# Cohomology of certain $p$-adic Groups

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# Contents

Some extra stuff.

# Chapter 1

# Cohomology of Unipotent Groups

cha:cohunigps

## 1.1 Introduction

So far some of the details are still skipped, but I have tried to write pretty much everything that's not already written in results I cite.

### Notation and setup

Let $p$ be a prime and let $k = \mathbb{Z}_p$. Also note that the following is true for any integral domain $k$ (in particular also for $\mathbb{F}_p$).

Let $\mathcal{G}_\mathbb{Z}$ be a split and connected reductive algebraic $\mathbb{Z}$-group and let $\mathcal{G} = (\mathcal{G}_\mathbb{Z})_k$ (the base change from $\mathbb{Z}$ to $k$). Let $\mathcal{T}_\mathbb{Z}$ be a split maximal torus of $\mathcal{G}_\mathbb{Z}$ and set $\mathcal{T} = (\mathcal{T}_\mathbb{Z})_k$. Let $\Phi = \Phi(\mathcal{G}, \mathcal{T})$ be the root system of $\mathcal{G}$ with respect to $\mathcal{T}$ and note that $\Phi$ can be identified with the root system of $\mathcal{G}_\mathbb{Z}$ with respect to $\mathcal{T}_\mathbb{Z}$. Also note that $\mathrm{Lie}(\mathcal{G}) = \mathrm{Lie}(\mathcal{G}_\mathbb{Z}) \otimes_\mathbb{Z} k$ and for any $\alpha \in \Phi$ we have the root subgroup $\mathcal{N}_\alpha \subseteq \mathcal{G}$ with $\mathrm{Lie}\,\mathcal{N}_\alpha = (\mathrm{Lie}\,\mathcal{G})_\alpha = (\mathrm{Lie}\,\mathcal{G}_\mathbb{Z})_\alpha \otimes_\mathbb{Z} k$. Now fix a $k$-basis $X_\alpha$ of the Lie algebra of $\mathcal{N}_\alpha$. This choice gives rise to a unique isomorphism isomorphism of group schemes $x_\alpha\colon \mathcal{G}_a \xrightarrow{\cong} \mathcal{N}_\alpha$ such that $(dx_\alpha)(1) = X_\alpha$. We furthermore fix a basis $\Delta \subseteq \Phi$ of the root system such that we get a decomposition $\Phi = \Phi^+ \cup \Phi^-$ into positive and negative roots. Let $\mathcal{B} = \mathcal{T}\mathcal{N}$ and $\mathcal{B}^+ = \mathcal{T}\mathcal{N}^+$ denote the Borel subgroups of $\mathcal{G}$ corresponding to $\Phi^-$ and $\Phi^+$, respectively, with unipotent radicals $\mathcal{N}$ and $\mathcal{N}^+$. (Here we also have corresponding algebraic $\mathbb{Z}$-groups.)

DK Note: We might be able to avoid going through $\mathbb{Z}$ at first with some work. Also, we may need to assume that $\mathcal{G}$ is simple.

For any total ordering of $\Phi^-$ the multiplication induces an isomorphism of schemes $\prod_{\alpha\in\Phi^-} \mathcal{N}_\alpha \xrightarrow{\cong} \mathcal{N}$. For convenience we fix in the following such a total ordering which has the additional property that $\alpha_1 \geq \alpha_2$ if $\mathrm{ht}(\alpha_1) \leq \mathrm{ht}(\alpha_2)$. All products indexed by $\Phi^-$ are meant to be taken according to this ordering. Here we have the height function $\mathrm{ht}\colon \mathbb{Z}[\Delta] \to \mathbb{Z}$ given by $\sum_{\alpha\in\Delta} m_\alpha \alpha \mapsto \sum_{\alpha\in\Delta} m_\alpha$. In particular, since $\Phi \subseteq \mathbb{Z}[\Delta]$ the height $\mathrm{ht}(\beta)$ of any root $\beta \in \Phi$ is defined.

Let furthermore $\rho$ be the half-sum of the elements of $\Phi^+$, let $X = X(\mathcal{T}) \cong X(\mathcal{T}_\mathbb{Z})$ be the character group of $\mathcal{T}$, let

$$X^+ = \{\lambda \in X \mid \langle \lambda, \alpha^\vee \rangle \geq 0 \text{ for all } \alpha \in \Phi^+\},$$

and let $h$ be the Coxeter number of $\mathcal{G}$ and assume from now on that $p \geq h-1$. For any $\lambda \in X^+$, let $V_\mathbb{Z}(\lambda)$ be the Weyl module for $\mathcal{G}_\mathbb{Z}$ over $\mathbb{Z}$ with highest weight $\lambda$, and let $V_k(\lambda) = V_\mathbb{Z}(\lambda) \otimes_\mathbb{Z} k$.

