Bachelorprojekt

Title

(subtitle)

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Abstract

Some text

Contents

Chapter 1

Harish-Chandra modules over $\mathfrak{sl}(2,\mathbf{C}) \times \mathfrak{sl}(2,\mathbf{C})$

Let G be a semisimple Lie group and let G_k be its maximal compact subgroup. Denote by L the semisimple Lie algebra of G and denote by L_k the Lie subalgebra corresponding to G_k.

Definition 1.1. *An L-module M is a Harish-Chandra module if, regarded as an L_k-module, it can be written as a sum*

$$M = \bigoplus_i M_i$$

of finite dimensional irreducible L_k-submodules M_i, where for each M_{i_0} only finitely many L_k-submodules equivalent to M_{i_0} occur in the decomposition of M.

A Harish-Chandra module M is indecomposable if it cannot be decomposed into the direct sum of L-submodules.

Our goal is to classify all indecomposable Harish-Chandra modules over $\mathfrak{sl}(2,\mathbf{C}) \times \mathfrak{sl}(2,\mathbf{C})$, where we by $\mathfrak{sl}(2,\mathbf{C}) \times \mathfrak{sl}(2,\mathbf{C})$ mean the following:

For L, L' Lie algebras over F, we consider $L \times L' = L \oplus L'$ as a Lie algebra over F with pointwise addition, multiplication given by $\alpha(a,b) = (\alpha a, \alpha b)$ for $\alpha \in F, a \in L, b \in L'$, and with Lie bracket $[(a_1,b_1),(a_2,b_2)] = ([a_1,a_2],[b_1,b_2])$ for $a_1, a_2 \in L, b_1, b_2 \in L'$.

Remark 1.2. Note that $L \times 0$ and $0 \times L'$ are ideals in $L \times L'$ as given above. Thus we see that $\mathfrak{sl}(2,\mathbf{C}) \times 0$ and $0 \times \mathfrak{sl}(2,\mathbf{C})$ are ideals in $\mathfrak{sl}(2,\mathbf{C}) \times \mathfrak{sl}(2,\mathbf{C})$ with

$$(\mathfrak{sl}(2,\mathbf{C}) \times 0) \oplus (0 \times \mathfrak{sl}(2,\mathbf{C})) = \mathfrak{sl}(2,\mathbf{C}) \times \mathfrak{sl}(2,\mathbf{C}),$$

so $\mathfrak{sl}(2,\mathbf{C}) \times \mathfrak{sl}(2,\mathbf{C})$ is semisimple. Hence it makes sense to talk about Harish-Chandra modules over $\mathfrak{sl}(2,\mathbf{C}) \times \mathfrak{sl}(2,\mathbf{C})$. $\triangle$

We fix the following as a standard basis for $\mathfrak{sl}(2,F)$:

$$x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Giving us the relations:

$$[x,y] = h, \qquad [h,x] = 2x, \qquad [h,y] = -2y, \tag{1.1}$$

cf. [Jan16, p. 35] or [Hum72, p. 6].

We claim now that

$$(x,x), \quad (y,y), \quad \tfrac{1}{2}(h,h), \quad (ix,-ix), \quad (iy,-iy), \quad \tfrac{1}{2}(ih,-ih)$$

is a basis of $\mathfrak{sl}(2,\mathbf{C}) \times \mathfrak{sl}(2,\mathbf{C})$. This is clearly the case since $\dim_{\mathbf{C}} \mathfrak{sl}(2,\mathbf{C}) = 3$, so $\dim_{\mathbf{C}} \mathfrak{sl}(2,\mathbf{C}) \times \mathfrak{sl}(2,\mathbf{C}) = 6$, and we see that the above elements span $\mathfrak{sl}(2,\mathbf{C}) \times \mathfrak{sl}(2,\mathbf{C})$; we have $\frac{1}{2}(x,x) - \frac{i}{2}(ix,-ix) = (x,0)$ and $\frac{1}{2}(x,x) + \frac{i}{2}(ix,-ix) = (0,x)$ and likewise with h and y.

Putting

$$\begin{aligned} h_+ &= (x,x), & h_- &= (y,y), & h_3 &= \tfrac{1}{2}(h,h), \\ f_+ &= (ix,-ix), & f_- &= (iy,-iy), & f_3 &= \tfrac{1}{2}(ih,-ih) \end{aligned}$$

we get the following commutation relations between these basis elements:

$$\begin{aligned} [h_+,h_3] &= \tfrac{1}{2}([x,h],[x,h]) = \tfrac{1}{2}(-2x,-2x) = -(x,x) = -h_+, \\ [h_-,h_3] &= \tfrac{1}{2}([y,h],[y,h]) = \tfrac{1}{2}(2y,2y) = (y,y) = h_-, \\ [h_+,h_-] &= ([x,y],[x,y]) = (h,h) = 2h_3, \\ [h_+,f_+] &= ([x,ix],[x,-ix]) = 0, \\ [h_-,f_-] &= ([y,iy],[y,-iy]) = 0, \\ [h_3,f_3] &= \tfrac{1}{4}([h,ih],[h,-ih]) = 0, \\ [h_+,f_3] &= \tfrac{1}{2}([x,ih],[x,-ih]) = \tfrac{1}{2}(-2ix,2ix) = -(ix,-ix) = -f_+, \\ [h_-,f_3] &= \tfrac{1}{2}([y,ih],[y,-ih]) = \tfrac{1}{2}(2iy,-2iy) = (iy,-iy) = f_-, \\ [h_+,f_-] &= ([x,iy],[x,-iy]) = (ih,-ih) = 2f_3, \\ [h_3,f_-] &= \tfrac{1}{2}([h,iy],[h,-iy]) = \tfrac{1}{2}(-2iy,2iy) = -(iy,-iy) = -f_-, \\ [h_-,f_+] &= ([y,ix],[y,-ix]) = (-ih,ih) = -(ih,-ih) = -2f_3, \\ [h_3,f_+] &= \tfrac{1}{2}([h,ix],[h,-ix]) = \tfrac{1}{2}(2ix,-2ix) = (ix,-ix) = f_+, \\ [f_+,f_3] &= \tfrac{1}{2}([ix,ih],[-ix,-ih]) = \tfrac{1}{2}(2x,2x) = (x,x) = h_+, \\ [f_-,f_3] &= \tfrac{1}{2}([iy,ih],[-iy,-ih]) = \tfrac{1}{2}(-2y,-2y) = -(y,y) = -h_-, \\ [f_+,f_-] &= ([ix,iy],[-ix,-iy]) = (-h,-h) = -(h,h) = -2h_3. \end{aligned} \tag{1.2}$$

