PhD Defense

On the mod p cohomology of pro-p Iwahori subgroups

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Overview

1. Introduction
   - Cohomology of compact Lie groups
   - Lazard Theory
   - $E_1^{s,t} = H^{s,t}(\mathfrak{g}, \mathbb{F}_p) \Longrightarrow H^{s+t}(G, \mathbb{F}_p)$
2. On the mod p cohomology of unipotent groups
3. On the mod p cohomology of pro-p Iwahori subgroups
   - $I \subseteq \mathrm{SL}_2(\mathbb{Z}_p)$
   - $I \subseteq \mathrm{GL}_2(\mathbb{Z}_p)$
   - Other calculations
   - Nilpotency index
4. Future work
   - Division quaternion algebras
   - Central division algebras
   - Serre spectral sequence

Overview

Cohomology of connected compact Lie groups (1)

Given a connected compact Lie group G with Lie algebra $\mathfrak{g}$ with $\ell = \mathrm{rank}(G)$.

Theorem (Chevalley and Eilenberg, 1948)

$$H^*(G, \mathbb{R}) \cong H^*(\mathfrak{g}, \mathbb{R}),$$

or more explicitly $H^*(G, \mathbb{R})$ is an exterior algebra $\bigwedge(\xi_1, \dots, \xi_\ell)$ on generators ξ_i of various odd degrees $2d_i - 1$.

Cohomology of connected compact Lie groups (2)

Theorem (Kac, 1985)

$$H^*(G, \mathbb{F}_p) \cong \mathbb{F}_p[x_1, \dots, x_r]/(x_1^{p^{k_1}}, \dots, x_r^{p^{k_r}}) \otimes_{\mathbb{F}_p} \bigwedge(\xi_1, \dots, \xi_\ell)$$

for $p > 2$. Here $\deg(\xi_i) = 2d_{i,p} - 1$ and $\deg(x_i) = 2d_{i,p}$, where Kac defines $d_{i,p}$ along with r and the k_i.

Cohomology of connected compact Lie groups (2)

Theorem (Kac, 1985)

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Note: The above cohomology is the cohomology of G as a topological space, and not continuous group cohomology. Continuous group cohomology can be thought of as the cohomology of the classifying space BG. One can identify $H^*(BG, \mathbb{R})$ with a polynomial algebra $\mathbb{R}[x_1, \dots, x_\ell]$ in variables of even degrees.

Mod p cohomology of p-adic Lie groups G

Let G be a p-valued compact p-adic Lie group, and denote by $\mathfrak{g} = \mathbb{F}_p \otimes_{\mathbb{F}_p[\pi]} \operatorname{gr} G$ the Lazard Lie algebra attached to G.

Theorem (Lazard, 1965)

If G is equi-p-valued, then there is an isomorphism of algebras

$$H^*(G, \mathbb{F}_p) \cong \bigwedge \mathfrak{g}^*.$$

Mod p cohomology of p-adic Lie groups G

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If G is equi-p-valued, then there is an isomorphism of algebras

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Note: Any compact p-adic Lie group contains an open equi-p-valuable subgroup. So distinction between p-valued and equi-p-valued groups is somewhat nuanced.

Other newer results

Let $\mathcal{N}$ be the unipotent radical of a Borel in a split reductive $\mathbb{Z}_p$-group.

Theorem (Ronchetti, 2020)

There is a $\mathcal{T}(\mathbb{Z}_p)$-equivariant isomorphism

$$H^*(\mathfrak{n}_{\mathbb{Z}_p}, \mathbb{Z}_p) \cong \operatorname{gr} H^*(N, \mathbb{Z}_p),$$

where $\mathfrak{n}_{\mathbb{Z}_p} = \operatorname{Lie}(\mathcal{N})$ and $N = \mathcal{N}(\mathbb{Z}_p)$.

Interesting p-valuable groups G

There are many examples of naturally occurring p-valuable groups G which are not equi-p-valuable, where detailed information about $H^*(G, \mathbb{F}_p)$ is important. E.g.

- unipotent groups (i.e., the $\mathbb{Z}_p$-points of the unipotent radical of a Borel in a split reductive group),
- Serre's standard groups with $e > 1$,
- pro-p Iwahori subgroups for large enough p,
- $1 + \mathfrak{m}_D$ where D is the quaternion division algebra over $\mathbb{Q}_p$ for $p > 3$ (or more generally a central division algebra over $\mathbb{Q}_p$).

p-adic integers

$$\mathbb{Z}_p = \left\{ \sum_{n=0}^{\infty} a_n p^n \,\middle|\, a_n \in \{0, 1, \ldots, p-1\} \right\} \supseteq \mathbb{Z}$$

is a commutative ring on which we have a p-adic valuation $v_p \colon \mathbb{Z}_p \to \mathbb{N} \cup \{\infty\}$ given by $v_p(0) = \infty$ and $v_p(a) = \min\{n \in \mathbb{N} \mid a_n \neq 0\}$ for $a = \sum_{n \in \mathbb{N}} a_n p^n \neq 0$ and satisfying

(a) $v_p(a) = \infty \iff a = 0$,

(b) $v_p(ab) = v_p(a) + v_p(b)$,

(c) $v_p(a+b) \geq \min\big(v_p(a), v_p(b)\big)$

p-valuations

Let G be any group.

Definition

A p-valuation ω on G is a function

$$\omega\colon G \setminus \{1\} \to (0, \infty)$$

which, with the convention that $\omega(1) = \infty$, satisfies

(a) $\omega(g) > \frac{1}{p-1}$,

(b) $\omega(g^{-1}h) \geq \min\big(\omega(g), \omega(h)\big)$,

(c) $\omega([g,h]) \geq \omega(g) + \omega(h)$ (where $[g,h] = ghg^{-1}h^{-1}$),

(d) $\omega(g^p) = \omega(g) + 1$

for any $g, h \in G$.

Filtration of G

Let G be a p-valued group. For any real number $\nu > 0$ put

$$G_\nu := \{g \in G : \omega(g) \geq \nu\} \quad \text{and} \quad G_{\nu+} := \{g \in G : \omega(g) > \nu\},$$

and note that these are normal subgroups.
The subgroups G_ν form a decreasing exhaustive and separated filtration of G with the properties

$$G_\nu = \bigcap_{\nu' < \nu} G_{\nu'} \quad \text{and} \quad [G_\nu, G_{\nu'}] \subseteq G_{\nu+\nu'}.$$

There is a unique (Hausdorff) topological group structure on G for which the G_ν form a fundamental system of open neighborhoods of the identity element. We assume that G is profinite, so that $G = \varprojlim_{\nu>0} G/G_\nu$ is a pro-p group since $\omega(g^p) = \omega(g) + 1$ implies that G/G_ν is a p-group (finite since G_ν is open).

Grading on G

Consider the graded abelian group (denoted additively)

$$\operatorname{gr} G = \bigoplus_{\nu > 0} \operatorname{gr}_\nu G,$$

where $\operatorname{gr}_\nu G := G_\nu / G_{\nu+}$ for $\nu > 0$.
An element $\xi \in \operatorname{gr} G$ is called homogeneous (of degree ν) if it lies in $\operatorname{gr}_\nu G$. Note that $p\xi = 0$ for any homogeneous $\xi \in \operatorname{gr} G$ since $\omega(g^p) = \omega(g) + 1$. Hence $\operatorname{gr} G$ is an $\mathbb{F}_p$-vector space.

Lie bracket on gr G

Bilinearly extending the map

$$\begin{aligned} \mathrm{gr}_\nu G \times \mathrm{gr}_{\nu'} G &\to \mathrm{gr}_{\nu+\nu'} G \\ (\xi, \eta) &\mapsto [\xi, \eta] := [g, h] G_{(\nu+\nu')+}, \end{aligned}$$

we obtain a graded $\mathbb{F}_p$-bilinear map

$$[-,-]\colon \ \mathrm{gr}\, G \times \mathrm{gr}\, G \to \mathrm{gr}\, G.$$

One can check that $[-,-]$ makes gr G a graded Lie algebra over $\mathbb{F}_p$.

