \right) w_2 \otimes v_k \\
&= (k+2) w_0 \otimes v_{k+2} - \frac{n-2(k+1)}{2} w_1 \otimes v_{k+1} + (k-n) w_2 \otimes v_k,
\end{aligned}
$$

where we in the last equality use that $(k+1)(k-n) = k^2 - nk + k - n = k^2 - k - nk - n + 2k$. We likewise see that for $k = n$ the first term vanishes so

$$
t_n = \frac{n}{2} w_1 \otimes v_n - w_2 \otimes v_{n-1}.
$$

Thus we altogether get the description in eq. (1.20).

Suppose now that $n \geq 2$. We want to describe the u_k's of eq. (1.21) more explicitely. We have that

$$u_0 \coloneqq w_0 \otimes v_2 - \frac{n-1}{2} w_1 \otimes v_1 + \frac{n(n-1)}{2} w_2 \otimes v_0$$

and $u_k = \frac{1}{k!} y^k . u_0$. We see inductively that

$$\begin{aligned} u_k = &\frac{(k+1)(k+2)}{2} w_0 \otimes v_{k+2} - \frac{(k+1)(n-k-1)}{2} w_1 \otimes v_{k+1} \\ &+ \frac{(n-k)(n-k-1)}{2} w_2 \otimes v_k \end{aligned}$$

for $0 \leq k \leq n-2$, since the base case holds and given the equality for $k < n-2$ we get

$$\begin{aligned} u_{k+1} &= \frac{1}{k+1} y.u_k \\ &= \frac{k+2}{2} y.w_0 \otimes v_{k+2} + \frac{k+2}{2} w_0 \otimes y.v_{k+2} \\ &\quad - \frac{n-k-1}{2} y.w_1 \otimes v_{k+1} - \frac{n-k-1}{2} w_1 \otimes y.v_{k+1} \\ &\quad + \frac{(n-k)(n-k-1)}{2(k+1)} w_2 \otimes y.v_k \\ &= \frac{k+2}{2} w_1 \otimes v_{k+2} + \frac{(k+2)(k+3)}{2} w_0 \otimes v_{k+3} \\ &\quad - (n-k-1) w_2 \otimes v_{k+1} - \frac{(n-k-1)(k+2)}{2} w_1 \otimes v_{k+2} \\ &\quad + \frac{(n-k)(n-k-1)}{2} w_2 \otimes v_{k+1} \\ &= \frac{(k+2)(k+3)}{2} w_0 \otimes v_{k+3} \\ &\quad - \frac{(n-k-1)(k+2) - (k+2)}{2} w_1 \otimes v_{k+2} \\ &\quad + \frac{(n-k)(n-k-1) - 2(n-k-1)}{2} w_2 \otimes v_{k+1} \\ &= \frac{(k+2)(k+3)}{2} w_0 \otimes v_{k+3} \\ &\quad - \frac{(k+2)(n-k-2)}{2} w_1 \otimes v_{k+2} \\ &\quad + \frac{(n-k-1)(n-k-2)}{2} w_2 \otimes v_{k+1}. \end{aligned}$$

Thus we altogether get the description in eq. (1.22).

A.2 Finding $w_1 \otimes v_k$

Using the bases $(s_k \mid 0 \le k \le n+2)$ of eq. (1.18), $(t_k \mid 0 \le k \le n)$ of eq. (1.20), and $(u_k \mid 0 \le k \le n-2)$ of eq. (1.22), we see that

$$
\begin{aligned}
&\frac{2(k+1)(n+1-k)}{(n+1)(n+2)} s_{k+1} - \frac{2(n-2k)}{n(n+2)} t_k - \frac{4}{n(n+1)} u_{k-1} \\
&= \frac{2(k+1)(n+1-k)}{(n+1)(n+2)} \Big(w_0 \otimes v_{k+1} + w_1 \otimes v_k + w_2 \otimes v_{k-1} \Big) \\
&\quad - \frac{2(n-2k)}{n(n+2)} \Big((k+1) w_0 \otimes v_{k+1} - \frac{n-2k}{2} w_1 \otimes v_k \\
&\qquad + (k-1-n) w_2 \otimes v_{k-1} \Big) \\
&\quad - \frac{4}{n(n+1)} \Big(\frac{k(k+1)}{2} w_0 \otimes v_{k+1} - \frac{k(n-k)}{2} w_1 \otimes v_k \\
&\qquad + \frac{(n-k+1)(n-k)}{2} w_2 \otimes v_{k-1} \Big) \\
&= \frac{\begin{pmatrix} 2(k+1)(n+1-k)n - 2(n-2k)(k+1)(n+1) \\ \qquad - 2k(k+1)(n+2) \end{pmatrix}}{n(n+1)(n+2)} w_0 \otimes v_{k+1} \\
&\quad + \frac{\begin{pmatrix} 2(k+1)(n+1-k)n + (n-2k)(n-2k)(n+1) \\ \qquad + 2k(n-k)(n+2) \end{pmatrix}}{n(n+1)(n+2)} w_1 \otimes v_k \\
&\quad + \frac{\begin{pmatrix} 2(k+1)(n+1-k)n - 2(n-2k)(k-1-n)(n+1) \\ \qquad - 2(n-k+1)(n-k)(n+2) \end{pmatrix}}{n(n+1)(n+2)} w_2 \otimes v_{k-1} \\
&= 2(k+1) \frac{(n+1-k)n - (n-2k)(n+1) - k(n+2)}{n(n+1)(n+2)} w_0 \otimes v_{k+1} \\
&\quad + \frac{\begin{pmatrix} 2(k+1)(n+1-k)n + (n-2k)(n-2k)(n+1) \\ \qquad + 2k(n-k)(n+2) \end{pmatrix}}{n(n+1)(n+2)} w_1 \otimes v_k \\
&\quad + 2(n+1-k) \frac{(k+1)n + (n-2k)(n+1) - (n-k)(n+2)}{n(n+1)(n+2)} w_2 \otimes v_{k-1}.
\end{aligned}
$$

Now straightforward algebraic manipulation (easily done in e.g. Mathematica) shows that

$$
\begin{aligned}
(n+1-k)n - (n-2k)(n+1) - k(n+2) &= 0, \\
(k+1)n + (n-2k)(n+1) - (n-k)(n+2) &= 0,
\end{aligned}
$$

and

$$
\begin{aligned}
&2(k+1)(n+1-k)n + (n-2k)(n-2k)(n+1) + 2k(n-k)(n+2) \\
&= n(n+1)(n+2).
\end{aligned}
$$

Thus we see that

$$\begin{aligned}
&\frac{2(k+1)(n+1-k)}{(n+1)(n+2)} s_{k+1} - \frac{2(n-2k)}{n(n+2)} t_k - \frac{4}{n(n+1)} u_{k-1} \\
&= 0 + \frac{n(n+1)(n+2)}{n(n+1)(n+2)} w_1 \otimes v_k + 0 \\
&= w_1 \otimes v_k
\end{aligned}$$

giving us eq. (1.23).