Remark 1.3. Note that these are the same relations as for the complexification of the Lie algebra L of the proper Lorentz group in [GP67b, p. 5], so L is isomorphic to $\mathfrak{sl}(2,\mathbf{C}) \times \mathfrak{sl}(2,\mathbf{C})$. This explains the equivalence of the work in this paper and the work in [GP67a; GP67b; GP67c]. △

Now let $L = \mathfrak{sl}(2, \mathbf{C}) \times \mathfrak{sl}(2, \mathbf{C})$ and denote by L_k the Lie subalgebra of L with basis h_+, h_-, h_3. One can show that this corresponds to a maximal compact subgroup in the way described in definition 1.1, but that is beyond what we will do in this paper. Note that the above commutation relations gives us that

$$[h_+, h_-] = 2h_3, \qquad [2h_3, h_+] = 2h_+, \qquad [2h_3, h_-] = -2h_-$$

Comparing with eq. (1.1) we see that we have an isomorphism

$$\begin{aligned} L_k &\to \mathfrak{sl}(2, \mathbf{C}) \\ h_+ &\mapsto x \\ h_- &\mapsto y \\ 2h_3 &\mapsto h, \end{aligned} \tag{1.3}$$

so we can use $\mathfrak{sl}(2, \mathbf{C})$-theory when we want to describe L_k-modules.

1.1 Representations of L_k

Let V be a $\mathbf{C}$ vector space and $\rho\colon L_k \to \mathfrak{gl}(V)$ a representation of L_k. We will use the notation $\rho(a) = A$ for $a \in L_k$ switching to upper case letters when we talk about the representation corresponding to a given element. Note that we will switch freely between the language of representations of L_k and the language of L_k-modules.

We will start out by describing the finite dimensional simple[1] L_k-modules. Recall cf. [Jan16, p. 36] that we know from $\mathfrak{sl}(2, \mathbf{C})$-theory that for integers $n \geq 0$ there exists a unique simple $\mathfrak{sl}(2, \mathbf{C})$-module $V(n)$ of dimension $n + 1$, and $V(n)$ has a basis $(v_0, v_1, \ldots, v_n)$ such that for all i, $0 \leq i \leq n$

$$\begin{aligned} h.v_i &= (n - 2i)v_i, \\ x.v_i &= \begin{cases} (n - i + 1)v_{i-1} & \text{if } i > 0, \\ 0 & \text{if } i = 0, \end{cases} \\ y.v_i &= \begin{cases} (i + 1)v_{i+1} & \text{if } i < n, \\ 0 & \text{if } i = n. \end{cases} \end{aligned} \tag{1.4}$$

Now using the isomorphism from eq. (1.3) we see that for integers $n \geq 0$ there exists a unique simple L_k-module $M(n)$ of dimension $n + 1$, and $M(n)$

[1] In [GP67b] the word irreducible is used instead of simple, but we will only use irreducible when talking about representations in this paper.

has a basis $(v_0, v_1, \ldots, v_n)$ such that for all i, $0 \le i \le n$

$$\begin{aligned}
H_3 v_i &= (\tfrac{1}{2}n - i)v_i,\\
H_+ v_i &= \begin{cases} (n-i+1)v_{i-1} & \text{if } i > 0,\\ 0 & \text{if } i = 0,\end{cases}\\
H_- v_i &= \begin{cases} (i+1)v_{i+1} & \text{if } i < n,\\ 0 & \text{if } i = n.\end{cases}
\end{aligned}$$

From this we build a new basis by taking

$$w_i = \frac{1}{\sqrt{\binom{n}{i}}} v_i,$$

Note that

$$H_3 w_i = \frac{1}{\sqrt{\binom{n}{i}}} H_3 v_i = \frac{1}{\sqrt{\binom{n}{i}}} (\tfrac{1}{2}n - i)v_i = (\tfrac{1}{2}n - i)w_i$$

for all i, $0 \le i \le n$, and clearly still

$$\begin{aligned}
H_+ w_0 &= 0,\\
H_- w_n &= 0.
\end{aligned}$$

But for i, $0 < i \le n$

$$\begin{aligned}
H_+ w_i &= \frac{1}{\sqrt{\binom{n}{i}}} H_+ v_i = \frac{1}{\sqrt{\binom{n}{i}}} (n-i+1)v_{i-1}\\
&= \sqrt{\frac{\binom{n}{i-1}}{\binom{n}{i}}}(n-i+1)\frac{1}{\sqrt{\binom{n}{i-1}}} v_{i-1}\\
&= \sqrt{\frac{i}{n-i+1}}(n-i+1)w_{i-1} = \sqrt{(n-i+1)i}\,w_{i-1},
\end{aligned}$$

and for i, $0 \le i < n$

$$\begin{aligned}
H_- w_i &= \frac{1}{\sqrt{\binom{n}{i}}} H_- v_i = \frac{1}{\sqrt{\binom{n}{i}}} (i+1)v_{i+1}\\
&= \sqrt{\frac{\binom{n}{i+1}}{\binom{n}{i}}}(i+1)\frac{1}{\sqrt{\binom{n}{i+1}}} v_{i+1}\\
&= \sqrt{\frac{n-i}{i+1}}(i+1)w_{i+1} = \sqrt{(n-i)(i+1)}\,w_{i+1}.
\end{aligned}$$

Finally write $\ell = \frac{1}{2}n$. We will re-index with $m = \frac{1}{2}n - i = \ell - i$ by setting

$$e_m = w_{\ell-m}$$

for $m \in \{-\ell, -\ell+1, \dots, \ell-1, \ell\}$. Thus we get

$$H_3 e_m = H_3 w_{\ell-m} = (\ell - (\ell - m))w_{\ell-m} = m e_m,$$

and since $e_\ell = w_0$ and $e_{-\ell} = w_n$ also

$$\begin{aligned} H_+ e_\ell &= 0, \\ H_- e_{-\ell} &= 0. \end{aligned}$$

And for $m \in \{-\ell, -\ell+1, \dots, \ell-2, \ell-1\}$ we get

$$\begin{aligned} H_+ e_m = H_+ w_{\ell-m} &= \sqrt{(n-(\ell-m)+1)(\ell-m)}\, w_{\ell-m-1} \\ &= \sqrt{(\ell+m+1)(\ell-m)}\, e_{m+1}, \end{aligned}$$

while for $m \in \{-\ell+1, -\ell+2, \dots, \ell-1, \ell\}$ we get

$$\begin{aligned} H_- e_m = H_- w_{\ell-m} &= \sqrt{(n-(\ell-m))(\ell-m+1)}\, w_{\ell-m+1} \\ &= \sqrt{(\ell+m)(\ell-m+1)}\, e_{m-1}. \end{aligned}$$

Thus we get the following Lemma:

