# Bachelorprojekt

Title

(subtitle)

Daniel Kongsgaard

Vejleder: Jens Carsten Jantzen

Aarhus Universitet

**Abstract**

Some text

# Contents

# Chapter 1

# Harish-Chandra modules over $\mathfrak{sl}(2, \mathbf{C}) \times \mathfrak{sl}(2, \mathbf{C})$

Let $G$ be a semisimple Lie group and let $G_k$ be its maximal compact subgroup. Denote by $L$ the semisimple Lie algebra of $G$ and denote by $L_k$ the Lie subalgebra corresponding to $G_k$.

**Definition 1.1.** *An $L$-module $M$ is a Harish-Chandra module if, regarded as an $L_k$-module, it can be written as a sum*

$$M = \bigoplus_i M_i$$

*of finite dimensional irreducible $L_k$-submodules $M_i$, where for each $M_{i_0}$ only finitely many $L_k$-submodules equivalent to $M_{i_0}$ occur in the decomposition of $M$.*

*A Harish-Chandra module $M$ is indecomposable if it cannot be decomposed into the direct sum of $L$-submodules.*

Our goal is to classify all indecomposable Harish-Chandra modules over $\mathfrak{sl}(2, \mathbf{C}) \times \mathfrak{sl}(2, \mathbf{C})$, where we by $\mathfrak{sl}(2, \mathbf{C}) \times \mathfrak{sl}(2, \mathbf{C})$ mean the following:

For $L, L'$ Lie algebras over $F$, we consider $L \times L' = L \oplus L'$ as a Lie algebra over $F$ with pointwise addition, multiplication given by $\alpha(a, b) = (\alpha a, \alpha b)$ for $\alpha \in F, a \in L, b \in L'$, and with Lie bracket $[(a_1, b_1), (a_2, b_2)] = ([a_1, a_2], [b_1, b_2])$ for $a_1, a_2 \in L, b_1, b_2 \in L'$.

**Remark 1.2.** Note that $L \times 0$ and $0 \times L'$ are ideals in $L \times L'$ as given above. Thus we see that $\mathfrak{sl}(2, \mathbf{C}) \times 0$ and $0 \times \mathfrak{sl}(2, \mathbf{C})$ are ideals in $\mathfrak{sl}(2, \mathbf{C}) \times \mathfrak{sl}(2, \mathbf{C})$ with

$$(\mathfrak{sl}(2, \mathbf{C}) \times 0) \oplus (0 \times \mathfrak{sl}(2, \mathbf{C})) = \mathfrak{sl}(2, \mathbf{C}) \times \mathfrak{sl}(2, \mathbf{C}),$$

so $\mathfrak{sl}(2, \mathbf{C}) \times \mathfrak{sl}(2, \mathbf{C})$ is semisimple. Hence it makes sense to talk about Harish-Chandra modules over $\mathfrak{sl}(2, \mathbf{C}) \times \mathfrak{sl}(2, \mathbf{C})$. $\triangle$

We fix the following as a standard basis for $\mathfrak{sl}(2,F)$:

$$x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Giving us the relations:

$$[x,y] = h, \qquad [h,x] = 2x, \qquad [h,y] = -2y, \tag{1.1}$$

cf. [Jan16, p. 35] or [Hum72, p. 6].

We claim now that

$$(x,x), \quad (y,y), \quad \tfrac{1}{2}(h,h), \quad (ix,-ix), \quad (iy,-iy), \quad \tfrac{1}{2}(ih,-ih)$$

is a basis of $\mathfrak{sl}(2,\mathbf{C}) \times \mathfrak{sl}(2,\mathbf{C})$. This is clearly the case since $\dim_{\mathbf{C}} \mathfrak{sl}(2,\mathbf{C}) = 3$, so $\dim_{\mathbf{C}} \mathfrak{sl}(2,\mathbf{C}) \times \mathfrak{sl}(2,\mathbf{C}) = 6$, and we see that the above elements span $\mathfrak{sl}(2,\mathbf{C}) \times \mathfrak{sl}(2,\mathbf{C})$; we have $\frac{1}{2}(x,x) - \frac{i}{2}(ix,-ix) = (x,0)$ and $\frac{1}{2}(x,x) + \frac{i}{2}(ix,-ix) = (0,x)$ and likewise with $h$ and $y$.

Putting

$$\begin{aligned} h_+ &= (x,x), & h_- &= (y,y), & h_3 &= \tfrac{1}{2}(h,h), \\ f_+ &= (ix,-ix), & f_- &= (iy,-iy), & f_3 &= \tfrac{1}{2}(ih,-ih) \end{aligned}$$

we get the following commutation relations between these basis elements:

$$\begin{aligned}
[h_+,h_3] &= \tfrac{1}{2}([x,h],[x,h]) = \tfrac{1}{2}(-2x,-2x) = -(x,x) = -h_+, \\
[h_-,h_3] &= \tfrac{1}{2}([y,h],[y,h]) = \tfrac{1}{2}(2y,2y) = (y,y) = h_-, \\
[h_+,h_-] &= ([x,y],[x,y]) = (h,h) = 2h_3, \\
[h_+,f_+] &= ([x,ix],[x,-ix]) = 0, \\
[h_-,f_-] &= ([y,iy],[y,-iy]) = 0, \\
[h_3,f_3] &= \tfrac{1}{4}([h,ih],[h,-ih]) = 0, \\
[h_+,f_3] &= \tfrac{1}{2}([x,ih],[x,-ih]) = \tfrac{1}{2}(-2ix,2ix) = -(ix,-ix) = -f_+, \\
[h_-,f_3] &= \tfrac{1}{2}([y,ih],[y,-ih]) = \tfrac{1}{2}(2iy,-2iy) = (iy,-iy) = f_-, \\
[h_+,f_-] &= ([x,iy],[x,-iy]) = (ih,-ih) = 2f_3, \\
[h_3,f_-] &= \tfrac{1}{2}([h,iy],[h,-iy]) = \tfrac{1}{2}(-2iy,2iy) = -(iy,-iy) = -f_-, \\
[h_-,f_+] &= ([y,ix],[y,-ix]) = (-ih,ih) = -(ih,-ih) = -2f_3, \\
[h_3,f_+] &= \tfrac{1}{2}([h,ix],[h,-ix]) = \tfrac{1}{2}(2ix,-2ix) = (ix,-ix) = f_+, \\
[f_+,f_3] &= \tfrac{1}{2}([ix,ih],[-ix,-ih]) = \tfrac{1}{2}(2x,2x) = (x,x) = h_+, \\
[f_-,f_3] &= \tfrac{1}{2}([iy,ih],[-iy,-ih]) = \tfrac{1}{2}(-2y,-2y) = -(y,y) = -h_-, \\
[f_+,f_-] &= ([ix,iy],[-ix,-iy]) = (-h,-h) = -(h,h) = -2h_3.
\end{aligned} \tag{1.2}$$

**Remark 1.3.** Note that these are the same relations as for the complexification of the Lie algebra $L$ of the proper Lorentz group in [GP67b, p. 5], so $L$ is isomorphic to $\mathfrak{sl}(2,\mathbf{C}) \times \mathfrak{sl}(2,\mathbf{C})$. This explains the equivalence of the work in this paper and the work in [GP67a; GP67b; GP67c]. △

Now let $L = \mathfrak{sl}(2, \mathbf{C}) \times \mathfrak{sl}(2, \mathbf{C})$ and denote by $L_k$ the Lie subalgebra of $L$ with basis $h_+, h_-, h_3$. One can show that this corresponds to a maximal compact subgroup in the way described in definition 1.1, but that is beyond what we will do in this paper. Note that the above commutation relations gives us that

$$[h_+, h_-] = 2h_3, \qquad [2h_3, h_+] = 2h_+, \qquad [2h_3, h_-] = -2h_-$$

Comparing with eq. (1.1) we see that we have an isomorphism

$$\begin{aligned} L_k &\to \mathfrak{sl}(2, \mathbf{C}) \\ h_+ &\mapsto x \\ h_- &\mapsto y \\ 2h_3 &\mapsto h, \end{aligned} \tag{1.3}$$

so we can use $\mathfrak{sl}(2, \mathbf{C})$-theory when we want to describe $L_k$-modules.

## 1.1 Representations of $L_k$

Let $V$ be a $\mathbf{C}$ vector space and $\rho\colon L_k \to \mathfrak{gl}(V)$ a representation of $L_k$. We will use the notation $\rho(a) = A$ for $a \in L_k$ switching to upper case letters when we talk about the representation corresponding to a given element. Note that we will switch freely between the language of representations of $L_k$ and the language of $L_k$-modules.

We will start out by describing the finite dimensional simple[1] $L_k$-modules. Recall cf. [Jan16, p. 36] that we know from $\mathfrak{sl}(2, \mathbf{C})$-theory that for integers $n \geq 0$ there exists a unique simple $\mathfrak{sl}(2, \mathbf{C})$-module $V(n)$ of dimension $n + 1$, and $V(n)$ has a basis $(v_0, v_1, \ldots, v_n)$ such that for all $i$, $0 \leq i \leq n$

$$\begin{aligned} h.v_i &= (n - 2i)v_i, \\ x.v_i &= \begin{cases} (n - i + 1)v_{i-1} & \text{if } i > 0, \\ 0 & \text{if } i = 0, \end{cases} \\ y.v_i &= \begin{cases} (i + 1)v_{i+1} & \text{if } i < n, \\ 0 & \text{if } i = n. \end{cases} \end{aligned} \tag{1.4}$$

Now using the isomorphism from eq. (1.3) we see that for integers $n \geq 0$ there exists a unique simple $L_k$-module $M(n)$ of dimension $n + 1$, and $M(n)$

[1] In [GP67b] the word irreducible is used instead of simple, but we will only use irreducible when talking about representations in this paper.

has a basis $(v_0, v_1, \dots, v_n)$ such that for all $i$, $0 \leq i \leq n$

$$\begin{aligned} H_3 v_i &= (\tfrac{1}{2}n - i)v_i, \\ H_+ v_i &= \begin{cases} (n-i+1)v_{i-1} & \text{if } i > 0, \\ 0 & \text{if } i = 0, \end{cases} \\ H_- v_i &= \begin{cases} (i+1)v_{i+1} & \text{if } i < n, \\ 0 & \text{if } i = n. \end{cases} \end{aligned} \tag{1.5}$$

From this we build a new basis by taking

$$w_i = \frac{1}{\sqrt{\binom{n}{i}}} v_i,$$

Note that

$$H_3 w_i = \frac{1}{\sqrt{\binom{n}{i}}} H_3 v_i = \frac{1}{\sqrt{\binom{n}{i}}} (\tfrac{1}{2}n - i)v_i = (\tfrac{1}{2}n - i)w_i$$

for all $i$, $0 \leq i \leq n$, and clearly still

$$\begin{aligned} H_+ w_0 &= 0, \\ H_- w_n &= 0. \end{aligned}$$

But for $i$, $0 < i \leq n$

$$\begin{aligned} H_+ w_i &= \frac{1}{\sqrt{\binom{n}{i}}} H_+ v_i = \frac{1}{\sqrt{\binom{n}{i}}} (n-i+1)v_{i-1} \\ &= \sqrt{\frac{\binom{n}{i-1}}{\binom{n}{i}}} (n-i+1) \frac{1}{\sqrt{\binom{n}{i-1}}} v_{i-1} \\ &= \sqrt{\frac{i}{n-i+1}} (n-i+1) w_{i-1} = \sqrt{(n-i+1)i}\, w_{i-1}, \end{aligned}$$

and for $i$, $0 \leq i < n$

$$\begin{aligned} H_- w_i &= \frac{1}{\sqrt{\binom{n}{i}}} H_- v_i = \frac{1}{\sqrt{\binom{n}{i}}} (i+1)v_{i+1} \\ &= \sqrt{\frac{\binom{n}{i+1}}{\binom{n}{i}}} (i+1) \frac{1}{\sqrt{\binom{n}{i+1}}} v_{i+1} \\ &= \sqrt{\frac{n-i}{i+1}} (i+1) w_{i+1} = \sqrt{(n-i)(i+1)}\, w_{i+1}. \end{aligned}$$