Likewise for $n \geq 1$, we get that

$$\begin{aligned}
\frac{2}{n+2}(s_1 - t_0) &= \frac{2}{n+2}\Big(w_0 \otimes v_1 + w_1 \otimes v_0 - w_0 \otimes v_1 + \frac{n}{2} w_1 \otimes v_0\Big) \\
&= \frac{2}{n+2} \frac{n+2}{2} w_1 \otimes v_0 \\
&= w_1 \otimes v_0
\end{aligned}$$

and

$$\begin{aligned}
\frac{2}{n+2}(s_{n+1} + t_n) &= \frac{2}{n+2}\Big(w_2 \otimes v_{n+1} + w_1 \otimes v_n + \frac{n}{2} w_1 \otimes v_n - w_2 \otimes v_{n-1}\Big) \\
&= \frac{2}{n+2} \frac{n+2}{2} w_1 \otimes v_n \\
&= w_1 \otimes v_n
\end{aligned}$$

giving us eq. (1.24).

A.3 Inner products in $V(2) \otimes V(n)$

Given $s_0 = w_0 \otimes v_0$, $t_0 = w_0 \otimes v_1 - \frac{n}{2} w_1 \otimes v_0$, and $u_0 = w_0 \otimes v_2 - \frac{n-1}{2} w_1 \otimes v_1 + \frac{n(n-1)}{2} w_2 \otimes v_0$ from eq. (1.18), eq. (1.20), and eq. (1.22), we want to find $\langle s_0, s_0 \rangle$, $\langle t_0, t_0 \rangle$, and $\langle u_0, u_0 \rangle$ using the inner products of eq. (1.25) and eq. (1.26). Noting that all terms with $\langle w_i \otimes v_j, w_k \otimes v_\ell \rangle$ with $i \neq k$ or $j \neq \ell$ vanish since then either $\langle w_i, w_k \rangle = 0$ or $\langle v_j, v_\ell \rangle = 0$, we see that

$$\begin{aligned}
\langle u_0, u_0 \rangle &= \Big\langle w_0 \otimes v_2 - \frac{n-1}{2} w_1 \otimes v_1 + \frac{n(n-1)}{2} w_2 \otimes v_0, \\
&\qquad w_0 \otimes v_2 - \frac{n-1}{2} w_1 \otimes v_1 + \frac{n(n-1)}{2} w_2 \otimes v_0 \Big\rangle \\
&= \langle w_0 \otimes v_2, w_0 \otimes v_2 \rangle + \frac{(n-1)^2}{4} \langle w_1 \otimes v_1, w_1 \otimes v_1 \rangle \\
&\quad + \frac{n^2(n-1)^2}{4} \langle w_2 \otimes v_0, w_2 \otimes v_0 \rangle \\
&= \langle w_0, w_0 \rangle \cdot \langle v_2, v_2 \rangle + \frac{(n-1)^2}{4} \langle w_1, w_1 \rangle \cdot \langle v_1, v_1 \rangle \\
&\quad + \frac{n^2(n-1)^2}{4} \langle w_2, w_2 \rangle \cdot \langle v_0, v_0 \rangle
\end{aligned}$$

$$\begin{aligned}
&= \binom{2}{0}\cdot\binom{n}{2} + \frac{(n-1)^2}{4}\binom{2}{1}\binom{n}{1} + \frac{n^2(n-1)^2}{4}\binom{2}{2}\cdot\binom{n}{0} \\
&= \frac{n(n-1)}{2} + \frac{n(n-1)^2}{2} + \frac{n^2(n-1)^2}{4} \\
&= n(n-1)\frac{2+2(n-1)+n(n-1)}{4} \\
&= n(n-1)\frac{n^2+n}{4} = \frac{n^2(n+1)(n-1)}{4}.
\end{aligned}$$

Similarly we get that

$$\langle s_0, s_0\rangle = 1$$

and

$$\langle t_0, t_0\rangle = \frac{n(n+2)}{2}.$$

Thus we indeed get eq. (1.27).

Now we want to show that we also have eq. (1.28). Working with the basis $(v_0,\dots,v_n)$ of $V(n)$ from eq. (1.5) first note that by eq. (1.25) $\langle h.v_j, v_k\rangle = (n-2j)\delta_{jk}\binom{n}{k} = \langle v_j, h.v_k\rangle$. Also for $j \neq k+1$, we have $\langle x.v_j, v_k\rangle = 0 = \langle v_j, y.v_k\rangle$, while $\langle x.v_{k+1}, v_k\rangle = (n-k)\binom{n}{k} = (k+1)\binom{n}{k+1} = \langle v_{k+1}, y.v_k\rangle$, so $\langle x.v_j, v_k\rangle = \langle v_j, y.v_k\rangle$ for all j. By symmetry also $\langle y.v_j, v_k\rangle = \langle v_j, x.v_k\rangle$, and so for all $v, w \in V(n)$

$$\langle X.v, w\rangle = \langle v, X^H.w\rangle \qquad \text{for all } X \in \mathfrak{sl}(2,\mathbf{C}), \tag{A.1}$$

since $h^H = h$, $x^H = y$, and $y^H = x$ by the definitions in eq. (1.1). The property of eq. (A.1) determines the inner product up to a positive factor. To see this note that $h^H = h$, so by linear algebra its eigenvalues are in $\mathbf{R}$ and distinct eigenspaces of h are orthogonal. Therefore since $(v_0,\dots,v_n)$ is a basis with eigenvectors of h from distinct eigenspaces the property eq. (A.1) implies that $\langle v_k, v_j\rangle = 0$ for $j \neq k$. Also the property implies that $\langle y.v_k, v_{k+1}\rangle = \langle v_k, x.v_{k+1}\rangle$, so

$$(k+1)\langle v_{k+1}, v_{k+1}, =\rangle\langle y.v_k, v_{k+1}\rangle = \langle v_k, x.v_{k+1}\rangle = (n-k)\langle v_k, v_k\rangle,$$

and inductively we see that $\langle v_k, v_k\rangle$ is determined by $\langle v_0, v_0\rangle$, so by the above the inner product is determined by $\langle v_0, v_0\rangle$. More precisely since

$$\frac{n-k}{k+1}\binom{n}{k} = \binom{n}{k+1}$$

we get inductively that

$$\langle v_k, v_k\rangle = \langle v_0, v_0\rangle\binom{n}{k}.$$

Since the inner products of $V(2)$ and $V(n)$ satisfy eq. (A.1), it is clear by eq. (1.26) that also the inner product of $V(2) \otimes V(n)$ satisfy eq. (A.1), and so clearly also the restrictions to the submodules corresponding to either $V(n-2)$, $V(n)$, or $V(n+2)$ also satisfy this property. Therefore as above we get e.g.

$$\langle s_k, s_k\rangle = \langle s_0, s_0\rangle \binom{n+2}{k}$$

by working in $V(n+2)$ instead of $V(n)$. Similarly we get the analogous results for t_k and u_k giving us eq. (1.28).

