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Chapter 1

Cohomology of Unipotent Groups

cha:cohunigps

1.1 Introduction

sec:cohunigps-intro

In this chapter we show that the cohomology of certain unipotent groups can be found via a simpler cohomology calculation for related Lie algebras. This is done using a spectral sequence due to [Sør].

1.1.1 Background and motivation

The cohomology of Lie groups has a long history. In particular, the mod p cohomology of a connected compact real Lie group has been well understood by Kac since the eighties, and the mod p cohomology $H^*(G, \mathbb{F}_p)$ of a equi-p-valued compact p-adic Lie group G was already described by Lazard in the sixties. This chapters work will build on several ideas of Lazard and Serre in their more general (but not yet finished) description of the case when G is not equi-p-valued, but we will focus only on unipotent groups originating from split and connected reductive $\mathbb{Z}_p$-groups, which is similar to recent work in the case of $\mathbb{Z}_p$ coefficients by Ronchetti.

It's worth noting that this work started out as an attempt to better understand the proof of [Gro14, Theorem 7.1], in particular the part using the result of

Grünenfelder, but has since develop in a different direction, where the coefficients are more restricted, but we obtain a more precise description.

DK Note: Write more background and motivation later.

1.1.2 Notation and setup

Let p be an odd prime.

Algebraic groups. We will work with schemes using the functorial approach and notation described in [Jan03]. In particular, given an integral domain R, we note that a *R-group functor* is a functor from the category of all R-algebras to the category of groups, a *R-group scheme* is a R-group functor that is an affine scheme over R when considered as a R-functor, and an *algebraic R-group* is a R-group scheme that is algebraic as an affine scheme. For more in depth introduction to these concepts, we refer to [Con14b] and [Jan03].

Base change. If R' is a R-algebra, then any R'-algebra A is in a natural way a R-algebra by combining the structural homomorphisms $R \to R'$ and $R' \to A$. We can therefore associate to each R-functor X a R'-functor $X_{R'}$ by $X_{R'}(A) = X(A)$ for any R'-algebra A. For any morphism $f\colon X \to X'$ of R-functors, we get a morphism $f_{R'}\colon X_{R'} \to X'_{R'}$ of k'-functors by $f_{R'}(A) = f(A)$ for any R'-algebra A. In this way we get a functor $X \mapsto X_{R'}$, $f \mapsto f_{R'}$ from the category of R-functors to the category of R'-functors, which we call the *base change* from R to R'.

Fixed $\mathbb{Z}_p$-groups and roots. We fix a split and connected reductive algebraic $\mathbb{Z}_p$-group $\mathcal{G}$ as well as a split maximal torus $\mathcal{T} \subseteq \mathcal{G}$. Let $\Phi = \Phi(\mathcal{G}, \mathcal{T})$ be the *root system* of $\mathcal{G}$ with respect to $\mathcal{T}$. For any $\alpha \in \Phi$ we have the root subgroup $\mathcal{N}_\alpha \subseteq \mathcal{G}$ with Lie algebra $\operatorname{Lie}\mathcal{N}_\alpha = (\operatorname{Lie}\mathcal{G})_\alpha$. We fix a $\mathbb{Z}_p$-basis $(X_\alpha)_{\alpha\in\Phi}$ of $\operatorname{Lie}\mathcal{N}_\alpha$, and note that this choice gives rise to unique isomorphisms of group schemes $x_\alpha\colon \mathbb{G}_a \xrightarrow{\cong} \mathcal{N}_\alpha$ such that $(dx_\alpha)(1) = X_\alpha$. We furthermore fix a basis $\Delta \subseteq \Phi$ of the root system, so we get a decomposition $\Phi = \Phi^+ \cup \Phi^-$ into positive and negative roots. Let $\mathcal{B} = \mathcal{T}\mathcal{N}$ and $\mathcal{B}^+ = \mathcal{T}\mathcal{N}^+$ denote the Borel subgroups of $\mathcal{G}$ corresponding to Φ^- and Φ^+, respectively, with unipotent radicals $\mathcal{N}$ and $\mathcal{N}^+$. Finally let $N = \mathcal{N}(\mathbb{Z}_p)$ and let $\mathfrak{n} = \mathrm{Lie}(\mathcal{N}_{\mathbb{F}_p})$ be the Lie algebra of $\mathcal{N}_{\mathbb{F}_p}$ over $\mathbb{F}_p$.

$\mathbb{Z}$-models. Let $\mathcal{G}_\mathbb{Z}$ be the Chevalley group over $\mathbb{Z}$ corresponding to $\mathcal{G}$ (cf. [Con14a, §1]), and consider the subgroups $\mathcal{T}_\mathbb{Z}, \mathcal{B}_\mathbb{Z}, \mathcal{N}_\mathbb{Z}$ corresponding to $\mathcal{T}, \mathcal{B}, \mathcal{N}$. Let furthermore $\mathfrak{n}_\mathbb{Z} = \mathrm{Lie}(\mathcal{N}_\mathbb{Z})$ be the Lie algebra of $\mathcal{N}_\mathbb{Z}$ over $\mathbb{Z}$, and note that $N = \mathcal{N}_\mathbb{Z}(\mathbb{Z}_p)$ and $\mathfrak{n} = \mathfrak{n}_\mathbb{Z} \otimes \mathbb{F}_p$. (Note also that $(\mathcal{G}_\mathbb{Z})_{\mathbb{Z}_p} = \mathcal{G}$, so although we abuse notation a bit here, it wont be a problem.)

Total ordering of Φ^-. For any total ordering of Φ^- the multiplication induces an isomorphism of schemes $\prod_{\alpha \in \Phi^-} \mathcal{N}_\alpha \xrightarrow{\cong} \mathcal{N}$. For convenience we fix a total ordering which has the additional property that $\alpha_1 \geq \alpha_2$ if $\mathrm{ht}(\alpha_1) \leq \mathrm{ht}(\alpha_2)$. All products indexed by Φ^- are meant to be taken according to this ordering. Here we have the height function $\mathrm{ht}\colon \mathbb{Z}[\Delta] \to \mathbb{Z}$ given by $\sum_{\alpha\in\Delta} m_\alpha \alpha \mapsto \sum_{\alpha\in\Delta} m_\alpha$. In particular, since $\Phi \subseteq \mathbb{Z}[\Delta]$ the height $\mathrm{ht}(\beta)$ of any root $\beta \in \Phi$ is defined.

Coxeter number and p. Let h be the Coxeter number of $\mathcal{G}$ and assume from now on that $p \geq h - 1$.

Weyl group and module. Let $\Phi^\vee$ be the dual root system of Φ and let W be the corresponding Weyl group with length function ℓ on W. Let furthermore $X = X(\mathcal{T}) \cong X(\mathcal{T}_\mathbb{Z})$ be the character group of $\mathcal{T}$, and set

$$X^+ = \{\lambda \in X \mid \langle \lambda, \alpha^\vee \rangle \geq 0 \text{ for all } \alpha \in \Phi^+\}.$$

For any $\lambda \in X^+$, let $V_\mathbb{Z}(\lambda)$ be the Weyl module for $\mathcal{G}_\mathbb{Z}$ over $\mathbb{Z}$ with highest weight λ, and let $V_{\mathbb{F}_p}(\lambda) = V_\mathbb{Z}(\lambda) \otimes_\mathbb{Z} \mathbb{F}_p$.

Lazard theory. We will introduce concepts from Lazard theory in next subsection, but we note now that we will let $\mathfrak{g} = \mathbb{F}_p \otimes_{\mathbb{F}_p[\pi]} \operatorname{gr} N$ be the Lazard Lie algebra corresponding to N.

Cohomology. For any ring R, we denote (using the Chevalley-Eilenberg complex) the Lie algebra cohomology of any R-Lie algebra $\mathfrak{g}$ by $H^\bullet(\mathfrak{g}, \,\cdot\,)$, while we write $H^\bullet_{\mathrm{dsc}}(G, \,\cdot\,)$ and $H^\bullet_{\mathrm{cts}}(H, \,\cdot\,)$ for the discrete (resp. continuous) group cohomology of a topological group G. Later we will introduce filtrations and then gradings on the cohomology, in which case we always use the notation $H^{s,t} = \operatorname{gr}^s H^{s+t}$ for any type of cohomology H.

Spectral sequences. Given a ring R, a cohomological spectral sequence is a choice of $r_0 \in \mathbb{N}$ and a collection of

- R-modules $E_r^{s,t}$ for each $s, t \in \mathbb{Z}$ and all integers $r \geq r_0$
- differentials $d_r^{s,t} \colon E_r^{s,t} \to E_r^{s+r,t+1-r}$ such that $d_r^2 = 0$ and E_{r+1} is isomorphic to the homology of (E_r, d_r), i.e.,

$$E_{r+1}^{s,t} = \frac{\ker(d_r^{s,t} \colon E_r^{s,t} \to E_r^{s+r,t+1-r})}{\operatorname{im}(d_r^{s-r,t+r-1} \colon E_r^{s-r,t+r-1} \to E_r^{s,t})}.$$

For a given r, the collection $(E_r^{s,t}, d_r^{s,t})_{s,t\in\mathbb{Z}}$ is called the r-th page. A spectral sequence *converges* if d_r vanishes on $E_r^{s,t}$ for any s, t when $r \gg 0$. In this case $E_r^{s,t}$ is independent of r for sufficiently large r, we denote it by $E_\infty^{s,t}$ and write

$$E_r^{s,t} \Longrightarrow E_\infty^{s+t}.$$

Also, we say that the spectral sequence collapses at the r'-th page if $E_r = E_\infty$ for all $r \geq r'$, but not for $r < r'$. Finally, when we have terms E_∞^n with a natural filtration $F^\bullet E_\infty^n$ (but no natural double grading), we set $E_\infty^{s,t} = \operatorname{gr}^s E_\infty^{s,t} = F^s E_\infty^{s+t} / F^{s+1} E_\infty^{s+t}$.

subsec:Laz-theory

1.1.3 Lazard theory

In this subsection we will briefly introduce elements of Lazard theory as presented in [Sch11a].

Let G be any abstract group and let the commutator be normalized to as $[g,h] = ghg^{-1}h^{-1}$.

Definition 1.1. A *p-valuation* ω on G is a real valued function

$$\omega\colon G \setminus \{1\} \to (0, \infty)$$

which, with the convention that $\omega(1) = \infty$, satisfies

(a) $\omega(g) > \frac{1}{p-1}$,

(b) $\omega(g^{-1}h) \geq \min(\omega(g), \omega(h))$,

(c) $\omega([g,h]) \geq \omega(g) + \omega(h)$,

(d) $\omega(g^p) = \omega(g) + 1$

for any $g, h \in G$. ♠

For the rest of this subsection, let (G, ω) be a p-valued group, i.e., a group with a p-valuation.

For any real number $\nu > 0$ put

$$G_\nu \coloneqq \{g \in G : \omega(g) \geq \nu\} \quad \text{and} \quad G_{\nu+} \coloneqq \{g \in G : \omega(g) > \nu\},$$

and note that these are normal subgroups, cf. [Sch11a, Sect. 23].

The subgroups G_ν form a decreasing exhaustive and separated filtration of G with the additional properties

$$G_\nu = \bigcap_{\nu' < \nu} G_{\nu'} \quad \text{and} \quad [G_\nu, G_{\nu'}] \subseteq G_{\nu+\nu'}.$$

There is a unique (Hausdorff) topological group structure on G for which the G_ν form a fundamental system of open neighborhoods of the identity element. It will be called the *topology defined by* ω. We will assume that G is profinte in the topology defined by ω. Hence $G = \varprojlim_{\nu>0} G/G_\nu$ as topological groups, and thus G must be a pro-p-group since $\omega(g^p) = \omega(g) + 1$ implies that G/G_ν is a p-group (finite since G_ν is open).

We now form, for each $\nu > 0$, the subquotient group

$$\operatorname{gr}_\nu G \coloneqq G_\nu / G_{\nu+}.$$

It is commutative by (c) and therefore will be denoted additively. We now consider the graded abelian group

$$\operatorname{gr} G \coloneqq \bigoplus_{\nu>0} \operatorname{gr}_\nu G.$$

An element $\xi \in \operatorname{gr} G$ is called, as usual, homogeneous (of degree ν) if it lies in $\operatorname{gr}_\nu G$. Furthermore, in this case any $g \in G_\nu$ such that $\xi = gG_{\nu+}$ is called a representative of ξ.

Note that $p\xi = 0$ for any homogeneous element $\xi \in \operatorname{gr} G$ since $\omega(g^p) = \omega(g)+1$. Hence $\operatorname{gr} G$ in fact is an $\mathbb{F}_p$-vector space. Furthermore, by bilinear extension of the map

$$\begin{aligned} \operatorname{gr}_\nu G \times \operatorname{gr}_{\nu'} G &\to \operatorname{gr}_{\nu+\nu'} G \\ (\xi, \eta) &\mapsto [\xi, \eta] \coloneqq [g, h] G_{\nu+\nu'} +, \end{aligned}$$

for $\nu, \nu' > 0$, we obtain a graded $\mathbb{F}_p$-bilinear map

$$[\,\cdot\,,\,\cdot\,]\colon \operatorname{gr} G \times \operatorname{gr} G \to \operatorname{gr} G$$

which satisfies

$$[\xi, \xi] = 0 \qquad \text{for any } \xi \in \operatorname{gr} G.$$

One can check that $[\,\cdot\,,\,\cdot\,]$ satisfies the Jacobi identity, and thus $\operatorname{gr} G$ is a graded Lie algebra over $\mathbb{F}_p$, cf. [Sch11a, Sect. 23].

Now, noticing that the map

$$\begin{aligned} \operatorname{gr}_\nu G &\to \operatorname{gr}_{\nu+1} G \\ gG_{\nu+} &\mapsto g^p G_{(\nu+1)+} \end{aligned}$$

is well defined and $\mathbb{F}_p$-linear, by considering for varying ν the direct sum of these maps, we can introduce an $\mathbb{F}_p$-linear map of degree one

$$\pi\colon \operatorname{gr} G \to \operatorname{gr} G.$$

We can and will therefore view $\operatorname{gr} G$ as a graded module over the polynomial ring $\mathbb{F}_p[\pi]$ in one variable over $\mathbb{F}_p$. Furthermore the Lie bracket on $\operatorname{gr} G$ is bilinear for the $\mathbb{F}_p[\pi]$-module structure, i.e., $\operatorname{gr} G$ is a Lie algebra over the ring $\mathbb{F}_p[\pi]$. For more details, we refer to [Sch11a, Sect. 25].

Definition 1.2. The pair (G, ω) is called of finite rank if $\operatorname{gr} G$ is finitely generated as an $\mathbb{F}_p[\pi]$-module. ♠

Note that G being of finite rank does not depend on the choice of the p-valuation, and assume from now on that (G, ω) is of finite rank. Note that $\operatorname{gr} G$ is finitely generated and torsionfree over the principal ideal domain $\mathbb{F}_p[\pi]$, and thus by the elementary divisor theorem $\operatorname{gr} G$ is free. We call

$$\operatorname{rank}(G, \omega) := \operatorname{rank}_{\mathbb{F}_p[P]} \operatorname{gr} G$$

the *rank* of the pair (G, ω).

For any $g \in G$ note that we then have a group homomorphism

$$\begin{aligned} c\colon \mathbb{Z} &\to G \\ m &\mapsto g^m. \end{aligned}$$

Since G/N, for any $N \lhd G$, is a p-group, we obtain $c^{-1}(N) = p^{a_N}\mathbb{Z}$ for some $a_N \geq 0$. It follows that c extends uniquely to a continuous group homomorphism

$$\tilde{c}\colon \mathbb{Z}_p \to \varprojlim_{N \lhd G} \mathbb{Z}/p^{a_N}\mathbb{Z} \xrightarrow{c} \varprojlim_{N} G/N = G$$

which we always will write as $g^x := \tilde{c}(x)$. More generally, for any finitely many elements $g_1, \dots, g_r \in G$, we have the continuous map

$$\begin{aligned} \mathbb{Z}_p^r &\to G \\ (x_1, \dots, x_r) &\mapsto g_1^{x_1} \cdots g_r^{x_r} \end{aligned} \tag{1.1}$$

{eq:ZprtoG}

which depends on the order of the g_i and therefore is not a group homomorphism. However we introduce the following notation, where v_p denotes the usual p-adic valuation on $\mathbb{Q}_p$.

Definition 1.3. The sequence of elements $(g_1, \dots, g_r)$ in G is called an *ordered basis* of (G, ω) if the map (1.1) is a bijection (and hence, by compactness, a homeomorphism) and

$$\omega(g_1^{x_1} \cdots g_r^{x_r}) = \min_{1 \le i \le r} (\omega(g_i) + v(x_i)) \qquad \text{for any } x_1, \dots, x_r \in \mathbb{Z}_p. \qquad \spadesuit$$

Definition 1.4. For any $g \in G \setminus \{1\}$, we put $\sigma(g) := gG_{\omega(g)+} \in \operatorname{gr} G$. $\spadesuit$

By [Sch11a, Remark 26.3], we note that for $g \in G \setminus \{1\}$ and $x \in \mathbb{Z}_p \setminus \{0\}$

$$\omega(g^x) = \omega(g) + v_p(x) \quad \text{and} \quad \sigma(g^x) = \bar{x}\pi^{v_p(x)} . \sigma(g), \tag{1.2}$$

{eq:sigma-gx}

where $\bar{x}$ is the image of $p^{-v_p(x)}x$ in $\mathbb{F}_p^\times$ (i.e., the first non-zero coefficient of $x = \sum_{k=0}^\infty a_k p^k$). We note that an ordered basis $(g_1, \dots, g_d)$ of (G, ω) corresponds to an ordered $\mathbb{F}_p[\pi]$-basis $\big(\sigma(g_1), \dots, \sigma(g_d)\big)$ of $\operatorname{gr} G$, cf. [Sch11a, Prop. 26.5].

Finally we let $\mathfrak{g} = \mathbb{F}_p \otimes_{\mathbb{F}_p[\pi]} \operatorname{gr} G = \mathbb{F}_p \otimes_{\mathbb{F}_p[\pi]} \operatorname{gr} G / \pi \operatorname{gr} G$, and note that this is a Lie algebra over $\mathbb{F}_p$ with an $\mathbb{F}_p$-basis of vectors $\xi_i = 1 \otimes \sigma(g_i)$.

1.1.4 Cohomology theories and the spectral sequence

ec:coh-and-spec-seq

One of the main results we use in this chapter is the spectral sequence introduced in [Sør, §6.1], so in this subsection we aim to introduce the concepts needed to use this spectral sequence. We also look into an important translation between continuous and discrete group cohomology that we will need later.

Let R be a ring and $\mathfrak{g}$ be a R-Lie algebra with R a trivial (left) $\mathfrak{g}$-module. Then we use the cochain complex $C^\bullet(\mathfrak{g}, R) = \mathrm{Hom}_R(\bigwedge^\bullet \mathfrak{g}, R)$, i.e.,

$$0 \longrightarrow R \xrightarrow{\partial_1} \mathrm{Hom}_R(\mathfrak{g}, R) \xrightarrow{\partial_2} \mathrm{Hom}_R\Big(\bigwedge^2 \mathfrak{g}, R\Big) \xrightarrow{\partial_3} \cdots,$$

where the coboundary map ∂_n is given by

$$\partial_n(f)(x_1, \dots, x_n) = \sum_{i<j} (-1)^{i+j} f([x_i, x_j], x_1, \dots, \widehat{x}_i, \dots, \widehat{x}_j, \dots, x_n),$$

where $\widehat{x}_i$ means excluding x_i. For more details we refer to [Car56, Thm. 7.1] or [Fuk86, §3], and note that we are considering the trivial action on R, which simplifies the formula slightly (cf. [Fuk86, §3.2]).

Now consider $R = \mathbb{F}_p$ in the following and suppose that $\mathfrak{g} = \mathfrak{g}^0 \oplus \mathfrak{g}^1 \oplus \cdots$ is a graded Lie algebra. Then $\bigwedge^n \mathfrak{g}$ is also graded by letting

$$\mathrm{gr}^j\Big(\bigwedge^n \mathfrak{g}\Big) = \bigoplus_{j_1 + \cdots + j_n = j} \mathfrak{g}^{j_1} \wedge \cdots \wedge \mathfrak{g}^{j_n}.$$

Letting $\mathbb{F}_p$ be a $\mathbb{Z}$-graded (concentrated in degree 0) $\mathfrak{g}$-module, we get a grading

$$\mathrm{Hom}_{\mathbb{F}_p}\Big(\bigwedge^n \mathfrak{g}, \mathbb{F}_p\Big) = \bigoplus_{s \in \mathbb{Z}} \mathrm{Hom}^s_{\mathbb{F}_p}\Big(\bigwedge^n \mathfrak{g}, \mathbb{F}_p\Big)$$

where $\mathrm{Hom}^s_{\mathbb{F}_p}$ denotes the homogeneous $\mathbb{F}_p$-linear maps of degree s, cf. [FF74, Lem. 4.2]. One can check that this passes to bigrading of Lie algebra cohomology

$$H^{s,t}(\mathfrak{g}, \mathbb{F}_p) = H^{s+t}\big(\mathrm{gr}^s \mathrm{Hom}_{\mathbb{F}_p}(\bigwedge^\bullet \mathfrak{g}, \mathbb{F}_p)\big).$$

See [Fuk86, §3] for more details.

In the spectral sequence described in [Sør, §6.1], we take $r_0 = 1$ (i.e., the spectral sequence start from the first page) and $E_1^{s,t} = H^{s,t}(\mathfrak{g}, \mathbb{F}_p)$, where $\mathfrak{g} = \mathbb{F}_p \otimes \mathrm{gr}\, G$ indeed is (positively) $\mathbb{Z}$-graded.

Let now G be a topological group and $\mathbb{F}_p$ a G-module. Then we will define two types of group cohommology: continuous and discrete.

Continuous group cohomology $H^n_{\text{cts}}(G, \mathbb{F}_p)$ is the cohomology of the complex $C^\bullet(G, \mathbb{F}_p) = \mathcal{C}(G^\bullet, \mathbb{F}_p)$, i.e.,

$$0 \longrightarrow \mathbb{F}_p \xrightarrow{\partial_1} \mathcal{C}(G, \mathbb{F}_p) \xrightarrow{\partial_2} \mathcal{C}(\mathbb{G}^2, \mathbb{F}_p) \xrightarrow{\partial_3} \mathcal{C}(G^3, \mathbb{F}_p) \xrightarrow{\partial_4} \cdots,$$

where the coboundary map ∂_n is given by

$$\partial_n(f)(g_1, \ldots, g_n) = \sum_{i=1}^{n} (-1)^i f(g_1, \ldots, g_i g_{i+1}, \ldots, g_n),$$

where the n-th term is interpreted as $(-1)^n f(g_1, \ldots, g_{n-1})$, cf. [Sør, §3] and note again that our formula is slightly simpler since we only consider the trivial action on $\mathbb{F}_p$.

Discrete group cohomology $H^n_{\text{dsc}}(G, \mathbb{F}_p)$ is the cohomology of the complex $C^\bullet(G, \mathbb{F}_p) = \operatorname{Hom}_G(\mathbb{Z}[G^\bullet], \mathbb{F}_p)$ as follows. One can check that

$$\cdots \xrightarrow{d_4} \mathbb{Z}[G^3] \xrightarrow{d_3} \mathbb{Z}[G^2] \xrightarrow{d_2} \mathbb{Z}[G] \xrightarrow{d_1} \mathbb{Z} \longrightarrow 0$$

with boundary maps

$$d_n \colon (g_0, g_1, \ldots, g_n) \mapsto \sum_{i=0}^{n} (-1)^i (g_0, \ldots, \widehat{g}_i, \ldots, g_n)$$

is a chain complex, and thus we get a cochain complex $C^\bullet(G, \mathbb{F}_p) = \operatorname{Hom}_G(C_\bullet, \mathbb{F}_p)$,

$$\begin{array}{ccc} 0 \longrightarrow \operatorname{Hom}_G(\mathbb{Z}, \mathbb{F}_p) & \xrightarrow{\partial_1} & \operatorname{Hom}_G(\mathbb{Z}[G^2], \mathbb{F}_p) \xrightarrow{\partial_2} \cdots \\ f & \longmapsto & f \circ d_1 \end{array}$$

Note that this discrete cohomology can be viewed as continuous cohomology if we equip G with the discrete topology.

Note that [Sør] gets the spectral sequence we are interested in by using an isomorphism to translate $H^\bullet_{\text{cts}}(G, \mathbb{F}_p)$ to $HH^\bullet(G, \mathbb{F}_p)$ (essentially what's known as Mac Lane isomorphism) and introducing a $\mathbb{Z}$-filtration and grading on $HH^\bullet(G, \mathbb{F}_p)$, which is used in the spectral sequence. We will skip the full details of this translation and just note that we get a $\mathbb{Z}$-filtration and grading on $H^{\bullet}(G, \mathbb{F}_p)$, which with $k = \mathbb{F}_p$ gives us the following, cf. [Sør, Thm. 5.5–§6.1].

thm:spec-seq **Theorem 1.5.** Let (G, ω) be a p-valuable group and $\mathfrak{g} = \mathbb{F}_p \otimes_{\mathbb{F}_p[\pi]} \operatorname{gr} G$ its Lazard Lie algebra. Then there is a convergent spectral sequence collapsing at a finite stage,

$$E_1^{s,t} = H^{s,t}(\mathfrak{g}, \mathbb{F}_p) \Longrightarrow H^{s+t}(G, \mathbb{F}_p).$$

This means that each sheet E_r has a multiplication $E_r \otimes E_r \to E_r$ compatible with the (s,t)-bigrading and satisfying Leibniz formula. Furthermore $H^*(E_r) \cong E_{r+1}$ as algebras. I.e., the multiplication on E_∞ is compatible with the cup product on $H^*(G, \mathbb{F}_p)$ in the sense that the following diagram commutes.

$$\begin{tikzcd} E_\infty^{s,n-s} \otimes E_\infty^{s',n'-s'} \arrow[r] \arrow[d, "\cong"'] & E_\infty^{s+s',n+n'-s-s'} \arrow[d, "\cong"] \\ \operatorname{gr}^s H^n(G, \mathbb{F}_p) \otimes \operatorname{gr}^{s'} H^{n'}(G, \mathbb{F}_p) \arrow[r] & \operatorname{gr}^{s+s'} H^{n+n'}(G, \mathbb{F}_p) \end{tikzcd}$$ ♣

Finally we note that [Fer+07, Thm. 2.10] implies that $H^n_{\text{cts}}(N, \mathbb{F}_p) \cong H^n_{\text{dsc}}(N, \mathbb{F}_p)$ for all n (with $N = \mathcal{N}(\mathbb{Z}_p)$ as above), if we can show that N is a pro-p group which is poly-$\mathbb{Z}_p$ by finite.

