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Contents

Some extra stuff.

Chapter 1

Cohomology of Unipotent Groups

cha:cohunigps

1.1 Introduction

sec:cohunigps-intro

So far some of the details are still skipped, but I have tried to write pretty much everything that's not already written in results I cite.

Notation and setup

Let p be a prime and let $k = \mathbb{Z}_p$. Also note that the following is true for any integral domain k (in particular also for $\mathbb{F}_p$).

Let $\mathcal{G}_{\mathbb{Z}}$ be a split and connected reductive algebraic $\mathbb{Z}$-group and let $\mathcal{G} = (\mathcal{G}_{\mathbb{Z}})_k$ (the base change from $\mathbb{Z}$ to k). Let $\mathcal{T}_{\mathbb{Z}}$ be a split maximal torus of $\mathcal{G}_{\mathbb{Z}}$ and set $\mathcal{T} = (\mathcal{T}_{\mathbb{Z}})_k$. Let $\Phi = \Phi(\mathcal{G}, \mathcal{T})$ be the root system of $\mathcal{G}$ with respect to $\mathcal{T}$ and note that Φ can be identified with the root system of $\mathcal{G}_{\mathbb{Z}}$ with respect to $\mathcal{T}_{\mathbb{Z}}$. Also note that $\mathrm{Lie}(\mathcal{G}) = \mathrm{Lie}(\mathcal{G}_{\mathbb{Z}}) \otimes_{\mathbb{Z}} k$ and for any $\alpha \in \Phi$ we have the root subgroup $\mathcal{N}_\alpha \subseteq \mathcal{G}$ with $\mathrm{Lie}\,\mathcal{N}_\alpha = (\mathrm{Lie}\,\mathcal{G})_\alpha = (\mathrm{Lie}\,\mathcal{G}_{\mathbb{Z}})_\alpha \otimes_{\mathbb{Z}} k$. Now fix a k-basis X_α of the Lie algebra of $\mathcal{N}_\alpha$. This choice gives rise to a unique isomorphism isomorphism of group schemes $x_\alpha \colon \mathcal{G}_a \xrightarrow{\cong} \mathcal{N}_\alpha$ such that $(dx_\alpha)(1) = X_\alpha$. We furthermore fix a basis $\Delta \subseteq \Phi$ of the root system such that we get a decomposition $\Phi = \Phi^+ \cup \Phi^-$ into positive and negative roots. Let $\mathcal{B} = \mathcal{T}\mathcal{N}$ and $\mathcal{B}^+ = \mathcal{T}\mathcal{N}^+$ denote the

DK Note: We might be able to avoid going through $\mathbb{Z}$ at first with some work. Also, we may need to assume that $\mathcal{G}$ is simple.

Borel subgroups of $\mathcal{G}$ corresponding to Φ^- and Φ^+, respectively, with unipotent radicals $\mathcal{N}$ and $\mathcal{N}^+$. (Here we also have corresponding algebraic $\mathbb{Z}$-groups.)

For any total ordering of Φ^- the multiplication induces an isomorphism of schemes $\prod_{\alpha\in\Phi^-}\mathcal{N}_\alpha \xrightarrow{\cong} \mathcal{N}$. For convenience we fix in the following such a total ordering which has the additional property that $\alpha_1 \geq \alpha_2$ if $\mathrm{ht}(\alpha_1) \leq \mathrm{ht}(\alpha_2)$. All products indexed by Φ^- are meant to be taken according to this ordering. Here we have the height function $\mathrm{ht}\colon \mathbb{Z}[\Delta] \to \mathbb{Z}$ given by $\sum_{\alpha\in\Delta} m_\alpha\alpha \mapsto \sum_{\alpha\in\Delta} m_\alpha$. In particular, since $\Phi \subseteq \mathbb{Z}[\Delta]$ the height $\mathrm{ht}(\beta)$ of any root $\beta\in\Phi$ is defined.

Let furthermore ρ be the half-sum of the elements of Φ^+, let $X = X(\mathcal{T}) \cong X(\mathcal{T}_\mathbb{Z})$ be the character group of $\mathcal{T}$, let

$$X^+ = \{\lambda \in X \mid \langle \lambda, \alpha^\vee\rangle \geq 0 \text{ for all } \alpha\in\Phi^+\},$$

and let h be the Coxeter number of $\mathcal{G}$ and assume from now on that $p \geq h-1$. For any $\lambda \in X^+$, let $V_\mathbb{Z}(\lambda)$ be the Weyl module for $\mathcal{G}_\mathbb{Z}$ over $\mathbb{Z}$ with highest weight λ, and let $V_k(\lambda) = V_\mathbb{Z}(\lambda)\otimes_\mathbb{Z} k$.