A.4 Finding $\overline{w}_1 \otimes \overline{v}_k$

We want to find $\overline{w}_1 \otimes \overline{v}_k$ in terms of $\overline{s}_k$, $\overline{t}_k$, and $\overline{u}_k$ from eqs. (1.29) and (1.30). First we note that for $0 < k < n$

$$\begin{aligned}
\sqrt{2\binom{n}{k}}\overline{w}_1 \otimes \overline{v}_k &= \sqrt{\binom{2}{1}}\overline{w}_1 \otimes \sqrt{\binom{n}{k}}\overline{v}_k \\
&= w_1 \otimes v_k \\
&= \frac{2(k+1)(n+1-k)}{(n+1)(n+2)} s_{k+1} - \frac{2(n-2k)}{n(n+2)} t_k - \frac{4}{n(n+1)} u_{k-1} \\
&= \frac{2(k+1)(n+1-k)}{(n+1)(n+2)} \sqrt{\binom{n+2}{k+1}} \overline{s}_{k+1} \\
&\quad - \frac{2(n-2k)}{n(n+2)} \sqrt{\frac{n(n+2)}{2}\binom{n}{k}} \overline{t}_k \\
&\quad - \frac{4}{n(n+1)} \sqrt{\frac{n^2(n+1)(n-1)}{4}\binom{n-2}{k-1}} \overline{u}_{k-1} \\
&= \frac{2(k+1)(n+1-k)}{(n+1)(n+2)} \sqrt{\binom{n+2}{k+1}} \overline{s}_{k+1} \\
&\quad - \frac{\sqrt{2}(n-2k)}{\sqrt{n(n+2)}} \sqrt{\binom{n}{k}} \overline{t}_k \\
&\quad - \frac{2\sqrt{(n-1)}}{\sqrt{(n+1)}} \sqrt{\binom{n-2}{k-1}} \overline{u}_{k-1}.
\end{aligned}$$

Now since

$$\frac{\binom{n+2}{k+1}}{\binom{n}{k}} = \frac{(n+2)(n+1)}{(k+1)(n+1-k)}, \qquad \frac{\binom{n-2}{k-1}}{\binom{n}{k}} = \frac{k(n-k)}{n(n-1)},$$

we see that

$$\begin{aligned}
\overline{w}_1 \otimes \overline{v}_k &= \frac{\sqrt{2}(k+1)(n+1-k)}{(n+1)(n+2)}\sqrt{\frac{(n+2)(n+1)}{(k+1)(n+1-k)}}\overline{s}_{k+1} \\
&\quad - \frac{(n-2k)}{\sqrt{n(n+2)}}\overline{t}_k \\
&\quad - \frac{\sqrt{2(n-1)}}{\sqrt{(n+1)}}\sqrt{\frac{k(n-k)}{n(n-1)}}\overline{u}_{k-1} \\
&= \sqrt{\frac{2(k+1)(n+1-k)}{(n+1)(n+2)}}\overline{s}_{k+1} - \frac{(n-2k)}{\sqrt{n(n+2)}}\overline{t}_k \\
&\quad - \sqrt{\frac{2k(n-k)}{n(n+1)}}\overline{u}_{k-1}.
\end{aligned}$$

Also since eq. (1.24) is a special case of eq. (1.23) the above formula also holds for $k \in \{0, n\}$ if we take the coefficient in front of $\overline{u}_{k-1}$ to be 0. Thus we indeed get eq. (1.31)

A.5 F_3, F_+, F_- in terms of E_+, E_-, D_0, D_+, D_-

We have already seen that

$$F_3\xi = \sqrt{\ell^2 - m^2}D_-\xi - mD_0\xi - \sqrt{(\ell+1)^2 - m^2}D_+\xi$$

for $\xi \in R_{\ell,m}$ by using eq. (1.33) and the definition of how we expanded D_0, D_+, and D_- to maps on all of M. Now we get by eqs. (1.3) and (1.11) and the commutative diagrams in eq. (1.12) that

$$\begin{aligned}
F_+\xi &= [F_3, H_+]\xi = F_3H_+\xi - H_+F_3\xi \\
&= \sqrt{(\ell+m+1)(\ell-m)}F_3E_+\xi - \sqrt{\ell^2-m^2}H_+D_-\xi + mH_+D_0\xi \\
&\quad + \sqrt{(\ell+1)^2-m^2}H_+D_+\xi \\
&= \sqrt{(\ell+m+1)(\ell-m)}\Big(\sqrt{\ell^2-(m+1)^2}D_-E_+\xi - (m+1)D_0E_+\xi \\
&\qquad\qquad - \sqrt{(\ell+1)^2-(m+1)^2}D_+E_+\xi\Big) \\
&\quad - \sqrt{\ell^2-m^2}\sqrt{((\ell-1)+m+1)((\ell-1)-m)}E_+D_-\xi \\
&\quad + m\sqrt{(\ell+m+1)(\ell-m)}E_+D_0\xi \\
&\quad + \sqrt{(\ell+1)^2-m^2}\sqrt{((\ell+1)+m+1)((\ell+1)-m)}E_+D_+\xi
\end{aligned}$$

$$
\begin{aligned}
&= \sqrt{(\ell+m+1)(\ell-m)}\Big(\sqrt{\ell^2-(m+1)^2}D_-E_+\xi - (m+1)D_0E_+\xi \\
&\qquad\qquad - \sqrt{(\ell+1)^2-(m+1)^2}D_+E_+\xi\Big) \\
&\quad - \sqrt{\ell^2-m^2}\sqrt{(\ell+m)(\ell-m-1)}D_-E_+\xi \\
&\quad + m\sqrt{(\ell+m+1)(\ell-m)}D_0E_+\xi \\
&\quad + \sqrt{(\ell+1)^2-m^2}\sqrt{(\ell+m+2)(\ell-m+1)}D_+E_+\xi \\
&= \Big(\sqrt{(\ell+m+1)(\ell-m)(\ell^2-(m+1)^2)} \\
&\qquad - \sqrt{(l^2-m^2)(l+m)(l-m-1)}\Big)D_-E_+\xi \\
&\quad - \sqrt{(\ell+m+1)(\ell-m)}D_0E_+\xi \\
&\quad + \Big(\sqrt{((\ell+1)^2-m^2)(\ell+m+2)(\ell-m+1)} \\
&\qquad - \sqrt{(\ell+m+1)(\ell-m)((\ell+1)^2-(m+1)^2)}\Big)D_+E_+\xi
\end{aligned}
$$

