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Some extra stuff.

Chapter 1

Cohomology of Unipotent Groups

cha:cohunigps

1.1 Introduction

sec:cohunigps-intro

So far some of the details are still skipped, but I have tried to write pretty much everything that's not already written in results I cite.

Notation and setup

Let p be a prime and let $k = \mathbb{Z}_p$. Also note that the following is true for any integral domain k (in particular also for \mathbb{F}_p).

Let $\mathcal{G}_{\mathbb{Z}}$ be a split and connected reductive algebraic \mathbb{Z} -group and let $\mathcal{G} = (\mathcal{G}_{\mathbb{Z}})_k$ (the base change from \mathbb{Z} to k). Let $\mathcal{T}_{\mathbb{Z}}$ be a split maximal torus of $\mathcal{G}_{\mathbb{Z}}$ and set $\mathcal{T} = (\mathcal{T}_{\mathbb{Z}})_k$. Let $\Phi = \Phi(\mathcal{G}, \mathcal{T})$ be the root system of \mathcal{G} with respect to \mathcal{T} and note that Φ can be identified with the root system of $\mathcal{G}_{\mathbb{Z}}$ with respect to $\mathcal{T}_{\mathbb{Z}}$. Also note that $\mathrm{Lie}(\mathcal{G}) = \mathrm{Lie}(\mathcal{G}_{\mathbb{Z}}) \otimes_{\mathbb{Z}} k$ and for any $\alpha \in \Phi$ we have the root subgroup $\mathcal{N}_{\alpha} \subseteq \mathcal{G}$ with $\mathrm{Lie} \mathcal{N}_{\alpha} = (\mathrm{Lie} \mathcal{G})_{\alpha} = (\mathrm{Lie} \mathcal{G}_{\mathbb{Z}})_{\alpha} \otimes_{\mathbb{Z}} k$. Now fix a k -basis X_{α} of the Lie algebra of \mathcal{N}_{α} . This choice gives rise to a unique isomorphism of group schemes $x_{\alpha}: \mathcal{G}_{\alpha} \xrightarrow{\cong} \mathcal{N}_{\alpha}$ such that $(dx_{\alpha})(1) = X_{\alpha}$. We furthermore fix a basis $\Delta \subseteq \Phi$ of the root system such that we get a decomposition $\Phi = \Phi^+ \cup \Phi^-$ into positive and negative roots. Let $\mathcal{B} = \mathcal{T}\mathcal{N}$ and $\mathcal{B}^+ = \mathcal{T}\mathcal{N}^+$ denote the

DK Note: We might be able to avoid going through \mathbb{Z} at first with some work. Also, we may need to assume that \mathcal{G} is simple.

Borel subgroups of \mathcal{G} corresponding to Φ^- and Φ^+ , respectively, with unipotent radicals \mathcal{N} and \mathcal{N}^+ . (Here we also have corresponding algebraic \mathbb{Z} -groups.)

For any total ordering of Φ^- the multiplication induces an isomorphism of schemes $\prod_{\alpha \in \Phi^-} \mathcal{N}_\alpha \xrightarrow{\cong} \mathcal{N}$. For convenience we fix in the following such a total ordering which has the additional property that $\alpha_1 \geq \alpha_2$ if $\text{ht}(\alpha_1) \leq \text{ht}(\alpha_2)$. All products indexed by Φ^- are meant to be taken according to this ordering. Here we have the height function $\text{ht}: \mathbb{Z}[\Delta] \rightarrow \mathbb{Z}$ given by $\sum_{\alpha \in \Delta} m_\alpha \alpha \mapsto \sum_{\alpha \in \Delta} m_\alpha$. In particular, since $\Phi \subseteq \mathbb{Z}[\Delta]$ the height $\text{ht}(\beta)$ of any root $\beta \in \Phi$ is defined.

Let furthermore ρ be the half-sum of the elements of Φ^+ , let $X = X(\mathcal{T}) \cong X(\mathcal{T}_{\mathbb{Z}})$ be the character group of \mathcal{T} , let

$$X^+ = \{\lambda \in X \mid \langle \lambda, \alpha^\vee \rangle \geq 0 \text{ for all } \alpha \in \Phi^+\},$$

and let h be the Coxeter number of \mathcal{G} and assume from now on that $p \geq h - 1$. For any $\lambda \in X^+$, let $V_{\mathbb{Z}}(\lambda)$ be the Weyl module for $\mathcal{G}_{\mathbb{Z}}$ over \mathbb{Z} with highest weight λ , and let $V_k(\lambda) = V_{\mathbb{Z}}(\lambda) \otimes_{\mathbb{Z}} k$.