Lemma 1.4. *Every simple finite dimensional L_k-module is uniquely given by a number $\ell \in \frac{1}{2}\mathbf{Z}_{\geq 0}$. For such ℓ the unique simple L_k-module $M(2\ell)$ has dimension $2\ell+1$, and $M(2\ell)$ has a basis $(e_{-\ell}, e_{-\ell+1}, \dots, e_{\ell-1}, e_\ell)$ such that for all $m \in \{-\ell, -\ell+1, \dots, \ell-1, \ell\}$ we have*

$$\begin{aligned} H_3 e_m &= m e_m, \\ H_+ e_m &= \begin{cases} \sqrt{(\ell+m+1)(\ell-m)}\, e_{m+1} & \text{if } m \neq \ell, \\ 0 & \text{if } m = \ell, \end{cases} \\ H_- e_m &= \begin{cases} \sqrt{(\ell+m)(\ell-m+1)}\, e_{m-1} & \text{if } m \neq -\ell, \\ 0 & \text{if } m = -\ell. \end{cases} \end{aligned} \tag{1.5}$$

1.1.1 Formulae for the operators $H_+, H_-, H_3, F_+, F_-, F_3$

Let M be a Harish-Chandra L-module. Then we have linear operators $H_+, H_-, H_3, F_+, F_-, F_3 \colon M \to M$ satisfying commutation relations as in eq. (1.2), and we want to give expressions for these in terms of other linear operators $E_+, E_-, D_+, D_-, D_0 \colon M \to M$.

We will denote by R_ℓ a finite dimensional L-module which is a (finite) direct sum of L_k-modules $M(2\ell+1)$ for the same number $\ell \in \frac{1}{2}\mathbf{Z}_{\geq 0}$. Then M is a direct sum of the subspaces R_ℓ since M is Harish-Chandra, and from lemma 1.4

we know that R_ℓ can be written as the direct sum of subspaces $R_{\ell,m}$, where $R_{\ell,m}$ are eigenspaces for H_3 such that

$$H_3\xi = m\xi \tag{1.6}$$

for $m\in\{-\ell,-\ell+1,\dots,\ell-1,\ell\}$ and $\xi\in R_{l,m}$. We will use the decomposition

$$M = \bigoplus_{\substack{\ell\in\frac{1}{2}\mathbf{Z}_{\geq 0}\\ m\in\{-\ell,-\ell+1,\dots,\ell-1,\ell\}}} R_{\ell,m} = \bigoplus_{\ell,m} R_{\ell,m}$$

throughout this paper.

By lemma 1.4 we also have that H_+ and H_- maps the $R_{\ell,m}$ into each other as follows:

$$\begin{aligned} &H_+\colon R_{\ell,m}\to R_{\ell,m+1} &&\text{if } -\ell\leq m<\ell, &&H_+\colon R_{\ell,\ell}\to 0,\\ &H_-\colon R_{\ell,m}\to R_{\ell,m-1} &&\text{if } -\ell< m\leq\ell, &&H_-\colon R_{\ell,-\ell}\to 0.\end{aligned}$$

Hence we have linear operators $H_+H_-, H_-H_+\colon R_{\ell,m}\to R_{\ell,m}$, and by eq. (1.5) we see that

$$\begin{aligned} H_+H_-\xi &= \sqrt{(\ell+(m-1)+1)(\ell-(m-1))}\sqrt{(\ell+m)(\ell-m+1)}\xi\\ &= (\ell+m)(\ell-m+1)\xi,\\ H_-H_+\xi &= \sqrt{(\ell+(m+1))(\ell-(m+1)+1)}\sqrt{(\ell+m+1)(\ell-m)}\xi\\ &= (\ell+m+1)(\ell-m)\xi.\end{aligned} \tag{1.7}$$

Note that this also covers the cases $m=\ell$ and $m=-\ell$.

Now we define $E_+\colon R_{\ell,m}\to R_{\ell,m+1}$ and $E_-\colon R_{\ell,m}\to R_{\ell,m-1}$ to be the linear maps satisfying

$$\begin{aligned} H_+\xi &= \begin{cases}\sqrt{(\ell+m+1)(\ell-m)}E_+\xi & \text{if } m\neq\ell\\ 0 & \text{if } m=\ell,\end{cases}\\ H_-\xi &= \begin{cases}\sqrt{(\ell+m)(\ell-m+1)}E_-\xi & \text{if } m\neq-\ell\\ 0 & \text{if } m=\ell\end{cases}\end{aligned} \tag{1.8}$$

for $\xi\in R_{\ell,m}$. Comparing eq. (1.8) and eq. (1.7) we see that

$$\begin{aligned} E_+E_-\xi &= \xi &&\text{if } m\neq-\ell\\ E_-E_+\xi &= \xi &&\text{if } m\neq\ell.\end{aligned}$$

Thus $E_+\colon R_{\ell,m}\to R_{\ell,m+1}$ and $E_-\colon R_{\ell,m+1}\to R_{\ell,m}$ are isomorphisms for $m\neq\ell$ and they are each others inverse.

Now note that H_+, H_-, and H_3 are completely determined by eq. (1.6) and eq. (1.8), so we just need to find maps to determine F_+, F_-, and F_3 now, while making sure that we get commutation relations as in eq. (1.2).

Consider maps D_0 and D_+ defined on $M=\bigoplus_{\ell,m}R_{\ell,m}$ and D_- defined on the direct sum without the summands $R_{\ell,\ell}$ and $R_{\ell,-\ell}$ such that $D_0R_{\ell,m}\subset R_{\ell,m}$, $D_+R_{\ell,m}\subset R_{\ell+1,m}$, and $D_-R_{\ell,m}\subset R_{\ell-1,m}$ and the diagrams

$$
\begin{tikzcd}
R_{\ell-1,m+1} & R_{\ell,m+1} \arrow[l, "D_-"'] \\
R_{\ell-1,m} \arrow[u, "E_+"] & R_{\ell,m} \arrow[l, "D_-"] \arrow[u, "E_+"']
\end{tikzcd}
\qquad
\begin{tikzcd}
R_{\ell,m+1} \arrow[r, "D_0"] & R_{\ell,m+1} \\
R_{\ell,m} \arrow[u, "E_+"] \arrow[r, "D_0"'] & R_{\ell,m} \arrow[u, "E_+"']
\end{tikzcd}
$$

$$
\begin{tikzcd}
R_{\ell,m+1} \arrow[r, "D_+"] & R_{\ell+1,m+1} \\
R_{\ell,m} \arrow[u, "E_+"] \arrow[r, "D_+"'] & R_{\ell+1,m+1} \arrow[u, "E_+"']
\end{tikzcd}
$$

commute, when $-\ell+1 \leq m < \ell-1$ in the top left diagram, $-\ell \leq m < \ell$ in the other two diagrams. We need quite a lot of work to find a way to describe F_+, F_-, and F_3 from these maps.