Let $\Phi^\vee$ be the dual root system of $\Phi$ and let $W$ be the corresponding Weyl group with length function $\ell$ on $W$. Let $\mathfrak{n}_\mathbb{Z} = \mathrm{Lie}(\mathcal{N}_\mathbb{Z})$ be the Lie algebra of $\mathcal{N}_\mathbb{Z}$ over $\mathbb{Z}$ and $\mathfrak{n} = \mathfrak{n}_{\mathbb{F}_p} = \mathrm{Lie}(\mathcal{N}_{\mathbb{F}_p}) = \mathfrak{n}_\mathbb{Z} \otimes \mathbb{F}_p$ be the Lie algebra of $\mathcal{N}_{\mathbb{F}_p}$ over $\mathbb{F}_p$.

Finally let $G = N = \mathcal{N}(\mathbb{Z}_p) = \mathcal{N}_\mathbb{Z}(\mathbb{Z}_p)$ and let $\mathfrak{g} = \mathbb{F}_p \otimes_{\mathbb{F}_p[\pi]} \mathrm{gr}\, G$.

## 1.2 The $p$-valuation

sec:pval

This section is mainly based on some unpublished notes by Schneider.

In this section we will write $N$ for $\mathcal{N}(\mathbb{Z}_p)$, and we note that as a set $N$ is the direct product $N = \prod_{\alpha\in\Phi^-} x_\alpha(\mathbb{Z}_p)$, which allows us to introduce the function

$$\begin{aligned} \omega\colon N \setminus \{1\} &\to \mathbb{N} \\ \prod_{\alpha\in\Phi^-} x_\alpha(a_\alpha) &\mapsto \min_{\alpha\in\Phi^-} \big(v_p(a_\alpha) - \mathrm{ht}(\alpha)\big), \end{aligned}$$

where $v_p$ denotes the usual $p$-adic valuation on $\mathbb{Z}_p$. Here it is important to note that we write any $g \in N$ uniquely as product

$$g = \prod_{\alpha\in\Phi^-} x_\alpha(a_\alpha)$$

by taking the product following the total ordering $\geq$ of $\Phi^-$ defined above. Now, with the convention that $\omega(1) \coloneqq \infty$, we define the descending sequence of subsets

$$N_m \coloneqq \{g \in N \mid \omega(g) \geq m\}$$

in $N$ for $m \geq 0$. The main goal of this section is to show that $\omega$ is a $p$-valuation by a careful analysis of the sequence of subsets given by $N_m$.

We first note that clearly $N_1 = N$, $\bigcap_m N_m = \{1\}$, and

$$\begin{aligned} N_m &= \prod_{\alpha\in\Phi^-} x_\alpha(p^{\max(0,m+\mathrm{ht}(\alpha))}\mathbb{Z}_p) \\ &= \prod_{\substack{\alpha\in\Phi^- \\ \mathrm{ht}(\alpha)=-1}} x_\alpha(p^{m-1}\mathbb{Z}_p)\cdots \prod_{\substack{\alpha\in\Phi^- \\ \mathrm{ht}(\alpha)=-(m-1)}} x_\alpha(p\mathbb{Z}_p) \prod_{\substack{\alpha\in\Phi^- \\ \mathrm{ht}(\alpha)\le -m}} x_\alpha(\mathbb{Z}_p). \end{aligned} \tag{1.1}$$ {eq:N_m}

In our analysis of this sequence we will also need two other filtrations of $N$. Firstly we will consider the filtration by congruence subgroups

$$\begin{aligned} N(m) &:= \ker\big(\mathcal{N}(\mathbb{Z}_p) \to \mathcal{N}(\mathbb{Z}/p^m\mathbb{Z})\big) \\ &= \prod_{\alpha\in\Phi^-} x_\alpha(p^m\mathbb{Z}_p) \end{aligned} \tag{1.2}$$ {eq:N(m)}

for $m \ge 0$. Secondly, using the descending central series of the group $\mathcal{G}(\mathbb{Q}_p)$ defined by $C^1\mathcal{G}(\mathbb{Q}_p) := \mathcal{G}(\mathbb{Q}_p)$ and $C^{m+1}\mathcal{G}(\mathbb{Q}_p) := [C^m\mathcal{G}(\mathbb{Q}_p), \mathcal{G}(\mathbb{Q}_p)]$, we consider the filtration given by

$$N_{(m)} := N \cap C^m\mathcal{G}(\mathbb{Q}_p)$$

for $m \ge 1$. By BT we have that

DK Note: Check reference

$$N_{(m)} = \prod_{\substack{\alpha\in\Phi^- \\ \mathrm{ht}(\alpha)\le -m}} x_\alpha(\mathbb{Z}_p). \tag{1.3}$$ {eq:N_(m)}

We note that the natural map

$$\prod_{\substack{\alpha\in\Phi^- \\ \mathrm{ht}(\alpha)=-m}} x_\alpha(\mathbb{Z}_p) \to N_{(m)}/N_{(m+1)}$$

is an isomorphism of abelian groups, and that all the subgroups $N(m)$ and $N_{(m)}$ are normal in $N$.

