Bachelorprojekt

Title

(subtitle)

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Abstract

Some text

Contents

Chapter 1

Harish-Chandra modules over $\mathfrak{sl}(2, \mathbf{C}) \times \mathfrak{sl}(2, \mathbf{C})$

Let G be a semisimple Lie group and let G_k be its maximal compact subgroup. Denote by L the semisimple Lie algebra of G and denote by L_k the Lie subalgebra corresponding to G_k.

Definition 1.1. *An L-module M is a Harish-Chandra module if, regarded as an L_k-module, it can be written as a sum*

$$M = \bigoplus_i M_i$$

of finite dimensional irreducible L_k-submodules M_i, where for each M_{i_0} only finitely many L_k-submodules equivalent to M_{i_0} occur in the decomposition of M.

A Harish-Chandra module M is indecomposable if it cannot be decomposed into the direct sum of L-submodules.

Our goal is to classify all indecomposable Harish-Chandra modules over $\mathfrak{sl}(2, \mathbf{C}) \times \mathfrak{sl}(2, \mathbf{C})$, where we by $\mathfrak{sl}(2, \mathbf{C}) \times \mathfrak{sl}(2, \mathbf{C})$ mean the following:

For L, L' Lie algebras over F, we consider $L \times L' = L \oplus L'$ as a Lie algebra over F with pointwise addition, multiplication given by $\alpha(a, b) = (\alpha a, \alpha b)$ for $\alpha \in F, a \in L, b \in L'$, and with Lie bracket $[(a_1, b_1), (a_2, b_2)] = ([a_1, a_2], [b_1, b_2])$ for $a_1, a_2 \in L, b_1, b_2 \in L'$.

Remark 1.2. Note that $L \times 0$ and $0 \times L'$ are ideals in $L \times L'$ as given above. Thus we see that $\mathfrak{sl}(2, \mathbf{C}) \times 0$ and $0 \times \mathfrak{sl}(2, \mathbf{C})$ are ideals in $\mathfrak{sl}(2, \mathbf{C}) \times \mathfrak{sl}(2, \mathbf{C})$ with

$$(\mathfrak{sl}(2, \mathbf{C}) \times 0) \oplus (0 \times \mathfrak{sl}(2, \mathbf{C})) = \mathfrak{sl}(2, \mathbf{C}) \times \mathfrak{sl}(2, \mathbf{C}),$$

so $\mathfrak{sl}(2, \mathbf{C}) \times \mathfrak{sl}(2, \mathbf{C})$ is semisimple. Hence it makes sense to talk about Harish-Chandra modules over $\mathfrak{sl}(2, \mathbf{C}) \times \mathfrak{sl}(2, \mathbf{C})$. $\triangle$

We fix the following as a standard basis for $\mathfrak{sl}(2,F)$:

$$x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Giving us the relations:

$$[x,y] = h, \qquad [h,x] = 2x, \qquad [h,y] = -2y, \tag{1.1}$$

cf. [Jan16, p. 35] or [Hum72, p. 6].

We claim now that

$$(x,x), \quad (y,y), \quad \tfrac{1}{2}(h,h), \quad (ix,-ix), \quad (iy,-iy), \quad \tfrac{1}{2}(ih,-ih)$$

is a basis of $\mathfrak{sl}(2,\mathbf{C}) \times \mathfrak{sl}(2,\mathbf{C})$. This is clearly the case since $\dim_{\mathbf{C}} \mathfrak{sl}(2,\mathbf{C}) = 3$, so $\dim_{\mathbf{C}} \mathfrak{sl}(2,\mathbf{C}) \times \mathfrak{sl}(2,\mathbf{C}) = 6$, and we see that the above elements span $\mathfrak{sl}(2,\mathbf{C}) \times \mathfrak{sl}(2,\mathbf{C})$; we have $\frac{1}{2}(x,x) - \frac{i}{2}(ix,-ix) = (x,0)$ and $\frac{1}{2}(x,x) + \frac{i}{2}(ix,-ix) = (0,x)$ and likewise with h and y.

Putting

$$\begin{aligned} h_+ &= (x,x), & h_- &= (y,y), & h_3 &= \tfrac{1}{2}(h,h), \\ f_+ &= (ix,-ix), & f_- &= (iy,-iy), & f_3 &= \tfrac{1}{2}(ih,-ih) \end{aligned}$$

we get the following commutation relations between these basis elements:

$$\begin{aligned}
[h_+,h_3] &= \tfrac{1}{2}([x,h],[x,h]) = \tfrac{1}{2}(-2x,-2x) = -(x,x) = -h_+, \\
[h_-,h_3] &= \tfrac{1}{2}([y,h],[y,h]) = \tfrac{1}{2}(2y,2y) = (y,y) = h_-, \\
[h_+,h_-] &= ([x,y],[x,y]) = (h,h) = 2h_3, \\
[h_+,f_+] &= ([x,ix],[x,-ix]) = 0, \\
[h_-,f_-] &= ([y,iy],[y,-iy]) = 0, \\
[h_3,f_3] &= \tfrac{1}{4}([h,ih],[h,-ih]) = 0, \\
[h_+,f_3] &= \tfrac{1}{2}([x,ih],[x,-ih]) = \tfrac{1}{2}(-2ix,2ix) = -(ix,-ix) = -f_+, \\
[h_-,f_3] &= \tfrac{1}{2}([y,ih],[y,-ih]) = \tfrac{1}{2}(2iy,-2iy) = (iy,-iy) = f_-, \\
[h_+,f_-] &= ([x,iy],[x,-iy]) = (ih,-ih) = 2f_3, \\
[h_3,f_-] &= \tfrac{1}{2}([h,iy],[h,-iy]) = \tfrac{1}{2}(-2iy,2iy) = -(iy,-iy) = -f_-, \\
[h_-,f_+] &= ([y,ix],[y,-ix]) = (-ih,ih) = -(ih,-ih) = -2f_3, \\
[h_3,f_+] &= \tfrac{1}{2}([h,ix],[h,-ix]) = \tfrac{1}{2}(2ix,-2ix) = (ix,-ix) = f_+, \\
[f_+,f_3] &= \tfrac{1}{2}([ix,ih],[-ix,-ih]) = \tfrac{1}{2}(2x,2x) = (x,x) = h_+, \\
[f_-,f_3] &= \tfrac{1}{2}([iy,ih],[-iy,-ih]) = \tfrac{1}{2}(-2y,-2y) = -(y,y) = -h_-, \\
[f_+,f_-] &= ([ix,iy],[-ix,-iy]) = (-h,-h) = -(h,h) = -2h_3.
\end{aligned} \tag{1.2}$$