$\mathbb{F}_p[\pi]$-Lie algebra structure on gr G

The map

$$\begin{aligned} \operatorname{gr}_\nu G &\to \operatorname{gr}_{\nu+1} G \\ gG_{\nu+} &\mapsto g^p G_{(\nu+1)+} \end{aligned}$$

is well-defined and $\mathbb{F}_p$-linear, so it induces an $\mathbb{F}_p$-linear map of degree one

$$\pi\colon \operatorname{gr} G \to \operatorname{gr} G.$$

We view gr G as a graded module over $\mathbb{F}_p[\pi]$, and note that the Lie bracket on gr G is bilinear for the $\mathbb{F}_p[\pi]$-module structure, i.e., gr G is a Lie algebra over $\mathbb{F}_p[\pi]$.

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Definition

The pair (G, ω) is called of finite rank if gr G is finitely generated as an $\mathbb{F}_p[\pi]$-module.

Ordered basis of (G, ω)

Assume from now on that (G, ω) is of finite rank d.
For any finitely many $g_1, \ldots, g_r \in G$ we have a continuous map (not group homomorphism)

$$\begin{aligned} \mathbb{Z}_p^r &\to G \\ (x_1, \ldots, x_r) &\mapsto g_1^{x_1} \cdots g_r^{x_r}. \end{aligned} \tag{1}$$

Definition

The sequence of elements $(g_1, \ldots, g_r)$ in G is called an ordered basis of (G, ω) if the map (1) is a bijection and

$$\omega(g_1^{x_1} \cdots g_r^{x_r}) = \min_{1 \leq i \leq r} \left(\omega(g_i) + v_p(x_i)\right) \quad \text{for any } x_1, \ldots, x_r \in \mathbb{Z}_p.$$

$\mathbb{F}_p[\pi]$-basis of gr G

Definition

For any $g \in G \setminus \{1\}$, we put $\sigma(g) := gG_{\omega(g)+} \in \operatorname{gr} G$.

Note that for $g \in G \setminus \{1\}$ and $x \in \mathbb{Z}_p \setminus \{0\}$

$$\omega(g^x) = \omega(g) + v_p(x) \quad \text{and} \quad \sigma(g^x) = \bar{x}\pi^{v_p(x)} . \sigma(g),$$

where $\bar{x}$ is the image of $p^{-v_p(x)}x$ in $\mathbb{F}_p^\times$.
An ordered basis $(g_1, \ldots, g_d)$ of (G, ω) corresponds to an $\mathbb{F}_p[\pi]$-basis $\big(\sigma(g_1), \ldots, \sigma(g_d)\big)$ of gr G.

Lazard Lie algebra $\mathfrak{g} = \mathbb{F}_p \otimes_{\mathbb{F}_p[\pi]} \mathrm{gr}\, G$

Let

$$\mathfrak{g} := \mathbb{F}_p \otimes_{\mathbb{F}_p[\pi]} \mathrm{gr}\, G,$$

and note that this an $\mathbb{F}_p$-Lie algebra with an $\mathbb{F}_p$-basis of vectors $\xi_i = 1 \otimes \sigma(g_i)$ for $i = 1, \ldots, d$.

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Note that the commutators $[g_i, g_j]$, allow us to calculate $\sigma\left([g_i, g_j]\right) = \left[\sigma(g_i), \sigma(g_j)\right]$ and thus $[\xi_i, \xi_j] = 1 \otimes \left[\sigma(g_i), \sigma(g_j)\right]$.

Continuous group cohomology (over $\mathbb{F}_p$)

Let G be a topological group and $\mathbb{F}_p$ a trivial G-module. Continuous group cohomology $H^*(G, \mathbb{F}_p)$ is the cohomology of the complex $C^\bullet(G, \mathbb{F}_p) = \mathcal{C}(G^\bullet, \mathbb{F}_p)$ of continuous maps $G \times G \times \cdots \times G \to \mathbb{F}_p$, i.e.,

$$0 \longrightarrow \mathbb{F}_p \xrightarrow{\partial_1} \mathcal{C}(G, \mathbb{F}_p) \xrightarrow{\partial_2} \mathcal{C}(G^2, \mathbb{F}_p) \xrightarrow{\partial_3} \cdots,$$

where the coboundary maps ∂_n are given by

$$\partial_n(f)(g_1, \ldots, g_n) = f(g_2, \ldots, g_n) + \sum_{i=1}^{n} (-1)^i f(g_1, \ldots, g_i g_{i+1}, \ldots, g_n),$$

with n-th term $(-1)^n f(g_1, \ldots, g_{n-1})$.

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with n-th term $(-1)^n f(g_1, \ldots, g_{n-1})$. (Discrete is a special case.)

Lie algebra cohomology (over $\mathbb{F}_p$) (1)

Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{F}_p$ with $\mathbb{F}_p$ a trivial (left) $\mathfrak{g}$-module. Lie algebra cohomology $H^*(\mathfrak{g}, \mathbb{F}_p)$ is the cohomology of the complex $C^\bullet(\mathfrak{g}, \mathbb{F}_p) = \mathrm{Hom}_{\mathbb{F}_p}(\bigwedge^\bullet \mathfrak{g}, \mathbb{F}_p)$, i.e.,

$$0 \longrightarrow \mathbb{F}_p \xrightarrow{\partial_1} \mathrm{Hom}_{\mathbb{F}_p}(\mathfrak{g}, \mathbb{F}_p) \xrightarrow{\partial_2} \mathrm{Hom}_{\mathbb{F}_p}\left(\bigwedge^2 \mathfrak{g}, \mathbb{F}_p\right) \xrightarrow{\partial_3} \cdots,$$

where the coboundary maps ∂_n are given by

$$\partial_n(f)(x_1, \ldots, x_n) = \sum_{i<j} (-1)^{i+j} f\left([x_i, x_j], x_1, \ldots, \widehat{x_i}, \ldots, \widehat{x_j}, \ldots, x_n\right),$$

where $\widehat{x_i}$ means excluding x_i.

Lie algebra cohomology (over $\mathbb{F}_p$) (2)

Note: The cochain complex corresponds to the chain complex $C_\bullet(\mathfrak{g}, \mathbb{F}_p) = \bigwedge^\bullet \mathfrak{g}$, i.e.,

$$\cdots \longrightarrow \bigwedge^3 \mathfrak{g} \xrightarrow{d_3} \bigwedge^2 \mathfrak{g} \xrightarrow{d_2} \mathfrak{g} \xrightarrow{d_1} \mathbb{F}_p \longrightarrow 0,$$

where the boundary maps d_n are given by

$$\begin{aligned} &d_n(x_1 \wedge \cdots \wedge x_n) \\ &= \sum_{i<j} (-1)^{i+j} [x_i, x_j] \wedge x_1 \wedge \cdots \wedge \widehat{x_i} \wedge \cdots \wedge \widehat{x_j} \wedge \cdots \wedge x_n, \end{aligned}$$

where $\widehat{x_i}$ means excluding x_i.

Bigrading of the Lie algebra cohomology

Suppose that $\mathfrak{g} = \mathfrak{g}^1 \oplus \mathfrak{g}^2 \cdots$ is a graded Lie algebra. Then $\bigwedge^n \mathfrak{g}$ is also graded by letting

$$\mathrm{gr}^j\Big(\bigwedge^n \mathfrak{g}\Big) = \bigoplus_{j_1+\cdots+j_n=j} \mathfrak{g}^{j_1} \wedge \cdots \wedge \mathfrak{g}^{j_n}.$$

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Letting $\mathbb{F}_p$ be a $\mathbb{Z}$-graded (concentrated in degree 0) $\mathfrak{g}$-module, we get a grading

$$\mathrm{Hom}_{\mathbb{F}_p}\Big(\bigwedge^n \mathfrak{g}, \mathbb{F}_p\Big) = \bigoplus_{s\in\mathbb{Z}} \mathrm{Hom}^s_{\mathbb{F}_p}\Big(\bigwedge^n \mathfrak{g}, \mathbb{F}_p\Big)$$

where $\mathrm{Hom}^s_{\mathbb{F}_p}$ denotes the homogeneous $\mathbb{F}_p$-linear maps of degree s. This passes to bigrading of Lie algebra cohomology

$$H^{s,t}(\mathfrak{g}, \mathbb{F}_p) = H^{s+t}\Big(\mathrm{gr}^s \mathrm{Hom}_{\mathbb{F}_p}(\bigwedge^{\bullet} \mathfrak{g}, \mathbb{F}_p)\Big).$$

Spectral sequences

A cohomological spectral sequence is a choice of $r_0 \in \mathbb{N}$ and a collection of

- $\mathbb{F}_p$-modules $E_r^{s,t}$ for each $s, t \in \mathbb{Z}$ and all integers $r \geq r_0$
- differentials $d_r^{s,t} \colon E_r^{s,t} \to E_r^{s+r,t+1-r}$ such that $d_r^2 = 0$ and E_{r+1} is (isomorphic to) the homology of (E_r, d_r), i.e.,

$$E_{r+1}^{s,t} = \frac{\ker(d_r^{s,t} \colon E_r^{s,t} \to E_r^{s+r,t+1-r})}{\mathrm{im}(d_r^{s-r,t+r-1} \colon E_r^{s-r,t+r-1} \to E_r^{s,t})}.$$

For a given r, the collection $(E_r^{s,t}, d_r^{s,t})_{s,t\in\mathbb{Z}}$ is called the r-th page.