Let Φ^\vee be the dual root system of Φ and let W be the corresponding Weyl group with length function ℓ on W . Let $\mathfrak{n}_{\mathbb{Z}} = \text{Lie}(\mathcal{N}_{\mathbb{Z}})$ be the Lie algebra of $\mathcal{N}_{\mathbb{Z}}$ over \mathbb{Z} and $\mathfrak{n} = \mathfrak{n}_{\mathbb{F}_p} = \text{Lie}(\mathcal{N}_{\mathbb{F}_p}) = \mathfrak{n}_{\mathbb{Z}} \otimes \mathbb{F}_p$ be the Lie algebra of $\mathcal{N}_{\mathbb{F}_p}$ over \mathbb{F}_p .

Finally let $G = N = \mathcal{N}(\mathbb{Z}_p) = \mathcal{N}_{\mathbb{Z}}(\mathbb{Z}_p)$ and let $\mathfrak{g} = \mathbb{F}_p \otimes_{\mathbb{F}_p[\pi]} \text{gr } G$.

1.2 The p -valuation

sec:pval

This section is mainly based on some unpublished notes by Schneider.

In this section we will write N for $\mathcal{N}(\mathbb{Z}_p)$, and we note that as a set N is the direct product $N = \prod_{\alpha \in \Phi^-} x_\alpha(\mathbb{Z}_p)$, which allows us to introduce the function

$$\begin{aligned} \omega: N \setminus \{1\} &\rightarrow \mathbb{N} \\ \prod_{\alpha \in \Phi^-} x_\alpha(a_\alpha) &\mapsto \min_{\alpha \in \Phi^-} (v_p(a_\alpha) - \text{ht}(\alpha)), \end{aligned}$$

where v_p denotes the usual p -adic valuation on \mathbb{Z}_p . Here it is important to note that we write any $g \in N$ uniquely as product

$$g = \prod_{\alpha \in \Phi^-} x_\alpha(a_\alpha)$$

by taking the product following the total ordering \geq of Φ^- defined above. Now, with the convention that $\omega(1) := \infty$, we define the descending sequence of subsets

$$N_m := \{g \in N \mid \omega(g) \geq m\}$$

in N for $m \geq 0$. The main goal of this section is to show that ω is a p -valuation by a careful analysis of the sequence of subsets given by N_m .

We first note that clearly $N_1 = N$, $\bigcap_m N_m = \{1\}$, and

$$\begin{aligned} N_m &= \prod_{\alpha \in \Phi^-} x_\alpha(p^{\max(0, m + \text{ht}(\alpha))} \mathbb{Z}_p) \\ &= \prod_{\substack{\alpha \in \Phi^- \\ \text{ht}(\alpha) = -1}} x_\alpha(p^{m-1} \mathbb{Z}_p) \cdots \prod_{\substack{\alpha \in \Phi^- \\ \text{ht}(\alpha) = -(m-1)}} x_\alpha(p \mathbb{Z}_p) \prod_{\substack{\alpha \in \Phi^- \\ \text{ht}(\alpha) \leq -m}} x_\alpha(\mathbb{Z}_p). \end{aligned} \quad (1.1) \quad \boxed{\text{eq:N_m}}$$

In our analysis of this sequence we will also need two other filtrations of N . Firstly we will consider the filtration by congruence subgroups

$$\begin{aligned} N(m) &:= \ker(\mathcal{N}(\mathbb{Z}_p) \rightarrow \mathcal{N}(\mathbb{Z}/p^m \mathbb{Z})) \\ &= \prod_{\alpha \in \Phi^-} x_\alpha(p^m \mathbb{Z}_p) \end{aligned} \quad (1.2) \quad \boxed{\text{eq:N(m)}}$$

for $m \geq 0$. Secondly, using the descending central series of the group $\mathcal{G}(\mathbb{Q}_p)$ defined by $C^1 \mathcal{G}(\mathbb{Q}_p) := \mathcal{G}(\mathbb{Q}_p)$ and $C^{m+1} \mathcal{G}(\mathbb{Q}_p) := [C^m \mathcal{G}(\mathbb{Q}_p), \mathcal{G}(\mathbb{Q}_p)]$, we consider the filtration given by

$$N_{(m)} := N \cap C^m \mathcal{G}(\mathbb{Q}_p)$$

for $m \geq 1$. By BT we have that

$$N_{(m)} = \prod_{\substack{\alpha \in \Phi^- \\ \text{ht}(\alpha) \leq -m}} x_\alpha(\mathbb{Z}_p). \quad (1.3) \quad \boxed{\text{eq:N_(m)}} \quad \begin{array}{l} \text{DK Note:} \\ \text{Check} \\ \text{reference.} \end{array}$$

We note that the natural map

$$\prod_{\substack{\alpha \in \Phi^- \\ \text{ht}(\alpha) = -m}} x_\alpha(\mathbb{Z}_p) \rightarrow N_{(m)} / N_{(m+1)}$$

is an isomorphism of abelian groups, and that all the subgroups $N(m)$ and $N_{(m)}$ are normal in N .