We already have that $L_k = \operatorname{span}_{\mathbf{C}}(h_+, h_-, h_3)$, but now we will also consider $L_p = \operatorname{span}_{\mathbf{C}}(f_+, f_-, f_3)$. Equation (1.2) gives us that $[L_k, L_p] \subset L_p$, so by the adjoint representation we can see L_p as an L_k-module, and again by eq. (1.2) we see that L_p is a simple L_k-module: If V is an L_k-submodule and we have a non-zero element $f = af_+ + bf_- + cf_3 \in V$ for some $a, b, c \in \mathbf{C}$ not all zero. Then

$$
\begin{aligned}
[h_+, af_+ + bf_- + cf_3] &= 2bf_3 - cf_+, \\
[h_-, af_+ + bf_- + cf_3] &= -2af_3 + cf_-, \\
[h_3, af_+ + bf_- + cf_3] &= af_+ - bf_-.
\end{aligned}
$$

If $c \neq 0$, we get that

$$
\begin{aligned}
[h_3, [h_+, f]] &= [h_3, 2bf_3 - cf_+] = -cf_+, \\
[h_3, [h_-, f]] &= [h_3, -2af_3 + cf_-] = -cf_-,
\end{aligned}
$$

so we see that $f_+, f_- \in V$, and thus also $[h_+, \frac{1}{2}f_-] = f_3 \in V$, so $V = L_p$. If on the other hand $c = 0$, then

$$
\begin{aligned}
[h_-, f] &= -2af_3, \\
[h_+, f] &= 2bf_3,
\end{aligned}
$$

so since either $a \neq 0$ or $b \neq 0$, we see that $f_3 \in V$, and thus also $[h_+, -f_3] = f_+ \in V$ and $[h_-, f_3] = f_- \in V$, so $V = L_p$. Hence L_p is indeed a simple L_k-module. Now since L_p is a simple finite dimensional L_k-module of dimension 3, we have that $L_p \simeq M(2)$ as L_k-modules.

In general given two L-modules V and W, we consider the tensor product $V \otimes W$ over $\mathbf{C}$ of the underlying vector spaces as an L-module via the action

$$
x.(v \otimes w) = x.v \otimes w + v \otimes x.w,
$$

cf. [Hum72, p. 26].

Now we are interested in the L_k-module $L_p \otimes M$, where M is a Harish-Chandra L-module as before. Specifically we will show that

$$\begin{aligned} \psi\colon L_p \otimes M &\to M \\ x \otimes v &\mapsto x.v \end{aligned} \tag{1.9}$$

is a homomorphism of L_k-modules. It is clear that ψ is linear, and for $y \in L_k$ we see that

$$y.(x \otimes v) = y.x \otimes v + x \otimes y.v = [y, x] \otimes v + x \otimes y.v,$$

for $x \otimes v \in L_p \otimes M$, since the action in L_p is by the adjoint representation. So

$$\begin{aligned} \psi(y.(x \otimes v)) &= \psi([y, x] \otimes v) + \psi(x \otimes y.v) = [y, x].v + x.(y.v) \\ &= y.(x.v) - x.(y.v) + x.(y.v) = y.(x.v) = y.\psi(x \otimes v), \end{aligned}$$

i.e. ψ is indeed a homomorphism of L_k-modules.

Now we note that $M = \bigoplus_\ell R_\ell$, so

$$L_p \otimes M = L_p \otimes \big(\bigoplus_\ell R_\ell\big) \simeq \bigoplus_\ell (L_p \otimes R_\ell),$$

as L_k-modules, and since also R_ℓ is direct sum of finitely many copies of $M(2\ell)$, we see that

$$\begin{aligned} L_p \otimes R_\ell &\simeq M(2) \otimes \big(M(2\ell)^1 \oplus M(2\ell)^2 \oplus \cdots \oplus M(2\ell)^r\big) \\ &\simeq (M(2) \otimes M(2\ell)^1) \oplus (M(2) \otimes M(2\ell)^2) \oplus \cdots \oplus (M(2) \otimes M(2\ell)^r), \end{aligned}$$

as L_k-modules, since $L_p \simeq M(2)$. Here the superscripts are just indices for the different $M(2\ell)$. Thus we want to describe the L_k-modules $M(2) \otimes M(2\ell)$, which we will do by first describing the $\mathfrak{sl}(2, \mathbf{C})$-modules $V(2) \otimes V(2\ell)$ and then translating back to a solution to our problem by the isomorphism of eq. (1.3).

1.1.2 Describing $V(2) \otimes V(n)$

Let $2\ell = n \in \mathbf{N}$. We want to show that

$$V(2) \otimes V(n) \simeq \begin{cases} V(n-2) \oplus V(n) \oplus V(n+2) & \text{if } n \geq 2, \\ V(3) \oplus V(1) & \text{if } n = 1, \\ V(2) & \text{if } n = 0. \end{cases} \tag{1.10}$$

Note that in all cases there is a summand $V(n+2)$. We can show the above by considerations using formal characters. We will use the notation of [Jan16, Chapter 8], specifically we will do calculations with the functions $e(\lambda)\colon H^* \to \mathbf{Z}$ for $\lambda \in H^*$. Firstly note that in general

$$\mathrm{ch}_V = \sum_{\lambda \in H^*} (\dim V_\lambda) e(\lambda),$$

and use the notation $V(n)_k$ for $V(\lambda)_\mu$ and $e(n)$ for $e(\lambda)$ with $\lambda, \mu \in H^*$ such that $\lambda(h) = n$ and $\mu(h) = k$. We get that

$$\mathrm{ch}_{V(2)} = e(-2) + e(0) + e(2)$$

and

$$\mathrm{ch}_{V(n)} = \sum_{i=0}^{n} e(n-2i),$$

since

$$\dim V(n)_k = \begin{cases} 1 & \text{if } k = n-2i \text{ for some } i \in \{0,1,\ldots,n\}, \\ 0 & \text{otherwise.} \end{cases}$$