Finally write $\ell = \frac{1}{2}n$. We will re-index with $m = \frac{1}{2}(n-2i) = \ell - i$ by setting

$$e_m = w_{\ell - m}$$

for $m \in \{-\ell, -\ell+1, \ldots, \ell-1, \ell\}$. Thus we get

$$H_3 e_m = H_3 w_{\ell-m} = (\ell - (\ell - m))w_{\ell-m} = m e_m,$$

and since $e_\ell = w_0$ and $e_{-\ell} = w_n$ also

$$\begin{aligned} H_+ e_\ell &= 0, \\ H_- e_{-\ell} &= 0. \end{aligned}$$

And for $m \in \{-\ell, -\ell+1, \ldots, \ell-2, \ell-1\}$ we get

$$\begin{aligned} H_+ e_m = H_+ w_{\ell-m} &= \sqrt{(n-(\ell-m)+1)(\ell-m)}w_{\ell-m-1} \\ &= \sqrt{(\ell+m+1)(\ell-m)}e_{m+1}, \end{aligned}$$

while for $m \in \{-\ell+1, -\ell+2, \ldots, \ell-1, \ell\}$ we get

$$\begin{aligned} H_- e_m = H_- w_{\ell-m} &= \sqrt{(n-(\ell-m))(\ell-m+1)}w_{\ell-m+1} \\ &= \sqrt{(\ell+m)(\ell-m+1)}e_{m-1}. \end{aligned}$$

Thus we get the following Lemma:

**Lemma 1.4.** *Every simple finite dimensional $L_k$-module is uniquely given by a number $\ell \in \frac{1}{2}\mathbf{Z}_{\geq 0}$. For such $\ell$ the unique simple $L_k$-module $M(2\ell)$ has dimension $2\ell+1$, and $M(2\ell)$ has a basis $(e_{-\ell}, e_{-\ell+1}, \ldots, e_{\ell-1}, e_\ell)$ such that for all $m \in \{-\ell, -\ell+1, \ldots, \ell-1, \ell\}$ we have*

$$\begin{aligned} H_3 e_m &= m e_m, \\ H_+ e_m &= \begin{cases} \sqrt{(\ell+m+1)(\ell-m)}e_{m+1} & \text{if } m \neq \ell, \\ 0 & \text{if } m = \ell, \end{cases} \\ H_- e_m &= \begin{cases} \sqrt{(\ell+m)(\ell-m+1)}e_{m-1} & \text{if } m \neq -\ell, \\ 0 & \text{if } m = -\ell. \end{cases} \end{aligned} \tag{1.6}$$

### 1.1.1 Formulae for the operators $H_+, H_-, H_3, F_+, F_-, F_3$

Let $M$ be a Harish-Chandra $L$-module. Then we have linear operators $H_+, H_-, H_3, F_+, F_-, F_3 \colon M \to M$ satisfying commutation relations as in eq. (1.2), and we want to give expressions for these in terms of other linear operators $E_+, E_-, D_+, D_-, D_0 \colon M \to M$.

We will denote by $R_\ell$ a finite dimensional $L$-module which is a (finite) direct sum of $L_k$-modules $M(2\ell)$ for the same number $\ell \in \frac{1}{2}\mathbf{Z}_{\geq 0}$. Then $M$ is a direct sum of the subspaces $R_\ell$ since $M$ is Harish-Chandra, and from Lemma 1.4 we know that $R_\ell$ can be written as the direct sum of subspaces $R_{\ell,m}$, where $R_{\ell,m}$ are eigenspaces for $H_3$ such that

$$H_3\xi = m\xi \tag{1.7}$$

for $m \in \{-\ell, -\ell+1, \dots, \ell-1, \ell\}$ and $\xi \in R_{l,m}$. We will use the decomposition

$$M = \bigoplus_{\substack{\ell \in \frac{1}{2}\mathbf{Z}_{\geq 0} \\ m \in \{-\ell, -\ell+1, \dots, \ell-1, \ell\}}} R_{\ell,m} = \bigoplus_{\ell,m} R_{\ell,m}$$

throughout this paper.

By Lemma 1.4 we also have that $H_+$ and $H_-$ maps the $R_{\ell,m}$ into each other as follows:

$$\begin{aligned} &H_+ \colon R_{\ell,m} \to R_{\ell,m+1} && \text{if } -\ell \leq m < \ell, && H_+ \colon R_{\ell,\ell} \to 0, \\ &H_- \colon R_{\ell,m} \to R_{\ell,m-1} && \text{if } -\ell < m \leq \ell, && H_- \colon R_{\ell,-\ell} \to 0. \end{aligned}$$

Hence we have linear operators $H_+H_-, H_-H_+ \colon R_{\ell,m} \to R_{\ell,m}$, and by eq. (1.6) we see that

$$\begin{aligned} H_+H_-\xi &= \sqrt{(\ell + (m-1) + 1)(\ell - (m-1))}\sqrt{(\ell+m)(\ell-m+1)}\xi \\ &= (\ell+m)(\ell-m+1)\xi, \\ H_-H_+\xi &= \sqrt{(\ell + (m+1))(\ell - (m+1) + 1)}\sqrt{(\ell+m+1)(\ell-m)}\xi \\ &= (\ell+m+1)(\ell-m)\xi. \end{aligned} \tag{1.8}$$

Note that this also covers the cases $m = \ell$ and $m = -\ell$.

Now we define $E_+ \colon R_{\ell,m} \to R_{\ell,m+1}$ and $E_- \colon R_{\ell,m} \to R_{\ell,m-1}$ to be the linear maps satisfying

$$\begin{aligned} H_+\xi &= \begin{cases} \sqrt{(\ell+m+1)(\ell-m)}E_+\xi & \text{if } m \neq \ell \\ 0 & \text{if } m = \ell, \end{cases} \\ H_-\xi &= \begin{cases} \sqrt{(\ell+m)(\ell-m+1)}E_-\xi & \text{if } m \neq -\ell \\ 0 & \text{if } m = \ell \end{cases} \end{aligned} \tag{1.9}$$

for $\xi \in R_{\ell,m}$. Comparing eq. (1.9) and eq. (1.8) we see that

$$\begin{aligned} E_+E_-\xi &= \xi && \text{if } m \neq -\ell \\ E_-E_+\xi &= \xi && \text{if } m \neq \ell. \end{aligned}$$

Thus $E_+ \colon R_{\ell,m} \to R_{\ell,m+1}$ and $E_- \colon R_{\ell,m+1} \to R_{\ell,m}$ are isomorphisms for $m \neq \ell$ and they are each others inverse.

Now note that $H_+$, $H_-$, and $H_3$ are completely determined by eq. (1.7) and eq. (1.9), so we just need to find maps to determine $F_+$, $F_-$, and $F_3$ now, while making sure that we get commutation relations as in eq. (1.2).

Consider maps $D_0$ and $D_+$ defined on $M = \bigoplus_{\ell,m} R_{\ell,m}$ and $D_-$ defined on the direct sum without the summands $R_{\ell,\ell}$ and $R_{\ell,-\ell}$ such that $D_0 R_{\ell,m} \subset R_{\ell,m}$, $D_+ R_{\ell,m} \subset R_{\ell+1,m}$, and $D_- R_{\ell,m} \subset R_{\ell-1,m}$ and the diagrams

$$
\begin{tikzcd}
R_{\ell-1,m+1} & R_{\ell,m+1} \arrow[l, "D_-"'] \\
R_{\ell-1,m} \arrow[u, "E_+"] & R_{\ell,m} \arrow[l, "D_-"] \arrow[u, "E_+"']
\end{tikzcd}
\qquad
\begin{tikzcd}
R_{\ell,m+1} \arrow[r, "D_0"] & R_{\ell,m+1} \\
R_{\ell,m} \arrow[u, "E_+"] \arrow[r, "D_0"'] & R_{\ell,m} \arrow[u, "E_+"']
\end{tikzcd}
\tag{1.10}
$$

$$
\begin{tikzcd}
R_{\ell,m+1} \arrow[r, "D_+"] & R_{\ell+1,m+1} \\
R_{\ell,m} \arrow[u, "E_+"] \arrow[r, "D_+"'] & R_{\ell+1,m+1} \arrow[u, "E_+"']
\end{tikzcd}
$$

commute, when $-\ell + 1 \leq m < \ell - 1$ in the top left diagram, $-\ell \leq m < \ell$ in the other two diagrams. Also similar diagrams with $E_-$ replacing $E_+$ commute since $E_- \colon R_{\ell,m} \to R_{\ell,m-1}$ for $m \neq -\ell$ is inverse to $E_+ \colon R_{\ell,m-1} \to R_{\ell,m}$. Now we need quite a lot of work to find a way to describe $F_+$, $F_-$, and $F_3$ from such maps.

We already have that $L_k = \operatorname{span}_{\mathbf{C}}(h_+, h_-, h_3)$, but now we will also consider $L_p = \operatorname{span}_{\mathbf{C}}(f_+, f_-, f_3)$. Equation (1.2) gives us that $[L_k, L_p] \subset L_p$, so by the adjoint representation we can see $L_p$ as an $L_k$-module, and again by eq. (1.2) we see that $L_p$ is a simple $L_k$-module: If $V$ is an $L_k$-submodule and we have a non-zero element $f = af_+ + bf_- + cf_3 \in V$ for some $a, b, c \in \mathbf{C}$ not all zero. Then

$$
\begin{aligned}
[h_+, af_+ + bf_- + cf_3] &= 2bf_3 - cf_+, \\
[h_-, af_+ + bf_- + cf_3] &= -2af_3 + cf_-, \\
[h_3, af_+ + bf_- + cf_3] &= af_+ - bf_-.
\end{aligned}
$$

If $c \neq 0$, we get that

$$
\begin{aligned}
[h_3, [h_+, f]] &= [h_3, 2bf_3 - cf_+] = -cf_+, \\
[h_3, [h_-, f]] &= [h_3, -2af_3 + cf_-] = -cf_-,
\end{aligned}
$$

so we see that $f_+, f_- \in V$, and thus also $[h_+, \frac{1}{2}f_-] = f_3 \in V$, so $V = L_p$. If on the other hand $c = 0$, then

$$
\begin{aligned}
[h_-, f] &= -2af_3, \\
[h_+, f] &= 2bf_3,
\end{aligned}
$$

so since either $a \neq 0$ or $b \neq 0$, we see that $f_3 \in V$, and thus also $[h_+, -f_3] = f_+ \in V$ and $[h_-, f_3] = f_- \in V$, so $V = L_p$. Hence $L_p$ is indeed a simple $L_k$-module. Now since $L_p$ is a simple finite dimensional $L_k$-module of dimension 3, we have that $L_p \simeq M(2)$ as $L_k$-modules.

In general given two $L$-modules $V$ and $W$, we consider the tensor product $V \otimes W$ over $\mathbf{C}$ of the underlying vector spaces as an $L$-module via the action

$$x.(v \otimes w) = x.v \otimes w + v \otimes x.w,$$

cf. [Hum72, p. 26].