We are now ready to prove the following lemma, which will help us when showing that ω is a p -valuation.

Lemma 1.1.**lem:N_m**

(i) $N_m = \prod_{1 \leq i \leq m} N(m-i) \cap N_{(i)}$, for any $m \geq 1$, is a normal subgroup of N which is independent of the choices made.

item:N_m**item:N_mcom**

(ii) $[N_\ell, N_m] \subseteq N_{\ell+m}$ for any $\ell, m \geq 1$.

(iii) N_m/N_{m+1} , for any $m \geq 1$, is an \mathbb{F}_p -vector space of dimension equal to $|\{\alpha \in \Phi^- \mid \text{ht}(\alpha) \geq -m\}|$.

item:g~p

(iv) Let $g \in N_m$ for some $m \geq 1$. If $g^p \in N_{m+2}$, then $g \in N_{m+1}$.

Proof. (i) Using (1.2) and (1.3) we note that

$$\prod_{\substack{\alpha \in \Phi^- \\ \text{ht}(\alpha) = -i}} x_\alpha(p^{m-1}\mathbb{Z}_p) \subseteq N(m-i) \cap N_{(i)} \quad \text{and} \quad \prod_{\substack{\alpha \in \Phi^- \\ \text{ht}(\alpha) \leq -m}} x_\alpha(\mathbb{Z}_p) = N(0) \cap N_{(m)}$$

for $1 \leq i < m$, so by (1.1) it's clear that $N_m \subseteq \prod_{1 \leq i \leq m} N(m-i) \cap N_{(i)}$. We also note, by (1.2) and (1.3), that

$$\begin{aligned} & (N(m-i) \cap N_{(i)}) (N(m-i-1) \cap N_{(i+1)}) \\ & \subseteq \left(\prod_{\substack{\alpha \in \Phi^- \\ \text{ht}(\alpha) = -i}} x_\alpha(p^{m-i}\mathbb{Z}_p) \right) (N(m-i-1) \cap N_{(i+1)}) \end{aligned}$$

for any $1 \leq i < m$, so

$$\begin{aligned} & \prod_{1 \leq i \leq m} N(m-i) \cap N_{(i)} \\ & \subseteq \prod_{\substack{\alpha \in \Phi^- \\ \text{ht}(\alpha) = -1}} x_\alpha(p^{m-1}\mathbb{Z}_p) \cdots \prod_{\substack{\alpha \in \Phi^- \\ \text{ht}(\alpha) = -(m-1)}} x_\alpha(p\mathbb{Z}_p) (N(0) \cap N_{(m)}) \\ & = N_m \end{aligned}$$

by induction, (1.1) and (1.3). This shows the equality and that N_m is normal clearly follows.

(ii) We first recall the following formulas for commutators

$$[gh, k] = g[h, k]g^{-1}[g, k] \quad \text{and} \quad [g, hk] = [g, h]h[g, k]h^{-1}. \quad (1.4) \quad \{\text{eq:comformulas}\}$$

Now, using (1.4), (i) and the fact that all the involved subgroups are normal, it's enough to show that

$$[N(\ell) \cap N_{(i)}, N(m) \cap N_{(j)}] \subseteq N(\ell + m) \cap N_{(i+j)}.$$

This further reduces to showing that

$$[N(\ell), N(m)] \subseteq N(\ell + m) \quad \text{and} \quad [N_{(i)}, N_{(j)}] \subseteq N_{(i+j)}.$$

The right inclusion is a well known property of the descending central series, so it follows from our definition of $N_{(m)}$. For the left inclusion it suffices, by (1.2), to show that

$$[x_\alpha(p^\ell \mathbb{Z}_p), x_\beta(p^m \mathbb{Z}_p)] \subseteq N(\ell + m)$$

for any $\alpha, \beta \in \Phi^-$. To show this inclusion we recall Chevalley's commutator formula

$$[x_\alpha(a), x_\beta(b)] \in x_{\alpha+\beta}(ab\mathbb{Z}_p) \prod_{\substack{i,j \geq 1 \\ i+j > 2}} x_{i\alpha+j\beta}(a^i b^j \mathbb{Z}_p),$$

where on the right hand side the convention is that $x_{i\alpha+j\beta} \equiv 1$ if $i\alpha + j\beta \notin \Phi$ (cf. BT). From (1.2) and Chevalley's commutator formula the inclusion follows.

DK Note:
Check
reference.

(iii) We note that

$$N(m-i) \cap N_{(i)} = \prod_{\substack{\alpha \in \Phi^- \\ \text{ht}(\alpha) \leq -i}} x_\alpha(p^{m-i} \mathbb{Z}_p)$$

for $1 \leq i \leq m$, so the statement follows from (i) and (ii).

DK Note:
Write (iii)
better.