Let $\Phi^\vee$ be the dual root system of Φ and let W be the corresponding Weyl group with length function ℓ on W. Let $\mathfrak{n}_\mathbb{Z} = \mathrm{Lie}(\mathcal{N}_\mathbb{Z})$ be the Lie algebra of $\mathcal{N}_\mathbb{Z}$ over $\mathbb{Z}$ and $\mathfrak{n} = \mathfrak{n}_{\mathbb{F}_p} = \mathrm{Lie}(\mathcal{N}_{\mathbb{F}_p}) = \mathfrak{n}_\mathbb{Z}\otimes\mathbb{F}_p$ be the Lie algebra of $\mathcal{N}_{\mathbb{F}_p}$ over $\mathbb{F}_p$.

Finally let $G = N = \mathcal{N}(\mathbb{Z}_p) = \mathcal{N}_\mathbb{Z}(\mathbb{Z}_p)$ and let $\mathfrak{g} = \mathbb{F}_p \otimes_{\mathbb{F}_p[\pi]} \mathrm{gr}\, G$.

1.2 The p-valuation

sec:pval

This section is mainly based on some unpublished notes by Schneider.

In this section we will write N for $\mathcal{N}(\mathbb{Z}_p)$, and we note that as a set N is the direct product $N = \prod_{\alpha\in\Phi^-} x_\alpha(\mathbb{Z}_p)$, which allows us to introduce the function

$$\begin{aligned} \omega\colon N\setminus\{1\} &\to \mathbb{N} \\ \prod_{\alpha\in\Phi^-} x_\alpha(a_\alpha) &\mapsto \min_{\alpha\in\Phi^-}\big(v_p(a_\alpha) - \mathrm{ht}(\alpha)\big), \end{aligned}$$

where v_p denotes the usual p-adic valuation on $\mathbb{Z}_p$. Here it is important to note that we write any $g\in N$ uniquely as product

$$g = \prod_{\alpha\in\Phi^-} x_\alpha(a_\alpha)$$

by taking the product following the total ordering $\geq$ of Φ^- defined above. Now, with the convention that $\omega(1) \coloneqq \infty$, we define the descending sequence of subsets

$$N_m \coloneqq \{g \in N \mid \omega(g) \geq m\}$$

in N for $m \geq 0$. The main goal of this section is to show that ω is a p-valuation by a careful analysis of the sequence of subsets given by N_m.

We first note that clearly $N_1 = N$, $\bigcap_m N_m = \{1\}$, and

$$\begin{aligned} N_m &= \prod_{\alpha \in \Phi^-} x_\alpha(p^{\max(0, m+\mathrm{ht}(\alpha))}\mathbb{Z}_p) \\ &= \prod_{\substack{\alpha \in \Phi^- \\ \mathrm{ht}(\alpha) = -1}} x_\alpha(p^{m-1}\mathbb{Z}_p) \cdots \prod_{\substack{\alpha \in \Phi^- \\ \mathrm{ht}(\alpha) = -(m-1)}} x_\alpha(p\mathbb{Z}_p) \prod_{\substack{\alpha \in \Phi^- \\ \mathrm{ht}(\alpha) \leq -m}} x_\alpha(\mathbb{Z}_p). \end{aligned} \tag{1.1}$$

{eq:N_m}

In our analysis of this sequence we will also need two other filtrations of N. Firstly we will consider the filtration by congruence subgroups

$$\begin{aligned} N(m) &\coloneqq \ker\big(\mathcal{N}(\mathbb{Z}_p) \to \mathcal{N}(\mathbb{Z}/p^m\mathbb{Z})\big) \\ &= \prod_{\alpha \in \Phi^-} x_\alpha(p^m\mathbb{Z}_p) \end{aligned} \tag{1.2}$$

{eq:N(m)}

for $m \geq 0$. Secondly, using the descending central series of the group $\mathcal{G}(\mathbb{Q}_p)$ defined by $C^1\mathcal{G}(\mathbb{Q}_p) \coloneqq \mathcal{G}(\mathbb{Q}_p)$ and $C^{m+1}\mathcal{G}(\mathbb{Q}_p) \coloneqq [C^m\mathcal{G}(\mathbb{Q}_p), \mathcal{G}(\mathbb{Q}_p)]$, we consider the filtration given by

$$N_{(m)} \coloneqq N \cap C^m\mathcal{G}(\mathbb{Q}_p)$$

for $m \geq 1$. By BT we have that

DK Note: Check reference.

$$N_{(m)} = \prod_{\substack{\alpha \in \Phi^- \\ \mathrm{ht}(\alpha) \leq -m}} x_\alpha(\mathbb{Z}_p). \tag{1.3}$$

{eq:N_(m)}

We note that the natural map

$$\prod_{\substack{\alpha \in \Phi^- \\ \mathrm{ht}(\alpha) = -m}} x_\alpha(\mathbb{Z}_p) \to N_{(m)}/N_{(m+1)}$$

is an isomorphism of abelian groups, and that all the subgroups $N(m)$ and $N_{(m)}$ are normal in N.