Now we are interested in the $L_k$-module $L_p \otimes M$, where $M$ is a Harish-Chandra $L$-module as before. Specifically we will show that the linear map

$$\begin{aligned} \psi\colon L_p \otimes M &\to M \\ x \otimes v &\mapsto x.v \end{aligned} \tag{1.11}$$

is a homomorphism of $L_k$-modules. For $y \in L_k$ we see that

$$y.(x \otimes v) = y.x \otimes v + x \otimes y.v = [y,x] \otimes v + x \otimes y.v,$$

for $x \otimes v \in L_p \otimes M$, since the action in $L_p$ is by the adjoint representation. So

$$\begin{aligned} \psi(y.(x \otimes v)) &= \psi([y,x] \otimes v) + \psi(x \otimes y.v) = [y,x].v + x.(y.v) \\ &= y.(x.v) - x.(y.v) + x.(y.v) = y.(x.v) = y.\psi(x \otimes v), \end{aligned}$$

i.e. $\psi$ is indeed a homomorphism of $L_k$-modules.

Now we note that $M = \bigoplus_\ell R_\ell$, so

$$L_p \otimes M = L_p \otimes \big(\bigoplus_\ell R_\ell\big) \simeq \bigoplus_\ell (L_p \otimes R_\ell),$$

as $L_k$-modules, and since also $R_\ell$ is direct sum of finitely many copies of $M(2\ell)$, we see that

$$\begin{aligned} L_p \otimes R_\ell &\simeq M(2) \otimes \big(M(2\ell)^1 \oplus M(2\ell)^2 \oplus \cdots \oplus M(2\ell)^r\big) \\ &\simeq (M(2) \otimes M(2\ell)^1) \oplus (M(2) \otimes M(2\ell)^2) \oplus \cdots \oplus (M(2) \otimes M(2\ell)^r), \end{aligned}$$

as $L_k$-modules, since $L_p \simeq M(2)$. Here the superscripts are just indices for the different $M(2\ell)$. Thus we want to describe the $L_k$-modules $M(2) \otimes M(2\ell)$, which we will do by first describing the $\mathfrak{sl}(2,\mathbf{C})$-modules $V(2) \otimes V(2\ell)$ and then translating back to a solution to our problem.

### 1.1.2 Describing $V(2) \otimes V(n)$

Let $2\ell = n \in \mathbf{N}$. We want to show that

$$V(2) \otimes V(n) \simeq \begin{cases} V(n-2) \oplus V(n) \oplus V(n+2) & \text{if } n \geq 2, \\ V(3) \oplus V(1) & \text{if } n = 1, \\ V(2) & \text{if } n = 0. \end{cases} \tag{1.12}$$

Note that in all cases there is a summand $V(n+2)$. We can show the above by considerations using formal characters. We will use the notation of [Jan16, Chapter 8], specifically we will do calculations with the functions $e(\lambda)\colon H^* \to \mathbf{Z}$ for $\lambda \in H^*$. Firstly note that in general

$$\mathrm{ch}_V = \sum_{\lambda \in H^*} (\dim V_\lambda) e(\lambda),$$

and use the notation $V(n)_k$ for $V(\lambda)_\mu$ and $e(n)$ for $e(\lambda)$ with $\lambda, \mu \in H^*$ such that $\lambda(h) = n$ and $\mu(h) = k$. We get that

$$\mathrm{ch}_{V(2)} = e(-2) + e(0) + e(2)$$

and

$$\mathrm{ch}_{V(n)} = \sum_{i=0}^{n} e(n-2i),$$

since

$$\dim V(n)_k = \begin{cases} 1 & \text{if } k = n-2i \text{ for some } i \in \{0,1,\dots,n\}, \\ 0 & \text{otherwise.} \end{cases}$$

Now since $e(\lambda) * e(\mu) = e(\lambda + \mu)$ in general cf. [Jan16, p. 93], we see that for $n \geq 2$

$$\begin{aligned}
\mathrm{ch}_{V(2)\otimes V(n)} &= \mathrm{ch}_{V(2)} * \mathrm{ch}_{V(n)} = e(-2) * \mathrm{ch}_{V(n)} + e(0) * \mathrm{ch}_{V(n)} + e(2) * \mathrm{ch}_{V(n)} \\
&= \sum_{i=0}^{n} e(n-2-2i) + \mathrm{ch}_{V(n)} + \sum_{i=0}^{n} e(n+2-2i) \\
&= e(-n-2) + e(-n) + \sum_{i=0}^{n-2} e(n-2-2i) + \mathrm{ch}_{V(n)} \\
&\quad + \sum_{i=0}^{n} e(n+2-2i) \\
&= \mathrm{ch}_{V(n-2)} + \mathrm{ch}_{V(n)} + \sum_{i=0}^{n+2} e(n+2-2i) \\
&= \mathrm{ch}_{V(n-2)} + \mathrm{ch}_{V(n)} + \mathrm{ch}_{V(n+2)} = \mathrm{ch}_{V(n-2)\oplus V(n)\oplus V(n+2)},
\end{aligned}$$

where the first equality follows from the fact that $\mathrm{ch}_{V\otimes W} = \mathrm{ch}_V * \mathrm{ch}_W$ in general, cf. [Hum72, p. 125]. Thus since two $L$-modules $V$ and $V'$ are isomorphic if and only if $\mathrm{ch}_V = \mathrm{ch}_{V'}$, cf. [Jan16, p. 90], we see that $V(2) \otimes V(n) \simeq V(n-2) \oplus V(n) \oplus V(n+2)$ if $n \geq 2$.

Likewise we see that

$$\begin{aligned}
\mathrm{ch}_{V(2)\otimes V(1)} &= \mathrm{ch}_{V(2)} * \mathrm{ch}_{V(1)} \\
&= \big(e(-2) + e(0) + e(2)\big) * e(-1) + \big(e(-2) + e(0) + e(2)\big) * e(1) \\
&= e(-3) + e(-1) + e(1) + e(-1) + e(1) + e(3) \\
&= \big(e(-3) + e(-1) + e(1) + e(3)\big) + \big(e(-1) + e(1)\big) \\
&= \mathrm{ch}_{V(3)} + \mathrm{ch}_{V(1)} = \mathrm{ch}_{V(3)\oplus V(1)}
\end{aligned}$$

and

$$\mathrm{ch}_{V(2)\otimes V(0)} = \mathrm{ch}_{V(2)} * \mathrm{ch}_{V(0)} = \mathrm{ch}_{V(2)} * e(0) = \mathrm{ch}_{V(2)},$$

so indeed $V(2) \otimes V(1) \simeq V(3) \oplus V(1)$ and $V(2) \otimes V(0) \simeq V(2)$.

Now consider $(w_0, w_1, w_2)$ a basis for $V(2)$ and $(v_i \mid 0 \leq i \leq n)$ a basis for $V(n)$ such that both satisfies the conditions from eq. (1.4). Then for $w_i \otimes v_j \in V(2) \otimes V(n)$ with $i \in \{0, 1, 2\}$ and $j \in \{0, 1, \ldots, n\}$ we see that

$$\begin{aligned}
h.(w_i \otimes v_j) &= h.w_i \otimes v_j + w_i \otimes h.v_j = (2 - 2i)w_i \otimes v_j + (n - 2j)w_i \otimes v_j \\
&= (n - 2(i + j - 1))w_i \otimes v_j.
\end{aligned} \tag{1.13}$$

Hence $v_0 \otimes w_0$ is up to scalar multiple the only vector of weight $n + 2$ in $V(2) \otimes V(n)$, so it is necessarily a highest weight vector generating the direct summand isomorphic to $V(n + 2)$. Note that by eq. (1.12) we indeed have a direct summand isomorphic to $V(n + 2)$ for all $n \in \mathbf{N}$. By $\mathfrak{sl}(2, \mathbf{C})$-theory, cf. [Jan16, p. 36], we know that this summand has a basis $(s_k \mid 0 \leq k \leq n + 2)$ satisfying equations as in eq. (1.4), where

$$s_k := \frac{1}{k!} y^k.(w_0 \otimes v_0). \tag{1.14}$$

By straightforward calculations, cf. Appendix A.1, we get for $n > 0$ that

$$\begin{aligned}
s_0 &= w_0 \otimes v_0, && \\
s_1 &= w_1 \otimes v_0 + w_0 \otimes v_1 && \text{if } n > 0, \\
s_k &= w_2 \otimes v_{k-2} + w_1 \otimes v_{k-1} + w_0 \otimes v_k && \text{for } 2 \leq k \leq n, \\
s_{n+1} &= w_2 \otimes v_{n-1} + w_1 \otimes v_n && \text{if } n > 0, \\
s_{n+2} &= w_2 \otimes v_n. &&
\end{aligned} \tag{1.15}$$

In case $n = 0$ we likewise see that $s_1 = w_1 \otimes v_0$ and $s_2 = w_2 \otimes v_0$, and we note that $(s_0, s_1, s_2)$ is a basis for $V(2) \otimes V(0) \simeq V(2)$.

Suppose now that $n \geq 1$. Note that by eq. (1.12) we have a direct summand isomorphic to $V(n)$, and by eq. (1.13) the weight space of weight $n$ is spanned by $w_0 \otimes v_1$ and $w_1 \otimes v_0$, so the vector of highest weight $n$ generating the summand corresponding to $V(n)$ must be of the form $aw_0 \otimes v_1 + bw_1 \otimes v_0$

for some $a, b \in \mathbf{C}$. Furthermore we know that for this vector generating the summand corresponding to $V(n)$, we must have that

$$\begin{aligned}
0 &= x.(aw_0 \otimes v_1 + bw_1 \otimes v_0) \\
&= ax.w_0 \otimes v_1 + aw_0 \otimes x.v_1 + bx.w_1 \otimes v_0 + bw_1 \otimes x.v_0 \\
&= 0 + a(n-1+1)w_0 \otimes v_0 + b(2-1+1)w_0 \otimes v_0 + 0 \\
&= (an+2b)w_0 \otimes v_0,
\end{aligned}$$

i.e. $an + 2b = 0$ so $b = -\frac{n}{2}a$. This determines the vector generating the summand corresponding to $V(n)$ up to a scalar, so taking $a = 1$, we see that we can take

$$t_0 \coloneqq w_0 \otimes v_1 - \frac{n}{2} w_1 \otimes v_0$$

as our vector generating the summand corresponding to $V(n)$. As before $\mathfrak{sl}(2, \mathbf{C})$-theory now yields that this summand has a basis $(t_k \mid 0 \leq k \leq n)$ satisfying equations as in eq. (1.4), where

$$t_k \coloneqq \frac{1}{k!} y^k.t_0. \tag{1.16}$$

By straightforward calculations, cf. Appendix A.1, we get that

$$\begin{aligned}
t_0 &= w_0 \otimes v_1 - \frac{n}{2} w_1 \otimes v_0, \\
t_k &= (k+1)w_0 \otimes v_{k+1} - \frac{n-2k}{2} w_1 \otimes v_k \\
&\quad + (k-1-n)w_2 \otimes v_{k-1} \qquad \text{for } 1 \leq k \leq n-1, \\
t_n &= \frac{n}{2} w_1 \otimes v_n - w_2 \otimes v_{n-1}.
\end{aligned} \tag{1.17}$$

