Bachelorprojekt

Title

(subtitle)

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Abstract

Some text

Contents

Chapter 1

Harish-Chandra modules over $\mathfrak{sl}(2, \mathbf{C}) \times \mathfrak{sl}(2, \mathbf{C})$

Let L be a semisimple Lie algebra and let L_k be a Lie subalgebra.

Definition 1.1. *An L-module M is a Harish-Chandra module for the pair (L, L_k) if, regarded as an L_k-module, it can be written as a sum*

$$M = \bigoplus_i M_i$$

of finite dimensional simple[1] *L_k-submodules M_i, where for each M_{i_0} only finitely many L_k-submodules equivalent to M_{i_0} occur in the decomposition of M. If L and L_k are clear from the context we will just call M a Harish-Chandra module.*

A Harish-Chandra module M is indecomposable if it cannot be decomposed into the direct sum of non-zero L-submodules.

Our goal is to classify all indecomposable Harish-Chandra modules over (L, L_k) for $L = \mathfrak{sl}(2, \mathbf{C}) \times \mathfrak{sl}(2, \mathbf{C})$ and $L_k = \{(u, u) \mid u \in \mathfrak{sl}(2, \mathbf{C})\}$, where we by $\mathfrak{sl}(2, \mathbf{C}) \times \mathfrak{sl}(2, \mathbf{C})$ mean the following:

For L, L' Lie algebras over F, we consider $L \times L' = L \oplus L'$ as a Lie algebra over F with pointwise addition, multiplication given by $\alpha(a, b) = (\alpha a, \alpha b)$ for $\alpha \in F, a \in L, b \in L'$, and with Lie bracket $[(a_1, b_1), (a_2, b_2)] = ([a_1, a_2], [b_1, b_2])$ for $a_1, a_2 \in L, b_1, b_2 \in L'$.

Remark 1.2. Note that $L \times 0$ and $0 \times L'$ are ideals in $L \times L'$ as given above. Thus we see that $\mathfrak{sl}(2, \mathbf{C}) \times 0$ and $0 \times \mathfrak{sl}(2, \mathbf{C})$ are ideals in $\mathfrak{sl}(2, \mathbf{C}) \times \mathfrak{sl}(2, \mathbf{C})$ with

$$(\mathfrak{sl}(2, \mathbf{C}) \times 0) \oplus (0 \times \mathfrak{sl}(2, \mathbf{C})) = \mathfrak{sl}(2, \mathbf{C}) \times \mathfrak{sl}(2, \mathbf{C}),$$

[1] In [GP67b] the word irreducible is used instead of simple, but we will only use irreducible when talking about representations in this paper.

so $\mathfrak{sl}(2, \mathbf{C}) \times \mathfrak{sl}(2, \mathbf{C})$ is semisimple.

Now if we take $L = \mathfrak{sl}(2, \mathbf{C}) \times \mathfrak{sl}(2, \mathbf{C})$ and $L_k = \{(u, u) \mid u \in \mathfrak{sl}(2, \mathbf{C})\}$ as a Lie subalgebra, it makes sense to talk about Harish-Chandra modules over (L, L_k). Here L_k is clearly a Lie subalgebra since it is a subspace and the Lie bracket on $\mathfrak{sl}(2, \mathbf{C}) \times \mathfrak{sl}(2, \mathbf{C})$ preserves L_k by the definition of the Lie bracket on a product. △

We fix the following as a standard basis for $\mathfrak{sl}(2, F)$:

$$x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Giving us the relations:

$$[x, y] = h, \qquad [h, x] = 2x, \qquad [h, y] = -2y, \tag{1.1}$$

cf. [Jan16, p. 35] or [Hum72, p. 6].

We claim now that

$$(x, x), \quad (y, y), \quad \tfrac{1}{2}(h, h), \quad (ix, -ix), \quad (iy, -iy), \quad \tfrac{1}{2}(ih, -ih)$$

is a basis of $\mathfrak{sl}(2, \mathbf{C}) \times \mathfrak{sl}(2, \mathbf{C})$. This is clearly the case since $\dim_{\mathbf{C}} \mathfrak{sl}(2, \mathbf{C}) = 3$, so $\dim_{\mathbf{C}} \mathfrak{sl}(2, \mathbf{C}) \times \mathfrak{sl}(2, \mathbf{C}) = 6$, and we see that the above elements span $\mathfrak{sl}(2, \mathbf{C}) \times \mathfrak{sl}(2, \mathbf{C})$; we have $\frac{1}{2}(x, x) - \frac{i}{2}(ix, -ix) = (x, 0)$ and $\frac{1}{2}(x, x) + \frac{i}{2}(ix, -ix) = (0, x)$ and likewise with h and y.

Putting

$$\begin{aligned} h_+ &= (x, x), & h_- &= (y, y), & h_3 &= \tfrac{1}{2}(h, h), \\ f_+ &= (ix, -ix), & f_- &= (iy, -iy), & f_3 &= \tfrac{1}{2}(ih, -ih) \end{aligned}$$

we get the following commutation relations between these basis elements:

$$\begin{aligned}
[h_+, h_3] &= \tfrac{1}{2}([x, h], [x, h]) = \tfrac{1}{2}(-2x, -2x) = -(x, x) = -h_+, \\
[h_-, h_3] &= \tfrac{1}{2}([y, h], [y, h]) = \tfrac{1}{2}(2y, 2y) = (y, y) = h_-, \\
[h_+, h_-] &= ([x, y], [x, y]) = (h, h) = 2h_3, \\
[h_+, f_+] &= ([x, ix], [x, -ix]) = 0, \\
[h_-, f_-] &= ([y, iy], [y, -iy]) = 0, \\
[h_3, f_3] &= \tfrac{1}{4}([h, ih], [h, -ih]) = 0, \\
[h_+, f_3] &= \tfrac{1}{2}([x, ih], [x, -ih]) = \tfrac{1}{2}(-2ix, 2ix) = -(ix, -ix) = -f_+, \\
[h_-, f_3] &= \tfrac{1}{2}([y, ih], [y, -ih]) = \tfrac{1}{2}(2iy, -2iy) = (iy, -iy) = f_-, \\
[h_+, f_-] &= ([x, iy], [x, -iy]) = (ih, -ih) = 2f_3, \\
[h_3, f_-] &= \tfrac{1}{2}([h, iy], [h, -iy]) = \tfrac{1}{2}(-2iy, 2iy) = -(iy, -iy) = -f_-, \\
[h_-, f_+] &= ([y, ix], [y, -ix]) = (-ih, ih) = -(ih, -ih) = -2f_3, \\
[h_3, f_+] &= \tfrac{1}{2}([h, ix], [h, -ix]) = \tfrac{1}{2}(2ix, -2ix) = (ix, -ix) = f_+, \\
[f_+, f_3] &= \tfrac{1}{2}([ix, ih], [-ix, -ih]) = \tfrac{1}{2}(2x, 2x) = (x, x) = h_+, \\
[f_-, f_3] &= \tfrac{1}{2}([iy, ih], [-iy, -ih]) = \tfrac{1}{2}(-2y, -2y) = -(y, y) = -h_-, \\
[f_+, f_-] &= ([ix, iy], [-ix, -iy]) = (-h, -h) = -(h, h) = -2h_3.
\end{aligned} \tag{1.2}$$

Remark 1.3. Note that these are the same relations as for the complexification of the Lie algebra L of the proper Lorentz group in [GP67b, p. 5], so L is isomorphic to $\mathfrak{sl}(2, \mathbf{C}) \times \mathfrak{sl}(2, \mathbf{C})$. This explains the equivalence of the work in this paper and the work in [GP67a; GP67b; GP67c]. $\triangle$

Now let $L = \mathfrak{sl}(2, \mathbf{C}) \times \mathfrak{sl}(2, \mathbf{C})$ and $L_k = \{(u, u) \mid u \in \mathfrak{sl}(2, \mathbf{C})\}$. Note that L_k is the Lie subalgebra of L with basis h_+, h_-, h_3, and that the above commutation relations gives us that

$$[h_+, h_-] = 2h_3, \qquad [2h_3, h_+] = 2h_+, \qquad [2h_3, h_-] = -2h_-$$

Comparing with eq. (1.1) we see that we have an isomorphism

$$\mathfrak{sl}(2, \mathbf{C}) \to L_k, \qquad u \mapsto (u, u), \tag{1.3}$$

or more explicitly $x \mapsto h_+$, $h_- \mapsto y$, and $h \mapsto 2h_3$, so we can use $\mathfrak{sl}(2, \mathbf{C})$-theory when we want to describe L_k-modules.

1.1 Representations of L_k

Let V be a $\mathbf{C}$ vector space and $\rho\colon L_k \to \mathfrak{gl}(V)$ a representation of L_k. We will use the notation $\rho(a) = A$ for $a \in L_k$ switching to upper case letters when we talk about the representation corresponding to a given element. Note that we will switch freely between the language of representations of L_k and the language of L_k-modules.

We will start out by describing the finite dimensional simple L_k-modules. Recall, cf. [Jan16, p. 36], that we know from $\mathfrak{sl}(2, \mathbf{C})$-theory that for integers $n \geq 0$ there exists a unique simple $\mathfrak{sl}(2, \mathbf{C})$-module $V(n)$ of dimension $n + 1$, and $V(n)$ has a basis $(v_0, v_1, \dots, v_n)$ such that for all i, $0 \leq i \leq n$

$$\begin{aligned} h.v_i &= (n - 2i)v_i, \\ x.v_i &= \begin{cases} (n - i + 1)v_{i-1} & \text{if } i > 0, \\ 0 & \text{if } i = 0, \end{cases} \\ y.v_i &= \begin{cases} (i + 1)v_{i+1} & \text{if } i < n, \\ 0 & \text{if } i = n. \end{cases} \end{aligned} \tag{1.4}$$

Now using the isomorphism from eq. (1.3) we see that for integers $n \geq 0$ there exists a unique simple L_k-module $M(n)$ of dimension $n + 1$[2], and $M(n)$

[2] We will use the notation $V(n)$ when talking about $\mathfrak{sl}(2, \mathbf{C})$-modules and $M(n)$ when talking about L_k-modules to clarify what kind of module we are talking about, but as vector spaces $V(n)$ and $M(n)$ are isomorphic.

has a basis $(v_0, v_1, \dots, v_n)$ such that for all i, $0 \le i \le n$

$$\begin{aligned} h_3.v_i &= (\tfrac{1}{2}n - i)v_i, \\ h_+.v_i &= \begin{cases} (n-i+1)v_{i-1} & \text{if } i > 0, \\ 0 & \text{if } i = 0, \end{cases} \\ h_-.v_i &= \begin{cases} (i+1)v_{i+1} & \text{if } i < n, \\ 0 & \text{if } i = n. \end{cases} \end{aligned} \tag{1.5}$$

Now consider $M(n)$ as an inner product space over $\mathbf{C}$ with inner product given by

$$\langle v_k, v_j \rangle = \delta_{jk} \binom{n}{k}. \tag{1.6}$$

We will switch to the orthonormal basis $(\overline{v}_0, \overline{v}_1, \dots, \overline{v}_n)$, where $\overline{v}_i = v_i / \|v_i\|$. Here $\|\cdot\|$ is given by $\|v\| = \sqrt{\langle v, v \rangle}$ as usually, and we note that

$$\overline{v}_i = \frac{1}{\sqrt{\binom{n}{i}}} v_i.$$

Note furthermore that

$$h_3.\overline{v}_i = \frac{1}{\sqrt{\binom{n}{i}}} h_3.v_i = \frac{1}{\sqrt{\binom{n}{i}}} (\tfrac{1}{2}n - i)v_i = (\tfrac{1}{2}n - i)\overline{v}_i$$

for all i, $0 \le i \le n$, and clearly still

$$\begin{aligned} h_+.\overline{v}_0 &= 0, \\ h_-.\overline{v}_n &= 0. \end{aligned}$$

But for i, $0 < i \le n$

$$\begin{aligned} h_+.\overline{v}_i &= \frac{1}{\sqrt{\binom{n}{i}}} h_+.v_i = \frac{1}{\sqrt{\binom{n}{i}}} (n-i+1)v_{i-1} \\ &= \sqrt{\frac{\binom{n}{i-1}}{\binom{n}{i}}} (n-i+1) \frac{1}{\sqrt{\binom{n}{i-1}}} v_{i-1} \\ &= \sqrt{\frac{i}{n-i+1}} (n-i+1)\overline{v}_{i-1} = \sqrt{(n-i+1)i}\,\overline{v}_{i-1}, \end{aligned}$$

and for i, $0 \le i < n$

$$\begin{aligned}
h_-.\overline{v}_i &= \frac{1}{\sqrt{\binom{n}{i}}} h_-.v_i = \frac{1}{\sqrt{\binom{n}{i}}}(i+1)v_{i+1} \\
&= \sqrt{\frac{\binom{n}{i+1}}{\binom{n}{i}}}(i+1)\frac{1}{\sqrt{\binom{n}{i+1}}}v_{i+1} \\
&= \sqrt{\frac{n-i}{i+1}}(i+1)\overline{v}_{i+1} = \sqrt{(n-i)(i+1)}\overline{v}_{i+1}.
\end{aligned}$$

Finally write $\ell = \frac{1}{2}n$. We will re-index with $m = \frac{1}{2}(n-2i) = \ell - i$ by setting

$$e_m = \overline{v}_{\ell-m}$$

for $m \in \{-\ell, -\ell+1, \ldots, \ell-1, \ell\}$. Thus we get

$$h_3.e_m = h_3.\overline{v}_{\ell-m} = (\ell - (\ell - m))\overline{v}_{\ell-m} = me_m,$$

and since $e_\ell = \overline{v}_0$ and $e_{-\ell} = \overline{v}_n$ also

$$\begin{aligned}
h_+.e_\ell &= 0, \\
h_-.e_{-\ell} &= 0.
\end{aligned}$$

And for $m \in \{-\ell, -\ell+1, \ldots, \ell-2, \ell-1\}$ we get

$$\begin{aligned}
h_+.e_m = h_+.\overline{v}_{\ell-m} &= \sqrt{(n-(\ell-m)+1)(\ell-m)}\overline{v}_{\ell-m-1} \\
&= \sqrt{(\ell+m+1)(\ell-m)}e_{m+1},
\end{aligned}$$

while for $m \in \{-\ell+1, -\ell+2, \ldots, \ell-1, \ell\}$ we get

$$\begin{aligned}
h_-.e_m = h_-.\overline{v}_{\ell-m} &= \sqrt{(n-(\ell-m))(\ell-m+1)}\overline{v}_{\ell-m+1} \\
&= \sqrt{(\ell+m)(\ell-m+1)}e_{m-1}.
\end{aligned}$$

Thus we get the following Lemma:

Lemma 1.4. *Every simple finite dimensional L_k-module is uniquely given by a number $\ell \in \frac{1}{2}\mathbf{Z}_{\ge 0}$. For such ℓ the unique simple L_k-module $M(2\ell)$ has dimension $2\ell+1$, and $M(2\ell)$ has a basis $(e_{-\ell}, e_{-\ell+1}, \ldots, e_{\ell-1}, e_\ell)$ such that for all $m \in \{-\ell, -\ell+1, \ldots, \ell-1, \ell\}$ we have*

$$\begin{aligned}
h_3.e_m &= me_m, \\
h_+.e_m &= \begin{cases} \sqrt{(\ell+m+1)(\ell-m)}e_{m+1} & \text{if } m \ne \ell, \\ 0 & \text{if } m = \ell, \end{cases} \\
h_-.e_m &= \begin{cases} \sqrt{(\ell+m)(\ell-m+1)}e_{m-1} & \text{if } m \ne -\ell, \\ 0 & \text{if } m = -\ell. \end{cases}
\end{aligned} \tag{1.7}$$

1.1.1 Formulae for the operators $H_+, H_-, H_3, F_+, F_-, F_3$

Let M be a Harish-Chandra L-module. Then we have linear operators $H_+, H_-, H_3, F_+, F_-, F_3 \colon M \to M$ satisfying commutation relations as in eq. (1.2), and we want to give expressions for these in terms of other linear operators $E_+, E_-, D_+, D_-, D_0 \colon M \to M$.

We will denote by R_ℓ a finite dimensional L-module which is a (finite) direct sum of L_k-modules $M(2\ell)$ for the same number $\ell \in \frac{1}{2}\mathbf{Z}_{\geq 0}$. Then M is a direct sum of the subspaces R_ℓ since M is Harish-Chandra, and from Lemma 1.4 we know that R_ℓ can be written as the direct sum of subspaces $R_{\ell,m}$, where $R_{\ell,m}$ are eigenspaces for H_3 such that

$$H_3\xi = m\xi \tag{1.8}$$

for $m \in \{-\ell, -\ell+1, \ldots, \ell-1, \ell\}$ and $\xi \in R_{l,m}$. We will use the decomposition

$$M = \bigoplus_{\substack{\ell \in \frac{1}{2}\mathbf{Z}_{\geq 0} \\ m \in \{-\ell,-\ell+1,\ldots,\ell-1,\ell\}}} R_{\ell,m} = \bigoplus_{\ell,m} R_{\ell,m}$$

throughout this paper.

By Lemma 1.4 we also have that H_+ and H_- maps the $R_{\ell,m}$ into each other as follows:

$$\begin{aligned}
&H_+ \colon R_{\ell,m} \to R_{\ell,m+1} && \text{if } -\ell \leq m < \ell, && H_+ \colon R_{\ell,\ell} \to 0,\\
&H_- \colon R_{\ell,m} \to R_{\ell,m-1} && \text{if } -\ell < m \leq \ell, && H_- \colon R_{\ell,-\ell} \to 0.
\end{aligned}$$

Hence we have linear operators $H_+H_-, H_-H_+ \colon R_{\ell,m} \to R_{\ell,m}$, and by eq. (1.7) we see that

$$\begin{aligned}
H_+H_-\xi &= \sqrt{(\ell+(m-1)+1)(\ell-(m-1))}\sqrt{(\ell+m)(\ell-m+1)}\xi\\
&= (\ell+m)(\ell-m+1)\xi,\\
H_-H_+\xi &= \sqrt{(\ell+(m+1))(\ell-(m+1)+1)}\sqrt{(\ell+m+1)(\ell-m)}\xi\\
&= (\ell+m+1)(\ell-m)\xi.
\end{aligned} \tag{1.9}$$

Note that this also covers the cases $m = \ell$ and $m = -\ell$.

Now we define $E_+ \colon R_{\ell,m} \to R_{\ell,m+1}$ and $E_- \colon R_{\ell,m} \to R_{\ell,m-1}$ to be the linear maps satisfying

$$\begin{aligned}
H_+\xi &= \begin{cases} \sqrt{(\ell+m+1)(\ell-m)}E_+\xi & \text{if } m \neq \ell \\ 0 & \text{if } m = \ell, \end{cases}\\
H_-\xi &= \begin{cases} \sqrt{(\ell+m)(\ell-m+1)}E_-\xi & \text{if } m \neq -\ell \\ 0 & \text{if } m = \ell \end{cases}
\end{aligned} \tag{1.10}$$

for $\xi \in R_{\ell,m}$. Comparing eq. (1.10) and eq. (1.9) we see that

$$\begin{aligned} E_+E_-\xi &= \xi && \text{if } m \neq -\ell \\ E_-E_+\xi &= \xi && \text{if } m \neq \ell. \end{aligned}$$

Thus $E_+\colon R_{\ell,m} \to R_{\ell,m+1}$ and $E_-\colon R_{\ell,m+1} \to R_{\ell,m}$ are isomorphisms for $m \neq \ell$ and they are each others inverse.