We are now ready to prove the following lemma, which will help us when showing that $\omega$ is a $p$-valuation.

lem:N_m
item:N_m
**Lemma 1.1.** (i) $N_m = \prod_{1\le i\le m} N(m-i)\cap N_{(i)}$, for any $m \ge 1$, is a normal subgroup of $N$ which is independent of the choices made.

item:N_mcom
(ii) $[N_\ell, N_m] \subseteq N_{\ell+m}$ for any $\ell, m \ge 1$.

(iii) $N_m/N_{m+1}$, for any $m \ge 1$, is an $\mathbb{F}_p$-vector space of dimension equal to $|\{\alpha\in\Phi^- \mid \mathrm{ht}(\alpha) \ge -m\}|$.

item:g^p (iv) Let $g \in N_m$ for some $m \geq 1$. If $g^p \in N_{m+2}$, then $g \in N_{m+1}$.

*Proof.* (i) Using (1.2) and (1.3) we note that

$$\prod_{\substack{\alpha\in\Phi^- \\ \mathrm{ht}(\alpha)=-i}} x_\alpha(p^{m-1}\mathbb{Z}_p) \subseteq N(m-i)\cap N_{(i)} \quad \text{and} \quad \prod_{\substack{\alpha\in\Phi^- \\ \mathrm{ht}(\alpha)\leq -m}} x_\alpha(\mathbb{Z}_p) = N(0)\cap N_{(m)}$$

for $1 \leq i < m$, so by (1.1) it's clear that $N_m \subseteq \prod_{1\leq i\leq m} N(m-i)\cap N_{(i)}$. We also note, by (1.2) and (1.3), that

$$\begin{aligned}\big(N(m-i)\cap N_{(i)}\big)&\big(N(m-i-1)\cap N_{(i+1)}\big)\\ &\subseteq \Big(\prod_{\substack{\alpha\in\Phi^- \\ \mathrm{ht}(\alpha)=-i}} x_\alpha(p^{m-i}\mathbb{Z}_p)\Big)\big(N(m-i-1)\cap N_{(i+1)}\big)\end{aligned}$$

for any $1 \leq i < m$, so

$$\begin{aligned}&\prod_{1\leq i\leq m} N(m-i)\cap N_{(i)}\\ &\qquad\subseteq \prod_{\substack{\alpha\in\Phi^- \\ \mathrm{ht}(\alpha)=-1}} x_\alpha(p^{m-1}\mathbb{Z}_p)\cdots \prod_{\substack{\alpha\in\Phi^- \\ \mathrm{ht}(\alpha)=-(m-1)}} x_\alpha(p\mathbb{Z}_p)\big(N(0)\cap N_{(m)}\big)\\ &\qquad= N_m\end{aligned}$$

by induction, (1.1) and (1.3). This shows the equality and that $N_m$ is normal clearly follows.

(ii) We first recall the following formulas for commutators

$$[gh,k] = g[h,k]g^{-1}[g,k] \quad \text{and} \quad [g,hk] = [g,h]h[g,k]h^{-1}. \tag{1.4}$$ {eq:comformulas}

Now, using (1.4), (i) and the fact that all the involved subgroups are normal, it's enough to show that

$$[N(\ell)\cap N_{(i)}, N(m)\cap N_{(j)}] \subseteq N(\ell+m)\cap N_{(i+j)}.$$

This further reduces to showing that

$$[N(\ell), N(m)] \subseteq N(\ell+m) \quad \text{and} \quad [N_{(i)}, N_{(j)}] \subseteq N_{(i+j)}.$$

The right inclusion is a well known property of the descending central series, so it follows from our defintion of $N_{(m)}$. For the left inclusion it suffices, by (1.2), to show that

$$[x_\alpha(p^\ell\mathbb{Z}_p), x_\beta(p^m\mathbb{Z}_p)] \subseteq N(\ell+m)$$

for any $\alpha, \beta \in \Phi^-$. To show this inclusion we recall Chevalley's commutator formula

$$[x_\alpha(a), x_\beta(b)] \in x_{\alpha+\beta}(ab\mathbb{Z}_p) \prod_{\substack{i,j\geq 1\\ i+j>2}} x_{i\alpha+j\beta}(a^i b^j \mathbb{Z}_p),$$

where on the right hand side the convention is that $x_{i\alpha+j\beta} \equiv 1$ if $i\alpha + j\beta \notin \Phi$ (cf. BT). From (1.2) and Chevalley's commutator formula the inclusion follows. DK Note: Check reference.

(iii) We note that

$$N(m-i) \cap N_{(i)} = \prod_{\substack{\alpha\in\Phi^-\\ \mathrm{ht}(\alpha)\leq -i}} x_\alpha(p^{m-i}\mathbb{Z}_p)$$

for $1 \leq i \leq m$, so the statement follows from (i) and (ii). DK Note: Write (iii) better.