E_1 (page 1)

$$
\begin{array}{ccccccccc}
\ddots & & \vdots & & \vdots & & \vdots & & \iddots \\
\cdots & \longrightarrow & E_1^{s-1,t+1} & \longrightarrow & E_1^{s,t+1} & \longrightarrow & E_1^{s+1,t+1} & \longrightarrow & \cdots \\
\cdots & \longrightarrow & E_1^{s-1,t} & \longrightarrow & E_1^{s,t} & \longrightarrow & E_1^{s,t} & \longrightarrow & \cdots \\
\cdots & \longrightarrow & E_1^{s-1,t-1} & \longrightarrow & E_1^{s,t-1} & \longrightarrow & E_1^{s,t-1} & \longrightarrow & \cdots \\
\iddots & & \vdots & & \vdots & & \vdots & & \ddots
\end{array}
$$

E_2 (page 2)

$$
\begin{tikzcd}
\ddots & \vdots & \vdots & \vdots & \iddots \\
\cdots & E_2^{s-1,t+1} \arrow[drr] & E_2^{s,t+1} \arrow[drr] & E_2^{s+1,t+1} & \cdots \\
\cdots & E_2^{s-1,t} \arrow[drr] & E_2^{s,t} \arrow[drr] & E_2^{s,t} & \cdots \\
\cdots & E_2^{s-1,t-1} & E_2^{s,t-1} & E_2^{s,t-1} & \cdots \\
\iddots & \vdots & \vdots & \vdots & \ddots
\end{tikzcd}
$$

Convergent spectral sequences

A spectral sequence *converges* if d_r vanishes on $E_r^{s,t}$ for any s, t when $r \gg 0$.
In this case $E_r^{s,t}$ is independent of r for sufficiently large r, we denote it by $E_\infty^{s,t}$ and write

$$E_r^{s,t} \Longrightarrow E_\infty^{s+t}.$$

If we have terms E_∞^n with a natural filtration $F^\bullet E_\infty^n$ (but no natural double grading), we set $E_\infty^{s,t} = \mathrm{gr}^s E_\infty^{s,t} = F^s E_\infty^{s+t} / F^{s+t} E_\infty^{s+t}$.

$E_1^{s,t} = H^{s,t}(\mathfrak{g}, \mathbb{F}_p) \Longrightarrow H^{s+t}(G, \mathbb{F}_p)$

Theorem (Sørensen, 2021)

Let (G, ω) be a p-valuable group and $\mathfrak{g} = \mathbb{F}_p \otimes_{\mathbb{F}_p[\pi]} \operatorname{gr} G$ its Lazard Lie algebra. Then there is a multiplicative spectral sequence collapsing at a finite stage,

$$E_1^{s,t} = H^{s,t}(\mathfrak{g}, \mathbb{F}_p) \Longrightarrow H^{s+t}(G, \mathbb{F}_p).$$

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$$E_1^{s,t} = H^{s,t}(\mathfrak{g}, \mathbb{F}_p) \Longrightarrow H^{s+t}(G, \mathbb{F}_p).$$

I.e., the multiplication on E_∞ is compatible with the cup product on $H^*(G, \mathbb{F}_p)$ in the sense that the following diagram commutes.

$$\begin{tikzcd}
E_\infty^{s,n-s} \otimes E_\infty^{s',n'-s'} \arrow[r] \arrow[d, "\cong"'] & E_\infty^{s+s',n+n'-s-s'} \arrow[d, "\cong"] \\
\operatorname{gr}^s H^n(G,k) \otimes \operatorname{gr}^{s'} H^{n'}(G,k) \arrow[r] & \operatorname{gr}^{s+s'} H^{n+n'}(G,k)
\end{tikzcd}$$

Proof idea:

Theorem (Sørensen, 2021)

There is a convergent spectral sequence collapsing at a finite stage,

$$E_1^{s,t} = HH^{s,t}(\operatorname{gr}\Omega(G), \operatorname{gr} W) \Longrightarrow HH^{s+t}(\Omega(G), W),$$

where W is a finite filtered $\Omega(G)$-bimodule.

$$\begin{aligned} H^*(G, W^{\mathrm{ad}}) \leftrightsquigarrow HH^*(\Omega(G), W) \leftrightsquigarrow HH^*(\operatorname{gr}\Omega(G), \operatorname{gr} W) \\ \cong HH^*(U(\mathfrak{g}), \operatorname{gr} W) \leftrightsquigarrow H^*(\mathfrak{g}, (\operatorname{gr} W)^{\mathrm{ad}}) \end{aligned}$$

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2. On the mod p cohomology of unipotent groups
3. On the mod p cohomology of pro-p Iwahori subgroups
   - $I \subseteq \mathrm{SL}_2(\mathbb{Z}_p)$
   - $I \subseteq \mathrm{GL}_2(\mathbb{Z}_p)$
   - Other calculations
   - Nilpotency index
4. Future work
   - Division quaternion algebras
   - Central division algebras
   - Serre spectral sequence

Main result

Let $N = \mathcal{N}(\mathbb{Z}_p)$ be the $\mathbb{Z}_p$-points of $\mathcal{N}$, where $\mathcal{N}$ is the unipotent radical of a Borel in a split and connected reductive $\mathbb{Z}_p$-group, and let $\mathfrak{n} = \mathrm{Lie}(\mathcal{N}_{\mathbb{F}_p})$. Then (for $p \geq h_{\mathcal{G}} - 1$ an odd prime)

- N is p-valuable with a p-valuation such that $\mathfrak{g} \cong \mathfrak{n}$ (as graded Lie algebras) for $\mathfrak{g} = \mathbb{F}_p \otimes_{\mathbb{F}_p[\pi]} \operatorname{gr} N$,
- $\operatorname{gr}^s H^{s+t}(N, \mathbb{F}_p) \cong H^{s,t}(\mathfrak{g}, \mathbb{F}_p) \cong H^{s,t}(\mathfrak{n}, \mathbb{F}_p)$, and
- the cup product on $H^*(\mathfrak{g}, \mathbb{F}_p)$ is compatible with the cup product on $H^*(N, \mathbb{F}_p)$ in the sense that the following diagram commutes.

$$\begin{tikzcd}
H^{s,n-s}(\mathfrak{n}, \mathbb{F}_p) \otimes H^{s',n'-s'}(\mathfrak{n}, \mathbb{F}_p) \arrow[r] \arrow[d, "\cong"'] & H^{s+s',n+n'-s-s'}(\mathfrak{n}, \mathbb{F}_p) \arrow[d, "\cong"] \\
\operatorname{gr}^s H^n(N, \mathbb{F}_p) \otimes \operatorname{gr}^{s'} H^{n'}(N, \mathbb{F}_p) \arrow[r] & \operatorname{gr}^{s+s'} H^{n+n'}(N, \mathbb{F}_p)
\end{tikzcd}$$

Example

For $\mathcal{N}$ think of the group

$$\mathcal{N} = \left\{ \begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix} \right\} \subseteq \mathrm{GL}_n \text{ or } \mathrm{SL}_n \, .$$

Overview

Pro-p Iwahori subgroups of SL_n and GL_n (1)

Let $\mathcal{G} = \mathrm{SL}_n$ or $\mathcal{G} = \mathrm{GL}_n$ and let h be the Coxeter number of $\mathcal{G}$ (i.e., $h = n$). Let furthermore

- $F/\mathbb{Q}_p$ a finite extension with ramification index $e = e(F/\mathbb{Q}_p)$ and inertia degree $f = f(F/\mathbb{Q}_p)$,
- $p - 1 > eh$,
- $\mathcal{O}_F$ the valuation ring of F with maximal ideal $\mathfrak{m}_F = (\varpi_F)$,
- exp and log the two mutually inverse isomorphisms (and homeomorphisms)

$$\mathfrak{m}_F \underset{\log}{\overset{\exp}{\rightleftarrows}} U_F^{(1)}.$$

Note that exp transfers a $\mathbb{Z}_p$-basis of $\mathfrak{m}_F$ to a $\mathbb{Z}_p$-basis of $U_F^{(1)} = 1 + \mathfrak{m}_F$.