Remark 1.3. Note that these are the same relations as for the complexification of the Lie algebra L of the proper Lorentz group in [GP67b, p. 5], so L is isomorphic to $\mathfrak{sl}(2,\mathbf{C}) \times \mathfrak{sl}(2,\mathbf{C})$. This explains the equivalence of the work in this paper and the work in [GP67a; GP67b; GP67c]. △

Now let $L = \mathfrak{sl}(2, \mathbf{C}) \times \mathfrak{sl}(2, \mathbf{C})$ and denote by L_k the Lie subalgebra of L with basis h_+, h_-, h_3. One can show that this corresponds to a maximal compact subgroup in the way described in definition 1.1, but that is beyond what we will do in this paper. Note that the above commutation relations gives us that

$$[h_+, h_-] = 2h_3, \qquad [2h_3, h_+] = 2h_+, \qquad [2h_3, h_-] = -2h_-$$

Comparing with eq. (1.1) we see that we have an isomorphism

$$\begin{aligned} L_k &\to \mathfrak{sl}(2, \mathbf{C}) \\ h_+ &\mapsto x \\ h_- &\mapsto y \\ 2h_3 &\mapsto h, \end{aligned} \tag{1.3}$$

so we can use $\mathfrak{sl}(2, \mathbf{C})$-theory when we want to describe L_k-modules.

1.1 Representations of L_k

Let V be a $\mathbf{C}$ vector space and $\rho\colon L_k \to \mathfrak{gl}(V)$ a representation of L_k. We will use the notation $\rho(a) = A$ for $a \in L_k$ switching to upper case letters when we talk about the representation corresponding to a given element. Note that we will switch freely between the language of representations of L_k and the language of L_k-modules.

We will start out by describing the finite dimensional simple[1] L_k-modules. Recall cf. [Jan16, p. 36] that we know from $\mathfrak{sl}(2, \mathbf{C})$-theory that for integers $n \geq 0$ there exists a unique simple $\mathfrak{sl}(2, \mathbf{C})$-module $V(n)$ of dimension $n + 1$, and $V(n)$ has a basis $(v_0, v_1, \ldots, v_n)$ such that for all i, $0 \leq i \leq n$

$$\begin{aligned} h.v_i &= (n - 2i)v_i, \\ x.v_i &= \begin{cases} (n - i + 1)v_{i-1} & \text{if } i > 0, \\ 0 & \text{if } i = 0, \end{cases} \\ y.v_i &= \begin{cases} (i + 1)v_{i+1} & \text{if } i < n, \\ 0 & \text{if } i = n. \end{cases} \end{aligned} \tag{1.4}$$

Now using the isomorphism from eq. (1.3) we see that for integers $n \geq 0$ there exists a unique simple L_k-module $M(n)$ of dimension $n + 1$, and $M(n)$

[1] In [GP67b] the word irreducible is used instead of simple, but we will only use irreducible when talking about representations in this paper.

has a basis $(v_0, v_1, \ldots, v_n)$ such that for all i, $0 \le i \le n$

$$\begin{aligned} H_3 v_i &= (\tfrac{1}{2}n - i)v_i, \\ H_+ v_i &= \begin{cases} (n-i+1)v_{i-1} & \text{if } i > 0, \\ 0 & \text{if } i = 0, \end{cases} \\ H_- v_i &= \begin{cases} (i+1)v_{i+1} & \text{if } i < n, \\ 0 & \text{if } i = n. \end{cases} \end{aligned}$$

From this we build a new basis by taking

$$w_i = \frac{1}{\sqrt{\binom{n}{i}}} v_i,$$

Note that

$$H_3 w_i = \frac{1}{\sqrt{\binom{n}{i}}} H_3 v_i = \frac{1}{\sqrt{\binom{n}{i}}} (\tfrac{1}{2}n - i)v_i = (\tfrac{1}{2}n - i)w_i$$

for all i, $0 \le i \le n$, and clearly still

$$\begin{aligned} H_+ w_0 &= 0, \\ H_- w_n &= 0. \end{aligned}$$

But for i, $0 < i \le n$

$$\begin{aligned} H_+ w_i &= \frac{1}{\sqrt{\binom{n}{i}}} H_+ v_i = \frac{1}{\sqrt{\binom{n}{i}}} (n-i+1)v_{i-1} \\ &= \sqrt{\frac{\binom{n}{i-1}}{\binom{n}{i}}} (n-i+1) \frac{1}{\sqrt{\binom{n}{i-1}}} v_{i-1} \\ &= \sqrt{\frac{i}{n-i+1}} (n-i+1) w_{i-1} = \sqrt{(n-i+1)i}\, w_{i-1}, \end{aligned}$$

and for i, $0 \le i < n$

$$\begin{aligned} H_- w_i &= \frac{1}{\sqrt{\binom{n}{i}}} H_- v_i = \frac{1}{\sqrt{\binom{n}{i}}} (i+1)v_{i+1} \\ &= \sqrt{\frac{\binom{n}{i+1}}{\binom{n}{i}}} (i+1) \frac{1}{\sqrt{\binom{n}{i+1}}} v_{i+1} \\ &= \sqrt{\frac{n-i}{i+1}} (i+1) w_{i+1} = \sqrt{(n-i)(i+1)}\, w_{i+1}. \end{aligned}$$

Finally write $\ell = \frac{1}{2}n$. We will re-index with $m = \frac{1}{2}n - i = \ell - i$ by setting

$$e_m = w_{\ell-m}$$

for $m \in \{-\ell, -\ell+1, \dots, \ell-1, \ell\}$. Thus we get

$$H_3 e_m = H_3 w_{\ell-m} = (\ell - (\ell - m))w_{\ell-m} = m e_m,$$

and since $e_\ell = w_0$ and $e_{-\ell} = w_n$ also

$$\begin{aligned} H_+ e_\ell &= 0, \\ H_- e_{-\ell} &= 0. \end{aligned}$$

And for $m \in \{-\ell, -\ell+1, \dots, \ell-2, \ell-1\}$ we get

$$\begin{aligned} H_+ e_m = H_+ w_{\ell-m} &= \sqrt{(n - (\ell - m) + 1)(\ell - m)}\, w_{\ell-m-1} \\ &= \sqrt{(\ell + m + 1)(\ell - m)}\, e_{m+1}, \end{aligned}$$

while for $m \in \{-\ell+1, -\ell+2, \dots, \ell-1, \ell\}$ we get

$$\begin{aligned} H_- e_m = H_- w_{\ell-m} &= \sqrt{(n - (\ell - m))(\ell - m + 1)}\, w_{\ell-m+1} \\ &= \sqrt{(\ell + m)(\ell - m + 1)}\, e_{m-1}. \end{aligned}$$