(iv) For any $1 \leq \ell \leq m$ we consider the chain of normal subgroups

$$N_{m+2}(N_m \cap N_{(\ell+1)}) \subseteq N_{m+1}(N_m \cap N_{(\ell+1)}) \subseteq N_{m+1}(N_m \cap N_{(\ell)})$$

between $N_{m+2}$ and $N_m$. By (1.4) and an argument like in (ii), we get that

$$[N_{m+1}(N_m \cap N_{(\ell)}), N_{m+1}(N_m \cap N_{(\ell)})] \subseteq N_{m+2}(N_m \cap N_{(\ell+1)}),$$

so the quotient group

$$N_{m+1}(N_m \cap N_{(\ell)})/N_{m+2}(N_m \cap N_{(\ell+1)})$$

is abelian. Now looking carefully at the groups as sets, we see that

$$N_m \cap N_{(\ell)} = \prod_{\substack{\alpha\in\Phi^-\\ \mathrm{ht}(\alpha)\leq -\ell}} x_\alpha(p^{\max(0,m+\mathrm{ht}(\alpha))}\mathbb{Z}_p)$$

and thus (using Chevalley's commutator formula and the fact that $\mathrm{ht}(i\alpha+j\beta) \leq \mathrm{ht}(\alpha+\beta) < \mathrm{ht}(\alpha), \mathrm{ht}(\beta)$ to move the products for the $\mathrm{ht}(\alpha) = -\ell$ term)

$$\begin{aligned} N_{m+1}(N_m \cap N_{(\ell)}) = & \prod_{\substack{\alpha\in\Phi^-\\ \mathrm{ht}(\alpha)>-\ell}} x_\alpha(p^{\max(0,m+1+\mathrm{ht}(\alpha))}\mathbb{Z}_p) \\ & \cdot \prod_{\substack{\alpha\in\Phi^-\\ \mathrm{ht}(\alpha)=-\ell}} x_\alpha(p^{m-\ell}\mathbb{Z}_p) \\ & \cdot \prod_{\substack{\alpha\in\Phi^-\\ \mathrm{ht}(\alpha)<-\ell}} x_\alpha(p^{\max(0,m+\mathrm{ht}(\alpha))}\mathbb{Z}_p). \end{aligned}$$

Similarly

$$N_{m+2}(N_m \cap N_{(\ell+1)}) = \prod_{\substack{\alpha\in\Phi^- \\ \mathrm{ht}(\alpha)>-\ell}} x_\alpha(p^{\max(0,m+2+\mathrm{ht}(\alpha))}\mathbb{Z}_p) \cdot \prod_{\substack{\alpha\in\Phi^- \\ \mathrm{ht}(\alpha)=-\ell}} x_\alpha(p^{m+2-\ell}\mathbb{Z}_p) \cdot \prod_{\substack{\alpha\in\Phi^- \\ \mathrm{ht}(\alpha)\leq-(\ell+1)}} x_\alpha(p^{\max(0,m+\mathrm{ht}(\alpha))}\mathbb{Z}_p),$$

and since the quotient group

$$N_{m+1}(N_m \cap N_{(\ell)})/N_{m+2}(N_m \cap N_{(\ell+1)})$$

is abelian, we see that it is isomorphic to

$$\prod_{\substack{\alpha\in\Phi^- \\ \mathrm{ht}(\alpha)>-\ell}} \frac{x_\alpha(p^{\max(0,m+1+\mathrm{ht}(\alpha))}\mathbb{Z}_p)}{x_\alpha(p^{\max(m+2+\mathrm{ht}(\alpha))}\mathbb{Z}_p)} \times \prod_{\mathrm{ht}(\alpha)=-\ell} \frac{x_\alpha(p^{m-\ell}\mathbb{Z}_p)}{x_\alpha(p^{m+2-\ell}\mathbb{Z}_p)}.$$

Here the subgroup

$$N_{m+1}(N_m \cap N_{(\ell+1)})/N_{m+2}(N_m \cap N_{(\ell+1)})$$

corresponds to

$$\prod_{\mathrm{ht}(\alpha)>-\ell} \frac{x_\alpha(p^{\max(0,m+1+\mathrm{ht}(\alpha))}\mathbb{Z}_p)}{x_\alpha(p^{\max(0,m+2+\mathrm{ht}(\alpha))}\mathbb{Z}_p)} \times \prod_{\mathrm{ht}(\alpha)=-\ell} \frac{x_\alpha(p^{m+1-\ell}\mathbb{Z}_p)}{x_\alpha(p^{m+2-\ell}\mathbb{Z}_p)}.$$

It follows that $N_{m+1}(N_m\cap N_{(\ell+1)})/N_{m+2}(N_m\cap N_{(\ell+1)})$ is the $p$-torsion subgroup of $N_{m+1}(N_m \cap N_{(\ell)})/N_{m+2}(N_m \cap N_{(\ell+1)})$.