Pro-p Iwahori subgroups of SL_n and GL_n (2)

When $\mathcal{G} = \mathrm{SL}_n$ or $\mathcal{G} = \mathrm{GL}_n$, we can always take the pro-p Iwahori subgroup I of $\mathcal{G}(F)$ to be the subgroup of $\mathcal{G}(\mathcal{O}_F)$ which is upper triangular and unipotent modulo ϖ_F.

Pro-p Iwahori subgroups of SL_n and GL_n (2)

When $\mathcal{G} = \mathrm{SL}_n$ or $\mathcal{G} = \mathrm{GL}_n$, we can always take the pro-p Iwahori subgroup I of $\mathcal{G}(F)$ to be the subgroup of $\mathcal{G}(\mathcal{O}_F)$ which is upper triangular and unipotent modulo ϖ_F.
When $\mathcal{G} = \mathrm{SL}_n$, we have roots $\Phi = \{\varepsilon_i - \varepsilon_j \mid 1 \leq i \neq j \leq n\}$ and can take

$$\Delta = \{\alpha_1 = \varepsilon_1 - \varepsilon_2, \alpha_2 = \varepsilon_2 - \varepsilon_3, \ldots, \alpha_{n-1} = \varepsilon_{n-1} - \varepsilon_n\},$$

where ε_i is the map that takes a diagonal matrix to its i-th diagonal entry. In this case

$$\alpha_i^\vee(u) = \mathrm{diag}(1, \ldots, 1, u, u^{-1}, 1, \ldots, 1) = \mathrm{diag}_{i,i+1}(u).$$

Ordered basis of I in $\mathrm{SL}_n(F)$

Let $\{b_1, \dots, b_\ell\}$ be a $\mathbb{Z}_p$-basis of $\mathcal{O}_F$, where $\ell = [F : \mathbb{Q}_p]$. With a chosen ordering of $\{(i,j) : 1 \leq i,j \leq n\}$, we get for $\mathcal{G} = \mathrm{SL}_n$:

Proposition (Lahiri and Sørensen, 2022)

- $\left(1_n + \varpi_F b_1 E_{ij}, \dots, 1_n + \varpi_F b_\ell E_{ij}\right)_{1 \leq j < i \leq n}$,
- $\left(\mathrm{diag}_{i,i+1}(\exp(\varpi_F b_1)), \dots, \mathrm{diag}_{i,i+1}(\exp(\varpi_F b_\ell))\right)_{i=1,\dots,n-1}$,
- $\left(1_n + b_1 E_{ij}, \dots, 1_n + b_\ell E_{ij}\right)_{1 \leq i < j \leq n}$

is an ordered basis of I.

Here E_{ij} denotes the matrix with 1 in the (i,j)-entry and zeroes in all other entries, and 1_n is the identity matrix in $M_n(F)$.

p-valuation on I

On this basis we have a p-valuation given by:

- $\omega\big(1_n + \varpi_F b_m E_{ij}\big) = \frac{1}{e} + \frac{j-i}{eh}$ for $j < i$,
- $\omega\big(\mathrm{diag}_{i,i+1}(\exp(\varpi_F b_m))\big) = \frac{1}{e}$ for $i = 1, \ldots, n-1$,
- $\omega\big(1_n + b_m E_{ij}\big) = \frac{j-i}{eh}$ for $i < j$.

Ordered basis of I in GL_n

Given the ordered basis of I in SL_n, it is straightforward to obtain an ordered basis of GL_n by simply adding the elements $\left(\exp(\varpi_F b_1)1_n, \ldots, \exp(\varpi_F b_\ell)1_n\right)$ to the middle item in the earlier list (corresponding to adding the root $\varepsilon_1 + \cdots + \varepsilon_n$).
The p-valuation of all of these are clearly also $\frac{1}{e}$.

The ordered basis of $I \subseteq \mathrm{SL}_2(\mathbb{Z}_p)$

Let I be the pro-p Iwahori subgroup of $\mathrm{SL}_2(\mathbb{Q}_p)$, so we can take I of the form

$$I = \begin{pmatrix} 1 + p\mathbb{Z}_p & \mathbb{Z}_p \\ p\mathbb{Z}_p & 1 + p\mathbb{Z}_p \end{pmatrix}^{\det=1} \subseteq \mathrm{SL}_2(\mathbb{Z}_p),$$

and

$$g_1 = \begin{pmatrix} 1 & 0 \\ p & 1 \end{pmatrix}, \quad g_2 = \begin{pmatrix} \exp(p) & 0 \\ 0 & \exp(-p) \end{pmatrix}, \quad g_3 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

is an ordered basis of I.

Commutators $[g_i, g_j]$

Calculating $[g_i, g_j]$ for $i, j = 1, 2, 3$, we can find $x_1, x_2, x_3 \in \mathbb{Z}_p$ such that

$$[g_i, g_j] = g_1^{x_1} g_2^{x_2} g_3^{x_3},$$

and thus

$$\left[\sigma(g_i), \sigma(g_j)\right] = \sigma\left([g_i, g_j]\right) = \sum_{\ell=1}^{d} \overline{x}_\ell \pi^{v_p(x_\ell)} . \sigma(g_\ell).$$

Letting $\{\ell_1, \dots, \ell_r\}$ be the subset of $\{1, \dots, d\}$ such that $v_p(x_{\ell_s}) = 0$ and $v_p(x_\ell) > 0$ for $\ell \notin \{\ell_1, \dots, \ell_r\}$, we get that

$$[\xi_i, \xi_j] = \sum_{s=1}^{r} \overline{x}_{\ell_s} \xi_{\ell_s}.$$

$g_1^{x_1} g_2^{x_2} g_3^{x_3}$

Note that

$$
\begin{aligned}
g_1^{x_1} g_2^{x_2} g_3^{x_3} &= \begin{pmatrix} \exp(px_2) & x_3 \exp(px_2) \\ px_1 \exp(px_2) & px_1 x_3 \exp(px_2) + \exp(-px_2) \end{pmatrix} \\
&= \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.
\end{aligned}
$$

$[g_1, g_2] = g_1^{x_1} g_2^{x_2} g_3^{x_3}$

$$[g_1, g_2] = \begin{pmatrix} 1 & 0 \\ p\big(1 - \exp(-2p)\big) & 1 \end{pmatrix} = g_1^{x_1} g_2^{x_2} g_3^{x_3}$$

implies that $x_2 = x_3 = 0$ and

$$a_{21} = px_1 = p\big(1 - \exp(-2p)\big) = 2p^2 + O(p^3).$$

So $x_1 = 2p + O(p^2)$, and thus $\sigma([g_1, g_2]) = 2\pi \,.\, \sigma(g_1)$, which implies that

$$[\xi_1, \xi_2] = 0.$$

$[g_1, g_3] = g_1^{x_1} g_2^{x_2} g_3^{x_3}$

$$[g_1, g_3] = \begin{pmatrix} 1-p & p \\ -p^2 & 1+p+p^2 \end{pmatrix} = g_1^{x_1} g_2^{x_2} g_3^{x_3}$$

implies that

$$\begin{aligned} a_{11} &= \exp(px_2) = 1-p, \\ a_{12} &= x_3 \exp(px_2) = x_3(1-p) = p, \\ a_{21} &= px_1 \exp(px_2) = px_1(1-p) = -p^2. \end{aligned}$$