for $\xi \in R_{\ell,m}$ and $-\ell+1 \leq m < \ell-1$. In the case where $m=-\ell$ the only problem is at the term with E_+D_-, but this is not a problem because the term vanishes since there is $\ell+m$ as part of the coefficient, so the formula also holds true in this case. In case $m=\ell-1$ the only problem is at the term with D_-E_+, but here we have $\ell^2-(m+1)^2$ as part of the coefficient, so this term also vanishes, and the formula also hold true in this case. Finally in case $m=\ell$ the terms with D_-E_+, D_0E_+, D_+E_+, E_+D_-, and E_+D_0 all cause problems, but again all of these terms vanish, so the formula still holds true in this case. Now by noting that $\ell^2-m^2=(\ell+m)(\ell-m)$ and $\ell^2-(m+1)^2=(\ell+m+1)(\ell-m-1)$, we see that

$$
\begin{aligned}
&\sqrt{(\ell+m+1)(\ell-m)(\ell^2-(m+1)^2)} - \sqrt{(l^2-m^2)(l+m)(l-m-1)} \\
&= \sqrt{(\ell+m+1)(\ell-m)(\ell+m+1)(\ell-m-1)} \\
&\quad - \sqrt{(\ell+m)(\ell-m)(\ell+m)(\ell-m-1)} \\
&= (\ell+m+1)\sqrt{(\ell-m)(\ell-m-1)} - (\ell+m)\sqrt{(\ell-m)(\ell-m-1)} \\
&= \sqrt{(\ell-m)(\ell-m-1)}
\end{aligned}
$$

and similarly

$$
\begin{aligned}
&\sqrt{((\ell+1)^2-m^2)(\ell+m+2)(\ell-m+1)} \\
&\qquad - \sqrt{(\ell+m+1)(\ell-m)((\ell+1)^2-(m+1)^2)} \\
&= \sqrt{(\ell+m+1)(\ell+m+2)}.
\end{aligned}
$$

So we get that

$$
\begin{aligned}
F_+\xi &= \sqrt{(\ell-m)(\ell-m-1)}D_-E_+\xi - \sqrt{(\ell+m+1)(\ell-m)}D_0E_+\xi \\
&\quad - \sqrt{(\ell+m+1)(\ell+m+2)}D_+E_+\xi
\end{aligned}
$$

for $\xi \in R_{\ell,m}$ and $-\ell \leq m \leq \ell$.

Similarly we get that

$$\begin{aligned} F_-\xi = &-\sqrt{(\ell+m)(\ell+m-1)}D_-E_-\xi - \sqrt{(\ell+m)(\ell-m+1)}D_0E_-\xi \\ &- \sqrt{(\ell-m+1)(\ell-m+2)}D_+E_-\xi \end{aligned}$$

for $\xi \in R_{\ell,m}$, and thus indeed we get eq. (1.34).

A.6 Relations for D_0, D_+, D_-

We want to show that the formulae eq. (1.34) for the linear operators F_+, F_-, and F_3 together with the formulae eqs. (1.9) and (1.11) for H_+, H_-, and H_3 define a representation of L, i.e. they satisfy the commutation relations of eq. (1.3), if and only if D_0, D_+, and D_- satisfy eq. (1.35).

We claim that all commutators containing H's already satisfy the relations. We already have by construction that all relations with only H's satisfy the relations, so we just need to check the relations for commutators with one F and one H. We will here check that $[H_-, F_+] = -2F_3$ as we want, and then we just note that the rest follow by similar considerations.

To simplify the otherwise very long calculations, we will only check the equility on $R_{\ell,\ell}$ and just note that similar calculations show the result on general $R_{\ell,m}$. By eqs. (1.10) and (1.34) first note that for $\xi \in R_{\ell,\ell}$ with $\ell \neq 0$ we have that $F_+\xi = \sqrt{(2\ell+1)(2\ell+2)}E_+D_+\xi \in R_{\ell+1,\ell+1}$, so

$$\begin{aligned} H_-F_+\xi &= \sqrt{(2\ell+2)\cdot 1}\sqrt{(2\ell+1)(2\ell+2)}E_-E_+D_+\xi \\ &= 2(\ell+1)\sqrt{2\ell+1}D_+\xi \\ &= 2\ell\sqrt{2\ell+1}D_+\xi + 2\sqrt{2\ell+1}D_+\xi, \end{aligned}$$

while $H_-\xi = \sqrt{2\ell}E_-\xi \in R_{\ell,\ell-1}$, so

$$\begin{aligned} F_+H_-\xi &= -\sqrt{1\cdot 2\ell}\sqrt{2\ell}D_0E_+E_-\xi + \sqrt{2\ell(2\ell+1)}\sqrt{2\ell}E_+D_+E_-\xi \\ &= -2\ell D_0\xi + 2\ell\sqrt{2\ell+1}E_+E_-D_+\xi \\ &= -2\ell D_0\xi + 2\ell\sqrt{2\ell+1}D_+\xi. \end{aligned}$$

Therefore we get that

$$\begin{aligned} [H_-, F_+]\xi &= H_-F_+\xi - F_+H_-\xi \\ &= 2\ell D_0\xi + 2\sqrt{2\ell+1}D_+\xi. \end{aligned}$$

Also

$$\begin{aligned} -2F_3\xi &= 2\ell D_0\xi + 2\sqrt{(\ell+1)^2-\ell^2}D_+\xi \\ &= 2\ell D_0\xi + 2\sqrt{2\ell+1}D_+\xi, \end{aligned}$$

and thus indeed $[H_-, F_+]\xi = -F_3\xi$ for $\xi \in R_{\ell,\ell}$.

Now we claim that $\ell D_+ D_0 \xi = (\ell+2) D_0 D_+ \xi$ corresponds to the relation $[F_+, F_3] = F_+$, while $(\ell+1) D_- D_0 \xi = (\ell-1) D_0 D_- \xi$ corresponds to the relation $[F_-, F_3] = -H_-$, and $\xi = (2\ell-1) D_+ D_- \xi - (2\ell+3) D_- D_+ \xi - D_0^2 \xi$ corresponds to the relation $[F_+, F_-] = -2H_3$. Here will only show the last claim and note that the others are similar (although a little simpler), and that the first and second cliam can be shown first without assuming the third.