Now let $g \in N_m$ for some $m \geq 1$ and assume that $g^p \in N_{m+2}$. For $\ell = 1$ we have $g \in N_m = N_{m+1}(N_m \cap N_{(1)})$, since $N_{(1)} = N$, and clearly $g^p \in N_{m+2}(N_m \cap N_{(2)})$. Since $N_{m+1}(N_m \cap N_{(2)})/N_{m+2}(N_m \cap N_{(2)})$ is the $p$-torsion subgroup of $N_{m+1}(N_m \cap N_{(1)})/N_{m+2}(N_m \cap N_{(2)})$, it follows that $g \in N_{m+1}(N_m \cap N_{(2)})$ and $g^p \in N_{m+2}(N_m \cap N_{(3)})$. By induction on $\ell$, we thus get that $g \in N_{m+1}(N_m \cap N_{(m+1)}) = N_{m+1}$. Here the last equality follows from the fact that $N_{(m+1)} \subseteq N_{m+1}$ by (1.1) and (1.3). $\square$

**Proposition 1.2.** The function $\omega$ is a $p$-valuation on $N$, i.e., it satisfies for any $g, h \in N$:

(a) $\omega(g) > \frac{1}{p-1}$,

(b) $\omega(g^{-1}h) \geq \min(\omega(g), \omega(h))$,

(c) $\omega([g,h]) \geq \omega(g) + \omega(h)$,

(d) $\omega(g^p) = \omega(g) + 1$.

*Proof.* We note that (a) is obvious by our definition of $\omega$, (c) follows from Lemma 1.1 (ii) and (d) follows from Lemma 1.1 (iv).

It only remains to show (b), which we will do by following the proof idea of Lemma 1 from [Zab], i.e., we are going to use triple induction. Also, for the sake of this proof (and only during this proof), we will take all product $\prod_{\alpha\in\Phi^-} x_\alpha(a_\alpha)$ to be in descending order in $\Phi^-$.

At first by induction on the number of non-zero coordinates among $(a_\alpha)_{\alpha\in\Phi^-}$ in $\prod_{\alpha\in\Phi^-} x_\alpha(a_\alpha)$ we are reduced to the case where $g$ is of the form $g = x_\alpha(a_\alpha)$ for some $\alpha \in \Phi^-$ and $a_\alpha \in \mathbb{Z}_p$. To see this let $g \in N \setminus \{1\}$ and write $g = \prod_{\alpha\in\Phi^-} x_\alpha(a_\alpha)$ in our unique way (according to the descending ordering of $\Phi^-$), and let $\beta$ be the smallest element of $\Phi^-$ for which $a_\alpha \neq 0$ so that $g = g' \cdot x_\beta(a_\beta)$. Then $g^{-1}h = x_\beta(a_\beta)^{-1}((g')^{-1}h)$ and thus strong induction will imply that

$$\begin{aligned}\omega(g^{-1}h) &\geq \min\bigl(v(a_\beta) - \mathrm{ht}(\beta), \omega((g')^{-1}h)\bigr)\\ &\geq \min\bigl(v(a_\beta) - \mathrm{ht}(\alpha), \omega(g'), \omega(h)\bigr) = \min\bigl(\omega(g), \omega(h)\bigr).\end{aligned}$$

Let now $h$ be of the form $h = \prod_{k=1}^{r} x_{\beta_k}(a_{\beta_k})$ with $\beta_1 > \beta_2 > \cdots > \beta_r$ in $\Phi^-$. If $\alpha \geq \beta_1$, then $g^{-1}h = x_\alpha(-a_\alpha) \cdot x_{\beta_1}(a_{\beta_1}) \prod_{k=2}^{r} x_{\beta_k}(a_{\beta_k})$, so (b) is clearly true if $\alpha > \beta_1$ (by the definition of $\omega$), and if $\alpha = \beta_1$, then $x_\alpha(-a_\alpha) \cdot x_{\beta_1}(a_{\beta_1}) = x_\alpha(-a_\alpha + a_{\beta_1})$ and (b) follows from $v_p(a+b) \geq \min(v_p(a), v_p(b))$ for $a, b \in \mathbb{Z}_p$.