Now note that H_+, H_-, and H_3 are completely determined by eq. (1.8) and eq. (1.10), so we just need to find maps to determine F_+, F_-, and F_3 now, while making sure that we get commutation relations as in eq. (1.2).

We already have that $L_k = \operatorname{span}_{\mathbf{C}}(h_+, h_-, h_3)$, but now we will also consider $L_p = \operatorname{span}_{\mathbf{C}}(f_+, f_-, f_3)$. We will show shortly that $u.R_\ell \subset R_{\ell-1} \oplus R_\ell \oplus R_{\ell+1}$ for all $u \in L_p$. This implies that there are maps $D_-^u\colon R_\ell \to R_{\ell-1}$, $D_0^u\colon R_\ell \to R_\ell$, and $D_+^u\colon R_\ell \to R_{\ell+1}$ such that $u.v = D_-^u(v) + D_0^u(v) + D_+^u(v)$ for all $u \in L_p$ and $v \in R_\ell$. In the following we will find maps D_-, D_0, and D_+ independent of u such that we can express D_-^u, D_0^u, and D_+^u in terms of these and the maps E_- and E_+ from above, thus we will also be able to express F_+, F_-, and F_3 in terms of D_-, D_0, D_+, E_-, and E_+. To be more precise we will find maps D_-, D_0, and D_+ such that we can express F_3 in terms of just these (and multiplication by some constant), and then we can get F_+ and F_- by the commutation relations.

For reasons that will be clearer later, we want the maps D_0 and D_+ to be defined on $M = \bigoplus_{\ell,m} R_{\ell,m}$ and D_- defined on the direct sum without the summands $R_{\ell,\ell}$ and $R_{\ell,-\ell}$ to be such that $D_0 R_{\ell,m} \subset R_{\ell,m}$, $D_+ R_{\ell,m} \subset R_{\ell+1,m}$, and $D_- R_{\ell,m} \subset R_{\ell-1,m}$ and the diagrams

Maybe move this to later

$$\begin{tikzcd} R_{\ell-1,m+1} & R_{\ell,m+1} \arrow[l, "D_-"'] \\ R_{\ell-1,m} \arrow[u, "E_+"] & R_{\ell,m} \arrow[l, "D_-"] \arrow[u, "E_+"'] \end{tikzcd} \qquad \begin{tikzcd} R_{\ell,m+1} \arrow[r, "D_0"] & R_{\ell,m+1} \\ R_{\ell,m} \arrow[u, "E_+"] \arrow[r, "D_0"'] & R_{\ell,m} \arrow[u, "E_+"'] \end{tikzcd}$$
$$\begin{tikzcd} R_{\ell,m+1} \arrow[r, "D_+"] & R_{\ell+1,m+1} \\ R_{\ell,m} \arrow[u, "E_+"] \arrow[r, "D_+"'] & R_{\ell+1,m+1} \arrow[u, "E_+"'] \end{tikzcd} \tag{1.11}$$

commute, when $-\ell + 1 \leq m < \ell - 1$ in the top left diagram, $-\ell \leq m < \ell$ in the other two diagrams. Also similar diagrams with E_- replacing E_+ commute since $E_-\colon R_{\ell,m} \to R_{\ell,m-1}$ for $m \neq -\ell$ is inverse to $E_+\colon R_{\ell,m-1} \to R_{\ell,m}$. Before we can get to the final description of these maps we need quite a lot of work.

* * * * *

Note that eq. (1.2) gives us that $[L_k, L_p] \subset L_p$, so by the adjoint representation we can see L_p as an L_k-module, and again by eq. (1.2) we see that L_p is a simple L_k-module: If V is an L_k-submodule and we have a non-zero element $f = af_+ + bf_- + cf_3 \in V$ for some $a, b, c \in \mathbf{C}$ not all zero, then

$$\begin{aligned}[][h_+, af_+ + bf_- + cf_3] &= 2bf_3 - cf_+, \\ [h_-, af_+ + bf_- + cf_3] &= -2af_3 + cf_-, \\ [h_3, af_+ + bf_- + cf_3] &= af_+ - bf_-.\end{aligned}$$

If $c \neq 0$, we get that

$$\begin{aligned}[][h_3, [h_+, f]] &= [h_3, 2bf_3 - cf_+] = -cf_+, \\ [h_3, [h_-, f]] &= [h_3, -2af_3 + cf_-] = -cf_-,\end{aligned}$$

so we see that $f_+, f_- \in V$, and thus also $[h_+, \frac{1}{2}f_-] = f_3 \in V$, so $V = L_p$. If on the other hand $c = 0$, then

$$\begin{aligned}[][h_-, f] &= -2af_3, \\ [h_+, f] &= 2bf_3,\end{aligned}$$

so since either $a \neq 0$ or $b \neq 0$, we see that $f_3 \in V$, and thus also $[h_+, -f_3] = f_+ \in V$ and $[h_-, f_3] = f_- \in V$, so $V = L_p$. Hence L_p is indeed a simple L_k-module. Now since L_p is a simple finite dimensional L_k-module of dimension 3, we have that $L_p \simeq M(2)$ as L_k-modules.

In general given two L-modules V and W, we consider the tensor product $V \otimes W$ over $\mathbf{C}$ of the underlying vector spaces as an L-module via the action

$$x.(v \otimes w) = x.v \otimes w + v \otimes x.w,$$

for $x \in L$ and $v \otimes w \in V \otimes W$, cf. [Hum72, p. 26].

Now we are interested in the L_k-module $L_p \otimes M$, where M is a Harish-Chandra L-module as before. Specifically we will show that the linear map

$$\begin{aligned}\psi \colon L_p \otimes M &\to M \\ x \otimes v &\mapsto x.v\end{aligned} \tag{1.12}$$

is a homomorphism of L_k-modules. For $y \in L_k$ we see that

$$y.(x \otimes v) = y.x \otimes v + x \otimes y.v = [y, x] \otimes v + x \otimes y.v,$$

for $x \otimes v \in L_p \otimes M$, since the action in L_p is by the adjoint representation. So

$$\begin{aligned}\psi(y.(x \otimes v)) &= \psi([y, x] \otimes v) + \psi(x \otimes y.v) = [y, x].v + x.(y.v) \\ &= y.(x.v) - x.(y.v) + x.(y.v) = y.(x.v) = y.\psi(x \otimes v),\end{aligned}$$

i.e. ψ is indeed a homomorphism of L_k-modules.

Now we note that $M = \bigoplus_\ell R_\ell$, so

$$L_p \otimes M = L_p \otimes \big(\bigoplus_\ell R_\ell\big) \simeq \bigoplus_\ell (L_p \otimes R_\ell),$$

as L_k-modules, and since R_ℓ is direct sum of finitely many copies of $M(2\ell)$, we see that

$$\begin{aligned} L_p \otimes R_\ell &\simeq M(2) \otimes \big(M(2\ell)^1 \oplus M(2\ell)^2 \oplus \cdots \oplus M(2\ell)^r\big) \\ &\simeq (M(2) \otimes M(2\ell)^1) \oplus (M(2) \otimes M(2\ell)^2) \oplus \cdots \oplus (M(2) \otimes M(2\ell)^r), \end{aligned}$$

as L_k-modules, since $L_p \simeq M(2)$. Here the superscripts are just indices for the different $M(2\ell)$. Thus we want to describe the L_k-modules $M(2) \otimes M(2\ell)$, which we will do by first describing the $\mathfrak{sl}(2, \mathbf{C})$-modules $V(2) \otimes V(2\ell)$ and then translating back to a solution to our problem.

1.1.2 Describing $V(2) \otimes V(n)$

Let $2\ell = n \in \mathbf{N}$. We want to show that

$$V(2) \otimes V(n) \simeq \begin{cases} V(n-2) \oplus V(n) \oplus V(n+2) & \text{if } n \geq 2, \\ V(3) \oplus V(1) & \text{if } n = 1, \\ V(2) & \text{if } n = 0. \end{cases} \tag{1.13}$$

Note that in all cases there is a summand $V(n+2)$. We can show the above by considerations using formal characters. We will use the notation of [Jan16, Chapter 8], specifically we will do calculations with the functions $e(\lambda)\colon H^* \to \mathbf{Z}$ for $\lambda \in H^*$. Firstly note that in general

$$\mathrm{ch}_V = \sum_{\lambda \in H^*} (\dim V_\lambda) e(\lambda),$$

and use the notation $V(n)_k$ for $V(\lambda)_\mu$ and $e(n)$ for $e(\lambda)$ with $\lambda, \mu \in H^*$ such that $\lambda(h) = n$ and $\mu(h) = k$. We get that

$$\mathrm{ch}_{V(2)} = e(-2) + e(0) + e(2)$$

and

$$\mathrm{ch}_{V(n)} = \sum_{i=0}^{n} e(n - 2i),$$

since

$$\dim V(n)_k = \begin{cases} 1 & \text{if } k = n - 2i \text{ for some } i \in \{0, 1, \dots, n\}, \\ 0 & \text{otherwise.} \end{cases}$$

Now since $e(\lambda) * e(\mu) = e(\lambda+\mu)$ in general cf. [Jan16, p. 93], we see that for $n \geq 2$

$$\begin{aligned}
\mathrm{ch}_{V(2)\otimes V(n)} &= \mathrm{ch}_{V(2)} * \mathrm{ch}_{V(n)} = e(-2) * \mathrm{ch}_{V(n)} + e(0) * \mathrm{ch}_{V(n)} + e(2) * \mathrm{ch}_{V(n)} \\
&= \sum_{i=0}^{n} e(n-2-2i) + \mathrm{ch}_{V(n)} + \sum_{i=0}^{n} e(n+2-2i) \\
&= e(-n-2) + e(-n) + \sum_{i=0}^{n-2} e(n-2-2i) + \mathrm{ch}_{V(n)} \\
&\quad + \sum_{i=0}^{n} e(n+2-2i) \\
&= \mathrm{ch}_{V(n-2)} + \mathrm{ch}_{V(n)} + \sum_{i=0}^{n+2} e(n+2-2i) \\
&= \mathrm{ch}_{V(n-2)} + \mathrm{ch}_{V(n)} + \mathrm{ch}_{V(n+2)} = \mathrm{ch}_{V(n-2)\oplus V(n)\oplus V(n+2)},
\end{aligned}$$

where the first equality follows from the fact that $\mathrm{ch}_{V\otimes W} = \mathrm{ch}_V * \mathrm{ch}_W$ in general, cf. [Hum72, p. 125]. Thus since two L-modules V and V' are isomorphic if and only if $\mathrm{ch}_V = \mathrm{ch}_{V'}$, cf. [Jan16, p. 90], we see that $V(2) \otimes V(n) \simeq V(n-2) \oplus V(n) \oplus V(n+2)$ if $n \geq 2$.

Likewise we see that

$$\begin{aligned}
\mathrm{ch}_{V(2)\otimes V(1)} &= \mathrm{ch}_{V(2)} * \mathrm{ch}_{V(1)} \\
&= \big(e(-2) + e(0) + e(2)\big) * e(-1) + \big(e(-2) + e(0) + e(2)\big) * e(1) \\
&= e(-3) + e(-1) + e(1) + e(-1) + e(1) + e(3) \\
&= \big(e(-3) + e(-1) + e(1) + e(3)\big) + \big(e(-1) + e(1)\big) \\
&= \mathrm{ch}_{V(3)} + \mathrm{ch}_{V(1)} = \mathrm{ch}_{V(3)\oplus V(1)}
\end{aligned}$$

and

$$\mathrm{ch}_{V(2)\otimes V(0)} = \mathrm{ch}_{V(2)} * \mathrm{ch}_{V(0)} = \mathrm{ch}_{V(2)} * e(0) = \mathrm{ch}_{V(2)},$$

so indeed $V(2) \otimes V(1) \simeq V(3) \oplus V(1)$ and $V(2) \otimes V(0) \simeq V(2)$.

Now consider (w_0, w_1, w_2) a basis for $V(2)$ and $(v_i \mid 0 \leq i \leq n)$ a basis for $V(n)$ such that both satisfies the conditions from eq. (1.4). Then for $w_i \otimes v_j \in V(2) \otimes V(n)$ with $i \in \{0,1,2\}$ and $j \in \{0,1,\dots,n\}$ we see that

$$\begin{aligned}
h.(w_i \otimes v_j) &= h.w_i \otimes v_j + w_i \otimes h.v_j = (2-2i) w_i \otimes v_j + (n-2j) w_i \otimes v_j \\
&= (n - 2(i+j-1)) w_i \otimes v_j. \qquad (1.14)
\end{aligned}$$

Hence $v_0 \otimes w_0$ is up to scalar multiple the only vector of weight $n+2$ in $V(2) \otimes V(n)$, so it is necessarily a highest weight vector generating the direct summand isomorphic to $V(n+2)$. Note that by eq. (1.13) we indeed have a direct summand isomorphic to $V(n+2)$ for all $n \in \mathbf{N}$. By $\mathfrak{sl}(2,\mathbf{C})$-theory, cf. [Jan16, p. 36], we know that this summand has a basis $(s_k \mid 0 \le k \le n+2)$ satisfying equations as in eq. (1.4), where

$$s_k := \frac{1}{k!} y^k.(w_0 \otimes v_0). \tag{1.15}$$

By straightforward calculations, cf. Appendix A.1, we get for $n > 0$ that

$$\begin{aligned}
s_0 &= w_0 \otimes v_0, & & \\
s_1 &= w_1 \otimes v_0 + w_0 \otimes v_1 & &\text{if } n > 0, \\
s_k &= w_2 \otimes v_{k-2} + w_1 \otimes v_{k-1} + w_0 \otimes v_k & &\text{for } 2 \le k \le n, \\
s_{n+1} &= w_2 \otimes v_{n-1} + w_1 \otimes v_n & &\text{if } n > 0, \\
s_{n+2} &= w_2 \otimes v_n. & &
\end{aligned} \tag{1.16}$$

In case $n = 0$ we likewise see that $s_1 = w_1 \otimes v_0$ and $s_2 = w_2 \otimes v_0$, and we note that (s_0, s_1, s_2) is a basis for $V(2) \otimes V(0) \simeq V(2)$.

Suppose now that $n \ge 1$. Note that by eq. (1.13) we have a direct summand isomorphic to $V(n)$, and by eq. (1.14) the weight space of weight n is spanned by $w_0 \otimes v_1$ and $w_1 \otimes v_0$, so the vector of highest weight n generating the summand corresponding to $V(n)$ must be of the form $a w_0 \otimes v_1 + b w_1 \otimes v_0$ for some $a, b \in \mathbf{C}$. Furthermore we know that for this vector generating the summand corresponding to $V(n)$, we must have that

$$\begin{aligned}
0 &= x.(a w_0 \otimes v_1 + b w_1 \otimes v_0) \\
&= a x.w_0 \otimes v_1 + a w_0 \otimes x.v_1 + b x.w_1 \otimes v_0 + b w_1 \otimes x.v_0 \\
&= 0 + a(n-1+1) w_0 \otimes v_0 + b(2-1+1) w_0 \otimes v_0 + 0 \\
&= (an + 2b) w_0 \otimes v_0,
\end{aligned}$$

i.e. $an + 2b = 0$ so $b = -\frac{n}{2}a$. This determines the vector generating the summand corresponding to $V(n)$ up to a scalar, so taking $a = 1$, we see that we can take

$$t_0 := w_0 \otimes v_1 - \frac{n}{2} w_1 \otimes v_0$$

as our vector generating the summand corresponding to $V(n)$. As before $\mathfrak{sl}(2,\mathbf{C})$-theory now yields that this summand has a basis $(t_k \mid 0 \le k \le n)$ satisfying equations as in eq. (1.4), where

$$t_k := \frac{1}{k!} y^k.t_0. \tag{1.17}$$

By straightforward calculations, cf. Appendix A.1, we get that

$$\begin{aligned} t_0 &= w_0 \otimes v_1 - \frac{n}{2} w_1 \otimes v_0, \\ t_k &= (k+1) w_0 \otimes v_{k+1} - \frac{n-2k}{2} w_1 \otimes v_k \\ &\quad + (k-1-n) w_2 \otimes v_{k-1} \qquad\qquad \text{for } 1 \le k \le n-1, \\ t_n &= \frac{n}{2} w_1 \otimes v_n - w_2 \otimes v_{n-1}. \end{aligned} \tag{1.18}$$

Suppose now that $n \geq 2$. By eq. (1.13) we have a direct summand isomorphic to $V(n-2)$, and by eq. (1.14) the weight space of weight $n-2$ is spanned by $w_0 \otimes v_2$, $w_1 \otimes v_1$, and $w_2 \otimes v_0$, so the vector of highest weight $n-2$ generating the summand corresponding to $V(n)$ must be of the form $aw_0 \otimes v_2 + bw_1 \otimes v_1 + cw_2 \otimes v_0$ for some $a, b, c \in \mathbf{C}$. Furthermore we know that for this vector generating the summand corresponding to $V(n-2)$, we must have

$$\begin{aligned} 0 &= x.(aw_0 \otimes v_2 + bw_1 \otimes v_1 + cw_2 \otimes v_0) \\ &= aw_0 \otimes x.v_2 + bx.w_1 \otimes v_1 + bw_1 \otimes x.v_1 + cx.w_2 \otimes v_0 \\ &= a(n-2+1) w_0 \otimes v_1 + b(2-1+1) w_0 \otimes v_1 + b(n-1+1) w_1 \otimes v_0 \\ &\quad + c(2-2+1) w_1 \otimes v_0 \\ &= \big((n-1)a + 2b\big) w_0 \otimes v_1 + \big(bn + c\big) w_1 \otimes v_0, \end{aligned}$$

i.e. $a(n-1) + 2b = 0$ and $bn + c = 0$. Giving us $c = -bn$ and $b = -\frac{n-1}{2}a$, so

$$c = \frac{n(n-1)}{2} a.$$

This determines the vector generating the summand corresponding to $V(n-2)$ up to a scalar, so taking $a = 1$, we see that we can take

$$u_0 \coloneqq w_0 \otimes v_2 - \frac{n-1}{2} w_1 \otimes v_1 + \frac{n(n-1)}{2} w_2 \otimes v_0$$

as our vector generating the summand corresponding to $V(n-2)$. Again $\mathfrak{sl}(2,\mathbf{C})$-theory now yields that this summand has a basis $(u_k \mid 0 \leq k \leq n-2)$ satisfying equations as in eq. (1.4), where

$$u_k \coloneqq \frac{1}{k!} y^k . u_0. \tag{1.19}$$

By straightforward calculations, cf. Appendix A.1, we get that

$$\begin{aligned} u_k &= \frac{(k+1)(k+2)}{2} w_0 \otimes v_{k+2} - \frac{(k+1)(n-k-1)}{2} w_1 \otimes v_{k+1} \\ &\quad + \frac{(n-k)(n-k-1)}{2} w_2 \otimes v_k \end{aligned} \tag{1.20}$$

for $0 \leq k \leq n-2$.

Now we want to express $w_1 \otimes v_k$ for $0 \leq k \leq n$ in terms of the bases $(s_k \mid 0 \leq k \leq n+2)$, $(t_k \mid 0 \leq k \leq n)$, and $(u_k \mid 0 \leq k \leq n-2)$. A straightforward but long calculation, cf. Appendix A.2, yields that

$$w_1 \otimes v_k = \frac{2(k+1)(n+1-k)}{(n+1)(n+2)} s_{k+1} - \frac{2(n-2k)}{n(n+2)} t_k - \frac{4}{n(n+1)} u_{k-1} \tag{1.21}$$

for $0 < k < n$, while

$$w_1 \otimes v_0 = \frac{2}{n+2}(s_1 - t_0) \qquad \text{and} \qquad w_1 \otimes v_n = \frac{2}{n+2}(s_{n+1} + t_n) \tag{1.22}$$

if $n \geq 1$. If $n = 0$ we have already seen (just after eq. (1.16)) that $w_1 \otimes v_0 = s_1$. Note that eq. (1.22) is a special case of eq. (1.21) if we set $u_{-1} = u_{n-1} = 0$.