Definition 1.6. A group G is poly-$\mathbb{Z}_p$ if it has a normal series

$$G = G_1 \supseteq G_2 \supseteq \cdots \supseteq G_n = 1$$

such that each factor group G_i/G_{i+1} is isomorphic to $\mathbb{Z}_p$.

A group is poly-$\mathbb{Z}_p$ by finite (virtually poly-$\mathbb{Z}_p$) if it contains a poly-$\mathbb{Z}_p$ subgroup of finite index. ♠

Note that [Con14b, Prop. 5.1.16(2) and Cor. 5.2.5] (as seen in the proof of [Con14b, Cor. 5.2.13] or [Con14b, Thm. 5.4.3]) gives us a composition series of $\mathcal{N}$

such that the successive quotients are $\mathbb{G}_a$, which implies that $N = \mathcal{N}(\mathbb{Z}_p)$ is poly-$\mathbb{Z}_p$ by finite since $\mathbb{G}_a(\mathbb{Z}_p) = \mathbb{Z}_p$. Thus, assuming that $\mathcal{N}(\mathbb{Z}_p)$ is a pro-p group, we get that

DK Note: Can probably remove this.

$$H^n_{\mathrm{cts}}(N, \mathbb{F}_p) \cong H^n_{\mathrm{dsc}}(N, \mathbb{F}_p) \qquad \text{for all } n. \tag{1.3}$$

{eq:coh-comp}

1.1.5 Main result

subsec:main-res

We show first that N is p-valuable, which implies by [Sør, §6.1] that we get a convergent multiplicative spectral sequence

DK Note: Rewrite to state Theorem precisely later.

$$E_1^{s,t} = H^{s,t}(\mathfrak{g}, \mathbb{F}_p) \Longrightarrow H^{s+t}(N, \mathbb{F}_p). \tag{1.4}$$

{eq:spec-seq}

We note that $\mathfrak{g} \cong \mathfrak{n}$ and then use ideas of [Gro14, §7] to transfer results from [PT18] about (the dimension of) $H^n(\mathfrak{n}_{\mathbb{Z}}, \mathbb{F}_p)$ and $H^n(\mathcal{N}_{\mathbb{Z}}(\mathbb{Z}), \mathbb{F}_p)$ to $H^n(\mathfrak{n}, \mathbb{F}_p)$ and $H^n(\mathcal{N}(\mathbb{Z}_p), \mathbb{F}_p)$, giving us that $\sum_{s+t=n} \dim_{\mathbb{F}_p} H^{s,t}(\mathfrak{g}, \mathbb{F}_p) = \dim_{\mathbb{F}_p} H^n(\mathfrak{n}, \mathbb{F}_p) = \dim_{\mathbb{F}_p} H^n(N, \mathbb{F}_p)$. This implies that (1.4) collapses on the first page, and thus $H^{s,n-s}(\mathfrak{n}, \mathbb{F}_p) \cong \mathrm{gr}^s H^n(N, \mathbb{F}_p)$. Noting that $E_\infty^{s,t} = E_1^{s,t}$, we get that the cup product on $E_1^{s,t} = H^{s,t}(\mathfrak{n}, \mathbb{F}_p)$ (from $H^*(\mathfrak{n}, \mathbb{F}_p)$) is compatible with the cup product on $H^*(N, \mathbb{F}_p)$ in the sense that the following diagram commutes.

$$\begin{tikzcd}
H^{s,n-s}(\mathfrak{n}, \mathbb{F}_p) \otimes H^{s',n'-s'}(\mathfrak{n}, \mathbb{F}_p) \arrow[r] \arrow[d, "\cong"'] & H^{s+s',n+n'-s-s'}(\mathfrak{n}, \mathbb{F}_p) \arrow[d, "\cong"] \\
\mathrm{gr}^s H^n(N, \mathbb{F}_p) \otimes \mathrm{gr}^{s'} H^{n'}(N, \mathbb{F}_p) \arrow[r] & \mathrm{gr}^{s+s'} H^{n+n'}(N, \mathbb{F}_p)
\end{tikzcd}$$

1.2 The p-valuation

sec:pval

In this section we will prove that N is p-valuable group, which we will need in multiple arguments later. It should be noted that this section, with the exception of Proposition 1.9, is a slightly rewritten version of [Sch11b] which expands on some of the arguments. Also, the proof of Proposition 1.9 is based on [Zab10, Lem. 1].

Note that as a set N is the direct product $N = \prod_{\alpha\in\Phi^-} x_\alpha(\mathbb{Z}_p)$, which allows us to introduce the function

$$\begin{aligned}\omega\colon N \setminus \{1\} &\to \mathbb{N}\\ \prod_{\alpha\in\Phi^-} x_\alpha(a_\alpha) &\mapsto \min_{\alpha\in\Phi^-}\big(v_p(a_\alpha) - \mathrm{ht}(\alpha)\big),\end{aligned} \tag{1.5}$$

{eq:p-val}

where v_p denotes the usual p-adic valuation on $\mathbb{Z}_p$. Here it is important to note that we write any $g \in N$ uniquely as product

$$g = \prod_{\alpha\in\Phi^-} x_\alpha(a_\alpha)$$

by taking the product following the total ordering $\geq$ of Φ^- defined above. Now, with the convention that $\omega(1) \coloneqq \infty$, we define the descending sequence of subsets

$$N_m \coloneqq \{g \in N \mid \omega(g) \geq m\}$$

in N for $m \geq 0$, following the notation used for p-valuable groups. The goal of this section is to show that this ω is a p-valuation by a careful analysis of the sequence of subsets given by N_m.

Remark 1.7. If we are willing to restrict from $p+1 \geq h$ to $p-1 > h$, then we can restrict the p-valuation of the pro-p Iwahori subgroup of $\mathcal{G}$ introduced in Section 2.1 to a p-valuation on N. We prefer the above p-valuation because it will introduce a grading on $\mathfrak{g}$ that will directly correspond to the grading (by height) on $\mathfrak{n}$, whereas the restricted p-valuation is a scalar multiple of this p-valuation on a basis. △

DK Note: Write this better.

We first note that clearly $N_1 = N$, $\bigcap_m N_m = \{1\}$, and

DK Note: Decide whether to use mathclap or not.

$$\begin{aligned}N_m &= \prod_{\alpha\in\Phi^-} x_\alpha(p^{\max(0,m+\mathrm{ht}(\alpha))}\mathbb{Z}_p)\\ &= \prod_{\substack{\alpha\in\Phi^-\\ \mathrm{ht}(\alpha)=-1}} x_\alpha(p^{m-1}\mathbb{Z}_p)\cdots\prod_{\substack{\alpha\in\Phi^-\\ \mathrm{ht}(\alpha)=-(m-1)}} x_\alpha(p\mathbb{Z}_p)\prod_{\substack{\alpha\in\Phi^-\\ \mathrm{ht}(\alpha)\leq -m}} x_\alpha(\mathbb{Z}_p).\end{aligned} \tag{1.6}$$

{eq:N_m}

In our analysis of this sequence it will be helpful to introduce the following two other filtrations of N. Firstly we will consider the filtration by congruence subgroups

$$N(m) \coloneqq \ker\big(\mathcal{N}(\mathbb{Z}_p) \to \mathcal{N}(\mathbb{Z}/p^m\mathbb{Z})\big) = \prod_{\alpha\in\Phi^-} x_\alpha(p^m\mathbb{Z}_p) \tag{1.7}$$

{eq:N-par-m}

for $m \geq 0$. Secondly, using the descending central series of the group $\mathcal{G}(\mathbb{Q}_p)$ defined by $C^1\mathcal{G}(\mathbb{Q}_p) \coloneqq \mathcal{G}(\mathbb{Q}_p)$ and $C^{m+1}\mathcal{G}(\mathbb{Q}_p) \coloneqq [C^m\mathcal{G}(\mathbb{Q}_p), \mathcal{G}(\mathbb{Q}_p)]$, we consider the filtration given by

$$N_{(m)} \coloneqq N \cap C^m\mathcal{G}(\mathbb{Q}_p)$$

for $m \geq 1$. By [BT73, Prop. 4.7(iii)] we have that

$$N_{(m)} = \prod_{\substack{\alpha\in\Phi^- \\ \mathrm{ht}(\alpha)\leq -m}} x_\alpha(\mathbb{Z}_p), \tag{1.8}$$

{eq:N_par-m}

and we note that the natural map

$$\prod_{\substack{\alpha\in\Phi^- \\ \mathrm{ht}(\alpha)= -m}} x_\alpha(\mathbb{Z}_p) \to N_{(m)}/N_{(m+1)}$$

is an isomorphism of abelian groups, and that all the subgroups $N(m)$ and $N_{(m)}$ are normal in N.

We are now ready to prove the following lemma, which will help us when showing that ω is a p-valuation.

Lemma 1.8.

lem:N_m

item:N_m

(i) $N_m = \prod_{1\leq i\leq m} N(m-i) \cap N_{(i)}$, for any $m \geq 1$, is a normal subgroup of N which is independent of the choices made.

item:N_mcom

(ii) $[N_\ell, N_m] \subseteq N_{\ell+m}$ for any $\ell, m \geq 1$.

(iii) N_m/N_{m+1}, for any $m \geq 1$, is an $\mathbb{F}_p$-vector space of dimension equal to $|\{\alpha \in \Phi^- \mid \mathrm{ht}(\alpha) \geq -m\}|$.

item:g^p

(iv) Let $g \in N_m$ for some $m \geq 1$. If $g^p \in N_{m+2}$, then $g \in N_{m+1}$. ♣

Proof. (i) Using (1.7) and (1.8) we note that

$$\prod_{\substack{\alpha\in\Phi^- \\ \mathrm{ht}(\alpha)=-i}} x_\alpha(p^{m-i}\mathbb{Z}_p) \subseteq N(m-i)\cap N_{(i)} \quad \text{and} \quad \prod_{\substack{\alpha\in\Phi^- \\ \mathrm{ht}(\alpha)\leq -m}} x_\alpha(\mathbb{Z}_p) = N(0)\cap N_{(m)}$$

for $1 \leq i < m$, so by (1.6) it's clear that $N_m \subseteq \prod_{1\leq i\leq m} N(m-i)\cap N_{(i)}$. We also note, by (1.7) and (1.8), that

$$\begin{aligned}
&\big(N(m-i)\cap N_{(i)}\big)\big(N(m-i-1)\cap N_{(i+1)}\big) \\
&\subseteq \Big(\prod_{\substack{\alpha\in\Phi^- \\ \mathrm{ht}(\alpha)=-i}} x_\alpha(p^{m-i}\mathbb{Z}_p)\Big)\big(N(m-i-1)\cap N_{(i+1)}\big)
\end{aligned}$$

for any $1 \leq i < m$, so

$$\begin{aligned}
&\prod_{1\leq i\leq m} N(m-i)\cap N_{(i)} \\
&\subseteq \prod_{\substack{\alpha\in\Phi^- \\ \mathrm{ht}(\alpha)=-1}} x_\alpha(p^{m-1}\mathbb{Z}_p)\cdots \prod_{\substack{\alpha\in\Phi^- \\ \mathrm{ht}(\alpha)=-(m-1)}} x_\alpha(p\mathbb{Z}_p)\big(N(0)\cap N_{(m)}\big) \\
&= N_m
\end{aligned}$$

by induction, (1.6) and (1.8). This shows the equality and that N_m is normal clearly follows.

(ii) We first recall the following formulas for commutators

$$[gh,k] = g[h,k]g^{-1}[g,k] \quad \text{and} \quad [g,hk] = [g,h]h[g,k]h^{-1}. \tag{1.9}$$

{eq:comformulas}

Now, using (1.9), (i) and the fact that all the involved subgroups are normal, it's enough to show that

$$[N(\ell)\cap N_{(i)}, N(m)\cap N_{(j)}] \subseteq N(\ell+m)\cap N_{(i+j)}.$$

This further reduces to showing that

$$[N(\ell), N(m)] \subseteq N(\ell+m) \quad \text{and} \quad [N_{(i)}, N_{(j)}] \subseteq N_{(i+j)}.$$

The right inclusion is a well known property of the descending central series, so it follows from our definition of $N_{(m)}$. For the left inclusion it suffices, by (1.7) and (1.9), to show that

$$[x_\alpha(p^\ell \mathbb{Z}_p), x_\beta(p^m \mathbb{Z}_p)] \subseteq N(\ell + m)$$

for any $\alpha, \beta \in \Phi^-$. To show this inclusion we recall Chevalley's commutator formula, cf. [Con14b, Prop. 5.1.14],

$$[x_\alpha(a), x_\beta(b)] \in x_{\alpha+\beta}(c_{\alpha,\beta,1,1} ab\mathbb{Z}_p) \prod_{\substack{i,j\geq 1 \\ i+j>2}} x_{i\alpha+j\beta}(c_{\alpha,\beta,i,j} a^i b^j \mathbb{Z}_p),$$

where $c_{\alpha,\beta,i,j} \in \mathbb{Z}_p$ and on the right hand side we use the convention is that $x_{i\alpha+j\beta} \equiv 1$ if $i\alpha + j\beta \notin \Phi$. From (1.7) and Chevalley's commutator formula the inclusion follows.

(iii) We note that

$$N(m-i) \cap N_{(i)} = \prod_{\substack{\alpha\in\Phi^- \\ \mathrm{ht}(\alpha)\leq -i}} x_\alpha(p^{m-i}\mathbb{Z}_p)$$

for $1 \leq i \leq m$, so the statement follows from (i) and (ii). DK Note: Write (iii) better.

(iv) For any $1 \leq \ell \leq m$ we consider the chain of normal subgroups

$$N_{m+2}(N_m \cap N_{(\ell+1)}) \subseteq N_{m+1}(N_m \cap N_{(\ell+1)}) \subseteq N_{m+1}(N_m \cap N_{(\ell)})$$

between N_{m+2} and N_m. By (1.9) and an argument like in (ii), we get that

$$[N_{m+1}(N_m \cap N_{(\ell)}), N_{m+1}(N_m \cap N_{(\ell)})] \subseteq N_{m+2}(N_m \cap N_{(\ell+1)}),$$

so the quotient group

$$N_{m+1}(N_m \cap N_{(\ell)})/N_{m+2}(N_m \cap N_{(\ell+1)})$$

is abelian. Now looking carefully at the groups as sets, we see that

$$N_m \cap N_{(\ell)} = \prod_{\substack{\alpha\in\Phi^- \\ \mathrm{ht}(\alpha)\leq -\ell}} x_\alpha(p^{\max(0,m+\mathrm{ht}(\alpha))}\mathbb{Z}_p)$$

and thus (using Chevalley's commutator formula and the fact that $\mathrm{ht}(i\alpha + j\beta) \leq \mathrm{ht}(\alpha+\beta) < \mathrm{ht}(\alpha), \mathrm{ht}(\beta)$ to move the products for the $\mathrm{ht}(\alpha) = -\ell$ term) DK Note: More detail here?

$$N_{m+1}(N_m \cap N_{(\ell)}) = \prod_{\substack{\alpha\in\Phi^- \\ \mathrm{ht}(\alpha)>-\ell}} x_\alpha(p^{\max(0,m+1+\mathrm{ht}(\alpha))}\mathbb{Z}_p) \cdot \prod_{\substack{\alpha\in\Phi^- \\ \mathrm{ht}(\alpha)=-\ell}} x_\alpha(p^{m-\ell}\mathbb{Z}_p) \cdot \prod_{\substack{\alpha\in\Phi^- \\ \mathrm{ht}(\alpha)<-\ell}} x_\alpha(p^{\max(0,m+\mathrm{ht}(\alpha))}\mathbb{Z}_p).$$

Similarly

$$N_{m+2}(N_m \cap N_{(\ell+1)}) = \prod_{\substack{\alpha\in\Phi^- \\ \mathrm{ht}(\alpha)>-\ell}} x_\alpha(p^{\max(0,m+2+\mathrm{ht}(\alpha))}\mathbb{Z}_p) \cdot \prod_{\substack{\alpha\in\Phi^- \\ \mathrm{ht}(\alpha)=-\ell}} x_\alpha(p^{m+2-\ell}\mathbb{Z}_p) \cdot \prod_{\substack{\alpha\in\Phi^- \\ \mathrm{ht}(\alpha)\leq-(\ell+1)}} x_\alpha(p^{\max(0,m+\mathrm{ht}(\alpha))}\mathbb{Z}_p),$$

and since the quotient group

$$N_{m+1}(N_m \cap N_{(\ell)})/N_{m+2}(N_m \cap N_{(\ell+1)})$$

is abelian, we see that it is isomorphic to

$$\prod_{\substack{\alpha\in\Phi^- \\ \mathrm{ht}(\alpha)>-\ell}} \frac{x_\alpha(p^{\max(0,m+1+\mathrm{ht}(\alpha))}\mathbb{Z}_p)}{x_\alpha(p^{\max(m+2+\mathrm{ht}(\alpha))}\mathbb{Z}_p)} \times \prod_{\substack{\alpha\in\Phi^- \\ \mathrm{ht}(\alpha)=-\ell}} \frac{x_\alpha(p^{m-\ell}\mathbb{Z}_p)}{x_\alpha(p^{m+2-\ell}\mathbb{Z}_p)}.$$

Here the subgroup

$$N_{m+1}(N_m \cap N_{(\ell+1)})/N_{m+2}(N_m \cap N_{(\ell+1)})$$

corresponds to

$$\prod_{\substack{\alpha\in\Phi^- \\ \mathrm{ht}(\alpha)>-\ell}} \frac{x_\alpha(p^{\max(0,m+1+\mathrm{ht}(\alpha))}\mathbb{Z}_p)}{x_\alpha(p^{\max(0,m+2+\mathrm{ht}(\alpha))}\mathbb{Z}_p)} \times \prod_{\substack{\alpha\in\Phi^- \\ \mathrm{ht}(\alpha)=-\ell}} \frac{x_\alpha(p^{m+1-\ell}\mathbb{Z}_p)}{x_\alpha(p^{m+2-\ell}\mathbb{Z}_p)}.$$

It follows that $N_{m+1}(N_m \cap N_{(\ell+1)})/N_{m+2}(N_m \cap N_{(\ell+1)})$ is the p-torsion subgroup of $N_{m+1}(N_m \cap N_{(\ell)})/N_{m+2}(N_m \cap N_{(\ell+1)})$.

Now let $g \in N_m$ for some $m \geq 1$. For $\ell = 1$ we have $g \in N_m = N_{m+1}(N_m \cap N_{(1)})$, since $N_{(1)} = N$, and clearly $g^p \in N_{m+2}(N_m \cap N_{(2)})$ because $g^p \in N_{(2)}$ by Chevalley's commutator formula and (1.8). Since $N_{m+1}(N_m \cap N_{(2)})/N_{m+2}(N_m \cap N_{(2)})$ is the p-torsion subgroup of $N_{m+1}(N_m \cap N_{(1)})/N_{m+2}(N_m \cap N_{(2)})$, it follows that $g \in N_{m+1}(N_m \cap N_{(2)})$ and thus $g^p \in N_{m+2}(N_m \cap N_{(3)})$ by Chevalley's commutator formula and (1.8). By induction on ℓ, we thus get that $g \in N_{m+1}(N_m \cap N_{(m+1)}) = N_{m+1}$. Here the last equality follows from the fact that $N_{(m+1)} \subseteq N_{m+1}$ by (1.6) and (1.8). $\square$

With this lemma, we are now ready to prove that ω is a p-valuation on N.

prop:N-p-val

Proposition 1.9. The function ω is a p-valuation on N, i.e., it satisfies for any $g, h \in N$:

(a) $\omega(g) > \frac{1}{p-1}$,

(b) $\omega(g^{-1}h) \geq \min(\omega(g), \omega(h))$,

(c) $\omega([g,h]) \geq \omega(g) + \omega(h)$,

(d) $\omega(g^p) = \omega(g) + 1$. ♣

Proof. We note that (a) is obvious by our definition of ω, (c) follows from Lemma 1.8 (ii) and (d) follows from Lemma 1.8 (iv).

It only remains to show (b), which we will do by following the proof idea of [Zab10, Lem. 1], i.e., we are going to use triple induction. Here we note that all products $\prod_{\alpha\in\Phi^-} x_\alpha(a_\alpha)$ are in ascending order in Φ^- (so descending in height). For ease of notation, we prove equivalently that $\omega(gh^{-1}) \geq \min(\omega(g), \omega(h))$ for $g, h \in N$.

At first by induction on the number of non-zero coordinates among $(a_\beta)_{\beta\in\Phi^-}$ in $\prod_{\beta\in\Phi^-} x_\beta(a_\beta)$ we are reduced to the case where h is of the form $h = x_\beta(a_\beta)$ for some $\beta \in \Phi^-$ and $a_\beta \in \mathbb{Z}_p$. To see this let $h \in N \setminus \{1\}$ and write $h = \prod_{\beta\in\Phi^-} x_\beta(a_\beta)$ in our unique way (according to the ordering of Φ^-), and let α be the smallest element of Φ^- for which $a_\alpha \neq 0$ so that $h = x_\alpha(a_\alpha) \cdot h'$. Then $gh^{-1} = g(h')^{-1} \cdot x_\alpha(a_\alpha)^{-1}$ and thus strong induction will imply that

$$\begin{aligned} \omega(gh^{-1}) &\geq \min\bigl(\omega(g(h')^{-1}), v(a_\alpha) - \mathrm{ht}(\alpha)\bigr) \\ &\geq \min\bigl(\omega(g), \omega(h'), v(a_\alpha) - \mathrm{ht}(\alpha)\bigr) = \min\bigl(\omega(g), \omega(h)\bigr). \end{aligned}$$

Fix $h = x_\beta(a_\beta)$ and let now g be of the form $g = \prod_{k=1}^r x_{\alpha_k}(a_{\alpha_k})$ with $\alpha_1 < \alpha_2 < \cdots < \alpha_r$ in Φ^-. If $\beta > \alpha_r$, then $gh^{-1} = \prod_{k=1}^{r-1} x_{\alpha_k}(a_{\alpha_k}) \cdot x_{\alpha_r}(a_{\alpha_r}) x_\beta(-a_\beta)$, so (b) is clearly true if $\beta > \alpha_1$ (by the definition of ω), and if $\beta = \alpha_r$, then $x_{\alpha_r}(a_{\alpha_r}) x_\beta(-a_\beta) = x_\beta(a_{\alpha_r} - a_\beta)$ and (b) follows from $v_p(a-b) \geq \min(v_p(a), v_p(b))$ for $a, b \in \mathbb{Z}_p$.

On the other hand, if $\beta < \alpha_r$, then we write

$$\begin{aligned} gh^{-1} &= \prod_{k=1}^{r} x_{\alpha_k}(a_{\alpha_k}) \cdot x_\beta(-a_\beta) \\ &= \prod_{k=1}^{r-1} x_{\alpha_k}(a_{\alpha_k}) \cdot x_\beta(-a_\beta) \cdot x_{\alpha_r}(a_{\alpha_r}) \cdot [x_{\alpha_r}(-a_{\alpha_r}), x_\beta(a_\beta)]. \end{aligned}$$

Now we use descending induction on β in the chosen ordering of Φ^- and suppose that the statement (b) is true for any g and any h' of the form $h' = x_{\beta'}(a_{\beta'})$ with $\beta' > \beta$. Note that the base case is trivial and recall that Φ^- is finite and totally ordered. Note furthermore that Chevalley's commutator formula gives us

$$[x_{\alpha'}(a_{\alpha'}), x_{\beta'}(a_{\beta'})] = \prod_{\substack{i\alpha'+j\beta'\in\Phi^- \\ i,j>0}} x_{i\alpha'+j\beta'}(c_{\alpha',\beta',i,j} a_{\alpha'}^i a_{\beta'}^j) \tag{1.10}$$ {eq:Chevalley}

for any $\alpha', \beta' \in \Phi^-$, where $c_{\alpha',\beta',i,j} \in \mathbb{Z}_p$. Also, we have $\mathrm{ht}(i\alpha' + j\beta') \leq \mathrm{ht}(\alpha' + \beta') < \mathrm{ht}(\alpha'), \mathrm{ht}(\beta')$, so we can apply the induction hypothesis for $x_{\alpha_r}(a_{\alpha_r})$ and each

$x_{i\alpha_r+j\beta}(c_{\alpha_r,\beta,i,j}(-a_{\alpha_r})^i a_\beta^j)$ in $[x_{\alpha_r}(-a_{\alpha_r}, x_\beta(a_\beta))]$, since $\alpha_r > \beta$ and all terms on the right side of (1.10) are larger than β (and α_r) in the ordering of Φ^-. We thus obtain

$$\begin{aligned}\omega(gh^{-1}) \geq \min\Bigg(&\min_{\substack{i\alpha_r+j\beta\in\Phi^- \\ i,j>0}} \omega(x_{i\alpha_r+j\beta}(c_{\alpha_r,\beta,i,j}(-a_{\alpha_r})^i a_\beta^j)), \\ &\omega(x_{\alpha_r}(a_{\alpha_r})), \omega\Big(\prod_{k=1}^{r-1} x_{\alpha_k}(a_{\alpha_k}) \cdot x_\beta(-a_\beta)\Big)\Bigg).\end{aligned} \tag{1.11}$$

{eq:omega-par-ginvh

Now, for $i, j > 0$ with $i\alpha' + j\beta' \in \Phi^-$,

$$\begin{aligned}\omega(x_{i\alpha'+j\beta'}(c_{\alpha',\beta',i,j}a_{\alpha'}^i a_{\beta'}^j)) &= v_p(c_{\alpha',\beta',i,j}a_{\alpha'}^i a_{\beta'}^j) - \mathrm{ht}(i\alpha' + j\beta') \\ &\geq v_p(c_{\alpha',\beta',i,j}) + v_p(a_{\alpha'}^i) + v_p(a_{\beta'}^j) - \mathrm{ht}(\alpha' + \beta') \\ &\geq v_p(a_{\alpha'}) - \mathrm{ht}(\alpha') + v_p(a_{\beta'}) - \mathrm{ht}(\beta') \\ &= \omega(x_{\alpha'}(a_{\alpha'})) + \omega(x_{\beta'}(a_{\beta'})) \\ &\geq \min\big(\omega(x_{\alpha'}(a_{\alpha'})), \omega(x_{\beta'}(a_{\beta'}))\big).\end{aligned} \tag{1.12}$$

{eq:omega-par-Chev}

So taking $\alpha' = \alpha_r$ and $\beta' = \beta$ and using (1.12) in (1.11), we get that

$$\omega(gh^{-1}) \geq \min\Bigg(\omega(x_{\alpha_r}(a_{\alpha_r})), \omega(x_\beta(a_\beta)), \omega\Big(\prod_{k=1}^{r-1} x_{\alpha_k}(a_{\alpha_k}) \cdot x_\beta(-a_\beta)\Big)\Bigg). \tag{1.13}$$

{eq:omega-par-ginvh

Finally induction on r will imply that

$$\begin{aligned}\omega\Big(\prod_{k=1}^{r-1} x_{\alpha_k}(a_{\alpha_k}) \cdot x_\beta(-a_\beta)\Big) &\geq \min\Bigg(\omega\Big(\prod_{k=1}^{r-1} x_{\alpha_k}(a_{\alpha_k})\Big), \omega(x_\beta(a_\beta))\Bigg) \\ &= \min\big(\min_{1\leq k\leq r-1} \omega(x_{\alpha_k}(a_{\alpha_k})), \omega(x_\beta(a_\beta))\big),\end{aligned}$$

which by (1.13) implies that

$$\begin{aligned}\omega(gh^{-1}) &\geq \min\big(\min_{1\leq k\leq r} \omega(x_{\alpha_k}(a_{\alpha_k})), \omega(x_\beta(a_\beta))\big) \\ &= \min\big(\omega(g), \omega(h)\big),\end{aligned}$$

thus finishing the proof. $\square$

We have now shown that $N = \mathcal{N}(\mathbb{Z}_p)$ is a p-valuable group with the p-valuation ω introduced in (1.5), which is the main result of this section. Before continuing, we will clarify what this means based on Lazard theory as described in Section 1.1.