(iv) For any $1 \leq \ell \leq m$ we consider the chain of normal subgroups

$$N_{m+2}(N_m \cap N_{(\ell+1)}) \subseteq N_{m+1}(N_m \cap N_{(\ell+1)}) \subseteq N_{m+1}(N_m \cap N_{(\ell)})$$

between N_{m+2} and N_m . By (1.4) and an argument like in (ii), we get that

$$[N_{m+1}(N_m \cap N_{(\ell)}), N_{m+1}(N_m \cap N_{(\ell)})] \subseteq N_{m+2}(N_m \cap N_{(\ell+1)}),$$

so the quotient group

$$N_{m+1}(N_m \cap N_{(\ell)}) / N_{m+2}(N_m \cap N_{(\ell+1)})$$

is abelian. Now looking carefully at the groups as sets, we see that

$$N_m \cap N_{(\ell)} = \prod_{\substack{\alpha \in \Phi^- \\ \text{ht}(\alpha) \leq -\ell}} x_\alpha(p^{\max(0, m + \text{ht}(\alpha))} \mathbb{Z}_p)$$

and thus (using Chevalley's commutator formula and the fact that $\text{ht}(i\alpha + j\beta) \leq \text{ht}(\alpha + \beta) < \text{ht}(\alpha), \text{ht}(\beta)$ to move the products for the $\text{ht}(\alpha) = -\ell$ term)

$$\begin{aligned} N_{m+1}(N_m \cap N_{(\ell)}) &= \prod_{\substack{\alpha \in \Phi^- \\ \text{ht}(\alpha) > -\ell}} x_\alpha(p^{\max(0, m+1 + \text{ht}(\alpha))} \mathbb{Z}_p) \\ &\cdot \prod_{\substack{\alpha \in \Phi^- \\ \text{ht}(\alpha) = -\ell}} x_\alpha(p^{m-\ell} \mathbb{Z}_p) \\ &\cdot \prod_{\substack{\alpha \in \Phi^- \\ \text{ht}(\alpha) < -\ell}} x_\alpha(p^{\max(0, m + \text{ht}(\alpha))} \mathbb{Z}_p). \end{aligned}$$

Similarly

$$\begin{aligned} N_{m+2}(N_m \cap N_{(\ell+1)}) &= \prod_{\substack{\alpha \in \Phi^- \\ \text{ht}(\alpha) > -\ell}} x_\alpha(p^{\max(0, m+2 + \text{ht}(\alpha))} \mathbb{Z}_p) \\ &\cdot \prod_{\substack{\alpha \in \Phi^- \\ \text{ht}(\alpha) = -\ell}} x_\alpha(p^{m+2-\ell} \mathbb{Z}_p) \\ &\cdot \prod_{\substack{\alpha \in \Phi^- \\ \text{ht}(\alpha) \leq -(\ell+1)}} x_\alpha(p^{\max(0, m + \text{ht}(\alpha))} \mathbb{Z}_p), \end{aligned}$$

and since the quotient group

$$N_{m+1}(N_m \cap N_{(\ell)}) / N_{m+2}(N_m \cap N_{(\ell+1)})$$

is abelian, we see that it is isomorphic to

$$\prod_{\substack{\alpha \in \Phi^- \\ \text{ht}(\alpha) > -\ell}} \frac{x_\alpha(p^{\max(0, m+1 + \text{ht}(\alpha))} \mathbb{Z}_p)}{x_\alpha(p^{\max(0, m+2 + \text{ht}(\alpha))} \mathbb{Z}_p)} \times \prod_{\text{ht}(\alpha) = -\ell} \frac{x_\alpha(p^{m-\ell} \mathbb{Z}_p)}{x_\alpha(p^{m+2-\ell} \mathbb{Z}_p)}.$$

Here the subgroup

$$N_{m+1}(N_m \cap N_{(\ell+1)}) / N_{m+2}(N_m \cap N_{(\ell+1)})$$

corresponds to

$$\prod_{\text{ht}(\alpha) > -\ell} \frac{x_\alpha(p^{\max(0, m+1+\text{ht}(\alpha))} \mathbb{Z}_p)}{x_\alpha(p^{\max(0, m+2+\text{ht}(\alpha))} \mathbb{Z}_p)} \times \prod_{\text{ht}(\alpha) = -\ell} \frac{x_\alpha(p^{m+1-\ell} \mathbb{Z}_p)}{x_\alpha(p^{m+2-\ell} \mathbb{Z}_p)}.$$

It follows that $N_{m+1}(N_m \cap N_{(\ell+1)})/N_{m+2}(N_m \cap N_{(\ell+1)})$ is the p -torsion subgroup of $N_{m+1}(N_m \cap N_{(\ell)})/N_{m+2}(N_m \cap N_{(\ell+1)})$.