Now since $e(\lambda) * e(\mu) = e(\lambda + \mu)$ in general cf. [Jan16, p. 93], we see that for $n \geq 2$

$$\begin{aligned}
\mathrm{ch}_{V(2)\otimes V(n)} &= \mathrm{ch}_{V(2)} * \mathrm{ch}_{V(n)} = e(-2) * \mathrm{ch}_{V(n)} + e(0) * \mathrm{ch}_{V(n)} + e(2) * \mathrm{ch}_{V(n)} \\
&= \sum_{i=0}^{n} e(n-2-2i) + \mathrm{ch}_{V(n)} + \sum_{i=0}^{n} e(n+2-2i) \\
&= e(-n-2) + e(-n) + \sum_{i=0}^{n-2} e(n-2-2i) + \mathrm{ch}_{V(n)} \\
&\quad + \sum_{i=0}^{n} e(n+2-2i) \\
&= \mathrm{ch}_{V(n-2)} + \mathrm{ch}_{V(n)} + \sum_{i=0}^{n+2} e(n+2-2i) \\
&= \mathrm{ch}_{V(n-2)} + \mathrm{ch}_{V(n)} + \mathrm{ch}_{V(n+2)} = \mathrm{ch}_{V(n-2)\oplus V(n)\oplus V(n+2)},
\end{aligned}$$

where the first equality follows from the fact that $\mathrm{ch}_{V\otimes W} = \mathrm{ch}_V * \mathrm{ch}_W$ in general, cf. [Hum72, p. 125]. Thus since two L-modules V and V' are isomorphic if and only if $\mathrm{ch}_V = \mathrm{ch}_{V'}$, cf. [Jan16, p. 90], we see that $V(2) \otimes V(n) \simeq V(n-2) \oplus V(n) \oplus V(n+2)$ if $n \geq 2$.

Likewise we see that

$$\begin{aligned}
\mathrm{ch}_{V(2)\otimes V(1)} &= \mathrm{ch}_{V(2)} * \mathrm{ch}_{V(1)} \\
&= \big(e(-2) + e(0) + e(2)\big) * e(-1) + \big(e(-2) + e(0) + e(2)\big) * e(1) \\
&= e(-3) + e(-1) + e(1) + e(-1) + e(1) + e(3) \\
&= \big(e(-3) + e(-1) + e(1) + e(3)\big) + \big(e(-1) + e(1)\big) \\
&= \mathrm{ch}_{V(3)} + \mathrm{ch}_{V(1)} = \mathrm{ch}_{V(3)\oplus V(1)}
\end{aligned}$$

and

$$\mathrm{ch}_{V(2)\otimes V(0)} = \mathrm{ch}_{V(2)} * \mathrm{ch}_{V(0)} = \mathrm{ch}_{V(2)} * e(0) = \mathrm{ch}_{V(2)},$$

so indeed $V(2) \otimes V(1) \simeq V(3) \oplus V(1)$ and $V(2) \otimes V(0) \simeq V(2)$.

Now consider (w_0, w_1, w_2) a basis for $V(2)$ and $(v_i \mid 0 \leq i \leq n)$ a basis for $V(n)$ such that both satisfies the conditions from eq. (1.4). Then for $w_i \otimes v_j \in V(2) \otimes V(n)$ with $i \in \{0,1,2\}$ and $j \in \{0,1,\dots,n\}$ we see that

$$\begin{aligned} h.(w_i \otimes v_j) &= h.w_i \otimes v_j + w_i \otimes h.v_j = (2-2i)w_i \otimes v_j + (n-2j)w_i \otimes v_j \\ &= (n - 2(i+j-1))w_i \otimes v_j. \end{aligned} \tag{1.11}$$

Hence $v_0 \otimes w_0$ is up to scalar multiple the only vector of weight $n+2$ in $V(2) \otimes V(n)$, so it is necessarily a highest weight vector generating the direct summand isomorphic to $V(n+2)$. Note that by eq. (1.10) we indeed have a direct summand isomorphic to $V(n+2)$ for all $n \in \mathbf{N}$. By $\mathfrak{sl}(2,\mathbf{C})$-theory, cf. [Jan16, p. 36], we know that this summand has a basis $(s_k \mid 0 \leq k \leq n+2)$ satisfying equations as in eq. (1.4), where

$$s_k \coloneqq \frac{1}{k!} y^k.(w_0 \otimes v_0). \tag{1.12}$$

By straightforward calculations, cf. Appendix A.1, we get for $n > 0$ that

$$\begin{aligned} s_0 &= w_0 \otimes v_0, && \\ s_1 &= w_1 \otimes v_0 + w_0 \otimes v_1 && \text{if } n > 0, \\ s_k &= w_2 \otimes v_{k-2} + w_1 \otimes v_{k-1} + w_0 \otimes v_k && \text{for } 2 \leq k \leq n, \\ s_{n+1} &= w_2 \otimes v_{n-1} + w_1 \otimes v_n && \text{if } n > 0, \\ s_{n+2} &= w_2 \otimes v_n. && \end{aligned} \tag{1.13}$$

In case $n = 0$ we likewise see that $s_1 = w_1 \otimes v_0$ and $s_2 = w_2 \otimes v_0$, and we note that (s_0, s_1, s_2) is a basis for $V(2) \otimes V(0) \simeq V(2)$.

Suppose now that $n \geq 1$. Note that by eq. (1.10) we have a direct summand isomorphic to $V(n)$, and by eq. (1.11) the weight space of weight n is spanned by $w_0 \otimes v_1$ and $w_1 \otimes v_0$, so the vector of highest weight n generating the summand corresponding to $V(n)$ must be of the form $aw_0 \otimes v_1 + bw_1 \otimes v_0$ for some $a, b \in \mathbf{C}$. Furthermore we know that for this vector generating the summand corresponding to $V(n)$, we must have that

$$\begin{aligned} 0 &= x.(aw_0 \otimes v_1 + bw_1 \otimes v_0) \\ &= ax.w_0 \otimes v_1 + aw_0 \otimes x.v_1 + bx.w_1 \otimes v_0 + bw_1 \otimes x.v_0 \\ &= 0 + a(n-1+1)w_0 \otimes v_0 + b(2-1+1)w_0 \otimes v_0 + 0 \\ &= (an+2b)w_0 \otimes v_0, \end{aligned}$$

i.e. $an + 2b = 0$ so $b = -\frac{n}{2}a$. This determines the vector generating the summand corresponding to $V(n)$ up to a scalar, so taking $a = 1$, we see that we can take

$$t_0 \coloneqq w_0 \otimes v_1 - \frac{n}{2} w_1 \otimes v_0$$

as our vector generating the summand corresponding to $V(n)$. As before $\mathfrak{sl}(2, \mathbf{C})$-theory now yields that this summand has a basis $(t_k \mid 0 \leq k \leq n)$ satisfying equations as in eq. (1.4), where

$$t_k \coloneqq \frac{1}{k!} y^k . t_0. \tag{1.14}$$

By straightforward calculations, cf. appendix A.1, we get that

$$\begin{aligned} t_0 &= w_0 \otimes v_1 - \frac{n}{2} w_1 \otimes v_0, \\ t_k &= (k+1) w_0 \otimes v_{k+1} - \frac{n-2k}{2} w_1 \otimes v_k \\ &\quad + (k-1-n) w_2 \otimes v_{k-1} \qquad\qquad \text{for } 1 \leq k \leq n-1, \\ t_n &= \frac{n}{2} w_1 \otimes v_n - w_2 \otimes v_{n-1}. \end{aligned} \tag{1.15}$$