We note that

$$\operatorname{gr} N := \bigoplus_{m\geq 1} N_m/N_{m+1}$$

is a graded $\mathbb{F}_p$-vector space, and recall the following well known result, cf. [Laz65] or [Sch11a, Sect. 25].

Proposition 1.10. $\operatorname{gr} N$ is a Lie algebra over the polynomial ring $\mathbb{F}_p[\pi]$ in one variable π where

$$[gN_{\ell+1}, hN_{m+1}] := [g,h]N_{\ell+m+1} \quad \text{and} \quad \pi(gN_{m+1}) := g^p N_{m+2},$$

and as an $\mathbb{F}_p[\pi]$-module $\operatorname{gr} N$ is free of rank $|\Phi^-|$. ♣

1.3 Spectral sequence and cohomology

sec:specsec

Recall that $N = \mathcal{N}(\mathbb{Z}_p)$, $\mathfrak{g} = \mathbb{F}_p \otimes_{\mathbb{F}_p[\pi]} \operatorname{gr} G$ and $\mathfrak{n} = \operatorname{Lie}(\mathcal{N}_{\mathbb{F}_p})$. In this section we will first look at the spectral sequence from [Sør] (cf. Theorem 1.5), i.e.,

$$E_1^{s,t} = H^{s,t}(\mathfrak{g}, \mathbb{F}_p) \Longrightarrow H^{s+t}_{\text{cts}}(N, \mathbb{F}_p),$$

and note that we can work with the left side using that $H^{s,t}(\mathfrak{g}, \mathbb{F}_p) \cong H^{s,t}(\mathfrak{n}, \mathbb{F}_p)$ and for the right side $H^{s+t}_{\text{cts}}(N, \mathbb{F}_p) \cong H^{s+t}_{\text{dsc}}(N, \mathbb{F}_p)$. Afterwards, we will use results from [PT18] to argue that the spectral sequence collapses on the first page.

We will start by showing that $\mathfrak{g} \cong \mathfrak{n}$, for which we will need the following lemma.

Lemma 1.11. $\operatorname{gr} N \cong \mathbb{F}_p[\pi] \otimes_{\mathbb{F}_p} \mathfrak{n}$ as graded Lie algebras (where π has degree 1).♣

Proof. We first note that the elements X_α, where X_α is our fixed $\mathbb{Z}_p$-basis of $\operatorname{Lie} \mathcal{N}_\alpha$, reduce modulo p to an $\mathbb{F}_p$-basis $\{\overline{X}_\alpha\}_{\alpha\in\Phi^-}$ of $\mathfrak{n}$. On the other hand all

$$\sigma\big(x_\alpha(1)\big) \in \operatorname{gr}_{-\operatorname{ht}(\alpha)} N,$$

with $x_\alpha(1) \in N_{-\operatorname{ht}(\alpha)}$, form an $\mathbb{F}_p[\pi]$-basis of $\operatorname{gr} N$, cf. [Sch11a] Proposition 26.5. Hence the map

$$\begin{aligned} \mathbb{F}_p[\pi] \otimes_{\mathbb{F}_p} \mathfrak{n} &\to \operatorname{gr} N \\ f \otimes \overline{X}_\alpha &\mapsto f \,.\, \sigma\big(x_\alpha(1)\big) \end{aligned}$$

is an isomorphism of graded modules. Chevalley's commutator formula says that there are p-adic integers $c_{\alpha,\beta}$ such that $[X_\alpha, X_\beta] = c_{\alpha,\beta} X_{\alpha+\beta}$ and DK Note: clarify

$$[x_\alpha(1), x_\beta(1)] \in x_{\alpha+\beta}(c_{\alpha,\beta}) N_{-\operatorname{ht}(\alpha)-\operatorname{ht}(\beta)+1} = x_{\alpha+\beta}(1)^{c_{\alpha,\beta}} N_{-\operatorname{ht}(\alpha)-\operatorname{ht}(\beta)+1},$$

where $X_{\alpha+\beta} = 0$ and $x_{\alpha+\beta} \equiv 1$ if $\alpha + \beta \notin \Phi$. This implies that the image of the above map is a Lie subalgebra, and thus that the map is an isomorphism of Lie algebras. $\square$

Now $\operatorname{gr} N \cong \mathbb{F}_p[\pi] \otimes_{\mathbb{F}_p} \mathfrak{n}$ implies that $\mathfrak{g} \cong \mathbb{F}_p \otimes_{\mathbb{F}_p[\pi]} \mathbb{F}_p[\pi] \otimes_{\mathbb{F}_p} \mathfrak{n} \cong \mathfrak{n}$, where both $\mathfrak{g}$ and $\mathfrak{n}$ is graded by the height function. From this it clearly follows that $H^{s,t}(\mathfrak{g}, \mathbb{F}_p) \cong H^{s,t}(\mathfrak{n}, \mathbb{F}_p)$. Note that this can also be seen directly by looking at the Chevalley constants. Finally, since we proved in the previous section that N is a pro-p group, we get (as noted in (1.3)) that $H^n_{\mathrm{cts}}(N, \mathbb{F}_p) \cong H^n_{\mathrm{dsc}}(N, \mathbb{F}_p)$ for all n.

By [PT18, §2.10] (using that $p \geq h-1$) and the Universal Coefficient Theorem (as used in [PT18, §3.8]), we get a $\mathbb{F}_p$-vector space isomorphism

$$H^n(\mathfrak{n}_\mathbb{Z}, \mathbb{F}_p) = H^n(\mathfrak{n}_\mathbb{Z}, V_{\mathbb{F}_p}(0)) \cong \bigoplus_{\substack{w \in W \\ \ell(w)=n}} V_{\mathbb{F}_p}(w \cdot 0),$$

where $V_{\mathbb{F}_p}(0) = \mathbb{F}_p$ with the trivial action (concentrated in degree 0). Similarly, by the corollary in [PT18, §3.8], we have a $\mathbb{F}_p$-vector space isomorphism

$$\operatorname{gr} H^n_{\mathrm{dsc}}(\mathcal{N}_\mathbb{Z}(\mathbb{Z}), \mathbb{F}_p) = \operatorname{gr} H^n_{\mathrm{dsc}}(\mathcal{N}_\mathbb{Z}(\mathbb{Z}), V_{\mathbb{F}_p}(0)) \cong \bigoplus_{\substack{w \in W \\ \ell(w)=n}} V_{\mathbb{F}_p}(w \cdot 0).$$

Here the grading on cohomology won't be important, since we just need that

$$\dim_{\mathbb{F}_p} H^n(\mathfrak{n}_\mathbb{Z}, \mathbb{F}_p) = \dim_{\mathbb{F}_p} H^n_{\mathrm{dsc}}(\mathcal{N}_\mathbb{Z}(\mathbb{Z}), \mathbb{F}_p). \tag{1.14}$$ {eq:PT-dims}

We now equip $\mathcal{N}_{\mathbb{Z}}(\mathbb{Z})$ with the discrete topology and claim that

$$H^n_{\text{dsc}}(\mathcal{N}_{\mathbb{Z}}(\mathbb{Z}), \mathbb{F}_p) = H^n_{\text{cts}}(\mathcal{N}_{\mathbb{Z}}(\mathbb{Z}), \mathbb{F}_p) \cong H^n_{\text{cts}}(\mathcal{N}(\mathbb{Z}_p), \mathbb{F}_p).$$

Here the first equality is clear since $\mathcal{N}_{\mathbb{Z}}(\mathbb{Z})$ is equipped with the discrete topology. To see the isomorphism, first note that $\mathbb{Z}$ is a discrete group, $\mathbb{Z}_p$ is a profinite group, and the homomorphism $\mathbb{Z} \to \mathbb{Z}_p$ has dense image in $\mathbb{Z}_p$. So we have homomorphisms

$$H^n_{\text{cts}}(\mathbb{Z}_p, \mathbb{F}_p) \to H^n_{\text{cts}}(\mathbb{Z}, \mathbb{F}_p)$$

for all $n \geq 0$ from [Ser02, Sect. I §2.6]. Now both $H^0_{\text{cts}}(\mathbb{Z}, \,\cdot\,)$ and $H^0_{\text{cts}}(\mathbb{Z}_p, \,\cdot\,)$ are the functor of taking invariant, both $H^1_{\text{cts}}(\mathbb{Z}, \,\cdot\,)$ and $H^1_{\text{cts}}(\mathbb{Z}_p, \,\cdot\,)$ are the functor of taking "coinvariants", and all $H^n(\mathbb{Z}, \,\cdot\,)$ and $H^n(\mathbb{Z}_p, \,\cdot\,)$ vanish for $n \geq 2$, so $\mathbb{Z}$ is "good" in the sense of [Ser02, Section I §2.6 Exercise 2]. Thus [Ser02, Section I §2.6 Exercise 2(d)] implies that the homomorphisms

DK Note: Rewrite without "coinvariants"

DK Note: Rewrite this more like in the introduction.

$$H^n_{\text{cts}}(\mathcal{N}(\mathbb{Z}_p), \mathbb{F}_p) \to H^n_{\text{cts}}(\mathcal{N}(\mathbb{Z}), \mathbb{F}_p) \qquad n \geq 0,$$

induced by the homomorphism $\mathcal{N}(\mathbb{Z}) \to \mathcal{N}(\mathbb{Z}_p)$, are all isomorphisms.

Hence

$$\dim_{\mathbb{F}_p} H^n(\mathfrak{n}_{\mathbb{Z}}, \mathbb{F}_p) = \dim_{\mathbb{F}_p} H^n_{\text{dsc}}(\mathcal{N}_{\mathbb{Z}}(\mathbb{Z}), \mathbb{F}_p) = \dim_{\mathbb{F}_p} H^n_{\text{cts}}(\mathcal{N}(\mathbb{Z}_p), \mathbb{F}_p).$$

Now $\mathfrak{n} = \mathfrak{n}_{\mathbb{Z}} \otimes \mathbb{F}_p$, and $H^n(\mathfrak{g}, \mathbb{F}_p) \cong H^n(\mathfrak{n}, \mathbb{F}_p)$ (since $\mathfrak{g} \cong \mathfrak{n}$) is the cohomology of the complex

$$C^\bullet(\mathfrak{n}, \mathbb{F}_p) = \operatorname{Hom}_{\mathbb{F}_p}\Big(\bigwedge^\bullet \mathfrak{n}, \mathbb{F}_p\Big)$$

while $H^n(\mathfrak{n}_{\mathbb{Z}}, \mathbb{F}_p)$ is the homology of the complex

$$C^\bullet(\mathfrak{n}_{\mathbb{Z}}, \mathbb{F}_p) = \operatorname{Hom}_{\mathbb{F}_p}\Big(\bigwedge^\bullet \mathfrak{n}_{\mathbb{Z}}, \mathbb{F}_p\Big).$$

Here $\bigwedge^\bullet \mathfrak{n}_{\mathbb{Z}}$ is a free $\mathbb{Z}$-module and $(\bigwedge^\bullet \mathfrak{n}_{\mathbb{Z}}) \otimes \mathbb{F}_p \cong \bigwedge^\bullet(\mathfrak{n}_{\mathbb{Z}} \otimes \mathbb{F}_p) \cong \bigwedge^\bullet \mathfrak{n}$, so we have natural isomorphisms

$$\operatorname{Hom}_{\mathbb{F}_p}\Big(\bigwedge^\bullet \mathfrak{n}_{\mathbb{Z}}, \mathbb{F}_p\Big) \cong \operatorname{Hom}_{\mathbb{F}_p}\Big(\Big(\bigwedge^\bullet \mathfrak{n}_{\mathbb{Z}}\Big) \otimes \mathbb{F}_p, \mathbb{F}_p\Big) \cong \operatorname{Hom}_{\mathbb{F}_p}\Big(\bigwedge^\bullet \mathfrak{n}, \mathbb{F}_p\Big).$$

These isomorphisms are clearly compatible with the differentials, so $C^\bullet(\mathfrak{n}, \mathbb{F}_p) \cong C^\bullet(\mathfrak{n}_\mathbb{Z}, \mathbb{F}_p)$, and thus $H^n(\mathfrak{n}, \mathbb{F}_p) \cong H^n(\mathfrak{n}_\mathbb{Z}, \mathbb{F}_p)$. Hence

$$\dim_{\mathbb{F}_p} H^n(\mathfrak{n}, \mathbb{F}_p) = \dim_{\mathbb{F}_p} H^n(\mathfrak{n}_\mathbb{Z}, \mathbb{F}_p) = \dim_{\mathbb{F}_p} H^n(\mathcal{N}(\mathbb{Z}_p), \mathbb{F}_p).$$

Now $\dim_{\mathbb{F}_p} H^n(\mathfrak{n}, \mathbb{F}_p) = \dim^n_{\mathbb{F}_p}(\mathfrak{g}, \mathbb{F}_p)$ and $N = \mathcal{N}(\mathbb{Z}_p)$ implies that

$$\sum_{s+t=n} \dim_{\mathbb{F}_p} H^{s,t}(\mathfrak{g}, \mathbb{F}_p) = \dim_{\mathbb{F}_p} H^n(\mathfrak{g}, \mathbb{F}_p) = \dim_{\mathbb{F}_p} H^n(N, \mathbb{F}_p),$$

so the multiplicative spectral sequence

$$E_1^{s,t} = H^{s,t}(\mathfrak{g}, \mathbb{F}_p) \Longrightarrow H^{s+t}(N, \mathbb{F}_p)$$

collapses on the first page, since the dimension of $E_r^{s,t}$ is non-increasing as r increases. Since the spectral sequence collapses on the first page, we get that $E_1^{s,t} = E_\infty^{s,t}$, so

$$\mathrm{gr}^s H^n(N, \mathbb{F}_p) \cong H^n(\mathfrak{g}, \mathbb{F}_p) \cong H^n(\mathfrak{n}, \mathbb{F}_p),$$

giving us a good description of $H^n(\mathcal{N}(\mathbb{Z}_p), \mathbb{F}_p)$. Furthermore, we can describe the cup product, by calculating it in $H^*(\mathfrak{g}, \mathbb{F}_p)$ or $H^*(\mathfrak{n}, \mathbb{F}_p)$, cf. Theorem 1.5 for the details.

DK Note: Rewrite theorem nicely here.

1.4 Example: $N \subseteq \mathrm{SL}_3(\mathbb{Z}_p)$

sec:ex-N-in-SL3

In the case of $\mathcal{G} = \mathrm{SL}_3$ (in this case $h = 4$, so $p \geq 3$), we can take $\mathcal{T}$ to be the diagonal matrices in SL_3 ($\det = 1$), $\mathcal{B}$ upper triangular matrices in SL_3 and

$$\mathcal{N} = \left\{ \begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix} \right\} \subseteq \mathrm{SL}_n .$$

Furthermore we can take $\Phi^- = \{\alpha_1, \alpha_2, \alpha_3 = \alpha_1 + \alpha_2\}$ with

$$X_{\alpha_1} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad x_{\alpha_1}(A)(a) = \begin{pmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$X_{\alpha_2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \qquad x_{\alpha_2}(A)(a) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & a \\ 0 & 0 & 1 \end{pmatrix},$$

$$X_{\alpha_3} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad x_{\alpha_3}(A)(a) = \begin{pmatrix} 1 & 0 & a \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

for $\mathbb{Z}_p$-algebra A and $a \in A$. Here $\mathrm{ht}(\alpha_1) = \mathrm{ht}(\alpha_2) = -1$ and $\mathrm{ht}(\alpha_3) = -2$, and explicit calculations show that, in $N = \mathcal{N}(\mathbb{Z}_p)$, $g_1 = x_{\alpha_1}(1), g_2 = x_{\alpha_2}(1), g_3 = x_{\alpha_3}(1)$ is an ordered basis of (N, ω). Thus (cf. [Sch11a, Prop. 26.5]) $\sigma(g_1), \sigma(g_2), \sigma(g_3)$ is a basis of the $\mathbb{F}_p[\pi]$-module $\mathrm{gr}\, N$, and ξ_1, ξ_2, ξ_3 is a basis of $\mathfrak{g} = \mathbb{F}_p \otimes_{\mathbb{F}_p[\pi]} \mathrm{gr}\, N$, where $\xi_i = 1 \otimes \sigma(g_i)$. Furthermore $\mathfrak{g} = \mathfrak{g}^1 \oplus \mathfrak{g}^2$, where $\mathfrak{g}^1 = \mathrm{span}(\xi_1, \xi_2)$ and $\mathfrak{g}^2 = \mathrm{span}(\xi_3)$.

DK Note: Maybe show the calculations.

The only non-trivial commutator among the g_i's is $[g_1, g_2] = x_{\alpha_3}(-1)$, which implies (cf. [Sch11a, Rem. 26.3]) that $\sigma([g_1, g_2]) = -\sigma(g_3)$ and thus $[\xi_1, \xi_2] = -\xi_3$. So $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}^2$.

Now $H^1(\mathfrak{g}, \mathbb{F}_p) = \mathrm{Hom}_k(\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}], \mathbb{F}_p) = H^{-1,2}(\mathfrak{g}, \mathbb{F}_p)$, and, since $\bigwedge^3 \mathfrak{g} = \mathfrak{g}^1 \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^2$ is degree 4, $H^3(\mathfrak{g}, \mathbb{F}_p) = H^{-4,7}(\mathfrak{g}, \mathbb{F}_p)$. And a version of Poincaré duality (cf. [Fuk86]) gives us that $H^1 \times H^2 \to H^3$ with $H^{-1,2} \times H^{s,t} \to H^{-4,7}$ only works for $(s,t) = (-3, 5)$, so $H^2(\mathfrak{g}, \mathbb{F}_p) = H^{-3,5}(\mathfrak{g}, \mathbb{F}_p)$. This gives us a description of $H^*(N, \mathbb{F}_p)$, and we note (either by explicit calculation or by considering properties of the wedge product) that the only non-trivial cup product is $H^1(N, \mathbb{F}_p) \times H^2(N, \mathbb{F}_p) \to H^3(N, \mathbb{F}_p)$.

DK Note: Write more details here.

Chapter 2

Cohomology of pro-p Iwahori Subgroups

cha:cohiwagps

2.1 Intoduction

sec:cohiwagps-intro

In this chapter we will calculate the cohomology over perfect fields k of a collection of pro-p Iwahori subgroups of SL_n and GL_n over $\mathbb{Z}_p$ or (low degree) extensions of $\mathbb{Z}_p$.

DK Note: Maybe change perfect fields to just $\mathbb{F}_p$.

2.1.1 Background and motivation

subsec:background-iwa

Write later.

2.1.2 Setup and notation

subsec:setup-iwa

Let p be an odd prime (further restricted later) and let k be a perfect field of characteristic p.

Field extension of $\mathbb{Q}_p$. We fix a finite extension of $F/\mathbb{Q}_p$ of degree D with valuation ring $\mathcal{O}_F$ and maximal ideal $\mathfrak{m}_F = (\varpi_F) \subseteq \mathcal{O}_F$. Let $e = e(F/\mathbb{Q}_p)$ be the *ramification index* and $f = f(F/\mathbb{Q}_p)$ the *inertia degree* of the extension $F/\mathbb{Q}_p$. Let furthermore v be the valuation on F for which $v(p) = 1$, and thus $v(\varpi_F) = \frac{1}{e}$.

exp **and** log**.** Given a $\mathfrak{m}$-adic number field F with valuation ring $\mathcal{O}_F$ and maximal ideal $\mathfrak{m}_F$ with $p\mathcal{O}_F = \mathfrak{m}_F^e$, we get by [Neu99, Prop. (5.5)] (noting that we will ensure that $1 > \frac{e}{p-1}$ later) that the power series

$$\exp(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \quad \text{and} \quad \log(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \cdots ,$$

are two mutually inverse isomorphisms (and homeomorphisms)

$$\mathfrak{m}_F \underset{\log}{\overset{\exp}{\rightleftarrows}} U_F^{(1)}.$$

Note that this implies that a $\mathbb{Z}_p$-basis of $\mathfrak{m}$ translates to a $\mathbb{Z}_p$-basis of $U_F^{(1)} = 1 + \mathfrak{m}_F$ via exp.

Big-O notation. For elements of $\mathcal{O}_F$ we write $x = y + O(p^r)$ if and only if $x - y \in p^r\mathcal{O}_F$.

Matrices. Let E_{ij} denote the matrix with 1 in the (i,j) entry, and zeroes in all other entries, and write 1_n for the identity matrix in $M_n(F)$. Let $A = (a_{ij})$. We write $A = \operatorname{diag}(a_1, \dots, a_n)$ for the diagonal matrix in $M_n(F)$ with entries $a_{ii} = a_i$ in the diagonal, and $A = \operatorname{diag}_{i_1,\dots,i_k}(a_1, \dots, a_k)$ for the diagonal matrix in $M_n(F)$ with entries $a_{i_\ell i_\ell} = a_\ell$ for $\ell = 1, \dots, k$ and zeroes in all other entries. Finally, we write $A^\top$ for the transpose matrix of A.

DK Note: Use correct name for E_{ij}.

Dual basis. Let V be a k-vector space with basis $\mathcal{B} = (e_1, \dots, e_d)$. Then we let $\mathcal{B}^* = (e_1^*, \dots, e_d^*)$ be the dual basis of $\operatorname{Hom}_k(V, k)$ defined by $e_i^*(e_i) = \delta_{ij}$, where δ_{ij} is the Kronecker delta function. Now consider two vector spaces V and W with bases $\mathcal{B}_V$ and $\mathcal{B}_W$. Given a linear map $d\colon V \to W$ with matrix A when described in these bases, it's a well known fact from linear algebra that the dual map

$d^*\colon \operatorname{Hom}_k(W,k) \to \operatorname{Hom}_k(V,k)$ has matrix $A^\top$ when described in the dual bases $\mathcal{B}_V^*$ and $\mathcal{B}_W^*$. We will often use this without mention and abuse notation writing d and $d^\top$ for these matrices.

Smith normal form. Let R be an integral domain and consider only non-zero matrices over R in this paragraph. Given an $n \times m$ matrix A, there exist invertible $m \times m$ and $n \times n$ matrices S and T such that

$$SAT = \begin{pmatrix} a_1 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & a_2 & \ddots & & & \vdots \\ \vdots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & a_r & \ddots & \vdots \\ \vdots & & & \ddots & 0 & \ddots \\ \vdots & & & & \ddots & 0 \\ 0 & \cdots & \cdots & \cdots & 0 & 0 \end{pmatrix}$$

and the diagonal entries a_i satisfy $a_i \mid a_{i+1}$ for $i = 1, \ldots, r-1$. This matrix is called the Smith normal form of the matrix A. Given $n \times m$ matrices A, B, we write $A \overset{\mathsf{SNF}}{\sim} B$ if A and B have the same Smith normal form. This notation will mainly be used when B is already a matrix in Smith normal form. Finally we introduce the notation $A = \mathrm{SNF}_{n\times m}(a_1, \ldots, a_r, 0, \ldots, 0)$ for the $n \times m$ matrix with $a_{ii} = a_i$ for $i = 1, \ldots, r$ and zeroes in all other entries as above. In next subsection, we will note that the Smith normal form will be useful for our cohomology calculations.

Remark 2.1. In the case $R = \mathbb{Z}$, the Smith normal form of a matrix can be found using the following row and column operations, which are invertible over $\mathbb{Z}$.

(R1): swap rows R_i and R_j (C1): swap columns C_i and C_j

(R2): multiply row R_i by -1 (C2): multiply column C_i by -1

(R3): replace row R_i by $R_i + kR_j$ for some row $R_j \neq R_i$ and $k \in \mathbb{Z}$

(C3): replace column C_i by $C_i + kC_j$ for some column $C_j \neq C_i$ and $k \in \mathbb{Z}$.