Thus we get the following Lemma:

Lemma 1.4. *Every simple finite dimensional L_k-module is uniquely given by a number $\ell \in \frac{1}{2}\mathbf{Z}_{\geq 0}$. For such ℓ the unique simple L_k-module $M(2\ell)$ has dimension $2\ell+1$, and $M(2\ell)$ has a basis $(e_{-\ell}, e_{-\ell+1}, \dots, e_{\ell-1}, e_\ell)$ such that for all $m \in \{-\ell, -\ell+1, \dots, \ell-1, \ell\}$ we have*

$$\begin{aligned} H_3 e_m &= m e_m, \\ H_+ e_m &= \begin{cases} \sqrt{(\ell+m+1)(\ell-m)}\, e_{m+1} & \text{if } m \neq \ell, \\ 0 & \text{if } m = \ell, \end{cases} \\ H_- e_m &= \begin{cases} \sqrt{(\ell+m)(\ell-m+1)}\, e_{m-1} & \text{if } m \neq -\ell, \\ 0 & \text{if } m = -\ell. \end{cases} \end{aligned} \tag{1.5}$$

Formulae for the operators $H_+, H_-, H_3, F_+, F_-, F_3$

Let M be a Harish-Chandra L-module. Then we have linear operators $H_+, H_-, H_3, F_+, F_-, F_3 \colon M \to M$ satisfying commutation relations as in eq. (1.2), and we want to give expressions for these in terms of other linear operators $E_+, E_-, D_+, D_-, D_0 \colon M \to M$.

We will denote by R_ℓ a finite dimensional L-module which is a (finite) direct sum of L_k-modules $M(2\ell+1)$ for the same number $\ell \in \frac{1}{2}\mathbf{Z}_{\geq 0}$. Then M is a direct sum of the subspaces R_ℓ since M is Harish-Chandra, and from lemma 1.4

we know that R_ℓ can be written as the direct sum of subspaces $R_{\ell,m}$, where $R_{\ell,m}$ are eigenspaces for H_3 such that

$$H_3\xi = m\xi \tag{1.6}$$

for $m \in \{-\ell, -\ell+1, \ldots, \ell-1, \ell\}$ and $\xi \in R_{l,m}$. We will use the decomposition

$$M = \bigoplus_{\substack{\ell\in\frac{1}{2}\mathbf{Z}_{\geq 0}\\ m\in\{-\ell,-\ell+1,\ldots,\ell-1,\ell\}}} R_{\ell,m} = \bigoplus_{\ell,m} R_{\ell,m}$$

throughout this paper.

By lemma 1.4 we also have that H_+ and H_- maps the $R_{\ell,m}$ into each other as follows:

$$\begin{aligned} &H_+\colon R_{\ell,m}\to R_{\ell,m+1} &&\text{if } -\ell\leq m<\ell, &&H_+\colon R_{\ell,\ell}\to 0,\\ &H_-\colon R_{\ell,m}\to R_{\ell,m-1} &&\text{if } -\ell< m\leq\ell, &&H_-\colon R_{\ell,-\ell}\to 0.\end{aligned}$$

Hence we have linear operators $H_+H_-, H_-H_+\colon R_{\ell,m}\to R_{\ell,m}$, and by eq. (1.5) we see that

$$\begin{aligned} H_+H_-\xi &= \sqrt{(\ell+(m-1)+1)(\ell-(m-1))}\sqrt{(\ell+m)(\ell-m+1)}\xi\\ &= (\ell+m)(\ell-m+1)\xi,\\ H_-H_+\xi &= \sqrt{(\ell+(m+1))(\ell-(m+1)+1)}\sqrt{(\ell+m+1)(\ell-m)}\xi\\ &= (\ell+m+1)(\ell-m)\xi. \end{aligned} \tag{1.7}$$

Note that this also covers the cases $m=\ell$ and $m=-\ell$.

Now we define $E_+\colon R_{\ell,m}\to R_{\ell,m+1}$ and $E_-\colon R_{\ell,m}\to R_{\ell,m-1}$ to be the linear maps satisfying

$$\begin{aligned} H_+\xi &= \begin{cases}\sqrt{(\ell+m+1)(\ell-m)}E_+\xi & \text{if } m\neq\ell\\ 0 & \text{if } m=\ell,\end{cases}\\ H_-\xi &= \begin{cases}\sqrt{(\ell+m)(\ell-m+1)}E_-\xi & \text{if } m\neq-\ell\\ 0 & \text{if } m=\ell\end{cases} \end{aligned} \tag{1.8}$$

for $\xi\in R_{\ell,m}$. Comparing eq. (1.8) and eq. (1.7) we see that

$$\begin{aligned} E_+E_-\xi &= \xi &&\text{if } m\neq-\ell\\ E_-E_+\xi &= \xi &&\text{if } m\neq\ell. \end{aligned}$$

Thus $E_+\colon R_{\ell,m}\to R_{\ell,m+1}$ and $E_-\colon R_{\ell,m+1}\to R_{\ell,m}$ are isomorphisms for $m\neq\ell$ and they are each others inverse.

Now note that H_+, H_-, and H_3 are completely determined by eq. (1.6) and eq. (1.8), so we just need to find maps to determine F_+, F_-, and F_3 now, while making sure that we get commutation relations as in eq. (1.2).

Consider maps D_0 and D_+ defined on $M=\bigoplus_{\ell,m}R_{\ell,m}$ and D_- defined on the direct sum without the summands $R_{\ell,\ell}$ and $R_{\ell,-\ell}$ such that $D_0R_{\ell,m}\subset R_{\ell,m}$, $D_+R_{\ell,m}\subset R_{\ell+1,m}$, and $D_-R_{\ell,m}\subset R_{\ell-1,m}$ and the diagrams

$$\begin{tikzcd}
R_{\ell-1,m+1} & R_{\ell,m+1} \arrow[l, "D_-"'] \\
R_{\ell-1,m} \arrow[u, "E_+"] & R_{\ell,m} \arrow[l, "D_-"] \arrow[u, "E_+"']
\end{tikzcd}
\qquad
\begin{tikzcd}
R_{\ell,m+1} \arrow[r, "D_0"] & R_{\ell,m+1} \\
R_{\ell,m} \arrow[u, "E_+"] \arrow[r, "D_0"'] & R_{\ell,m} \arrow[u, "E_+"']
\end{tikzcd}$$

$$\begin{tikzcd}
R_{\ell,m+1} \arrow[r, "D_+"] & R_{\ell+1,m+1} \\
R_{\ell,m} \arrow[u, "E_+"] \arrow[r, "D_+"'] & R_{\ell+1,m+1} \arrow[u, "E_+"']
\end{tikzcd}$$

commute, when $-\ell+1 \leq m < \ell-1$ in the top left diagram, $-\ell \leq m < \ell$ in the other two diagrams. We need quite a lot of work to find a way to describe F_+, F_-, and F_3 from these maps.