Now let $g \in N_m$ for some $m \geq 1$ and assume that $g^p \in N_{m+2}$. For $\ell = 1$ we have $g \in N_m = N_{m+1}(N_m \cap N_{(1)})$, since $N_{(1)} = N$, and clearly $g^p \in N_{m+2}(N_m \cap N_{(2)})$. Since $N_{m+1}(N_m \cap N_{(2)})/N_{m+2}(N_m \cap N_{(2)})$ is the p -torsion subgroup of $N_{m+1}(N_m \cap N_{(1)})/N_{m+2}(N_m \cap N_{(2)})$, it follows that $g \in N_{m+1}(N_m \cap N_{(2)})$ and $g^p \in N_{m+2}(N_m \cap N_{(3)})$. By induction on ℓ , we thus get that $g \in N_{m+1}(N_m \cap N_{(m+1)}) = N_{m+1}$. Here the last equality follows from the fact that $N_{(m+1)} \subseteq N_{m+1}$ by (1.1) and (1.3). \square

Proposition 1.2. The function ω is a p -valuation on N , i.e., it satisfies for any $g, h \in N$:

- (a) $\omega(g) > \frac{1}{p-1}$,
- (b) $\omega(g^{-1}h) \geq \min(\omega(g), \omega(h))$,
- (c) $\omega([g, h]) \geq \omega(g) + \omega(h)$,
- (d) $\omega(g^p) = \omega(g) + 1$.

Proof. We note that (a) is obvious by our definition of ω , (c) follows from Lemma 1.1 (ii) and (d) follows from Lemma 1.1 (iv).

It only remains to show (b), which we will do by following the proof idea of Lemma 1 from [Zab], i.e., we are going to use triple induction. Here we note that all products $\prod_{\alpha \in \Phi^-} x_\alpha(a_\alpha)$ are in ascending order in Φ^- (so descending in height). For notational ease, we prove equivalently that $\omega(gh^{-1}) \geq \min(\omega(g), \omega(h))$ for $g, h \in N$. DK Note: add ref

At first by induction on the number of non-zero coordinates among $(a_\beta)_{\beta \in \Phi^-}$ in $\prod_{\beta \in \Phi^-} x_\beta(a_\beta)$ we are reduced to the case where h is of the form $h = x_\beta(a_\beta)$ for some $\beta \in \Phi^-$ and $a_\beta \in \mathbb{Z}_p$. To see this let $h \in N \setminus \{1\}$ and write

$h = \prod_{\beta \in \Phi^-} x_\beta(a_\beta)$ in our unique way (according to the ordering of Φ^-), and let α be the smallest element of Φ^- for which $a_\alpha \neq 0$ so that $h = x_\alpha(a_\alpha) \cdot h'$. Then $gh^{-1} = g(h')^{-1} \cdot x_\alpha(a_\alpha)^{-1}$ and thus strong induction will imply that

$$\begin{aligned} \omega(gh^{-1}) &\geq \min(\omega(g(h')^{-1}), v(a_\alpha) - \text{ht}(\alpha)) \\ &\geq \min(\omega(g), \omega(h'), v(a_\alpha) - \text{ht}(\alpha)) = \min(\omega(g), \omega(h)). \end{aligned}$$

Fix $h = x_\beta(a_\beta)$ and let now g be of the form $g = \prod_{k=1}^r x_{\alpha_k}(a_{\alpha_k})$ with $\alpha_1 < \alpha_2 < \dots < \alpha_r$ in Φ^- . If $\beta > \alpha_r$, then $gh^{-1} = \prod_{k=1}^{r-1} x_{\alpha_k}(a_{\alpha_k}) \cdot x_{\alpha_r}(a_{\alpha_r})x_\beta(-a_\beta)$, so (b) is clearly true if $\beta > \alpha_1$ (by the definition of ω), and if $\beta = \alpha_r$, then $x_{\alpha_r}(a_{\alpha_r})x_\beta(-a_\beta) = x_\beta(a_{\alpha_r} - a_\beta)$ and (b) follows from $v_p(a - b) \geq \min(v_p(a), v_p(b))$ for $a, b \in \mathbb{Z}_p$.

On the other hand, if $\beta < \alpha_r$, then we write

$$\begin{aligned} gh^{-1} &= \prod_{k=1}^r x_{\alpha_k}(a_{\alpha_k}) \cdot x_\beta(-a_\beta) \\ &= \prod_{k=1}^{r-1} x_{\alpha_k}(a_{\alpha_k}) \cdot x_\beta(-a_\beta) \cdot x_{\alpha_r}(a_{\alpha_r}) \cdot [x_{\alpha_r}(-a_{\alpha_r}), x_\beta(a_\beta)]. \end{aligned}$$