We are now ready to prove the following lemma, which will help us when showing that ω is a p-valuation.

Lemma 1.1.

lem:N_m (i) $N_m = \prod_{1\le i\le m} N(m-i)\cap N_{(i)}$, for any $m\ge 1$, is a normal subgroup of
item:N_m N which is independent of the choices made.

item:N_mcom (ii) $[N_\ell, N_m] \subseteq N_{\ell+m}$ for any $\ell, m \ge 1$.

(iii) N_m/N_{m+1}, for any $m\ge 1$, is an $\mathbb{F}_p$-vector space of dimension equal to $|\{\alpha\in\Phi^- \mid \mathrm{ht}(\alpha)\ge -m\}|$.

item:g^p (iv) Let $g\in N_m$ for some $m\ge 1$. If $g^p\in N_{m+2}$, then $g\in N_{m+1}$.

Proof. (i) Using (1.2) and (1.3) we note that

$$\prod_{\substack{\alpha\in\Phi^-\\ \mathrm{ht}(\alpha)=-i}} x_\alpha(p^{m-1}\mathbb{Z}_p) \subseteq N(m-i)\cap N_{(i)} \quad \text{and} \quad \prod_{\substack{\alpha\in\Phi^-\\ \mathrm{ht}(\alpha)\le -m}} x_\alpha(\mathbb{Z}_p) = N(0)\cap N_{(m)}$$

for $1\le i<m$, so by (1.1) it's clear that $N_m \subseteq \prod_{1\le i\le m} N(m-i)\cap N_{(i)}$. We also note, by (1.2) and (1.3), that

$$\begin{aligned}&\big(N(m-i)\cap N_{(i)}\big)\big(N(m-i-1)\cap N_{(i+1)}\big)\\ &\qquad\subseteq \Big(\prod_{\substack{\alpha\in\Phi^-\\ \mathrm{ht}(\alpha)=-i}} x_\alpha(p^{m-i}\mathbb{Z}_p)\Big)\big(N(m-i-1)\cap N_{(i+1)}\big)\end{aligned}$$

for any $1\le i<m$, so

$$\begin{aligned}&\prod_{1\le i\le m} N(m-i)\cap N_{(i)}\\ &\quad\subseteq \prod_{\substack{\alpha\in\Phi^-\\ \mathrm{ht}(\alpha)=-1}} x_\alpha(p^{m-1}\mathbb{Z}_p)\cdots \prod_{\substack{\alpha\in\Phi^-\\ \mathrm{ht}(\alpha)=-(m-1)}} x_\alpha(p\mathbb{Z}_p)\big(N(0)\cap N_{(m)}\big)\\ &\quad= N_m\end{aligned}$$

by induction, (1.1) and (1.3). This shows the equality and that N_m is normal clearly follows.

(ii) We first recall the following formulas for commutators

$$[gh,k] = g[h,k]g^{-1}[g,k] \quad \text{and} \quad [g,hk] = [g,h]h[g,k]h^{-1}. \tag{1.4}$$ {eq:comformulas}

Now, using (1.4), (i) and the fact that all the involved subgroups are normal, it's enough to show that

$$[N(\ell) \cap N_{(i)}, N(m) \cap N_{(j)}] \subseteq N(\ell + m) \cap N_{(i+j)}.$$

This further reduces to showing that

$$[N(\ell), N(m)] \subseteq N(\ell + m) \quad \text{and} \quad [N_{(i)}, N_{(j)}] \subseteq N_{(i+j)}.$$

The right inclusion is a well known property of the descending central series, so it follows from our defintion of $N_{(m)}$. For the left inclusion it suffices, by (1.2), to show that

$$[x_\alpha(p^\ell \mathbb{Z}_p), x_\beta(p^m \mathbb{Z}_p)] \subseteq N(\ell + m)$$

for any $\alpha, \beta \in \Phi^-$. To show this inclusion we recall Chevalley's commutator formula

$$[x_\alpha(a), x_\beta(b)] \in x_{\alpha+\beta}(ab\mathbb{Z}_p) \prod_{\substack{i,j \geq 1 \\ i+j > 2}} x_{i\alpha + j\beta}(a^i b^j \mathbb{Z}_p),$$

where on the right hand side the convention is that $x_{i\alpha+j\beta} \equiv 1$ if $i\alpha + j\beta \notin \Phi$ (cf. BT). From (1.2) and Chevalley's commutator formula the inclusion follows. DK Note: Check reference.

(iii) We note that

$$N(m - i) \cap N_{(i)} = \prod_{\substack{\alpha \in \Phi^- \\ \mathrm{ht}(\alpha) \leq -i}} x_\alpha(p^{m-i} \mathbb{Z}_p)$$

for $1 \leq i \leq m$, so the statement follows from (i) and (ii). DK Note: Write (iii) better.