We will not do these calculations by hand in this chapter, and will instead utilize implementations in Sage and SymPy that can find the Smith normal fom of a matrix over $\mathbb{Z}$. Here it's important to note that the SymPy implementation doesn't allow the use of the rules (R2) and (C2), so we get a small difference between the results of the calculations in SymPy and Sage, but it will only be a difference of sign on some entries in the diagonal. $\triangle$

Lazard theory. For an introduction to Lazard theory see Section 1.1, or [Sch11a] for more details. In particular, note that the Lazard Lie algebra generalizes from $\mathbb{F}_p$ to general k of characteristic p. We will let $\mathfrak{g} = k \otimes_{\mathbb{F}_p[\pi]} \operatorname{gr} I$ be the Lazard Lie algebra corresponding to the pro-p Iwahori subgroup I. Furthermore, recall that a sequence of elements $(g_1, \dots, g_r)$ in G is called an *ordered basis* of (G, ω) if the map $\mathbb{Z}_p^r \to G$ given by $(x_1, \dots, x_r) \mapsto g_1^{x_1} \cdots g_r^{x_r}$ is a bijection (and hence, by compactness, a homeomorphism) and

$$\omega(g_1^{x_1} \cdots g_r^{x_r}) = \min_{1 \leq i \leq r} (\omega(g_i) + v(x_i)) \qquad \text{for any } x_1, \dots, x_r \in \mathbb{Z}_p.$$

Algebraic groups. We will work with schemes using the functorial approach and notation described in [Jan03]. In particular, given an integral domain R, we note that a *R-group functor* is a functor from the category of all R-algebras to the category of groups, a *R-group scheme* is a R-group functor that is an affine scheme over R when considered as a R-functor, and an *algebraic R-group* is a R-group scheme that is algebraic as an affine scheme. For more in depth introduction to these concepts, we refer to [Con14b] and [Jan03].

Fixed groups and roots. We fix a split and connected reductive algebraic F-group $\mathcal{G}$, and consider the locally profinite group $G = \mathcal{G}(F)$. We then fix split maximal torus $\mathcal{T} \subseteq \mathcal{G}$ and let $T = \mathcal{T}F$. In T we have a maximal compact subgroup T^0 and its Sylow pro-p subgroup T^1.

Let $\Phi = \Phi(\mathcal{G}, \mathcal{T})$ be the *root system* of $\mathcal{G}$ with respect to $\mathcal{T}$, and let $(X^*(T), \Phi, X_*(T), \Phi^\vee)$ be the associated root datum. Fix a system of positive roots Φ^+ and let $\Pi \subseteq \Phi^+$ be the simple roots. For any $\alpha \in \Phi$ we have the root subgroup $\mathcal{U}_\alpha \subseteq \mathcal{G}$ with Lie algebra $\mathrm{Lie}\,\mathcal{U}_\alpha = (\mathrm{Lie}\,\mathcal{G})_\alpha$. We let $U_\alpha = \mathcal{U}_\alpha(F)$ and choose an isomorphism $x_\alpha \colon F \xrightarrow{\cong} U_\alpha$ such that $t x_\alpha(x) t^{-1} = x_\alpha(\alpha(t)x)$ for $t \in T$ and $x \in F$. For $r \in \mathbb{Z}_{\geq 0}$ we let $U_{\alpha,r} = x_\alpha(\mathfrak{m}_F^r)$.

DK Note: Write Δ instead of Π?

Coxeter number and p. Let h be the Coxeter number of $\mathcal{G}$ and assume from now on that $p - 1 > eh$.

Pro-p Iwahori subgroups. We follow the definitions of [OS19] with $\mathcal{G}, \mathcal{T}$ and $(U)_\alpha$ as above. Let I be the pro-p Iwahori subgroup of G (associated with a positive chamber as in [OS19], but we don't need the exact definition). We note by [OS19, Lem. 2.1(i)] and the proof of [OS19, Lem. 2.3] that I has the following factorization: Multiplication defines a homeomorphism

$$\prod_{\alpha \in \Phi^-} U_{\alpha,1} \times T^1 \times \prod_{\alpha \in \Phi^+} U_{\alpha,0} \xrightarrow{\cong} I,$$

where the products are ordered in an arbitrarily chosen way. For a more detailed introduction to these pro-p groups we refer to [OS19].

DK Note: Maybe add a more general reference too.

Pro-p Iwahori subgroups of $\mathrm{GL}_n(\mathbb{Z}_p)$ and $\mathrm{SL}_n(\mathbb{Z}_p)$. In this chapter, we will only work with pro-p Iwahori subgroups of $\mathrm{GL}_n(F)$ or $\mathrm{SL}_n(F)$, which simplifies the definitions. When $\mathcal{G} = \mathrm{GL}_n$ or $\mathcal{G} = \mathrm{SL}_n$, we can always take $\mathcal{T}$ the diagonal maximal torus, and we can take I to be the subgroup of $\mathcal{G}(\mathcal{O}_F)$ which is upper triangular and unipotent modulo ϖ. In this case we have that $U_{\alpha,1}$ for $\alpha \in \Phi^-$ correspond to entries below the diagonal and $U_{\alpha,0}$ for $\alpha \in \Phi^+$ corresponds to the entries above the diagonal.

p-valuation on I. By a recent pre-print by Lahiri and Sørensen (not yet published), we know (since $p-1 > eh$) that I admits a p-valuation ω with the property

DK Note: Figure out exactly how to write this. Probably cite as (unpublished).

$$\omega(x_\alpha(x)) = v(x) + \frac{\mathrm{ht}(\alpha)}{eh} \qquad \begin{cases} x \in \mathfrak{m}_F & \text{if } \alpha \in \Phi^-, \\ x \in \mathcal{O}_F & \text{if } \alpha \in \Phi^+. \end{cases} \tag{2.1}$$

{eq:Iwa-p-val}

Ordered basis of I. Let $\{b_1, \dots, b_D\}$ be a $\mathbb{Z}_p$-basis of $\mathcal{O}_F$ and let $\{u_1, \dots, u_D\}$ be a $\mathbb{Z}_p$-basis of $U_F^{(1)} = 1 + \mathfrak{m}_F$, where $D = [F : \mathbb{Q}_p]$. Then $\big(x_\alpha(b_1), \dots, x_\alpha(b_D)\big)$ is an ordered basis for $U_{\alpha,0}$ when $\alpha \in \Phi^+$, and $\big(x_\alpha(\varpi_F b_1), \dots, x_\alpha(\varpi_F b_D)\big)$ is an ordered basis for $U_{\alpha,1}$ when $\alpha \in \Phi^-$. Furthermore, when G is semisimple and simply connected, we have that the simple coroots $\{\alpha^\vee : \alpha \in \Pi\}$ form a $\mathbb{Z}$-basis of $X_*(T)$, and thus $\big(\alpha^\vee(u_1), \dots, \alpha^\vee(u_D)\big)_{\alpha\in\Pi}$ form an ordered basis of T^1. By [LS22, Prop. 3.1], given orderings of Φ^+ and Φ^-, and assuming that G is semisimple and simply connected, we now get: the sequence of elements

- $\big(x_\alpha(\varpi_F b_1), \dots, x_\alpha(\varpi_F b_D)\big)_{\alpha\in\Phi^-}$,
- $\big(\alpha^\vee(u_1), \dots, \alpha^\vee(u_D)\big)_{\alpha\in\Pi}$,
- $\big(x_\alpha(b_1), \dots, x_\alpha(b_D)\big)_{\alpha\in\Phi^+}$

forms an ordered basis of (I, ω) (with ω from (2.1)) which is a saturated p-valued group. Here, [LS22] notes that the p-valuation from (2.1) on this basis is given by

$$\begin{cases} \omega\big(x_\alpha(\varpi_F b_\ell)\big) = \dfrac{1}{e}\Big(1 + \dfrac{\mathrm{ht}(\alpha)}{h}\Big) & \alpha \in \Phi^- \\ \omega\big(\alpha^\vee(u_i)\big) = \dfrac{1}{e} & \alpha \in \Pi \\ \omega\big(x_\alpha(b_\ell)\big) = \dfrac{\mathrm{ht}(\alpha)}{eh} & \alpha \in \Phi^+. \end{cases} \tag{2.2}$$

{eq:Iwa-p-val-basis

Recalling from above that $\exp\colon \mathfrak{m}_F = (\varpi_F) \to U_F^{(1)} = 1 + \mathfrak{m}_F$ takes a basis to a basis, and noting that $\{\varpi_F b_1, \dots, \varpi_F b_D\}$ is a $\mathbb{Z}_p$-basis of $\mathfrak{m}_F = \varpi_F \mathcal{O}_F$, we see that we can take $u_i = \exp(\varpi_F b_i)$ for $i = 1, \dots, D$. When $\mathcal{G} = \mathrm{SL}_n$, we have that $\Phi = \{\varepsilon_i - \varepsilon_j \mid 1 \leq i, j \leq n, i \neq j\}$ and can take

DK Note: Reference for this being simply connected.

$$\Pi = \{\alpha_1 = \varepsilon_1 - \varepsilon_2, \alpha_2 = \varepsilon_2 - \varepsilon_3, \dots, \alpha_{n-1} = \varepsilon_{n-1} - \varepsilon_n\},$$

where ε_i is the map that takes a diagonal matrix to its i-th diagonal entry. In this case $\alpha_i^\vee(u) = \mathrm{diag}(0, \dots, 0, u, -u, 0, \dots, 0) = \mathrm{diag}_{i,i+1}(u, -u)$, where the non-zero entries are the i-th and $(i+1)$-th entries. This together with the above gives us the following ordered basis (in the listed order and with a chosen ordering of $\{(i,j) : 1 \leq i, j \leq n\}$) in the case $\mathcal{G} = \mathrm{SL}_n$:

DK Note: Standard Lie algebra theory, add a reference.

- $\big(1_n + \varpi_F b_1 E_{ij}, \dots, 1_n + \varpi_F b_D E_{ij}\big)_{1 \leq j < i \leq n}$,
- $\big(\mathrm{diag}_{i,i+1}(\exp(\varpi_F b_1)), \dots, \mathrm{diag}_{i,i+1}(\exp(\varpi_F b_D))\big)_{i=1,\dots,n-1}$,
- $\big(1_n + b_1 E_{ij}, \dots, 1_n + b_D E_{ij}\big)_{1 \leq i < j \leq n}$.

Here the p-valuation described in (2.1) and (2.2) is given by

$$\begin{cases} \omega\big(1_n + \varpi_F b_\ell E_{ij}\big) = \dfrac{1}{e}\Big(1 + \dfrac{j-i}{h}\Big) & j < i, \\ \omega\big(\mathrm{diag}_{i,i+1}(\exp(\varpi_F b_\ell))\big) = \dfrac{1}{e} & i = 1, \dots, n-1, \\ \omega\big(1_n + b_\ell E_{ij}\big) = \dfrac{j-i}{eh} & i < j \end{cases} \tag{2.3}$$

{eq:Iwa-p-val-basis

on the above ordered basis.

Finally note that an ordered basis of GL_n can be obtained from an ordered basis of SL_n by adding non-trivial elements of the center, which in the above corresponds to adding $\big(\exp(\varpi_F b_1)1_n, \dots, \exp(\varpi_F b_D)1_n\big)$ to the middle item above (adding the root $\varepsilon_1 + \dots + \varepsilon_n$), and the p-valuation on these is still $\frac{1}{e}$.

Cohomology. We denote (using the Chevalley-Eilenberg complex) the Lie algebra cohomology of any k-Lie algebra $\mathfrak{g}$ by $H^\bullet(\mathfrak{g}, \cdot\,)$, while we write $H^\bullet(G, \cdot\,)$ for the continuous group cohomology of a topological group G. Here we let the entries distinguish between different types of cohomology without any ambiguity. As in Section 1.1, we introduce filtrations and then gradings on the cohomology and use the notation $H^{s,t} = \operatorname{gr}^s H^{s+t}$ for any type of cohomology H.

Spectral sequences. Given a ring R, a cohomological spectral sequence is a choice of $r_0 \in \mathbb{N}$ and a collection of

- R-modules $E_r^{s,t}$ for each $s, t \in \mathbb{Z}$ and all integers $r \geq r_0$
- differentials $d_r^{s,t} \colon E_r^{s,t} \to E_r^{s+r,t+1-r}$ such that $d_r^2 = 0$ and E_{r+1} is isomorphic to the homology of (E_r, d_r), i.e.,

$$E_{r+1}^{s,t} = \frac{\ker(d_r^{s,t} \colon E_r^{s,t} \to E_r^{s+r,t+1-r})}{\operatorname{im}(d_r^{s-r,t+r-1} \colon E_r^{s-r,t+r-1} \to E_r^{s,t})}.$$

For a given r, the collection $(E_r^{s,t}, d_r^{s,t})_{s,t\in\mathbb{Z}}$ is called the r-th page. A spectral sequence *converges* if d_r vanishes on $E_r^{s,t}$ for any s, t when $r \gg 0$. In this case $E_r^{s,t}$ is independent of r for sufficiently large r, we denote it by $E_\infty^{s,t}$ and write

$$E_r^{s,t} \Longrightarrow E_\infty^{s+t}.$$

Also, we say that the spectral sequence collapses at the r'-th page if $E_r = E_\infty$ for all $r \geq r'$, but not for $r < r'$. Finally, when we have terms E_∞^n with a natural filtration $F^\bullet E_\infty^n$ (but no natural double grading), we set $E_\infty^{s,t} = \operatorname{gr}^s E_\infty^{s,t} = F^s E_\infty^{s+t} / F^{s+1} E_\infty^{s+t}$.

2.1.3 Smith normal form and cohomology

subsec:SNF-coh

It's well known that the Smith normal form of a matrices are useful when calculating (co)homology over $\mathbb{Z}$ as follows.

fact:SNF-Z-coh **Fact 2.2.** *Given a complex*

$$\mathbb{Z}^n \xrightarrow{d_1} \mathbb{Z}^m \xrightarrow{d_2} \mathbb{Z}^\ell,$$

where d_1 and d_2 are $\mathbb{Z}$-linear maps with $d_2 \circ d_1 = 0$, the homology at the middle term is given by

$$\ker(d_2)/\operatorname{im}(d_1) \cong \bigoplus_{i=1}^{r} \mathbb{Z}/a_i\mathbb{Z} \oplus \mathbb{Z}^{m-r-s}.$$

Here $r = \operatorname{rank}(d_1)$, $s = \operatorname{rank}(d_2)$ and $a_1, \ldots, a_r$ are the non-zero diagonal elements of the Smith normal form of d_1.

DK Note: Fact is still italic.

We will not directly use this result, but instead we will follow the same ideas but reduce modulo p to get matrices over k (using the natural embedding $\mathbb{F}_p \hookrightarrow k$). Assuming that the non-zero diagonal entries a_i of the Smith normal form of a matrix d are in $\{1, 2, \ldots, p-1\}$ (or more generally $\gcd(a_i, p) = 1$), we note that $a_i \pmod{p} \in k^\times$. So, given an $n \times m$ matrix d with integer entries such that

$$d \overset{\mathsf{SNF}}{\sim} \mathrm{SNF}_{n\times m}(a_1, \ldots, a_r, 0, \ldots, 0),$$

where $a_1, \ldots, a_r$ are non-zero and $\gcd(a_i, p) = 1$, we get by considering d as a matrix over k that

$$\begin{aligned} \dim_k \ker(d) &= m - r, \\ \dim_k \operatorname{im}(d) &= r, \\ \dim_k \operatorname{coker}(d) &= n - r. \end{aligned} \tag{2.4}$$

{eq:snf-dims}

Remark 2.3. Note that finding the Smith normal form of all matrices used in our (co)homology calculations, will thus allow us to calculated (co)homological dimensions for p relatively prime to all non-zero diagonal entries of the Smith normal form matrices. This is what makes this method preferable to just calculating the rank of the matrices directly, since that would just allow us to find (co)homological dimensions for $p \gg 0$, but not give us the precise p it will work for. △

So, when calculating dimensions of homology over k of the middle term in a given complex

$$k^n \xrightarrow{d_1} k^m \xrightarrow{d_2} k^\ell,$$

where d_1 and d_2 can be described by matrices with integer entries, and $d_1 \overset{\mathsf{SNF}}{\sim} \mathrm{SNF}_{m\times n}(a_1, \dots, a_r, 0, \dots, 0)$ and $d_2 \overset{\mathsf{SNF}}{\sim} \mathrm{SNF}_{\ell\times m}(b_1, \dots, b_r, 0, \dots, 0)$, then we can do it as follows:

$n = 0$: The dimension of the homology of the middle term is $\dim_k \ker(d_2) = m - s$.

$\ell = 0$: The dimension of the homology of the middle term is $\dim_k \operatorname{coker}(d_1) = r$.

$n, \ell \neq 0$: The dimension of the homology of the middle term is $\dim_k \frac{\ker(d_2)}{\mathrm{im}(d_1)} = (m - s) - (r) = m - s - r$.

Remark 2.4. Here the general formula is obviously just that the dimension of the homology of the middle term is $m - s - r$, but when implementing this, it will be useful to split it into the above cases. Also, note that this is what we directly get from Fact 2.2 in the case $k = \mathbb{F}_p$, recalling that $\mathbb{Z}/m\mathbb{Z} \otimes \mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}/\gcd(n, m)\mathbb{Z}$. △

Remark 2.5. When the dimensions of the vector spaces we work with get sufficiently large, the runtime of calculating the full Smith normal form of integer matrices becomes prohibitively high, so we will have to use an alternative solution. In this case, we can utilize that we know such a form exists, and that $\mathrm{rank}_{\mathbb{Z}}(A) = \mathrm{rank}_{\mathbb{Z}}(B)$ when $A \overset{\mathsf{SNF}}{\sim} B$. Considering the $n \times m$ matrix A with integer entries as a matrix over $\mathbb{R}$, we can then find the Singular value decomposition (SVD) of A, i.e., complex matrices U, Σ, V such that $A = U\Sigma V^*$. Here U is an $n \times n$ unitary matrix, Σ is a rectangular diagonal $n \times m$ matrix (a matrix like in the Smith normal form) with non-negative real numbers on the diagonal, and V is an $m \times m$ unitary matrix. Now $\mathrm{rank}_{\mathbb{R}} \Sigma = \mathrm{rank}_{\mathbb{Z}} A$ allows us to find dimensions of (co)homology as in the case where we know the Smith normal form, but we use information about which p exactly the calculations work for. Thus we will only be able to find the (co)homological dimensions for $p \gg 0$ in this case. $\triangle$

2.2 Techniques

sec:tech-iwa

In this section we will describe how to calculate information about the cohomology of a p-valuable group by using it's Lazard Lie algebra. Note that this section uses a lot of concepts and notation from Section 1.1.3.

Let (G, ω) be a p-valuable group and let k be a perfect field of characteristic p. In this section we will describe how the spectral sequence

$$E_1^{s,t} = H^{s,t}(\mathfrak{g}, k) \Longrightarrow H^{s+t}(G, k) \tag{2.5}$$ {eq:spec-sec-tech}

from [Sør, §6.1] can be used to calculate information about the dimensions of $H^n(G, k)$ for varying n and information about the cup product on $H^*(G, k)$. After this, we will then briefly discuss how this applies to pro-p Iwahori subgroups I of GL_n or SL_n.

Recall that $\mathfrak{g}$ in the above spectral sequence is given by $\mathfrak{g} = k \otimes_{\mathbb{F}_p[\pi]} \operatorname{gr} G$, so to describe $\mathfrak{g}$, we first need a good description of the $\mathbb{F}_p[\pi]$-Lie algebra $\operatorname{gr} G$. To get this description, suppose that we have an ordered basis $(g_1, \ldots, g_d)$ of G, so that $\omega(g) = \min_{i=1,\ldots,d}\big(\omega(g_i) + v_p(x_i)\big)$ for $g = g_1^{x_1} \cdots g_d^{x_d}$, and recall that $\big(\sigma(g_1), \ldots, \sigma(g_d)\big)$ is a basis of $\operatorname{gr} G$, where $\sigma(g) = gG_{\omega(g)+} \in \operatorname{gr} G$ for $g \neq 1$.

To understand the $\mathbb{F}_p[\pi]$-Lie algebra, we need to find $[\sigma(g_i), \sigma(g_j)] = \sigma\big([g_i, g_j]\big)$ for all $i, j = 1, \ldots, d$. We recall from (1.2) that $\sigma(g^x) = \overline{x}\pi^{v_p(x)} \,.\, \sigma(g)$ for $g \in G \setminus \{1\}$ and $x \in \mathbb{Z}_p \setminus \{0\}$. Now, calculating $[g_i, g_j]$ for all $i, j = 1, \ldots, d$, we can find $x_1, \ldots, x_d \in \mathbb{Z}_p$ such that

$$[g_i, g_j] = g_1^{x_1} \cdots g_d^{x_d},$$

and thus

$$\big[\sigma(g_i), \sigma(g_j)\big] = \sigma\big([g_i, g_j]\big) = \sum_{\ell=1}^{d} \overline{x}_\ell \pi^{v_p(x_\ell)} \,.\, \sigma(g_\ell).$$

See the proofs of [Sch11a, Lem. 26.4 and Prop. 26.5] for more details.

Let $\{\ell_1, \dots, \ell_r\}$ be the subset of $\{1, \dots, d\}$ such that $v_p(x_{\ell_s}) = 0$ and $v_p(x_\ell) > 0$ for $\ell \notin \{\ell_1, \dots, \ell_r\}$, and recall that $\mathfrak{g} = k \otimes_{\mathbb{F}_p[\pi]} \operatorname{gr} G$ has basis $1 \otimes \sigma(g_i)$. Since π acts trivially on k here, we see that

$$[\xi_i, \xi_j] = \big[1 \otimes \sigma(g_i), 1 \otimes \sigma(g_j)\big] = \sum_{s=1}^{r} \overline{x}_{\ell_s} \sigma(g_{\ell_s}).$$

Now we have a basis $(\xi_1, \dots, \xi_d)$ of $\mathfrak{g} = k \otimes_{\mathbb{F}_p[\pi]} \operatorname{gr} G$, and we know all the structure constants.

m:struc-consts-lift *Remark 2.6.* Note that the structure constants are in $\mathbb{F}_p \subseteq k$ by the above, so we can lift them to structure constants in $\{0, 1, \dots, p-1\} \subseteq \mathbb{Z}$. This will be useful later.$\triangle$

Assume from now on that the Lie algebra $\mathfrak{g}$ is unitary, i.e., that $[\xi_i, \xi_j] = \sum_{\ell=1}^{d} c_{ij\ell} \xi_\ell$ has $\sum_{j=1}^{d} c_{ijj} = 0$. This will be the case for all Lie algebras, we will work with in this chapter. Suppose furthermore that $\mathfrak{g}$ is a graded Lie algebra, graded by finitely many positive integers, $\mathfrak{g} = \mathfrak{g}^1 \oplus \mathfrak{g}^2 \oplus \cdots \oplus \mathfrak{g}^m$, which will also be the case for all Lie algebras we are work with in this chapter.

rem:g-Z-grading *Remark 2.7.* Note that any p-valuable group G admits a p-valuation ω with values in $\frac{1}{m}\mathbb{Z}$ for some $m \in \mathbb{N}$, cf. [Sch11a, Cor. 33.3]. Thus we can reindex the filtration of G by letting $G^i = G_{\frac{i}{e}}$ for $i = 0, 1, \dots,$ and this translates to $\operatorname{gr}^i G = \operatorname{gr}_{\frac{i}{e}} G$ and $\mathfrak{g}^i = \mathfrak{g}_{\frac{i}{e}}$ in general. In the cases we care about there will be no zero graded part, which allows us to make the above assumption. $\triangle$

Then $\bigwedge^n \mathfrak{g}$ is graded as well by letting

$$\operatorname{gr}^j \Big(\bigwedge^n \mathfrak{g}\Big) = \bigoplus_{j_1 + \cdots + j_n = j} \mathfrak{g}^{j_1} \wedge \cdots \wedge \mathfrak{g}^{j_n}.$$

We note that, since $\mathfrak{g}$ is finite dimensional, there are only finitely many non-zero $\operatorname{gr}^j \big(\bigwedge^n \mathfrak{g}\big)$ we are interested in, and we can find a basis of each of these using our basis $(\xi_1, \dots, \xi_d)$ of $\mathfrak{g}$.