Now we use descending induction on β in the chosen ordering of Φ^- and suppose that the statement (b) is true for any g and any h' of the form $h' = x_{\beta'}(a_{\beta'})$ with $\beta' > \beta$. Note that the base case is trivial and recall that Φ^- is finite and totally ordered. Note furthermore that Chevalley's commutator formula gives us

$$[x_{\alpha'}(a_{\alpha'}), x_{\beta'}(a_{\beta'})] = \prod_{\substack{i\alpha' + j\beta' \in \Phi^- \\ i, j > 0}} x_{i\alpha' + j\beta'}(c_{\alpha', \beta', i, j} a_{\alpha'}^i a_{\beta'}^j) \quad (1.5) \quad \boxed{\text{\{eq:Chevalley\}}}$$

for any $\alpha', \beta' \in \Phi^-$, where $c_{\alpha', \beta', i, j} \in \mathbb{Z}_p$. Also, we have $\text{ht}(i\alpha' + j\beta') \leq \text{ht}(\alpha' + \beta') < \text{ht}(\alpha'), \text{ht}(\beta')$, so we can apply the induction hypothesis for $x_{\alpha_r}(a_{\alpha_r})$ and each $x_{i\alpha_r + j\beta}(c_{\alpha_r, \beta, i, j}(-a_{\alpha_r})^i a_\beta^j)$ in $[x_{\alpha_r}(-a_{\alpha_r}), x_\beta(a_\beta)]$, since $\alpha_r > \beta$ and all terms on the right side of (1.5) are larger than β (and α_r) in the ordering of

Φ^- . We thus obtain

$$\omega(gh^{-1}) \geq \min \left(\min_{\substack{i\alpha_r + j\beta \in \Phi^- \\ i,j > 0}} \omega(x_{i\alpha_r + j\beta}(c_{\alpha_r, \beta, i, j}(-a_{\alpha_r})^i a_{\beta}^j)), \right. \\ \left. \omega(x_{\alpha_r}(a_{\alpha_r})), \omega \left(\prod_{k=1}^{r-1} x_{\alpha_k}(a_{\alpha_k}) \cdot x_{\beta}(-a_{\beta}) \right) \right). \quad (1.6) \quad \boxed{\text{eq:omega(ginvh)}}$$

Now, for $i, j > 0$ with $i\alpha' + j\beta' \in \Phi^-$,

$$\begin{aligned} \omega(x_{i\alpha' + j\beta'}(c_{\alpha', \beta', i, j} a_{\alpha'}^i a_{\beta'}^j)) &= v_p(c_{\alpha', \beta', i, j} a_{\alpha'}^i a_{\beta'}^j) - \text{ht}(i\alpha' + j\beta') \\ &\geq v_p(c_{\alpha', \beta', i, j}) + v_p(a_{\alpha'}^i) + v_p(a_{\beta'}^j) - \text{ht}(\alpha' + \beta') \\ &\geq v_p(a_{\alpha'}) - \text{ht}(\alpha') + v_p(a_{\beta'}) - \text{ht}(\beta') \\ &= \omega(x_{\alpha'}(a_{\alpha'})) + \omega(x_{\beta'}(a_{\beta'})) \\ &\geq \min(\omega(x_{\alpha'}(a_{\alpha'})), \omega(x_{\beta'}(a_{\beta'}))). \end{aligned} \quad (1.7) \quad \boxed{\text{eq:omega(Chev)}}$$

So taking $\alpha' = \alpha_r$ and $\beta' = \beta$ and using (1.7) in (1.6), we get that

$$\omega(gh^{-1}) \geq \min \left(\omega(x_{\alpha_r}(a_{\alpha_r})), \omega(x_{\beta}(a_{\beta})), \omega \left(\prod_{k=1}^{r-1} x_{\alpha_k}(a_{\alpha_k}) \cdot x_{\beta}(-a_{\beta}) \right) \right). \quad (1.8) \quad \boxed{\text{eq:omega(ginvh)2}}$$

Finally induction on r will imply that

$$\begin{aligned} \omega \left(\prod_{k=1}^{r-1} x_{\alpha_k}(a_{\alpha_k}) \cdot x_{\beta}(-a_{\beta}) \right) &\geq \min \left(\omega \left(\prod_{k=1}^{r-1} x_{\alpha_k}(a_{\alpha_k}) \right), \omega(x_{\beta}(a_{\beta})) \right) \\ &= \min \left(\min_{1 \leq k \leq r-1} \omega(x_{\alpha_k}(a_{\alpha_k})), \omega(x_{\beta}(a_{\beta})) \right), \end{aligned}$$

which by (1.8) implies that

$$\begin{aligned} \omega(gh^{-1}) &\geq \min \left(\min_{1 \leq k \leq r} \omega(x_{\alpha_k}(a_{\alpha_k})), \omega(x_{\beta}(a_{\beta})) \right) \\ &= \min(\omega(g), \omega(h)), \end{aligned}$$

thus finishing the proof. \square

1.3 A multiplicative spectral sequence

$\boxed{\text{sec:specsec}}$

In this section we will write G for $\mathcal{N}(\mathbb{Z}_p)$, and we let $\mathfrak{g} = \mathbb{F}_p \otimes_{\mathbb{F}_p[\pi]} \text{gr } G$.