So $x_2 = \frac{1}{p}\log(1-p) = \frac{1}{p}\left((-p) + O(p^2)\right) = -1 + O(p)$ and $x_1, x_3 \in p\mathbb{Z}_p$. Hence

$$[\xi_1, \xi_3] = -\xi_2.$$

$[g_2, g_3]$

$$[g_2, g_3] = \begin{pmatrix} 1 & \exp(2p) - 1 \\ 0 & 1 \end{pmatrix} = g_1^{x_1} g_2^{x_2} g_3^{x_3}$$

implies that $x_1 = x_2 = 0$ and $a_{12} = x_3 = \exp(2p) - 1 = 2p + O(p^2)$. So

$$[\xi_1, \xi_2] = 0.$$

Commutators in $\mathfrak{g} = \mathbb{F}_p \otimes_{\mathbb{F}_p[\pi]} \operatorname{gr} I$ and grading on $\mathfrak{g}$

Altogether we have that ξ_1, ξ_2, ξ_3 is a basis of the Lazard Lie algebra $\mathfrak{g} = \mathbb{F}_p \otimes_{\mathbb{F}_p[\pi]} \operatorname{gr} I$ with commutators

$$[\xi_1, \xi_2] = 0, \qquad [\xi_1, \xi_3] = -\xi_2, \qquad [\xi_2, \xi_3] = 0.$$

Commutators in $\mathfrak{g} = \mathbb{F}_p \otimes_{\mathbb{F}_p[\pi]} \operatorname{gr} I$ and grading on $\mathfrak{g}$

Altogether we have that ξ_1, ξ_2, ξ_3 is a basis of the Lazard Lie algebra $\mathfrak{g} = \mathbb{F}_p \otimes_{\mathbb{F}_p[\pi]} \operatorname{gr} I$ with commutators

$$[\xi_1, \xi_2] = 0, \qquad [\xi_1, \xi_3] = -\xi_2, \qquad [\xi_2, \xi_3] = 0.$$

Note: By the general formula for ω on SL_n (with $e = 1$ and $h = 2$), we see that

$$\omega(g_1) = 1 - \frac{1}{2} = \frac{1}{2}, \qquad \omega(g_2) = 1, \qquad \omega(g_3) = \frac{1}{2},$$

so

$$\mathfrak{g}^1 = \mathfrak{g}_{\frac{1}{2}} = \operatorname{span}_{\mathbb{F}_p}(\xi_1, \xi_3), \qquad \mathfrak{g}^2 = \mathfrak{g}_1 = \operatorname{span}_{\mathbb{F}_p}(\xi_2).$$

Grading on $\bigwedge^n \mathfrak{g}$

For $n = 0$: $\mathbb{F}_p = \mathrm{gr}^0 \mathbb{F}_p$ with $\mathbb{F}_p$-basis 1.

For $n = 1$: $\mathfrak{g} = \mathrm{gr}^1 \mathfrak{g} \oplus \mathrm{gr}^2 \mathfrak{g}$, where $\mathrm{gr}^1 \mathfrak{g} = \mathfrak{g}^1$ has basis ξ_1, ξ_3 and $\mathrm{gr}^2 \mathfrak{g} = \mathfrak{g}^2$ has basis ξ_2.

For $n = 2$: $\bigwedge^2 \mathfrak{g} = \mathrm{gr}^2(\bigwedge^2 \mathfrak{g}) \oplus \mathrm{gr}^3(\bigwedge^2 \mathfrak{g})$, where $\mathrm{gr}^2(\bigwedge^2 \mathfrak{g}) = \mathfrak{g}^1 \wedge \mathfrak{g}^1$ has basis $\xi_1 \wedge \xi_3$ and $\mathrm{gr}^3(\bigwedge^2 \mathfrak{g}) = \mathfrak{g}^1 \wedge \mathfrak{g}^2$ has basis $\xi_1 \wedge \xi_2, \xi_3 \wedge \xi_2$.

For $n = 3$: $\bigwedge^3 \mathfrak{g} = \mathrm{gr}^4(\bigwedge^3 \mathfrak{g})$, where $\mathrm{gr}^4(\bigwedge^3 \mathfrak{g}) = \mathfrak{g}^1 \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^2$ has basis $\xi_1 \wedge \xi_3 \wedge \xi_2$.

For $n > 3$: $\bigwedge^n \mathfrak{g} = 0$.

Grading on $\mathrm{Hom}_{\mathbb{F}_p}(\bigwedge^n \mathfrak{g}, \mathbb{F}_p)$

Recall that

$$\mathrm{Hom}_{\mathbb{F}_p}\Big(\bigwedge^n \mathfrak{g}, \mathbb{F}_p\Big) = \bigoplus_{s\in\mathbb{Z}} \mathrm{Hom}^s_{\mathbb{F}_p}\Big(\bigwedge^n \mathfrak{g}, \mathbb{F}_p\Big),$$

and let

$$e_{i_1,\dots,i_n} = (\xi_{i_1} \wedge \cdots \wedge \xi_{i_n})^*$$

be the element of the dual basis of $\mathrm{Hom}_{\mathbb{F}_p}(\bigwedge^n \mathfrak{g}, \mathbb{F}_p)$ corresponding to $\xi_{i_1} \wedge \cdots \wedge \xi_{i_n}$ in the basis of $\bigwedge^n \mathfrak{g}$.

Grading on $\mathrm{Hom}_{\mathbb{F}_p}(\bigwedge^n \mathfrak{g}, \mathbb{F}_p)$

Recall that

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be the element of the dual basis of $\mathrm{Hom}_{\mathbb{F}_p}(\bigwedge^n \mathfrak{g}, \mathbb{F}_p)$ corresponding to $\xi_{i_1} \wedge \cdots \wedge \xi_{i_n}$ in the basis of $\bigwedge^n \mathfrak{g}$.
We can transfer the previous grading and bases to $\mathrm{Hom}_{\mathbb{F}_p}(\bigwedge^n \mathfrak{g}, \mathbb{F}_p)$ using this.

Finding $H^{s,t} = H^{s,t}(\mathfrak{g}, \mathbb{F}_p)$

We will now calculate all maps

$$\mathrm{gr}^j\Big(\bigwedge^n \mathfrak{g}\Big) \to \mathrm{gr}^j\Big(\bigwedge^{n-1} \mathfrak{g}\Big)$$

$$x_1 \wedge \cdots \wedge x_n \mapsto \sum_{i<j} (-1)^{i+j}[x_i, x_j] \wedge x_1 \wedge \cdots \wedge \widehat{x_i} \wedge \cdots \wedge \widehat{x_j} \wedge \cdots \wedge x_n$$

in the chain complex and transfer them to the cochain complex

$$\mathrm{Hom}^s_{\mathbb{F}_p}\Big(\bigwedge^{n-1} \mathfrak{g}, \mathbb{F}_p\Big) \to \mathrm{Hom}^s_{\mathbb{F}_p}\Big(\bigwedge^{n} \mathfrak{g}, \mathbb{F}_p\Big).$$

$\mathrm{gr}^0 H^n(\mathfrak{g}, \mathbb{F}_p)$

In grade 0 we have the chain complex

$$0 \longrightarrow \mathbb{F}_p \longrightarrow 0,$$

which gives us the grade 0 cochain complex

$$0 \longleftarrow \mathrm{Hom}^0_{\mathbb{F}_p}(\mathbb{F}_p, \mathbb{F}_p) \longleftarrow 0.$$

So $H^0 = H^{0,0}$ with $\dim H^{0,0} = 1$.

$\mathrm{gr}^{-1} H^n(\mathfrak{g}, \mathbb{F}_p)$

In grade 1 we have the chain complex

$$0 \longrightarrow \mathfrak{g}^1 \longrightarrow 0,$$

which gives us the grade -1 cochain complex

$$0 \longleftarrow \mathrm{Hom}_{\mathbb{F}_p}^{-1}(\mathfrak{g}, \mathbb{F}_p) \longleftarrow 0.$$

So $\dim H^{-1,2} = 2$ and $H^{-1,2} = \mathbb{F}_p[e_1, e_3]$.