Again to simplify the otherwise very long calculation, we will only check the result on $R_{\ell,\ell}$. Note that for $\xi \in R_{\ell,\ell}$ with $\ell \neq 0$ we have that $F_+\xi = \sqrt{(2\ell+1)(2\ell+2)} E_+ D_+ \xi \in R_{\ell+1,\ell+1}$, so

$$\begin{aligned}
F_- F_+ \xi &= -\sqrt{(2\ell+2)(2\ell+1)}\sqrt{(2\ell+1)(2\ell+2)} D_- E_- E_+ D_+ \xi \\
&\quad - \sqrt{2\ell+2}\sqrt{(2\ell+1)(2\ell+2)} D_0 E_- E_+ D_+ \xi \\
&\quad - \sqrt{2}\sqrt{(2\ell+1)(2\ell+2)} E_- D_+ E_+ D_+ \xi \\
&= -(2\ell+1)(2\ell+2) D_- D_+ \xi - (2\ell+2)\sqrt{2\ell+1} D_0 D_+ \xi \\
&\quad - 2\sqrt{(2\ell+1)(\ell+1)} E_+ E_- D_+ D_+ \xi \\
&= -(2\ell+1)(2\ell+2) D_- D_+ \xi - (2\ell+2)\sqrt{2\ell+1} D_0 D_+ \xi \\
&\quad - 2\sqrt{(2\ell+1)(\ell+1)} D_+ D_+ \xi.
\end{aligned}$$

Likewise we have that

$$F_- \xi = -\sqrt{2\ell(2\ell-1)} D_- E_- \xi - \sqrt{2\ell} D_0 E_- \xi - \sqrt{2} E_- D_+ \xi,$$

where $D_- E_- \xi \in R_{\ell-1,\ell-1}$, $D_0 E_- \xi \in R_{\ell,\ell-1}$, and $E_- D_+ \xi \in R_{\ell+1,\ell-1}$, so

$$\begin{aligned}
F_+ F_- \xi &= -\sqrt{(2\ell-1)2\ell}\sqrt{2\ell(2\ell-1)} E_+ D_+ D_- E_- \xi \\
&\quad + \sqrt{2\ell}\sqrt{2\ell} D_0 E_+ D_0 E_- \xi - \sqrt{2\ell(2\ell+1)}\sqrt{2\ell} E_+ D_+ D_0 E_- \xi \\
&\quad - \sqrt{2}\sqrt{2} D_- E_+ E_- D_+ \xi + \sqrt{2(2\ell+1)}\sqrt{2} D_0 E_+ E_- D_+ \xi \\
&\quad - \sqrt{(2\ell+1)(2\ell+2)}\sqrt{2} E_+ D_+ E_- D_+ \xi \\
&= -2\ell(2\ell-1) E_+ D_+ D_- E_- \xi + 2\ell D_0^2 \xi - 2\ell\sqrt{2\ell+1} D_+ D_0 \xi \\
&\quad - 2 D_- D_+ \xi + 2\sqrt{2\ell+1} D_0 D_+ - 2\sqrt{(2\ell+1)(\ell+1)} D_+^2 \xi.
\end{aligned}$$

Hence we get that

$$\begin{aligned}
[F_+, F_-]\xi &= F_+ F_- \xi - F_- F_+ \xi \\
&= -2\ell(2\ell-1) E_+ D_+ D_- E_- \xi + 2\ell D_0^2 \xi - 2\ell\sqrt{2\ell+1} D_+ D_0 \xi \\
&\quad - \big(2 - (2\ell+1)(2\ell+2)\big) D_- D_+ \xi + (2 + (2\ell+2))\sqrt{2\ell+1} D_0 D_+ \\
&= -2\ell(2\ell-1) E_+ D_+ D_- E_- \xi + 2\ell D_0^2 \xi - 2\ell\sqrt{2\ell+1} D_+ D_0 \xi \\
&\quad + 2\ell(2\ell+3) D_- D_+ \xi + 2(\ell+2)\sqrt{2\ell+1} D_0 D_+ \xi
\end{aligned}$$

$$
\begin{aligned}
&= -2\ell(2\ell-1)E_+D_+D_-E_-\xi + 2\ell D_0^2\xi \\
&\quad + 2\sqrt{2\ell+1}\big((\ell+2)D_0D_+\xi - \ell D_+D_0\xi\big) \\
&\quad + 2\ell(2\ell+3)D_-D_+\xi + 2(\ell+2)\sqrt{2\ell+1}D_0D_+\xi \\
&= -2\ell(2\ell-1)E_+D_+D_-E_-\xi + 2\ell D_0^2\xi + 2\ell(2\ell+3)D_-D_+\xi,
\end{aligned}
$$

since $2-(2\ell+1)(2\ell+2) = 2-4\ell^2-6\ell-2 = -4\ell^2-6\ell = -2\ell(2\ell+3)$ and $(\ell+2)D_0D_+\xi = \ell D_+D_0\xi$. Also

$$-2H_3\xi = -2\ell\xi,$$

so dividing by -2ℓ (noting that $\ell \neq 0$) we see that $[F_+,F_-]\xi = -2H_3\xi$ for $\xi \in R_{\ell,\ell}$ if and only if

$$\xi = (2\ell-1)E_+D_+D_-E_-\xi - (2\ell+3)D_-D_+\xi - D_0^2\xi.$$

Here we note that for $\xi \in R_{\ell,m}$, $m \neq \pm\ell$, the term $E_+D_+D_-E_-\xi$ simplifies to $D_+D_-\xi$, while for $m=-\ell$ it becomes $E_-D_+D_-E_+\xi$.

A.7 Finding d_ℓ^-

We want to find d_ℓ^- in general given that we already know that $d_{\ell_0}^- = 0$ and

$$(2\ell-1)d_\ell^- - (2\ell+3)d_{\ell+1}^- = 1 - \frac{\ell_0^2\ell_1^2}{\ell^2(\ell+1)^2}.$$

Multiplying the left side of the above equation by $2\ell+1$ we get

$$(4\ell^2-1)d_\ell^- - (4\ell^1+2\ell-3)d_{\ell+1}^- = (4\ell^2-1)d_\ell^- - (4(\ell+1)^2-1)d_{\ell+1}^-$$

and multiplying the right side by $2\ell+1$ we get

$$2\ell+1-\ell_0^2\ell_1^2\frac{2\ell+1}{\ell^2(\ell+1)^2} = 2\ell+1-\ell_0^2\ell_1^2\Big(\frac{1}{\ell^2}-\frac{1}{(\ell+1)^2}\Big),$$

so we see that

$$(4\ell^2-1)d_\ell^- - (4(\ell+1)^2-1)d_{\ell+1}^- = 2\ell+1-\ell_0^2\ell_1^2\Big(\frac{1}{\ell^2}-\frac{1}{(\ell+1)^2}\Big). \tag{A.2}$$

Now we know that $d_{\ell_0}^- = 0$, so

$$
\begin{aligned}
-(4(\ell_0+1)^2-1)d_{\ell_0+1}^- &= 2\ell_0+1-\ell_1^2\Big(1-\frac{\ell_0^2}{(\ell_0+1)^2}\Big) \\
&= (\ell_0+1)^2-\ell_0^2-\ell_1^2\frac{(\ell_0+1)^2-\ell_0^2}{(\ell_0+1)^2} \\
&= \frac{\big((\ell_0+1)^2-\ell_1^2\big)\big((\ell_0+1)^2-\ell_0^2\big)}{(\ell_0+1)^2},
\end{aligned}
$$

and thus

$$d^-_{\ell_0+1} = -\frac{\big((\ell_0+1)^2-\ell_1^2\big)\big((\ell_0+1)^2-\ell_0^2\big)}{(\ell_0+1)^2(4(\ell_0+1)^2-1)}.$$