Now consider $V(2)$ and $V(n)$ as inner product spaces over $\mathbf{C}$ with inner products given by

$$\langle w_k, w_j \rangle = \delta_{jk} \binom{2}{k} \qquad \text{and} \qquad \langle v_k, v_j \rangle = \delta_{jk} \binom{n}{k}. \tag{1.23}$$

Then we can also consider $V(2) \otimes V(n)$ an inner product space with inner product given by

$$\langle w \otimes v, w' \otimes v' \rangle = \langle w, w' \rangle \cdot \langle v, v' \rangle \tag{1.24}$$

for $w, w' \in V(2)$ and $v, v' \in V(n)$. Now by straightforward calculations, cf. Appendix A.3, we get that

Maybe write about why this is an inner product

$$\langle s_0, s_0 \rangle = 1, \qquad \langle t_0, t_0 \rangle = \frac{n(n+2)}{2}, \qquad \langle u_0, u_0 \rangle = \frac{n^2(n+1)(n-1)}{4}. \tag{1.25}$$

Now set $\overline{w}_k = w_k / \|w_k\|$, $\overline{v}_k = v_k / \|v_k\|$, $\overline{s}_k = s_k / \|s_k\|$, $\overline{t}_k = t_k / \|t_k\|$, and $\overline{s}_k = s_k / \|s_k\|$ for all possible k, where $\|\cdot\|$ is given by $\|v\| = \sqrt{\langle v, v \rangle}$ as usually in an inner product space. Note that

Show the following equations — I can show these by long calculations, but I think there is an easier way

$$\begin{aligned}
\langle w_k, w_k \rangle &= \binom{2}{k} \\
\langle v_k, v_k \rangle &= \binom{n}{k} \\
\langle s_k, s_k \rangle &= \langle s_0, s_0 \rangle \binom{n+2}{k} = \binom{n+2}{k} \\
\langle t_k, t_k \rangle &= \langle t_0, t_0 \rangle \binom{n}{k} = \frac{n(n+2)}{2} \binom{n}{k} \\
\langle u_k, u_k \rangle &= \langle u_0, u_0 \rangle \binom{n-2}{k} = \frac{n^2(n+1)(n-1)}{4} \binom{n-2}{k}
\end{aligned}$$

for k where it makes sense, so we see that

$$w_k = \sqrt{\binom{2}{k}}\overline{w}_k, \qquad v_k = \sqrt{\binom{n}{k}}\overline{v}_k, \qquad s_k = \sqrt{\binom{n+2}{k}}\overline{s}_k, \tag{1.26}$$

and

$$t_k = \sqrt{\frac{n(n+2)}{2}\binom{n}{k}}\overline{t}_k, \qquad u_k = \sqrt{\frac{n^2(n+1)(n-1)}{4}\binom{n-2}{k}}\overline{u}_k. \tag{1.27}$$

Remark 1.5. Since

$$\overline{v}_k = \frac{1}{\sqrt{\binom{n}{k}}}v_k,$$

we note that we just need to change indices to go to the basis (e_m) from the basis of (v_k) as in the work leading to Lemma 1.4. △

By a simple calculation, cf. Appendix A.4, we get that

$$\begin{aligned}\overline{w}_1 \otimes \overline{v}_k = &\sqrt{\frac{2(k+1)(n+1-k)}{(n+1)(n+2)}}\overline{s}_{k+1} - \frac{(n-2k)}{\sqrt{n(n+2)}}\overline{t}_k \\ &- \sqrt{\frac{2k(n-k)}{n(n+1)}}\overline{u}_{k-1}.\end{aligned} \tag{1.28}$$

for $0 \leq k \leq n$. Now changing indices as mentioned in Remark 1.5 to $\ell = \frac{1}{2}n$ and $m = \frac{1}{2}(n-2k) = \ell - k$ as we did to get to Lemma 1.4, i.e. $n = 2\ell$ and $k = \ell - m$, we get that

$$\begin{aligned}\overline{w}_1 \otimes e_m &= \overline{w}_1 \otimes \overline{v}_k \\ &= \sqrt{\frac{2(\ell-m+1)(2\ell+1)-(\ell-m)}{(2\ell+1)(2\ell+2)}}\overline{s}_{k+1} - \frac{(2\ell-2(\ell-m))}{\sqrt{2\ell(2\ell+2)}}\overline{t}_k \\ &\quad - \sqrt{\frac{2(\ell-m)(2\ell-(\ell-m))}{2\ell(2\ell+1)}}\overline{u}_{k-1} \\ &= \sqrt{\frac{(\ell-m+1)(\ell+1+m)}{(2\ell+1)(\ell+1)}}\overline{s}_{k+1} - \frac{m}{\sqrt{\ell(\ell+1)}}\overline{t}_k \\ &\quad - \sqrt{\frac{(\ell-m)(\ell+m)}{\ell(2\ell+1)}}\overline{u}_{k-1},\end{aligned}$$

where e_m is as in the work we did to get Lemma 1.4 except for the fact that we consider $\mathfrak{sl}(2,\mathbf{C})$-modules still. Now setting

$$\widetilde{D}_+(\overline{v}_k) = -\frac{\overline{s}_{k+1}}{\sqrt{(\ell+1)(2\ell+1)}}, \quad \widetilde{D}_0(\overline{v}_k) = \frac{\overline{t}_k}{\sqrt{\ell(\ell+1)}}, \quad \widetilde{D}_-(\overline{v}_k) = -\frac{\overline{u}_{k-1}}{\sqrt{\ell(2\ell+1)}},$$

we see that

$$\begin{aligned}
\overline{w}_1 \otimes e_m &= \overline{w}_1 \otimes \overline{v}_k \\
&= \sqrt{(\ell+1)^2 - m^2}\frac{\overline{s}_{k+1}}{\sqrt{(\ell+1)(2\ell+1)}} - m\frac{\overline{t}_k}{\sqrt{\ell(\ell+1)}} \\
&\quad - \sqrt{\ell^2 - m^2}\frac{\overline{u}_{k-1}}{\ell(2\ell+1)} \\
&= \sqrt{\ell^2 - m^2}\widetilde{D}_-(\overline{v}_k) - m\widetilde{D}_0(\overline{v}_k) - \sqrt{(\ell+1)^2 - m^2}\widetilde{D}_+(\overline{v}_k).
\end{aligned} \tag{1.29}$$

Note that for $m \in \{\pm\ell\}$ the $\widetilde{D}_-$ term vanishes, so the formula works here although D_- is not well-defined in these edge cases.

* * * * *

Getting back to the problem at the end of Section 1.1.1, we want to give the maps D_0, D_+, and D_- such that $D_0R_{\ell,m} \subset R_{\ell,m}$, $D_+R_{\ell,m} \subset R_{\ell+1,m}$, and $D_-R_{\ell,m} \subset R_{\ell-1,m}$, the diagrams of eq. (1.11) commute, and we can describe F_3, F_+, F_- by the maps D_0, D_+, D_-, E_+, and E_-. Now consider the $\mathfrak{sl}(2,\mathbf{C})$-modules $V(n)$ as L_k-modules $M(n)$ via the isomorphism of eq. (1.3), and note that since

$$R_\ell = M(2\ell)^1 \oplus M(2\ell)^2 \oplus \cdots \oplus M(2\ell)^r$$

and each $M(2\ell)^i$ has a basis $(e^i_{-\ell}, e^i_{-\ell+1}, \ldots, e^i_{\ell-1}, e^i_\ell)$ with $H_3e^i_m = me^i_m$ for all m, we have that $R_{\ell,m}$ has basis $(e^1_m, e^2_m, \ldots, e^r_m)$ by definition. So when describing the maps D_0, D_+, and D_-, we just need to describe what the maps should do to each e^i_m. We already know that $E_+e^i_m = e^i_{m+1}$ and $E_-e^i_m = e^i_{m-1}$ where it makes sense, so if the maps D_0, D_+, and D_- do not depend on m or i, we get the commutative diagrams of eq. (1.11), thus we want to describe what each map does to $M(2\ell)$ in general, so we will stop writing the superscripts.

Since we want to describe the maps F_3, F_+, and F_-, we are actually interested in the actions of L_p, so by using ψ of eq. (1.12) and the considerations at the end of Section 1.1.1, we can start out by describing $M(2)\otimes M(2\ell)$, i.e. we can use the description of $V(2)\otimes V(n)$ from above. Note that we have already seen that $L_p \simeq M(2)$ as L_k-modules, but we would like to better understand how the basis (f_+, f_3, f_-) of L_p corresponds to the basis (w_0, w_1, w_2) of $M(2)$ as in eq. (1.5). In the basis (w_0, w_1, w_2) we have that $h_+.w_0 = 0$ (since this is what corresponds to $x.w_0 = 0$ in $V(2)$ by eq. (1.3)), so by checking eq. (1.2) we see that w_0 must correspond to a multiple of f_3, but the basis is chosen up to scalar, so we can take w_0 to be $-\frac{\sqrt{2}}{2}f_3$. Now we get w_1 by taking $h_-.w_0$ (corresponding to $y.w_0$ in $V(2)$ by eq. (1.3)), thus we get that

$$w_1 = h_-.w_0 = -\tfrac{\sqrt{2}}{2}h_-.f_+ = -\tfrac{\sqrt{2}}{2}[h_-, f_+] = \sqrt{2}f_3.$$

Likewise we get that $w_2 = [h_-, \sqrt{2}f_3] = \sqrt{2}f_-$, so we can take our basis to be $(w_0, w_1, w_2) = (-\frac{\sqrt{2}}{2}f_+, \sqrt{2}f_3, \sqrt{2}f_-)$ when thinking of L_p as the L_k-module $M(2)$. Normalizing as in eq. (1.26), we get that $(\overline{w}_0, \overline{w}_1, \overline{w}_2) =$

$(-\frac{\sqrt{2}}{2}f_+, f_3, \sqrt{2}f_-)$. So by eq. (1.29), we see that in $L_p \otimes M(2\ell)$

$$f_3 \otimes e_m = \sqrt{\ell^2 - m^2}\widetilde{D}_-(e_m) - m\widetilde{D}_0(e_m) - \sqrt{(\ell+1)^2 - m^2}\widetilde{D}_+(e_m),$$

where $e_m = \overline{v}_k$ for $k = \ell - m$ and $f_3 = \overline{w}_1$.

Remark 1.6. Note that if we have bases $(e^{(2\ell+2)}_{-\ell-1}, e^{(2\ell+2)}_{-\ell}, \dots, e^{(2\ell+2)}_{\ell}, e^{(2\ell+2)}_{\ell+1})$ for $M(2\ell+2)$, $(e^{(2\ell)}_{-\ell}, e^{(2\ell)}_{-\ell+1}, \dots, e^{(2\ell)}_{\ell-1}, e^{(2\ell)}_{\ell})$ for $M(2\ell)$, and $(e^{(2\ell-2)}_{-\ell+1}, e^{(2\ell-2)}_{-\ell+2}, \dots, e^{(2\ell-2)}_{\ell-2}, e^{(2\ell-2)}_{\ell-1})$ for $M(2\ell-2)$ (if $\ell \geq 1$) as in Lemma 1.4, then as above changing indices with $k = \ell + 1 - m$ we see that $e^{(2\ell+2)}_m$ corresponds to $\overline{s}_k$. Likewise changing indices with $k = \ell - m$ we see that $e^{(2\ell)}_m$ corresponds to $\overline{t}_k$, and with $k = \ell - 1 - m$ we see that $e^{(2\ell-2)}_m$ corresponds to $\overline{u}_k$. △

Now using ψ from eq. (1.12), we see that

$$\begin{aligned} F_3 e_m &= f_3.e_m = \psi(f_3 \otimes e_m) \\ &= \sqrt{\ell^2 - m^2}\psi\widetilde{D}_-(e_m) - m\psi\widetilde{D}_0(e_m) - \sqrt{(\ell+1)^2 - m^2}\psi\widetilde{D}_+(e_m). \end{aligned} \tag{1.30}$$

So we can take $D_0 = \psi\widetilde{D}_0$, $D_+ = \psi\widetilde{D}_+$, and $D_- = \psi\widetilde{D}_-$ to get three linear maps with which we can describe the map F_3. So far this is just maps on $M(2\ell)$, but we can expand to maps on R_ℓ by using the maps on each summand of $R_\ell = M(2\ell)^1 \oplus \cdots \oplus M(2\ell)^r$, and likewise we can expand further to maps on $M = \bigoplus_\ell R_\ell$ by using the maps on each summand. Also indeed $D_0 R_{\ell,m} \subset R_{\ell,m}$, $D_+ R_{\ell,m} \subset R_{\ell+1,m}$, and $D_- R_{\ell,m} \subset R_{\ell-1,m}$, since for $\xi \in R_{\ell,m}$ we have that

$$\begin{aligned} H_3 D_0(\xi) &= h_3.\psi\widetilde{D}_0(\xi) = \psi h_3.\widetilde{D}_0(\xi) = \psi H_3\widetilde{D}_0(\xi) = m\psi\widetilde{D}_0(\xi) \\ &= mD_0(\xi), \end{aligned}$$

since ψ is an L_k-module homomorphism and by Remark 1.6 we see that $\widetilde{D}_0(e_m)$ is a scalar multiple of $\overline{t}_k = \overline{t}_{\ell-m} = e^{(2\ell)}_m$, and indeed $H_3 e^{(2\ell)}_m = m e^{(2\ell)}_m$. The same reasoning with $\overline{s}_{k+1}$ for D_+ and $\overline{u}_{k-1}$ for D_- yields the other two inclusions. Also note that the diagrams of eq. (1.11) commute by the defintion of D_0, D_+, and D_-, since the maps independent of m and E_+ and E_- are isomorphisms.

Write this a little more clearly

Now simple calculations, cf. Appendix A.5, gives us that

$$\begin{aligned} F_3\xi &= \sqrt{\ell^2 - m^2}D_-\xi - mD_0\xi - \sqrt{(\ell+1)^2 - m^2}D_+\xi, \\ F_+\xi &= \sqrt{(\ell-m)(\ell-m-1)}D_-E_+\xi - \sqrt{(\ell-m)((\ell+m+1))}D_0E_+\xi \\ &\quad + \sqrt{(\ell+m+1)(\ell+m+2)}E_+D_+\xi, \\ F_-\xi &= -\sqrt{(\ell+m)(\ell+m-1)}D_-E_-\xi - \sqrt{(\ell+m)(\ell-m+1)}D_0E_-\xi \\ &\quad - \sqrt{(\ell-m+1)(\ell-m+2)}E_-D_+\xi \end{aligned} \tag{1.31}$$

for $\xi \in R_{\ell,m}$. Note here that although D_- is not defined on $R_{\ell,\ell}$ and $R_{\ell,-\ell}$ the above still makes sense since in these cases the terms with D_- vanish, either by the coefficient being zero or by E_+ or E_- mapping to zero.

We claim now that the formulae eq. (1.31) for the linear operators F_+, F_-, and F_3 together with the formulae eqs. (1.8) and (1.10) for H_+, H_-, and H_3 define a representation of L, i.e. they satisfy the commutation relations of eq. (1.2), if and only if D_0, D_+, and D_- satisfy

$$\begin{aligned} \ell D_+ D_0 \xi &= (\ell+2) D_0 D_+ \xi, \\ (\ell+1) D_- D_0 \xi &= (\ell-1) D_0 D_- \xi, \\ \xi &= (2\ell-1) D_+ D_- \xi - (2\ell+3) D_- D_+ \xi - D_0^2 \xi \end{aligned} \tag{1.32}$$

for $\xi \in R_{\ell,m}$.

I haven't shown this properly yet — I guess it should follow from looking at the relations but the calculations are very long, so I skipped it for now

1.1.3 Simple Harish-Chandra modules for the pair (L, L_k)

We want to classify the simple Harish-Chandra modules for the pair (L, L_k) for later use. Before most of the work we need some basic results.

Let M be an simple Harish-Chandra module over L and suppose that each non-trivial subspace $R_{\ell,m}$ in $M = \bigoplus_{\ell,m} R_{\ell,m}$ is one dimensional. In this case each L_k-module $R_\ell \simeq M(2\ell)$ is simple. We will later show that actually all simple Harish-Chandra modules are of this kind, so we indeed get a classification of the simple Harish-Chandra modules in the following.

Denote by ℓ_0 the minimal index ℓ in the decomposition $M = \bigoplus_\ell R_\ell$. Note that

$$M' = \bigoplus_{\ell' \in \{\ell_0, \ell_0+1, \dots\}} R_{\ell'}$$

is invariant under E_+, E_-, D_0, D_+, and D_-, so by the formulae eq. (1.31) for F_+, F_-, and F_3, we see that M' is a submodule since we already know that it is an L_k-submodule because $R_{\ell'}$ all are L_k-submodules. Thus $M' = M$ since M is simple and hence the index ℓ in $M = \bigoplus_\ell R_\ell$ range over only integral values or only half-integral values.

Additionally we want to show that the kernel of the map $D_- \colon M \to M$ is R_{ℓ_0}. To do this assume for contradiction that $D_- R_{\ell',m_0} = 0$ for some index $\ell' > \ell_0$ and $m_0 \in \{-\ell_0, -\ell_0+1, \dots, \ell_0-1, \ell_0\}$. Then by the commutative diagram in eq. (1.11) with D_-, i.e. $D_- E_+ = E_+ D_-$, and the fact that $E_+ \colon R_{\ell',m} \to R_{\ell',m+1}$ is an isomorphism for $m < \ell'$, we see that $D_- R_{\ell',m} = 0$ for all $m \in \{-\ell', -\ell'+1, \dots, \ell'-1, \ell'\}$. But then

$$M'' = \bigoplus_{\ell'' \in \{\ell', \ell'+1, \dots\}} R_{\ell''}$$

is a proper L-submodule of M, which contradicts the simplicity of M. Thus indeed $\ker D_- = R_{\ell_0}$.

Likewise we see that if M is infinite dimensional, then $D_+\colon M\to M$ has trivial kernel since if $D_+R_{\ell'}=0$, then $M=\bigoplus_{\ell\in\{\ell_0,\ell_0+1,\dots\}}R_\ell$ is finite dimensional. This is the case since all terms with $\ell>\ell'$ must be trivial since otherwise

$$M''=\bigoplus_{\ell''\in\{\ell_0,\ell_0+1,\dots,\ell'\}}R_{\ell''}$$

is a proper L-submodule of M, which contradicts the simplicity of M.