On the other hand, if $\alpha < \beta_1$, then we write

$$\begin{aligned}g^{-1}h &= x_\alpha(-a_\alpha) \cdot \prod_{k=1}^{r} x_{\beta_k}(a_{\beta_k})\\ &= [x_\alpha(-a_\alpha), x_{\beta_1}(a_{\beta_1})] x_{\beta_1}(a_{\beta_1}) \cdot x_\alpha(-a_\alpha) \prod_{k=2}^{r} x_{\beta_k}(a_{\beta_k}).\end{aligned}$$

Now we use descending induction on $\alpha$ in the chosen ordering of $\Phi^-$ and suppose that the statement (b) is true for any $h$ and any $g'$ of the form

$g' = x_{\alpha'}(a_{\alpha'})$ with $\alpha' > \alpha$. Note that we already implicitly described the base case earlier and recall that $\Phi^-$ is finite and totally ordered. Note furthermore that Chevalley's commutator formula gives us

$$[x_\alpha(-a_\alpha), x_\beta(a_\beta)] = \prod_{\substack{i\alpha+j\beta\in\Phi^- \\ i,j>0}} x_{i\alpha+j\beta}(c_{\alpha,\beta,i,j}(-a_\alpha)^i a_\beta^j) \tag{1.5}$$

{eq:Chevalley}

for any $\alpha, \beta \in \Phi^-$, where $c_{\alpha,\beta,i,j} \in \mathbb{Z}_p$. Also, we have $\mathrm{ht}(i\alpha+j\beta) \le \mathrm{ht}(\alpha+\beta) < \mathrm{ht}(\alpha), \mathrm{ht}(\beta)$, so we can apply the inductional hypothesis for $x_{\beta_1}(a_{\beta_1})$ (since $\beta_1 > \alpha$) and for all terms on the right side of (1.5) (recalling that $\alpha' \ge \beta'$ if $\mathrm{ht}(\alpha') \le \mathrm{ht}(\beta')$ and with the choice $\beta = \beta_1$) in order to obtain

$$\begin{aligned} \omega(g^{-1}h) \ge \min\Bigg( &\min_{\substack{i\alpha+j\beta\in\Phi^- \\ i,j>0}} \omega(x_{i\alpha+j\beta}(c_{\alpha,\beta,i,j}(-a_\alpha)^i a_\beta^j)), \\ &\omega(x_{\beta_1}(a_{\beta_1})), \omega\Big(x_\alpha(-a_\alpha)\prod_{k=2}^{r} x_{\beta_k}(a_{\beta_k})\Big)\Bigg). \end{aligned} \tag{1.6}$$

{eq:omega(ginvh)}

Now, for $i, j > 0$ with $i\alpha + j\beta \in \Phi$,

$$\begin{aligned} \omega(x_{i\alpha+j\beta}(c_{\alpha,\beta,i,j}(-a_\alpha)^i a_\beta^j)) &= v_p(c_{\alpha,\beta,i,j}(-a_\alpha)^i a_\beta^j) - \mathrm{ht}(i\alpha+j\beta) \\ &\ge v_p(c_{\alpha,\beta,i,j}) + v_p((-a_\alpha)^i) + v_p(a_\beta^j) - \mathrm{ht}(\alpha+\beta) \\ &\ge v_p(a_\alpha) - \mathrm{ht}(\alpha) + v_p(a_\beta) - \mathrm{ht}(\beta) \\ &= \omega(x_\alpha(a_\alpha)) + \omega(x_\beta(a_\beta)) \\ &\ge \min\big(\omega(x_\alpha(a_\alpha)), \omega(x_\beta(a_\beta))\big). \end{aligned} \tag{1.7}$$

{eq:omega(Chev)}

So taking $\beta = \beta_1$ and using (1.7) in (1.6), we get that

$$\omega(g^{-1}h) \ge \min\Bigg(\omega(x_\alpha(a_\alpha)), \omega(x_{\beta_1}(a_{\beta_1})), \omega\Big(x_\alpha(-a_\alpha)\prod_{k=2}^{r} x_{\beta_k}(a_{\beta_k})\Big)\Bigg). \tag{1.8}$$

{eq:omega(ginvh)2}

Finally induction on $r$ will imply that

$$\begin{aligned} \omega\Big(x_\alpha(-a_\alpha)\prod_{k=2}^{r} x_{\beta_k}(a_{\beta_k})\Big) &\ge \min\Bigg(\omega(x_\alpha(a_\alpha)), \omega\Big(\prod_{k=2}^{r} x_{\beta_k}(a_{\beta_k})\Big)\Bigg) \\ &= \min\big(\omega(x_\alpha(a_\alpha)), \min_{2\le k\le r} \omega(x_{\beta_k}(a_{\beta_k}))\big), \end{aligned}$$

which by (1.8) implies that

$$\begin{aligned} \omega(g^{-1}h) &\ge \min\big(\omega(x_\alpha(a_\alpha)), \min_{1\le k\le r} \omega(x_{\beta_k}(a_{\beta_k}))\big) \\ &= \min\big(\omega(g), \omega(h)\big), \end{aligned}$$

thus finishing the proof. □

## 1.3 A multiplicative spectral sequence

sec:specsec

In this section we will write $G$ for $\mathcal{N}(\mathbb{Z}_p)$, and we let $\mathfrak{g} = \mathbb{F}_p \otimes_{\mathbb{F}_p[\pi]} \operatorname{gr} G$.