Suppose now that $n \geq 2$. By eq. (1.12) we have a direct summand isomorphic to $V(n-2)$, and by eq. (1.13) the weight space of weight $n-2$ is spanned by $w_0 \otimes v_2$, $w_1 \otimes v_1$, and $w_2 \otimes v_0$, so the vector of highest weight $n-2$ generating the summand corresponding to $V(n)$ must be of the form $aw_0 \otimes v_2 + bw_1 \otimes v_1 + cw_2 \otimes v_0$ for some $a, b, c \in \mathbf{C}$. Furthermore we know that for this vector generating the summand corresponding to $V(n-2)$, we must have

$$\begin{aligned}
0 &= x.(aw_0 \otimes v_2 + bw_1 \otimes v_1 + cw_2 \otimes v_0) \\
&= aw_0 \otimes x.v_2 + bx.w_1 \otimes v_1 + bw_1 \otimes x.v_1 + cx.w_2 \otimes v_0 \\
&= a(n-2+1)w_0 \otimes v_1 + b(2-1+1)w_0 \otimes v_1 + b(n-1+1)w_1 \otimes v_0 \\
&\quad + c(2-2+1)w_1 \otimes v_0 \\
&= \big((n-1)a + 2b\big)w_0 \otimes v_1 + \big(bn + c\big)w_1 \otimes v_0,
\end{aligned}$$

i.e. $a(n-1) + 2b = 0$ and $bn + c = 0$. Giving us $c = -bn$ and $b = -\frac{n-1}{2}a$, so

$$c = \frac{n(n-1)}{2} a.$$

This determines the vector generating the summand corresponding to $V(n-2)$ up to a scalar, so taking $a = 1$, we see that we can take

$$u_0 := w_0 \otimes v_2 - \frac{n-1}{2} w_1 \otimes v_1 + \frac{n(n-1)}{2} w_2 \otimes v_0$$

as our vector generating the summand corresponding to $V(n-2)$. Again $\mathfrak{sl}(2, \mathbf{C})$-theory now yields that this summand has a basis $(u_k \mid 0 \leq k \leq n-2)$ satisfying equations as in eq. (1.4), where

$$u_k := \frac{1}{k!} y^k . u_0. \tag{1.18}$$

By straightforward calculations, cf. Appendix A.1, we get that

$$\begin{aligned} u_k = & \frac{(k+1)(k+2)}{2} w_0 \otimes v_{k+2} - \frac{(k+1)(n-k-1)}{2} w_1 \otimes v_{k+1} \\ & + \frac{(n-k)(n-k-1)}{2} w_2 \otimes v_k \end{aligned} \tag{1.19}$$

for $0 \leq k \leq n-2$.

Now we want to express $w_1 \otimes v_k$ for $0 \leq k \leq n$ in terms of the bases $(s_k \mid 0 \leq k \leq n+2)$, $(t_k \mid 0 \leq k \leq n)$, and $(u_k \mid 0 \leq k \leq n-2)$. A straightforward but long calculation, cf. Appendix A.2, yields that

$$w_1 \otimes v_k = \frac{2(k+1)(n+1-k)}{(n+1)(n+2)} s_{k+1} - \frac{2(n-2k)}{n(n+2)} t_k - \frac{4}{n(n+1)} u_{k-1} \tag{1.20}$$

for $0 < k < n$, while

$$w_1 \otimes v_0 = \frac{2}{n+2}(s_1 - t_0) \qquad \text{and} \qquad w_1 \otimes v_n = \frac{2}{n+2}(s_{n+1} + t_n) \tag{1.21}$$

if $n \geq 1$. If $n = 0$ we have already seen (just after eq. (1.15)) that $w_1 \otimes v_0 = s_1$. Note that eq. (1.21) is a special case of eq. (1.20) if we set $u_{-1} = u_{n-1} = 0$.

Now consider $V(2)$ and $V(n)$ as inner product spaces over $\mathbf{C}$ with inner products given by

$$\langle w_k, w_j \rangle = \delta_{jk} \binom{2}{k} \qquad \text{and} \qquad \langle v_k, v_j \rangle = \delta_{jk} \binom{n}{k}. \tag{1.22}$$

Then we can also consider $V(2) \otimes V(n)$ an inner product space with inner product given by

$$\langle w \otimes v, w' \otimes v' \rangle = \langle w, w' \rangle \cdot \langle v, v' \rangle \tag{1.23}$$

Maybe write about why this is an inner product

for $w, w' \in V(2)$ and $v, v' \in V(n)$. Now by straightforward calculations, cf. Appendix A.3, we get that

$$\langle s_0, s_0 \rangle = 1, \qquad \langle t_0, t_0 \rangle = \frac{n(n+2)}{2}, \qquad \langle u_0, u_0 \rangle = \frac{n^2(n+1)(n-1)}{4}. \tag{1.24}$$

Now set $\overline{w}_k = w_k/\|w_k\|$, $\overline{v}_k = v_k/\|v_k\|$, $\overline{s}_k = s_k/\|s_k\|$, $\bar{t}_k = t_k/\|t_k\|$, and $\bar{s}_k = s_k/\|s_k\|$ for all possible $k$, where $\|\cdot\|$ is given by $\|v\| = \sqrt{\langle v, v\rangle}$ for $v \in V(2) \otimes V(n)$ as usually in an inner product space. Note that

ow the following
uations — I can
ow these by long
culations, but I
nk there is an
sier way

$$\begin{aligned}
\langle w_k, w_k\rangle &= \binom{2}{k} \\
\langle v_k, v_k\rangle &= \binom{n}{k} \\
\langle s_k, s_k\rangle &= \langle s_0, s_0\rangle \binom{n+2}{k} = \binom{n+2}{k} \\
\langle t_k, t_k\rangle &= \langle t_0, t_0\rangle \binom{n}{k} = \frac{n(n+2)}{2}\binom{n}{k} \\
\langle u_k, u_k\rangle &= \langle u_0, u_0\rangle \binom{n-2}{k} = \frac{n^2(n+1)(n-1)}{4}\binom{n-2}{k}
\end{aligned}$$

for $k$ where it makes sense, so we see that

$$w_k = \sqrt{\binom{2}{k}}\,\overline{w}_k, \qquad v_k = \sqrt{\binom{n}{k}}\,\overline{v}_k, \qquad s_k = \sqrt{\binom{n+2}{k}}\,\overline{s}_k, \tag{1.25}$$

and

$$t_k = \sqrt{\frac{n(n+2)}{2}\binom{n}{k}}\,\bar{t}_k, \qquad u_k = \sqrt{\frac{n^2(n+1)(n-1)}{4}\binom{n-2}{k}}\,\overline{u}_k. \tag{1.26}$$

**Remark 1.5.** Since

$$\overline{v}_k = \frac{1}{\sqrt{\binom{n}{k}}} v_k,$$

we note that we just need to change indices to go to the basis $(e_m)$ from the basis of $(v_k)$ as in the work leading to Lemma 1.4. △

By a simple calculation, cf. Appendix A.4, we get that

$$\begin{aligned}
\overline{w}_1 \otimes \overline{v}_k = \sqrt{\frac{2(k+1)(n+1-k)}{(n+1)(n+2)}}\,\overline{s}_{k+1} - \frac{(n-2k)}{\sqrt{n(n+2)}}\bar{t}_k \\
- \sqrt{\frac{2k(n-k)}{n(n+1)}}\,\overline{u}_{k-1}.
\end{aligned} \tag{1.27}$$

for $0 \le k \le n$. Now changing indices as mentioned in Remark 1.5 to $\ell = \frac{1}{2}n$ and $m = \frac{1}{2}(n-2k) = \ell - k$ as we did to get to Lemma 1.4, i.e. $n = 2\ell$ and

$k = \ell - m$, we get that

$$\begin{aligned}
\overline{w}_1 \otimes e_m &= \overline{w}_1 \otimes \overline{v}_k \\
&= \sqrt{\frac{2(\ell-m+1)(2\ell+1)-(\ell-m)}{(2\ell+1)(2\ell+2)}}\overline{s}_{k+1} - \frac{(2\ell-2(\ell-m))}{\sqrt{2\ell(2\ell+2)}}\overline{t}_k \\
&\quad - \sqrt{\frac{2(\ell-m)(2\ell-(\ell-m))}{2\ell(2\ell+1)}}\overline{u}_{k-1} \\
&= \sqrt{\frac{(\ell-m+1)(\ell+1+m)}{(2\ell+1)(\ell+1)}}\overline{s}_{k+1} - \frac{m}{\sqrt{\ell(\ell+1)}}\overline{t}_k \\
&\quad - \sqrt{\frac{(\ell-m)(\ell+m)}{\ell(2\ell+1)}}\overline{u}_{k-1},
\end{aligned}$$

where $e_m$ is as in the work we did to get Lemma 1.4 except for the fact that we consider $\mathfrak{sl}(2,\mathbf{C})$-modules still. Now setting

$$\widetilde{D}_+(\overline{v}_k) = -\frac{\overline{s}_{k+1}}{\sqrt{(\ell+1)(2\ell+1)}}, \quad \widetilde{D}_0(\overline{v}_k) = \frac{\overline{t}_k}{\sqrt{\ell(\ell+1)}}, \quad \widetilde{D}_-(\overline{v}_k) = -\frac{\overline{u}_{k-1}}{\sqrt{\ell(2\ell+1)}},$$

we see that

$$\begin{aligned}
\overline{w}_1 \otimes e_m &= \overline{w}_1 \otimes \overline{v}_k \\
&= \sqrt{(\ell+1)^2-m^2}\frac{\overline{s}_{k+1}}{\sqrt{(\ell+1)(2\ell+1)}} - m\frac{\overline{t}_k}{\sqrt{\ell(\ell+1)}} \\
&\quad - \sqrt{\ell^2-m^2}\frac{\overline{u}_{k-1}}{\ell(2\ell+1)} \\
&= \sqrt{\ell^2-m^2}\widetilde{D}_-(\overline{v}_k) - m\widetilde{D}_0(\overline{v}_k) - \sqrt{(\ell+1)^2-m^2}\widetilde{D}_+(\overline{v}_k).
\end{aligned} \tag{1.28}$$

Note that for $m \in \{\pm\ell\}$ the $\widetilde{D}_-$ term vanishes, so the formula works here although $D_-$ is not well-defined in these edge cases.

* * * * *

Getting back to the problem at the end of Section 1.1.1, we want to give the maps $D_0$, $D_+$, and $D_-$ such that $D_0R_{\ell,m} \subset R_{\ell,m}$, $D_+R_{\ell,m} \subset R_{\ell+1,m}$, and $D_-R_{\ell,m} \subset R_{\ell-1,m}$, the diagrams of eq. (1.10) commute, and we can describe $F_3$, $F_+$, $F_-$ by the maps $D_0$, $D_+$, $D_-$, $E_+$, and $E_-$. Now consider the $\mathfrak{sl}(2,\mathbf{C})$-modules $V(n)$ as $L_k$-modules $M(n)$ via the isomorphism of eq. (1.3), and note that since

$$R_\ell = M(2\ell)^1 \oplus M(2\ell)^2 \oplus \cdots \oplus M(2\ell)^r$$

and each $M(2\ell)^i$ has a basis $(e^i_{-\ell}, e^i_{-\ell+1}, \ldots, e^i_{\ell-1}, e^i_\ell)$ with $H_3e^i_m = me^i_m$ for all $m$, we have that $R_{\ell,m}$ has basis $(e^1_m, e^2_m, \ldots, e^r_m)$ by definition. So when describing the maps $D_0$, $D_+$, and $D_-$, we just need to describe what the maps should do to each $e^i_m$. To get the commutative diagrams of eq. (1.10) we will consider maps such that $D_* M(2\ell)^i \subset M(2\ell)^i$, i.e. we will just describe what each map does to $M(2\ell)$ in general and stop writing superscripts.