Infinite dimensional simple modules

Assume that M is a Harish-Chandra module of the above kind and is infinite dimensional. Because all $R_{\ell,m}$ are one dimensional, the diagram with E_+ and D_+ in eq. (1.11) commute, i.e. $D_+E_+=E_+D_+$, and D_+ has trivial kernel, while E_+ is an isomorphism for $m\neq\ell$, we see that we can choose a basis $\{\xi_{\ell,m}\}$ of M such that $\xi_{\ell,m}\in R_{\ell,m}$ and

$$\begin{aligned}
E_+\xi_{\ell,m}&=\xi_{\ell,m+1} &&\text{for } -\ell\leq m<\ell,\\
D_+\xi_{\ell,m}&=\xi_{\ell+1,m} &&\text{for } \ell\in\{\ell_0,\ell_0+1,\dots\}.
\end{aligned}$$

In this basis we get that

$$\begin{aligned}
E_-\xi_{\ell,m}&=\xi_{\ell,m-1} &&\text{for } -\ell<m\leq\ell,\\
D_0\xi_{\ell,m}&=d^0_\ell\xi_{\ell,m} &&\text{for } \ell\in\{\ell_0,\ell_0+1,\dots\},\\
D_-\xi_{\ell,m}&=d^-_\ell\xi_{\ell-1,m} &&\text{for } \ell\in\{\ell_0+1,\ell_0+2,\dots\},\\
D_-\xi_{\ell_0,m}&=0,
\end{aligned}\tag{1.33}$$

where the first equation comes from the fact that $E_-\colon R_{\ell,m}\to R_{\ell,m-1}$ for $m\neq-\ell$ is the inverse of $E_+\colon R_{\ell,m-1}\to R_{\ell,m}$, while the independence of m in the other equations comes from the commutativity of the diagrams of eq. (1.11).

Now eqs. (1.32) and (1.33) implies that

$$\begin{aligned}
\ell d^0_\ell&=(\ell+2)d^0_{\ell+1},\\
(\ell+1)d^-_\ell d^0_\ell&=(\ell-1)d^0_{\ell-1}d^-_\ell,\\
1&=(2\ell-1)d^-_\ell-(2\ell+3)d^-_{\ell+1}-(d^0_\ell)^2,\\
d^-_{\ell_0}&=0,
\end{aligned}\tag{1.34}$$

for $\ell\in\{\ell_0,\ell_0+1,\dots\}$ except in the second equation where we also demand that $\ell>\ell_0$. We see that

$$d^0_{\ell+1}=\frac{\ell}{\ell+2}d^0_\ell.$$

So if $\ell_0\neq0$, then for some constant c

$$d^0_{\ell_0}=\frac{c}{\ell_0(\ell_0+1)},$$

so we see inductively that if

$$d_\ell^0 = \frac{c}{\ell(\ell+1)}, \tag{1.35}$$

then

$$\begin{aligned} d_{\ell+1}^0 &= \frac{\ell}{\ell+2} d_\ell^0 = \frac{\ell}{\ell+2} \frac{c}{\ell(\ell+1)} \\ &= \frac{c}{(\ell+1)(\ell+2)}. \end{aligned}$$

Thus if $\ell_0 \neq 0$ eq. (1.35) holds true in general for some constant c.

If on the other hand $\ell_0 = 0$, then we see that

$$2d_{\ell_0+1}^0 = 0,$$

so $d_{\ell_0+1}^0 = 0$, and thus

$$d_\ell^0 = \frac{\ell-1}{\ell+1} d_{\ell-1}^0 = 0$$

for all $\ell \in \{1, 2, \ldots\}$. Also in this case have $d_0^0 = c_1$, where c_1 is some constant.

To unify these two cases we set $c = i\ell_0\ell_1$ and $c_1 = i\ell_1$ for some real constant ℓ_1 such that

This should be for complex ℓ_1 because of how I want to use it later, but this is a problem by the next comment

$$d_\ell^0 = \frac{i\ell_0\ell_1}{\ell(\ell+1)} \tag{1.36}$$

for $\ell \in \{\ell_0, \ell_0+1, \ldots\}$. Substituting this expression with d_ℓ^0 in the third equation of eq. (1.34) we get that

$$(2\ell-1)d_\ell^- - (2\ell+3)d_{\ell+1}^- = 1 - \frac{\ell_0^2\ell_1^2}{\ell^2(\ell+1)^2},$$

and a simple calculation, cf. Appendix A.7, yields that

$$d_\ell^- = -\frac{\left(\ell^2 - \ell_1^2\right)\left(\ell^2 - \ell_0^2\right)}{\ell^2(4\ell^2-1)}, \tag{1.37}$$

for $\ell > \ell_0$.

Since we showed in the beginning of this subsection that the kernel of D_- is R_{ℓ_0}, we must have that $d_\ell^- \neq 0$ for all $\ell > \ell_0$. Thus $\ell^2 - \ell_1^2 \neq 0$ for $\ell > \ell_0$, so $|\ell_1| - \ell_0$ cannot be a positive integer, because if that was the case then $|\ell_1| = \ell_0 + (|\ell_1| - \ell_0) \in \{\ell_0+1, \ell_0+2, \ldots\}$, but $|\ell_1|^2 - \ell_1^2 = 0$ since $\ell_1 \in \mathbf{R}$.

How to do this if ℓ_1 is complex

Hence altogether by eqs. (1.10) and (1.31) in the basis $\{\xi_{\ell,m}\}$ the operators H_+, H_-, H_3, F_+, F_-, and F_3 are given by the formulae

$$
\begin{aligned}
H_3\xi_{\ell,m} &= m\xi_{\ell,m}, \\
H_+\xi_{\ell,m} &= \sqrt{(\ell+m+1)(\ell-m)}\xi_{\ell,m+1}, \\
H_-\xi_{\ell,m} &= \sqrt{(\ell+m)(\ell-m+1)}\xi_{\ell,m-1}, \\
F_3\xi_{\ell,m} &= \sqrt{\ell^2-m^2}d_\ell^-\xi_{\ell-1,m} - md_\ell^0\xi_{\ell,m} - \sqrt{(\ell+1)^2-m^2}d_\ell^+\xi_{\ell+1,m}, \\
F_+\xi_{\ell,m} &= \sqrt{(\ell-m)(\ell-m-1)}d_\ell^-\xi_{\ell-1,m+1} - \sqrt{(\ell-m)((\ell+m+1))}d_\ell^0\xi_{\ell,m+1} \\
&\quad + \sqrt{(\ell+m+1)(\ell+m+2)}d_\ell^+\xi_{\ell+1,m+1}, \\
F_-\xi_{\ell,m} &= -\sqrt{(\ell+m)(\ell+m-1)}\xi_{\ell-1,m-1} - \sqrt{(\ell+m)(\ell-m+1)}\xi_{\ell,m-1} \\
&\quad - \sqrt{(\ell-m+1)(\ell-m+2)}\xi_{\ell+1,m-1},
\end{aligned}
\tag{1.38}
$$

where

$$
d_\ell^0 = \frac{i\ell_0\ell_1}{\ell(\ell+1)}, \qquad d_\ell^- = -\frac{\left(\ell^2-\ell_1^2\right)\left(\ell^2-\ell_0^2\right)}{\ell^2(4\ell^2-1)}, \qquad d_\ell^+ = 1, \tag{1.39}
$$

for $\ell \in \{\ell_0, \ell_0+1, \ldots\}$, and where ℓ_1 is a real number such that $|\ell_1| - \ell_0$ is not a positive integer. Here we use the convention that $\xi_{\ell',m'} = 0$ for pairs ℓ', m' where there is no such basis element.

Finite dimensional simple modules

Assume that M is a Harish-Chandra module of the above kind and that M is finite dimensional, i.e. $M = \bigoplus_{\ell,m} R_{\ell,m}$ where $R_{\ell,m}$ are one dimensional subspaces for $\ell_0 \leq \ell < |\ell_1|$. Here ℓ_1 is some real number such that $|\ell_1| \geq \ell_0$ and $|\ell_1| - \ell_0$ is integral. We can choose a basis $\{\xi_{\ell,m}\}$ as in the infinite dimensional case and we still get the formulae eqs. (1.38) and (1.39) describing the actions of H_+, H_-, H_3, F_+, F_-, and F_3, though now in this basis we only consider $\ell \in \{\ell_0, \ell_0+1, \ldots, |\ell_1|-1\}$.

Maybe describe a little more.

1.2 Decomposition of modules into indecomposables

Now we want to continue our work using our knowledge of the classification of simple Harish-Chandra modules for the pair (L, L_k) to begin our classification of indecomposable Harish-Chandra modules for the pair (L, L_k). To do this we will first need to some work with Laplace operators.

1.2.1 Laplace operators

Let $U(L)$ be the universal enveloping algebra of L, cf. [Jan16, Appendix E]. We know, cf. [Jan16, p. E-9], that M is an L-module if and only if it is an

$U(L)$-module, so we can describe L-modules by describing $U(L)$-modules. To do this we will first need to have an explicit description of the center $Z(U(L))$ of $U(L)$. We will begin this description by first noting that $Z(U(\mathfrak{sl}(2,\mathbf{C}) \times \mathfrak{sl}(2,\mathbf{C}))) \simeq Z(U(\mathfrak{sl}(2,\mathbf{C}))) \otimes Z(U(\mathfrak{sl}(2,\mathbf{C})))$, which follows from the fact that $Z(U(L_1 \times L_2)) \simeq Z(U(L_1)) \otimes Z(U(L_2))$ for Lie algebras L_1 and L_2 in general cf. Appendix B.1.

We have seen in Exercise 11 in the Lie algebra course that $Z(U(\mathfrak{sl}(2,\mathbf{C})))$ is the algebra of polynomials in $C = h^2 + 2h + 4yx$, i.e. $Z(U(L)) = \mathbf{C}[C]$. Thus we see that $Z(U(\mathfrak{sl}(2,\mathbf{C}))) \otimes Z(U(\mathfrak{sl}(2,\mathbf{C})))$ is the algebra of polynomials in $C \otimes 1$ and $1 \otimes C$, or equivalently the algebra of polynomials in $C \otimes 1 - 1 \otimes C$ and $C \otimes 1 + 1 \otimes C$. Translating back to $Z(U(L))$ with the isomorphism ψ from eq. (B.1), noting that actually we have used the notation $\iota_1(C) = C$ in $U(L_1)$ and $\iota_2(C) = C$ in $U(L_2)$ above, we see that

Maybe write the argument in an appendix or find better reference

$$\begin{aligned}\psi(\iota_1(C) \otimes 1 - 1 \otimes \iota_2(C)) &= \psi_1\iota_1(C)\psi_2(1) - \psi_1(1)\psi_2\iota_2(C) \\ &= \iota(C,0) - \iota(0,C) = \iota(C,-C).\end{aligned}$$

Now we will again use the notation $\iota(u,v) = (u,v)$ in $U(L)$ for $(u,v) \in L$ and likewise with ι_1 and ι_2, so the above says that $\psi(C \otimes 1 - 1 \otimes C) = (C,-C)$. Likewise we get that $\psi(C \otimes 1 + 1 \otimes C) = (C,C)$, so we want to describe $(C,-C) = (h^2 + 2h + 4yx, -h^2 - 2h - 4yx)$ and $(C,C) = (h^2 + 2h + 4yx, h^2 + 2h + 4yx)$ in terms of our basis $h_+, h_-, h_3, f_+, f_-, f_3$. We note that

$$\begin{aligned}&\tfrac{1}{2}(h_-f_+ + f_-h_+) + h_3f_3 + f_3 \\ &= \tfrac{1}{2}\big((y,y)(ix,-ix) + (iy,-iy)(x,x)\big) + \tfrac{1}{4}(h,h)(ih,-ih) + \tfrac{1}{2}(ih,-ih) \\ &= \tfrac{1}{2}(2iyx,-2iyx) + \tfrac{1}{4}(ih^2,-ih^2) + \tfrac{1}{2}(ih,-ih) \\ &= \tfrac{i}{4}(h^2 + 4yx + 2h, -h^2 - 4yx - 2h) \\ &= \tfrac{i}{4}(C,-C)\end{aligned}$$

and

$$\begin{aligned}&h_-h_+ - f_-f_+ + h_3^2 - f_3^2 + 2h_3 \\ &= (y,y)(x,x) - (iy,-iy)(ix,-ix) + \tfrac{1}{4}(h,h)^2 - \tfrac{1}{4}(ih,-ih)^2 + (h,h) \\ &= (yx,yx) + (yx,yx) + \tfrac{1}{4}(h^2,h^2) + \tfrac{1}{4}(h^2,h^2) + (h,h) \\ &= \tfrac{1}{2}(h^2 + 2h + 4yx, h^2 + 2h + 4yx) \\ &= \tfrac{1}{2}(C,C).\end{aligned}$$

Thus since the constants don't matter when we look at the algebra of polynomials in $(C,-C)$ and (C,C), we see that setting

$$\Delta_1 = \tfrac{1}{2}(h_-f_+ + f_-h_+) + h_3f_3 + f_3, \quad \Delta_2 = h_-h_+ - f_-f_+ + h_3^2 - f_3^2 + 2h_3, \tag{1.40}$$

we have that $Z(U(L))$ is the algebra of polynomials in Δ_1 and Δ_2. Thus in term of the corresponding linear operators on a Harish-Chandra module M for the pair (L, L_k), we define linear operators

$$\begin{aligned} \Delta_1 &\coloneqq \tfrac{1}{2}(H_-F_+ + F_-H_+) + H_3F_3 + F_3 \\ \Delta_2 &\coloneqq H_-H_+ - F_-F_+ + H_3^2 - F_3^2 + 2H_3, \end{aligned} \tag{1.41}$$

which are called Laplace operators. Note that by eqs. (1.8), (1.9) and (1.31), cf. Appendix A.8, we get that

$$\begin{aligned} \Delta_1\xi &= -\ell(\ell+1)D_0\xi \\ \Delta_2\xi &= (\ell^2-1)\xi - (\ell+1)^2D_0^2\xi + (4\ell^2-1)D_+D_-\xi \end{aligned} \tag{1.42}$$

for $\xi \in R_\ell$. Alternatively by eq. (1.32), cf. Appendix A.8, we also get that

$$\Delta_2\xi = \big((\ell+1)^2-1\big)\xi + \ell^2D_0^2\xi + \big(4(\ell+1)^2-1\big)D_-D_+\xi \tag{1.43}$$

for $\xi \in R_\ell$, which will sometimes be more useful.

Now by noting that D_0, D_+D_-, and D_0^2 all preserve $R_{\ell,m}$ eq. (1.42) gives us the following Lemma:

Lemma 1.7. *Each subspace $R_{\ell,m}$ is invariant under the Laplace operators Δ_1 and Δ_2.*

Additionally we are ready to prove the Lemma:

Lemma 1.8. *The linear operators D_+, D_-, D_0, E_+, and E_- commute with the Laplace operators Δ_1 and Δ_2.*

Proof. Denote by $(\Delta_i)_{\ell,m}$ the restriction of Δ_i to $R_{\ell,m}$ for $i = 1, 2$. Lemma 1.7 implies that $\Delta_i = \bigoplus_{\ell,m}(\Delta_i)_{\ell,m}$ for $i = 1, 2$, so it is enough to check that $(\Delta)_{\ell,m}$ commutes with the operators for all ℓ and m. Therefore eqs. (1.42) and (1.43) implies that Δ_i commute with E_+ and E_- since D_+, D_-, and D_0 commute with E_+ and E_- where it makes sense and using eq. (1.43) for Δ_2 it makes sense for all $R_{\ell,m}$.

Now multiplying the first equation of eq. (1.32) with $\ell + 1$, we see that

$$\ell(\ell+1)D_+D_0\xi = (\ell+1)(\ell+2)D_0D_+\xi$$

for $\xi \in R_{\ell,m}$, so by eq. (1.42), we see that

$$D_+\Delta_1\xi = -\ell(\ell+1)D_+D_0\xi = -(\ell+1)(\ell+2)D_0D_+\xi = \Delta_1D_+\xi$$

for $\xi \in R_{\ell,m}$. Thus Δ_1 indeed commutes with D_+. Similarly the second equation of eq. (1.32) imply that Δ_1 commutes with D_-, and also it is obvious from eq. (1.42) that Δ_1 commutes with D_0.

Likewise the first equation of eq. (1.32) together with eqs. (1.42) and (1.43) implies that

$$\begin{aligned}\Delta_2 D_+\xi &= \big((\ell+1)^2-1\big)D_+\xi - (\ell+2)^2 D_0^2 D_+\xi + \big(4(\ell+1)^2-1\big)D_+D_-D_+\xi \\ &= \big((\ell+1)^2-1\big)D_+\xi - \ell^2 D_+ D_0^2\xi + \big(4(\ell+1)^2-1\big)D_+D_-D_+\xi \\ &= D_+\Delta_2\xi\end{aligned}$$

for $\xi \in R_{\ell,m}$. Thus Δ_2 commutes with D_+, and similarly using the second equation of eq. (1.32) we get that Δ_2 commutes with D_-. Finally it is clear that D_0 commutes with the first two terms of Δ_2, so we just need to show that $D_0(D_+D_-)\xi = (D_+D_-)D_0\xi$ for $\xi \in R_{\ell,m}$ where it makes sense. But now the first and second equation of eq. (1.32) imply that

$$(\ell+1)D_0D_+D_-\xi = (\ell-1)D_+D_0D_-\xi = (\ell+1)D_+D_-D_0\xi$$

for $\xi \in R_{\ell,m}$, so since $\ell \geq 0$ and thus $\ell \neq -1$, we get that $D_0(D_+D_-)\xi = (D_+D_-)D_0\xi$. Hence Δ_2 indeed commutes with D_0 also. □

1.2.2 Properties of the Laplace operators in indecomposable modules

Now we are finally ready to begin considering the properties of Δ_1 and Δ_2 in indecomposable Harish-Chandra modules, which will end up being an important part of our characterization of indecomposable Harish-Chandra modules for the pair (L, L_k).

Proposition 1.9. *A Harish-Chandra module M for the pair (L, L_k) is decomposable into the direct sum of a countable number of indecomposable modules. On each indecomposable module the Laplace operators Δ_1 and Δ_2 have each one eigenvalue, λ_1 and λ_2 respectively.*

Proof. Since each of the subspaces $R_{\ell,m}$ is invariant under Δ_1 and Δ_2 by Lemma 1.7 and since these operators commute with each other, we get that $R_{\ell,m}$ can be written as a direct sum of subspaces $R_{\ell,m}(\lambda_1^i, \lambda_2^i)$ on each of which each of the operators Δ_1 and Δ_2 has one eigenvalue λ_1^i and λ_2^i respectively. Note that here the index set of i is finite since $R_{\ell,m}$ is finite dimensional.