We already have that $L_k = \mathrm{span}_{\mathbf{C}}(h_+, h_-, h_3)$, but now we will also consider $L_p = \mathrm{span}_{\mathbf{C}}(f_+, f_-, f_3)$. Equation (1.2) gives us that $[L_k, L_p] \subset L_p$, so by the adjoint representation we can see L_p as an L_k-module, and again by eq. (1.2) we see that L_p is a simple L_k-module: If V is an L_k-submodule and we have a non-zero element $f = af_+ + bf_- + cf_3 \in V$ for some $a, b, c \in \mathbf{C}$ not all zero. Then

$$\begin{aligned}
[h_+, af_+ + bf_- + cf_3] &= 2bf_3 - cf_+, \\
[h_-, af_+ + bf_- + cf_3] &= -2af_3 + cf_-, \\
[h_3, af_+ + bf_- + cf_3] &= af_+ - bf_-.
\end{aligned}$$

If $c \neq 0$, we get that

$$\begin{aligned}
[h_3, [h_+, f]] &= [h_3, 2bf_3 - cf_+] = -cf_+, \\
[h_3, [h_-, f]] &= [h_3, -2af_3 + cf_-] = -cf_-,
\end{aligned}$$

so we see that $f_+, f_- \in V$, and thus also $[h_+, \frac{1}{2}f_-] = f_3 \in V$, so $V = L_p$. If on the other hand $c = 0$, then

$$\begin{aligned}
[h_-, f] &= -2af_3, \\
[h_+, f] &= 2bf_3,
\end{aligned}$$

so since either $a \neq 0$ or $b \neq 0$, we see that $f_3 \in V$, and thus also $[h_+, -f_3] = f_+ \in V$ and $[h_-, f_3] = f_- \in V$, so $V = L_p$. Hence L_p is indeed a simple L_k-module. Now since L_p is a simple finite dimensional L_k-module of dimension 3, we have that $L_p \simeq M(2)$ as L_k-modules.

In general given two L-modules V and W, we consider the tensor product $V \otimes W$ over $\mathbf{C}$ of the underlying vector spaces as an L-module via the action

$$x.(v \otimes w) = x.v \otimes w + v \otimes x.w,$$

cf. [Hum72, p. 26].

Now we are interested in the L_k-module $L_p \otimes M$, where M is a Harish-Chandra L-module as before. Specifically we will show that

$$\begin{aligned} \psi\colon L_p \otimes M &\to M \\ x \otimes v &\mapsto x.v \end{aligned} \tag{1.9}$$

is a homomorphism of L_k-modules. It is clear that ψ is linear, and for $y \in L_k$ we see that

$$y.(x \otimes v) = y.x \otimes v + x \otimes y.v = [y, x] \otimes v + x \otimes y.v,$$

for $x \otimes v \in L_p \otimes M$, since the action in L_p is by the adjoint representation. So

$$\begin{aligned} \psi(y.(x \otimes v)) &= \psi([y, x] \otimes v) + \psi(x \otimes y.v) = [y, x].v + x.(y.v) \\ &= y.(x.v) - x.(y.v) + x.(y.v) = y.(x.v) = y.\psi(x \otimes v), \end{aligned}$$

i.e. ψ is indeed a homomorphism of L_k-modules.

Now we note that $M = \bigoplus_\ell R_\ell$, so

$$L_p \otimes M = L_p \otimes \big(\bigoplus_\ell R_\ell\big) \simeq \bigoplus_\ell (L_p \otimes R_\ell),$$

as L_k-modules, and since also R_ℓ is direct sum of finitely many copies of $M(2\ell)$, we see that

$$\begin{aligned} L_p \otimes R_\ell &\simeq M(2) \otimes \big(M(2\ell)^1 \oplus M(2\ell)^2 \oplus \cdots \oplus M(2\ell)^r\big) \\ &\simeq (L_p \otimes M(2\ell)^1) \oplus (L_p \otimes M(2\ell)^2) \oplus \cdots \oplus (L_p \otimes M(2\ell)^r), \end{aligned}$$

as L_k-modules, since $L_p \simeq M(2)$. Here the superscripts are just indices for the different $M(2\ell)$. Thus we want to describe the L_k-modules $M(2) \otimes M(2\ell)$, which we will do by first describing the $\mathfrak{sl}(2,\mathbf{C})$-modules $V(2) \otimes V(2\ell)$ and then translating back to a solution to our problem by the isomorphism of eq. (1.3).

Let $2\ell = n \in \mathbf{N}$. We want to show that

$$V(2) \otimes V(n) = \begin{cases} V(n-2) \oplus V(n) \oplus V(n+2) & \text{if } n \geq 2, \\ V(3) \oplus V(1) & \text{if } n = 1, \\ V(2) & \text{if } n = 0. \end{cases} \tag{1.10}$$

Note that in all cases there is a summand $V(n+2)$. We can show the above by considerations using formal characters. We will use the notation of [Jan16, Chapter 8], specifically we will do calculations with the functions $e(\lambda)\colon H^* \to \mathbf{Z}$ for $\lambda \in H^*$. Firstly note that in general

$$\mathrm{ch}_V = \sum_{\lambda \in H^*} (\dim V_\lambda) e(\lambda),$$

and use the notation $V(n)_k$ for $V(\lambda)_\mu$ and $e(n)$ for $e(\lambda)$ with $\lambda, \mu \in H^*$ such that $\lambda(h) = n$ and $\mu(h) = k$. We get that

$$\mathrm{ch}_{V(2)} = e(-2) + e(0) + e(2)$$

and

$$\mathrm{ch}_{V(n)} = \sum_{i=0}^{n} e(n-2i),$$

since

$$\dim V(n)_k = \begin{cases} 1 & \text{if } k = n-2i \text{ for some } i \in \{0,1,\dots,n\}, \\ 0 & \text{otherwise.} \end{cases}$$