We get inductively that

$$d^-_\ell = -\frac{\big(\ell^2-\ell_1^2\big)\big(\ell^2-\ell_0^2\big)}{\ell^2(4\ell^2-1)},$$

for $\ell > \ell_0$, since we already have the base case, and assuming the equality for $\ell > \ell_0$ we get by eq. (A.2) that

$$\begin{aligned}
&-(4(\ell{+}1)^2-1)d^-_{\ell+1}\\
&\qquad= \frac{\big(\ell^2-\ell_1^2\big)\big(\ell^2-\ell_0^2\big)}{\ell^2} + 2\ell + 1 - \ell_0^2\ell_1^2\Big(\frac{1}{\ell^2} - \frac{1}{(\ell+1)^2}\Big)\\
&\qquad= \frac{(\ell+1)^2(\ell^2-\ell_1^2)(\ell^2-\ell_0^2) + \ell^2(\ell+1)^2(2\ell+1) - \ell_0^2\ell_1^2(2\ell+1)}{\ell^2(\ell+1)^2}.
\end{aligned}$$

So since simple algebraic manipulation (easily done in e.g. Mathematica) show that

$$\begin{aligned}
&(\ell+1)^2(\ell^2-\ell_1^2)(\ell^2-\ell_0^2) + \ell^2(\ell+1)^2(2\ell+1) - \ell_0^2\ell_1^2(2\ell+1)\\
&= \ell^2\Big((\ell^2-\ell_1^2)(\ell^2-\ell_0^2) + (2\ell+1)\big(\ell^2-\ell_0^2-\ell_1^2+(\ell+1)^2\big)\Big)
\end{aligned}$$

and

$$\begin{aligned}
&\big((\ell+1)^2-\ell_1^2\big)\big((\ell+1)^2-\ell_0^2\big)\\
&= (\ell^2-\ell_1^2)(\ell^2-\ell_0^2) + (2\ell+1)\big((\ell+1)^2-\ell_0^2+\ell^2-\ell_1^2\big),
\end{aligned}$$

we see that

$$-(4(\ell+1)^2-1)d^-_{\ell+1} = \frac{\big((\ell+1)^2-\ell_1^2\big)\big((\ell+1)^2-\ell_0^2\big)}{(\ell+1)^2},$$

and thus indeed

$$d^-_{\ell+1} = -\frac{\big((\ell+1)^2-\ell_1^2\big)\big((\ell+1)^2-\ell_0^2\big)}{(\ell+1)^2(4(\ell+1)^2-1)}.$$

A.8 Finding $\Delta_1\xi$ and $\Delta_2\xi$

We have

$$\begin{aligned}
\Delta_1 &\coloneqq \tfrac{1}{2}(H_-F_+ + F_-H_+) + H_3F_3 + F_3\\
\Delta_2 &\coloneqq H_-H_+ - F_-F_+ + H_3^2 - F_3^2 + 2H_3
\end{aligned}$$

as in eq. (1.44), and we want to find $\Delta_1\xi$ and $\Delta_2\xi$ for $\xi \in R_{\ell,m}$.

By eqs. (1.9), (1.10) and (1.34) we see that for $\xi \in R_{\ell,\ell}$ (noting that $H_+\xi = 0$)

$$\begin{aligned}
\Delta_1\xi &= \tfrac{1}{2}(H_-F_+\xi + F_-H_+\xi) + H_3F_3\xi + F_3\xi \\
&= \tfrac{1}{2}\sqrt{(2\ell+1)(2\ell+2)}H_-E_+D_+\xi - \ell H_3D_0\xi - \sqrt{2\ell+1}H_3D_+\xi \\
&\quad - \ell D_0\xi - \sqrt{2\ell+1}D_+\xi \\
&= \tfrac{1}{2}\sqrt{(2\ell+1)(2\ell+2)}\sqrt{2\ell+2}E_-E_+D_+\xi - \ell^2D_0\xi - \ell\sqrt{2\ell+1}D_+\xi \\
&\quad - \ell D_0\xi - \sqrt{2\ell+1}D_+\xi \\
&= (\ell+1)\sqrt{2\ell+1}D_+\xi - \ell\sqrt{2\ell+1}D_+\xi - \sqrt{2\ell+1}D_+\xi \\
&\quad - (\ell^2+\ell)D_0\xi \\
&= -\ell(\ell+1)D_0\xi.
\end{aligned}$$

Now by Lemma 1.10 we have that E_- commute with Δ_1, and we also already have that E_- and D_0 commute, so

$$\Delta_1E_-^r\xi = E_-^r\Delta_1\xi = -\ell(\ell+1)E_-^rD_0\xi = -\ell(\ell+1)D_0E_-^r\xi.$$

Thus since $E_-\colon R_{\ell,m} \to R_{\ell,m-1}$ are isomorphisms for all $m \neq -\ell$, we get that indeed $\Delta_1 = -\ell(\ell+1)D_0$ on all $R_{\ell,m}$.

Similar calculations show that

$$\Delta_2\xi = (\ell^2-1)\xi - (\ell+1)^2D_0^2\xi + (4\ell^2-1)D_+D_-\xi$$

for all $\xi \in R_\ell$.

Additionally by eq. (1.35) we have that $\xi = (2\ell-1)D_+D_-\xi - (2\ell+3)D_-D_+\xi - D_0^2\xi$, so we get that

$$\begin{aligned}
(4\ell^2-1)D_+D_-\xi &= (2\ell+1)(2\ell-1)D_+D_-\xi \\
&= (2\ell+1)\xi + (2\ell+1)(2\ell+3)D_-D_+\xi + (2\ell+1)D_0^2\xi \\
&= (2\ell+1)\xi + (4(\ell+1)^2-1)D_-D_+\xi + (2\ell+1)D_0^2\xi
\end{aligned}$$

for $\xi \in R_\ell$ since $(2\ell+1)(2\ell+3) = (2(\ell+1)-1)(2(\ell+1)+1) = 4(\ell+1)^2-1$, and therefore also

$$\begin{aligned}
\Delta_2\xi &= (\ell^2-1)\xi - (\ell+1)^2D_0^2\xi + (2\ell+1)\xi + (4(\ell+1)^2-1)D_-D_+\xi + (2\ell+1)D_0^2\xi \\
&= \big((\ell+1)^2-1\big)\xi + \ell^2D_0^2\xi + \big(4(\ell+1)^2-1\big)D_-D_+\xi
\end{aligned}$$

for $\xi \in R_\ell$.