Since we want to describe the maps $F_3$, $F_+$, and $F_-$, we are actually interested in the actions of $L_p$, so by using $\psi$ of eq. (1.11) and the considerations at the end of Section 1.1.1, we can start out by describing $M(2)\otimes M(2\ell)$, i.e. we can use the description of $V(2)\otimes V(n)$ from above. Note that we have already seen that $L_p \simeq M(2)$ as $L_k$-modules, but we would like to better understand how the basis $(f_+, f_3, f_-)$ of $L_p$ corresponds to the basis $(w_0, w_1, w_2)$ of $M(2)$ as in eq. (1.5). In the basis $(w_0, w_1, w_2)$ we have that $h_+.w_0 = 0$ (since this is what corresponds to $x.w_0 = 0$ in $V(2)$ by eq. (1.3)), so by checking eq. (1.2) we see that $w_0$ must correspond to a multiple $f_3$, but the basis is chosen up to scalar, so we can take $w_0$ to be $-\frac{\sqrt{2}}{2} f_3$. Now we get $w_1$ by taking $h_-.w_0$ (corresponding to $y.w_0$ in $V(2)$ by eq. (1.3)), thus we get that

$$w_1 = h_-.w_0 = -\tfrac{\sqrt{2}}{2} h_-.f_+ = -\tfrac{\sqrt{2}}{2}[h_-, f_+] = \sqrt{2} f_3.$$

Likewise we get that $w_2 = [h_-, \sqrt{2} f_3] = \sqrt{2} f_-$, so we can take our basis to be $(w_0, w_1, w_2) = (-\frac{\sqrt{2}}{2} f_+, \sqrt{2} f_3, \sqrt{2} f_-)$ when thinking of $L_p$ as the $L_k$-module $M(2)$. Normalizing as in eq. (1.25), we get that $(\overline{w}_0, \overline{w}_1, \overline{w}_2) = (-\frac{\sqrt{2}}{2} f_+, f_3, \sqrt{2} f_-)$. So by eq. (1.31), we see that in $L_p \otimes M(2\ell)$

$$f_3 \otimes e_m = \sqrt{\ell^2 - m^2}\,\widetilde{D}_-(e_m) - m\widetilde{D}_0(e_m) - \sqrt{(\ell+1)^2 - m^2}\,\widetilde{D}_+(e_m),$$

where $e_m = \overline{v}_k$ for $k = \ell - m$ and $f_3 = \overline{w}_1$.

**Remark 1.6.** Note that if we have bases $(e^{(2\ell+2)}_{-\ell-1}, e^{(2\ell+2)}_{-\ell}, \dots, e^{(2\ell+2)}_{\ell}, e^{(2\ell+2)}_{\ell+1})$ for $M(2\ell+2)$, $(e^{(2\ell)}_{-\ell}, e^{(2\ell)}_{-\ell+1}, \dots, e^{(2\ell)}_{\ell-1}, e^{(2\ell)}_{\ell})$ for $M(2\ell)$, and $(e^{(2\ell-2)}_{-\ell+1}, e^{(2\ell-2)}_{-\ell+2}, \dots, e^{(2\ell-2)}_{\ell-2}, e^{(2\ell-2)}_{\ell-1})$ for $M(2\ell-2)$ (if $\ell \geq 1$) as in Lemma 1.4, then as above changing indices with $k = \ell + 1 - m$ we see that $e^{(2\ell+2)}_m$ corresponds to $\overline{s}_k$. Likewise changing indices with $k = \ell - m$ we see that $e^{(2\ell)}_m$ corresponds to $\overline{t}_k$, and with $k = \ell - 1 - m$ we see that $e^{(2\ell-2)}_m$ corresponds to $\overline{u}_k$. △

Now using $\psi$ from eq. (1.11), we see that

$$\begin{aligned} F_3 e_m &= f_3.e_m = \psi(f_3 \otimes e_m) \\ &= \sqrt{\ell^2 - m^2}\,\psi\widetilde{D}_-(e_m) - m\psi\widetilde{D}_0(e_m) - \sqrt{(\ell+1)^2 - m^2}\,\psi\widetilde{D}_+(e_m). \end{aligned} \tag{1.29}$$

So we can take $D_0 = \psi\widetilde{D}_0$, $D_+ = \psi\widetilde{D}_+$, and $D_- = \psi\widetilde{D}_-$ to get three linear maps with which we can describe the map $F_3$. So far this is just maps on $M(2\ell)$, but we can expand to maps on $R_\ell$ by using the maps on each summand of $R_\ell = M(2\ell)^1 \oplus \cdots \oplus M(2\ell)^r$, and likewise we can expand further to maps on

$M = \bigoplus_\ell R_\ell$ by using the maps on each summand. Also indeed $D_0 R_{\ell,m} \subset R_{\ell,m}$, $D_+ R_{\ell,m} \subset R_{\ell+1,m}$, and $D_- R_{\ell,m} \subset R_{\ell-1,m}$, since for $\xi \in R_{\ell,m}$ we have that

$$\begin{aligned} H_3 D_0(\xi) &= h_3.\psi \widetilde{D}_0(\xi) = \psi h_3.\widetilde{D}_0(\xi) = \psi H_3 \widetilde{D}_0(\xi) = m\psi\widetilde{D}_0(\xi) \\ &= m D_0(\xi), \end{aligned}$$

since $\psi$ is an $L_k$-module homomorphism and by Remark 1.6 we see that $\widetilde{D}_0(e_m)$ is a scalar multiple of $\bar{t}_k = \bar{t}_{\ell-m} = e_m^{(2\ell)}$, and indeed $H_3 e_m^{(2\ell)} = m e_m^{(2\ell)}$. The same reasoning with $\bar{s}_{k+1}$ for $D_+$ and $\bar{u}_{k-1}$ for $D_-$ yields the other two inclusions. Also note that the diagrams of eq. (1.10) commute by the defintion of $D_0$, $D_+$, and $D_-$, since the maps independent of $m$ and $E_+$ and $E_-$ are isomorphisms.

Write this a little more clearly

Now simple calculations, cf. Appendix A.5, gives us that

$$\begin{aligned} F_3\xi &= \sqrt{\ell^2 - m^2} D_-\xi - m D_0 \xi - \sqrt{(\ell+1)^2 - m^2} D_+\xi, \\ F_+\xi &= \sqrt{(\ell-m)(\ell-m-1)} D_- E_+ \xi - \sqrt{(\ell-m)((\ell+m+1))} D_0 E_+ \xi \\ &\quad + \sqrt{(\ell+m+1)(\ell+m+2)} E_+ D_+ \xi, \\ F_-\xi &= -\sqrt{(\ell+m)(\ell+m-1)} D_- E_- \xi - \sqrt{(\ell+m)(\ell-m+1)} D_0 E_- \xi \\ &\quad - \sqrt{(\ell-m+1)(\ell-m+2)} E_- D_+ \xi \end{aligned} \tag{1.30}$$

for $\xi \in R_{\ell,m}$. Note here that although $D_-$ is not defined on $R_{\ell,\ell}$ and $R_{\ell,-\ell}$ the above still makes sense since in these cases the terms with $D_-$ vanish, either by the coefficient being zero or by $E_+$ or $E_-$ mapping to zero.

We claim now that the formulae eq. (1.30) for the linear operators $F_+$, $F_-$, and $F_3$ together with the formulae eqs. (1.7) and (1.9) for $H_+$, $H_-$, and $H_3$ define a representation of $L$, i.e. they satisfy the commutation relations of eq. (1.2), if and only if $D_0$, $D_+$, and $D_-$ satisfy

$$\begin{aligned} \ell D_+ D_0 \xi &= (\ell+2) D_0 D_+ \xi, \\ (\ell+1) D_- D_0 \xi &= (\ell-1) D_0 D_- \xi, \\ \xi &= (2\ell-1) D_+ D_- \xi - (2\ell+3) D_- D_+ \xi - D_0^2 \xi \end{aligned} \tag{1.31}$$

for $\xi \in R_{\ell,m}$.

I haven't shown this properly yet — I guess it should follow from looking at the relations but the calculations are very long, so I skipped it for now

### 1.1.3 Simple Harish-Chandra modules over $L$

We want to classify the simple Harish-Chandra modules over $L$ for later use. Before most of the work we need some basic results.

Let $M$ be an simple Harish-Chandra module over $L$ and suppose that each non-trivial subspace $R_{\ell,m}$ in $M = \bigoplus_{\ell,m} R_{\ell,m}$ is one dimensional. In this case each $L_k$-module $R_\ell \simeq M(2\ell)$ is simple. We will later show that actually all simple Harish-Chandra modules are of this kind, so we indeed get a classification of the simple Harish-Chandra modules in the following.

Denote by $\ell_0$ the minimal index $\ell$ in the decomposition $M = \bigoplus_\ell R_\ell$. Note that

$$M' = \bigoplus_{\ell' \in \{\ell_0, \ell_0+1, \dots\}} R_{\ell'}$$

is invariant under $E_+$, $E_-$, $D_0$, $D_+$, and $D_-$, so by the formulae eq. (1.30) for $F_+$, $F_-$, and $F_3$, we see that $M'$ is a submodule since we already know that it is an $L_k$-submodule, because $R_{\ell'}$ all are $L_k$-submodules. Thus $M' = M$ since $M$ is simple and hence the index $\ell$ in $M = \bigoplus_\ell R_\ell$ range over only integral values or only half-integral values.

Additionally we want to show that the kernel of the map $D_- \colon M \to M$ is $R_{\ell_0}$. To do this assume for contradiction that $D_- R_{\ell', m_0} = 0$ for some index $\ell' > \ell_0$ and $m_0 \in \{-\ell_0, -\ell_0 + 1, \dots, \ell_0 - 1, \ell_0\}$. Then by the commutative diagram in eq. (1.10) with $D_-$, i.e. $D_- E_+ = E_+ D_-$, and the fact that $E_+ \colon R_{\ell', m} \to R_{\ell', m+1}$ is an isomorphism for $m < \ell'$, we see that $D_- R_{\ell', m} = 0$ for all $m \in \{-\ell', -\ell' + 1, \dots, \ell' - 1, \ell'\}$. But then

$$M'' = \bigoplus_{\ell'' \in \{\ell', \ell'+1, \dots\}} R_{\ell''}$$

is a proper $L$-submodule of $M$, which contradicts the simplicity of $M$. Thus indeed $\ker D_- = R_{\ell_0}$.

Likewise we see that if $M$ is infinite dimensional, then $D_+ \colon M \to M$ has trivial kernel since if $D_+ R_{\ell'} = 0$, then $M = \bigoplus_{\ell \in \{\ell_0, \ell_0+1, \dots\}} R_\ell$ is finite dimensional. This is the case since all terms with $\ell > \ell'$ must be trivial since otherwise

$$M'' = \bigoplus_{\ell'' \in \{\ell_0, \ell_0+1, \dots, \ell'\}} R_{\ell''}$$

is a proper $L$-submodule of $M$, which contradicts the simplicity of $M$.