Now since $e(\lambda) * e(\mu) = e(\lambda+\mu)$ in general cf. [Jan16, p. 93], we see that for $n \geq 2$

$$\begin{aligned} \mathrm{ch}_{V(2)\otimes V(n)} &= \mathrm{ch}_{V(2)} * \mathrm{ch}_{V(n)} = e(-2) * \mathrm{ch}_{V(n)} + e(0) * \mathrm{ch}_{V(n)} + e(2) * \mathrm{ch}_{V(n)} \\ &= \sum_{i=0}^{n} e(n-2-2i) + \mathrm{ch}_{V(n)} + \sum_{i=0}^{n} e(n+2-2i) \\ &= e(-n-2) + e(-n) + \sum_{i=0}^{n-2} e(n-2i) + \mathrm{ch}_{V(n)} + \sum_{i=0}^{n} e(n+2-2i) \\ &= \mathrm{ch}_{V(n-2)} + \mathrm{ch}_{V(n)} + \sum_{i=0}^{n+2} e(n+2-2i) \\ &= \mathrm{ch}_{V(n-2)} + \mathrm{ch}_{V(n)} + \mathrm{ch}_{V(n+2)} = \mathrm{ch}_{V(n-2)\oplus V(n)\oplus V(n+2)}, \end{aligned}$$

where the first equality follows from the fact that $\mathrm{ch}_{V\otimes W} = \mathrm{ch}_V * \mathrm{ch}_W$ in general, cf. [Hum72, p. 125]. Thus since two L-modules V and V' are isomorphic if and only if $\mathrm{ch}_V = \mathrm{ch}_{V'}$, cf. [Jan16, p. 90], we see that $V(2) \otimes V(n) \simeq V(n-2) \oplus V(n) \oplus V(n+2)$ if $n \geq 2$.

Likewise we see that

$$\begin{aligned} \mathrm{ch}_{V(2)\otimes V(1)} &= \mathrm{ch}_{V(2)} * \mathrm{ch}_{V(1)} \\ &= (e(-2) + e(0) + e(2)) * e(-1) + (e(-2) + e(0) + e(2)) * e(1) \\ &= e(-3) + e(-1) + e(1) + e(-1) + e(1) + e(3) \\ &= \big(e(-3) + e(-1) + e(1) + e(3)\big) + \big(e(-1) + e(1)\big) \\ &= \mathrm{ch}_{V(3)} + \mathrm{ch}_{V(1)} = \mathrm{ch}_{V(3)\oplus V(1)} \end{aligned}$$

and

$$\mathrm{ch}_{V(2)\otimes V(0)} = \mathrm{ch}_{V(2)} * \mathrm{ch}_{V(0)} = \mathrm{ch}_{V(2)} * e(0) = \mathrm{ch}_{V(2)},$$

so indeed $V(2) \otimes V(1) \simeq V(3) \oplus V(1)$ and $V(2) \otimes V(0) \simeq V(2)$.

Now consider (w_0, w_1, w_2) a basis for $V(2)$ and $(v_i \mid 0 \leq i \leq n)$ a basis for $V(n)$ such that both satisfies the conditions from eq. (1.4). Then for $w_i \otimes v_j \in V(2) \otimes V(n)$ with $i \in \{0,1,2\}$ and $j \in \{0,1,\dots,n\}$ we see that

$$\begin{aligned} h.(w_i \otimes v_j) &= h.w_i \otimes v_j + w_i \otimes h.v_j = (2-2i)w_i \otimes v_j + (n-2j)w_i \otimes v_j \\ &= (n - 2(i+j-1))w_i \otimes v_j. \end{aligned} \tag{1.11}$$

Hence $v_0 \otimes w_0$ is up to scalar multiple the only vector of weight $n+2$ in $V(2) \otimes V(n)$, so it is necessarily a highest weight vector generating the direct summand isomorphic to $V(n+2)$. Note that by eq. (1.10) we indeed have a direct summand isomorphic to $V(n+2)$ for all $n \in \mathbf{N}$. By $\mathfrak{sl}(2,\mathbf{C})$-theory, cf. [Jan16, p. 36], we know that this summand has a basis $(s_k \mid 0 \leq k \leq n+2)$ satisfying equations as in eq. (1.4), where

$$s_k := \frac{1}{k!} y^k.(w_0 \otimes v_0).$$

Maybe I will move some calculations of s_k, t_k, and u_k to an appendix

We want to describe the s_k's more explicitly. First note that if $n > 0$

$$\begin{aligned} s_1 &= y.(w_0 \otimes v_0) = y.w_0 \otimes v_0 + w_0 \otimes y.v_0 \\ &= w_1 \otimes v_0 + w_0 \otimes v_1 \end{aligned}$$

and

$$\begin{aligned} s_2 &= \tfrac{1}{2} y.s_1 \\ &= \tfrac{1}{2} y.w_1 \otimes v_0 + \tfrac{1}{2} w_1 \otimes y.v_0 + \tfrac{1}{2} y.w_0 \otimes v_1 + w_0 \otimes \tfrac{1}{2} y.v_1 \\ &= w_2 \otimes v_0 + \tfrac{1}{2} w_1 \otimes v_1 + \tfrac{1}{2} w_1 \otimes v_1 + w_0 \otimes v_2 \\ &= w_2 \otimes v_0 + w_1 \otimes v_1 + w_0 \otimes v_2. \end{aligned}$$

Inductively we see that

$$s_k = w_2 \otimes v_{k-2} + w_1 \otimes v_{k-1} + w_0 \otimes v_k$$

for $k \leq n$, since the base case holds and given the equality for $k < n$ we get

$$\begin{aligned} s_{k+1} &= \frac{1}{k+1} y.s_k \\ &= w_2 \otimes \frac{1}{k+1} y.v_{k-2} + \frac{1}{k+1} y.w_1 \otimes v_{k-1} + w_1 \otimes \frac{1}{k+1} y.v_{k-1} \\ &\quad + \frac{1}{k+1} y.w_0 \otimes v_k + w_0 \otimes \frac{1}{k+1} y.v_k \\ &= \frac{k-1}{k+1} w_2 \otimes v_{k-1} + \frac{2}{k+1} w_2 \otimes v_{k-1} + \frac{k}{k+1} w_1 \otimes v_k + \frac{1}{k+1} w_1 \otimes v_k \\ &\quad + w_0 \otimes v_{k+1} \\ &= w_2 \otimes v_{k-1} + w_1 \otimes v_k + w_0 \otimes v_{k+1}. \end{aligned}$$