Consider now fixed λ_1 and λ_2 and the set S of those (ℓ, m) for which there exists subspaces $R_{\ell,m}(\lambda_1^i, \lambda_2^i)$ with $\lambda_1 = \lambda_1^i$ and $\lambda_2 = \lambda_2^i$. Denote by $M(\lambda_1, \lambda_2)$ the subspace of M with $M(\lambda_1, \lambda_2) = \bigoplus_{(\ell,m)\in S} R_{\ell,m}(\lambda_1, \lambda_2)$ such that in $M(\lambda_1, \lambda_2)$ each of the operators Δ_1 and Δ_2 has one eigenvalue, λ_1 and λ_2 respectively. We want to show that $M(\lambda_1, \lambda_2)$ is a submodule of M, i.e. that it is invariant under H_+, H_-, H_3, F_+, F_-, and F_3, but this is clearly the case since $M(\lambda_1, \lambda_2)$ is invariant under E_+, E_-, D_+, D_-, and D_0 because Δ_1 and Δ_2 commute with these operators by Lemma 1.8. Finally note that the number of $M(\lambda_1, \lambda_2)$ in the decomposition of M is countable since the number of $R_{\ell,m}$ is countable and the number of $R_{\ell,m}(\lambda_1^i, \lambda_2^i)$ in a given $R_{\ell,m}$ is finite, and note that $M(\lambda_1, \lambda_2)$ is indecomposable since

Why indecomposable

Proposition 1.10. *Let M be a Harish-Chandra module in which each of the Laplace operators Δ_1 and Δ_2 has one eigenvalue. Then there exists an integral or half-integral number $\ell_0 \geq 0$ and a complex number ℓ_1 such that the eigenvalues λ_1 and λ_2 have the form*

$$\lambda_1 = -i\ell_0\ell_1, \qquad \lambda_2 = \ell_0^2 + \ell_1^2 - 1. \tag{1.44}$$

Proof. Denote by ℓ_0 the minimal index in the decomposition $M = \bigoplus_\ell R_\ell$ of M into L_k-submodules of R_ℓ. By the definition of D_- it maps R_{ℓ_0} to zero, so by eq. (1.42) we get that

$$\begin{aligned} \Delta_1\xi &= -\ell_0(\ell_0+1)D_0\xi \\ \Delta_2\xi &= (\ell_0^2 - 1)\xi - (\ell_0+1)D_0^2\xi \end{aligned} \tag{1.45}$$

for $\xi \in R_{\ell_0}$. Now the subspace R_{ℓ_0} is invariant under D_0, so we can find an eigenvector ξ_0 for D_0 such that $D_0\xi_0 = \mu\xi_0$ for some $\mu \in \mathbf{C}$. Thus we see that

$$\begin{aligned} \Delta_1\xi &= -\ell_0(\ell_0+1)\mu\xi_0 \\ \Delta_2\xi &= (\ell_0^2 - 1)\xi - (\ell_0+1)\mu^2\xi, \end{aligned}$$

so we get eigenvalues λ_1 and λ_2 of Δ_1 and Δ_2 with

$$\lambda_1 = -\ell_0(\ell_0+1)\mu, \qquad \lambda_2 = (\ell_0^2 - 1) - (\ell_0+1)\mu^2.$$

Hence putting $(\ell_0+1)\mu = i\ell_1$, we get that

$$\lambda_1 = -i\ell_0\ell_1, \qquad \lambda_2 = \ell_0^2 + \ell_1^2 - 1.$$

Now by assumption each of Δ_1 and Δ_2 has only one eigenvalue on M, and thus these eigenvalues are expressed in terms of the ℓ_0 and ℓ_1 as in eq. (1.44). $\square$

Note that all such eigenvalues λ_1 and λ_2 for Δ_1 and Δ_2 are possible, since in the case of finite dimensional simple modules N as in Section 1.1.3 we have by eqs. (1.39) and (1.42) that

$$\begin{aligned} \Delta_1\xi_{\ell,m} &= -\ell(\ell+1)d_\ell^0\xi_{\ell,m} = -i\ell_0\ell_1, \\ \Delta_2\xi_{\ell,m} &= (\ell^2-1)\xi_{\ell,m} - (\ell+1)^2(d_\ell^0)^2\xi_{\ell,m} + (4\ell^2-1)d_{\ell+1}^+ d_\ell^- \xi_{\ell,m} \\ &= (\ell^2-1)\xi_{\ell,m} + \frac{\ell_0^2\ell_1^2}{\ell^2}\xi_{\ell,m} - \frac{(\ell^2-\ell_1^2)(\ell^2-\ell_0^2)}{\ell^2}\xi_{\ell,m} \\ &= (\ell^2-1)\xi_{\ell,m} - (\ell^2 - \ell_0^2 - \ell_1^2)\xi_{\ell,m} \\ &= (\ell_0^2 + \ell_1^2 - 1)\xi_{\ell,m}, \end{aligned}$$

for $\xi_{\ell,m} \in R_{\ell,m}$, where $\ell_0 \geq 0$ is an integral or half-integral number and ℓ_1 is a complex number such that we can construct N such that ℓ_0 and ℓ_1 are as we want.

I have only showed the case with ℓ_1 some real number, see earlier comments

Proposition 1.11. *Let M be a Harish-Chandra module in which the Laplace operators Δ_1 and Δ_2 have only one eigenvalue λ_1 and λ_2 respectively. Then on each subspace R_ℓ the operators D_+D_-, D_-D_+, and D_0 have only one eigenvalue d_ℓ^-, d_ℓ^+, and d_ℓ^+ respectively. Here the numbers d_ℓ^-, d_ℓ^+, and d_ℓ^+ are expressed in terms of ℓ_0 and ℓ_1 in the following way:*

$$\begin{aligned}
d_0^- &= d_{1/2}^- = 0, \\
d_\ell^- &= \frac{(\ell^2-\ell_0^2)(\ell_1^2-\ell^2)}{(4\ell^2-1)\ell^2} && \text{if } \ell \neq 0, \tfrac{1}{2}, \\
d_\ell^+ &= \frac{((\ell+1)^2-\ell_0^2)(\ell_1^2-(\ell+1)^2)}{(4(\ell+1)^2-1)(\ell+1)^2} \\
d_0^0 &= i\ell_1, \\
d_\ell^0 &= \frac{i\ell_0\ell_1}{\ell(\ell+1)} && \text{if } \ell \neq 0.
\end{aligned} \tag{1.46}$$

I might want to change the result slightly. What happens if $R_{\ell+1} = 0$, then we should have $d_\ell^+ = 0$, I either need to change the result or go back through the paper and correct some things and argue that this will not happen/will not matter

Proof. By eq. (1.42) we see that

$$\begin{aligned}
(4\ell^2-1)D_+D_-\xi &= \Delta_2\xi - (\ell^2-1)\xi + (\ell+1)^2 D_0^2\xi, \\
(\ell+1)D_0\xi &= -\frac{\Delta_1}{\ell}\xi
\end{aligned}$$

for $\xi \in R_\ell$ with $\ell > \ell_0$ such that $\ell \neq 0$. Thus

$$(4\ell^2-1)D_+D_-\xi = \Delta_2\xi - (\ell^2-1)\xi + \frac{\Delta_1^2}{\ell^2}\xi$$

for $\xi \in R_\ell$ with $\ell > \ell_0$. Hence since Δ_1 and Δ_2 each only have one eigenvalue on R_ℓ so thus D_+D_-, and we see by eq. (1.44) that

$$\begin{aligned}
d_\ell^- &= \frac{1}{(4\ell^2-1)}\Big(\lambda_2 - (\ell-1)^2 + \frac{\lambda_1^2}{\ell^2}\Big) \\
&= \frac{(\ell_0^2+\ell_1^2-1)\ell^2 - (\ell-1)^2\ell^2 - \ell_0^2\ell_1^2}{(4\ell^2-1)\ell^2} \\
&= \frac{(\ell_0^2+\ell_1^2-\ell^2)\ell^2 - \ell_0^2\ell_1^2}{(4\ell^2-1)\ell^2} \\
&= \frac{(\ell^2-\ell_0^2)(\ell_1^2-\ell^2)}{(4\ell^2-1)\ell^2}
\end{aligned}$$

for $\ell \neq 0, \frac{1}{2}$. Now since D_- by definition maps R_0 and $R_{1/2}$ to zero if they occur in the decomposition of M, we see that $d_0^- = 0$ and $d_{1/2}^- = 0$, and likewise we know that D_- maps R_{ℓ_0} to zero so $d_{\ell_0}^- = 0$, which also holds true with the formula above. Thus we have proven the formulae for d_ℓ^-, and the other formulae are proven similarly. ∎

Now denote by $C_s(\lambda_1, \lambda_2)$ for $s = 1$ or $s = \frac{1}{2}$ the set of all Harish-Chandra modules for the pair (L, L_k) in which the Laplace operators have the eigenvalues λ_1 and λ_2, and in which if $s = 1$ every $M \in C_1(\lambda_1, \lambda_2)$ has only integral numbers as indices in the decomposition $M = \bigoplus_\ell R_\ell$, and if $s = \frac{1}{2}$ every $M \in C_{1/2}(\lambda_1, \lambda_2)$ has only half-integral numbers as indices in the decomposition $M = \bigoplus_\ell R_\ell$.

Proposition 1.12. *Let $M \in C_s(\lambda_1, \lambda_2)$, $M' \in C_{s'}(\lambda_1', \lambda_2')$, where $(s, \lambda_1, \lambda_2) \neq (s', \lambda_1', \lambda_2')$. Then $\operatorname{Hom}_L(M, M') = 0$.*

Proof. Let $\gamma \in \operatorname{Hom}_L(M, M')$ and assume that $\gamma \neq 0$. First we will show that $\gamma R_\ell \subset R_\ell'$, where R_ℓ comes from the decomposition of M and R_ℓ' from the decomposition of M'. To see this note that R_ℓ and R_ℓ' are direct sums of finitely many L_k-modules $M(2\ell)$, where each $M(2\ell)$ is generated by a maximal vector of weight ℓ (weight w.r.t. h_3 in L_k). So since γ takes a maximal vector of weight ℓ to either zero or another maximal vector of weight ℓ, we see that indeed $\gamma R_\ell \subset R_\ell'$.

Denoting by Δ_i the Laplace operators in M and by Δ_i' the Laplace operators in M', we also have that $\Delta_i'\gamma = \gamma\Delta_i$ for $i = 1, 2$, since in both cases Δ_i and Δ_i' correspond to the actions in $U(L)$ of eq. (1.40) and $\gamma \in \operatorname{Hom}_L(M, M') = \operatorname{Hom}_{U(L)}(M, M')$. Now since $\gamma \neq 0$ we get that $\gamma R_\ell \subset R_\ell'$ implies that $s = s'$ and $\Delta_i'\gamma = \gamma\Delta_i$ implies that $\lambda_i = \lambda_i'$ for $i = 1, 2$, but this is a contradiction since $(s, \lambda_1, \lambda_2) \neq (s', \lambda_1', \lambda_2')$. Hence we must have that $\gamma = 0$, and thus indeed $\operatorname{Hom}_L(M, M') = 0$. ☐

Remark 1.13. From now on in places where the index s is not important, we will simply denote $C_s(\lambda_1, \lambda_2)$ by $C(\lambda_1, \lambda_2)$. △

Definition 1.14. *The category of modules $C(\lambda_1, \lambda_2)$ is called singular if the numbers ℓ_0 and ℓ_1 constructed from λ_1 and λ_2 as in eq.* (1.44) *are such that $\ell_1 - \ell_0$ is an integer. Otherwise it is called non-singular.*

We will see that the study of the non-singular categories $C(\lambda_1, \lambda_2)$ is simpler than that of the singular ones, and we are now ready to begin our description of the category of the singular categories $C(\lambda_1, \lambda_2)$ which will be important to our classification of indecomposable Harish-Chandra modules.

1.3 The non-singular category $C(\lambda_1, \lambda_2)$

Let $M \in C(\lambda_1, \lambda_2)$ be an L-module, where (λ_1, λ_2) is a non-singular pair, i.e. $\ell_1 - \ell_0$ is not an integer. We now want to that this module M is completely determined by a finite dimensional vector space and a nilpotent map a on this vector space, where an isomorphism of the modules is equivalent to similarity of the linear maps a.

Define on the finite dimensional linear subspace R_{ℓ_0,m_0} for some $m_0 \in \{-\ell_0, -\ell_0+1, \dots, \ell_0-1, \ell_0\}$ a linear map $a\colon R_{\ell_0,m_0} \to R_{\ell_0,m_0}$ by

$$a\xi = D_0\xi - \frac{i\ell_1}{\ell_0+1}\xi \tag{1.47}$$

for $\xi \in R_{\ell_0,m_0}$. This map is nilpotent since by Proposition 1.11 the only eigenvalue of D_0 on R_{ℓ_0} is

$$d^0_{\ell_0} = \frac{i\ell_1}{\ell_0+1}.$$

We want to show that the module M is completely determined by the finite dimensional vector space R_{ℓ_0,m_0} and the linear map $a\colon R_{\ell_0,m_0} \to R_{\ell_0,m_0}$ when $C(\lambda_1, \lambda_2)$ is non-singular. To do this we first need some lemmas.

Lemma 1.15. *In a non-singular module $M \in C(\lambda_1, \lambda_2)$, the maps*

$$\begin{aligned} D_+\colon R_{\ell,m} &\to R_{\ell+1,m} \\ D_-\colon R_{\ell+1,m} &\to R_{\ell,m} \end{aligned}$$

for $\ell \geq \ell_0$ are isomorphisms.

Proof. By Proposition 1.11 the eigenvalues of D_+D-- and D_-D_+ on R_ℓ for $\ell \neq 0, \frac{1}{2}$ are

$$\begin{aligned} d^-_\ell &= \frac{(\ell^2-\ell_0^2)(\ell_1^2-\ell^2)}{(4\ell^2-1)\ell^2} && \text{for } \ell > \ell_0, \\ d^+_\ell &= \frac{((\ell+1)^2-\ell_0^2)(\ell_1^2-(\ell+1)^2)}{(4(\ell+1)^2-1)(\ell+1)^2}. \end{aligned}$$

Now by assumption M is non-singular, i.e. $\ell_1 - \ell_0$ is not an integer, and we want to show that $\ell_1^2 - \ell^2 \neq 0$ for all $\ell \in \{\ell_0, \ell_0+1, \dots\}$.

Assume that $\ell_1^2 - \ell^2 = 0$ for some $\ell = \ell_0 + k$, where k is a non-negative integer. Since $\ell_1 - \ell_0$ is not an integer, we also have that $\ell_1 - (\ell_0 + k)$ is not an integer and hence not equal to zero, so we must have that $\ell_1 + \ell_0 + k = 0$ since $\ell_1^2 - \ell^2 = (\ell_1 - \ell)(\ell_1 + \ell)$. But this would imply that $\ell_1 = -\ell_0 - k$ and hence $\ell_1 - \ell_0 = -2\ell_0 - k$ is an integer, which is a contradiction with the non-singularity of M.

Thus we see that $\ell_1^2 - \ell^2$ is non-zero for all $\ell \in \{\ell_0, \ell_0+1, \dots\}$, and therefore the eigenvalues d^+_ℓ and d^-_ℓ are different from zero for all ℓ except ℓ_0 in the case of d^-_ℓ. Hence the maps $D_+D_-\colon R_{\ell,m} \to R_{\ell,m}$ for $\ell \neq \ell_0$ and $D_-D_+\colon R_{\ell,m} \to R_{\ell,m}$ correspond to multiplication by a non-zero constant, so $D_-\colon R_{\ell+1,m} \to R_{\ell,m}$ and $D_+\colon R_{\ell,m} \to R_{\ell+1,m}$ are injective, and thus the $\dim R_{\ell,m} = \dim R_{\ell+1,m}$, which again implies that D_- and D_+ as above are actually isomorphisms. $\square$

Lemma 1.16. *In a non-singular module $M \in C(\lambda_1, \lambda_2)$ the Laplace operators Δ_1 and Δ_2 are such that each operator $(\Delta_i)_{\ell,m}$ is similar to $(\Delta_i)_{\ell_0,m_0}$ for $i = 1, 2$.*

Proof. Recall that the maps $E_+ : R_{\ell_0,m} \to R_{\ell_0,m+1}$ for $-\ell_0 \leq m < \ell_0$ and $E_- : R_{\ell_0,m} \to R_{\ell_0,m-1}$ for $-\ell_0 < m \leq \ell_0$ are isomorphisms, and the Laplace operators Δ_1 and Δ_2 commute with these maps by Lemma 1.8, so $(\Delta_i)_{\ell_0,m}$ is similar to $(\Delta_i)_{\ell_0,m_0}$ for each m and $i = 1, 2$.

Likewise the $D_+ : R_{\ell,m} \to R_{\ell+1,m}$ are also isomorphisms for all ℓ and commute with both Δ_1 and Δ_2, so the map $(\Delta_i)_{\ell_0+1,m}$ is similar to $(\Delta_i)_{\ell_0,m}$, and inductively $(\Delta_i)_{\ell,m}$ is similar to $(\Delta_i)_{\ell_0,m}$ for all $\ell \in \{\ell_0, \ell_0 + 1, \ldots\}$. Hence indeed $(\Delta_i)_{\ell,m}$ is similar to $(\Delta_i)_{\ell_0,m_0}$. □

Lemma 1.17. *If $M \in C(\lambda_1, \lambda_2)$ is a non-singular module, then the Laplace operators Δ_1 and Δ_2 are connected on the whole of M by the relation*

$$\Delta_1^2 + \ell_0^2 \Delta_2 - \ell_0^2(\ell_0^2 - 1)\,\mathrm{id} = 0. \tag{1.48}$$

Proof. Suppose that $\ell_0 \neq 0$. By eq. (1.45) we get that

$$\begin{aligned}\Delta_2 \xi + \frac{\Delta_1^2}{\ell_0^2}\xi - (\ell_0^2 - 1)\,\mathrm{id}\,\xi = (\ell_0^2 - 1)\xi - (\ell_0 + 1)^2 D_0^2 \xi \\ + (\ell_0 + 1)^2 D_0^2 \xi - (\ell_0^2 - 1)\xi = 0,\end{aligned}$$

for $\xi \in R_{\ell_0,m_0}$, so multiplying by ℓ_0^2 we get eq. (1.48) on R_{ℓ_0,m_0}. By Lemma 1.16 $(\Delta_i)_{\ell,m}$ is similar to $(\Delta_i)_{\ell_0,m_0}$, so the relation holds true for any $\xi \in R_{\ell,m}$, and thus on all of M.

Suppose otherwise that $\ell_0 = 0$. Then eq. (1.45) implies that $(\Delta_1)_{0,0}$ is zero, and thus the relation follows easily on $R_{0,0}$, and we can expand to all of M as above. □

Remark 1.18. Note that Lemmas 1.15 to 1.17 are not true in the singular case. △

We have seen above that to each non-singular module $M \in C(\lambda_1, \lambda_2)$ there corresponds a finite dimensional vector space $P = R_{\ell_0,m_0}$ and a nilpotent linear map $a : P \to P$ given by

$$a\xi = \Big(D_0 - \frac{i\ell_1}{\ell_0 + 1}\,\mathrm{id}\Big)\xi$$

for $\xi \in R_{\ell_0,m_0} = P$. Denote now by $\tilde{A}$ the pair (P, a) consisting of a finite dimensional vector space P and a nilpotent mapping $a : P \to P$.

Proposition 1.19. *To each pair $\tilde{A}$ and non-singular pair (λ_1, λ_2) of numbers there is a corresponding L-module $M \in C(\lambda_1, \lambda_2)$ such that $P = R_{\ell_0,m_0}$ and a is related to D_0 by eq. (1.47).*

Proof. Denote by R_{ℓ_0,m_0} the space P and consider the linear transformation

$$D_0\xi = a\xi + \frac{i\ell_1}{\ell_0+1}\xi$$

for $\xi \in R_{\ell_0,m_0}$. Consider the space

$$M = \bigoplus_{\substack{\ell\in\{\ell_0,\ell_0+1,\dots\}\\ m\in\{-\ell,-\ell+1,\dots,\ell-1,\ell\}}} R_{\ell,m},$$

which is a direct sum of vector spaces with $\dim R_{\ell,m} = \dim P$ for all ℓ and m.

Now take an isomorphism $E_+\colon R_{\ell,m} \to R_{\ell,m+1}$ for $m \neq \ell$ and put $E_+\colon R_{\ell,\ell} \to 0$, which we can do since $\dim R_{\ell,m} = \dim R_{\ell,m+1}$. Define an isomorphism $E_-\colon R_{\ell,m+1} \to R_{\ell,m}$ such that it is inverse to $E_+\colon R_{\ell,m} \to R_{\ell,m+1}$ and put $E_-\colon R_{\ell,-\ell} \to 0$. Take now isomorphisms $D_+\colon R_{\ell,m_0} \to R_{\ell+1,m_0}$ for some fixed m_0, and define on all the remaining $R_{\ell,m}$ linear maps $D_+\colon R_{\ell,m} \to R_{\ell+1,m}$ such that the diagram

$$\begin{tikzcd}
R_{\ell,m+1} \arrow[r, "D_+"] & R_{\ell+1,m+1} \\
R_{\ell,m} \arrow[u, "E_+"] \arrow[r, "D_+"'] & R_{\ell+1,m} \arrow[u, "E_+"']
\end{tikzcd}$$

commutes for $-\ell \leq m < \ell$. Now we only need to construct linear maps D_0 and D_- on M satisfying properties as we have seen earlier, but to do this we will first define linear maps Δ_1 and Δ_2 corresponding to the Laplace operators.