Here $\operatorname{gr} G \cong \mathbb{F}_p[\pi] \otimes_{\mathbb{F}_p} \mathfrak{n}$ by Proposition 3.2 of Schneider's notes, so $\mathfrak{g} \cong \mathbb{F}_p \otimes_{\mathbb{F}_p[\pi]} \mathbb{F}_p[\pi] \otimes_{\mathbb{F}_p} \mathfrak{n} \cong \mathfrak{n}$. (Which can also be shown by looking at the Chevalley constants.)

Note that $G$ is a pro-$p$-group and by Corollary 2.2 of Schneider's notes $G$ is $p$-valuable, so by Theorem 29.8 of [Sch] $G$ is a (compact) $p$-adic Lie group.

DK Note: This actually takes quite a lot of work to write the argument for, but it's mostly written in Schneider's notes already.

Now we have a $p$-valued group $(G, \omega)$, so by [Sør] we get a multiplicative convergent spectral sequence

$$E_1^{s,t} = H^{s,t}(\mathfrak{g}, \mathbb{F}_p) \Longrightarrow H^{s+t}(G, \mathbb{F}_p).$$

Here $H^{s,t}(\mathfrak{g}, \mathbb{F}_p) = H^{s+t}(\operatorname{gr}^s C^\bullet(\mathfrak{g}, \mathbb{F}_p))$ by definition, where the Lie algebra $\mathfrak{g} \cong \mathfrak{n}$ is graded by the height function.

## 1.4 Dimension of cohomology of $\mathfrak{n}$ and $N = \mathcal{N}(\mathbb{Z}_p)$

sec:dimofcoh

By Corollary 2.10 and Corollary 3.8 of [PT] and the Universal Coefficient Theorem there is a finite, natural $\mathcal{T}_{\mathbb{Z}}(\mathbb{Z})$-filtration such that we get isomorphisms of $\mathbb{F}_p$-vector spaces[1]

$$H^n(\mathfrak{n}_{\mathbb{Z}}, V_{\mathbb{F}_p}(0)) \cong \bigoplus_{\substack{w \in W \\ \ell(w)=n}} V_{\mathbb{F}_p}(w \cdot 0) \cong \operatorname{gr} H^n(\mathcal{N}_{\mathbb{Z}}(\mathbb{Z}), V_{\mathbb{F}_p}(0))$$

for any $n \geq 0$ if $p \geq h-1$ (which we assumed to be the case). (Here $V_{\mathbb{F}_p}(\lambda) \cong \mathbb{F}_p$ with $\mathcal{T}_{\mathbb{Z}}(\mathbb{F}_p) = \mathcal{T}(\mathbb{F}_p) = \mathcal{T}_{\mathbb{F}_p}(\mathbb{F}_p)$ acting via $\lambda$.)

Furthermore

$$H^n(\mathcal{N}_{\mathbb{Z}}(\mathbb{Z}), V_{\mathbb{F}_p}(0)) \cong H^n(\mathcal{N}(\mathbb{Z}_p), V_{\mathbb{F}_p}(0)).$$

To see this, first note that $\mathbb{Z}$ is a discrete group, $\mathbb{Z}_p$ is a profinite group, and the homomorphism $\mathbb{Z} \to \mathbb{Z}_p$ has dense image in $\mathbb{Z}_p$. So we have homomorphisms

$$H^n(\mathbb{Z}_p, \mathbb{F}_p) \to H^n(\mathbb{Z}, \mathbb{F}_p)$$

for all $n \geq 0$ from [Ser, Section I §2.6]. Now both $H^0(\mathbb{Z}, \cdot\,)$ and $H^0(\mathbb{Z}_p, \cdot\,)$ are the functor of taking invariant, both $H^1(\mathbb{Z}, \cdot\,)$ and $H^1(\mathbb{Z}_p, \cdot\,)$ are the functor of

[1] You get more than this, but we don't need more here.

taking coinvariants, and all $H^n(\mathbb{Z}, \,\cdot\,)$ and $H^n(\mathbb{Z}_p, \,\cdot\,)$ vanish for $n \geq 2$, so $\mathbb{Z}$ is "good" in the sense of [Ser, Section I §2.6 Exercise 2]. Thus [Ser, Section I §2.6 Exercise 2(d)] implies that the homomorphisms

$$H^n(\mathcal{N}(\mathbb{Z}_p), \mathbb{F}_p) \to H^n(\mathcal{N}(\mathbb{Z}), \mathbb{F}_p) \qquad n \geq 0,$$

induced by the homomorphism $\mathcal{N}(\mathbb{Z}) \to \mathcal{N}(\mathbb{Z}_p)$, are all isomorphisms.