(iv) For any $1 \leq \ell \leq m$ we consider the chain of normal subgroups

$$N_{m+2}(N_m \cap N_{(\ell+1)}) \subseteq N_{m+1}(N_m \cap N_{(\ell+1)}) \subseteq N_{m+1}(N_m \cap N_{(\ell)})$$

between N_{m+2} and N_m. By (1.4) and an argument like in (ii), we get that

$$[N_{m+1}(N_m \cap N_{(\ell)}), N_{m+1}(N_m \cap N_{(\ell)})] \subseteq N_{m+2}(N_m \cap N_{(\ell+1)}),$$

so the quotient group

$$N_{m+1}(N_m \cap N_{(\ell)}) / N_{m+2}(N_m \cap N_{(\ell+1)})$$

is abelian. Now looking carefully at the groups as sets, we see that

$$N_m \cap N_{(\ell)} = \prod_{\substack{\alpha\in\Phi^- \\ \mathrm{ht}(\alpha)\leq -\ell}} x_\alpha(p^{\max(0,m+\mathrm{ht}(\alpha))}\mathbb{Z}_p)$$

and thus (using Chevalley's commutator formula and the fact that $\mathrm{ht}(i\alpha+j\beta) \leq \mathrm{ht}(\alpha+\beta) < \mathrm{ht}(\alpha), \mathrm{ht}(\beta)$ to move the products for the $\mathrm{ht}(\alpha) = -\ell$ term)

$$\begin{aligned} N_{m+1}(N_m \cap N_{(\ell)}) = &\prod_{\substack{\alpha\in\Phi^- \\ \mathrm{ht}(\alpha)> -\ell}} x_\alpha(p^{\max(0,m+1+\mathrm{ht}(\alpha))}\mathbb{Z}_p) \\ &\cdot \prod_{\substack{\alpha\in\Phi^- \\ \mathrm{ht}(\alpha)= -\ell}} x_\alpha(p^{m-\ell}\mathbb{Z}_p) \\ &\cdot \prod_{\substack{\alpha\in\Phi^- \\ \mathrm{ht}(\alpha)< -\ell}} x_\alpha(p^{\max(0,m+\mathrm{ht}(\alpha))}\mathbb{Z}_p). \end{aligned}$$

Similarly

$$\begin{aligned} N_{m+2}(N_m \cap N_{(\ell+1)}) = &\prod_{\substack{\alpha\in\Phi^- \\ \mathrm{ht}(\alpha)> -\ell}} x_\alpha(p^{\max(0,m+2+\mathrm{ht}(\alpha))}\mathbb{Z}_p) \\ &\cdot \prod_{\substack{\alpha\in\Phi^- \\ \mathrm{ht}(\alpha)= -\ell}} x_\alpha(p^{m+2-\ell}\mathbb{Z}_p) \\ &\cdot \prod_{\substack{\alpha\in\Phi^- \\ \mathrm{ht}(\alpha)\leq -(\ell+1)}} x_\alpha(p^{\max(0,m+\mathrm{ht}(\alpha))}\mathbb{Z}_p), \end{aligned}$$

and since the quotient group

$$N_{m+1}(N_m \cap N_{(\ell)})/N_{m+2}(N_m \cap N_{(\ell+1)})$$

is abelian, we see that it is isomorphic to

$$\prod_{\substack{\alpha\in\Phi^- \\ \mathrm{ht}(\alpha)> -\ell}} \frac{x_\alpha(p^{\max(0,m+1+\mathrm{ht}(\alpha))}\mathbb{Z}_p)}{x_\alpha(p^{\max(m+2+\mathrm{ht}(\alpha))}\mathbb{Z}_p)} \times \prod_{\mathrm{ht}(\alpha)=-\ell} \frac{x_\alpha(p^{m-\ell}\mathbb{Z}_p)}{x_\alpha(p^{m+2-\ell}\mathbb{Z}_p)}.$$

Here the subgroup

$$N_{m+1}(N_m \cap N_{(\ell+1)})/N_{m+2}(N_m \cap N_{(\ell+1)})$$

corresponds to

$$\prod_{\mathrm{ht}(\alpha)>-\ell} \frac{x_\alpha(p^{\max(0,m+1+\mathrm{ht}(\alpha))}\mathbb{Z}_p)}{x_\alpha(p^{\max(0,m+2+\mathrm{ht}(\alpha))}\mathbb{Z}_p)} \times \prod_{\mathrm{ht}(\alpha)=-\ell} \frac{x_\alpha(p^{m+1-\ell}\mathbb{Z}_p)}{x_\alpha(p^{m+2-\ell}\mathbb{Z}_p)}.$$

It follows that $N_{m+1}(N_m \cap N_{(\ell+1)})/N_{m+2}(N_m \cap N_{(\ell+1)})$ is the p-torsion subgroup of $N_{m+1}(N_m \cap N_{(\ell)})/N_{m+2}(N_m \cap N_{(\ell+1)})$.