On R_{ℓ_0,m_0} set

$$\begin{aligned}
\Delta_1\xi &= -\ell_0(\ell_0+1)D_0\xi \\
&= -\ell_0(\ell_0+1)a\xi - i\ell_1\ell_0\xi, \\
\Delta_2\xi &= (\ell_0^2-1)\xi - (\ell_0+1)^2D_0^2\xi \\
&= (\ell_0^2-1)\xi + \ell_1^2\xi - (\ell_0+1)i\ell_1 a\xi - (\ell_0+1)^2a^2\xi \\
&= (\ell_0^2+\ell_1^2-1)\xi - (\ell_0+1)^2\Big(a^2\xi + 2\frac{i\ell_1}{\ell_0+1}a\xi\Big)
\end{aligned} \tag{1.49}$$

for $\xi \in R_{\ell_0,m_0}$. Now note that for arbitrary $R_{\ell,m}$ the linear map $J_{\ell,m} = (E_+)^{m-m_0}(D_+)^{\ell-\ell_0}$ is a composition of isomorphisms and hence an isomorphism, so we can define

$$(\Delta_i)_{\ell,m}\xi = J_{\ell,m}(\Delta_i)_{\ell_0,m_0}(J_{\ell,m})^{-1}$$

for $\xi \in R_{\ell,m}$ and $i = 1, 2$. Thus we have defined Δ_1 and Δ_2 on all of M.

Now define $D_0\colon R_{\ell,m} \to R_{\ell,m}$ by

$$D_0\xi = -\frac{1}{\ell(\ell+1)}\Delta_1\xi$$

for $\xi \in R_{\ell,m}$, and $D_+D_- \colon R_{\ell,m} \to R_{\ell,m}$ by

$$D_+D_-\xi = \frac{1}{4\ell^2-1}(\Delta_2\xi - (\ell^2-1)\xi + (\ell+1)^2 D_0^2\xi) = \frac{1}{4\ell^2-1}\Big(\Delta_2\xi - (\ell^2-1)\xi + \frac{\Delta_1^2}{\ell^2}\xi\Big)$$

for $\xi \in R_{\ell,m}$, $\ell \neq \ell_0$, which we can do since D_+ is an isomorphism. Using this we define $D_- \colon R_{\ell,m} \to R_{\ell-1,m}$ to be the map $(D_+)^{-1}(D_+D_-)$ for $\ell \neq \ell_0$, and equal to zero for $\ell = \ell_0$.

Write this more explicitely

Now the maps E_+, E_-, D_0, D_+, and D_- constructed above satisfy the relations of Section 1.1.1, so the operators F_+, F_-, F_3, H_+, H_-, and H_3 constructed from these maps as in eqs. (1.8), (1.10) and (1.32) gives M an L-module structure. Finally we get $P = R_{\ell_0,m_0}$ and eq. (1.47) by construction, and we note that M is non-singular since the pair (λ_1, λ_2) with the corresponding ℓ_0 and ℓ_1 is non-singular by assumption. □

Corollary 1.20. *For modules M and M' from the non-singular category $C(\lambda_1, \lambda_2)$ to be equivalent it is necessary and sufficient that the subspaces R_{ℓ_0,m_0} and R'_{ℓ_0,m_0} in these modules have the same dimension, and that their maps $D_0 \colon R_{\ell_0,m_0} \to R_{\ell_0,m_0}$ and $D'_0 \colon R_{\ell_0,m_0}$ are similar.*

Write a little before this

Chapter 2

Linear relations

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Appendix A

Calculations

Throughout the paper there are situations where we need to do some straightforward but rather long calculations, so to clean up the exposition somewhat we will collect most of these calculations in this appendix and then just use the results in the paper.

A.1 Bases of $V(2) \otimes V(n)$

We want to describe the s_k's of eq. (1.15) more explicitly. We have that $s_0 = w_0 \otimes v_0$ and $s_k = \frac{1}{k!} y^k . s_0$, and we note that if $n > 0$ then

$$\begin{aligned} s_1 &= y.(w_0 \otimes v_0) = y.w_0 \otimes v_0 + w_0 \otimes y.v_0 \\ &= w_1 \otimes v_0 + w_0 \otimes v_1 \end{aligned}$$

and

$$\begin{aligned} s_2 &= \tfrac{1}{2} y.s_1 \\ &= \tfrac{1}{2} y.w_1 \otimes v_0 + \tfrac{1}{2} w_1 \otimes y.v_0 + \tfrac{1}{2} y.w_0 \otimes v_1 + w_0 \otimes \tfrac{1}{2} y.v_1 \\ &= w_2 \otimes v_0 + \tfrac{1}{2} w_1 \otimes v_1 + \tfrac{1}{2} w_1 \otimes v_1 + w_0 \otimes v_2 \\ &= w_2 \otimes v_0 + w_1 \otimes v_1 + w_0 \otimes v_2. \end{aligned}$$

Inductively we see that

$$s_k = w_2 \otimes v_{k-2} + w_1 \otimes v_{k-1} + w_0 \otimes v_k$$

for $k \leq n$, since the base case holds and given the equality for $k < n$ we get

$$\begin{aligned} s_{k+1} = &\frac{1}{k+1} y.s_k \\ &= w_2 \otimes \frac{1}{k+1} y.v_{k-2} + \frac{1}{k+1} y.w_1 \otimes v_{k-1} + w_1 \otimes \frac{1}{k+1} y.v_{k-1} \\ &\quad + \frac{1}{k+1} y.w_0 \otimes v_k + w_0 \otimes \frac{1}{k+1} y.v_k \end{aligned}$$

$$
\begin{aligned}
&= \frac{k-1}{k+1} w_2 \otimes v_{k-1} + \frac{2}{k+1} w_2 \otimes v_{k-1} + \frac{k}{k+1} w_1 \otimes v_k + \frac{1}{k+1} w_1 \otimes v_k \\
&\quad + w_0 \otimes v_{k+1} \\
&= w_2 \otimes v_{k-1} + w_1 \otimes v_k + w_0 \otimes v_{k+1}.
\end{aligned}
$$

We likewise see that for $k = n+1$ the last term vanishes, so we have $s_{k+1} = w_2 \otimes v_{n-1} + w_1 \otimes v_n$, and for $k = n+2$ the two last terms vanish, so we get $s_{k+2} = w_2 \otimes v_n$. Thus altogether we get the description in eq. (1.16).

Suppose now that $n \geq 1$. We want to describe the t_k's of eq. (1.17) more explicitly. We have that $t_0 = w_0 \otimes v_1 - \frac{n}{2} w_1 \otimes v_0$ and $t_k = \frac{1}{k!} y^k.t_0$. We see that

$$
\begin{aligned}
t_1 &= y.\big(w_0 \otimes v_1 - \frac{n}{2} w_1 \otimes v_0\big) \\
&= y.w_0 \otimes v_1 + w_0 \otimes y.v_1 - \frac{n}{2} y.w_1 \otimes v_0 + \frac{n}{2} w_1 \otimes y.v_0 \\
&= w_1 \otimes v_1 + 2w_0 \otimes v_2 - n w_2 \otimes v_0 - \frac{n}{2} w_1 \otimes v_1 \\
&= 2w_0 \otimes v_2 - \frac{n-2}{2} w_1 \otimes v_1 - n w_2 \otimes v_0,
\end{aligned}
$$

and inductively we get that

$$
t_k = (k+1) w_0 \otimes v_{k+1} - \frac{n-2k}{2} w_1 \otimes v_k + (k-1-n) w_2 \otimes v_{k-1}
$$

for $1 \leq k \leq n-1$, since the base case holds and given the equality for $k < n-1$ we get

$$
\begin{aligned}
t_{k+1} &= \frac{1}{k+1} y.t_k \\
&= y.w_0 \otimes v_{k+1} + w_0 \otimes y.v_{k+1} - \frac{n-2k}{2(k+1)} y.w_1 \otimes v_k \\
&\quad - \frac{n-2k}{2(k+1)} w_1 \otimes y.v_k + \frac{k-1-n}{k+1} w_2 \otimes y.v_{k-1} \\
&= w_1 \otimes v_{k+1} + (k+2) w_0 \otimes v_{k+2} - \frac{n-2k}{k+1} w_2 \otimes v_k \\
&\quad - \frac{n-2k}{2} w_1 \otimes v_{k+1} + \frac{(k-1-n)k}{k+1} w_2 \otimes v_k \\
&= (k+2) w_0 \otimes v_{k+2} - \frac{n-2(k+1)}{2} w_1 \otimes v_{k+1} \\
&\quad + \left(\frac{k^2 - k - nk - n + 2k}{k+1} \right) w_2 \otimes v_k \\
&= (k+2) w_0 \otimes v_{k+2} - \frac{n-2(k+1)}{2} w_1 \otimes v_{k+1} + (k-n) w_2 \otimes v_k,
\end{aligned}
$$

where we in the last equality use that $(k+1)(k-n) = k^2 - nk + k - n = k^2 - k - nk - n + 2k$. We likewise see that for $k = n$ the first term vanishes so

$$
t_n = \frac{n}{2} w_1 \otimes v_n - w_2 \otimes v_{n-1}.
$$

Thus we altogether get the description in eq. (1.18).

Suppose now that $n \geq 2$. We want to describe the u_k's of eq. (1.19) more explicitely. We have that

$$u_0 \coloneqq w_0 \otimes v_2 - \frac{n-1}{2} w_1 \otimes v_1 + \frac{n(n-1)}{2} w_2 \otimes v_0$$

and $u_k = \frac{1}{k!} y^k . u_0$. We see inductively that

$$\begin{aligned} u_k = {} & \frac{(k+1)(k+2)}{2} w_0 \otimes v_{k+2} - \frac{(k+1)(n-k-1)}{2} w_1 \otimes v_{k+1} \\ & + \frac{(n-k)(n-k-1)}{2} w_2 \otimes v_k \end{aligned}$$

for $0 \leq k \leq n-2$, since the base case holds and given the equality for $k < n-2$ we get

$$\begin{aligned} u_{k+1} &= \frac{1}{k+1} y.u_k \\ &= \frac{k+2}{2} y.w_0 \otimes v_{k+2} + \frac{k+2}{2} w_0 \otimes y.v_{k+2} \\ &\quad - \frac{n-k-1}{2} y.w_1 \otimes v_{k+1} - \frac{n-k-1}{2} w_1 \otimes y.v_{k+1} \\ &\quad + \frac{(n-k)(n-k-1)}{2(k+1)} w_2 \otimes y.v_k \\ &= \frac{k+2}{2} w_1 \otimes v_{k+2} + \frac{(k+2)(k+3)}{2} w_0 \otimes v_{k+3} \\ &\quad - (n-k-1) w_2 \otimes v_{k+1} - \frac{(n-k-1)(k+2)}{2} w_1 \otimes v_{k+2} \\ &\quad + \frac{(n-k)(n-k-1)}{2} w_2 \otimes v_{k+1} \\ &= \frac{(k+2)(k+3)}{2} w_0 \otimes v_{k+3} \\ &\quad - \frac{(n-k-1)(k+2) - (k+2)}{2} w_1 \otimes v_{k+2} \\ &\quad + \frac{(n-k)(n-k-1) - 2(n-k-1)}{2} w_2 \otimes v_{k+1} \\ &= \frac{(k+2)(k+3)}{2} w_0 \otimes v_{k+3} \\ &\quad - \frac{(k+2)(n-k-2)}{2} w_1 \otimes v_{k+2} \\ &\quad + \frac{(n-k-1)(n-k-2)}{2} w_2 \otimes v_{k+1}. \end{aligned}$$

Thus we altogether get the description in eq. (1.20).

A.2 Finding $w_1 \otimes v_k$

Using the bases $(s_k \mid 0 \leq k \leq n+2)$ of eq. (1.16), $(t_k \mid 0 \leq k \leq n)$ of eq. (1.18), and $(u_k \mid 0 \leq k \leq n-2)$ of eq. (1.20), we see that

$$
\begin{aligned}
&\frac{2(k+1)(n+1-k)}{(n+1)(n+2)} s_{k+1} - \frac{2(n-2k)}{n(n+2)} t_k - \frac{4}{n(n+1)} u_{k-1} \\
&= \frac{2(k+1)(n+1-k)}{(n+1)(n+2)} \Big(w_0 \otimes v_{k+1} + w_1 \otimes v_k + w_2 \otimes v_{k-1} \Big) \\
&\quad - \frac{2(n-2k)}{n(n+2)} \Big((k+1) w_0 \otimes v_{k+1} - \frac{n-2k}{2} w_1 \otimes v_k \\
&\qquad + (k-1-n) w_2 \otimes v_{k-1} \Big) \\
&\quad - \frac{4}{n(n+1)} \Big(\frac{k(k+1)}{2} w_0 \otimes v_{k+1} - \frac{k(n-k)}{2} w_1 \otimes v_k \\
&\qquad + \frac{(n-k+1)(n-k)}{2} w_2 \otimes v_{k-1} \Big) \\
&= \frac{\begin{pmatrix} 2(k+1)(n+1-k)n - 2(n-2k)(k+1)(n+1) \\ \hfill - 2k(k+1)(n+2) \end{pmatrix}}{n(n+1)(n+2)} w_0 \otimes v_{k+1} \\
&\quad + \frac{\begin{pmatrix} 2(k+1)(n+1-k)n + (n-2k)(n-2k)(n+1) \\ \hfill + 2k(n-k)(n+2) \end{pmatrix}}{n(n+1)(n+2)} w_1 \otimes v_k \\
&\quad + \frac{\begin{pmatrix} 2(k+1)(n+1-k)n - 2(n-2k)(k-1-n)(n+1) \\ \hfill - 2(n-k+1)(n-k)(n+2) \end{pmatrix}}{n(n+1)(n+2)} w_2 \otimes v_{k-1} \\
&= 2(k+1) \frac{(n+1-k)n - (n-2k)(n+1) - k(n+2)}{n(n+1)(n+2)} w_0 \otimes v_{k+1} \\
&\quad + \frac{\begin{pmatrix} 2(k+1)(n+1-k)n + (n-2k)(n-2k)(n+1) \\ \hfill + 2k(n-k)(n+2) \end{pmatrix}}{n(n+1)(n+2)} w_1 \otimes v_k \\
&\quad + 2(n+1-k) \frac{(k+1)n + (n-2k)(n+1) - (n-k)(n+2)}{n(n+1)(n+2)} w_2 \otimes v_{k-1}.
\end{aligned}
$$

Now we note that

$$
\begin{aligned}
(n+1-k)n &- (n-2k)(n+1) - k(n+2) \\
&= n\Big((n+1-k) - (n-2k) - k \Big) - (n-2k) - 2k \\
&= n - (n-2k) - 2k = 0,
\end{aligned}
$$

and

$$\begin{aligned}
&(k+1)n + (n-2k)(n+1) - (n-k)(n+2) \\
&\qquad = n\Big((k+1) + (n-2k) - (n-k)\Big) + (n-2k) - 2(n-k) \\
&\qquad = n + n - 2k - 2n + 2k = 0,
\end{aligned}$$

while

$$\begin{aligned}
&2(k+1)(n+1-k)n + (n-2k)(n-2k)(n+1) + 2k(n-k)(n+2) \\
&= n\Big(2(k+1)(n+1-k) + (n-2k)(n+1) + 2k(n-k)\Big) \\
&\quad - 2k(n-2k)(n+1) + 4k(n-k) \\
&= n\Big(2(k+1)(n+1-k) + (n-2k)(n+1) + 2k(n-k)\Big) \\
&\quad - 2kn(n-2k) - 2k(n-2k) + 4k(n-k) \\
&= n\Big(2(k+1)(n+1-k) + (n-2k)(n+1) + 2k(n-k)\Big) \\
&\quad - 2kn(n-2k) + 2kn \\
&= n\Big(2(k+1)(n+1-k) + (n-2k)(n+1) + 2k(n-k) - 2k(n-2k) \\
&\qquad + 2k\Big),
\end{aligned}$$

where

$$\begin{aligned}
&2(k+1)(n+1-k) + (n-2k)(n+1) + 2k(n-k) - 2k(n-2k) + 2k \\
&= (n+1)\Big(2(k+1) + (n-2k)\Big) - 2k(k+1) \\
&\quad + 2k\Big((n-k) - (n-2k) + 1\Big) \\
&= (n+1)(n+2) - 2k(k+1) + 2k(k+1) \\
&= (n+1)(n+2),
\end{aligned}$$

so

$$\begin{aligned}
&2(k+1)(n+1-k)n + (n-2k)(n-2k)(n+1) + 2k(n-k)(n+2) \\
&= n(n+1)(n+2).
\end{aligned}$$

Thus we see that

I will probably remove some of this and just say that algebraic manipulation shows that ...

$$\begin{aligned}
&\frac{2(k+1)(n+1-k)}{(n+1)(n+2)} s_{k+1} - \frac{2(n-2k)}{n(n+2)} t_k - \frac{4}{n(n+1)} u_{k-1} \\
&= 0 + \frac{n(n+1)(n+2)}{n(n+1)(n+2)} w_1 \otimes v_k + 0 \\
&= w_1 \otimes v_k
\end{aligned}$$

giving us eq. (1.21).

Likewise for $n \geq 1$, we get that

$$\begin{aligned}
\frac{2}{n+2}(s_1 - t_0) &= \frac{2}{n+2}\Big(w_0 \otimes v_1 + w_1 \otimes v_0 - w_0 \otimes v_1 + \frac{n}{2} w_1 \otimes v_0\Big) \\
&= \frac{2}{n+2}\frac{n+2}{2} w_1 \otimes v_0 \\
&= w_1 \otimes v_0
\end{aligned}$$

and

$$\begin{aligned}
\frac{2}{n+2}(s_{n+1} + t_n) &= \frac{2}{n+2}\Big(w_2 \otimes v_{n+1} + w_1 \otimes v_n + \frac{n}{2} w_1 \otimes v_n - w_2 \otimes v_{n-1}\Big) \\
&= \frac{2}{n+2}\frac{n+2}{2} w_1 \otimes v_n \\
&= w_1 \otimes v_n
\end{aligned}$$

giving us eq. (1.22).