Remark 2.8. When ordering the basis of $\mathrm{gr}^j\big(\bigwedge^n \mathfrak{g}\big) = \bigoplus_{j_1+\cdots+j_n=j} \mathfrak{g}^{j_1} \wedge \cdots \wedge \mathfrak{g}^{j_n}$, we will do it as follows. First we order the $\mathfrak{g}^{j_1} \wedge \cdots \wedge \mathfrak{g}^{j_n}$ by the lexicographical order on $(j_1, \ldots, j_n)$. Then we order the basis of each $\mathfrak{g}^{j_1} \wedge \cdots \wedge \mathfrak{g}^{j_n}$ by the lexicographical order on equal j_ℓ's, i.e., if $\mathfrak{g}^1 = \mathrm{span}_k(\xi_1, \xi_3)$ and $\mathfrak{g}^2 = \mathrm{span}_k(\xi_2, \xi_4)$, then $\mathfrak{g}^1 \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^2$ has basis $\xi_1 \wedge \xi_3 \wedge \xi_2, \xi_1 \wedge \xi_2 \wedge \xi_4$. △

Assuming furthermore that k is $\mathbb{Z}$-graded (concentrated in degree 0), the space $\mathrm{Hom}_k\big(\bigwedge^n \mathfrak{g}, k\big)$ inherits the $\mathbb{Z}$-grading

$$\mathrm{Hom}_k\Big(\bigwedge^n \mathfrak{g}, k\Big) = \bigoplus_{s \in \mathbb{Z}} \mathrm{Hom}_k^s\Big(\bigwedge^n \mathfrak{g}, k\Big),$$

where Hom_k^s denotes the homogeneous k-linear maps of degree s, cf. [FF74, Lem. 4.2]. We note, by [Fuk86, §3.7], that these gradings on the chain and cochain complexes transfer to gradings on the homology and cohomology. We write

$$H^{s,t} = H^{s,t}(\mathfrak{g}, k) = H^{s+t}\Big(\mathrm{gr}^s \,\mathrm{Hom}_k\big(\bigwedge^\bullet \mathfrak{g}, k\big)\Big).$$

Remark 2.9. We don't spend effort to describe the homology for a few reasons. First, we need the cohomology, not the homology, in our spectral sequence. Second, by [Fuk86, §3.6], we have a version of Poincaré duality for Lie algebra cohomology, i.e., $H^n(\mathfrak{g}, k) \cong H_{n-d}(\mathfrak{g}, k)$, so we can easily describe the homology using the cohomology. Third, we will care about the cup product later, and we don't get a nice product in homology, cf. [Fuk86, §3.2]. △

Now we have bases of all $\mathrm{gr}^j\big(\bigwedge^n \mathfrak{g}\big)$, and by [Fuk86, §3.7] we get graded chain complexes that we can use to find the homology of $\mathfrak{g}$. Here the boundary maps of

$$\cdots \xrightarrow{d_4} \bigwedge^3 \mathfrak{g} \xrightarrow{d_3} \bigwedge^2 \mathfrak{g} \xrightarrow{d_2} \mathfrak{g} \xrightarrow{d_1} k \longrightarrow 0,$$

are given by

$$d_n(x_1 \wedge \cdots \wedge x_n) = \sum_{i<j} (-1)^{i+j} [x_i, x_j] \wedge x_1 \wedge \cdots \wedge \widehat{x}_i \wedge \cdots \wedge \widehat{x}_j \wedge \cdots \wedge x_n,$$

and the coboundary maps

$$\cdots \xleftarrow{\partial_3} \operatorname{Hom}_k\Big(\bigwedge^2 \mathfrak{g}, k\Big) \xleftarrow{\partial_2} \operatorname{Hom}_k(\mathfrak{g}, k) \xleftarrow{\partial_1} \operatorname{Hom}_k(k,k) = k \longleftarrow 0,$$

are the dual maps of the boundary maps (see [Fuk86, §3.1] for more details). Thus, if we use the dual basis of $\bigwedge^n \mathfrak{g}$ in $\operatorname{Hom}_k\big(\bigwedge^n, k\big)$, we get that $\partial_n = d_n^\top$ as matrices, where $(\,\cdot\,)^\top$ is the transpose (cf. Section 2.1). Since we know bases and linear maps explicitly, and we know that the linear maps restrict to graded linear maps, we can now find matrices describing all graded linear maps

$$\operatorname{gr}^j\Big(\bigwedge^n \mathfrak{g}\Big) \to \operatorname{gr}^j\Big(\bigwedge^{n-1} \mathfrak{g}\Big),$$

and thus we can find matrices describing all graded linear maps

$$\operatorname{Hom}_k^s\Big(\bigwedge^{n-1} \mathfrak{g}, k\Big) \to \operatorname{Hom}_k^s\Big(\bigwedge^{n} \mathfrak{g}, k\Big).$$

Noting that all the structure constants can be lifted to $\mathbb{Z}$ and looking at the formula for the boundary maps, it's clear that the above matrices describing the (co)boundary maps can be lifted to $\mathbb{Z}$. Finding the Smith normal form of these lifts, we can calculate the cohomology over k by Section 2.1.3 for p large enough. (In most examples, $p \geq 5$ will be enough.)

Suppose now that we have found the dimensions of $H^{s,t} = H^{s,t}(\mathfrak{g}, k)$ for all s, t. To get information about the cohomology $H^n(G,k)$, we need to use the multiplicative spectral sequence (2.5), i.e.,

$$E_1^{s,t} = H^{s,t}(\mathfrak{g}, k) \Longrightarrow H^{s+t}(G,k)$$

and information about spectral sequences in general. We already know that this spectral sequence collapses at a finite page, and one can hope is that it will actually collapse at the first page. One way we can verify that the spectral sequence (in certain cases) collapses at the first page, is by considering the exact bidegrees of the differentials. We know that the differentials $d_r^{s,t} \colon E_r^{s,t} \to E_r^{s+r,t+1-r}$ has bidegree

$(r, 1-r)$, so if the non-zero modules on the first pages is distributed in such a way that all differentials $d_r^{s,t}$ are trivial for all s, t and $r \geq 1$, then we can be sure that the spectral sequence collapses on the first page. This will become clearer when we look at examples in the next few sections.

Now note, by [Fuk86, §3.7], that the cup product is compatible with the gradings on the Lie algebra cohomology, in particular $H^{s,t} \cup H^{s',t'} \subseteq H^{s+s',t+t'}$ (where $H^{s,t} = H^{s,t}(\mathfrak{g}, k)$). Thus, since the spectral sequence is multiplicative, we can describe the cup product on $H^*(G, k)$ when the spectral sequence collapses on the first page.

Even when the spectral sequence doesn't (necessarily) collapse on the first page, we can still get some bounds on the dimensions of $H^n(G, k)$ that will allow us to draw some conclusions about the structure of $H^*(G, k)$.

In the rest of this chapter we will focus on using the techniques described in this section to get as much as possible information about the cohomology of $H^*(I, k)$, where I is the pro-p Iwahori subgroup of $\mathrm{SL}_n(\mathbb{Z}_p)$ or $\mathrm{GL}_n(\mathbb{Z}_p)$.

2.3 $I \subseteq \mathrm{SL}_2(\mathbb{Z}_p)$

sec:Iwa-SL2

In this section we will describe the continuous group cohomology of the pro-p Iwahori subgroup I of $\mathrm{SL}_2(\mathbb{Q}_p)$.

When I is the pro-p Iwahori subgroup in $\mathrm{SL}_2(\mathbb{Q}_p)$, we know by Section 2.1 that we can take it to be of the form

$$I = \begin{pmatrix} 1 + p\mathbb{Z}_p & \mathbb{Z}_p \\ p\mathbb{Z}_p & 1 + p\mathbb{Z}_p \end{pmatrix}^{\det=1} \subseteq \mathrm{SL}_2(\mathbb{Z}_p).$$

In this case, an obvious guess for an ordered basis (using that $(1+p)^{\mathbb{Z}_p} = 1 + p\mathbb{Z}_p$) is

$$g_1' = \begin{pmatrix} 1 & 0 \\ p & 1 \end{pmatrix}, \qquad g_2' = \begin{pmatrix} 1+p & 0 \\ 0 & (1+p)^{-1} \end{pmatrix}, \qquad g_3' = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Because we want to be able to describe the commutators using this ordered basis, we will at one point need to solve for x in equation of the form $(1+p)^x = y$. For this reason a better choice of ordered basis is (as described in Section 2.1)

$$g_1 = \begin{pmatrix} 1 & 0 \\ p & 1 \end{pmatrix}, \qquad g_2 = \begin{pmatrix} \exp(p) & 0 \\ 0 & \exp(-p) \end{pmatrix}, \qquad g_3 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \tag{2.6}$$ {eq:gis-SL2}

In this case the above equations to solve translate to solving for x in $\exp(x) = y$, which we can easily do, as $x = \log(y)$, cf. Section 2.1.

2.3.1 Finding the commutators $[\xi_i, \xi_j]$

sec:non-id-xi_ij-SL2

Now write

$$g_1^{x_1} g_2^{x_2} g_3^{x_3} = \begin{pmatrix} \exp(px_2) & x_3 \exp(px_2) \\ px_1 \exp(px_2) & px_1 x_3 \exp(px_2) + \exp(-px_2) \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}. \tag{2.7}$$ {eq:gixi-SL2}

Furthermore, write $g_{ij} = [g_i, g_j]$ and $\xi_{ij} = [\xi_i, \xi_j]$. Then we are ready to find x_1, x_2, x_3 such that $g_{ij} = g_1^{x_1} g_2^{x_2} g_3^{x_3}$ for different $i < j$. (In the following we use that $\frac{1}{p-1} = 1 + p + p^2 + \cdots$ and $\log(1-p) = -p - \frac{p^2}{2} - \frac{p^3}{3} - \cdots$.)

We now list all non-identitiy commutators $g_{ij} = [g_i, g_j]$ and find $\xi_{ij} = [\xi_i, \xi_j]$ based on these. (For $g_{ij} = 1_2$ it's clear that $x_1 = x_2 = x_3 = 0$, and thus $\xi_{ij} = 0$.)

$g_{12} = \begin{pmatrix} 1 & 0 \\ p\big(1 - \exp(-2p)\big) & 1 \end{pmatrix}$**:** Comparing g_{12} with (2.7), we see that $x_2 = x_3 = 0$.

This leaves $a_{21} = px_1 = p\big(1 - \exp(-2p)\big) = 2p^2 + O(p^3)$, which implies that $x_1 = 2p + O(p^2)$. Hence $\sigma(g_{12}) = 2\pi \,.\, \sigma(g_1)$, which implies that $\xi_{12} = 0$.

$g_{13} = \begin{pmatrix} 1-p & p \\ -p^2 & 1+p+p^2 \end{pmatrix}$**:** Comparing g_{13} with (2.7), we see that

$$\begin{aligned} a_{11} &= \exp(px_2) = 1 - p, \\ a_{12} &= x_3 \exp(px_2) = x_3(1-p) = p, \\ a_{21} &= px_1 \exp(px_2) = px_1(1-p) = -p^2, \end{aligned}$$

and thus

$$\begin{aligned} x_2 &= \frac{1}{p}\log(1-p) = \frac{1}{p}\big((-p)+O(p^2)\big) = -1+O(p), \\ x_3 &= \frac{p}{1-p} = p + O(p^2), \\ x_1 &= \frac{-p^2}{p(1-p)} = -p + O(p^2). \end{aligned}$$

Hence $\sigma(g_{13}) = -\pi\,.\,\sigma(g_1) - \sigma(g_2) - \pi\,.\,\sigma(g_3)$, which implies that $\xi_{13} = -\xi_2$.

$g_{23} = \begin{pmatrix} 1 & \exp(2p)-1 \\ 0 & 1 \end{pmatrix}$**:** Comparing g_{23} with (2.7), we see that $x_1 = x_2 = 0$. This leaves $a_{12} = x_3 = \exp(2p) - 1 = 2p + O(p^2)$. Hence $\sigma(g_{23}) = 2\pi\,.\,\sigma(g_3)$, which implies that $\xi_{23} = 0$.

To clarify, we found that

$$\begin{aligned} \sigma(g_{12}) &= 2\pi\,.\,\sigma(g_1), \\ \sigma(g_{13}) &= -\pi\,.\,\sigma(g_1) - \sigma(g_2) - \pi\,.\,\sigma(g_3), \\ \sigma(g_{23}) &= 2\pi\,.\,\sigma(g_3), \end{aligned}$$

and recalling that $\xi_i = 1 \otimes \sigma(g_i)$ in $k \otimes_{\mathbb{F}_p[\pi]} \operatorname{gr} G$, where π acts trivially on k, we get that

$$\xi_{12} = 0, \qquad \xi_{13} = -\xi_2, \qquad \xi_{23} = 0, \tag{2.8}$$ {eq:xi_ij-SL2}

where $\xi_{ij} = [\xi_i, \xi_j]$.

2.3.2 Describing the graded chain complex, $\mathrm{gr}^j\left(\bigwedge^n \mathfrak{g}\right)$

:graded-complex-SL2

Looking at (2.3) (with $e = 1$ and $h = 2$), we see that

$$\omega(g_1) = 1 - \frac{1}{2} = \frac{1}{2}, \qquad \omega(g_2) = 1, \qquad \omega(g_3) = \frac{1}{2}.$$

Hence $\mathfrak{g}^1 = \mathfrak{g}_{\frac{1}{2}} = \mathrm{span}_k(\xi_1, \xi_3)$ and $\mathfrak{g}^2 = \mathfrak{g}_1 = \mathrm{span}_k(\xi_2)$, cf. Remark 2.7.

Now we are ready to describe the graded chain complex

$$\mathrm{gr}^j\Big(\bigwedge^n \mathfrak{g}\Big) = \bigoplus_{j_1+\cdots+j_n=j} \mathfrak{g}^{j_1} \wedge \cdots \wedge \mathfrak{g}^{j_n}$$

and its bases. We list the grading of $\bigwedge^n \mathfrak{g}$ for all n.

$n = 0$:

$$\mathrm{gr}^j(k) = \begin{cases} k & j = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Bases:

$$k : \quad 1.$$

$n = 1$:

$$\mathrm{gr}^j(\mathfrak{g}) = \begin{cases} \mathfrak{g}^2 & j = 2, \\ \mathfrak{g}^1 & j = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Bases:

$$\mathfrak{g}^1 : \quad \xi_1, \xi_3,$$
$$\mathfrak{g}^2 : \quad \xi_2.$$

$n = 2$:

$$\mathrm{gr}^j\Big(\bigwedge^2 \mathfrak{g}\Big) = \begin{cases} \mathfrak{g}^1 \wedge \mathfrak{g}^2 & j = 3, \\ \mathfrak{g}^1 \wedge \mathfrak{g}^1 & j = 2, \\ 0 & \text{otherwise.} \end{cases}$$

Bases:

$$\mathfrak{g}^1 \wedge \mathfrak{g}^2 : \quad \xi_1 \wedge \xi_2, \xi_3 \wedge \xi_2,$$
$$\mathfrak{g}^1 \wedge \mathfrak{g}^1 : \quad \xi_1 \wedge \xi_3.$$

$n = 3$:

$$\mathrm{gr}^j\Big(\bigwedge^3 \mathfrak{g}\Big) = \begin{cases} \mathfrak{g}^1 \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^2 & j = 4, \\ 0 & \text{otherwise.} \end{cases}$$

Bases:

$$\mathfrak{g}^1 \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^2 : \quad \xi_1 \wedge \xi_3 \wedge \xi_2.$$

$n \geq 4$:

$$\mathrm{gr}^j\Big(\bigwedge^n \mathfrak{g}\Big) = 0 \text{ for all } j.$$

Table 2.1: Dimensions of $\mathrm{gr}^j\big(\bigwedge^n \mathfrak{g}\big)$ for the $I \subseteq \mathrm{SL}_2(\mathbb{Z}_p)$ case. tab:graded-dims-SL2

$n \backslash j$	0	1	2	3	4
0	1				
1		2	1		
2			1	2	
3					1

We collect the above information about the dimensions of the chain complex of $\mathfrak{g}$ in Table 2.1, and note that we only need to consider non-zero (non-empty) entries of the table, when we calculate $H^{s,t} = H^{s,n-s}$ (where $H^{s,t} = H^{s,t}(\mathfrak{g}, k)$). Also, recalling that

$$\mathrm{Hom}_k\Big(\bigwedge^n \mathfrak{g}, k\Big) = \bigoplus_{s \in \mathbb{Z}} \mathrm{Hom}_k^s\Big(\bigwedge^n \mathfrak{g}, k\Big),$$

we see that, with $j = -s$, we get the same table for dimensions of the graded hom-spaces in the cochain complex.

2.3.3 Finding the graded Lie algebra cohomology, $H^{s,t}(\mathfrak{g}, k)$

subsec:graded-coh-SL2

Remark 2.10. In this section we will calculate the cohomology directly instead of using the method described in Section 2.1.3, since the calculations are only with small matrices. To see how Section 2.1.3 is used, we refer to Section 2.5. △

We will now go through all different graded chain complexes one by one, using that gr^j in the chain complex corresponds to gr^s with $s = -j$ in the cochain complex. We note that the graded chain complex corresponds to vertical downwards arrows in Table 2.1, while the cochain complex corresponds to vertical upwards arrows. And finally, we reiterate that $H^n = H^n(\mathfrak{g}, k)$ and $H^{s,t} = H^{s,t}(\mathfrak{g}, k)$ in the following.

In grade 0 we have the chain complex

$$0 \longrightarrow k \longrightarrow 0,$$

which gives us the grade 0 cochain complex

$$0 \longleftarrow \mathrm{Hom}_k^0(k, k) \longleftarrow 0.$$

So $H^0 = H^{0,0}$ with $\dim H^{0,0} = 1$.

In grade 1 we have the chain complex

$$0 \longrightarrow \mathfrak{g}^1 \longrightarrow 0,$$

which gives us the grade -1 cochain complex

$$0 \longleftarrow \mathrm{Hom}_k^{-1}(\mathfrak{g}, k) \longleftarrow 0.$$

So $\dim H^{-1,2} = 2$ by Table 2.1.

In grade 2 we have the chain complex

$$0 \longrightarrow \mathfrak{g}^1 \wedge \mathfrak{g}^1 \xrightarrow{(1)} \mathfrak{g}^2 \longrightarrow 0,$$

since

$$\begin{aligned} &\mathfrak{g}^1 \wedge \mathfrak{g}^1 \to \mathfrak{g}^2 \\ &\xi_1 \wedge \xi_3 \mapsto -[\xi_1, \xi_3] = \xi_2. \end{aligned}$$

This gives us the grade -2 cochain complex

$$0 \longleftarrow \mathrm{Hom}_k^{-2}\big(\textstyle\bigwedge^2 \mathfrak{g}, k\big) \xleftarrow{(1)} \mathrm{Hom}_k^{-2}(\mathfrak{g}, k) \longleftarrow 0.$$

So with $d = (1)$, and comparing with Table 2.1,

$$\dim H^{-2,3} = \dim \ker(d) = 0,$$
$$\dim H^{-2,4} = \dim \operatorname{coker}(d) = 0.$$

In grade 3 we have the chain complex

$$0 \longrightarrow \mathfrak{g}^1 \wedge \mathfrak{g}^2 \longrightarrow 0,$$

which gives us the grade -3 cochain complex

$$0 \longleftarrow \operatorname{Hom}_k^{-3}\big(\textstyle\bigwedge^2 \mathfrak{g}, k\big) \longleftarrow 0.$$

So $\dim H^{-3,5} = 2$ by Table 2.1.

In grade 4 we have the chain complex

$$0 \longrightarrow \mathfrak{g}^1 \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^2 \longrightarrow 0,$$

which gives us the grade -4 cochain complex

$$0 \longleftarrow \operatorname{Hom}_k^{-4}\big(\textstyle\bigwedge^3 \mathfrak{g}, k\big) \longleftarrow 0.$$

So $\dim H^{-4,7} = 1$ by Table 2.1.

Altogether, we see that

$$\begin{aligned} H^0 &= H^{0,0}, \\ H^1 &= H^{-1,2}, \\ H^2 &= H^{-3,5}, \\ H^3 &= H^{-4,7}, \end{aligned} \tag{2.9}$$

{eq:Hn-to-Hst-SL2}

with dimension as described in Table 2.2.

Table 2.2: Dimensions of $E_1^{s,t} = H^{s,t}(\mathfrak{g}, k)$ for the $I \subseteq \mathrm{SL}_2(\mathbb{Z}_p)$ case. tab:graded-coh-dims

$t \backslash s$	0	-1	-2	-3	-4
0	1				
1					
2		2			
3					
4					
5				2	
6					
7					1

2.3.4 Describing the group cohomology, $H^n(I, k)$

ubsec:group-coh-SL2

We note that all differentials $d_r^{s,t} \colon E_r^{s,t} \to E_r^{s+r,t+1-r}$ in Table 2.2 has bidegree $(r, 1-r)$, i.e., they are all below the $(r, -r)$ arrow going r to the left and r up in the table, where $r \geq 1$. Looking at Table 2.2, this clearly means that all differentials for $r \geq 1$ are trivial, and thus the spectral sequence collapses on the first page. Hence $H^{s,t}(\mathfrak{g}, k) = E_1^{s,t} \cong E_\infty^{s,t} = \mathrm{gr}^s H^{s+t}(I, k)$, and by (2.9) and Table 2.2 we get that

$$\dim H^n(I, k) = \begin{cases} 1 & n = 0, \\ 2 & n = 1, \\ 2 & n = 2, \\ 1 & n = 3. \end{cases} \tag{2.10}$$

{eq:dim-HnI-SL2}

Recalling that the spectral sequence is multiplicative, we also note, by Table 2.2, that $H^{s,t} \cup H^{s',t'} \subseteq H^{s+s',t+t'}$ implies that the cup products

$$\mathrm{gr}^s H^n(I, k) \otimes \mathrm{gr}^{s'} H^{n'}(I, k) \to \mathrm{gr}^{s+s'} H^{n+n'}(I, k)$$

are trivial. But, since the spectral sequence collapses on the first page, we also have (2.9) for $H^n(I, k)$, and thus the cup product is trivial.

2.3.5 Lower p-series of I

:lower-p-series-SL2

Possibly do later. I can show that $I_2 = [I, I]$ so $I_3 \subseteq I^p$. I'm having a few problems with $I^p \subseteq I_3$.

DK Note: Check back later.

2.4 $I \subseteq \mathrm{GL}_2(\mathbb{Z}_p)$

sec:Iwa-GL2

In this section we will describe the continuous group cohomology of the pro-p Iwahori subgroup I of $\mathrm{GL}_2(\mathbb{Q}_p)$.

When I is the pro-p Iwahori subgroup in $\mathrm{GL}_2(\mathbb{Q}_p)$, we know by Section 2.1 that we can take it to be of the form

$$I = \begin{pmatrix} 1 + p\mathbb{Z}_p & \mathbb{Z}_p \\ p\mathbb{Z}_p & 1 + p\mathbb{Z}_p \end{pmatrix} \subseteq \mathrm{GL}_2(\mathbb{Z}_p),$$

and, by Section 2.1, we have an ordered basis

$$\begin{aligned} g_1 &= \begin{pmatrix} 1 & 0 \\ p & 1 \end{pmatrix}, \qquad g_2 = \begin{pmatrix} \exp(p) & 0 \\ 0 & \exp(-p) \end{pmatrix}, \\ g_3 &= \begin{pmatrix} \exp(p) & 0 \\ 0 & \exp(p) \end{pmatrix}, \qquad g_4 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \end{aligned} \tag{2.11}$$

{eq:gis-GL2}

Since we just renamed some elements and added an element of the center of $\mathrm{GL}(\mathbb{Z}_p)$ when comparing to the ordered basis of $I \subseteq \mathrm{SL}_2(\mathbb{Z}_p)$ from Section 2.3, it's clear from Equation (2.8) that the only non-zero commutator in $\mathfrak{g} = k \otimes \operatorname{gr} G$ is

$$[\xi_1, \xi_4] = -\xi_2,$$

where $\xi_i = 1 \otimes \sigma(g_i)$ as usual.

2.4.1 Describing the graded chain complex, $\mathrm{gr}^j\left(\bigwedge^n \mathfrak{g}\right)$

:graded-complex-GL2

Looking at (2.3) (with $e = 1$ and $h = 2$) and the note about the GL_n case after (2.3), we see that

$$\omega(g_1) = \frac{1}{2}, \qquad \omega(g_2) = 1,$$
$$\omega(g_3) = 1 \qquad \omega(g_4) = \frac{1}{2}.$$

Hence $\mathfrak{g}^1 = \mathfrak{g}_{\frac{1}{2}} = \mathrm{span}_k(\xi_1, \xi_4)$ and $\mathfrak{g}^2 = \mathfrak{g}_1 = \mathrm{span}_k(\xi_2, \xi_3)$, cf. Remark 2.7.

Now we are ready to describe the graded chain complex

$$\mathrm{gr}^j\left(\bigwedge^n \mathfrak{g}\right) = \bigoplus_{j_1+\cdots+j_n=j} \mathfrak{g}^{j_1} \wedge \cdots \wedge \mathfrak{g}^{j_n}$$

and its bases.

$n = 0:$

$$\mathrm{gr}^j(k) = \begin{cases} k & j = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Bases:

$$k: \quad 1.$$

$n = 1:$

$$\mathrm{gr}^j(\mathfrak{g}) = \begin{cases} \mathfrak{g}^2 & j = 2, \\ \mathfrak{g}^1 & j = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Bases:

$$\mathfrak{g}^1: \quad \xi_1, \xi_4,$$
$$\mathfrak{g}^2: \quad \xi_2, \xi_3.$$

$n = 2:$

$$\mathrm{gr}^j\Big(\bigwedge^2 \mathfrak{g}\Big) = \begin{cases} \mathfrak{g}^2 \wedge \mathfrak{g}^2 & j = 4, \\ \mathfrak{g}^1 \wedge \mathfrak{g}^2 & j = 3, \\ \mathfrak{g}^1 \wedge \mathfrak{g}^1 & j = 2, \\ 0 & \text{otherwise.} \end{cases}$$

Bases:

$$\begin{aligned} \mathfrak{g}^2 \wedge \mathfrak{g}^2 : &\quad \xi_2 \wedge \xi_3, \\ \mathfrak{g}^1 \wedge \mathfrak{g}^2 : &\quad \xi_1 \wedge \xi_2, \xi_1 \wedge \xi_3, \xi_4 \wedge \xi_2, \xi_4 \wedge \xi_3, \\ \mathfrak{g}^1 \wedge \mathfrak{g}^1 : &\quad \xi_1 \wedge \xi_4. \end{aligned}$$

$n = 3:$

$$\mathrm{gr}^j\Big(\bigwedge^3 \mathfrak{g}\Big) = \begin{cases} \mathfrak{g}^1 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^2 & j = 5, \\ \mathfrak{g}^1 \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^2 & j = 4, \\ 0 & \text{otherwise.} \end{cases}$$

Bases:

$$\begin{aligned} \mathfrak{g}^1 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^2 : &\quad \xi_1 \wedge \xi_2 \wedge \xi_3, \xi_4 \wedge \xi_2 \wedge \xi_3, \\ \mathfrak{g}^1 \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^2 : &\quad \xi_1 \wedge \xi_4 \wedge \xi_2, \xi_1 \wedge \xi_4 \wedge \xi_3. \end{aligned}$$

$n = 4:$

$$\mathrm{gr}^j\Big(\bigwedge^4 \mathfrak{g}\Big) = \begin{cases} \mathfrak{g}^1 \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^2 & j = 6, \\ 0 & \text{otherwise.} \end{cases}$$

Bases:

$$\mathfrak{g}^1 \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^4 : \quad \xi_1 \wedge \xi_4 \wedge \xi_2 \wedge \xi_3.$$

$n \geq 5:$

$$\mathrm{gr}^j\Big(\bigwedge^n \mathfrak{g}\Big) = 0 \text{ for all } j.$$

Table 2.3: Dimensions of $\mathrm{gr}^j\left(\bigwedge^n \mathfrak{g}\right)$ for the $I \subseteq \mathrm{GL}_2(\mathbb{Z}_p)$ case. tab:graded-dims-GL2

$n \backslash j$	0	1	2	3	4	5	6
0	1						
1		2	2				
2			1	4	1		
3					2	2	
4							1

We collect the above information about the dimensions of the chain complex of $\mathfrak{g}$ in Table 2.3, and note that we only need to consider non-zero (non-empty) entries of the table, when we calculate $H^{s,t} = H^{s,n-s}$ (where $H^{s,t} = H^{s,t}(\mathfrak{g}, k)$). Also, recalling that

$$\mathrm{Hom}_k\Big(\bigwedge^n \mathfrak{g}, k\Big) = \bigoplus_{s \in \mathbb{Z}} \mathrm{Hom}_k^s\Big(\bigwedge^n \mathfrak{g}, k\Big),$$

we see that, with $j = -s$, we get the same table for dimensions of the graded hom-spaces in the cochain complex.