Appendix B

Auxiliary results

In this appendix we will collect the proofs of some auxiliary results that we will need in the paper.

B.1 $Z(U(L_1 \times L_2)) \simeq Z(U(L_1)) \otimes Z(U(L_2))$

Let $L = L_1 \times L_2$ be a product of two Lie algebras, and let $\iota_1 \colon L_1 \to U(L_1)$, $\iota_2 \colon L_2 \to U(L_2)$, and $\iota \colon L \to U(L)$ be the canonical homomorphisms of Lie algebras, we get from the universal property of universal enveloping algebras. We want to show first that $U(L) \simeq U(L_1) \otimes U(L_2)$.

Consider the map

$$\rho \colon L \to U(L_1) \otimes U(L_2), \qquad (u_1, u_2) \mapsto \iota_1(u_1) \otimes 1 + 1 \otimes \iota_2(u_2),$$

which is a homomorphisms of Lie algebras since it is clearly linear and

$$\begin{aligned}
[\rho(u_1, u_2), \rho(v_1, v_2)] &= [u_1 \otimes 1 + 1 \otimes u_2, v_1 \otimes 1 + 1 \otimes v_2] \\
&= (u_1 \otimes 1 + 1 \otimes u_2)(v_1 \otimes 1 + 1 \otimes v_2) \\
&\quad - (v_1 \otimes 1 + 1 \otimes v_2)(u_1 \otimes 1 + 1 \otimes u_2) \\
&= u_1 v_1 \otimes 1 + u_1 \otimes v_2 + v_1 \otimes u_2 + 1 \otimes u_2 v_2 \\
&\quad - v_1 u_1 \otimes 1 - v_1 \otimes u_2 - u_1 \otimes v_2 - 1 \otimes v_2 u_2 \\
&= (u_1 v_1 - v_1 u_1) \otimes 1 + 1 \otimes (u_2 v_2 - v_2 u_2) \\
&= [u_1, v_1] \otimes 1 + 1 \otimes [u_2, v_2] \\
&= \rho([u_1, v_1], [u_2, v_2]) \\
&= \rho([(u_1, u_2), (v_1, v_2)])
\end{aligned}$$

for $(u_1, u_2), (v_1, v_2) \in L$ by the definition of the tensor product of an algebra. Thus by the universal property of $(U(L), \iota)$ we get a unique homomorphisms of associative algebras $\varphi \colon U(L) \to U(L_1) \otimes U(L_2)$ such that the following diagram commutes:

$$\begin{tikzcd} L \arrow[r, "\iota"] \arrow[dr, "\rho"'] & U(L) \arrow[d, dashed, "\varphi"] \\ & U(L_1) \otimes U(L_2) \end{tikzcd}$$

Now let $i_1 \colon L_1 \to L$ be the inclusion of L_1 into L given by $u \mapsto (u, 0)$ for $u \in L_1$. By the definition of the bracket on $L = L_1 \times L_2$ it is easy to see that i_1 is a Lie algebra homomorphism, and thus the map $\iota \circ i_1 \colon L_1 \to L \to U(L)$ is also a Lie algebra homomorphism. Hence by the universal property of $(U(L_1), \iota_1)$ we get a unique homomorphism of associative algebras $\psi_1 \colon U(L_1) \to U(L)$ such that the following diagram commutes:

$$\begin{tikzcd} L_1 \arrow[r, "\iota_1"] \arrow[dr, "\iota \circ i_1"'] & U(L_1) \arrow[d, dashed, "\psi_1"] \\ & U(L) \end{tikzcd}$$

Likewise we get a unique homomorphism of associative algebras $\psi_2 \colon U(L_2) \to U(L)$ such that $\iota \circ i_2 = \psi_1 \circ \iota_2$. Now since $[(u_1, 0), (0, u_2)] = ([u_1, 0], [0, u_2]) = 0$ for $u_1 \in L_1$ and $u_2 \in L_2$, we see that

$$\begin{aligned} 0 &= \iota([(u_1, 0), (0, u_2)]) = [\iota i_1(u_1), \iota i_2(u_2)] = [\psi_1 \iota_1(u_1), \psi_2 \iota_2(u_2)] \\ &= \psi_1 \iota_1(u_1) \psi_2 \iota_2(u_2) - \psi_2 \iota_2(u_2) \psi_1 \iota_1(u_1). \end{aligned}$$

Thus $\psi_1 \iota_1(u_1) \psi_2 \iota_2(u_2) = \psi_2 \iota_2(u_2) \psi_1 \iota_1(u_1)$ for all $u_1 \in L_1$ and $u_2 \in L_2$. Hence since the $\iota_j(u_j)$ for $u_j \in L_j$ generate $U(L_j)$ by the PBW theorem for $j = 1, 2$, cf. [Jan16, p. E-7], we get that $\psi_1(u_1)\psi_2(u_2) = \psi_2(u_2)\psi_1(u_1)$ for all $u_1 \in U(L_1)$ and $u_2 \in U(L_2)$. Therefore the map

$$\psi \colon U(L_1) \otimes U(L_2) \to U(L), \qquad u_1 \otimes u_2 \mapsto \psi_1(u_1)\psi_2(u_2), \tag{B.1}$$

is a homomorphism of associative algebras since

$$\begin{aligned} \psi((u_1 \otimes u_2)(v_1 \otimes v_2)) &= \psi(u_1 v_1 \otimes v_1 v_2) = \psi_1(u_1 v_1)\psi_2(u_2 v_2) \\ &= \psi_1(u_1)\psi_1(v_1)\psi_2(u_2)\psi_2(v_2) \\ &= \psi_1(u_1)\psi_2(u_2)\psi_1(v_1)\psi_2(v_2) \\ &= \psi(u_1 \otimes u_2)\psi(v_1 \otimes v_2). \end{aligned}$$

Note now that

$$\begin{aligned} \psi\varphi\iota(u_1, u_2) &= \psi\rho(u_1, u_2) = \psi(\iota_1(u_1) \otimes 1 + 1 \otimes \iota_2(u_2)) \\ &= \psi_1\iota_1(u_1)\psi_2(1) + \psi_1(1)\psi_2\iota_2(u_2) \\ &= \iota(u_1, 0) + \iota(0, u_2) = \iota(u_1, u_2) \end{aligned}$$

for all $(u_1, u_2) \in L$, so by the PBW theorem as above we get that $\psi\varphi = \mathrm{id}_{U(L)}$. Likewise

$$\begin{aligned}\varphi\psi(\iota_1(u_1) \otimes 1 + 1 \otimes \iota_2(u_2)) &= \varphi(\psi_1\iota_1(u_1)\psi_2(1) + \psi_1(1)\psi_2\iota_2(u_2)) \\ &= \varphi(\iota(u_1, 0) + \iota(0, u_2)) = \varphi\iota(u_1, u_2) \\ &= \rho(u_1, u_2) = \iota(u_1) \otimes 1 + 1 \otimes \iota_2(u_2)\end{aligned}$$

for all $u_1 \in L_1$ and $u_2 \in L_2$. Now by the PBW theorem the $\iota_1(u_1)$ for $u_1 \in L_1$ generate $U(L_1)$ and the $\iota_2(u_2)$ for $u_2 \in L_2$ generate $U(L_2)$, so we see that the $\iota_1(u_1) \otimes 1 + 1 \otimes \iota_2(u_2)$ for $u_1 \in L_1$ and $u_2 \in L_2$ generate $U(L_1) \otimes U(L_2)$ and thus $\varphi\psi = \mathrm{id}_{U(L_1)\otimes U(L_2)}$. Hence we see that φ and ψ are isomorphisms between $U(L)$ and $U(L_1) \otimes U(L_2)$, so indeed $U(L) \simeq U(L_1) \otimes U(L_2)$.