Now let $g \in N_m$ for some $m \geq 1$ and assume that $g^p \in N_{m+2}$. For $\ell = 1$ we have $g \in N_m = N_{m+1}(N_m \cap N_{(1)})$, since $N_{(1)} = N$, and clearly $g^p \in N_{m+2}(N_m \cap N_{(2)})$. Since $N_{m+1}(N_m \cap N_{(2)})/N_{m+2}(N_m \cap N_{(2)})$ is the p-torsion subgroup of $N_{m+1}(N_m \cap N_{(1)})/N_{m+2}(N_m \cap N_{(2)})$, it follows that $g \in N_{m+1}(N_m \cap N_{(2)})$ and $g^p \in N_{m+2}(N_m \cap N_{(3)})$. By induction on ℓ, we thus get that $g \in N_{m+1}(N_m \cap N_{(m+1)}) = N_{m+1}$. Here the last equality follows from the fact that $N_{(m+1)} \subseteq N_{m+1}$ by (1.1) and (1.3). $\square$

Proposition 1.2. The function ω is a p-valuation on N, i.e., it satisfies for any $g, h \in N$:

(a) $\omega(g) > \frac{1}{p-1}$,

(b) $\omega(g^{-1}h) \geq \min(\omega(g), \omega(h))$,

(c) $\omega([g,h]) \geq \omega(g) + \omega(h)$,

(d) $\omega(g^p) = \omega(g) + 1$.

Proof. We note that (a) is obvious by our definition of ω, (c) follows from Lemma 1.1 (ii) and (d) follows from Lemma 1.1 (iv).

It only remains to show (b), which we will do by following the proof idea of Lemma 1 from [Zab], i.e., we are going to use triple induction. Also, for the sake of this proof (and only during this proof), we will take all product $\prod_{\alpha\in\Phi^-} x_\alpha(a_\alpha)$ to be in descending order in Φ^-.

At first by induction on the number of non-zero coordinates among $(a_\alpha)_{\alpha\in\Phi^-}$ in $\prod_{\alpha\in\Phi^-} x_\alpha(a_\alpha)$ we are reduced to the case where g is of the form $g = x_\alpha(a_\alpha)$ for some $\alpha \in \Phi^-$ and $a_\alpha \in \mathbb{Z}_p$. To see this let $g \in N \setminus \{1\}$ and write $g = \prod_{\alpha\in\Phi^-} x_\alpha(a_\alpha)$ in our unique way (according to the descending ordering of Φ^-), and let β be the smallest element of Φ^- for which $a_\alpha \neq 0$ so that $g = g' \cdot x_\beta(a_\beta)$. Then $g^{-1}h = x_\beta(a_\beta)^{-1}((g')^{-1}h)$ and thus strong induction will imply that

$$\begin{aligned}\omega(g^{-1}h) &\geq \min\bigl(v(a_\beta) - \mathrm{ht}(\beta), \omega((g')^{-1}h)\bigr) \\ &\geq \min\bigl(v(a_\beta) - \mathrm{ht}(\alpha), \omega(g'), \omega(h)\bigr) = \min\bigl(\omega(g), \omega(h)\bigr).\end{aligned}$$

Let now h be of the form $h = \prod_{k=1}^r x_{\beta_k}(a_{\beta_k})$ with $\beta_1 > \beta_2 > \cdots > \beta_r$ in Φ^-. If $\alpha \geq \beta_1$, then $g^{-1}h = x_\alpha(-a_\alpha) \cdot x_{\beta_1}(a_{\beta_1}) \prod_{k=2}^r x_{\beta_k}(a_{\beta_k})$, so (b) is clearly true if $\alpha > \beta_1$ (by the definition of ω), and if $\alpha = \beta_1$, then $x_\alpha(-a_\alpha) \cdot x_{\beta_1}(a_{\beta_1}) = x_\alpha(-a_\alpha + a_{\beta_1})$ and (b) follows from $v_p(a+b) \geq \min(v_p(a), v_p(b))$ for $a, b \in \mathbb{Z}_p$.