2.4.2 Finding the graded Lie algebra cohomology, $H^{s,t}(\mathfrak{g}, k)$

bsec:graded-coh-GL2

We will now go through all different graded chain complexes one by one, using that gr^j in the chain complex corresponds to gr^s with $s = -j$ in the cochain complex. We note that the graded chain complex corresponds to vertical downwards arrows in Table 2.3, while the cochain complex corresponds to vertical upwards arrows. And finally, we reiterate that $H^n = H^n(\mathfrak{g}, k)$ and $H^{s,t} = H^{s,t}(\mathfrak{g}, k)$ in the following.

In grade 0 we have the chain complex

$$0 \longrightarrow k \longrightarrow 0,$$

which gives us the grade 0 cochain complex

$$0 \longleftarrow \mathrm{Hom}_k^0(k, k) \longleftarrow 0.$$

So $H^0 = H^{0,0}$ with $\dim H^{0,0} = 1$.

In grade 1 we have the chain complex

$$0 \longrightarrow \mathfrak{g}^1 \longrightarrow 0,$$

which gives us the grade -1 cochain complex

$$0 \longleftarrow \mathrm{Hom}_k^{-1}(\mathfrak{g}, k) \longleftarrow 0.$$

So $\dim H^{-1,2} = 2$ by Table 2.1.

In grade 2 we have the chain complex

$$0 \longrightarrow \mathfrak{g}^1 \wedge \mathfrak{g}^1 \xrightarrow{\begin{pmatrix} 1 \\ 0 \end{pmatrix}} \mathfrak{g}^2 \longrightarrow 0,$$

since

$$\begin{aligned} &\mathfrak{g}^1 \wedge \mathfrak{g}^1 \to \mathfrak{g}^2 \\ &\xi_1 \wedge \xi_4 \mapsto -[\xi_1, \xi_4] = \xi_2. \end{aligned}$$

This gives us the grade -2 cochain complex

$$0 \longleftarrow \mathrm{Hom}_k^{-2}\big(\textstyle\bigwedge^2 \mathfrak{g}, k\big) \xleftarrow{\begin{pmatrix} 1 & 0 \end{pmatrix}} \mathrm{Hom}_k^{-2}(\mathfrak{g}, k) \longleftarrow 0.$$

So with

$$d = \begin{pmatrix} 1 & 0 \end{pmatrix},$$

and comparing with Table 2.3,

$$\begin{aligned} \dim H^{-2,3} &= \dim \ker(d) = 1, \\ \dim H^{-2,4} &= \dim \operatorname{coker}(d) = 0. \end{aligned}$$

In grade 3 we have the chain complex

$$0 \longrightarrow \mathfrak{g}^1 \wedge \mathfrak{g}^2 \longrightarrow 0,$$

which gives us the grade -3 cochain complex

$$0 \longleftarrow \mathrm{Hom}_k^{-3}\big(\textstyle\bigwedge^2 \mathfrak{g}, k\big) \longleftarrow 0.$$

So $\dim H^{-3,5} = 2$ by Table 2.1.

In grade 4 we have the chain complex

$$0 \longrightarrow \mathfrak{g}^1 \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^2 \xrightarrow{\begin{pmatrix} 0 & 1 \end{pmatrix}} \mathfrak{g}^2 \wedge \mathfrak{g}^2 \longrightarrow 0,$$

since

$$\begin{aligned}
&\mathfrak{g}^1 \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^2 \to \mathfrak{g}^2 \wedge \mathfrak{g}^2 \\
&\xi_1 \wedge \xi_4 \wedge \xi_2 \mapsto -[\xi_1, \xi_4] \wedge \xi_2 + [\xi_1, \xi_2] \wedge \xi_4 - [\xi_4, \xi_2] \wedge \xi_1 = \xi_2 \wedge \xi_2 = 0, \\
&\xi_1 \wedge \xi_4 \wedge \xi_3 \mapsto -[\xi_1, \xi_4] \wedge \xi_3 + [\xi_1, \xi_3] \wedge \xi_4 - [\xi_4, \xi_3] \wedge \xi_1 = \xi_2 \wedge \xi_3.
\end{aligned}$$

This gives us the grade -4 cochain complex

$$0 \longleftarrow \mathrm{Hom}_k^{-4}\big(\textstyle\bigwedge^3 \mathfrak{g}, k\big) \xleftarrow{\begin{pmatrix} 0 \\ 1 \end{pmatrix}} \mathrm{Hom}_k^{-4}\big(\textstyle\bigwedge^2 \mathfrak{g}, k\big) \longleftarrow 0.$$

So with

$$d = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

and comparing with Table 2.3,

$$\begin{aligned}
\dim H^{-4,6} &= \dim \ker(d) = 0, \\
\dim H^{-4,7} &= \dim \operatorname{coker}(d) = 1.
\end{aligned}$$

In grade 5 we have the chain complex

$$0 \longrightarrow \mathfrak{g}^1 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^2 \longrightarrow 0,$$

which gives us the grade -5 cochain complex

$$0 \longleftarrow \mathrm{Hom}_k^{-5}\big(\textstyle\bigwedge^3 \mathfrak{g}, k\big) \longleftarrow 0.$$

So $\dim H^{-5,8} = 2$ by Table 2.1.

In grade 6 we have the chain complex

$$0 \longrightarrow \mathfrak{g}^1 \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^2 \longrightarrow 0,$$

which gives us the grade -6 cochain complex

$$0 \longleftarrow \mathrm{Hom}_k^{-6}\big(\textstyle\bigwedge^4 \mathfrak{g}, k\big) \longleftarrow 0.$$

So $\dim H^{-6,10} = 1$ by Table 2.1.

Table 2.4: Dimensions of $E_1^{s,t} = H^{s,t}(\mathfrak{g}, k)$ for the $I \subseteq \mathrm{GL}_2(\mathbb{Z}_p)$ case. tab:graded-coh-dims

$t \backslash s$	0	-1	-2	-3	-4	-5	-6
0	1						
1							
2		2					
3			1				
4							
5				4			
6							
7					1		
8						2	
9							
10							1

Altogether, we see that

$$\begin{aligned}
H^0 &= H^{0,0}, \\
H^1 &= H^{-1,2} \oplus H^{-2,3}, \\
H^2 &= H^{-3,5}, \\
H^3 &= H^{-4,7} \oplus H^{-5,8}, \\
H^4 &= H^{-6,10},
\end{aligned} \tag{2.12}$$

{eq:Hn-to-Hst-GL2}

with dimension as described in Table 2.4.

2.4.3 Describing the group cohomology, $H^n(I, k)$

ubsec:group-coh-GL2

We note that all differentials $d_r^{s,t} \colon E_r^{s,t} \to E_r^{s+r,t+1-r}$ in Table 2.2 has bidegree $(r, 1-r)$, i.e., they are all below the $(r, -r)$ arrow going r to the left and r up in the table, where $r \geq 1$. Looking at Table 2.4, this clearly means that all differentials for $r \geq 1$ are trivial, and thus the spectral sequence collapses on the first page. Hence $H^{s,t}(\mathfrak{g}, k) = E_1^{s,t} \cong E_\infty^{s,t} = \mathrm{gr}^s H^{s+t}(I, k)$, and by (2.12) and Table 2.4 we get that

$$\dim H^n(I,k) = \begin{cases} 1 & n = 0, \\ 3 & n = 1, \\ 4 & n = 2, \\ 3 & n = 3, \\ 1 & n = 4. \end{cases} \tag{2.13}$$

{eq:dim-HnI-GL2}

Recalling that the spectral sequence is multiplicative, we also note, by Table 2.4, that $H^{s,t} \cup H^{s',t'} \subseteq H^{s+s',t+t'}$ implies that the cup products

$$\mathrm{gr}^s H^n(I,k) \otimes \mathrm{gr}^{s'} H^{n'}(I,k) \to \mathrm{gr}^{s+s'} H^{n+n'}(I,k)$$

are trivial except for the cases

$$\begin{aligned}
H^{-1,2} \cup H^{-2,3} &\subseteq H^{-3,5},\\
H^{-1,2} \cup H^{-3,5} &\subseteq H^{-4,7},\\
H^{-1,2} \cup H^{-5,8} &\subseteq H^{-6,10},\\
H^{-2,3} \cup H^{-3,5} &\subseteq H^{-5,8},\\
H^{-2,3} \cup H^{-4,7} &\subseteq H^{-6,10},\\
H^{-3,5} \cup H^{-3,5} &\subseteq H^{-6,10},
\end{aligned} \tag{2.14}$$

{eq:non-triv-cups-G

and the reverse of the above (which we can find using graded commutativity).

Next we want to describe these cup products.

Let $e_{i_1,\dots,i_m} = (\xi_{i_1} \wedge \cdots \wedge \xi_{i_m})^*$ be the element of the dual basis of $\mathrm{Hom}_k\big(\bigwedge^m \mathfrak{g}, k\big)$ corresponding to $\xi_{i_1} \wedge \cdots \wedge \xi_{i_m}$ in the basis of $\bigwedge^m \mathfrak{g}$. Looking at the cochain complexes and descriptions of the maps above together with the known bases of the graded chain complexes, we get the following precise descriptions of the of the graded cohomology spaces $H^{s,t} = H^{s,t}(\mathfrak{g}, k)$:

$$\begin{aligned}
H^{-1,2} &= k[e_1, e_4],\\
H^{-2,3} &= \ker \begin{pmatrix} 1 & 0 \end{pmatrix} = k[e_3],\\
H^{-3,5} &= k[e_{1,2}, e_{1,3}, e_{4,2}, e_{4,3}],\\
H^{-4,7} &= \frac{k[e_{1,4,2}, e_{1,4,3}]}{\operatorname{im}\begin{pmatrix} 0 \\ 1 \end{pmatrix}} = k[e_{1,4,2}],\\
H^{-5,8} &= k[e_{1,2,3}, e_{4,2,3}],\\
H^{-6,10} &= k[e_{1,4,2,3}].
\end{aligned} \tag{2.15}$$

{eq:Hst-spaces-GL2}

For $f \in \mathrm{Hom}_k\big(\bigwedge^p \mathfrak{g}, k\big)$ and $g \in \mathrm{Hom}_k\big(\bigwedge^q \mathfrak{g}, k\big)$, we know from [Car56, Chap. XIII, Sect. 8], that the cup product in cohomology is induced by: $f \cup g \in \mathrm{Hom}_k\big(\bigwedge^{p+q} \mathfrak{g}, k\big)$ defined by

DK Note: Move to techniques.

$$(f \cup g)(x_1 \wedge \cdots \wedge x_{p+q}) = \sum_{\substack{\sigma \in S_{p+q} \\ \sigma(1) < \cdots < \sigma(p) \\ \sigma(p+1) < \cdots < \sigma(p+q)}} \operatorname{sign}(\sigma) f(x_{\sigma(1)} \wedge \cdots \wedge x_{\sigma(p)}) g(x_{\sigma(p+1)} \wedge \cdots \wedge x_{\sigma(p+q)}). \tag{2.16}$$

{eq:cup-prod-def}

We will now find all the cup products in (2.14) by working with our given bases and the (2.16).

We will start by finding

$$H^{-1,2} \otimes H^{-2,3} \xrightarrow{\cup} H^{-3,5}.$$

Looking at (2.15), we need to describe the maps $e_1 \cup e_3$ and $e_4 \cup e_3$ on the basis of $\mathrm{gr}^3 \bigwedge^2 \mathfrak{g}$, i.e., on $\mathcal{B} = (\xi_1 \wedge \xi_2, \xi_1 \wedge \xi_3, \xi_4 \wedge \xi_2, \xi_4 \wedge \xi_3)$. In the case of $e_1 \cup e_3$, (2.16) simplifies to

$$(e_1 \cup e_3)(x_1 \wedge x_2) = \sum_{\sigma \in S_2} \operatorname{sign}(\sigma) e_1(x_{\sigma(1)}) e_3(x_{\sigma(2)}),$$

which is zero on all of $\mathcal{B}$ except $\xi_1 \wedge \xi_3$ with $\sigma = (1)$, where we get (using that $\operatorname{sign}\big((1)\big) = 1$)

$$(e_1 \cup e_3)(\xi_1 \wedge \xi_3) = 1.$$

Hence $e_1 \cup e_3 = e_{1,3}$. In the case of $e_4 \cup e_3$, (2.16) simplifies to

$$(e_4 \cup e_3)(x_1 \wedge x_2) = \sum_{\sigma \in S_2} \operatorname{sign}(\sigma) e_4(x_{\sigma(1)}) e_3(x_{\sigma(2)}),$$

which is zero on all of $\mathcal{B}$ except $\xi_4 \wedge \xi_3$ with $\sigma = (1)$, where we get (using that $\operatorname{sign}\big((1)\big) = 1$)

$$(e_4 \cup e_3)(\xi_4 \wedge \xi_3) = 1.$$

Hence $e_4 \cup e_3 = e_{4,3}$. Looking at (2.15), we see that $e_{1,3}$ and $e_{4,3}$ are the second and forth basis elements of $H^{-3,5}$, so the above calculation caries over to the above cup product in cohomology.

We will now describe

$$H^{-1,2} \otimes H^{-3,5} \xrightarrow{\cup} H^{-4,7}.$$

Looking at (2.15), we need to describe the maps

$$e_1 \cup e_{1,2}, e_1 \cup e_{1,3}, e_1 \cup e_{4,2}, e_1 \cup e_{4,3},$$
$$e_4 \cup e_{1,2}, e_4 \cup e_{1,3}, e_4 \cup e_{4,2}, e_4 \cup e_{4,3}$$

on the basis of $\mathrm{gr}^4 \bigwedge^3 \mathfrak{g}$, i.e., on $\mathcal{B} = (\xi_1 \wedge \xi_4 \wedge \xi_2, \xi_1 \wedge \xi_4 \wedge \xi_3)$. In any case with repeat numbers, it's clear from (2.16) (and the fact that there are no repeats in $\mathcal{B}$) that the cup product will be zero, so we only need to consider $e_1 \cup e_{4,2}$, $e_1 \cup e_{4,3}$, $e_4 \cup e_{1,2}$ and $e_4 \cup e_{1,3}$. In all cases of $e_i \cup e_{j,k}$, (2.16) simplifies to

$$(e_i \cup e_{j,k})(x_2 \wedge x_2 \wedge x_3) = \sum_{\substack{\sigma \in S_3 \\ \sigma(2) < \sigma(3)}} \mathrm{sign}(\sigma) e_i(x_{\sigma(1)}) e_{j,k}(x_{\sigma(2)} \wedge x_{\sigma(3)}),$$

i.e., the sum is over $\sigma \in \{(1), (1,2), (1,3,2)\}$. When $(i,j,k) = (1,4,2)$ or $(i,j,k) = (1,4,3)$, this sum is zero on all of $\mathcal{B}$ except $\xi_i \wedge \xi_j \wedge \xi_k$ with $\sigma = (1)$, since σ needs to fix 1 and $\mathcal{B}$ uses the same ordering. Here we get (using that $\mathrm{sign}\big((1)\big) = 1$)

$$(e_i \cup e_{j,k})(\xi_i \wedge \xi_j \wedge \xi_k) = 1.$$

Hence $e_1 \cup e_{4,2} = e_{1,4,2}$ and $e_1 \cup e_{4,3} = e_{1,4,3}$. When $(i,j,k) = (4,1,2)$ or $(i,j,k) = (4,1,3)$, the sum is zero on all of $\mathcal{B}$ except $\xi_j \wedge \xi_i \wedge \xi_k$ with $\sigma = (1,2)$, since the order of the first and second elements of (i,j,k) are swapped compared to in $\mathcal{B}$. Here we get (using that $\mathrm{sign}\big((1,2)\big) = -1$)

$$(e_i \cup e_{j,k})(\xi_i \wedge \xi_j \wedge \xi_k) = -1.$$

Hence $e_4 \cup e_{1,2} = -e_{1,4,2}$ and $e_4 \cup e_{1,3} = -e_{1,4,3}$. Looking at (2.15), we see that $e_{1,4,3}$ reduces to zero in $H^{-4,7}$, while $e_{1,4,2}$ is part of the basis. So in the cup product on the cohomology, the only nontrivial products are $e_1 \cup e_{4,2} = e_{1,4,2}$ and $e_4 \cup e_{1,2} = -e_{1,4,2}$.

At this point, it should be clear how to skip some of the details, so we will proceed with less justification than above.

Now consider

$$H^{-1,2} \otimes H^{-5,8} \xrightarrow{\cup} H^{-6,10}.$$

Looking at (2.15), the only nontrivial maps we need to describe are $e_1 \cup e_{4,2,3}$ and $e_4 \cup e_{1,2,3}$ on the basis of $\mathrm{gr}^6 \bigwedge^4 \mathfrak{g}$, i.e., on $\mathcal{B} = (\xi_1 \wedge \xi_4 \wedge \xi_2 \wedge \xi_3)$. For $e_1 \cup e_{4,2,3}$ to be non-zero, we need $\sigma \in S_4$ that fixes 1 and satisfies $\sigma(2) < \sigma(3) < \sigma(4)$, which is only true for $\sigma = (1)$. For $e_4 \cup e_{1,2,3}$ to be non-zero, we need $\sigma \in S_4$ that swaps 1 and 2 and satisfies $\sigma(2) < \sigma(3) < \sigma(4)$, which is only true for $\sigma = (1, 2)$. Since $\mathrm{sign}\big((1)\big) = 1$ and $\mathrm{sign}\big((1, 2)\big) = -1$, we get that $e_1 \cup e_{4,2,3} = e_{1,4,2,3}$ and $e_4 \cup e_{1,2,3} = -e_{1,4,2,3}$. Looking at (2.15), we see that $e_{1,4,2,3}$ it the basis elements of $H^{-6,10}$, so the above calculation caries over to the above cup products in cohomology.

Continue with

$$H^{-2,3} \otimes H^{-3,5} \xrightarrow{\cup} H^{-5,8}.$$

Looking at (2.15), the only nontrivial maps we need to describe are $e_3 \cup e_{1,2}$ and $e_3 \cup e_{4,2}$ on the basis of $\mathrm{gr}^5 \bigwedge^3 \mathfrak{g}$, i.e., on $\mathcal{B} = (\xi_1 \wedge \xi_2 \wedge \xi_3, \xi_4 \wedge \xi_2 \wedge \xi_3)$. For $e_3 \cup e_{1,2}$ or $e_3 \cup e_{4,2}$ to be non-zero, we need $\sigma \in S_3$ that satisfies $\sigma(1) = 3$ (putting ξ_3 first) and $\sigma(2) < \sigma(3)$, which is only true for $\sigma = (1, 3, 2)$. Since $\mathrm{sign}\big(((1, 3, 2)\big) = 1$, we get that $e_3 \cup e_{1,2} = e_{1,2,3}$ and $e_3 \cup e_{4,2} = e_{4,2,3}$. Looking at (2.15), we see that $e_{1,2,3}$ and $e_{4,2,3}$ are the basis elements of $H^{-5,8}$, so the above calculation caries over to the above cup products in cohomology.

Continue with

$$H^{-2,3} \otimes H^{-4,7} \xrightarrow{\cup} H^{-6,10}.$$

Looking at (2.15), the only map we need to describe is $e_3 \cup e_{1,4,2}$ on the basis of $\mathrm{gr}^6 \bigwedge^4 \mathfrak{g}$, i.e., on $\mathcal{B} = (\xi_1 \wedge \xi_4 \wedge \xi_2 \wedge \xi_3)$. For $e_3 \cup e_{1,4,2}$ to be non-zero, we need $\sigma \in S_4$ that satisfies $\sigma(1) = 4$ (putting ξ_3 first) and $\sigma(2) < \sigma(3) < \sigma(4)$, which is only true for $\sigma = (1, 4, 3, 2)$. Since $\mathrm{sign}\big(((1, 4, 3, 2)\big) = -1$, we get that $e_3 \cup e_{1,4,2} = -e_{1,4,2,3}$. Looking at (2.15), we see that $e_{1,4,2,3}$ it the basis elements of $H^{-6,10}$, so the above calculation caries over to the above cup product in cohomology.

Finally, consider

$$H^{-3,5} \otimes H^{-3,5} \xrightarrow{\cup} H^{-6,10}.$$

Looking at (2.15), the only nontrivial maps we need to describe are $e_{1,2} \cup e_{4,3}$ and $e_{1,3} \cup e_{4,2}$ (getting the rest by graded-commutativity) on the basis of $\mathrm{gr}^6 \bigwedge^4 \mathfrak{g}$, i.e., on $\mathcal{B} = (\xi_1 \wedge \xi_4 \wedge \xi_2 \wedge \xi_3)$. For $e_{1,2} \cup e_{4,3}$ to be non-zero, we need $\sigma \in S_4$ that satisfies

- $\{\sigma(1), \sigma(2)\} = \{1, 3\}$ (putting ξ_1 and ξ_2 in $e_{1,2}$),
- $\{\sigma(3), \sigma(4)\} = \{2, 4\}$ (putting ξ_4 and ξ_3 in $e_{4,3}$),
- $\sigma(1) < \sigma(2)$ and $\sigma(3) < \sigma(4)$,

which is only true for $\sigma = (2, 3)$. Since $\mathrm{sign}\big((2, 3)\big) = -1$, we get that $e_{1,2} \cup e_{4,3} = -e_{1,4,2,3}$. For $e_{1,3} \cup e_{4,2}$ to be non-zero, we need $\sigma \in S_4$ that satisfies

- $\{\sigma(1), \sigma(2)\} = \{1, 4\}$ (putting ξ_1 and ξ_3 in $e_{1,3}$),
- $\{\sigma(3), \sigma(4)\} = \{2, 3\}$ (putting ξ_4 and ξ_2 in $e_{4,2}$),
- $\sigma(1) < \sigma(2)$ and $\sigma(3) < \sigma(4)$,

which doesn't exist. Hence $e_{1,3} \cup e_{4,2} = 0$. Looking at (2.15), we see that $e_{1,4,2,3}$ it the basis elements of $H^{-6,10}$, so the above calculation caries over to the above cup products in cohomology. Also, since $H^{-3,5} = H^2$, we get, by graded commutativity of the cup product, that $e_{4,2} \cup e_{1,3} = 0$ and $e_{4,3} \cup e_{1,2} = (-1)^{2\times 2} e_{1,2} \cup e_{4,3} = e_{1,4,2,3}$.

DK Note: Or did I miss it?

In conclusion, all the non-zero cup products (up to graded commutativity) are:

$$\begin{aligned} e_1 \cup e_3 &= e_{1,3}, \\ e_4 \cup e_3 &= e_{4,3}, \\ e_1 \cup e_{4,2} &= e_{1,4,2}, \\ e_4 \cup e_{1,2} &= -e_{1,4,2}, \\ e_1 \cup e_{4,2,3} &= e_{1,4,2,3}, \\ e_4 \cup e_{1,2,3} &= -e_{1,4,2,3}, \\ e_3 \cup e_{1,2} &= e_{1,2,3}, \\ e_3 \cup e_{4,2} &= e_{4,2,3}, \\ e_3 \cup e_{1,4,2} &= -e_{1,4,2,3}, \\ e_{1,2} \cup e_{4,3} &= -e_{1,4,2,3}. \end{aligned} \tag{2.17}$$

{eq:cup-products-GL

Now, since the spectral sequence collapses on the first page, all of the above work on the cup product of the Lie algebra cohomology transfers to the cup product on $H^*(I, k)$ as described above.

2.5 $I \subseteq \mathrm{SL}_3(\mathbb{Z}_p)$

sec:Iwa-SL3

In this section we will describe the continuous group cohomology of the pro-p Iwahori subgroup I of $\mathrm{SL}_3(\mathbb{Q}_p)$.

When I is the pro-p Iwahori subgroup in $\mathrm{SL}_3(\mathbb{Q}_p)$, we know by Section 2.1 that we can take it to be of the form

$$I = \begin{pmatrix} 1 + p\mathbb{Z}_p & \mathbb{Z}_p & \mathbb{Z}_p \\ p\mathbb{Z}_p & 1 + p\mathbb{Z}_p & \mathbb{Z}_p \\ p\mathbb{Z}_p & p\mathbb{Z}_p & 1 + p\mathbb{Z}_p \end{pmatrix}^{\det = 1} \subseteq \mathrm{SL}_3(\mathbb{Z}_p),$$

and, by Section 2.1, we have an ordered basis

$$g_1 = \begin{pmatrix} 1 & & \\ & 1 & \\ p & & 1 \end{pmatrix}, \quad g_2 = \begin{pmatrix} 1 & & \\ p & 1 & \\ & & 1 \end{pmatrix}, \quad g_3 = \begin{pmatrix} 1 & & \\ & 1 & \\ & p & 1 \end{pmatrix},$$
$$g_4 = \begin{pmatrix} \exp(p) & & \\ & \exp(-p) & \\ & & 1 \end{pmatrix}, \quad g_5 = \begin{pmatrix} 1 & & \\ & \exp(p) & \\ & & \exp(-p) \end{pmatrix}, \tag{2.18}$$
$$g_6 = \begin{pmatrix} 1 & & \\ & 1 & 1 \\ & & 1 \end{pmatrix}, \quad g_7 = \begin{pmatrix} 1 & 1 & \\ & 1 & \\ & & 1 \end{pmatrix}, \quad g_8 = \begin{pmatrix} 1 & & 1 \\ & 1 & \\ & & 1 \end{pmatrix}.$$

{eq:gis-SL3}

Here we write any zeros as blank space in matrices, to make the notation easier to read for the bigger matrices.