A.3 Inner products in $V(2) \otimes V(n)$

Given $s_0 = w_0 \otimes v_0$, $t_0 = w_0 \otimes v_1 - \frac{n}{2} w_1 \otimes v_0$, and $u_0 = w_0 \otimes v_2 - \frac{n-1}{2} w_1 \otimes v_1 + \frac{n(n-1)}{2} w_2 \otimes v_0$ from eq. (1.16), eq. (1.18), and eq. (1.20), we want to find $\langle s_0, s_0 \rangle$, $\langle t_0, t_0 \rangle$, and $\langle u_0, u_0 \rangle$ using the inner products of eq. (1.23) and eq. (1.24). We see that

$$\begin{aligned}
\langle s_0, s_0 \rangle &= \langle w_0 \otimes v_0, w_0 \otimes v_0 \rangle = \langle w_0, w_0 \rangle \cdot \langle v_0, v_0 \rangle \\
&= \binom{2}{0} \cdot \binom{n}{0} = 1.
\end{aligned}$$

Likewise we get that

$$\begin{aligned}
\langle t_0, t_0 \rangle &= \Big\langle w_0 \otimes v_1 - \frac{n}{2} w_1 \otimes v_0, w_0 \otimes v_1 - \frac{n}{2} w_1 \otimes v_0 \Big\rangle \\
&= \langle w_0 \otimes v_1, w_0 \otimes v_1 \rangle - \frac{n}{2}\langle w_0 \otimes v_1, w_1 \otimes v_0 \rangle - \frac{n}{2}\langle w_1 \otimes v_0, w_0 \otimes v_1 \rangle \\
&\quad + \frac{n^2}{4}\langle w_1 \otimes v_0, w_1 \otimes v_0 \rangle \\
&= \langle w_0, w_0 \rangle \cdot \langle v_1, v_1 \rangle - \frac{n}{2}\langle w_0, w_1 \rangle \langle v_1, v_0 \rangle - \frac{n}{2}\langle w_1, w_0 \rangle \cdot \langle v_0, v_1 \rangle \\
&\quad + \frac{n^2}{4}\langle w_1, w_1 \rangle \cdot \langle v_0, v_0 \rangle \\
&= \binom{2}{0} \cdot \binom{n}{1} - 0 - 0 + \frac{n^2}{4}\binom{2}{1} \cdot \binom{n}{0} \\
&= n + \frac{n^2}{2} = \frac{n(n+2)}{2},
\end{aligned}$$

and noting that as above all terms with $\langle w_i \otimes v_j, w_k \otimes v_\ell \rangle$ with $i \neq k$ or $j \neq \ell$ vanish since then either $\langle w_i, w_k \rangle = 0$ or $\langle v_j, v_\ell \rangle$, we see that

$$\begin{aligned}
\langle u_0, u_0 \rangle &= \Big\langle w_0 \otimes v_2 - \frac{n-1}{2} w_1 \otimes v_1 + \frac{n(n-1)}{2} w_2 \otimes v_0, \\
&\qquad w_0 \otimes v_2 - \frac{n-1}{2} w_1 \otimes v_1 + \frac{n(n-1)}{2} w_2 \otimes v_0 \Big\rangle \\
&= \langle w_0 \otimes v_2, w_0 \otimes v_2 \rangle + \frac{(n-1)^2}{4} \langle w_1 \otimes v_1, w_1 \otimes v_1 \rangle \\
&\quad + \frac{n^2(n-1)^2}{4} \langle w_2 \otimes v_0, w_2 \otimes v_0 \rangle \\
&= \langle w_0, w_0 \rangle \cdot \langle v_2, v_2 \rangle + \frac{(n-1)^2}{4} \langle w_1, w_1 \rangle \cdot \langle v_1, v_1 \rangle \\
&\quad + \frac{n^2(n-1)^2}{4} \langle w_2, w_2 \rangle \cdot \langle v_0, v_0 \rangle \\
&= \binom{2}{0} \cdot \binom{n}{2} + \frac{(n-1)^2}{4} \binom{2}{1} \binom{n}{1} + \frac{n^2(n-1)^2}{4} \binom{2}{2} \cdot \binom{n}{0} \\
&= \frac{n(n-1)}{2} + \frac{n(n-1)^2}{2} + \frac{n^2(n-1)^2}{4} \\
&= n(n-1) \frac{2 + 2(n-1) + n(n-1)}{4} \\
&= n(n-1) \frac{n^2+n}{4} = \frac{n^2(n+1)(n-1)}{4}.
\end{aligned}$$

Thus we get exactly the results of eq. (1.25).

Need to show $\langle s_k, s_k \rangle = \langle s_0, s_0 \rangle \binom{n+2}{k}$ and more

A.4 Finding $\overline{w}_1 \otimes \overline{v}_k$

We want to find $\overline{w}_1 \otimes \overline{v}_k$ in terms of $\overline{s}_k$, $\overline{t}_k$, and $\overline{u}_k$ from eqs. (1.26) and (1.27). First we note that for $0 < k < n$

$$\begin{aligned}
\sqrt{2\binom{n}{k}}\, \overline{w}_1 \otimes \overline{v}_k &= \sqrt{\binom{2}{1}}\, \overline{w}_1 \otimes \sqrt{\binom{n}{k}}\, \overline{v}_k \\
&= w_1 \otimes v_k \\
&= \frac{2(k+1)(n+1-k)}{(n+1)(n+2)} s_{k+1} - \frac{2(n-2k)}{n(n+2)} t_k - \frac{4}{n(n+1)} u_{k-1} \\
&= \frac{2(k+1)(n+1-k)}{(n+1)(n+2)} \sqrt{\binom{n+2}{k+1}}\, \overline{s}_{k+1} \\
&\quad - \frac{2(n-2k)}{n(n+2)} \sqrt{\frac{n(n+2)}{2} \binom{n}{k}}\, \overline{t}_k \\
&\quad - \frac{4}{n(n+1)} \sqrt{\frac{n^2(n+1)(n-1)}{4} \binom{n-2}{k-1}}\, \overline{u}_{k-1}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{2(k+1)(n+1-k)}{(n+1)(n+2)}\sqrt{\binom{n+2}{k+1}}\overline{s}_{k+1} \\
&\quad - \frac{\sqrt{2}(n-2k)}{\sqrt{n(n+2)}}\sqrt{\binom{n}{k}}\overline{t}_k \\
&\quad - \frac{2\sqrt{(n-1)}}{\sqrt{(n+1)}}\sqrt{\binom{n-2}{k-1}}\overline{u}_{k-1}.
\end{aligned}
$$

Now since

$$
\frac{\binom{n+2}{k+1}}{\binom{n}{k}} = \frac{(n+2)(n+1)}{(k+1)(n+1-k)}, \qquad \frac{\binom{n-2}{k-1}}{\binom{n}{k}} = \frac{k(n-k)}{n(n-1)},
$$

we see that

$$
\begin{aligned}
\overline{w}_1 \otimes \overline{v}_k &= \frac{\sqrt{2}(k+1)(n+1-k)}{(n+1)(n+2)}\sqrt{\frac{(n+2)(n+1)}{(k+1)(n+1-k)}}\overline{s}_{k+1} \\
&\quad - \frac{(n-2k)}{\sqrt{n(n+2)}}\overline{t}_k \\
&\quad - \frac{\sqrt{2(n-1)}}{\sqrt{(n+1)}}\sqrt{\frac{k(n-k)}{n(n-1)}}\overline{u}_{k-1} \\
&= \sqrt{\frac{2(k+1)(n+1-k)}{(n+1)(n+2)}}\overline{s}_{k+1} - \frac{(n-2k)}{\sqrt{n(n+2)}}\overline{t}_k \\
&\quad - \sqrt{\frac{2k(n-k)}{n(n+1)}}\overline{u}_{k-1}.
\end{aligned}
$$

Also since eq. (1.22) is a special case of eq. (1.21) the above formula also holds for $k \in \{0, n\}$ if we take the coefficient in front of $\overline{u}_{k-1}$ to be 0. Thus we indeed get eq. (1.28)

A.5 F_3, F_+, F_- in terms of E_+, E_-, D_0, D_+, D_-

We have already seen that

$$
F_3\xi = \sqrt{\ell^2 - m^2}D_-\xi - mD_0\xi - \sqrt{(\ell+1)^2 - m^2}D_+\xi
$$

for $\xi \in R_{\ell,m}$ by using eq. (1.30) and the definition of how we expanded D_0, D_+, and D_- to maps on all of M. Now we get by eqs. (1.2) and (1.10) and the commutative diagrams in eq. (1.11) that

$$
\begin{aligned}
F_+\xi &= [F_3, H_+]\xi = F_3 H_+\xi - H_+F_3\xi \\
&= \sqrt{(\ell+m+1)(\ell-m)}F_3E_+\xi - \sqrt{\ell^2-m^2}H_+D_-\xi + mH_+D_0\xi \\
&\quad + \sqrt{(\ell+1)^2-m^2}H_+D_+\xi \\
&= \sqrt{(\ell+m+1)(\ell-m)}\Big(\sqrt{\ell^2-(m+1)^2}D_-E_+\xi - (m+1)D_0E_+\xi \\
&\qquad\qquad\qquad\qquad\qquad - \sqrt{(\ell+1)^2-(m+1)^2}D_+E_+\xi\Big) \\
&\quad - \sqrt{\ell^2-m^2}\sqrt{((\ell-1)+m+1)((\ell-1)-m)}E_+D_-\xi \\
&\quad + m\sqrt{(\ell+m+1)(\ell-m)}E_+D_0\xi \\
&\quad + \sqrt{(\ell+1)^2-m^2}\sqrt{((\ell+1)+m+1)((\ell+1)-m)}E_+D_+\xi \\
&= \sqrt{(\ell+m+1)(\ell-m)}\Big(\sqrt{\ell^2-(m+1)^2}D_-E_+\xi - (m+1)D_0E_+\xi \\
&\qquad\qquad\qquad\qquad\qquad - \sqrt{(\ell+1)^2-(m+1)^2}D_+E_+\xi\Big) \\
&\quad - \sqrt{\ell^2-m^2}\sqrt{(\ell+m)(\ell-m-1)}D_-E_+\xi \\
&\quad + m\sqrt{(\ell+m+1)(\ell-m)}D_0E_+\xi \\
&\quad + \sqrt{(\ell+1)^2-m^2}\sqrt{(\ell+m+2)(\ell-m+1)}D_+E_+\xi \\
&= \Big(\sqrt{(\ell+m+1)(\ell-m)(\ell^2-(m+1)^2)} \\
&\qquad - \sqrt{(l^2-m^2)(l+m)(l-m-1)}\Big)D_-E_+\xi \\
&\quad - \sqrt{(\ell+m+1)(\ell-m)}D_0E_+\xi \\
&\quad + \Big(\sqrt{((\ell+1)^2-m^2)(\ell+m+2)(\ell-m+1)} \\
&\qquad - \sqrt{(\ell+m+1)(\ell-m)((\ell+1)^2-(m+1)^2)}\Big)D_+E_+\xi
\end{aligned}
$$

for $\xi \in R_{\ell,m}$ and $-\ell+1 \le m < \ell-1$. In the case where $m = -\ell$ the only problem is at the term with E_+D_-, but this is not a problem because the term vanishes since there is $\ell+m$ as part of the coefficient, so the formula also holds true in this case. In case $m = \ell-1$ the only problem is at the term with D_-E_+, but here we have $\ell^2-(m+1)^2$ as part of the coefficient, so this term also vanishes, and the formula also hold true in this case. Finally in case $m = \ell$ the terms with D_-E_+, D_0E_+, D_+E_+, E_+D_-, and E_+D_0 all cause problems, but again all of these terms vanish, so the formula still holds true in this case. Now by pure algebraic manipulation note that

I have checked this in Mathematica, but I would prefer not to write this out, although I can do it later if necessary

$$
\begin{aligned}
&\sqrt{(\ell+m+1)(\ell-m)(\ell^2-(m+1)^2)} - \sqrt{(l^2-m^2)(l+m)(l-m-1)} \\
&= \sqrt{(\ell-m)(\ell-m-1)}
\end{aligned}
$$

and

$$\begin{aligned}
&\sqrt{((\ell+1)^2-m^2)(\ell+m+2)(\ell-m+1)}\\
&\qquad-\sqrt{(\ell+m+1)(\ell-m)((\ell+1)^2-(m+1)^2)}\\
&=\sqrt{(\ell+m+1)((\ell+m+2))},
\end{aligned}$$

so we get that

$$\begin{aligned}
F_+\xi=&\sqrt{(\ell-m)(\ell-m-1)}D_-E_+\xi-\sqrt{(\ell+m+1)(\ell-m)}D_0E_+\xi\\
&-\sqrt{(\ell+m+1)(\ell+m+2)}D_+E_+\xi
\end{aligned}$$

for $\xi\in R_{\ell,m}$ and $-\ell\leq m\leq\ell$.

Likewise we get that

$$\begin{aligned}
F_-\xi&=[H_-,F_3]\xi=H_-F_3\xi-F_3H_-\xi\\
&=\sqrt{\ell^2-m^2}H_-D_-\xi-mH_-D_0\xi-\sqrt{(\ell+1)^2-m^2}H_-D_+\xi\\
&\quad-\sqrt{(\ell+m)(\ell-m+1)}F_3E_-\xi\\
&=\sqrt{\ell^2-m^2}\sqrt{((\ell-1)+m)((\ell-1)-m+1)}E_-D_-\\
&\quad-m\sqrt{(\ell+m)(\ell-m+1)}E_-D_0\xi\\
&\quad-\sqrt{(\ell+1)^2-m^2}\sqrt{((\ell+1)+m)((\ell+1)-m+1)}E_-D_+\\
&\quad-\sqrt{(\ell+m)(\ell-m+1)}\Big(\sqrt{\ell^2-(m-1)^2}D_-E_-\xi-(m-1)D_0E_-\xi\\
&\quad-\sqrt{(\ell+1)^2-(m-1)^2}D_+E_-\xi\Big)\\
&=\sqrt{\ell^2-m^2}\sqrt{(\ell+m-1)(\ell-m)}D_-E_--m\sqrt{(\ell+m)(\ell-m+1)}D_0E_-\xi\\
&\quad-\sqrt{(\ell+1)^2-m^2}\sqrt{(\ell+m+1)(\ell-m+2)}D_+E_-\\
&\quad-\sqrt{(\ell+m)(\ell-m+1)}\Big(\sqrt{\ell^2-(m-1)^2}D_-E_-\xi-(m-1)D_0E_-\xi\\
&\quad-\sqrt{(\ell+1)^2-(m-1)^2}D_+E_-\xi\Big)\\
&=-\Big(\sqrt{(\ell+m)(\ell-m+1)(\ell^2-(m-1)^2)}\\
&\qquad-\sqrt{(\ell^2-m^2)(\ell+m-1)(\ell-m)}\Big)D_-E_-\xi\\
&\quad-\sqrt{(\ell+m)(\ell-m+1)}D_0E_-\xi\\
&\quad-\Big(\sqrt{((\ell+1)^2-m^2)(\ell+m+1)(\ell-m+2)}\\
&\qquad-\sqrt{(\ell+m)(\ell-m+1)((\ell+1)^2-(m-1)^2)}\Big)D_+E_-\xi
\end{aligned}$$

for $\xi\in R_{\ell,m}$ and $-\ell+1<m\leq\ell-1$. Again note that by the problematic terms vanish in such a way that this formula holds true for all m with $-\ell\leq m\leq\ell$.

Also note that

$$\begin{aligned}&\sqrt{(\ell+m)(\ell-m+1)(\ell^2-(m-1)^2)}-\sqrt{(\ell^2-m^2)(\ell+m-1)(\ell-m)}\\&=\sqrt{(\ell+m)(\ell+m-1)}\end{aligned}$$

and

$$\begin{aligned}&\sqrt{((\ell+1)^2-m^2)(\ell+m+1)(\ell-m+2)}\\&\quad-\sqrt{(\ell+m)(\ell-m+1)((\ell+1)^2-(m-1)^2)}\\&=\sqrt{(\ell-m+1)(\ell-m+2)},\end{aligned}$$

so we get that

$$\begin{aligned}F_-\xi=&-\sqrt{(\ell+m)(\ell+m-1)}D_-E_-\xi-\sqrt{(\ell+m)(\ell-m+1)}D_0E_-\xi\\&-\sqrt{(\ell-m+1)(\ell-m+2)}D_+E_-\xi\end{aligned}$$

for $\xi\in R_{\ell,m}$. Thus indeed we get eq. (1.31).

A.6 Relations for D_0, D_+, D_-

We want to show that the formulae eq. (1.31) for the linear operators F_+, F_-, and F_3 together with the formulae eqs. (1.8) and (1.10) for H_+, H_-, and H_3 define a representation of L, i.e. they satisfy the commutation relations of eq. (1.2), if and only if D_0, D_+, and D_- satisfy eq. (1.32). By eqs. (1.2) and (1.10) ...

Write calculations here

A.7 Finding d_ℓ^-

We want to find d_ℓ^- in general given that we already know that $d_{\ell_0}^-=0$ and

$$(2\ell-1)d_\ell^--(2\ell+3)d_{\ell+1}^-=1-\frac{\ell_0^2\ell_1^2}{\ell^2(\ell+1)^2}.$$

Multiplying the left side of the above equation by $2\ell+1$ we get

$$(4\ell^2-1)d_\ell^--(4\ell^1+2\ell-3)d_{\ell+1}^-=(4\ell^2-1)d_\ell^--(4(\ell+1)^2-1)d_{\ell+1}^-$$

and multiplying the right side by $2\ell+1$ we get

$$2\ell+1-\ell_0^2\ell_1^2\frac{2\ell+1}{\ell^2(\ell+1)^2}=2\ell+1-\ell_0^2\ell_1^2\Big(\frac{1}{\ell^2}-\frac{1}{(\ell+1)^2}\Big),$$

so we see that

$$(4\ell^2-1)d_\ell^--(4(\ell+1)^2-1)d_{\ell+1}^-=2\ell+1-\ell_0^2\ell_1^2\Big(\frac{1}{\ell^2}-\frac{1}{(\ell+1)^2}\Big).\qquad\text{(A.1)}$$

Now we know that $d^-_{\ell_0} = 0$, so

$$\begin{aligned}
-(4(\ell_0+1)^2-1)d^-_{\ell_0+1} &= 2\ell_0 + 1 - \ell_1^2\Big(1 - \frac{\ell_0^2}{(\ell_0+1)^2}\Big) \\
&= (\ell_0+1)^2 - \ell_0^2 - \ell_1^2\frac{(\ell_0+1)^2-\ell_0^2}{(\ell_0+1)^2} \\
&= \frac{\big((\ell_0+1)^2-\ell_1^2\big)\big((\ell_0+1)^2-\ell_0^2\big)}{(\ell_0+1)^2},
\end{aligned}$$

and thus

$$d^-_{\ell_0+1} = -\frac{\big((\ell_0+1)^2-\ell_1^2\big)\big((\ell_0+1)^2-\ell_0^2\big)}{(\ell_0+1)^2(4(\ell_0+1)^2-1)}.$$

We get inductively that

$$d^-_\ell = -\frac{\big(\ell^2-\ell_1^2\big)\big(\ell^2-\ell_0^2\big)}{\ell^2(4\ell^2-1)},$$

for $\ell > \ell_0$, since we already have the base case, and assuming the equality for $\ell > \ell_0$ we get by eq. (A.1) that

$$\begin{aligned}
&-(4(\ell+1)^2-1)d^-_{\ell+1} \\
&\qquad= \frac{\big(\ell^2-\ell_1^2\big)\big(\ell^2-\ell_0^2\big)}{\ell^2} + 2\ell + 1 - \ell_0^2\ell_1^2\Big(\frac{1}{\ell^2} - \frac{1}{(\ell+1)^2}\Big) \\
&\qquad= \frac{(\ell+1)^2(\ell^2-\ell_1^2)(\ell^2-\ell_0^2) + \ell^2(\ell+1)^2(2\ell+1) - \ell_0^2\ell_1^2(2\ell+1)}{\ell^2(\ell+1)^2}.
\end{aligned}$$