On the other hand, if $\alpha < \beta_1$, then we write

$$\begin{aligned}g^{-1}h &= x_\alpha(-a_\alpha) \cdot \prod_{k=1}^r x_{\beta_k}(a_{\beta_k}) \\ &= [x_\alpha(-a_\alpha), x_{\beta_1}(a_{\beta_1})] x_{\beta_1}(a_{\beta_1}) \cdot x_\alpha(-a_\alpha) \prod_{k=2}^r x_{\beta_k}(a_{\beta_k}).\end{aligned}$$

Now we use descending induction on α in the chosen ordering of Φ^- and suppose that the statement (b) is true for any h and any g' of the form $g' = x_{\alpha'}(a_{\alpha'})$ with $\alpha' > \alpha$. Note that we already implicitly described the base case earlier and recall that Φ^- is finite and totally ordered. Note furthermore that Chevalley's commutator formula gives us

$$[x_\alpha(-a_\alpha), x_\beta(a_\beta)] = \prod_{\substack{i\alpha + j\beta \in \Phi^- \\ i,j>0}} x_{i\alpha+j\beta}(c_{\alpha,\beta,i,j}(-a_\alpha)^i a_\beta^j) \tag{1.5}$$ {eq:Chevalley}

for any $\alpha, \beta \in \Phi^-$, where $c_{\alpha,\beta,i,j} \in \mathbb{Z}_p$. Also, we have $\mathrm{ht}(i\alpha + j\beta) \leq \mathrm{ht}(\alpha+\beta) < \mathrm{ht}(\alpha), \mathrm{ht}(\beta)$, so we can apply the inductional hypothesis for $x_{\beta_1}(a_{\beta_1})$ (since $\beta_1 > \alpha$) and for all terms on the right side of (1.5) (recalling that $\alpha' \geq \beta'$ if $\mathrm{ht}(\alpha') \leq \mathrm{ht}(\beta')$ and with the choice $\beta = \beta_1$) in order to obtain

$$\begin{aligned}\omega(g^{-1}h) \geq \min\Biggl(&\min_{\substack{i\alpha+j\beta\in\Phi^- \\ i,j>0}} \omega(x_{i\alpha+j\beta}(c_{\alpha,\beta,i,j}(-a_\alpha)^i a_\beta^j)), \\ &\omega(x_{\beta_1}(a_{\beta_1})), \omega\Bigl(x_\alpha(-a_\alpha) \prod_{k=2}^r x_{\beta_k}(a_{\beta_k})\Bigr)\Biggr).\end{aligned} \tag{1.6}$$ {eq:omega(ginvh)}

Now, for $i, j > 0$ with $i\alpha + j\beta \in \Phi$,

$$\begin{aligned}
\omega(x_{i\alpha+j\beta}(c_{\alpha,\beta,i,j}(-a_\alpha)^i a_\beta^j)) &= v_p(c_{\alpha,\beta,i,j}(-a_\alpha)^i a_\beta^j) - \mathrm{ht}(i\alpha + j\beta) \\
&\geq v_p(c_{\alpha,\beta,i,j}) + v_p((-a_\alpha)^i) + v_p(a_\beta^j) - \mathrm{ht}(\alpha + \beta) \\
&\geq v_p(a_\alpha) - \mathrm{ht}(\alpha) + v_p(a_\beta) - \mathrm{ht}(\beta) \\
&= \omega(x_\alpha(a_\alpha)) + \omega(x_\beta(a_\beta)) \\
&\geq \min\big(\omega(x_\alpha(a_\alpha)), \omega(x_\beta(a_\beta))\big).
\end{aligned} \tag{1.7}$$

{eq:omega(Chev)}

So taking $\beta = \beta_1$ and using (1.7) in (1.6), we get that

$$\omega(g^{-1}h) \geq \min\Bigg(\omega(x_\alpha(a_\alpha)), \omega(x_{\beta_1}(a_{\beta_1})), \omega\Big(x_\alpha(-a_\alpha)\prod_{k=2}^{r} x_{\beta_k}(a_{\beta_k})\Big)\Bigg). \tag{1.8}$$

{eq:omega(ginvh)2}

Finally induction on r will imply that

$$\begin{aligned}
\omega\Big(x_\alpha(-a_\alpha)\prod_{k=2}^{r} x_{\beta_k}(a_{\beta_k})\Big) &\geq \min\Bigg(\omega(x_\alpha(a_\alpha)), \omega\Big(\prod_{k=2}^{r} x_{\beta_k}(a_{\beta_k})\Big)\Bigg) \\
&= \min\big(\omega(x_\alpha(a_\alpha)), \min_{2\leq k\leq r} \omega(x_{\beta_k}(a_{\beta_k}))\big),
\end{aligned}$$

which by (1.8) implies that

$$\begin{aligned}
\omega(g^{-1}h) &\geq \min\big(\omega(x_\alpha(a_\alpha)), \min_{1\leq k\leq r} \omega(x_{\beta_k}(a_{\beta_k}))\big) \\
&= \min\big(\omega(g), \omega(h)\big),
\end{aligned}$$

thus finishing the proof. □

1.3 A multiplicative spectral sequence

sec:specsec

In this section we will write G for $\mathcal{N}(\mathbb{Z}_p)$, and we let $\mathfrak{g} = \mathbb{F}_p \otimes_{\mathbb{F}_p[\pi]} \mathrm{gr}\, G$.