### Infinite dimensional simple modules

Assume that $M$ as above is infinite dimensional. Because all $R_{\ell,m}$ are one dimensional, the diagram with $E_+$ and $D_+$ in eq. (1.10) commute, i.e. $D_+ E_+ = E_+ D_+$, and $D_+$ has trivial kernel, while $E_+$ is an isomorphism for $m \neq \ell$, we see that we can choose a basis $\{\xi_{\ell,m}\}$ of $M$ such that $\xi_{\ell,m} \in R_{\ell,m}$ and

$$\begin{aligned} E_+ \xi_{\ell,m} &= \xi_{\ell,m+1} && \text{for } -\ell \le m < \ell, \\ D_+ \xi_{\ell,m} &= \xi_{\ell+1,m} && \text{for } \ell \in \{\ell_0, \ell_0 + 1, \dots\}. \end{aligned}$$

In this basis we get that

$$\begin{aligned} E_- \xi_{\ell,m} &= \xi_{\ell,m-1} && \text{for } -\ell < m \le \ell, \\ D_0 \xi_{\ell,m} &= d_\ell^0 \xi_{\ell,m} && \text{for } \ell \in \{\ell_0, \ell_0 + 1, \dots\}, \\ D_- \xi_{\ell,m} &= d_\ell^- \xi_{\ell-1,m} && \text{for } \ell \in \{\ell_0 + 1, \ell_0 + 2, \dots\}, \\ D_- \xi_{\ell_0,m} &= 0, \end{aligned} \tag{1.32}$$

where the first equation comes from the fact that $E_-\colon R_{\ell,m} \to R_{\ell,m-1}$ for $m \neq -\ell$ is the inverse of $E_+\colon R_{\ell,m-1} \to R_{\ell,m}$, while the independence of $m$ in the other equations comes from the commutativity of the diagrams of eq. (1.10).

Now eqs. (1.31) and (1.32) implies that

$$\begin{aligned} \ell d_\ell^0 &= (\ell+2)d_{\ell+1}^0, \\ (\ell+1)d_\ell^- d_\ell^0 &= (\ell-1)d_{\ell-1}^0 d_\ell^-, \\ 1 &= (2\ell-1)d_\ell^- - (2\ell+3)d_{\ell+1}^- - (d_\ell^0)^2, \\ d_{\ell_0}^- &= 0, \end{aligned}$$

for $\ell \in \{\ell_0, \ell_0+1, \ldots\}$ except in the second equation where we also demand that $\ell > \ell_0$. We see that

$$d_{\ell+1}^0 = \frac{\ell}{\ell+2} d_\ell^0.$$

So if $\ell_0 = 0$ choosing a constant $c$ such that

$$d_{\ell_0}^0 = \frac{c}{\ell_0(\ell_0+1)},$$

we see inductively that if

$$d_\ell^0 = \frac{c}{\ell(\ell+1)}, \tag{1.33}$$

then

$$\begin{aligned} d_{\ell+1}^0 = \frac{\ell}{\ell+2} d_\ell^0 &= \frac{\ell}{\ell+2} \frac{c}{\ell(\ell+1)} \\ &= \frac{c}{(\ell+1)(\ell+2)}. \end{aligned}$$

Thus if $\ell_0 \neq 0$ eq. (1.33) holds true in general for some constant $c$.

# Chapter 2

# Linear relations

# Bibliography


[GP67a] I. M. Gel'Fand and V. A. Ponomarev. 'Classification of Indecomposable Infinitesimal Representations of the Lorentz Group'. Trans. by Jack Ceder. In: *Dok1. Akad. Nauk SSSR* 8.5 (1967).

[GP67b] I. M. Gel'Fand and V. A. Ponomarev. *Indecomposable Representations of the Lorentz Group*. Trans. by B. Hartley. 1967.

[GP67c] I. M. Gel'Fand and V. A. Ponomarev. 'The Category of Harish-Chandra Modules over the Lie Algebra of the Lorentz Group'. Trans. by A. M. Scott. In: *Dok1. Akad. Nauk SSSR* 8.5 (1967).

[Hum72] James E. Humphreys. *Introduction to Lie Algebras and Representation Theory*. 1st ed. Vol. 9. Springer, 1972. ISBN: 978-0-387-90053-7.

[Jan16] Jens Carsten Jantzen. *Lie Algebras*. Lecture notes from the Lie algebra course. 2016.

# Appendix A

# Calculations

Throughout the paper there are situations where we need to do some straightforward but rather long calculations, so to clean up the exposition somewhat we will collect most of these calculations in this appendix and then just use the results in the paper.

## A.1 Bases of $V(2) \otimes V(n)$

We want to describe the $s_k$'s of eq. (1.14) more explicitly. We have that $s_0 = w_0 \otimes v_0$ and $s_k = \frac{1}{k!} y^k.s_0$, and we note that if $n > 0$ then

$$\begin{aligned} s_1 &= y.(w_0 \otimes v_0) = y.w_0 \otimes v_0 + w_0 \otimes y.v_0 \\ &= w_1 \otimes v_0 + w_0 \otimes v_1 \end{aligned}$$

and

$$\begin{aligned} s_2 &= \tfrac{1}{2} y.s_1 \\ &= \tfrac{1}{2} y.w_1 \otimes v_0 + \tfrac{1}{2} w_1 \otimes y.v_0 + \tfrac{1}{2} y.w_0 \otimes v_1 + w_0 \otimes \tfrac{1}{2} y.v_1 \\ &= w_2 \otimes v_0 + \tfrac{1}{2} w_1 \otimes v_1 + \tfrac{1}{2} w_1 \otimes v_1 + w_0 \otimes v_2 \\ &= w_2 \otimes v_0 + w_1 \otimes v_1 + w_0 \otimes v_2. \end{aligned}$$

Inductively we see that

$$s_k = w_2 \otimes v_{k-2} + w_1 \otimes v_{k-1} + w_0 \otimes v_k$$

for $k \leq n$, since the base case holds and given the equality for $k < n$ we get

$$\begin{aligned} s_{k+1} &= \frac{1}{k+1} y.s_k \\ &= w_2 \otimes \frac{1}{k+1} y.v_{k-2} + \frac{1}{k+1} y.w_1 \otimes v_{k-1} + w_1 \otimes \frac{1}{k+1} y.v_{k-1} \\ &\quad + \frac{1}{k+1} y.w_0 \otimes v_k + w_0 \otimes \frac{1}{k+1} y.v_k \end{aligned}$$

$$
\begin{aligned}
&= \frac{k-1}{k+1} w_2 \otimes v_{k-1} + \frac{2}{k+1} w_2 \otimes v_{k-1} + \frac{k}{k+1} w_1 \otimes v_k + \frac{1}{k+1} w_1 \otimes v_k \\
&\quad + w_0 \otimes v_{k+1} \\
&= w_2 \otimes v_{k-1} + w_1 \otimes v_k + w_0 \otimes v_{k+1}.
\end{aligned}
$$

We likewise see that for $k = n+1$ the last term vanishes, so we have $s_{k+1} = w_2 \otimes v_{n-1} + w_1 \otimes v_n$, and for $k = n+2$ the two last terms vanish, so we get $s_{k+2} = w_2 \otimes v_n$. Thus altogether we get the description in eq. (1.15).

Suppose now that $n \geq 1$. We want to describe the $t_k$'s of eq. (1.16) more explicitly. We have that $t_0 = w_0 \otimes v_1 - \frac{n}{2} w_1 \otimes v_0$ and $t_k = \frac{1}{k!} y^k . t_0$. We see that

$$
\begin{aligned}
t_1 &= y.\big(w_0 \otimes v_1 - \frac{n}{2} w_1 \otimes v_0\big) \\
&= y.w_0 \otimes v_1 + w_0 \otimes y.v_1 - \frac{n}{2} y.w_1 \otimes v_0 + \frac{n}{2} w_1 \otimes y.v_0 \\
&= w_1 \otimes v_1 + 2 w_0 \otimes v_2 - n w_2 \otimes v_0 - \frac{n}{2} w_1 \otimes v_1 \\
&= 2 w_0 \otimes v_2 - \frac{n-2}{2} w_1 \otimes v_1 - n w_2 \otimes v_0,
\end{aligned}
$$

and inductively we get that

$$
t_k = (k+1) w_0 \otimes v_{k+1} - \frac{n-2k}{2} w_1 \otimes v_k + (k-1-n) w_2 \otimes v_{k-1}
$$

for $1 \leq k \leq n-1$, since the base case holds and given the equality for $k < n-1$ we get

$$
\begin{aligned}
t_{k+1} &= \frac{1}{k+1} y.t_k \\
&= y.w_0 \otimes v_{k+1} + w_0 \otimes y.v_{k+1} - \frac{n-2k}{2(k+1)} y.w_1 \otimes v_k \\
&\quad - \frac{n-2k}{2(k+1)} w_1 \otimes y.v_k + \frac{k-1-n}{k+1} w_2 \otimes y.v_{k-1} \\
&= w_1 \otimes v_{k+1} + (k+2) w_0 \otimes v_{k+2} - \frac{n-2k}{k+1} w_2 \otimes v_k \\
&\quad - \frac{n-2k}{2} w_1 \otimes v_{k+1} + \frac{(k-1-n)k}{k+1} w_2 \otimes v_k \\
&= (k+2) w_0 \otimes v_{k+2} - \frac{n-2(k+1)}{2} w_1 \otimes v_{k+1} \\
&\quad + \left( \frac{k^2 - k - nk - n + 2k}{k+1} \right) w_2 \otimes v_k \\
&= (k+2) w_0 \otimes v_{k+2} - \frac{n-2(k+1)}{2} w_1 \otimes v_{k+1} + (k-n) w_2 \otimes v_k,
\end{aligned}
$$

where we in the last equality use that $(k+1)(k-n) = k^2 - nk + k - n = k^2 - k - nk - n + 2k$. We likewise see that for $k = n$ the first term vanishes so

$$
t_n = \frac{n}{2} w_1 \otimes v_n - w_2 \otimes v_{n-1}.
$$

Thus we altogether get the description in eq. (1.17).

Suppose now that $n \geq 2$. We want to describe the $u_k$'s of eq. (1.18) more explicitely. We have that

$$u_0 \coloneqq w_0 \otimes v_2 - \frac{n-1}{2} w_1 \otimes v_1 + \frac{n(n-1)}{2} w_2 \otimes v_0$$

and $u_k = \frac{1}{k!} y^k . u_0$. We see inductively that

$$\begin{aligned} u_k = {} & \frac{(k+1)(k+2)}{2} w_0 \otimes v_{k+2} - \frac{(k+1)(n-k-1)}{2} w_1 \otimes v_{k+1} \\ & + \frac{(n-k)(n-k-1)}{2} w_2 \otimes v_k \end{aligned}$$

for $0 \leq k \leq n-2$, since the base case holds and given the equality for $k < n-2$ we get

$$\begin{aligned} u_{k+1} &= \frac{1}{k+1} y . u_k \\ &= \frac{k+2}{2} y . w_0 \otimes v_{k+2} + \frac{k+2}{2} w_0 \otimes y . v_{k+2} \\ &\quad - \frac{n-k-1}{2} y . w_1 \otimes v_{k+1} - \frac{n-k-1}{2} w_1 \otimes y . v_{k+1} \\ &\quad + \frac{(n-k)(n-k-1)}{2(k+1)} w_2 \otimes y . v_k \\ &= \frac{k+2}{2} w_1 \otimes v_{k+2} + \frac{(k+2)(k+3)}{2} w_0 \otimes v_{k+3} \\ &\quad - (n-k-1) w_2 \otimes v_{k+1} - \frac{(n-k-1)(k+2)}{2} w_1 \otimes v_{k+2} \\ &\quad + \frac{(n-k)(n-k-1)}{2} w_2 \otimes v_{k+1} \\ &= \frac{(k+2)(k+3)}{2} w_0 \otimes v_{k+3} \\ &\quad - \frac{(n-k-1)(k+2) - (k+2)}{2} w_1 \otimes v_{k+2} \\ &\quad + \frac{(n-k)(n-k-1) - 2(n-k-1)}{2} w_2 \otimes v_{k+1} \\ &= \frac{(k+2)(k+3)}{2} w_0 \otimes v_{k+3} \\ &\quad - \frac{(k+2)(n-k-2)}{2} w_1 \otimes v_{k+2} \\ &\quad + \frac{(n-k-1)(n-k-2)}{2} w_2 \otimes v_{k+1} . \end{aligned}$$

Thus we altogether get the description in eq. (1.19).