Here $\mathrm{gr} G \cong \mathbb{F}_p[\pi] \otimes_{\mathbb{F}_p} \mathfrak{n}$ by Proposition 3.2 of Schneider's notes, so $\mathfrak{g} \cong \mathbb{F}_p \otimes_{\mathbb{F}_p[\pi]} \mathbb{F}_p[\pi] \otimes_{\mathbb{F}_p} \mathfrak{n} \cong \mathfrak{n}$. (Which can also be shown by looking at the Chevalley constants.)

Note that G is a pro- p -group and by Corollary 2.2 of Schneider's notes G is p -valuable, so by Theorem 29.8 of [Sch] G is a (compact) p -adic Lie group.

Now we have a p -valued group (G, ω) , so by [Sør] we get a multiplicative convergent spectral sequence

$$E_1^{s,t} = H^{s,t}(\mathfrak{g}, \mathbb{F}_p) \implies H^{s+t}(G, \mathbb{F}_p).$$

Here $H^{s,t}(\mathfrak{g}, \mathbb{F}_p) = H^{s+t}(\mathrm{gr}^s C^\bullet(\mathfrak{g}, \mathbb{F}_p))$ by definition, where the Lie algebra $\mathfrak{g} \cong \mathfrak{n}$ is graded by the height function.

DK Note:

This actually takes quite a lot of work to write the argument for, but it's mostly written in Schneider's notes already.

1.4 Dimension of cohomology of \mathfrak{n} and $N = \mathcal{N}(\mathbb{Z}_p)$

sec:dimofcoh

By Corollary 2.10 and Corollary 3.8 of [PT] and the Universal Coefficient Theorem there is a finite, natural $\mathcal{T}_{\mathbb{Z}}(\mathbb{Z})$ -filtration such that we get isomorphisms of \mathbb{F}_p -vector spaces¹

$$H^n(\mathfrak{n}_{\mathbb{Z}}, V_{\mathbb{F}_p}(0)) \cong \bigoplus_{\substack{w \in W \\ \ell(w)=n}} V_{\mathbb{F}_p}(w \cdot 0) \cong \mathrm{gr} H^n(\mathcal{N}_{\mathbb{Z}}(\mathbb{Z}), V_{\mathbb{F}_p}(0))$$

for any $n \geq 0$ if $p \geq h-1$ (which we assumed to be the case). (Here $V_{\mathbb{F}_p}(\lambda) \cong \mathbb{F}_p$ with $\mathcal{T}_{\mathbb{Z}}(\mathbb{F}_p) = \mathcal{T}(\mathbb{F}_p) = \mathcal{T}_{\mathbb{F}_p}(\mathbb{F}_p)$ acting via λ .)

Furthermore

$$H^n(\mathcal{N}_{\mathbb{Z}}(\mathbb{Z}), V_{\mathbb{F}_p}(0)) \cong H^n(\mathcal{N}(\mathbb{Z}_p), V_{\mathbb{F}_p}(0)).$$

To see this, first note that \mathbb{Z} is a discrete group, \mathbb{Z}_p is a profinite group, and the homomorphism $\mathbb{Z} \rightarrow \mathbb{Z}_p$ has dense image in \mathbb{Z}_p . So we have homomorphisms

$$H^n(\mathbb{Z}_p, \mathbb{F}_p) \rightarrow H^n(\mathbb{Z}, \mathbb{F}_p)$$

for all $n \geq 0$ from [Ser, Section I §2.6]. Now both $H^0(\mathbb{Z}, \cdot)$ and $H^0(\mathbb{Z}_p, \cdot)$ are the functor of taking invariant, both $H^1(\mathbb{Z}, \cdot)$ and $H^1(\mathbb{Z}_p, \cdot)$ are the functor of

¹You get more than this, but we don't need more here.

taking coinvariants, and all $H^n(\mathbb{Z}, \cdot)$ and $H^n(\mathbb{Z}_p, \cdot)$ vanish for $n \geq 2$, so \mathbb{Z} is “good” in the sense of [Ser, Section I §2.6 Exercise 2]. Thus [Ser, Section I §2.6 Exercise 2(d)] implies that the homomorphisms

$$H^n(\mathcal{N}(\mathbb{Z}_p), \mathbb{F}_p) \rightarrow H^n(\mathcal{N}(\mathbb{Z}), \mathbb{F}_p) \quad n \geq 0,$$

induced by the homomorphism $\mathcal{N}(\mathbb{Z}) \rightarrow \mathcal{N}(\mathbb{Z}_p)$, are all isomorphisms.