We likewise see that for $k = n+1$ the last term vanishes, so we have $s_{k+1} = w_2 \otimes v_{n-1} + w_1 \otimes v_n$, and for $k = n+2$ the two last terms vanish, so we get $s_{k+2} = w_2 \otimes v_n$. Thus altogether we have that

$$\begin{aligned} s_0 &= w_0 \otimes v_0, \\ s_1 &= w_1 \otimes v_0 + w_0 \otimes v_1 && \text{if } n > 0, \\ s_k &= w_2 \otimes v_{k-2} + w_1 \otimes v_{k-1} + w_0 \otimes v_k && \text{for } 2 \leq k \leq n, \\ s_{n+1} &= w_2 \otimes v_{n-1} + w_1 \otimes v_n && \text{if } n > 0, \\ s_{n+2} &= w_2 \otimes v_n. \end{aligned} \tag{1.12}$$

In case $n = 0$ we see that $s_1 = w_1 \otimes v_0$ and $s_2 = w_2 \otimes v_0$, and we note that (s_0, s_1, s_2) is a basis for $V(2) \otimes V(0) \simeq V(2)$.

Suppose now that $n \geq 1$. Note that by eq. (1.10) we have a direct summand isomorphic to $V(n)$, and by eq. (1.11) this weight space of weight n are spanned by $w_0 \otimes v_1$ and $w_1 \otimes v_0$, so the vector of highest weight n generating this summand must be of the form $aw_0 \otimes v_1 + bw_1 \otimes v_0$ for some $a, b \in \mathbf{C}$. Furthermore we know that for this vector generating the summand corresponding to $V(n)$, we must have that

$$\begin{aligned} 0 &= x.(aw_0 \otimes v_1 + bw_1 \otimes v_0) \\ &= ax.w_0 \otimes v_1 + aw_0 \otimes x.v_1 + bx.w_1 \otimes v_0 + bw_1 \otimes x.v_0 \\ &= 0 + a(n-1+1)w_0 \otimes v_0 + b(2-1+1)w_0 \otimes v_0 + 0 \\ &= (an + 2b)w_0 \otimes v_0, \end{aligned}$$

i.e. $an + 2b = 0$ so $b = -\frac{n}{2}a$. This determines the vector generating the summand corresponding to $V(n)$ up to a scalar, so taking $a = 1$, we see that we can take

$$t_0 \coloneqq w_0 \otimes v_1 - \frac{n}{2} w_1 \otimes v_0$$

as our vector generating the summand corresponding to $V(n)$. As before $\mathfrak{sl}(2, \mathbf{C})$-theory now yields that this summand has a basis $(t_k \mid 0 \leq k \leq n)$ satisfying equations as in eq. (1.4), where

$$t_k \coloneqq \frac{1}{k!} y^k . t_0.$$

We see that

$$\begin{aligned} t_1 &= y.\big(w_0 \otimes v_1 - \frac{n}{2} w_1 \otimes v_0\big) \\ &= y.w_0 \otimes v_1 + w_0 \otimes y.v_1 - \frac{n}{2} y.w_1 \otimes v_0 + \frac{n}{2} w_1 \otimes y.v_0 \\ &= w_1 \otimes v_1 + 2w_0 \otimes v_2 - nw_2 \otimes v_0 - \frac{n}{2} w_1 \otimes v_1 \\ &= 2w_0 \otimes v_2 - \frac{n-2}{2} w_1 \otimes v_1 - nw_2 \otimes v_0, \end{aligned}$$

and inductively we get that

$$t_k = (k+1)w_0\otimes v_{k+1} - \frac{n-2k}{2}w_1\otimes v_k + (k-1-n)w_2\otimes v_{k-1}$$

for $1\le k\le n-1$, since the base case holds and given the equality for $k<n-1$ we get

$$\begin{aligned}
t_{k+1} &= \frac{1}{k+1}y.t_k\\
&= y.w_0\otimes v_{k+1} + w_0\otimes y.v_{k+1} - \frac{n-2k}{2(k+1)}y.w_1\otimes v_k\\
&\quad - \frac{n-2k}{2(k+1)}w_1\otimes y.v_k + \frac{k-1-n}{k+1}w_2\otimes y.v_{k-1}\\
&= w_1\otimes v_{k+1} + (k+2)w_0\otimes v_{k+2} - \frac{n-2k}{k+1}w_2\otimes v_k\\
&\quad - \frac{n-2k}{2}w_1\otimes v_{k+1} + \frac{(k-1-n)k}{k+1}w_2\otimes v_k\\
&= (k+2)w_0\otimes v_{k+2} - \frac{n-2(k+1)}{2}w_1\otimes v_{k+1}\\
&\quad + \left(\frac{k^2-k-nk-n+2k}{k+1}\right)w_2\otimes v_k\\
&= (k+2)w_0\otimes v_{k+2} - \frac{n-2(k+1)}{2}w_1\otimes v_{k+1} + (k-n)w_2\otimes v_k,
\end{aligned}$$

where we in the last equality use that $(k+1)(k-n) = k^2-nk+k-n = k^2-k-nk-n+2k$. We likewise see that for $k=n$ the first term vanishes so

$$t_n = \frac{n}{2}w_1\otimes v_n - w_2\otimes v_{n-1}.$$

Suppose now that $n\ge 2$. By eq. (1.10) we have a direct summand isomorphic to $V(n-2)$, and by eq. (1.11) this weight space of weight $n-2$ is spanned by $w_0\otimes v_2$, $w_1\otimes v_1$, and $w_2\otimes v_0$, so the vector of highest weight $n-2$ generating this summand must be of the form $aw_0\otimes v_2 + bw_1\otimes v_1 + cw_2\otimes v_0$ for some $a,b,c\in\mathbf{C}$. Furthermore we know that for this vector generating the summand corresponding to $V(n-2)$, we must have

$$\begin{aligned}
0 &= x.(aw_0\otimes v_2 + bw_1\otimes v_1 + cw_2\otimes v_0)\\
&= aw_0\otimes x.v_2 + bx.w_1\otimes v_1 + bw_1\otimes x.v_1 + cx.w_2\otimes v_0\\
&= a(n-2+1)w_0\otimes v_1 + b(2-1+1)w_0\otimes v_1 + b(n-1+1)w_1\otimes v_0\\
&\quad + c(2-2+1)w_1\otimes v_0\\
&= \big((n-1)a+2b\big)w_0\otimes v_1 + \big(bn+c\big)w_1\otimes v_0,
\end{aligned}$$