## A.2 Finding $w_1 \otimes v_k$

Using the bases $(s_k \mid 0 \le k \le n+2)$ of eq. (1.15), $(t_k \mid 0 \le k \le n)$ of eq. (1.17), and $(u_k \mid 0 \le k \le n-2)$ of eq. (1.19), we see that

$$\begin{aligned}
&\frac{2(k+1)(n+1-k)}{(n+1)(n+2)} s_{k+1} - \frac{2(n-2k)}{n(n+2)} t_k - \frac{4}{n(n+1)} u_{k-1} \\
&= \frac{2(k+1)(n+1-k)}{(n+1)(n+2)} \Big( w_0 \otimes v_{k+1} + w_1 \otimes v_k + w_2 \otimes v_{k-1} \Big) \\
&\quad - \frac{2(n-2k)}{n(n+2)} \Big( (k+1) w_0 \otimes v_{k+1} - \frac{n-2k}{2} w_1 \otimes v_k \\
&\qquad + (k-1-n) w_2 \otimes v_{k-1} \Big) \\
&\quad - \frac{4}{n(n+1)} \Big( \frac{k(k+1)}{2} w_0 \otimes v_{k+1} - \frac{k(n-k)}{2} w_1 \otimes v_k \\
&\qquad + \frac{(n-k+1)(n-k)}{2} w_2 \otimes v_{k-1} \Big) \\
&= \frac{\begin{pmatrix} 2(k+1)(n+1-k)n - 2(n-2k)(k+1)(n+1) \\ \hfill - 2k(k+1)(n+2) \end{pmatrix}}{n(n+1)(n+2)} w_0 \otimes v_{k+1} \\
&\quad + \frac{\begin{pmatrix} 2(k+1)(n+1-k)n + (n-2k)(n-2k)(n+1) \\ \hfill + 2k(n-k)(n+2) \end{pmatrix}}{n(n+1)(n+2)} w_1 \otimes v_k \\
&\quad + \frac{\begin{pmatrix} 2(k+1)(n+1-k)n - 2(n-2k)(k-1-n)(n+1) \\ \hfill - 2(n-k+1)(n-k)(n+2) \end{pmatrix}}{n(n+1)(n+2)} w_2 \otimes v_{k-1} \\
&= 2(k+1) \frac{(n+1-k)n - (n-2k)(n+1) - k(n+2)}{n(n+1)(n+2)} w_0 \otimes v_{k+1} \\
&\quad + \frac{\begin{pmatrix} 2(k+1)(n+1-k)n + (n-2k)(n-2k)(n+1) \\ \hfill + 2k(n-k)(n+2) \end{pmatrix}}{n(n+1)(n+2)} w_1 \otimes v_k \\
&\quad + 2(n+1-k) \frac{(k+1)n + (n-2k)(n+1) - (n-k)(n+2)}{n(n+1)(n+2)} w_2 \otimes v_{k-1}.
\end{aligned}$$

Now we note that

$$\begin{aligned}
(n+1-k)n &- (n-2k)(n+1) - k(n+2) \\
&= n\Big( (n+1-k) - (n-2k) - k \Big) - (n-2k) - 2k \\
&= n - (n-2k) - 2k = 0,
\end{aligned}$$

and

$$\begin{aligned}
&(k+1)n + (n-2k)(n+1) - (n-k)(n+2) \\
&\qquad = n\Big((k+1) + (n-2k) - (n-k)\Big) + (n-2k) - 2(n-k) \\
&\qquad = n + n - 2k - 2n + 2k = 0,
\end{aligned}$$

while

$$\begin{aligned}
&2(k+1)(n+1-k)n + (n-2k)(n-2k)(n+1) + 2k(n-k)(n+2) \\
&= n\Big(2(k+1)(n+1-k) + (n-2k)(n+1) + 2k(n-k)\Big) \\
&\quad - 2k(n-2k)(n+1) + 4k(n-k) \\
&= n\Big(2(k+1)(n+1-k) + (n-2k)(n+1) + 2k(n-k)\Big) \\
&\quad - 2kn(n-2k) - 2k(n-2k) + 4k(n-k) \\
&= n\Big(2(k+1)(n+1-k) + (n-2k)(n+1) + 2k(n-k)\Big) \\
&\quad - 2kn(n-2k) + 2kn \\
&= n\Big(2(k+1)(n+1-k) + (n-2k)(n+1) + 2k(n-k) - 2k(n-2k) \\
&\qquad + 2k\Big),
\end{aligned}$$

where

$$\begin{aligned}
&2(k+1)(n+1-k) + (n-2k)(n+1) + 2k(n-k) - 2k(n-2k) + 2k \\
&= (n+1)\Big(2(k+1) + (n-2k)\Big) - 2k(k+1) \\
&\quad + 2k\Big((n-k) - (n-2k) + 1\Big) \\
&= (n+1)(n+2) - 2k(k+1) + 2k(k+1) \\
&= (n+1)(n+2),
\end{aligned}$$

so

$$\begin{aligned}
&2(k+1)(n+1-k)n + (n-2k)(n-2k)(n+1) + 2k(n-k)(n+2) \\
&= n(n+1)(n+2).
\end{aligned}$$

Thus we see that

$$\begin{aligned}
&\frac{2(k+1)(n+1-k)}{(n+1)(n+2)} s_{k+1} - \frac{2(n-2k)}{n(n+2)} t_k - \frac{4}{n(n+1)} u_{k-1} \\
&= 0 + \frac{n(n+1)(n+2)}{n(n+1)(n+2)} w_1 \otimes v_k + 0 \\
&= w_1 \otimes v_k
\end{aligned}$$

I will probably remove some of this and just say that algebraic manipulation shows that ...

giving us eq. (1.20).

Likewise for $n \geq 1$, we get that

$$\begin{aligned}\frac{2}{n+2}(s_1 - t_0) &= \frac{2}{n+2}\Big(w_0 \otimes v_1 + w_1 \otimes v_0 - w_0 \otimes v_1 + \frac{n}{2} w_1 \otimes v_0\Big)\\ &= \frac{2}{n+2}\frac{n+2}{2} w_1 \otimes v_0\\ &= w_1 \otimes v_0\end{aligned}$$

and

$$\begin{aligned}\frac{2}{n+2}(s_{n+1} + t_n) &= \frac{2}{n+2}\Big(w_2 \otimes v_{n+1} + w_1 \otimes v_n + \frac{n}{2} w_1 \otimes v_n - w_2 \otimes v_{n-1}\Big)\\ &= \frac{2}{n+2}\frac{n+2}{2} w_1 \otimes v_n\\ &= w_1 \otimes v_n\end{aligned}$$

giving us eq. (1.21).

## A.3 Inner products in $V(2) \otimes V(n)$

Given $s_0 = w_0 \otimes v_0$, $t_0 = w_0 \otimes v_1 - \frac{n}{2} w_1 \otimes v_0$, and $u_0 = w_0 \otimes v_2 - \frac{n-1}{2} w_1 \otimes v_1 + \frac{n(n-1)}{2} w_2 \otimes v_0$ from eq. (1.15), eq. (1.17), and eq. (1.19), we want to find $\langle s_0, s_0 \rangle$, $\langle t_0, t_0 \rangle$, and $\langle u_0, u_0 \rangle$ using the inner products of eq. (1.22) and eq. (1.23). We see that

$$\begin{aligned}\langle s_0, s_0 \rangle &= \langle w_0 \otimes v_0, w_0 \otimes v_0 \rangle = \langle w_0, w_0 \rangle \cdot \langle v_0, v_0 \rangle\\ &= \binom{2}{0} \cdot \binom{n}{0} = 1.\end{aligned}$$

Likewise we get that

$$\begin{aligned}\langle t_0, t_0 \rangle &= \Big\langle w_0 \otimes v_1 - \frac{n}{2} w_1 \otimes v_0, w_0 \otimes v_1 - \frac{n}{2} w_1 \otimes v_0 \Big\rangle\\ &= \langle w_0 \otimes v_1, w_0 \otimes v_1 \rangle - \frac{n}{2}\langle w_0 \otimes v_1, w_1 \otimes v_0 \rangle - \frac{n}{2}\langle w_1 \otimes v_0, w_0 \otimes v_1 \rangle\\ &\quad + \frac{n^2}{4}\langle w_1 \otimes v_0, w_1 \otimes v_0 \rangle\\ &= \langle w_0, w_0 \rangle \cdot \langle v_1, v_1 \rangle - \frac{n}{2}\langle w_0, w_1 \rangle \langle v_1, v_0 \rangle - \frac{n}{2}\langle w_1, w_0 \rangle \cdot \langle v_0, v_1 \rangle\\ &\quad + \frac{n^2}{4}\langle w_1, w_1 \rangle \cdot \langle v_0, v_0 \rangle\\ &= \binom{2}{0} \cdot \binom{n}{1} - 0 - 0 + \frac{n^2}{4}\binom{2}{1} \cdot \binom{n}{0}\\ &= n + \frac{n^2}{2} = \frac{n(n+2)}{2},\end{aligned}$$

and noting that as above all terms with $\langle w_i \otimes v_j, w_k \otimes v_\ell\rangle$ with $i \neq k$ or $j \neq \ell$ vanish since then either $\langle w_i, w_k\rangle = 0$ or $\langle v_j, v_\ell\rangle$, we see that

$$\begin{aligned}
\langle u_0, u_0\rangle &= \Big\langle w_0 \otimes v_2 - \frac{n-1}{2} w_1 \otimes v_1 + \frac{n(n-1)}{2} w_2 \otimes v_0, \\
&\qquad w_0 \otimes v_2 - \frac{n-1}{2} w_1 \otimes v_1 + \frac{n(n-1)}{2} w_2 \otimes v_0 \Big\rangle \\
&= \langle w_0 \otimes v_2, w_0 \otimes v_2\rangle + \frac{(n-1)^2}{4}\langle w_1 \otimes v_1, w_1 \otimes v_1\rangle \\
&\quad + \frac{n^2(n-1)^2}{4}\langle w_2 \otimes v_0, w_2 \otimes v_0\rangle \\
&= \langle w_0, w_0\rangle \cdot \langle v_2, v_2\rangle + \frac{(n-1)^2}{4}\langle w_1, w_1\rangle \cdot \langle v_1, v_1\rangle \\
&\quad + \frac{n^2(n-1)^2}{4}\langle w_2, w_2\rangle \cdot \langle v_0, v_0\rangle \\
&= \binom{2}{0} \cdot \binom{n}{2} + \frac{(n-1)^2}{4}\binom{2}{1}\binom{n}{1} + \frac{n^2(n-1)^2}{4}\binom{2}{2} \cdot \binom{n}{0} \\
&= \frac{n(n-1)}{2} + \frac{n(n-1)^2}{2} + \frac{n^2(n-1)^2}{4} \\
&= n(n-1)\frac{2 + 2(n-1) + n(n-1)}{4} \\
&= n(n-1)\frac{n^2+n}{4} = \frac{n^2(n+1)(n-1)}{4}.
\end{aligned}$$

Thus we get exactly the results of eq. (1.24).