Hence

$$\dim_{\mathbb{F}_p} H^n(\mathfrak{n}_{\mathbb{Z}}, \mathbb{F}_p) = \dim_{\mathbb{F}_p} H^n(\mathcal{N}_{\mathbb{Z}}(\mathbb{Z}), \mathbb{F}_p) = \dim_{\mathbb{F}_p} H^n(\mathcal{N}(\mathbb{Z}_p), \mathbb{F}_p).$$

Now $\mathfrak{n} = \mathfrak{n}_{\mathbb{Z}} \otimes \mathbb{F}_p$, and $H^n(\mathfrak{g}, \mathbb{F}_p) \cong H^n(\mathfrak{n}, \mathbb{F}_p)$ (since $\mathfrak{g} \cong \mathfrak{n}$) is the homology of the complex

$$C^\bullet(\mathfrak{n}, \mathbb{F}_p) = \text{Hom}_{\mathbb{F}_p} \left(\bigwedge^\bullet \mathfrak{n}, \mathbb{F}_p \right)$$

while $H^n(\mathfrak{n}_{\mathbb{Z}}, \mathbb{F}_p)$ is the homology of the complex

$$C^\bullet(\mathfrak{n}_{\mathbb{Z}}, \mathbb{F}_p) = \text{Hom}_{\mathbb{F}_p} \left(\bigwedge^\bullet \mathfrak{n}_{\mathbb{Z}}, \mathbb{F}_p \right).$$

Here $\bigwedge^\bullet \mathfrak{n}_{\mathbb{Z}}$ is a free \mathbb{Z} -module and $(\bigwedge^\bullet \mathfrak{n}_{\mathbb{Z}}) \otimes \mathbb{F}_p \cong \bigwedge^\bullet (\mathfrak{n}_{\mathbb{Z}} \otimes \mathbb{F}_p) \cong \bigwedge^\bullet \mathfrak{n}$, so we have natural isomorphisms

$$\text{Hom}_{\mathbb{F}_p} \left(\bigwedge^\bullet \mathfrak{n}_{\mathbb{Z}}, \mathbb{F}_p \right) \cong \text{Hom}_{\mathbb{F}_p} \left(\left(\bigwedge^\bullet \mathfrak{n}_{\mathbb{Z}} \right) \otimes \mathbb{F}_p, \mathbb{F}_p \right) \cong \text{Hom}_{\mathbb{F}_p} \left(\bigwedge^\bullet \mathfrak{n}, \mathbb{F}_p \right).$$

These isomorphisms are clearly compatible with the differentials, so $C^\bullet(\mathfrak{n}, \mathbb{F}_p) \cong C^\bullet(\mathfrak{n}_{\mathbb{Z}}, \mathbb{F}_p)$, and thus $H^n(\mathfrak{n}, \mathbb{F}_p) \cong H^n(\mathfrak{n}_{\mathbb{Z}}, \mathbb{F}_p)$. Hence

$$\dim_{\mathbb{F}_p} H^n(\mathfrak{n}, \mathbb{F}_p) = \dim_{\mathbb{F}_p} H^n(\mathfrak{n}_{\mathbb{Z}}, \mathbb{F}_p) = \dim_{\mathbb{F}_p} H^n(\mathcal{N}(\mathbb{Z}_p), \mathbb{F}_p).$$

1.5 Cohomology of $N = \mathcal{N}(\mathbb{Z}_p)$

Now Section 1.4 implies that

$$\sum_{s+t=n} \dim_{\mathbb{F}_p} H^{s,t}(\mathfrak{g}, \mathbb{F}_p) = \dim_{\mathbb{F}_p} H^n(\mathfrak{g}, \mathbb{F}_p) = \dim_{\mathbb{F}_p} H^n(G, \mathbb{F}_p),$$

so the multiplicative spectral sequence

$$E_1^{s,t} = H^{s,t}(\mathfrak{g}, \mathbb{F}_p) \implies H^{s+t}(G, \mathbb{F}_p)$$

from Section 1.3 converges on the first page. I.e.,

$$H^n(N, \mathbb{F}_p) = H^n(G, \mathbb{F}_p) \cong H^n(\mathfrak{g}, \mathbb{F}_p) \cong H^n(\mathfrak{n}, \mathbb{F}_p),$$

giving us a good description of $H^n(\mathcal{N}(\mathbb{Z}_p), \mathbb{F}_p)$. (Since the spectral sequence is multiplicative, can we also say that the cup product can be taken from the right hand side?)

DK Note:

How do we
argue this
purely from

looking at the
dimensions?

Do we need to
just look at
the page and
differentials in
more detail?

Chapter 2

Cohomology of Iwahori Subgroups

cha:cohiwagps

2.1 Introduction

sec:cohiwagps

Chapter 3

List-Decodable Mean Estimation and Clustering

cha:robstat

3.1 Introduction

sec:robstat-intro

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