$\mathrm{gr}^{-2} H^n(\mathfrak{g}, \mathbb{F}_p)$ (1)

In grade 2 we have the chain complex

$$0 \longrightarrow \mathfrak{g}^1 \wedge \mathfrak{g}^1 \xrightarrow{(1)} \mathfrak{g}^2 \longrightarrow 0,$$

since

$$\begin{aligned} \mathfrak{g}^1 \wedge \mathfrak{g}^1 &\to \mathfrak{g}^2 \\ \xi_1 \wedge \xi_3 &\mapsto -[\xi_1, \xi_3] = \xi_2. \end{aligned}$$

$\mathrm{gr}^{-2} H^n(\mathfrak{g}, \mathbb{F}_p)$ (2)

This gives us the grade -2 cochain complex

$$0 \longleftarrow \mathrm{Hom}_{\mathbb{F}_p}^{-2}\left(\bigwedge^2 \mathfrak{g}, \mathbb{F}_p\right) \xleftarrow{(1)} \mathrm{Hom}_{\mathbb{F}_p}^{-2}(\mathfrak{g}, \mathbb{F}_p) \longleftarrow 0.$$

So

$$\begin{aligned} \dim H^{-2,3} &= \dim \ker((1)) = 0, \\ \dim H^{-2,4} &= \dim \mathrm{coker}((1)) = 0. \end{aligned}$$

$\mathrm{gr}^{-3} H^n(\mathfrak{g}, \mathbb{F}_p)$

In grade 3 we have the chain complex

$$0 \longrightarrow \mathfrak{g}^1 \wedge \mathfrak{g}^2 \longrightarrow 0,$$

which gives us the grade -3 cochain complex

$$0 \longleftarrow \mathrm{Hom}_{\mathbb{F}_p}^{-3}\left(\bigwedge^2 \mathfrak{g}, \mathbb{F}_p\right) \longleftarrow 0.$$

So $\dim H^{-3,5} = 2$ and $H^{-3,5} = \mathbb{F}_p[e_{1,2}, e_{3,2}]$.

$\mathrm{gr}^{-4} H^n(\mathfrak{g}, \mathbb{F}_p)$

In grade 4 we have the chain complex

$$0 \longrightarrow \mathfrak{g}^1 \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^2 \longrightarrow 0,$$

which gives us the grade -4 cochain complex

$$0 \longleftarrow \mathrm{Hom}_k^{-4}\left(\bigwedge^3 \mathfrak{g}, k\right) \longleftarrow 0.$$

So $\dim H^{-4,7} = 1$ and $H^{-4,7} = \mathbb{F}_p[e_{1,3,2}]$.

$H^*(\mathfrak{g}, \mathbb{F}_p)$

Altogether we see that

$$\begin{aligned} H^0 &= H^{0,0} = \mathbb{F}_p, \\ H^1 &= H^{-1,2} = \mathbb{F}_p[e_1, e_3], \\ H^2 &= H^{-3,5} = \mathbb{F}_p[e_{1,2}, e_{3,2}], \\ H^3 &= H^{-4,7} = \mathbb{F}_p[e_{1,3,2}], \end{aligned}$$

with dimensions $1, 2, 2, 1$.

$E_1^{s,t} = H^{s,t}(\mathfrak{g}, \mathbb{F}_p) \Longrightarrow H^{s+t}(I, \mathbb{F}_p)$

$t \backslash s$	0	−1	−2	−3	−4
0	1				
1					
2		2			
3					
4					
5				2	
6					
7					1

Recall that all differential $d_r^{s,t}\colon E_r^{s,t} \to E_r^{s+r,t+1-r}$ has bidegree $(r, 1-r)$, i.e., they are all below the $(r, -r)$ arrow going r to the left and r up in the table to the left, where $r \geq 1$.

This means that all differentials for $r \geq 1$ are trivial, so the spectral sequence collapses on the first page.

$H^n(I, \mathbb{F}_p)$ dimensions

Hence $H^{s,t}(\mathfrak{g}, \mathbb{F}_p) = E_1^{s,t} \cong E_\infty^{s,t} = \mathrm{gr}^s H^{s+t}(I, \mathbb{F}_p)$, and we get that

$$\dim H^n(I, \mathbb{F}_p) = \begin{cases} 1 & n = 0, \\ 2 & n = 1, \\ 2 & n = 2, \\ 1 & n = 3. \end{cases}$$

$H^n(I, \mathbb{F}_p)$ dimensions

Hence $H^{s,t}(\mathfrak{g}, \mathbb{F}_p) = E_1^{s,t} \cong E_\infty^{s,t} = \mathrm{gr}^s H^{s+t}(I, \mathbb{F}_p)$, and we get that

$$\dim H^n(I, \mathbb{F}_p) = \begin{cases} 1 & n = 0, \\ 2 & n = 1, \\ 2 & n = 2, \\ 1 & n = 3. \end{cases}$$

Furthermore $H^{s,t} \cup H^{s',t'} \subseteq H^{s+s',t+t'}$ by a result of Fuks, so the cup products

$$\mathrm{gr}^s H^n(I, \mathbb{F}_p) \otimes \mathrm{gr}^{s'} H^{n'}(I, \mathbb{F}_p) \to \mathrm{gr}^{s+s'} H^{n+n'}(I, \mathbb{F}_p)$$

are trivial, except for $H^1(I, \mathbb{F}_p) \otimes H^2(I, \mathbb{F}_p) \to H^3(I, \mathbb{F}_p)$.

Cup product in Lie algebra cohomology

For $f \in \mathrm{Hom}_{\mathbb{F}_p}(\bigwedge^p \mathfrak{g}, \mathbb{F}_p)$ and $g \in \mathrm{Hom}_{\mathbb{F}_p}(\bigwedge^q \mathfrak{g}, \mathbb{F}_p)$, we recall that the cup product in cohomology is induced by:
$f \cup g \in \mathrm{Hom}_{\mathbb{F}_p}(\bigwedge^{p+q} \mathfrak{g}, \mathbb{F}_p)$ defined by

$$(f \cup g)(x_1 \wedge \cdots \wedge x_{p+q}) \\ = \sum_{\substack{\sigma \in S_{p+q} \\ \sigma(1)<\cdots<\sigma(p) \\ \sigma(p+1)<\cdots<\sigma(p+q)}} \mathrm{sign}(\sigma) f(x_{\sigma(1)} \wedge \cdots \wedge x_{\sigma(p)}) g(x_{\sigma(p+1)} \wedge \cdots \wedge x_{\sigma(p+q)}).$$

When finding $\cup\colon H^1 \otimes H^2 \to H^3$, where $H^1 = \mathbb{F}_p[e_1, e_3]$, $H^2 = \mathbb{F}_p[e_{1,2}, e_{3,2}]$ and $H^3 = \mathbb{F}_p[e_{1,3,2}]$, we need to calculate $e_i \cup e_{j,k}$ on the basis $\xi_1 \wedge \xi_3 \wedge \xi_2$ of $\mathrm{gr}^4(\bigwedge^3 \mathfrak{g})$.

Cup product on $H^*(\mathfrak{g}, \mathbb{F}_p)$

In this case, the sum simplifies to

$$\begin{aligned}&(e_i \cup e_{j,k})(x_1 \wedge x_2 \wedge x_3)\\ &= \sum_{\substack{\sigma \in S_3 \\ \sigma(2) < \sigma(3)}} \mathrm{sign}(\sigma) e_i(x_{\sigma(1)}) e_{j,k}(x_{\sigma(2)} \wedge x_{\sigma(3)}).\end{aligned}$$

The terms on the right are only non-zero if $x_{\sigma(1)} = \xi_i$ and $x_{\sigma(2)} \wedge x_{\sigma(3)} = \xi_j \wedge \xi_k$ (up to constants).

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The terms on the right are only non-zero if $x_{\sigma(1)} = \xi_i$ and $x_{\sigma(2)} \wedge x_{\sigma(3)} = \xi_j \wedge \xi_k$ (up to constants).