Suppose now that $n \geq 2$. By eq. (1.10) we have a direct summand isomorphic to $V(n-2)$, and by eq. (1.11) the weight space of weight $n-2$ is spanned by $w_0 \otimes v_2$, $w_1 \otimes v_1$, and $w_2 \otimes v_0$, so the vector of highest weight $n-2$ generating the summand corresponding to $V(n)$ must be of the form $aw_0 \otimes v_2 + bw_1 \otimes v_1 + cw_2 \otimes v_0$ for some $a, b, c \in \mathbf{C}$. Furthermore we know that for this vector generating the summand corresponding to $V(n-2)$, we must have

$$\begin{aligned} 0 &= x.(aw_0 \otimes v_2 + bw_1 \otimes v_1 + cw_2 \otimes v_0) \\ &= aw_0 \otimes x.v_2 + bx.w_1 \otimes v_1 + bw_1 \otimes x.v_1 + cx.w_2 \otimes v_0 \\ &= a(n-2+1) w_0 \otimes v_1 + b(2-1+1) w_0 \otimes v_1 + b(n-1+1) w_1 \otimes v_0 \\ &\quad + c(2-2+1) w_1 \otimes v_0 \\ &= \big((n-1)a + 2b\big) w_0 \otimes v_1 + \big(bn + c\big) w_1 \otimes v_0, \end{aligned}$$

i.e. $a(n-1) + 2b = 0$ and $bn + c = 0$. Giving us $c = -bn$ and $b = -\frac{n-1}{2}a$, so

$$c = \frac{n(n-1)}{2} a.$$

This determines the vector generating the summand corresponding to $V(n-2)$ up to a scalar, so taking $a = 1$, we see that we can take

$$u_0 \coloneqq w_0 \otimes v_2 - \frac{n-1}{2} w_1 \otimes v_1 + \frac{n(n-1)}{2} w_2 \otimes v_0$$

as our vector generating the summand corresponding to $V(n-2)$. Again $\mathfrak{sl}(2,\mathbf{C})$-theory now yields that this summand has a basis $(u_k \mid 0 \leq k \leq n-2)$ satisfying equations as in eq. (1.4), where

$$u_k \coloneqq \frac{1}{k!} y^k . u_0. \tag{1.16}$$

By straightforward calculations, cf. appendix A.1, we get that

$$\begin{aligned} u_k = {} & \frac{(k+1)(k+2)}{2} w_0 \otimes v_{k+2} - \frac{(k+1)(n-k-1)}{2} w_1 \otimes v_{k+1} \\ & + \frac{(n-k)(n-k-1)}{2} w_2 \otimes v_k \end{aligned} \tag{1.17}$$

for $0 \leq k \leq n-2$.

Now we want to express $w_1 \otimes v_k$ for $0 \leq k \leq n$ in terms of the bases $(s_k \mid 0 \leq k \leq n+2)$, $(t_k \mid 0 \leq k \leq n)$, and $(u_k \mid 0 \leq k \leq n-2)$. A straightforward but long calculation, cf. Appendix A.2, yields that

$$w_1 \otimes v_k = \frac{2(k+1)(n+1-k)}{(n+1)(n+2)} s_{k+1} - \frac{2(n-2k)}{n(n+2)} t_k - \frac{4}{n(n+1)} u_{k-1} \tag{1.18}$$

for $0 < k < n$, while

$$w_1 \otimes v_0 = \frac{2}{n+2}(s_1 - t_0) \qquad \text{and} \qquad w_1 \otimes v_n = \frac{2}{n+2}(s_{n+1} + t_n) \tag{1.19}$$

if $n \geq 1$. If $n = 0$ we have already seen (just after eq. (1.13)) that $w_1 \otimes v_0 = s_1$.

Chapter 2

Linear relations

Bibliography

[GP67a] I. M. Gel'Fand and V. A. Ponomarev. 'Classification of Indecomposable Infinitesimal Representations of the Lorentz Group'. Trans. by Jack Ceder. In: *Dok1. Akad. Nauk SSSR* 8.5 (1967).

[GP67b] I. M. Gel'Fand and V. A. Ponomarev. *Indecomposable Representations of the Lorentz Group*. Trans. by B. Hartley. 1967.

[GP67c] I. M. Gel'Fand and V. A. Ponomarev. 'The Category of Harish-Chandra Modules over the Lie Algebra of the Lorentz Group'. Trans. by A. M. Scott. In: *Dok1. Akad. Nauk SSSR* 8.5 (1967).

[Hum72] James E. Humphreys. *Introduction to Lie Algebras and Representation Theory*. 1st ed. Vol. 9. Springer, 1972. ISBN: 978-0-387-90053-7.

[Jan16] Jens Carsten Jantzen. *Lie Algebras*. Lecture notes from the Lie algebra course. 2016.

Appendix A

Calculations

Throughout the paper there are situations where we need to do some straightforward but rather long calculations, so to clean up the exposition somewhat we will collect most of these calculations in this appendix and then just use the results in the paper.

A.1 Bases of $V(2) \otimes V(n)$

We want to describe the s_k's of eq. (1.12) more explicitly. We have that $s_0 = w_0 \otimes v_0$ and $s_k = \frac{1}{k!} y^k.s_0$, and we note that if $n > 0$ then

$$\begin{aligned} s_1 &= y.(w_0 \otimes v_0) = y.w_0 \otimes v_0 + w_0 \otimes y.v_0 \\ &= w_1 \otimes v_0 + w_0 \otimes v_1 \end{aligned}$$

and

$$\begin{aligned} s_2 &= \tfrac{1}{2} y.s_1 \\ &= \tfrac{1}{2} y.w_1 \otimes v_0 + \tfrac{1}{2} w_1 \otimes y.v_0 + \tfrac{1}{2} y.w_0 \otimes v_1 + w_0 \otimes \tfrac{1}{2} y.v_1 \\ &= w_2 \otimes v_0 + \tfrac{1}{2} w_1 \otimes v_1 + \tfrac{1}{2} w_1 \otimes v_1 + w_0 \otimes v_2 \\ &= w_2 \otimes v_0 + w_1 \otimes v_1 + w_0 \otimes v_2. \end{aligned}$$