Hence

$$\dim_{\mathbb{F}_p} H^n(\mathfrak{n}_{\mathbb{Z}}, \mathbb{F}_p) = \dim_{\mathbb{F}_p} H^n(\mathcal{N}_{\mathbb{Z}}(\mathbb{Z}), \mathbb{F}_p) = \dim_{\mathbb{F}_p} H^n(\mathcal{N}(\mathbb{Z}_p), \mathbb{F}_p).$$

Now $\mathfrak{n} = \mathfrak{n}_{\mathbb{Z}} \otimes \mathbb{F}_p$, and $H^n(\mathfrak{g}, \mathbb{F}_p) \cong H^n(\mathfrak{n}, \mathbb{F}_p)$ (since $\mathfrak{g} \cong \mathfrak{n}$) is the homology of the complex

$$C^\bullet(\mathfrak{n}, \mathbb{F}_p) = \operatorname{Hom}_{\mathbb{F}_p}\Big(\bigwedge^\bullet \mathfrak{n}, \mathbb{F}_p\Big)$$

while $H^n(\mathfrak{n}_{\mathbb{Z}}, \mathbb{F}_p)$ is the homology of the complex

$$C^\bullet(\mathfrak{n}_{\mathbb{Z}}, \mathbb{F}_p) = \operatorname{Hom}_{\mathbb{F}_p}\Big(\bigwedge^\bullet \mathfrak{n}_{\mathbb{Z}}, \mathbb{F}_p\Big).$$

Here $\bigwedge^\bullet \mathfrak{n}_{\mathbb{Z}}$ is a free $\mathbb{Z}$-module and $(\bigwedge^\bullet \mathfrak{n}_{\mathbb{Z}}) \otimes \mathbb{F}_p \cong \bigwedge^\bullet(\mathfrak{n}_{\mathbb{Z}} \otimes \mathbb{F}_p) \cong \bigwedge^\bullet \mathfrak{n}$, so we have natural isomorphisms

$$\operatorname{Hom}_{\mathbb{F}_p}\Big(\bigwedge^\bullet \mathfrak{n}_{\mathbb{Z}}, \mathbb{F}_p\Big) \cong \operatorname{Hom}_{\mathbb{F}_p}\Big(\Big(\bigwedge^\bullet \mathfrak{n}_{\mathbb{Z}}\Big) \otimes \mathbb{F}_p, \mathbb{F}_p\Big) \cong \operatorname{Hom}_{\mathbb{F}_p}\Big(\bigwedge^\bullet \mathfrak{n}, \mathbb{F}_p\Big).$$

These isomorphisms are clearly compatible with the differentials, so $C^\bullet(\mathfrak{n}, \mathbb{F}_p) \cong C^\bullet(\mathfrak{n}_{\mathbb{Z}}, \mathbb{F}_p)$, and thus $H^n(\mathfrak{n}, \mathbb{F}_p) \cong H^n(\mathfrak{n}_{\mathbb{Z}}, \mathbb{F}_p)$. Hence

$$\dim_{\mathbb{F}_p} H^n(\mathfrak{n}, \mathbb{F}_p) = \dim_{\mathbb{F}_p} H^n(\mathfrak{n}_{\mathbb{Z}}, \mathbb{F}_p) = \dim_{\mathbb{F}_p} H^n(\mathcal{N}(\mathbb{Z}_p), \mathbb{F}_p).$$

## 1.5 Cohomology of $N = \mathcal{N}(\mathbb{Z}_p)$

Now Section 1.4 implies that

$$\sum_{s+t=n} \dim_{\mathbb{F}_p} H^{s,t}(\mathfrak{g}, \mathbb{F}_p) = \dim_{\mathbb{F}_p} H^n(\mathfrak{g}, \mathbb{F}_p) = \dim_{\mathbb{F}_p} H^n(G, \mathbb{F}_p),$$

so the multiplicative spectral sequence

$$E_1^{s,t} = H^{s,t}(\mathfrak{g}, \mathbb{F}_p) \Longrightarrow H^{s+t}(G, \mathbb{F}_p)$$

from Section 1.3 converges on the first page. I.e.,

$$H^n(N, \mathbb{F}_p) = H^n(G, \mathbb{F}_p) \cong H^n(\mathfrak{g}, \mathbb{F}_p) \cong H^n(\mathfrak{n}, \mathbb{F}_p),$$

giving us a good description of $H^n(\mathcal{N}(\mathbb{Z}_p), \mathbb{F}_p)$. (Since the spectral sequence is multiplicative, can we also say that the cup product can be taken from the right hand side?)

DK Note: How do we argue this purely from looking at the dimensions? Do we need to just look at the page and differentials in more detail?

# Bibliography


PT [PT] P. Polo and J. Tilouine. "Bernstein-Gelfand-Gelfand complexes and cohomology of nilpotent groups over $\mathbb{Z}_p$ for representations with $p$-small weights". In: ().

Sch [Sch] Peter Schneider. *$p$-Adic Lie Groups*. Springer, 2011.

GalCoh [Ser] Jean-Pierre Serre. *Galois Cohomology*. Trans. by Patrick Ion. Springer, 2002.

Sor [Sør] Claus Sørensen. "Hochschild Cohomology and $p$-Adic Lie Groups". In: ().

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