Here $\mathrm{gr}\, G \cong \mathbb{F}_p[\pi] \otimes_{\mathbb{F}_p} \mathfrak{n}$ by Proposition 3.2 of Schneider's notes, so $\mathfrak{g} \cong \mathbb{F}_p \otimes_{\mathbb{F}_p[\pi]} \mathbb{F}_p[\pi] \otimes_{\mathbb{F}_p} \mathfrak{n} \cong \mathfrak{n}$. (Which can also be shown by looking at the Chevalley constants.)

Note that G is a pro-p-group and by Corollary 2.2 of Schneider's notes G is p-valuable, so by Theorem 29.8 of [Sch] G is a (compact) p-adic Lie group.

DK Note: This actually takes quite a lot of work to write the argument for, but it's mostly written in Schneider's

Now we have a p-valued group (G, ω), so by [Sør] we get a multiplicative convergent spectral sequence

$$E_1^{s,t} = H^{s,t}(\mathfrak{g}, \mathbb{F}_p) \Longrightarrow H^{s+t}(G, \mathbb{F}_p).$$

Here $H^{s,t}(\mathfrak{g}, \mathbb{F}_p) = H^{s+t}(\operatorname{gr}^s C^\bullet(\mathfrak{g}, \mathbb{F}_p))$ by definition, where the Lie algebra $\mathfrak{g} \cong \mathfrak{n}$ is graded by the height function.

1.4 Dimension of cohomology of $\mathfrak{n}$ and $N = \mathcal{N}(\mathbb{Z}_p)$

sec:dimofcoh

By Corollary 2.10 and Corollary 3.8 of [PT] and the Universal Coefficient Theorem there is a finite, natural $\mathcal{T}_{\mathbb{Z}}(\mathbb{Z})$-filtration such that we get isomorphisms of $\mathbb{F}_p$-vector spaces[1]

$$H^n(\mathfrak{n}_{\mathbb{Z}}, V_{\mathbb{F}_p}(0)) \cong \bigoplus_{\substack{w \in W \\ \ell(w)=n}} V_{\mathbb{F}_p}(w \cdot 0) \cong \operatorname{gr} H^n(\mathcal{N}_{\mathbb{Z}}(\mathbb{Z}), V_{\mathbb{F}_p}(0))$$

for any $n \geq 0$ if $p \geq h-1$ (which we assumed to be the case). (Here $V_{\mathbb{F}_p}(\lambda) \cong \mathbb{F}_p$ with $\mathcal{T}_{\mathbb{Z}}(\mathbb{F}_p) = \mathcal{T}(\mathbb{F}_p) = \mathcal{T}_{\mathbb{F}_p}(\mathbb{F}_p)$ acting via λ.)

Furthermore

$$H^n(\mathcal{N}_{\mathbb{Z}}(\mathbb{Z}), V_{\mathbb{F}_p}(0)) \cong H^n(\mathcal{N}(\mathbb{Z}_p), V_{\mathbb{F}_p}(0)).$$

To see this, first note that $\mathbb{Z}$ is a discrete group, $\mathbb{Z}_p$ is a profinite group, and the homomorphism $\mathbb{Z} \to \mathbb{Z}_p$ has dense image in $\mathbb{Z}_p$. So we have homomorphisms

$$H^n(\mathbb{Z}_p, \mathbb{F}_p) \to H^n(\mathbb{Z}, \mathbb{F}_p)$$

for all $n \geq 0$ from [Ser, Section I §2.6]. Now both $H^0(\mathbb{Z}, \cdot\,)$ and $H^0(\mathbb{Z}_p, \cdot\,)$ are the functor of taking invariant, both $H^1(\mathbb{Z}, \cdot\,)$ and $H^1(\mathbb{Z}_p, \cdot\,)$ are the functor of taking coinvariants, and all $H^n(\mathbb{Z}, \cdot\,)$ and $H^n(\mathbb{Z}_p, \cdot\,)$ vanish for $n \geq 2$, so $\mathbb{Z}$ is "good" in the sense of [Ser, Section I §2.6 Exercise 2]. Thus [Ser, Section I §2.6 Exercise 2(d)] implies that the homomorphisms

$$H^n(\mathcal{N}(\mathbb{Z}_p), \mathbb{F}_p) \to H^n(\mathcal{N}(\mathbb{Z}), \mathbb{F}_p) \qquad n \geq 0,$$

[1] You get more than this, but we don't need more here.

induced by the homomorphism $\mathcal{N}(\mathbb{Z}) \to \mathcal{N}(\mathbb{Z}_p)$, are all isomorphisms.