Inductively we see that

$$s_k = w_2 \otimes v_{k-2} + w_1 \otimes v_{k-1} + w_0 \otimes v_k$$

for $k \leq n$, since the base case holds and given the equality for $k < n$ we get

$$\begin{aligned} s_{k+1} &= \frac{1}{k+1} y.s_k \\ &= w_2 \otimes \frac{1}{k+1} y.v_{k-2} + \frac{1}{k+1} y.w_1 \otimes v_{k-1} + w_1 \otimes \frac{1}{k+1} y.v_{k-1} \\ &\quad + \frac{1}{k+1} y.w_0 \otimes v_k + w_0 \otimes \frac{1}{k+1} y.v_k \end{aligned}$$

$$= \frac{k-1}{k+1} w_2 \otimes v_{k-1} + \frac{2}{k+1} w_2 \otimes v_{k-1} + \frac{k}{k+1} w_1 \otimes v_k + \frac{1}{k+1} w_1 \otimes v_k$$
$$+ w_0 \otimes v_{k+1}$$
$$= w_2 \otimes v_{k-1} + w_1 \otimes v_k + w_0 \otimes v_{k+1}.$$

We likewise see that for $k = n+1$ the last term vanishes, so we have $s_{k+1} = w_2 \otimes v_{n-1} + w_1 \otimes v_n$, and for $k = n+2$ the two last terms vanish, so we get $s_{k+2} = w_2 \otimes v_n$. Thus altogether we get the description in eq. (1.13).

Suppose now that $n \geq 1$. We want to describe the t_k's of eq. (1.14) more explicitly. We have that $t_0 = w_0 \otimes v_1 - \frac{n}{2} w_1 \otimes v_0$ and $t_k = \frac{1}{k!} y^k . t_0$. We see that

$$\begin{aligned} t_1 &= y.\big(w_0 \otimes v_1 - \frac{n}{2} w_1 \otimes v_0\big) \\ &= y.w_0 \otimes v_1 + w_0 \otimes y.v_1 - \frac{n}{2} y.w_1 \otimes v_0 + \frac{n}{2} w_1 \otimes y.v_0 \\ &= w_1 \otimes v_1 + 2w_0 \otimes v_2 - n w_2 \otimes v_0 - \frac{n}{2} w_1 \otimes v_1 \\ &= 2w_0 \otimes v_2 - \frac{n-2}{2} w_1 \otimes v_1 - n w_2 \otimes v_0, \end{aligned}$$

and inductively we get that

$$t_k = (k+1) w_0 \otimes v_{k+1} - \frac{n-2k}{2} w_1 \otimes v_k + (k-1-n) w_2 \otimes v_{k-1}$$

for $1 \leq k \leq n-1$, since the base case holds and given the equality for $k < n-1$ we get

$$\begin{aligned} t_{k+1} &= \frac{1}{k+1} y.t_k \\ &= y.w_0 \otimes v_{k+1} + w_0 \otimes y.v_{k+1} - \frac{n-2k}{2(k+1)} y.w_1 \otimes v_k \\ &\quad - \frac{n-2k}{2(k+1)} w_1 \otimes y.v_k + \frac{k-1-n}{k+1} w_2 \otimes y.v_{k-1} \\ &= w_1 \otimes v_{k+1} + (k+2) w_0 \otimes v_{k+2} - \frac{n-2k}{k+1} w_2 \otimes v_k \\ &\quad - \frac{n-2k}{2} w_1 \otimes v_{k+1} + \frac{(k-1-n)k}{k+1} w_2 \otimes v_k \\ &= (k+2) w_0 \otimes v_{k+2} - \frac{n-2(k+1)}{2} w_1 \otimes v_{k+1} \\ &\quad + \left(\frac{k^2 - k - nk - n + 2k}{k+1} \right) w_2 \otimes v_k \\ &= (k+2) w_0 \otimes v_{k+2} - \frac{n-2(k+1)}{2} w_1 \otimes v_{k+1} + (k-n) w_2 \otimes v_k, \end{aligned}$$

where we in the last equality use that $(k+1)(k-n) = k^2 - nk + k - n = k^2 - k - nk - n + 2k$. We likewise see that for $k = n$ the first term vanishes so

$$t_n = \frac{n}{2} w_1 \otimes v_n - w_2 \otimes v_{n-1}.$$

Thus we altogether get the description in eq. (1.15).

Suppose now that $n \geq 2$. We want to describe the u_k's of eq. (1.16) more explicitely. We have that

$$u_0 \coloneqq w_0 \otimes v_2 - \frac{n-1}{2} w_1 \otimes v_1 + \frac{n(n-1)}{2} w_2 \otimes v_0$$

and $u_k = \frac{1}{k!} y^k . u_0$. We see inductively that

$$\begin{aligned} u_k = {} & \frac{(k+1)(k+2)}{2} w_0 \otimes v_{k+2} - \frac{(k+1)(n-k-1)}{2} w_1 \otimes v_{k+1} \\ & + \frac{(n-k)(n-k-1)}{2} w_2 \otimes v_k \end{aligned}$$

for $0 \leq k \leq n-2$, since the base case holds and given the equality for $k < n-2$ we get

$$\begin{aligned} u_{k+1} &= \frac{1}{k+1} y . u_k \\ &= \frac{k+2}{2} y . w_0 \otimes v_{k+2} + \frac{k+2}{2} w_0 \otimes y . v_{k+2} \\ &\quad - \frac{n-k-1}{2} y . w_1 \otimes v_{k+1} - \frac{n-k-1}{2} w_1 \otimes y . v_{k+1} \\ &\quad + \frac{(n-k)(n-k-1)}{2(k+1)} w_2 \otimes y . v_k \\ &= \frac{k+2}{2} w_1 \otimes v_{k+2} + \frac{(k+2)(k+3)}{2} w_0 \otimes v_{k+3} \\ &\quad - (n-k-1) w_2 \otimes v_{k+1} - \frac{(n-k-1)(k+2)}{2} w_1 \otimes v_{k+2} \\ &\quad + \frac{(n-k)(n-k-1)}{2} w_2 \otimes v_{k+1} \\ &= \frac{(k+2)(k+3)}{2} w_0 \otimes v_{k+3} \\ &\quad - \frac{(n-k-1)(k+2) - (k+2)}{2} w_1 \otimes v_{k+2} \\ &\quad + \frac{(n-k)(n-k-1) - 2(n-k-1)}{2} w_2 \otimes v_{k+1} \\ &= \frac{(k+2)(k+3)}{2} w_0 \otimes v_{k+3} \\ &\quad - \frac{(k+2)(n-k-2)}{2} w_1 \otimes v_{k+2} \\ &\quad + \frac{(n-k-1)(n-k-2)}{2} w_2 \otimes v_{k+1}. \end{aligned}$$

Thus we altogether get the description in eq. (1.17).