When $x_1 \wedge x_2 \wedge x_3 = \xi_1 \wedge \xi_3 \wedge \xi_2$, we see that $e_1 \cup e_{3,2} = e_{1,3,2}$ (with $\sigma = (1)$) and $e_3 \cup e_{1,2} = -e_{1,3,2}$ (with $\sigma = (1, 2)$). This translates to a cup product on $H^*(I, \mathbb{F}_p)$.

$H^*\left((1+\mathfrak{m}_D)^{\mathrm{Nrd}=1}, \mathbb{F}_p\right)$ for D a div. quat. alg. over $\mathbb{Q}_p$

Let D be the division quaternion algebra over $\mathbb{Q}_p$ for a prime $p > 3$ and let $G = (1+\mathfrak{m}_D)^{\mathrm{Nrd}=1}$, where $\mathrm{Nrd} = \mathrm{Nrd}_{D/\mathbb{Q}_p}$ is the norm form. Sørensen and Henn have shown that

$$H^*(G, \mathbb{F}_p) \cong \mathbb{F}_p \oplus \mathbb{F}_D \oplus \mathbb{F}_D \oplus \mathbb{F}_p$$

of graded $\mathbb{F}_p$-algebras (where $\mathbb{F}_D \cong \mathbb{F}_{p^2}$). The only non-trivial cup product is $H^1(G, \mathbb{F}_p) \times H^2(G, \mathbb{F}_p) \to H^3(G, \mathbb{F}_p)$, which corresponds to the trace pairing

$$\mathbb{F}_D \times \mathbb{F}_D \to \mathbb{F}_p, \quad (x, y) \mapsto \mathrm{Tr}(xy).$$

Comparing $H^*(I, \mathbb{F}_p)$ and $H^*\left((1+\mathfrak{m}_D)^{\mathrm{Nrd}=1}, \mathbb{F}_p\right)$

When $p \equiv 3 \pmod 4$, we can write $\mathbb{F}_D = \mathbb{F}_p[\alpha]$ with $\alpha^2 = -1$, and see that

$$\mathrm{Tr}(1) = 2, \qquad \mathrm{Tr}(\alpha) = 0, \qquad \mathrm{Tr}(\alpha^2) = -2.$$

Comparing $H^*(I, \mathbb{F}_p)$ and $H^*\left((1+\mathfrak{m}_D)^{\mathrm{Nrd}=1}, \mathbb{F}_p\right)$

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$$\mathrm{Tr}(1) = 2, \qquad \mathrm{Tr}(\alpha) = 0, \qquad \mathrm{Tr}(\alpha^2) = -2.$$

Hence

$$H^*(I,\ \mathbb{F}_p) \cong H^*\left((1+\mathfrak{m}_D)^{\mathrm{Nrd}=1}, \mathbb{F}_p\right).$$

Note: We even have that $\mathfrak{g} \cong \mathfrak{g}_D$, where $\mathfrak{g}_D = \mathbb{F}_p \otimes_{\mathbb{F}_p[\pi]} \mathrm{gr}\, G$.

Question

Does this result generalize?

Sidenote: I is not uniformly powerful (1)

Definition

Let G be a finitely generated pro-p group. The *lower p-series* $\cdots \geq P_3(G) \geq P_2(G) \geq P_1(G)$ of G is given by $P_i(G)$, where $P_1(G) = G$ and

$$P_{i+1}(G) = P_i(G)^p \left[P_i(G), G\right]$$

for $i \geq 1$.

Sidenote: I is not uniformly powerful (2)

Definition

Let p be an odd prime. A pro-p group G is *uniformly powerful* (often written as *uniform*) if

(i) G is finitely generated,

(ii) G is *powerful*, i.e., $G/\overline{G^p}$ is abelian, and

(iii) for all i, $[P_i(G) : P_{i+1}(G)] = [G : P_2(G)]$.

Sidenote: I is not uniformly powerful (3)

Theorem (K, 2022)

Let I be the pro-p Iwahori subgroup of $\mathrm{SL}_2(\mathbb{Q}_p)$ and let g_1, g_2, g_3 be the ordered basis of I. Then the lower p-series is given by

$$P_i(I) = \begin{cases} I^{p^n} & \text{if } i = 2n+1, \\ [I, I]^{p^{n-1}} & \text{if } i = 2n, \end{cases}$$

where $P_{2n+1}(I) = I^{p^n}$ is the subgroup generated by $g_1^{p^n}, g_2^{p^n}, g_3^{p^n}$ and $P_{2n}(I) = [I, I]^{p^{n-1}}$ is the subgroup generated by $g_1^{p^n}, g_2^{p^{n-1}}, g_3^{p^n}$. Thus I is not uniformly powerful, since

$$[P_i(G) : P_{i+1}(G)] = \begin{cases} 1 & \text{if } i = 2n, \\ 2 & \text{if } i = 2n+1. \end{cases}$$

The ordered basis of $I \subseteq \mathrm{GL}_2(\mathbb{Z}_p)$

Let I be the pro-p Iwahori subgroup of $\mathrm{GL}_2(\mathbb{Q}_p)$, so we can take I of the form

$$I = \begin{pmatrix} 1 + p\mathbb{Z}_p & \mathbb{Z}_p \\ p\mathbb{Z}_p & 1 + p\mathbb{Z}_p \end{pmatrix} \subseteq \mathrm{GL}_2(\mathbb{Z}_p),$$

and

$$g_1 = \begin{pmatrix} 1 & 0 \\ p & 1 \end{pmatrix}, \qquad g_2 = \begin{pmatrix} \exp(p) & 0 \\ 0 & \exp(-p) \end{pmatrix},$$

$$g_3 = \begin{pmatrix} \exp(p) & 0 \\ 0 & \exp(p) \end{pmatrix}, \qquad g_4 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

is an ordered basis of I.

$E_1^{s,t} = H^{s,t}(\mathfrak{g}, \mathbb{F}_p) \Longrightarrow H^{s+t}(I, \mathbb{F}_p)$

Let $\mathfrak{g} = \mathbb{F}_p \otimes_{\mathbb{F}_p[\pi]} \operatorname{gr} I$ be the Lazard Lie algebra of I, and note that it has basis $\xi_1, \xi_2, \xi_3, \xi_4$ with $[\xi_1, \xi_4] = -\xi_2$ the only non-zero commutator.

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An argument similar to the $I \subseteq \mathrm{SL}_2(\mathbb{Z}_p)$ case allows us to find $E_1^{s,t} = H^{s,t}(\mathfrak{g}, \mathbb{F}_p)$ and show that

$$E_1^{s,t} = H^{s,t}(\mathfrak{g}, \mathbb{F}_p) \Longrightarrow H^{s+t}(I, \mathbb{F}_p)$$

collapses at the first page.

$H^n(I, \mathbb{F}_p)$ dimensions

We get that

$$\dim H^n(I, \mathbb{F}_p) = \begin{cases} 1 & n = 0, \\ 3 & n = 1, \\ 4 & n = 2, \\ 3 & n = 3, \\ 1 & n = 4. \end{cases}$$

This time the non-trivial cup products are

$$H^1 \otimes H^1 \to H^2, \qquad H^1 \otimes H^2 \to H^3,$$
$$H^1 \otimes H^3 \to H^4, \qquad H^2 \otimes H^2 \to H^4.$$

Compairing $H^*(I, \mathbb{F}_p)$ with $H^*(1 + \mathfrak{m}_D, \mathbb{F}_p)$

By explicit calculation, we can check that

$$H^*(I, \mathbb{F}_p) \cong H^*(1 + \mathfrak{m}_D, \mathbb{F}_p).$$

Proposition (Sørensen, 2021)

$H^*(1 + \mathfrak{m}_D, \mathbb{F}_p) \cong H^*\left((1 + \mathfrak{m}_D)^{\mathrm{Nrd}=1}, \mathbb{F}_p\right) \otimes_{\mathbb{F}_p} \mathbb{F}_p[\varepsilon]$ (where $\varepsilon^2 = 0$).