Hence

$$\dim_{\mathbb{F}_p} H^n(\mathfrak{n}_{\mathbb{Z}}, \mathbb{F}_p) = \dim_{\mathbb{F}_p} H^n(\mathcal{N}_{\mathbb{Z}}(\mathbb{Z}), \mathbb{F}_p) = \dim_{\mathbb{F}_p} H^n(\mathcal{N}(\mathbb{Z}_p), \mathbb{F}_p).$$

Now $\mathfrak{n} = \mathfrak{n}_{\mathbb{Z}} \otimes \mathbb{F}_p$, and $H^n(\mathfrak{g}, \mathbb{F}_p) \cong H^n(\mathfrak{n}, \mathbb{F}_p)$ (since $\mathfrak{g} \cong \mathfrak{n}$) is the homology of the complex

$$C^\bullet(\mathfrak{n}, \mathbb{F}_p) = \mathrm{Hom}_{\mathbb{F}_p}\Big(\bigwedge^\bullet \mathfrak{n}, \mathbb{F}_p\Big)$$

while $H^n(\mathfrak{n}_{\mathbb{Z}}, \mathbb{F}_p)$ is the homology of the complex

$$C^\bullet(\mathfrak{n}_{\mathbb{Z}}, \mathbb{F}_p) = \mathrm{Hom}_{\mathbb{F}_p}\Big(\bigwedge^\bullet \mathfrak{n}_{\mathbb{Z}}, \mathbb{F}_p\Big).$$

Here $\bigwedge^\bullet \mathfrak{n}_{\mathbb{Z}}$ is a free $\mathbb{Z}$-module and $(\bigwedge^\bullet \mathfrak{n}_{\mathbb{Z}}) \otimes \mathbb{F}_p \cong \bigwedge^\bullet(\mathfrak{n}_{\mathbb{Z}} \otimes \mathbb{F}_p) \cong \bigwedge^\bullet \mathfrak{n}$, so we have natural isomorphisms

$$\mathrm{Hom}_{\mathbb{F}_p}\Big(\bigwedge^\bullet \mathfrak{n}_{\mathbb{Z}}, \mathbb{F}_p\Big) \cong \mathrm{Hom}_{\mathbb{F}_p}\Big(\Big(\bigwedge^\bullet \mathfrak{n}_{\mathbb{Z}}\Big) \otimes \mathbb{F}_p, \mathbb{F}_p\Big) \cong \mathrm{Hom}_{\mathbb{F}_p}\Big(\bigwedge^\bullet \mathfrak{n}, \mathbb{F}_p\Big).$$

These isomorphisms are clearly compatible with the differentials, so $C^\bullet(\mathfrak{n}, \mathbb{F}_p) \cong C^\bullet(\mathfrak{n}_{\mathbb{Z}}, \mathbb{F}_p)$, and thus $H^n(\mathfrak{n}, \mathbb{F}_p) \cong H^n(\mathfrak{n}_{\mathbb{Z}}, \mathbb{F}_p)$. Hence

$$\dim_{\mathbb{F}_p} H^n(\mathfrak{n}, \mathbb{F}_p) = \dim_{\mathbb{F}_p} H^n(\mathfrak{n}_{\mathbb{Z}}, \mathbb{F}_p) = \dim_{\mathbb{F}_p} H^n(\mathcal{N}(\mathbb{Z}_p), \mathbb{F}_p).$$

1.5 Cohomology of $N = \mathcal{N}(\mathbb{Z}_p)$

Now Section 1.4 implies that

$$\sum_{s+t=n} \dim_{\mathbb{F}_p} H^{s,t}(\mathfrak{g}, \mathbb{F}_p) = \dim_{\mathbb{F}_p} H^n(\mathfrak{g}, \mathbb{F}_p) = \dim_{\mathbb{F}_p} H^n(G, \mathbb{F}_p),$$

so the multiplicative spectral sequence

$$E_1^{s,t} = H^{s,t}(\mathfrak{g}, \mathbb{F}_p) \Longrightarrow H^{s+t}(G, \mathbb{F}_p)$$

from Section 1.3 converges on the first page. I.e.,

$$H^n(N, \mathbb{F}_p) = H^n(G, \mathbb{F}_p) \cong H^n(\mathfrak{g}, \mathbb{F}_p) \cong H^n(\mathfrak{n}, \mathbb{F}_p),$$

giving us a good description of $H^n(\mathcal{N}(\mathbb{Z}_p), \mathbb{F}_p)$. (Since the spectral sequence is multiplicative, can we also say that the cup product can be taken from the right hand side?)

DK Note: How do we argue this purely from looking at the dimensions? Do we need to just look at the page and differentials in more detail?

Chapter 2

Cohomology of Iwahori Subgroups

cha:cohiwagps

sec:cohiwagps

2.1 Intoduction

Chapter 3

List-Decodable Mean Estimation and Clustering

cha:robstat

sec:robstat-intro

3.1 Introduction

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