A.2 Finding $w_1 \otimes v_k$

Using the bases $(s_k \mid 0 \leq k \leq n+2)$ of eq. (1.13), $(t_k \mid 0 \leq k \leq n)$ of eq. (1.15), and $(u_k \mid 0 \leq k \leq n-2)$ of eq. (1.17), we see that

$$
\begin{aligned}
&\frac{2(k+1)(n+1-k)}{(n+1)(n+2)} s_{k+1} - \frac{2(n-2k)}{n(n+2)} t_k - \frac{4}{n(n+1)} u_{k-1} \\
&= \frac{2(k+1)(n+1-k)}{(n+1)(n+2)} \Big(w_0 \otimes v_{k+1} + w_1 \otimes v_k + w_2 \otimes v_{k-1} \Big) \\
&\quad - \frac{2(n-2k)}{n(n+2)} \Big((k+1) w_0 \otimes v_{k+1} - \frac{n-2k}{2} w_1 \otimes v_k \\
&\qquad + (k-1-n) w_2 \otimes v_{k-1} \Big) \\
&\quad - \frac{4}{n(n+1)} \Big(\frac{k(k+1)}{2} w_0 \otimes v_{k+1} - \frac{k(n-k)}{2} w_1 \otimes v_k \\
&\qquad + \frac{(n-k+1)(n-k)}{2} w_2 \otimes v_{k-1} \Big) \\
&= \frac{\begin{pmatrix} 2(k+1)(n+1-k)n - 2(n-2k)(k+1)(n+1) \\ \hfill - 2k(k+1)(n+2) \end{pmatrix}}{n(n+1)(n+2)} w_0 \otimes v_{k+1} \\
&\quad + \frac{\begin{pmatrix} 2(k+1)(n+1-k)n + (n-2k)(n-2k)(n+1) \\ \hfill + 2k(n-k)(n+2) \end{pmatrix}}{n(n+1)(n+2)} w_1 \otimes v_k \\
&\quad + \frac{\begin{pmatrix} 2(k+1)(n+1-k)n - 2(n-2k)(k-1-n)(n+1) \\ \hfill - 2(n-k+1)(n-k)(n+2) \end{pmatrix}}{n(n+1)(n+2)} w_2 \otimes v_{k-1} \\
&= 2(k+1) \frac{(n+1-k)n - (n-2k)(n+1) - k(n+2)}{n(n+1)(n+2)} w_0 \otimes v_{k+1} \\
&\quad + \frac{\begin{pmatrix} 2(k+1)(n+1-k)n + (n-2k)(n-2k)(n+1) \\ \hfill + 2k(n-k)(n+2) \end{pmatrix}}{n(n+1)(n+2)} w_1 \otimes v_k \\
&\quad + 2(n+1-k) \frac{(k+1)n + (n-2k)(n+1) - (n-k)(n+2)}{n(n+1)(n+2)} w_2 \otimes v_{k-1}.
\end{aligned}
$$

Now we note that

$$
\begin{aligned}
&(n+1-k)n - (n-2k)(n+1) - k(n+2) \\
&\qquad = n\Big((n+1-k) - (n-2k) - k \Big) - (n-2k) - 2k \\
&\qquad = n - (n-2k) - 2k = 0,
\end{aligned}
$$

and

$$\begin{aligned}
&(k+1)n + (n-2k)(n+1) - (n-k)(n+2) \\
&\qquad = n\Big((k+1) + (n-2k) - (n-k)\Big) + (n-2k) - 2(n-k) \\
&\qquad = n + n - 2k - 2n + 2k = 0,
\end{aligned}$$

while

$$\begin{aligned}
&2(k+1)(n+1-k)n + (n-2k)(n-2k)(n+1) + 2k(n-k)(n+2) \\
&= n\Big(2(k+1)(n+1-k) + (n-2k)(n+1) + 2k(n-k)\Big) \\
&\quad - 2k(n-2k)(n+1) + 4k(n-k) \\
&= n\Big(2(k+1)(n+1-k) + (n-2k)(n+1) + 2k(n-k)\Big) \\
&\quad - 2kn(n-2k) - 2k(n-2k) + 4k(n-k) \\
&= n\Big(2(k+1)(n+1-k) + (n-2k)(n+1) + 2k(n-k)\Big) \\
&\quad - 2kn(n-2k) + 2kn \\
&= n\Big(2(k+1)(n+1-k) + (n-2k)(n+1) + 2k(n-k) - 2k(n-2k) \\
&\qquad + 2k\Big),
\end{aligned}$$

where

$$\begin{aligned}
&2(k+1)(n+1-k) + (n-2k)(n+1) + 2k(n-k) - 2k(n-2k) + 2k \\
&= (n+1)\Big(2(k+1) + (n-2k)\Big) - 2k(k+1) \\
&\quad + 2k\Big((n-k) - (n-2k) + 1\Big) \\
&= (n+1)(n+2) - 2k(k+1) + 2k(k+1) \\
&= (n+1)(n+2),
\end{aligned}$$

so

$$\begin{aligned}
&2(k+1)(n+1-k)n + (n-2k)(n-2k)(n+1) + 2k(n-k)(n+2) \\
&= n(n+1)(n+2).
\end{aligned}$$

Thus we see that

$$\begin{aligned}
&\frac{2(k+1)(n+1-k)}{(n+1)(n+2)} s_{k+1} - \frac{2(n-2k)}{n(n+2)} t_k - \frac{4}{n(n+1)} u_{k-1} \\
&= 0 + \frac{n(n+1)(n+2)}{n(n+1)(n+2)} w_1 \otimes v_k + 0 \\
&= w_1 \otimes v_k
\end{aligned}$$

giving us eq. (1.18).

Likewise for $n \geq 1$, we get that

$$\begin{aligned}
\frac{2}{n+2}(s_1 - t_0) &= \frac{2}{n+2}\Big(w_0 \otimes v_1 + w_1 \otimes v_0 - w_0 \otimes v_1 + \frac{n}{2} w_1 \otimes v_0\Big) \\
&= \frac{2}{n+2}\frac{n+2}{2} w_1 \otimes v_0 \\
&= w_1 \otimes v_0
\end{aligned}$$

and

$$\begin{aligned}
\frac{2}{n+2}(s_{n+1} + t_n) &= \frac{2}{n+2}\Big(w_2 \otimes v_{n+1} + w_1 \otimes v_n + \frac{n}{2} w_1 \otimes v_n - w_2 \otimes v_{n-1}\Big) \\
&= \frac{2}{n+2}\frac{n+2}{2} w_1 \otimes v_n \\
&= w_1 \otimes v_n
\end{aligned}$$

giving us eq. (1.19).