2.5.1 Finding the commutators $[\xi_i, \xi_j]$

ec:non-id-xi_ij-SL3

Now

$$g_1^{x_1} g_2^{x_2} g_3^{x_3} g_4^{x_4} g_5^{x_5} g_6^{x_6} g_7^{x_7} g_8^{x_8} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix},$$

where

$$
\begin{aligned}
a_{11} &= \exp(px_4),\\
a_{12} &= x_7 \exp(px_4),\\
a_{13} &= x_8 \exp(px_4),\\
a_{21} &= px_2 \exp(px_4),\\
a_{22} &= px_2x_7 \exp(px_4) + \exp\big(p(x_5 - x_4)\big),\\
a_{23} &= px_2x_8 \exp(px_4) + x_6 \exp\big(p(x_5 - x_4)\big),\\
a_{31} &= px_1 \exp(px_4),\\
a_{32} &= px_1x_7 \exp(px_4) + px_3 \exp\big(p(x_5 - x_4)\big),\\
a_{33} &= px_1x_8 \exp(px_4) + px_3x_6 \exp\big(p(x_5 - x_4)\big) + \exp(-px_5).
\end{aligned} \tag{2.19}
$$

{eq:gixi-SL3}

Furthermore, write $g_{ij} = [g_i, g_j]$ and $\xi_{ij} = [\xi_i, \xi_j]$. Then we are ready to find $x_1, \dots, x_8$ such that $g_{ij} = g_1^{x_1} \cdots g_8^{x_8}$ for different $i < j$. (In the following we use that $\frac{1}{p-1} = 1 + p + p^2 + \cdots$ and $\log(1-p) = -p - \frac{p^2}{2} - \frac{p^3}{3} - \cdots$.) Also, except in the first case, we will note that $x_i \in p\mathbb{Z}_p$ implies that the coefficient on ξ_k in ξ_{ij} is zero.

We now list all non-identitiy commutators $g_{ij} = [g_i, g_j]$ and find $\xi_{ij} = [\xi_i, \xi_j]$ based on these. (For $g_{ij} = 1_3$ it's clear that $x_1 = \cdots = x_8 = 0$, and thus $\xi_{ij} = 0$.)

$g_{14} = \begin{pmatrix} 1 & & \\ & & 1 \\ p\big(1-\exp(-p)\big) & & 1 \end{pmatrix}$: Comparing g_{14} with (2.19), we see that $x_2 = x_4 = x_7 = x_8 = 0$, and thus also $x_3 = x_5 = x_6 = 0$. This leaves $a_{31} = px_1 = p\big(1-\exp(-p)\big) = p^2 + O(p^3)$, which implies that $x_1 = p + O(p^2)$. Hence $\sigma(g_{14}) = \pi \,.\, \sigma(g_1)$, which implies that $\xi_{14} = 0$.

$g_{15} = \begin{pmatrix} 1 & & \\ & & 1 \\ p\big(1-\exp(-p)\big) & & 1 \end{pmatrix}$: Since $g_{15} = g_{14}$, the above calculation shows that $\xi_{15} = 0$.

$g_{16} = \begin{pmatrix} 1 & & \\ -p & 1 & \\ & & 1 \end{pmatrix}$: Comparing g_{16} with (2.19), we see that $x_1 = x_4 = x_7 = x_8 = 0$, and thus also $x_3 = x_5 = x_6 = 0$. This leaves $a_{21} = px_2 = -p$, which implies that $x_2 = -1$. Hence $\sigma(g_{16}) = -\sigma(g_2)$, which implies that $\xi_{16} = -\xi_2$.

$g_{17} = \begin{pmatrix} 1 & & \\ & 1 & \\ & p & 1 \end{pmatrix}$: Comparing g_{17} with (2.19), we see that $x_1 = x_2 = x_4 = x_7 = x_8 = 0$, and thus also $x_5 = x_6 = 0$. This leaves $a_{32} = px_3 = p$, which implies that $x_3 = 1$. Hence $\sigma(g_{17}) = \sigma(g_3)$, which implies that $\xi_{17} = \xi_3$.

$g_{18} = \begin{pmatrix} 1-p & & p \\ & 1 & \\ -p^2 & & 1+p+p^2 \end{pmatrix}$: Comparing g_{18} with (2.19), we see that $x_2 = x_7 = 0$, and thus also $x_3 = x_6 = 0$ and $x_4 = x_5$. Using

$$a_{11} = \exp(px_4) = 1-p,$$
$$a_{13} = x_8 \exp(px_4) = x_8(1-p) = p,$$
$$a_{31} = px_1 \exp(px_4) = px_1(1-p) = -p^2,$$

we get that

$$x_4 = \frac{1}{p}\log(1-p) = \frac{1}{p}\big((-p) + O(p^2)\big) = -1 + O(p),$$
$$x_8 = \frac{p}{1-p} = p + O(p^2),$$
$$x_1 = \frac{-p^2}{p(1-p)} = -p + O(p^2).$$

Hence $\sigma(g_{18}) = -\pi \,.\, \sigma(g_1) - \sigma(g_4) - \sigma(g_5) + \pi \,.\, \sigma(g_8)$, which implies that $\xi_{18} = -(\xi_4 + \xi_5)$.

$g_{23} = \begin{pmatrix} 1 & & \\ & 1 & \\ -p^2 & & 1 \end{pmatrix}$: Comparing g_{23} with (2.19), we see that $x_2 = x_4 = x_7 = x_8 = 0$, and thus also $x_3 = x_5 = x_6 = 0$. This leaves $a_{31} = px_1 = -p^2$, which implies that $x_1 = -p$. Hence $\sigma(g_{23}) = -\pi \,.\, \sigma(g_1)$, which implies that $\xi_{23} = 0$.

$g_{24} = \begin{pmatrix} 1 & & \\ p\big(1-\exp(-2p)\big) & 1 & \\ & & 1 \end{pmatrix}$: Comparing g_{24} with (2.19), we see that $x_1 = x_4 = x_7 = x_8 = 0$, and thus also $x_3 = x_5 = x_6 = 0$. This leaves $a_{21} = px_2 = p\big(1-\exp(-2p)\big) = p\Big(1-\big(1+(-2p)+O(p^2)\big)\Big) = 2p^2 + O(p^3)$, which implies that $x_2 = 2p + O(p^2)$. Hence $\sigma(g_{24}) = 2\pi \,.\, \sigma(g_1)$, which implies that $\xi_{24} = 0$.

$g_{25} = \begin{pmatrix} 1 & & \\ p\big(1-\exp(p)\big) & 1 & \\ & & 1 \end{pmatrix}$: Except a factor -2 in the exponential, which clearly doesn't change the final result, we have the same calculation as for g_{24}. Thus $\xi_{25} = 0$.

$g_{27} = \begin{pmatrix} 1-p & p & \\ -p^2 & 1+p+p^2 & \\ & & 1 \end{pmatrix}$: Comparing g_{27} with (2.19), we see that $x_1 = x_8 = 0$, and thus also $x_3 = x_6 = 0$, so $x_5 = 0$. Using

$$\begin{aligned} a_{11} &= \exp(px_4) = 1-p, \\ a_{12} &= x_7 \exp(px_4) = x_8(1-p) = p, \\ a_{21} &= px_2 \exp(px_4) = px_2(1-p) = -p^2, \end{aligned}$$

we get that

$$\begin{aligned} x_4 &= \frac{1}{p}\log(1-p) = \frac{1}{p}\big((-p)+O(p^2)\big) = -1 + O(p), \\ x_7 &= \frac{p}{1-p} = p + O(p^2), \end{aligned}$$

$$x_2 = \frac{-p^2}{p(1-p)} = -p + O(p^2).$$

Hence $\sigma(g_{27}) = -\pi \,.\, \sigma(g_2) - \sigma(g_4) + \pi \,.\, \sigma(g_7)$, which implies that $\xi_{27} = -\xi_4$.

$g_{28} = \begin{pmatrix} 1 & & \\ & 1 & p \\ & & 1 \end{pmatrix}$: Comparing g_{28} with (2.19), we see that $x_1 = x_2 = x_4 = x_7 = x_8 = 0$, and thus also $x_3 = x_5 = 0$. This leaves $a_{23} = x_6 = p$. Hence $\sigma(g_{28}) = \pi \,.\, \sigma(g_6)$, which implies that $\xi_{28} = 0$.

$g_{34} = \begin{pmatrix} 1 & & \\ & 1 & \\ & p\big(1-\exp(p)\big) & 1 \end{pmatrix}$: Comparing g_{34} with (2.19), we see that $x_1 = x_2 = x_4 = x_7 = x_8 = 0$, and thus also $x_5 = x_6 = 0$. This leaves $a_{32} = px_3 = p\big(1-\exp(p)\big) = p\big(1-\big(1+p+O(p^2)\big)\big) = -p^2 + O(p^3)$, which implies that $x_3 = -p + O(p^2)$. Hence $\sigma(g_{34}) = -\pi \,.\, \sigma(g_3)$, which implies that $\xi_{34} = 0$.

$g_{35} = \begin{pmatrix} 1 & & \\ & 1 & \\ & p\big(1-\exp(-2p)\big) & 1 \end{pmatrix}$: Except a factor -2 in the exponential, which clearly doesn't change the final result, we have the same calculation as for g_{34}. Thus $\xi_{35} = 0$.

$g_{36} = \begin{pmatrix} 1 & & \\ & 1-p & p \\ & -p^2 & 1+p+p^2 \end{pmatrix}$: Comparing g_{36} with (2.19), we see that $x_1 = x_2 = x_4 = x_7 = x_8 = 0$. Using

$$a_{22} = \exp(px_5) = 1-p,$$

$$a_{23} = x_6 \exp(px_5) = x_6(1-p) = p,$$

$$a_{32} = px_3 \exp(px_5) = px_3(1-p) = -p^2,$$

we get that

$$x_5 = \frac{1}{p}\log(1-p) = \frac{1}{p}\big((-p) + O(p^2)\big) = -1 + O(p),$$
$$x_6 = \frac{p}{1-p} = p + O(p^2),$$
$$x_3 = \frac{-p^2}{p(1-p)} = -p + O(p^2).$$

Hence $\sigma(g_{36}) = -\pi \,.\, \sigma(g_3) - \sigma(g_5) + \pi \,.\, \sigma(g_6)$, which implies that $\xi_{36} = -\xi_5$.

$g_{38} = \begin{pmatrix} 1 & -p & \\ & 1 & \\ & & 1 \end{pmatrix}$: Comparing g_{38} with (2.19), we see that $x_1 = x_2 = x_4 = x_8 = 0$, and thus also $x_3 = x_5 = x_6 = 0$. This leaves $a_{12} = x_7 = -p$. Hence $\sigma(g_{38}) = -\pi \,.\, \sigma(g_3)$, which implies that $\xi_{38} = 0$.

$g_{46} = \begin{pmatrix} 1 & & \\ & 1 & \exp(-p) - 1 \\ & & 1 \end{pmatrix}$: Comparing g_{46} with (2.19), we see that $x_1 = x_2 = x_4 = x_7 = x_8 = 0$, and thus also $x_3 = x_5 = 0$. This leaves $a_{23} = x_6 = \exp(-p) - 1 = -p + O(p^2)$. Hence $\sigma(g_{46}) = -\pi \,.\, \sigma(g_6)$, which implies that $\xi_{46} = 0$.

$g_{47} = \begin{pmatrix} 1 & \exp(2p) - 1 & \\ & 1 & \\ & & 1 \end{pmatrix}$: Comparing g_{47} with (2.19), we see that $x_1 = x_2 = x_4 = x_8 = 0$, and thus also $x_3 = x_5 = x_6 = 0$. This leaves $a_{12} = x_7 = \exp(2p) - 1 = 2p + O(p^2)$. Hence $\sigma(g_{47}) = 2\pi \,.\, \sigma(g_7)$, which implies that $\xi_{47} = 0$.

$g_{48} = \begin{pmatrix} 1 & & \exp(p) - 1 \\ & 1 & \\ & & 1 \end{pmatrix}$: Comparing g_{48} with (2.19), we see that $x_1 = x_2 = x_4 = x_7 = 0$, and thus also $x_3 = x_5 = x_6 = 0$. This leaves $a_{13} = x_8 = \exp(p) - 1 =$

$p + O(p^2)$. Hence $\sigma(g_{48}) = \pi \,.\, \sigma(g_8)$, which implies that $\xi_{48} = 0$.

$g_{56} = \begin{pmatrix} 1 & & \\ & 1 & \exp(2p) - 1 \\ & & 1 \end{pmatrix}$: Except a factor -2 in the exponential, which clearly doesn't change the final result, we have the same calculation as for g_{46}. Thus $\xi_{56} = 0$.

$g_{57} = \begin{pmatrix} 1 & \exp(-p) - 1 & \\ & 1 & \\ & & 1 \end{pmatrix}$: Except a factor -2 in the exponential, which clearly doesn't change the final result, we have the same calculation as for g_{47}. Thus $\xi_{57} = 0$.

$g_{58} = \begin{pmatrix} 1 & & \exp(p) - 1 \\ & 1 & \\ & & 1 \end{pmatrix}$: Since $g_{58} = g_{48}$, the above calculation shows that $\xi_{58} = 0$.

$g_{67} = \begin{pmatrix} 1 & & -1 \\ & 1 & \\ & & 1 \end{pmatrix}$: Comparing g_{67} with (2.19), we see that $x_1 = x_2 = x_4 = x_7 = 0$, and thus also $x_3 = x_5 = x_6 = 0$. This leaves $a_{13} = x_8 = -1$. Hence $\sigma(g_{67}) = -\sigma(g_8)$, which implies that $\xi_{67} = -\xi_8$.

Thus the non-zero commutators are:

$$\begin{aligned} &[\xi_1, \xi_6] = -\xi_2, \quad [\xi_1, \xi_7] = \xi_3, \quad\ \ [\xi_1, \xi_8] = -(\xi_4 + \xi_5), \\ &[\xi_2, \xi_7] = -\xi_4, \quad [\xi_3, \xi_6] = -\xi_5, \quad [\xi_6, \xi_7] = -\xi_8. \end{aligned} \tag{2.20}$$

{eq:xi_ij-SL3}

2.5.2 Describing the graded chain complex, $\mathrm{gr}^j\left(\bigwedge^n \mathfrak{g}\right)$

:graded-complex-SL3

Looking at (2.3) (with $e = 1$ and $h = 3$), we see that

$$\begin{aligned}
\omega(g_1) &= 1 - \frac{2}{3} = \frac{1}{3},\\
\omega(g_2) &= 1 - \frac{1}{3} = \frac{2}{3},\\
\omega(g_3) &= 1 - \frac{1}{3} = \frac{2}{3},\\
\omega(g_4) &= 1,\\
\omega(g_5) &= 1,\\
\omega(g_6) &= \frac{1}{3},\\
\omega(g_7) &= \frac{1}{3},\\
\omega(g_8) &= \frac{2}{3}.
\end{aligned}$$

Hence

$$\mathfrak{g} = k \otimes_{\mathbb{F}_p[\pi]} \mathrm{gr}\, I = \mathrm{span}_k(\xi_1, \ldots, \xi_8) = \mathfrak{g}^1 \oplus \mathfrak{g}^2 \oplus \mathfrak{g}^3,$$

where

$$\begin{aligned}
\mathfrak{g}^1 &= \mathfrak{g}_{\frac{1}{3}} = \mathrm{span}_k(\xi_1, \xi_6, \xi_7),\\
\mathfrak{g}^2 &= \mathfrak{g}_{\frac{2}{3}} = \mathrm{span}_k(\xi_2, \xi_3, \xi_8),\\
\mathfrak{g}^3 &= \mathfrak{g}_1 = \mathrm{span}_k(\xi_4, \xi_5).
\end{aligned}$$

See Remark 2.7 for more details.

Now we are ready to describe the graded chain complex

$$\mathrm{gr}^j\Big(\bigwedge^n \mathfrak{g}\Big) = \bigoplus_{j_1+\cdots+j_n=j} \mathfrak{g}^{j_1} \wedge \cdots \wedge \mathfrak{g}^{j_n},$$

but we will skip the description of the bases this time. For a description of the basis, we refer to the supplemental files of [Konne]. We list the grading of $\bigwedge^n \mathfrak{g}$ for all n.

$n = 0:$

$$\mathrm{gr}^j(k) = \begin{cases} k & j = 0, \\ 0 & \text{otherwise.} \end{cases}$$

$n = 1:$

$$\mathrm{gr}^j(\mathfrak{g}) = \begin{cases} \mathfrak{g}^3 & j = 3, \\ \mathfrak{g}^2 & j = 2, \\ \mathfrak{g}^1 & j = 1, \\ 0 & \text{otherwise.} \end{cases}$$

$n = 2:$

$$\mathrm{gr}^j\Big(\bigwedge^2 \mathfrak{g}\Big) = \begin{cases} \mathfrak{g}^3 \wedge \mathfrak{g}^3 & j = 6, \\ \mathfrak{g}^2 \wedge \mathfrak{g}^3 & j = 5, \\ \mathfrak{g}^1 \wedge \mathfrak{g}^3 \oplus \mathfrak{g}^2 \wedge \mathfrak{g}^2 & j = 4, \\ \mathfrak{g}^1 \wedge \mathfrak{g}^2 & j = 3, \\ \mathfrak{g}^1 \wedge \mathfrak{g}^1 & j = 2, \\ 0 & \text{otherwise.} \end{cases}$$

$n = 3$:

$$\mathrm{gr}^j\Big(\bigwedge^3 \mathfrak{g}\Big) = \begin{cases} \mathfrak{g}^2 \wedge \mathfrak{g}^3 \wedge \mathfrak{g}^3 & j = 8, \\ \begin{matrix} \mathfrak{g}^1 \wedge \mathfrak{g}^3 \wedge \mathfrak{g}^3 \\ \oplus\, \mathfrak{g}^2 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^3 \end{matrix} & j = 7, \\ \begin{matrix} \mathfrak{g}^1 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^3 \\ \oplus\, \mathfrak{g}^2 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^2 \end{matrix} & j = 6, \\ \begin{matrix} \mathfrak{g}^1 \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^3 \\ \oplus\, \mathfrak{g}^1 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^2 \end{matrix} & j = 5, \\ \mathfrak{g}^1 \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^2 & j = 4, \\ \mathfrak{g}^1 \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^1 & j = 3, \\ 0 & \text{otherwise.} \end{cases}$$

$n = 4$:

$$\mathrm{gr}^j\Big(\bigwedge^4 \mathfrak{g}\Big) = \begin{cases} \mathfrak{g}^2 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^3 \wedge \mathfrak{g}^3 & j = 10, \\ \begin{matrix} \mathfrak{g}^1 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^3 \wedge \mathfrak{g}^3 \\ \oplus\, \mathfrak{g}^2 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^3 \end{matrix} & j = 9, \\ \begin{matrix} \mathfrak{g}^1 \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^3 \wedge \mathfrak{g}^3 \\ \oplus\, \mathfrak{g}^1 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^3 \end{matrix} & j = 8, \\ \begin{matrix} \mathfrak{g}^1 \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^3 \\ \oplus\, \mathfrak{g}^1 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^2 \end{matrix} & j = 7, \\ \begin{matrix} \mathfrak{g}^1 \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^3 \\ \oplus\, \mathfrak{g}^1 \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^2 \end{matrix} & j = 6, \\ \mathfrak{g}^1 \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^2 & j = 5, \\ 0 & \text{otherwise.} \end{cases}$$

$n = 5:$

$$\mathrm{gr}^j\Big(\bigwedge^5 \mathfrak{g}\Big) = \begin{cases} \mathfrak{g}^2 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^3 \wedge \mathfrak{g}^3 & j = 12, \\ \mathfrak{g}^1 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^3 \wedge \mathfrak{g}^3 & j = 11, \\ \mathfrak{g}^1 \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^3 \wedge \mathfrak{g}^3 \\ \quad \oplus \mathfrak{g}^1 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^3 & j = 10, \\ \mathfrak{g}^1 \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^3 \wedge \mathfrak{g}^3 \\ \quad \oplus \mathfrak{g}^1 \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^3 & j = 9, \\ \mathfrak{g}^1 \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^3 \\ \quad \oplus \mathfrak{g}^1 \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^2 & j = 8, \\ \mathfrak{g}^1 \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^2 & j = 7, \\ 0 & \text{otherwise.} \end{cases}$$

$n = 6:$

$$\mathrm{gr}^j\Big(\bigwedge^6 \mathfrak{g}\Big) = \begin{cases} \mathfrak{g}^1 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^3 \wedge \mathfrak{g}^3 & j = 13, \\ \mathfrak{g}^1 \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^3 \wedge \mathfrak{g}^3 & j = 12, \\ \mathfrak{g}^1 \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^3 \wedge \mathfrak{g}^3 \\ \quad \oplus \mathfrak{g}^1 \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^3 & j = 11, \\ \mathfrak{g}^1 \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^3 & j = 10, \\ \mathfrak{g}^1 \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^2 & j = 9, \\ 0 & \text{otherwise.} \end{cases}$$

$n = 7:$

$$\mathrm{gr}^j\Big(\bigwedge^7 \mathfrak{g}\Big) = \begin{cases} \mathfrak{g}^1 \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^3 \wedge \mathfrak{g}^3 & j = 14, \\ \mathfrak{g}^1 \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^3 \wedge \mathfrak{g}^3 & j = 13, \\ \mathfrak{g}^1 \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^3 & j = 12, \\ 0 & \text{otherwise.} \end{cases}$$

$n = 8:$

$$\mathrm{gr}^j\Big(\bigwedge^8 \mathfrak{g}\Big) = \begin{cases} \mathfrak{g}^1 \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^3 \wedge \mathfrak{g}^3 & j = 15, \\ 0 & \text{otherwise.} \end{cases}$$

$n \geq 9:$

$$\mathrm{gr}^j\Big(\bigwedge^n \mathfrak{g}\Big) = 0 \text{ for all } j.$$

Table 2.5: Dimensions of $\mathrm{gr}^j\left(\bigwedge^n \mathfrak{g}\right)$.

tab:graded-dims-SL3

$n \backslash j$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	1															
1		3	3	2												
2			3	9	9	6	1									
3				1	9	15	19	9	3							
4						3	11	21	21	11	3					
5								3	9	19	15	9	1			
6										1	6	9	9	3		
7													2	3	3	
8																1

We collect the above information about the dimensions of the chain complex of $\mathfrak{g}$ in Table 2.5, and note that we only need to consider non-zero (non-empty) entries of the table, when we calculate $H^{s,t} = H^{s,n-s}$ (where $H^{s,t} = H^{s,t}(\mathfrak{g}, k)$). Also, recalling that

$$\mathrm{Hom}_k\Big(\bigwedge^n \mathfrak{g}, k\Big) = \bigoplus_{s \in \mathbb{Z}} \mathrm{Hom}_k^s\Big(\bigwedge^n \mathfrak{g}, k\Big),$$

we see that, with $j = -s$, we get the same table for dimensions of the graded hom-spaces in the cochain complex.

2.5.3 Finding the graded Lie algebra cohomology, $H^{s,t}(\mathfrak{g}, k)$

sec:graded-coh-SL3

We will now go through all different graded chain complexes one by one, using that gr^j in the chain complex corresponds to gr^s with $s = -j$ in the cochain complex. We note that the graded chain complex corresponds to vertical downwards arrows in

Table 2.5, while the cochain complex corresponds to vertical upwards arrows. And finally, we reiterate that $H^n = H^n(\mathfrak{g}, k)$ and $H^{s,t} = H^{s,t}(\mathfrak{g}, k)$ in the following.

Remark 2.11. We will repeatedly use that, if

$$d \overset{\mathsf{SNF}}{\sim} \mathrm{SNF}_{n\times m}(a_1, \dots, a_r, 0, \dots, 0)$$

with $a_1, \dots, a_r$ non-zero (in $\mathbb{F}_p$), then

$$\begin{aligned} \dim \ker(d) &= m - r, \\ \dim \operatorname{im}(d) &= r, \\ \dim \operatorname{coker}(d) &= n - r, \end{aligned}$$

as described in Section 2.1.3. △

In grade 0 we have the chain complex

$$0 \longrightarrow k \longrightarrow 0$$

which gives us the grade 0 cochain complex

$$0 \longleftarrow \mathrm{Hom}_k^0(k, k) \longleftarrow 0$$

So $H^0 = H^{0,0}$ with $\dim H^{0,0} = 1$.

In grade 1 we have the chain complex

$$0 \longrightarrow \mathfrak{g}^1 \longrightarrow 0$$

which gives us the grade -1 cochain complex

$$0 \longleftarrow \mathrm{Hom}_k^{-1}(\mathfrak{g}, k) \longleftarrow 0$$

So $\dim H^{-1,2} = 3$ by Table 2.5.