Thus

$$H^*(I_{\mathrm{GL}_2(\mathbb{Q}_p)}, \mathbb{F}_p) \cong H^*(I_{\mathrm{SL}_2(\mathbb{Q}_p)}, \mathbb{F}_p) \otimes_{\mathbb{F}_p} \mathbb{F}_p[\varepsilon].$$

Other pro-p Iwahori cohomology calculations

We can similarly find the mod p cohomology of the pro-p Iwahori I_G for $G =$

- $\mathrm{SL}_3(\mathbb{Q}_p)$,
- $\mathrm{GL}_3(\mathbb{Q}_p)$,
- $\mathrm{SL}_4(\mathbb{Q}_p)$ (partially),
- $\mathrm{GL}_4(\mathbb{Q}_p)$ (partially),
- $\mathrm{SL}_2(F)$ (for $F/\mathbb{Q}_p$ quadratic),
- $\mathrm{GL}_2(F)$ (for $F/\mathbb{Q}_p$ quadratic).

Consequences: Nilpotency index of mod p cohomology (1)

Given any (suitable) cohomology theory H (say over $\mathbb{F}_p$), we can think of the ring H^* with the cup product $H^* = \mathbb{F}_p \oplus H^+$, where $\mathbb{F}_p = H^0$ and $H^+ = \oplus_{n>0} H^n$.
Assuming that only finitely many H^n are non-zero and that each H^n is finite dimensional, one can note that H^+ must be nilpotent.

Question

What is the smallest positive integer m such that $(H^+)^m = 0$?

Question

What is the smallest positive integer m such that $(H^1)^m = 0$?

Consequences: Nilpotency index of mod p cohomology (2)

n	2	3	4
$I \subseteq \mathrm{SL}_n(\mathbb{Z}_p)$, H^1	$\mathbf{2}$	$\mathbf{2}$	3
$I \subseteq \mathrm{SL}_n(\mathbb{Z}_p)$, H^+	$\mathbf{3}$	5	8
$I \subseteq \mathrm{GL}_n(\mathbb{Z}_p)$, H^1	$\mathbf{3}$	4	5
$I \subseteq \mathrm{GL}_n(\mathbb{Z}_p)$, H^+	$\mathbf{4}$	7	11
$I \subseteq \mathrm{SL}_n(\mathcal{O}_F)$ (quadratic), H^1	3		
$I \subseteq \mathrm{SL}_n(\mathcal{O}_F)$ (quadratic), H^+	4		
$I \subseteq \mathrm{GL}_n(\mathcal{O}_F)$ (quadratic), H^1	5		
$I \subseteq \mathrm{GL}_n(\mathcal{O}_F)$ (quadratic), H^+	7		

Overview

1. Introduction
   - Cohomology of compact Lie groups
   - Lazard Theory
   - $E_1^{s,t} = H^{s,t}(\mathfrak{g}, \mathbb{F}_p) \Longrightarrow H^{s+t}(G, \mathbb{F}_p)$
2. On the mod p cohomology of unipotent groups
3. On the mod p cohomology of pro-p Iwahori subgroups
   - $I \subseteq \mathrm{SL}_2(\mathbb{Z}_p)$
   - $I \subseteq \mathrm{GL}_2(\mathbb{Z}_p)$
   - Other calculations
   - Nilpotency index
4. Future work
   - Division quaternion algebras
   - Central division algebras
   - Serre spectral sequence

Division quaternion algebras

Let $p > 5$ be a prime such that $p \equiv 3 \pmod 4$. Let D be a division quaternion algebra over $\mathbb{Q}_p$ and note that we can assume that $i^2 = -1$ and $j^2 = p$. Also $\mathcal{O}_D = \mathbb{Z}_p[i,j,k]$ (where $k = ij$) and $\mathfrak{m}_D = j\mathcal{O}_D = \mathcal{O}_D j$ has $\mathbb{Z}_p$-basis p, pi, j, k (by Voight's book). $D \subseteq M_2(\mathbb{Q}_p(i))$ gives the right diagram.

$$\begin{tikzcd}
(1+\mathfrak{m}_D)^{\mathrm{Nrd}=1} \arrow[r, hook] \arrow[d, hook] & I_{\mathrm{SL}_2(\mathbb{Q}_p(i))} \arrow[d, hook] & I_{\mathrm{SL}_2(\mathbb{Q}_p)} \arrow[l, hook'] \arrow[d, hook] \\
(\mathcal{O}_D^\times)^{\mathrm{Nrd}=1} \arrow[r, hook] \arrow[d, hook] & \mathrm{SL}_2(\mathbb{Z}_p[i]) \arrow[d, hook] & \mathrm{SL}_2(\mathbb{Z}_p) \arrow[l, hook'] \arrow[d, hook] \\
(D^\times)^{\mathrm{Nrd}=1} \arrow[r, hook] \arrow[d, hook] & \mathrm{SL}_2(\mathbb{Q}_p(i)) \arrow[d, hook] & \mathrm{SL}_2(\mathbb{Q}_p) \arrow[l, hook'] \arrow[d, hook] \\
D^\times \arrow[r, hook] & \mathrm{GL}_2(\mathbb{Q}_p(i)) & \mathrm{GL}_2(\mathbb{Q}_p) \arrow[l, hook'] \\
\mathcal{O}_D^\times \arrow[r, hook] \arrow[u, hook] & \mathrm{GL}_2(\mathbb{Z}_p[i]) \arrow[u, hook] & \mathrm{GL}_2(\mathbb{Z}_p) \arrow[l, hook'] \arrow[u, hook] \\
1+\mathfrak{m}_D \arrow[r, hook] \arrow[u, hook] & I_{\mathrm{GL}_2(\mathbb{Q}_p(i))} \arrow[u, hook] & I_{\mathrm{GL}_2(\mathbb{Q}_p)}. \arrow[l, hook'] \arrow[u, hook]
\end{tikzcd}$$

Central division algebras

Conjecture

Let D be the central division algebra over $\mathbb{Q}_p$ of dimension n^2 and invariant $\frac{1}{n}$. Let $\mathcal{O}_D$ be the maximal compact (local) subring of D with maximal ideal $\mathfrak{m}_D$ and residue field $\mathbb{F}_D \cong \mathbb{F}_{p^n}$. If $p > n+1$ then

- $H^*(I_{\mathrm{GL}_n(\mathbb{Q}_p)}, \mathbb{F}_p) \cong H^*(1+\mathfrak{m}_D, \mathbb{F}_p)$ as graded algebras, and
- $H^*(I_{\mathrm{SL}_n(\mathbb{Q}_p)}, \mathbb{F}_p) \cong H^*\left((1+\mathfrak{m}_D)^{\mathrm{Nrd}=1}, \mathbb{F}_p\right)$ as graded algebras.

In particular (due to Sørensen)

$$H^*(I_{\mathrm{GL}_n(\mathbb{Q}_p)}, \mathbb{F}_p) \cong H^*(I_{\mathrm{SL}_n(\mathbb{Q}_p)}, \mathbb{F}_p) \otimes_{\mathbb{F}_p} \mathbb{F}_p[\varepsilon]$$

as graded algebras, where $\varepsilon^2 = 0$.

Serre spectral sequence

Assume we have the "standard" setup over $\mathbb{Q}_p$ with $\mathcal{G} = \mathrm{SL}_n$, $\mathcal{U}$ unipotent upper triangular matrices and $\mathcal{T}$ diagonal matrices with determinant 1. Let

$$\begin{aligned} I &= \{g \in \mathcal{G}(\mathbb{Z}_p) : \mathrm{red}(g) \in \mathcal{U}(\mathbb{F}_p)\} \text{ (pro-}p\text{ Iwahori)}, \\ K &= \ker\big(\mathrm{red}\colon \mathcal{G}(\mathbb{Z}_p) \to \mathcal{G}(\mathbb{F}_p)\big) \lhd I. \end{aligned}$$

Then $I/K \cong \mathcal{U}(\mathbb{F}_p)$, and thus we get the (multiplicative) Serre spectral sequence

$$E_2^{s,t} = H^s\big(\mathcal{U}(\mathbb{F}_p), H^t(K, \mathbb{F}_p)\big) \Longrightarrow H^{s+t}(I, \mathbb{F}_p).$$

Since K is uniformly powerful, we know by Lazard that

$$H^t(K, \mathbb{F}_p) \cong \bigwedge^t \mathrm{Hom}_{\mathbb{F}_p}(K, \mathbb{F}_p).$$

Thank you