i.e. $a(n-1)+2b=0$ and $bn+c=0$. Giving us $c=-bn$ and $b=-\frac{n-1}{2}a$, so

$$c = \frac{n(n-1)}{2}a.$$

This determines the vector generating the summand generating the summand corresponding to $V(n-2)$ up to a scalar, so taking $a=1$, we see that we can take

$$u_0 \coloneqq w_0 \otimes v_2 - \frac{n-1}{2} w_1 \otimes v_1 + \frac{n(n-1)}{2} w_2 \otimes v_0$$

as our vector generating the summand corresponding to $V(n-2)$. Again $\mathfrak{sl}(2,\mathbf{C})$-theory now yields that this summand has a basis $(u_k \mid 0 \leq k \leq n-2)$ satisfying equations as in eq. (1.4), where

$$u_k \coloneqq \frac{1}{k!} y^k . u_0.$$

Inductively we get that

$$\begin{aligned} u_k = &\frac{(k+1)(k+2)}{2} w_0 \otimes v_{k+2} - \frac{(k+1)(n-k-1)}{2} w_1 \otimes v_{k+1} \\ &+ \frac{(n-k)(n-k-1)}{2} w_2 \otimes v_k \end{aligned}$$

for $0 \leq k \leq n-2$, since the base case holds and given the equality for $k < n-2$ we get

$$\begin{aligned} u_{k+1} &= \frac{1}{k+1} y . u_k \\ &= \frac{k+2}{2} y . w_0 \otimes v_{k+2} + \frac{k+2}{2} w_0 \otimes y . v_{k+2} \\ &\quad - \frac{n-k-1}{2} y . w_1 \otimes v_{k+1} - \frac{n-k-1}{2} w_1 \otimes y . v_{k+1} \\ &\quad + \frac{(n-k)(n-k-1)}{2(k+1)} w_2 \otimes y . v_k \\ &= \frac{k+2}{2} w_1 \otimes v_{k+2} + \frac{(k+2)(k+3)}{2} w_0 \otimes v_{k+3} \\ &\quad - (n-k-1) w_2 \otimes v_{k+1} - \frac{(n-k-1)(k+2)}{2} w_1 \otimes v_{k+2} \\ &\quad + \frac{(n-k)(n-k-1)}{2} w_2 \otimes v_{k+1} \\ &= \frac{(k+2)(k+3)}{2} w_0 \otimes v_{k+3} \\ &\quad - \frac{(n-k-1)(k+2) - (k+2)}{2} w_1 \otimes v_{k+2} \\ &\quad + \frac{(n-k)(n-k-1) - 2(n-k-1)}{2} w_2 \otimes v_{k+1} \\ &= \frac{(k+2)(k+3)}{2} w_0 \otimes v_{k+3} \\ &\quad - \frac{(k+2)(n-k-2)}{2} w_1 \otimes v_{k+2} \\ &\quad + \frac{(n-k-1)(n-k-2)}{2} w_2 \otimes v_{k+1}. \end{aligned}$$

This should probably be in an appendix

Now we want to express $w_1 \otimes v_k$ for $0 \le k \le n$ in terms of the bases $(s_k \mid 0 \le k \le n+2)$, $(t_k \mid 0 \le k \le n)$, and $(u_k \mid 0 \le k \le n-2)$. We see that for $0 < k < n$

$$\begin{aligned}
&\frac{2(k+1)(n+1-k)}{(n+1)(n+2)} s_{k+1} - \frac{2(n-2k)}{n(n+2)} t_k - \frac{4}{n(n+1)} u_{k-1} \\
&= \frac{2(k+1)(n+1-k)}{(n+1)(n+2)} \Big(w_0 \otimes v_{k+1} + w_1 \otimes v_k + w_2 \otimes v_{k-1} \Big) \\
&\quad - \frac{2(n-2k)}{n(n+2)} \Big((k+1) w_0 \otimes v_{k+1} - \frac{n-2k}{2} w_1 \otimes v_k \\
&\qquad + (k-1-n) w_2 \otimes v_{k-1} \Big) \\
&\quad - \frac{4}{n(n+1)} \Big(\frac{k(k+1)}{2} w_0 \otimes v_{k+1} - \frac{k(n-k)}{2} w_1 \otimes v_k \\
&\qquad + \frac{(n-k+1)(n-k)}{2} w_2 \otimes v_{k-1} \Big) \\
&= \frac{\begin{pmatrix} 2(k+1)(n+1-k)n - 2(n-2k)(k+1)(n+1) \\ \hfill - 2k(k+1)(n+2) \end{pmatrix}}{n(n+1)(n+2)} w_0 \otimes v_{k+1} \\
&\quad + \frac{\begin{pmatrix} 2(k+1)(n+1-k)n + (n-2k)(n-2k)(n+1) \\ \hfill + 2k(n-k)(n+2) \end{pmatrix}}{n(n+1)(n+2)} w_1 \otimes v_k \\
&\quad + \frac{\begin{pmatrix} 2(k+1)(n+1-k)n - 2(n-2k)(k-1-n)(n+1) \\ \hfill - 2(n-k+1)(n-k)(n+2) \end{pmatrix}}{n(n+1)(n+2)} w_2 \otimes v_{k-1} \\
&= 2(k+1) \frac{(n+1-k)n - (n-2k)(n+1) - k(n+2)}{n(n+1)(n+2)} w_0 \otimes v_{k+1} \\
&\quad + \frac{\begin{pmatrix} 2(k+1)(n+1-k)n + (n-2k)(n-2k)(n+1) \\ \hfill + 2k(n-k)(n+2) \end{pmatrix}}{n(n+1)(n+2)} w_1 \otimes v_k \\
&\quad + 2(n+1-k) \frac{(k+1)n + (n-2k)(n+1) - (n-k)(n+2)}{n(n+1)(n+2)} w_2 \otimes v_{k-1}.
\end{aligned}$$