In grade 2 we have the chain complex

$$0 \longrightarrow \mathfrak{g}^1 \wedge \mathfrak{g}^1 \xrightarrow{\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}} \mathfrak{g}^2 \longrightarrow 0$$

since

$$\begin{aligned}
&\mathfrak{g}^1 \wedge \mathfrak{g}^1 \to \mathfrak{g}^2 \\
&\xi_1 \wedge \xi_6 \mapsto -[\xi_1, \xi_6] = \xi_2 \\
&\xi_1 \wedge \xi_7 \mapsto -[\xi_1, \xi_7] = -\xi_3 \\
&\xi_6 \wedge \xi_7 \mapsto -[\xi_6, \xi_7] = \xi_8.
\end{aligned}$$

This gives us the grade -2 cochain complex

$$0 \longleftarrow \mathrm{Hom}_k^{-2}\big(\textstyle\bigwedge^2 \mathfrak{g}, k\big) \xleftarrow{\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}} \mathrm{Hom}_k^{-2}(\mathfrak{g}, k) \longleftarrow 0,$$

where

$$d = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \overset{\mathsf{SNF}}{\sim} \mathrm{SNF}_{3\times 3}(1, 1, 1).$$

So

$$\begin{aligned}
\dim H^{-2,3} &= \dim \ker(d) = 0, \\
\dim H^{-2,4} &= \dim \operatorname{coker}(d) = 0.
\end{aligned}$$

In grade 3 we have the chain complex

$$0 \longrightarrow \mathfrak{g}^1 \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^1 \xrightarrow{\begin{pmatrix} 0 & 0 & -1 & 0 & -1 & 0 & -1 & 0 & 0 \end{pmatrix}^{\top}} \mathfrak{g}^1 \wedge \mathfrak{g}^2 \xrightarrow[\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \end{pmatrix}]{} \mathfrak{g}^3 \longrightarrow 0$$

which gives us the grade -3 cochain complex

$$0 \leftarrow \mathrm{Hom}_k^{-3}\left(\bigwedge^3 \mathfrak{g}, k\right) \xleftarrow{\begin{pmatrix} 0 & 0 & -1 & 0 & -1 & 0 & -1 & 0 & 0 \end{pmatrix}} \mathrm{Hom}_k^{-3}\left(\bigwedge^2 \mathfrak{g}, k\right) \xleftarrow[\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \end{pmatrix}^{\top}]{} \mathrm{Hom}_k^{-3}(\mathfrak{g}, k) \leftarrow 0,$$

where

$$d_1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \\ 0 & 0 \\ 0 & -1 \\ 0 & 0 \\ 0 & -1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \overset{\mathsf{SNF}}{\sim} \mathrm{SNF}_{9\times 2}(1,1),$$

$$d_2 = \begin{pmatrix} 0 & 0 & -1 & 0 & -1 & 0 & -1 & 0 & 0 \end{pmatrix} \overset{\mathsf{SNF}}{\sim} \mathrm{SNF}_{1\times 9}(1).$$

So

$$\begin{aligned} \dim H^{-3,4} &= \dim \ker(d_1) = 2 - 2 = 0, \\ \dim H^{-3,5} &= \dim \frac{\ker(d_2)}{\mathrm{im}(d_1)} = (9-1) - 2 = 6, \\ \dim H^{-3,6} &= \dim \mathrm{coker}(d_2) = 1 - 1 = 0. \end{aligned}$$

In grade 4 we have the chain complex

$$0 \longrightarrow \mathfrak{g}^1 \wedge \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^2 \xrightarrow{d^\top} \begin{matrix} \mathfrak{g}^1 \wedge \mathfrak{g}^3 \\ \oplus\, \mathfrak{g}^2 \wedge \mathfrak{g}^2 \end{matrix} \longrightarrow 0$$

which gives us the grade -4 cochain complex

$$0 \longleftarrow \mathrm{Hom}_k^{-4}\big(\textstyle\bigwedge^3 \mathfrak{g}, k\big) \xleftarrow{d} \mathrm{Hom}_k^{-4}\big(\textstyle\bigwedge^2 \mathfrak{g}, k\big) \longleftarrow 0$$

where

$$d \overset{\mathsf{SNF}}{\sim} \mathrm{SNF}_{9\times 9}(1,1,1,1,1,1,0,0,0).$$

So

$$\dim H^{-4,6} = \dim\ker(d) = 9 - 6 = 3,$$

$$\dim H^{-4,7} = \dim\operatorname{coker}(d) = 9 - 6 = 3.$$

In grade 5 we have the chain complex

$$0 \longrightarrow \mathfrak{g}^1 \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^2 \xrightarrow{d_2^\top} \begin{matrix} \mathfrak{g}^1 \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^3 \\ \oplus\, \mathfrak{g}^1 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^2 \end{matrix} \xrightarrow{d_1^\top} \mathfrak{g}^2 \wedge \mathfrak{g}^3 \longrightarrow 0$$

which gives us the grade -5 cochain complex

$$0 \leftarrow \mathrm{Hom}_k^{-5}\big(\textstyle\bigwedge^4 \mathfrak{g}, k\big) \overset{d_2}{\leftarrow} \mathrm{Hom}_k^{-5}\big(\textstyle\bigwedge^3 \mathfrak{g}, k\big) \overset{d_1}{\leftarrow} \mathrm{Hom}_k^{-5}\big(\textstyle\bigwedge^2 \mathfrak{g}, k\big) \leftarrow 0,$$

where

$$d_1 \overset{\mathsf{SNF}}{\sim} \mathrm{SNF}_{15\times 6}(1,1,1,1,1,1),$$

$$d_2 \overset{\mathsf{SNF}}{\sim} \mathrm{SNF}_{3\times 15}(1,1,1).$$

So

$$\dim H^{-5,7} = \dim\ker(d_1) = 6 - 6 = 0,$$

$$\dim H^{-5,8} = \dim\frac{\ker(d_2)}{\mathrm{im}(d_1)} = (15-3) - 6 = 6,$$

$$\dim H^{-5,9} = \dim\operatorname{coker}(d_2) = 3 - 3 = 0.$$

In grade 6 we have the chain complex

$$0 \longrightarrow \begin{array}{c} \mathfrak{g}^1 \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^3 \\ \oplus\, \mathfrak{g}^1 \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^2 \end{array} \xrightarrow{d_2^\top} \begin{array}{c} \mathfrak{g}^1 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^3 \\ \oplus\, \mathfrak{g}^2 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^2 \end{array} \xrightarrow{d_1^\top} \mathfrak{g}^3 \wedge \mathfrak{g}^3 \longrightarrow 0$$

which gives us the grade -6 cochain complex

$$0 \leftarrow \mathrm{Hom}_k^{-6}\big(\textstyle\bigwedge^4 \mathfrak{g}, k\big) \overset{d_2}{\leftarrow} \mathrm{Hom}_k^{-6}\big(\textstyle\bigwedge^3 \mathfrak{g}, k\big) \overset{d_1}{\leftarrow} \mathrm{Hom}_k^{-6}\big(\textstyle\bigwedge^2 \mathfrak{g}, k\big) \leftarrow 0,$$

where

$$\begin{aligned} d_1 &\overset{\mathsf{SNF}}{\sim} \mathrm{SNF}_{19\times 1}(1), \\ d_2 &\overset{\mathsf{SNF}}{\sim} \mathrm{SNF}_{11\times 19}(1,1,1,1,1,1,1,1,1,1,2). \end{aligned}$$

So

$$\begin{aligned} \dim H^{-6,8} &= \dim \ker(d_1) = 1 - 1 = 0, \\ \dim H^{-6,9} &= \dim \frac{\ker(d_2)}{\mathrm{im}(d_1)} = (19-11) - 1 = 7, \\ \dim H^{-6,10} &= \dim \mathrm{coker}(d_2) = 11 - 11 = 0. \end{aligned}$$

In grade 7 we have the chain complex

$$0 \to \mathfrak{g}^1 \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^2 \overset{d_2^\top}{\to} \begin{array}{c} \mathfrak{g}^1 \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^3 \\ \oplus\, \mathfrak{g}^1 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^2 \end{array} \overset{d_1^\top}{\to} \begin{array}{c} \mathfrak{g}^1 \wedge \mathfrak{g}^3 \wedge \mathfrak{g}^3 \\ \oplus\, \mathfrak{g}^2 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^3 \end{array} \to 0$$

which gives us the grade -7 cochain complex

$$0 \leftarrow \mathrm{Hom}_k^{-7}\big(\textstyle\bigwedge^5 \mathfrak{g}, k\big) \overset{d_2}{\leftarrow} \mathrm{Hom}_k^{-7}\big(\textstyle\bigwedge^4 \mathfrak{g}, k\big) \overset{d_1}{\leftarrow} \mathrm{Hom}_k^{-7}\big(\textstyle\bigwedge^3 \mathfrak{g}, k\big) \leftarrow 0,$$

where

$$\begin{aligned} d_1 &\overset{\mathsf{SNF}}{\sim} \mathrm{SNF}_{21\times 9}(1,1,1,1,1,1,1,1,1), \\ d_2 &\overset{\mathsf{SNF}}{\sim} \mathrm{SNF}_{3\times 21}(1,1,1). \end{aligned}$$

So

$$\begin{aligned}
\dim H^{-7,10} &= \dim \ker(d_1) = 9 - 9 = 0, \\
\dim H^{-7,11} &= \dim \frac{\ker(d_2)}{\mathrm{im}(d_1)} = (21 - 3) - 9 = 9, \\
\dim H^{-7,12} &= \dim \mathrm{coker}(d_2) = 3 - 3 = 0.
\end{aligned}$$

By [Fuk86, §3.6 and §3.7], we can now find the rest of the cohomology using a version of Poincaré duality for Lie algebra cohomology. But we keep the sketch work to make it clear that nothing goes wrong. We refer to [Konne] for the calculations.

In grade -8 we get coboundary maps

$$\begin{aligned}
d_1 &\overset{\mathsf{SNF}}{\sim} \mathrm{SNF}_{21\times 3}(1,1,1), \\
d_2 &\overset{\mathsf{SNF}}{\sim} \mathrm{SNF}_{9\times 21}(1,1,1,1,1,1,1,1,1).
\end{aligned}$$

So

$$\begin{aligned}
\dim H^{-8,11} &= \dim \ker(d_1) = 3 - 3 = 0, \\
\dim H^{-8,12} &= \dim \frac{\ker(d_2)}{\mathrm{im}(d_1)} = (21 - 9) - 3 = 9, \\
\dim H^{-8,13} &= \dim \mathrm{coker}(d_2) = 9 - 9 = 0.
\end{aligned}$$

In grade -9 we get coboundary maps

$$\begin{aligned}
d_1 &\overset{\mathsf{SNF}}{\sim} \mathrm{SNF}_{19\times 11}(1,1,1,1,1,1,1,1,1,1,1), \\
d_2 &\overset{\mathsf{SNF}}{\sim} \mathrm{SNF}_{1\times 19}(1).
\end{aligned}$$

So

$$\begin{aligned}
\dim H^{-9,13} &= \dim \ker(d_1) = 11 - 11 = 0, \\
\dim H^{-9,14} &= \dim \frac{\ker(d_2)}{\mathrm{im}(d_1)} = (19 - 1) - 11 = 7, \\
\dim H^{-9,15} &= \dim \mathrm{coker}(d_2) = 1 - 1 = 0.
\end{aligned}$$

In grade -10 we get coboundary maps

$$d_1 \overset{\mathsf{SNF}}{\sim} \mathrm{SNF}_{15\times 3}(1,1,1),$$
$$d_2 \overset{\mathsf{SNF}}{\sim} \mathrm{SNF}_{6\times 15}(1,1,1,1,1,1).$$

So

$$\dim H^{-10,14} = \dim\ker(d_1) = 3-3 = 0,$$
$$\dim H^{-10,15} = \dim\frac{\ker(d_2)}{\mathrm{im}(d_1)} = (15-6)-3 = 6,$$
$$\dim H^{-10,16} = \dim\mathrm{coker}(d_2) = 6-6 = 0.$$

In grade -11 we get coboundary maps

$$d \overset{\mathsf{SNF}}{\sim} \mathrm{SNF}_{9\times 9}(1,1,1,1,1,1,0,0,0).$$

So

$$\dim H^{-11,16} = \dim\ker(d) = 9-6 = 3,$$
$$\dim H^{-11,17} = \dim\mathrm{coker}(d) = 9-6 = 3.$$

In grade -12 we get coboundary maps

$$d_1 \overset{\mathsf{SNF}}{\sim} \mathrm{SNF}_{9\times 1}(1),$$
$$d_2 \overset{\mathsf{SNF}}{\sim} \mathrm{SNF}_{2\times 9}(1,1).$$

So

$$\dim H^{-12,17} = \dim\ker(d_1) = 1-1 = 0,$$
$$\dim H^{-12,18} = \dim\frac{\ker(d_2)}{\mathrm{im}(d_1)} = (9-2)-1 = 6,$$
$$\dim H^{-12,19} = \dim\mathrm{coker}(d_2) = 2-2 = 0.$$

In grade -13 we get coboundary maps

$$d \overset{\mathsf{SNF}}{\sim} \mathrm{SNF}_{3\times 3}(1,1,1).$$

So

$$\dim H^{-13,19} = \dim \ker(d) = 3 - 3 = 0,$$

$$\dim H^{-13,20} = \dim \operatorname{coker}(d) = 3 - 3 = 0.$$

In grade 14 we have the chain complex

$$0 \longrightarrow \mathfrak{g}^1 \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^3 \wedge \mathfrak{g}^3 \longrightarrow 0$$

which gives us the grade -14 cochain complex

$$0 \longleftarrow \operatorname{Hom}_k^{-14}\big(\textstyle\bigwedge^7 \mathfrak{g}, k\big) \longleftarrow 0$$

So $\dim H^{-14,21} = 3$ by Table 2.5.

In grade 15 we have the chain complex

$$0 \longrightarrow \mathfrak{g}^1 \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^3 \wedge \mathfrak{g}^3 \longrightarrow 0$$

which gives us the grade -15 cochain complex

$$0 \longleftarrow \operatorname{Hom}_k^{-15}\big(\textstyle\bigwedge^8 \mathfrak{g}, k\big) \longleftarrow 0$$

So $H^8 = H^{-15,23}$ with $\dim H^{-15,23} = 1$ by Table 2.5.

Altogether, we see that

$$\begin{aligned}
H^0 &= H^{0,0}, \\
H^1 &= H^{-1,2}, \\
H^2 &= H^{-3,5} \oplus H^{-4,6}, \\
H^3 &= H^{-4,7} \oplus H^{-5,8} \oplus H^{-6,9}, \\
H^4 &= H^{-7,11} \oplus H^{-8,12}, \\
H^5 &= H^{-9,14} \oplus H^{-10,15} \oplus H^{-11,16}, \\
H^6 &= H^{-11,17} \oplus H^{-12,18}, \\
H^7 &= H^{-14,21}, \\
H^8 &= H^{-15,23}
\end{aligned} \tag{2.21}$$

{eq:Hn-to-Hst-SL3}

with dimension as described in Table 2.6.

Table 2.6: Dimensions of $E_1^{s,t} = H^{s,t} = \mathrm{gr}^s\, H^{s+t}(\mathfrak{g}, k)$ for the $I \subseteq \mathrm{SL}_3(\mathbb{Z}_p)$ case. tab:graded-coh-dims

t \ s	0	-1	-2	-3	-4	-5	-6	-7	-8	-9	-10	-11	-12	-13	-14	-15
0	1															
1																
2		3														
3																
4																
5				6												
6					3											
7					3											
8						6										
9							7									
10																
11								9								
12									9							
13																
14										7						
15											6					
16												3				
17												3				
18													6			
19																
20																
21															3	
22																
23																1

2.5.4 Describing the group cohomology, $H^n(I, k)$

subsec:group-coh-SL3

We note that all differentials $d_r^{s,t} \colon E_r^{s,t} \to E_r^{s+r,t+1-r}$ in Table 2.6 has bidegree $(r, 1-r)$, i.e., they are all below the $(r, -r)$ arrow going r to the left and r up in the table, where $r \geq 1$. Looking at Table 2.6, this clearly means that all differentials for $r \geq 1$ are trivial, and thus the spectral sequence collapses on the first page. Hence

$H^{s,t}(\mathfrak{g}, k) = E_1^{s,t} \cong E_\infty^{s,t} = \operatorname{gr}^s H^{s+t}(I, k)$, and by (2.21) and Table 2.6 we get that

$$\dim H^n(I,k) = \begin{cases} 1 & n = 0, \\ 3 & n = 1, \\ 9 & n = 2, \\ 16 & n = 3, \\ 18 & n = 4, \\ 16 & n = 5, \\ 9 & n = 6, \\ 3 & n = 7, \\ 1 & n = 8. \end{cases} \tag{2.22}$$

{eq:dim-HnI-SL3}

Recalling that the spectral sequence is multiplicative, we also note, by Table 2.6, that $H^{s,t} \cup H^{s',t'} \subseteq H^{s+s',t+t'}$ implies that the cup products

$$\operatorname{gr}^s H^n(I,k) \otimes \operatorname{gr}^{s'} H^{n'}(I,k) \to \operatorname{gr}^{s+s'} H^{n+n'}(I,k)$$

are trivial. But, since the spectral sequence collapses on the first page, we also have (2.21) for $H^n(I, k)$, and thus the cup product is trivial.

2.6 $I \subseteq \mathrm{GL}_3(\mathbb{Z}_p)$

sec:Iwa-GL3

$$
\begin{gathered}
g_1 = \begin{pmatrix} 1 & & \\ & 1 & \\ p & & 1 \end{pmatrix}, \quad g_2 = \begin{pmatrix} 1 & & \\ p & 1 & \\ & & 1 \end{pmatrix}, \quad g_3 = \begin{pmatrix} 1 & & \\ & 1 & \\ & p & 1 \end{pmatrix}, \\
g_4 = \begin{pmatrix} \exp(p) & & \\ & \exp(-p) & \\ & & 1 \end{pmatrix}, \quad g_5 = \begin{pmatrix} 1 & & \\ & \exp(p) & \\ & & \exp(-p) \end{pmatrix}, \\
g_6 = \begin{pmatrix} \exp(p) & & \\ & \exp(p) & \\ & & \exp(p) \end{pmatrix}, \\
g_7 = \begin{pmatrix} 1 & & \\ & 1 & 1 \\ & & 1 \end{pmatrix}, \quad g_8 = \begin{pmatrix} 1 & 1 & \\ & 1 & \\ & & 1 \end{pmatrix}, \quad g_9 = \begin{pmatrix} 1 & & 1 \\ & 1 & \\ & & 1 \end{pmatrix}.
\end{gathered} \tag{2.23}
$$

{eq:gis-GL3}

2.7 $I \subseteq \mathrm{SL}_4(\mathbb{Z}_p)$ and $I \subseteq \mathrm{GL}_4(\mathbb{Z}_p)$

sec:Iwa-SL4-GL4

2.8 Nilpotency index

sec:nilp-index

Before ending this chapter with a brief discussion of future research directions, we will mention an interesting consequence of our above calculations.

Given any cohomology theory H (say over k), one can always think of the ring H^* with the cup product as $H^* = k \oplus H^+$, where $k = H^0$ and $H^+ = \bigoplus_{n>0} H^n$. Assuming that only finitely many H^n are non-zero and that each H^n is finite dimensional, one can note that H^+ must be nilpotent. Thus an interesting question becomes: what is the nilpotency index of H^+? I.e., what's the smallest positive integer m such that $(H^+)^m = 0$? In continuation of this, another slightly easier

Table 2.7: Dimensions of $E_1^{s,t} = H^{s,t} = \operatorname{gr}^s H^{s+t}(\mathfrak{g}, k)$ for the $I \subseteq \mathrm{GL}_3(\mathbb{Z}_p)$ case. tab:graded-coh-dims

$t \backslash s$	0	−1	−2	−3	−4	−5	−6	−7	−8	−9	−10	−11	−12	−13	−14	−15	−16	−17	−18
0	1																		
1																			
2		3																	
3																			
4				1															
5				6															
6					6														
7					3														
8						6													
9							13												
10								3											
11								12											
12									15										
13										7									
14										7									
15											15								
16												12							
17												3							
18													13						
19														6					
20															3				
21															6				
22																6			
23																1			
24																			
25																		3	
26																			
27																			1

question to answer is, what is the nilpotency index of H^1? I.e., what's the smallest positive integer m such that $(H^1)^m = 0$.

Furthermore, one can consider the exterior algebra $\Lambda(H^*)$, which by definition is $T(H^*)/I$, where $T(H^*) = \bigoplus_{\ell=0}^{\infty} (H^*)^{\otimes \ell}$ is the tensor algebra with multiplication given by the canonical isomorphisms $(H^*)^{\otimes m} \otimes (H^*)^{\otimes \ell} \to (H^*)^{\otimes(m+\ell)}$, and I is the two-sided ideal generated by all elements of the form $x \otimes x$. Now another interesting question is, what's the smallest number of wedges needed to ensure that $H^* \wedge \cdots \wedge H^* = 0$? Or simpler, what's the smallest number of wedges needed to ensure that $H^1 \wedge \cdots \wedge H^1 = 0$?

DK Note: Is this the correct setup? Otherwise, how to change it? Answer this part later.

We will now try to answer the above questions for the group cohomology $H^*(I,k)$ in each of the cases we have discussed in this chapter. Before beginning, recall that

$$H^{s,t} \cup H^{s',t'} \subseteq H^{s+s',t+t'} \tag{2.24}$$ {eq:graded-coh-inc}

by [Fuk86, §3.7].

In the $I \subseteq \mathrm{SL}_2(\mathbb{Z}_p)$ case, we saw in Equation (2.10) that the cup product is trivial, so $H^* \cup H^* = 0$ and thus also $H^1 \cup H^1 = 0$.

In the $I \subseteq \mathrm{GL}_2(\mathbb{Z}_p)$ case, we completely described the (graded) cup product in Equation (2.13), which should be enough to answer the questions. Looking at Table 2.4 and using (2.24), we see that an upper bound for H^1 is that

$$H^1 \cup H^1 \cup H^1 \neq 0, H^1 \cup H^1 \cup H^1 \cup H^1 \qquad = 0,$$

by starting with $H^{-1,2} \cup H^{-2,3} \subseteq H^{-3,5} \neq 0$ and then using that $H^{-3,5} \cup H^{-1,2} \subseteq H^{-4,7} \neq 0$ or $H^{-3,5} \cup H^{-2,3} \subseteq H^{-5,8} \neq 0$, and finally $H^{-4,7} \cup H^{-2,3} \subseteq H^{-6,10} \neq 0$ or $H^{-5,8} \cup H^{-1,2} \subseteq H^{-6,10} \neq 0$. The question is whether we can follow those steps with non-zero cup products. We note by (2.17) that

$$e_1 \cup e_3 = e_{1,3}, \qquad e_4 \cup e_3 = e_{4,3},$$

are the only non-zero cup product we can do in $H^1 = H^{-1,2} \oplus H^{-2,3}$. But $H^{-1,2} = k[e_1, e_4]$ and $H^{-2,3} = k[e_3]$ by (2.15), and we already noted in Section 2.4.3 that $e_{i_1,\dots,i_m} \cup e_{j_1,\dots,j_\ell}$ if $\{i_1,\dots,i_m\} \cap \{j_1,\dots,j_\ell\} \neq \emptyset$, so we clearly can't cup with anything from H^1 without getting zero. Thus

$$H^1 \cup H^1 \neq 0, H^1 \cup H^1 \cup H^1 \qquad = 0.$$

Now, having only four possible numbers in the subscript and using the above equation, we note that we can only ever hope to have two cup products before getting zero (cf. (2.15)). By (2.17)

$$e_3 \cup (e_1 \cup e_{4,2}) = e_3 \cup e_{1,4,2} = -e_{1,4,2,3} \neq 0,$$

so

$$H^* \cup H^* \cup H^* \neq 0,$$
$$H^* \cup H^* \cup H^* \cup H^* = 0,$$

for $I \subseteq \mathrm{GL}_2(\mathbb{Z}_p)$.

In the $I \subseteq \mathrm{SL}_2(\mathbb{Z}_p)$ case, we haven't described the cup product in detail, but we can tell purely from (2.24) and Table 2.6, that $H^1 \cup H^1 = 0$. Going through Table 2.6, we also note that an upper bound for H^* is

$$H^* \cup H^* \cup H^* \neq 0,$$
$$H^* \cup H^* \cup H^* \cup H^* = 0,$$

DK Note: Check this one in more detail. Write a bit for the last few.

sec:future

2.9 Future work

An interesting research direction in the future is to try to work with the Serre spectral sequence in the following way.

Assume we have the "standard" setup with $\mathcal{G} = \mathrm{SL}_n$, $\mathcal{U}$ unipotent upper triangular matrices and $\mathcal{T}$ diagonal matrices with determinant 1. Let also $I \subseteq \mathrm{SL}_n(\mathbb{Z}_p)$ be the pro-p Iwahori subgroup of $\mathrm{SL}_n(\mathbb{Q}_p)$ which is upper triangular and unipotent modulo p, and let

$$K := \ker\bigl(\mathrm{red}\colon \mathcal{G}(\mathbb{Z}_p) \to \mathcal{G}(\mathbb{F}_p)\bigr),$$

where $\mathrm{red}\colon \mathcal{G}(\mathbb{Z}_p) \to \mathcal{G}(\mathbb{F}_p)$ is the reduction map. (Note that $I = \{g \in \mathcal{G}(\mathbb{Z}_p) : \mathrm{red}(g) \in \mathcal{U}(\mathbb{F}_p)\}$ in this case, cf. [CR16].) Then

$$I/K \cong \mathcal{U}(\mathbb{F}_p),$$

and thus we get the Serre spectral sequence

$$E_2^{i,j} = H^i\bigl(\mathcal{U}(\mathbb{F}_p), H^j(K, \mathbb{F}_p)\bigr) \Longrightarrow H^{i+j}(I, \mathbb{F}_p),$$

which is also a multiplicative spectral sequence. Since K is a uniformly powerful group (cf. [OS19, Prop. 7.6]), we know by [Laz65, p. 183] that

$$H^j(K, \mathbb{F}_p) \cong \bigwedge^j \mathrm{Hom}_{\mathbb{F}_p}(K, \mathbb{F}_p).$$

Now we can let $\mathrm{SL}_n(\mathbb{Z}_p)$ act by

$$(g \,.\, f)(x) = f(g^{-1}xg)$$

for $g \in \mathrm{SL}_n(\mathbb{Z}_p)$, $f\colon K \to \mathbb{F}_p$ and $x \in K$, and hope to split $\bigwedge^j \mathrm{Hom}_{\mathbb{F}_p}(K, \mathbb{F}_p)$ into a direct sum of Verma modules $\bigoplus_\lambda V(\lambda)$ for p-restricted λ (λ with $0 \le \langle \lambda, \alpha^\vee \rangle \le p-1$), similarly to what's done in [PT18] (as we used in Chapter 1). This description might generalize easier than what we have worked with in this chapter, but it's harder to get started with since the spectral sequence is more complicated. One can hope that the difference in the spectral sequence might make it so that it will collapse on the second page (the starting page in this case).

DK Note: Maybe add a note about dimension of Lie algebra center conjecture.

Appendix A

Calculations

A.1 $I \subseteq \mathrm{SL}_4(\mathbb{Z}_p)$

sec:SL4-calc

A.2 $I \subseteq \mathrm{GL}_4(\mathbb{Z}_p)$

sec:GL4-calc

Appendix B

Other research

cha:robstat

B.1 Introduction

sec:robstat-intro

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