Note that the above also gives us an isomorphism $Z(U(L)) \simeq Z(U(L_1) \otimes U(L_2))$. Now we want to show that we also have that $Z(U(L_1) \otimes U(L_2)) = Z(U(L_1)) \otimes Z(U(L_2))$ such that when describing $Z(U(L))$ we can instead describe $Z(U(L_1)) \otimes Z(U(L_2))$. For $z_1 \otimes z_2 \in Z(U(L_1)) \otimes Z(U(L_2))$ we get that

$$(z_1 \otimes z_2)(u_1 \otimes u_2) = z_1u_1 \otimes z_2u_2 = u_1z_1 \otimes u_2z_2 = (u_1 \otimes u_2)(z_1 \otimes z_2)$$

for all $u_1 \otimes u_2 \in U(L_1) \otimes U(L_2)$, so we have the inclusion $Z(U(L_1)) \otimes Z(U(L_2)) \subseteq Z(U(L_1) \otimes U(L_2))$.

To get the other inclusion let $z = \sum_i u_i \otimes v_i \in Z(U(L_1) \otimes U(L_2))$. By combining terms with linearly dependent v_i's, we can assume that the v_i's in the sum are linearly independent. Now for $u \otimes 1 \in U(L_1) \otimes U(L_2)$ we have that $z(u \otimes 1) = (u \otimes 1)z$, so

$$0 = z(u \otimes 1) - (u \otimes 1)z = \sum_i (u_iu - uu_i) \otimes v_i.$$

Thus since the v_i's are linearly independent, we must have that $u_iu - uu_i = 0$ for all i, i.e. $u_i \in Z(U(L_1))$ for all i. Likewise we get that $v_i \in Z(U(L_2))$ for all i, and hence $z = \sum_i u_i \otimes v_i \in Z(U(L_1)) \otimes Z(U(L_2))$. Therefore we get the inclusion $Z(U(L_1) \otimes U(L_2)) \subseteq Z(U(L_1)) \otimes Z(U(L_2))$, and thus indeed we have the equality $Z(U(L_1) \otimes U(L_2)) = Z(U(L_1)) \otimes Z(U(L_2))$. So altogether we have an isomorphism $Z(U(L)) \simeq Z(U(L_1)) \otimes Z(U(L_2))$.

B.2 Determining a linear map from its square and eigenvalue

Let $A\colon V \to V$ be a linear operator on a finite dimensional vector space V over $\mathbf{C}$, and assume that A only has only one eigenvalue, $\lambda \neq 0$. We claim then that A is uniquely determined by A^2 and λ. To see this first note that

$$\ker(A - \lambda\,\mathrm{id}_V)^r = \ker(A^2 - \lambda^2\,\mathrm{id}_V)^r$$

for all integers $r > 0$. This is the case since $(A^2 - \lambda^2 \operatorname{id}_V)^r = (A + \lambda \operatorname{id}_V)^r (A - \lambda \operatorname{id}_V)^r$, and since $A + \lambda \operatorname{id}_V$ is bijective for $\lambda \neq 0$ because $\det(A + \lambda \operatorname{id}_V)$ cannot be 0 since $-\lambda$ is not an eigenvalue of A.

Now choose a basis of $\ker(A^2 - \lambda^2 \operatorname{id}_V)$, expand to a basis of $\ker(A^2 - \lambda^2 \operatorname{id}_V)^2$, expand further to a basis of $\ker(A^2 - \lambda^2 \operatorname{id}_V)^3$, and so on. Since $A - \lambda \operatorname{id}_V$ takes $\ker(A^2 - \lambda^2 \operatorname{id}_V)^r = \ker(A - \lambda \operatorname{id}_V)^r$ to $\ker(A^2 - \lambda^2 \operatorname{id}_V)^{r-1} = \ker(A - \lambda \operatorname{id}_V)^{r-1}$, we see that the matrix of A with respect to this basis is upper triangular with all diagonal entries equal to λ. To see this more clearly suppose that $(v_1, \ldots, v_s)$ is the basis of $\ker(A^2 - \lambda^2 \operatorname{id}_V)^{r-1} = \ker(A - \lambda \operatorname{id}_V)^{r-1}$ and that $(v_1, \ldots, v_s, \ldots, v_n)$ is the basis of $\ker(A^2 - \lambda^2 \operatorname{id}_V)^r = \ker(A - \lambda \operatorname{id}_V)^r$. Then we have that $(A - \lambda \operatorname{id}_V) v_\ell \in \ker(A - \lambda \operatorname{id}_V)^{r-1}$ for all $\ell \in \{1, \ldots, n\}$, i.e. $A v_\ell - \lambda v_\ell = \sum_{k=1}^s \beta_k v_k$ for some $\beta_k \in \mathbf{C}$, so for $\ell > s$ we have that $A v_\ell = \sum_{k=1}^s \beta_k v_k + \lambda v_\ell$, so by induction in r we have the claim since for $v \in \ker(A - \lambda \operatorname{id}_V)$ we have that $Av = \lambda v$ which gives the base case. Write $A = (a_{ij})$ in this basis, where we now know that $a_{ij} = 0$ if $i > j$, and note that writing $m = \dim V$ we have that

$$(A^2)_{ij} = \sum_{k=1}^{m} a_{ik} a_{kj} = \sum_{k=i}^{j} a_{ik} a_{kj} = \lambda a_{ij} + \sum_{k=i+1}^{j-1} a_{ik} a_{kj},$$

since for $k < i$ $a_{ik} = 0$ and for $k > j$ $a_{kj} = 0$, and since $a_{ii} = a_{jj} = \lambda$. Hence by induction in $i - j$ we get that A is determined by A^2 and λ, since clearly $a_{ii} = \lambda$ satisfies this and inductively we can find a_{ij} from the above formuala knowing A^2, λ, and $a_{k\ell}$ with $k - \ell < i - j$.