Now we note that

$$\begin{aligned}
(n+1-k)n &- (n-2k)(n+1) - k(n+2) \\
&= n\big((n+1-k) - (n-2k) - k\big) - (n-2k) - 2k \\
&= n - (n-2k) - 2k = 0,
\end{aligned}$$

and

$$\begin{aligned}
(k+1)n &+ (n-2k)(n+1) - (n-k)(n+2) \\
&= n\big((k+1) + (n-2k) - (n-k)\big) + (n-2k) - 2(n-k) \\
&= n + n - 2k - 2n + 2k = 0,
\end{aligned}$$

while

$$\begin{aligned}2(k+1)(n+1-k)n+(n-2k)(n-2k)(n+1)+2k(n-k)(n+2)\\ &= n(n+1)\big(2(k+1)(n-2k)+2k\big)-2(k+1)kn\\ &\quad -2k(n-2k)(n+1)+2k(n-k)(n+1)+2k(n-k)\\ &= n(n+1)\big(2k(n-2k)+(n-2k)+2k)\\ &\quad -2k(n-2k)(n+1)-2k^2(n+1)+2k(n-k)\\ &= ... = n(n+1)(n+2)\end{aligned}$$

Finish this

Chapter 2

Linear relations

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Appendix A

Appendix

AUTO GENERATED TEXT

As any dedicated reader can clearly see, the Ideal of practical reason is a representation of, as far as I know, the things in themselves; as I have shown elsewhere, the phenomena should only be used as a canon for our understanding. The paralogisms of practical reason are what first give rise to the architectonic of practical reason. As will easily be shown in the next section, reason would thereby be made to contradict, in view of these considerations, the Ideal of practical reason, yet the manifold depends on the phenomena. Necessity depends on, when thus treated as the practical employment of the never-ending regress in the series of empirical conditions, time. Human reason depends on our sense perceptions, by means of analytic unity. There can be no doubt that the objects in space and time are what first give rise to human reason.

Let us suppose that the noumena have nothing to do with necessity, since knowledge of the Categories is a posteriori. Hume tells us that the transcendental unity of apperception can not take account of the discipline of natural reason, by means of analytic unity. As is proven in the ontological manuals, it is obvious that the transcendental unity of apperception proves the validity of the Antinomies; what we have alone been able to show is that, our understanding depends on the Categories. It remains a mystery why the Ideal stands in need of reason. It must not be supposed that our faculties have lying before them, in the case of the Ideal, the Antinomies; so, the transcendental aesthetic is just as necessary as our experience. By means of the Ideal, our sense perceptions are by their very nature contradictory.

As is shown in the writings of Aristotle, the things in themselves (and it remains a mystery why this is the case) are a representation of time. Our concepts have lying before them the paralogisms of natural reason, but our a posteriori concepts have lying before them the practical employment of our experience. Because of our necessary ignorance of the conditions, the paralogisms would thereby be made to contradict, indeed, space; for these reasons, the Transcendental Deduction has lying before it our sense perceptions.

(Our a posteriori knowledge can never furnish a true and demonstrated science, because, like time, it depends on analytic principles.) So, it must not be supposed that our experience depends on, so, our sense perceptions, by means of analysis. Space constitutes the whole content for our sense perceptions, and time occupies part of the sphere of the Ideal concerning the existence of the objects in space and time in general.

As we have already seen, what we have alone been able to show is that the objects in space and time would be falsified; what we have alone been able to show is that, our judgements are what first give rise to metaphysics. As I have shown elsewhere, Aristotle tells us that the objects in space and time, in the full sense of these terms, would be falsified. Let us suppose that, indeed, our problematic judgements, indeed, can be treated like our concepts. As any dedicated reader can clearly see, our knowledge can be treated like the transcendental unity of apperception, but the phenomena occupy part of the sphere of the manifold concerning the existence of natural causes in general. Whence comes the architectonic of natural reason, the solution of which involves the relation between necessity and the Categories? Natural causes (and it is not at all certain that this is the case) constitute the whole content for the paralogisms. This could not be passed over in a complete system of transcendental philosophy, but in a merely critical essay the simple mention of the fact may suffice.

Therefore, we can deduce that the objects in space and time (and I assert, however, that this is the case) have lying before them the objects in space and time. Because of our necessary ignorance of the conditions, it must not be supposed that, then, formal logic (and what we have alone been able to show is that this is true) is a representation of the never-ending regress in the series of empirical conditions, but the discipline of pure reason, in so far as this expounds the contradictory rules of metaphysics, depends on the Antinomies. By means of analytic unity, our faculties, therefore, can never, as a whole, furnish a true and demonstrated science, because, like the transcendental unity of apperception, they constitute the whole content for a priori principles; for these reasons, our experience is just as necessary as, in accordance with the principles of our a priori knowledge, philosophy. The objects in space and time abstract from all content of knowledge. Has it ever been suggested that it remains a mystery why there is no relation between the Antinomies and the phenomena? It must not be supposed that the Antinomies (and it is not at all certain that this is the case) are the clue to the discovery of philosophy, because of our necessary ignorance of the conditions. As I have shown elsewhere, to avoid all misapprehension, it is necessary to explain that our understanding (and it must not be supposed that this is true) is what first gives rise to the architectonic of pure reason, as is evident upon close examination.

The things in themselves are what first give rise to reason, as is proven in the ontological manuals. By virtue of natural reason, let us suppose that the transcendental unity of apperception abstracts from all content of knowledge;

in view of these considerations, the Ideal of human reason, on the contrary, is the key to understanding pure logic. Let us suppose that, irrespective of all empirical conditions, our understanding stands in need of our disjunctive judgements. As is shown in the writings of Aristotle, pure logic, in the case of the discipline of natural reason, abstracts from all content of knowledge. Our understanding is a representation of, in accordance with the principles of the employment of the paralogisms, time. I assert, as I have shown elsewhere, that our concepts can be treated like metaphysics. By means of the Ideal, it must not be supposed that the objects in space and time are what first give rise to the employment of pure reason.

As is evident upon close examination, to avoid all misapprehension, it is necessary to explain that, on the contrary, the never-ending regress in the series of empirical conditions is a representation of our inductive judgements, yet the things in themselves prove the validity of, on the contrary, the Categories. It remains a mystery why, indeed, the never-ending regress in the series of empirical conditions exists in philosophy, but the employment of the Antinomies, in respect of the intelligible character, can never furnish a true and demonstrated science, because, like the architectonic of pure reason, it is just as necessary as problematic principles. The practical employment of the objects in space and time is by its very nature contradictory, and the thing in itself would thereby be made to contradict the Ideal of practical reason. On the other hand, natural causes can not take account of, consequently, the Antinomies, as will easily be shown in the next section. Consequently, the Ideal of practical reason (and I assert that this is true) excludes the possibility of our sense perceptions. Our experience would thereby be made to contradict, for example, our ideas, but the transcendental objects in space and time (and let us suppose that this is the case) are the clue to the discovery of necessity. But the proof of this is a task from which we can here be absolved.