So since

$$\begin{aligned}
&(\ell+1)^2(\ell^2-\ell_1^2)(\ell^2-\ell_0^2) + \ell^2(\ell+1)^2(2\ell+1) - \ell_0^2\ell_1^2(2\ell+1) \\
&\quad= \ell^2(\ell^2-\ell_1^2)(\ell^2-\ell_0^2) \\
&\qquad+ (2\ell+1)\big((\ell^2-\ell_1^2)(\ell^2-\ell_0^2) + \ell^2(\ell+1)^2 - \ell_0^2\ell_1^2\big) \\
&\quad= \ell^2(\ell^2-\ell_1^2)(\ell^2-\ell_0^2) \\
&\qquad+ (2\ell+1)\big(\ell^4 - \ell^2\ell_0^2 - \ell^2\ell_1^2 + \ell^2(\ell+1)^2\big) \\
&\quad= \ell^2\Big((\ell^2-\ell_1^2)(\ell^2-\ell_0^2) \\
&\qquad\qquad+ (2\ell+1)\big(\ell^2 - \ell_0^2 - \ell_1^2 + (\ell+1)^2\big)\Big)
\end{aligned}$$

and

$$\begin{aligned}
\big((\ell+1)^2-\ell_1^2\big)\big((\ell+1)^2-\ell_0^2\big) &= \big(\ell^2-\ell_1^2+2\ell+1\big)\big(\ell^2-\ell_0^2+2\ell+1\big)\\
&= (\ell^2-\ell_1^2)(\ell^2-\ell_0^2)\\
&\quad +(2\ell+1)\big((\ell^2-\ell_0^2+2\ell+1)+(\ell^2-\ell_1^2)\big)\\
&= (\ell^2-\ell_1^2)(\ell^2-\ell_0^2)\\
&\quad +(2\ell+1)\big((\ell+1)^2-\ell_0^2+\ell^2-\ell_1^2\big),
\end{aligned}$$

we see that

$$-(4(\ell+1)^2-1)d^-_{\ell+1} = \frac{\big((\ell+1)^2-\ell_1^2\big)\big((\ell+1)^2-\ell_0^2\big)}{(\ell+1)^2},$$

and thus indeed

$$d^-_{\ell+1} = -\frac{\big((\ell+1)^2-\ell_1^2\big)\big((\ell+1)^2-\ell_0^2\big)}{(\ell+1)^2(4(\ell+1)^2-1)}.$$

A.8 Finding $\Delta_1\xi$ and $\Delta_2\xi$

We have

$$\begin{aligned}
\Delta_1 &\coloneqq \tfrac{1}{2}(H_-F_+ + F_-H_+) + H_3F_3 + F_3\\
\Delta_2 &\coloneqq H_-H_+ - F_-F_+ + H_3^2 - F_3^2 + 2H_3
\end{aligned}$$

as in eq. (1.41), and we want to find $\Delta_1\xi$ and $\Delta_2\xi$ for $\xi \in R_{\ell,m}$. By eqs. (1.8), (1.9) and (1.31) we see that

$$\begin{aligned}
&\Delta_1\xi\\
&= \tfrac{1}{2}H_-F_+\xi + \tfrac{1}{2}F_-H_+\xi + H_3F_3\xi + F_3\xi\\
&= \tfrac{1}{2}\sqrt{(\ell-m)(\ell-m-1)}H_-D_-E_+\xi\\
&\quad -\tfrac{1}{2}\sqrt{(\ell-m)((\ell+m+1))}H_-D_0E_+\xi\\
&\quad +\tfrac{1}{2}\sqrt{(\ell+m+1)(\ell+m+2)}H_-E_+D_+\xi\\
&\quad +\tfrac{1}{2}\sqrt{(\ell+m+1)(\ell-m)}F_-E_+\xi\\
&\quad +\sqrt{\ell^2-m^2}H_3D_-\xi - mH_3D_0\xi - \sqrt{(\ell+1)^2-m^2}H_3D_+\xi\\
&\quad +\sqrt{\ell^2-m^2}D_-\xi - mD_0\xi - \sqrt{(\ell+1)^2-m^2}D_+\xi
\end{aligned}$$

$$
\begin{aligned}
&= \tfrac{1}{2}\sqrt{(\ell-m)(\ell-m-1)} \\
&\qquad \cdot \sqrt{((\ell-1)+(m+1))((\ell-1)-(m+1)+1)}E_-D_-E_+\xi \\
&\quad - \tfrac{1}{2}\sqrt{(\ell-m)((\ell+m+1))}\sqrt{(\ell+(m+1))(\ell-(m+1)+1)}E_-D_0E_+\xi \\
&\quad + \tfrac{1}{2}\sqrt{(\ell+m+1)(\ell+m+2)} \\
&\qquad\quad \cdot \sqrt{((\ell+1)+(m+1))((\ell+1)-(m+1)+1)}E_-E_+D_+\xi \\
&\quad + \tfrac{1}{2}\sqrt{(\ell+m+1)(\ell-m)}\Big(-\sqrt{(\ell+(m+1))(\ell+(m+1)-1)}D_-E_-E_+\xi \\
&\qquad - \sqrt{(\ell+(m+1))(\ell-(m+1)+1)}D_0E_-E_+\xi \\
&\qquad - \sqrt{(\ell-(m+1)+1)(\ell-(m+1)+2)}E_-D_+E_+\xi\Big) \\
&\quad + \sqrt{\ell^2-m^2}mD_-\xi - m\cdot mD_0\xi - \sqrt{(\ell+1)^2-m^2}mD_+\xi \\
&\quad + \sqrt{\ell^2-m^2}D_-\xi - mD_0\xi - \sqrt{(\ell+1)^2-m^2}D_+\xi \\
&= \tfrac{1}{2}(\ell-m-1)\sqrt{\ell^2-m^2}D_-\xi - \tfrac{1}{2}(\ell-m)((\ell+m+1))D_0\xi \\
&\quad + \tfrac{1}{2}(\ell+m+2)\sqrt{(\ell+1)^2-m^2}D_+\xi \\
&\quad + \tfrac{1}{2}\sqrt{(\ell+m+1)(\ell-m)}\Big(-\sqrt{(\ell+m+1)(\ell+m)}D_-\xi \\
&\qquad - \sqrt{(\ell+m+1)(\ell-m)}D_0\xi - \sqrt{(\ell-m)(\ell-m+1)}D_+\xi\Big) \\
&\quad + \sqrt{\ell^2-m^2}mD_-\xi - m^2D_0\xi - \sqrt{(\ell+1)^2-m^2}mD_+\xi \\
&\quad + \sqrt{\ell^2-m^2}D_-\xi - mD_0\xi - \sqrt{(\ell+1)^2-m^2}D_+\xi \\
&= \tfrac{1}{2}(\ell-m-1)\sqrt{\ell^2-m^2}D_-\xi - \tfrac{1}{2}(\ell-m)(\ell+m+1)D_0\xi \\
&\quad + \tfrac{1}{2}(\ell+m+2)\sqrt{(\ell+1)^2-m^2}D_+\xi - \tfrac{1}{2}(\ell+m+1)\sqrt{\ell^2-m^2}D_-\xi \\
&\quad - \tfrac{1}{2}(\ell+m+1)(\ell-m)D_0\xi - \tfrac{1}{2}(\ell-m)\sqrt{(\ell+1)^2-m^2}D_+\xi \\
&\quad + \sqrt{\ell^2-m^2}mD_-\xi - m^2D_0\xi - \sqrt{(\ell+1)^2-m^2}mD_+\xi \\
&\quad + \sqrt{\ell^2-m^2}D_-\xi - mD_0\xi - \sqrt{(\ell+1)^2-m^2}D_+\xi \\
&= \Big(\tfrac{1}{2}(\ell-m-1) - \tfrac{1}{2}(\ell+m+1) + m + 1\Big)\sqrt{\ell^2-m^2}D_-\xi \\
&\quad + \Big(-\tfrac{1}{2}(\ell-m)(\ell+m+1) - \tfrac{1}{2}(\ell+m+1)(\ell-m) - m^2 - m\Big)D_0\xi \\
&\quad + \Big(\tfrac{1}{2}(\ell+m+2) - \tfrac{1}{2}(\ell-m) - m - 1\Big)\sqrt{(\ell+1)^2-m^2}D_+\xi \\
&= 0 + (-\ell^2 - \ell + m^2 + m - m^2 - m)D_0\xi + 0 \\
&= -\ell(\ell+1)D_0\xi
\end{aligned}
$$

for $\xi \in R_{\ell,m}$, where $-\ell+1 \le m \le \ell-1$. Now as in Appendix A.5, we note that the coefficients causing problems in the edge cases vanish, so we get the above equality for all m, and the formula is independent of m, we see that we actually have

$$\Delta_1\xi = -\ell(\ell+1)D_0\xi$$

for all $\xi \in R_\ell$.

Similar calculations show that

$$\Delta_2\xi = (\ell^2-1)\xi - (\ell+1)^2 D_0^2\xi + (4\ell^2-1)D_+D_-\xi$$

for all $\xi \in R_\ell$.

Additionally by eq. (1.32) we have that $\xi = (2\ell-1)D_+D_-\xi - (2\ell+3)D_-D_+\xi - D_0^2\xi$, so we get that

$$\begin{aligned}(4\ell^2-1)D_+D_-\xi &= (2\ell+1)(2\ell-1)D_+D_-\xi \\ &= (2\ell+1)\xi + (2\ell+1)(2\ell+3)D_-D_+\xi + (2\ell+1)D_0^2\xi \\ &= (2\ell+1)\xi + (4(\ell+1)^2-1)D_-D_+\xi + (2\ell+1)D_0^2\xi\end{aligned}$$

for $\xi \in R_\ell$ since $(2\ell+1)(2\ell+3) = (2(\ell+1)-1)(2(\ell+1)+1) = 4(\ell+1)^2-1$, and therefore also

$$\begin{aligned}\Delta_2\xi &= (\ell^2-1)\xi - (\ell+1)^2 D_0^2\xi + (2\ell+1)\xi + (4(\ell+1)^2-1)D_-D_+\xi + (2\ell+1)D_0^2\xi \\ &= \big((\ell+1)^2-1\big)\xi + \ell^2 D_0^2\xi + \big(4(\ell+1)^2-1\big)D_-D_+\xi\end{aligned}$$

for $\xi \in R_\ell$.

Appendix B

Auxiliary results

In this appendix we will collect the proofs of some auxiliary results that we will need in the paper.

B.1 $Z(U(L_1 \times L_2)) \simeq Z(U(L_1)) \otimes Z(U(L_2))$

Let $L = L_1 \times L_2$ be a product of two Lie algebras, and let $\iota_1 \colon L_1 \to U(L_1)$, $\iota_2 \colon L_2 \to U(L_2)$, and $\iota \colon L \to U(L)$ be the canonical homomorphisms of Lie algebras, we get from the universal property of universal enveloping algebras. We want to show first that $U(L) \simeq U(L_1) \otimes U(L_2)$.

Consider the map

$$\rho \colon L \to U(L_1) \otimes U(L_2), \qquad (u_1, u_2) \mapsto \iota_1(u_1) \otimes 1 + 1 \otimes \iota_2(u_2),$$

which is a homomorphisms of Lie algebras since it is clearly linear and

$$\begin{aligned}
[\rho(u_1, u_2), \rho(v_1, v_2)] &= [u_1 \otimes 1 + 1 \otimes u_2, v_1 \otimes 1 + 1 \otimes v_2] \\
&= (u_1 \otimes 1 + 1 \otimes u_2)(v_1 \otimes 1 + 1 \otimes v_2) \\
&\quad - (v_1 \otimes 1 + 1 \otimes v_2)(u_1 \otimes 1 + 1 \otimes u_2) \\
&= u_1 v_1 \otimes 1 + u_1 \otimes v_2 + v_1 \otimes u_2 + 1 \otimes u_2 v_2 \\
&\quad - v_1 u_1 \otimes 1 - v_1 \otimes u_2 - u_1 \otimes v_2 - 1 \otimes v_2 u_2 \\
&= (u_1 v_1 - v_1 u_1) \otimes 1 + 1 \otimes (u_2 v_2 - v_2 u_2) \\
&= [u_1, v_1] \otimes 1 + 1 \otimes [u_2, v_2] \\
&= \rho([u_1, v_1], [u_2, v_2]) \\
&= \rho([(u_1, u_2), (v_1, v_2)])
\end{aligned}$$

for $(u_1, u_2), (v_1, v_2) \in L$ by the definition of the tensor product of an algebra. Thus by the universal property of $(U(L), \iota)$ we get a unique homomorphisms of associative algebras $\varphi \colon U(L) \to U(L_1) \otimes U(L_2)$ such that the following diagram commutes:

$$\begin{tikzcd} L \arrow[r, "\iota"] \arrow[dr, "\rho"'] & U(L) \arrow[d, dashed, "\varphi"] \\ & U(L_1) \otimes U(L_2) \end{tikzcd}$$

Now let $i_1 \colon L_1 \to L$ be the inclusion of L_1 into L given by $u \mapsto (u, 0)$ for $u \in L_1$. By the definition of the bracket on $L = L_1 \times L_2$ it is easy to see that i_1 is a Lie algebra homomorphism, and thus the map $\iota \circ i_1 \colon L_1 \to L \to U(L)$ is also a Lie algebra homomorphism. Hence by the universal property of $(U(L_1), \iota_1)$ we get a unique homomorphism of associative algebras $\psi_1 \colon U(L_1) \to U(L)$ such that the following diagram commutes:

$$\begin{tikzcd} L_1 \arrow[r, "\iota_1"] \arrow[dr, "\iota \circ i_1"'] & U(L_1) \arrow[d, dashed, "\psi_1"] \\ & U(L) \end{tikzcd}$$

Likewise we get a unique homomorphism of associative algebras $\psi_2 \colon U(L_2) \to U(L)$ such that $\iota \circ i_2 = \psi_1 \circ \iota_2$. Now since $[(u_1, 0), (0, u_2)] = ([u_1, 0], [0, u_2]) = 0$ for $u_1 \in L_1$ and $u_2 \in L_2$, we see that

$$\begin{aligned} 0 &= \iota([(u_1, 0), (0, u_2)]) = [\iota i_1(u_1), \iota i_2(u_2)] = [\psi_1 \iota_1(u_1), \psi_2 \iota_2(u_2)] \\ &= \psi_1 \iota_1(u_1) \psi_2 \iota_2(u_2) - \psi_2 \iota_2(u_2) \psi_1 \iota_1(u_1). \end{aligned}$$

Thus $\psi_1 \iota_1(u_1) \psi_2 \iota_2(u_2) = \psi_2 \iota_2(u_2) \psi_1 \iota_1(u_1)$ for all $u_1 \in L_1$ and $u_2 \in L_2$. Hence since the $\iota_j(u_j)$ for $u_j \in L_j$ generate $U(L_j)$ by the PBW theorem for $j = 1, 2$, cf. [Jan16, p. E-7], we get that $\psi_1(u_1)\psi_2(u_2) = \psi_2(u_2)\psi_1(u_1)$ for all $u_1 \in U(L_1)$ and $u_2 \in U(L_2)$. Therefore the map

$$\psi \colon U(L_1) \otimes U(L_2) \to U(L), \qquad u_1 \otimes u_2 \mapsto \psi_1(u_1)\psi_2(u_2), \tag{B.1}$$

is a homomorphism of associative algebras since

$$\begin{aligned} \psi((u_1 \otimes u_2)(v_1 \otimes v_2)) &= \psi(u_1 v_1 \otimes v_1 v_2) = \psi_1(u_1 v_1)\psi_2(u_2 v_2) \\ &= \psi_1(u_1)\psi_1(v_1)\psi_2(u_2)\psi_2(v_2) \\ &= \psi_1(u_1)\psi_2(u_2)\psi_1(v_1)\psi_2(v_2) \\ &= \psi(u_1 \otimes u_2)\psi(v_1 \otimes v_2). \end{aligned}$$

Note now that

$$\begin{aligned} \psi\varphi\iota(u_1, u_2) &= \psi\rho(u_1, u_2) = \psi(\iota_1(u_1) \otimes 1 + 1 \otimes \iota_2(u_2)) \\ &= \psi_1 \iota_1(u_1)\psi_2(1) + \psi_1(1)\psi_2 \iota_2(u_2) \\ &= \iota(u_1, 0) + \iota(0, u_2) = \iota(u_1, u_2) \end{aligned}$$

for all $(u_1, u_2) \in L$, so by the PBW theorem as above we get that $\psi\varphi = \mathrm{id}_{U(L)}$. Likewise

$$\begin{aligned}\varphi\psi(\iota_1(u_1) \otimes 1 + 1 \otimes \iota_2(u_2)) &= \varphi(\psi_1\iota_1(u_1)\psi_2(1) + \psi_1(1)\psi_2\iota_2(u_2)) \\ &= \varphi(\iota(u_1, 0) + \iota(0, u_2)) = \varphi\iota(u_1, u_2) \\ &= \rho(u_1, u_2) = \iota(u_1) \otimes 1 + 1 \otimes \iota_2(u_2)\end{aligned}$$

for all $u_1 \in L_1$ and $u_2 \in L_2$. Now by the PBW theorem the $\iota_1(u_1)$ for $u_1 \in L_1$ generate $U(L_1)$ and the $\iota_2(u_2)$ for $u_2 \in L_2$ generate $U(L_2)$, so we see that the $\iota_1(u_1) \otimes 1 + 1 \otimes \iota_2(u_2)$ for $u_1 \in L_1$ and $u_2 \in L_2$ generate $U(L_1) \otimes U(L_2)$ and thus $\varphi\psi = \mathrm{id}_{U(L_1)\otimes U(L_2)}$. Hence we see that φ and ψ are isomorphisms between $U(L)$ and $U(L_1) \otimes U(L_2)$, so indeed $U(L) \simeq U(L_1) \otimes U(L_2)$.

Note that the above also gives us an isomorphism $Z(U(L)) \simeq Z(U(L_1) \otimes U(L_2))$. Now we want to show that we also have that $Z(U(L_1) \otimes U(L_2)) = Z(U(L_1)) \otimes Z(U(L_2))$ such that when describing $Z(U(L))$ we can instead describe $Z(U(L_1)) \otimes Z(U(L_2))$. For $z_1 \otimes z_2 \in Z(U(L_1)) \otimes Z(U(L_2))$ we get that

$$(z_1 \otimes z_2)(u_1 \otimes u_2) = z_1u_1 \otimes z_2u_2 = u_1z_1 \otimes u_2z_2 = (u_1 \otimes u_2)(z_1 \otimes z_2)$$

for all $u_1 \otimes u_2 \in U(L_1) \otimes U(L_2)$, so we have the inclusion $Z(U(L_1)) \otimes Z(U(L_2)) \subseteq Z(U(L_1) \otimes U(L_2))$.

To get the other inclusion let $z = \sum_i u_i \otimes v_i \in Z(U(L_1) \otimes U(L_2))$. By combining terms with linearly dependent v_i's, we can assume that the v_i's in the sum are linearly independent. Now for $u \otimes 1 \in U(L_1) \otimes U(L_2)$ we have that $z(u \otimes 1) = (u \otimes 1)z$, so

$$0 = z(u \otimes 1) - (u \otimes 1)z = \sum_i (u_iu - uu_i) \otimes v_i.$$

Thus since the v_i's are linearly independent, we must have that $u_iu - uu_i = 0$ for all i, i.e. $u_i \in Z(U(L_1))$ for all i. Likewise we get that $v_i \in Z(U(L_2))$ for all i, and hence $z = \sum_i u_i \otimes v_i \in Z(U(L_1)) \otimes Z(U(L_2))$. Therefore we get the inclusion $Z(U(L_1) \otimes U(L_2)) \subseteq Z(U(L_1)) \otimes Z(U(L_2))$, and thus indeed we have the equality $Z(U(L_1) \otimes U(L_2)) = Z(U(L_1)) \otimes Z(U(L_2))$. So altogether we have an isomorphism $Z(U(L)) \simeq Z(U(L_1)) \otimes Z(U(L_2))$.