Need to show $\langle s_k, s_k\rangle = \langle s_0, s_0\rangle\binom{n+2}{k}$ and more

## A.4 Finding $\overline{w}_1 \otimes \overline{v}_k$

We want to find $\overline{w}_1 \otimes \overline{v}_k$ in terms of $\overline{s}_k$, $\overline{t}_k$, and $\overline{u}_k$ from eqs. (1.25) and (1.26). First we note that for $0 < k < n$

$$\begin{aligned}
\sqrt{2\binom{n}{k}}\overline{w}_1 \otimes \overline{v}_k &= \sqrt{\binom{2}{1}}\overline{w}_1 \otimes \sqrt{\binom{n}{k}}\overline{v}_k \\
&= w_1 \otimes v_k \\
&= \frac{2(k+1)(n+1-k)}{(n+1)(n+2)} s_{k+1} - \frac{2(n-2k)}{n(n+2)} t_k - \frac{4}{n(n+1)} u_{k-1} \\
&= \frac{2(k+1)(n+1-k)}{(n+1)(n+2)} \sqrt{\binom{n+2}{k+1}}\overline{s}_{k+1} \\
&\quad - \frac{2(n-2k)}{n(n+2)} \sqrt{\frac{n(n+2)}{2}\binom{n}{k}}\overline{t}_k \\
&\quad - \frac{4}{n(n+1)} \sqrt{\frac{n^2(n+1)(n-1)}{4}\binom{n-2}{k-1}}\overline{u}_{k-1}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{2(k+1)(n+1-k)}{(n+1)(n+2)} \sqrt{\binom{n+2}{k+1}} \overline{s}_{k+1} \\
&\quad - \frac{\sqrt{2}(n-2k)}{\sqrt{n(n+2)}} \sqrt{\binom{n}{k}} \overline{t}_k \\
&\quad - \frac{2\sqrt{(n-1)}}{\sqrt{(n+1)}} \sqrt{\binom{n-2}{k-1}} \overline{u}_{k-1}.
\end{aligned}
$$

Now since

$$
\frac{\binom{n+2}{k+1}}{\binom{n}{k}} = \frac{(n+2)(n+1)}{(k+1)(n+1-k)}, \qquad \frac{\binom{n-2}{k-1}}{\binom{n}{k}} = \frac{k(n-k)}{n(n-1)},
$$

we see that

$$
\begin{aligned}
\overline{w}_1 \otimes \overline{v}_k &= \frac{\sqrt{2}(k+1)(n+1-k)}{(n+1)(n+2)} \sqrt{\frac{(n+2)(n+1)}{(k+1)(n+1-k)}} \overline{s}_{k+1} \\
&\quad - \frac{(n-2k)}{\sqrt{n(n+2)}} \overline{t}_k \\
&\quad - \frac{\sqrt{2(n-1)}}{\sqrt{(n+1)}} \sqrt{\frac{k(n-k)}{n(n-1)}} \overline{u}_{k-1} \\
&= \sqrt{\frac{2(k+1)(n+1-k)}{(n+1)(n+2)}} \overline{s}_{k+1} - \frac{(n-2k)}{\sqrt{n(n+2)}} \overline{t}_k \\
&\quad - \sqrt{\frac{2k(n-k)}{n(n+1)}} \overline{u}_{k-1}.
\end{aligned}
$$

Also since eq. (1.21) is a special case of eq. (1.20) the above formula also holds for $k \in \{0, n\}$ if we take the coefficient in front of $\overline{u}_{k-1}$ to be 0. Thus we indeed get eq. (1.27)

## A.5 $F_3, F_+, F_-$ in terms of $E_+, E_-, D_0, D_+, D_-$

We have already seen that

$$
F_3\xi = \sqrt{\ell^2 - m^2} D_-\xi - mD_0\xi - \sqrt{(\ell+1)^2 - m^2} D_+\xi
$$

for $\xi \in R_{\ell,m}$ by using eq. (1.29) and the definition of how we expanded $D_0$, $D_+$, and $D_-$ to maps on all of $M$. Now we get by eqs. (1.2) and (1.9) and the commutative diagrams in eq. (1.10) that

$$\begin{aligned}
F_+\xi &= [F_3, H_+]\xi = F_3H_+\xi - H_+F_3\xi \\
&= \sqrt{(\ell+m+1)(\ell-m)}F_3E_+\xi - \sqrt{\ell^2-m^2}H_+D_-\xi + mH_+D_0\xi \\
&\quad + \sqrt{(\ell+1)^2-m^2}H_+D_+\xi \\
&= \sqrt{(\ell+m+1)(\ell-m)}\Big(\sqrt{\ell^2-(m+1)^2}D_-E_+\xi - (m+1)D_0E_+\xi \\
&\qquad\qquad - \sqrt{(\ell+1)^2-(m+1)^2}D_+E_+\xi\Big) \\
&\quad - \sqrt{\ell^2-m^2}\sqrt{((\ell-1)+m+1)((\ell-1)-m)}E_+D_-\xi \\
&\quad + m\sqrt{(\ell+m+1)(\ell-m)}E_+D_0\xi \\
&\quad + \sqrt{(\ell+1)^2-m^2}\sqrt{((\ell+1)+m+1)((\ell+1)-m)}E_+D_+\xi \\
&= \sqrt{(\ell+m+1)(\ell-m)}\Big(\sqrt{\ell^2-(m+1)^2}D_-E_+\xi - (m+1)D_0E_+\xi \\
&\qquad\qquad - \sqrt{(\ell+1)^2-(m+1)^2}D_+E_+\xi\Big) \\
&\quad - \sqrt{\ell^2-m^2}\sqrt{(\ell+ml)(\ell-m-1)}D_-E_+\xi \\
&\quad + m\sqrt{(\ell+m+1)(\ell-m)}D_0E_+\xi \\
&\quad + \sqrt{(\ell+1)^2-m^2}\sqrt{(\ell+m+2)(\ell-m+1)}D_+E_+\xi \\
&= \Big(\sqrt{(\ell+m+1)(\ell-m)(\ell^2-(m+1)^2)} \\
&\qquad - \sqrt{(l^2-m^2)(l+m)(l-m-1)}\Big)D_-E_+\xi \\
&\quad - \sqrt{(\ell+m+1)(\ell-m)}D_0E_+\xi \\
&\quad + \Big(\sqrt{((\ell+1)^2-m^2)(\ell+m+2)(\ell-m+1)} \\
&\qquad - \sqrt{(\ell+m+1)(\ell-m)((\ell+1)^2-(m+1)^2)}\Big)D_+E_+\xi
\end{aligned}$$

for $\xi \in R_{\ell,m}$ and $-\ell+1 \le m < \ell-1$. By pure algebraic manipulation note that

I have checked this in Mathematica, but I would prefer not to write this out, although I can do it later if necessary

$$\begin{aligned}
&\sqrt{(\ell+m+1)(\ell-m)(\ell^2-(m+1)^2)} - \sqrt{(l^2-m^2)(l+m)(l-m-1)} \\
&= \sqrt{(\ell-m)(\ell-m-1)}
\end{aligned}$$

and

$$\begin{aligned}
&\sqrt{((\ell+1)^2-m^2)(\ell+m+2)(\ell-m+1)} \\
&\qquad - \sqrt{(\ell+m+1)(\ell-m)((\ell+1)^2-(m+1)^2)} \\
&= \sqrt{(\ell+m+1)((\ell+m+2))},
\end{aligned}$$

so we get that

$$\begin{aligned}F_+\xi = \sqrt{(\ell-m)(\ell-m-1)}D_-E_+\xi - \sqrt{(\ell+m+1)(\ell-m)}D_0E_+\xi\\ - \sqrt{(\ell+m+1)(\ell+m+2)}D_+E_+\xi\end{aligned}$$

for $\xi \in R_{\ell,m}$ and $-\ell+1 \le m < \ell-1$. Note that this formula also holds true for $m \in \{-\ell, \ell-1, \ell\}$.

Consider this more carefully

Likewise we get that

$$\begin{aligned}
F_-\xi &= [H_-, F_3]\xi = H_-F_3\xi - F_3H_-\xi\\
&= \sqrt{\ell^2-m^2}H_-D_-\xi - mH_-D_0\xi - \sqrt{(\ell+1)^2-m^2}H_-D_+\xi\\
&\quad - \sqrt{(\ell+m)(\ell-m+1)}F_3E_-\xi\\
&= \sqrt{\ell^2-m^2}\sqrt{((\ell-1)+m)((\ell-1)-m+1)}E_-D_-\\
&\quad - m\sqrt{(\ell+m)(\ell-m+1)}E_-D_0\xi\\
&\quad - \sqrt{(\ell+1)^2-m^2}\sqrt{((\ell+1)+m)((\ell+1)-m+1)}E_-D_+\\
&\quad - \sqrt{(\ell+m)(\ell-m+1)}\Big(\sqrt{\ell^2-(m-1)^2}D_-E_-\xi - (m-1)D_0E_-\xi\\
&\quad - \sqrt{(\ell+1)^2-(m-1)^2}D_+E_-\xi\Big)\\
&= \sqrt{\ell^2-m^2}\sqrt{(\ell+m-1)(\ell-m)}D_-E_- - m\sqrt{(\ell+m)(\ell-m+1)}D_0E_-\xi\\
&\quad - \sqrt{(\ell+1)^2-m^2}\sqrt{(\ell+m+1)(\ell-m+2)}D_+E_-\\
&\quad - \sqrt{(\ell+m)(\ell-m+1)}\Big(\sqrt{\ell^2-(m-1)^2}D_-E_-\xi - (m-1)D_0E_-\xi\\
&\quad - \sqrt{(\ell+1)^2-(m-1)^2}D_+E_-\xi\Big)\\
&= -\Big(\sqrt{(\ell+m)(\ell-m+1)(\ell^2-(m-1)^2)}\\
&\qquad - \sqrt{(\ell^2-m^2)(\ell+m-1)(\ell-m)}\Big)D_-E_-\xi\\
&\quad - \sqrt{(\ell+m)(\ell-m+1)}D_0E_-\xi\\
&\quad - \Big(\sqrt{((\ell+1)^2-m^2)(\ell+m+1)(\ell-m+2)}\\
&\qquad - \sqrt{(\ell+m)(\ell-m+1)((\ell+1)^2-(m-1)^2)}\Big)D_+E_-\xi
\end{aligned}$$

Which $m$

for $\xi \in R_{\ell,m}$ and . Again note that

$$\begin{aligned}&\sqrt{(\ell+m)(\ell-m+1)(\ell^2-(m-1)^2)} - \sqrt{(\ell^2-m^2)(\ell+m-1)(\ell-m)}\\ &= \sqrt{(\ell+m)(\ell+m-1)}\end{aligned}$$

and

$$\begin{aligned}&\sqrt{((\ell+1)^2-m^2)(\ell+m+1)(\ell-m+2)}\\ &\qquad - \sqrt{(\ell+m)(\ell-m+1)((\ell+1)^2-(m-1)^2)}\\ &= \sqrt{(\ell-m+1)(\ell-m+2)},\end{aligned}$$

so we get that

$$F_-\xi = -\sqrt{(\ell+m)(\ell+m-1)}D_-E_-\xi - \sqrt{(\ell+m)(\ell-m+1)}D_0E_-\xi \\ - \sqrt{(\ell-m+1)(\ell-m+2)}D_+E_-\xi$$

for $\xi \in R_{\ell,m}$. Thus indeed we get eq. (1.30).

## A.6 Relations for $D_0$, $D_+$, $D_-$

We want to show that the formulae eq. (1.30) for the linear operators $F_+$, $F_-$, and $F_3$ together with the formulae eqs. (1.7) and (1.9) for $H_+$, $H_-$, and $H_3$ define a representation of $L$, i.e. they satisfy the commutation relations of eq. (1.2), if and only if $D_0$, $D_+$, and $D_-$ satisfy eq. (1.31). By eqs. (1.2) and (1.9) ...