Dissertation

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Contents

Some extra stuff.

Chapter 1

Cohomology of Unipotent Groups

cha:cohunigps

1.1 Introduction

sec:cohunigps-intro

In this chapter we show that the cohomology of certain unipotent groups can be found via a simpler cohomology calculation for related Lie algebras. This is done using a spectral sequence due to [Sør].

Background and motivation

The cohomology of Lie groups has a long history. In particular, the mod p cohomology of a connected compact real Lie group has been well understood by Kac since the eighties, and the mod p cohomology $H^*(G, \mathbb{F}_p)$ of a equi-p-valued compact p-adic Lie group G was already described by Lazard in the sixties. This chapters work will build on several ideas of Lazard and Serre in their more general (but not yet finished) description of the case when G is not equi-p-valued, but we will focus only on unipotent groups originating from split and connected reductive $\mathbb{Z}_p$-groups, which is similar to recent work in the case of $\mathbb{Z}_p$ coefficients by Ronchetti.

It's worth noting that this work started out as an attempt to better understand the proof of [Gro, Theorem 7.1], in particular the part using the result of Grünenfelder, but has since develop in a different direction, where the coefficients are more restricted, but we obtain a more precise description.

DK Note: Write more background and motivation later.

Notation and setup

Let p be an odd prime.

Algebraic groups. We will work with schemes using the functorial approach and notation described in [Jan]. In particular, given an integral domain k, we note that a *k-group functor* is a functor from the category of all k-algebras to the category of groups, a *k-group scheme* is a k-group functor that is an affine scheme over k when considered as a k-functor, and an *algebraic k-group* is a k-group scheme that is algebraic as an affine scheme. For more in depth introduction to these concepts, we refer to [Conb] and [Jan].

Base change. If k' is a k-algebra, then any k'-algebra A is in a natural way a k-algebra by combining the structural homomorphisms $k \to k'$ and $k' \to A$. We can therefore associate to each k-functor X a k'-functor $X_{k'}$ by $X_{k'}(A) = X(A)$ for any k'-algebra A. For any morphism $f\colon X \to X'$ of k-functors, we get a morphism $f_{k'}\colon X_{k'} \to X'_{k'}$ of k'-functors by $f_{k'}(A) = f(A)$ for any k'-algebra A.

In this way we get a functor $X \mapsto X_{k'}$, $f \mapsto f_{k'}$ from the category of k-functors to the category of k'-functors, which we call the *base change* from k to k'.

Fixed $\mathbb{Z}_p$-groups and roots. We fix a split and connected reductive algebraic $\mathbb{Z}_p$-group $\mathcal{G}$ as well as a split maximal torus $\mathcal{T} \subseteq \mathcal{G}$. Let $\Phi = \Phi(\mathcal{G}, \mathcal{T})$ be the root system of $\mathcal{G}$ with respect to $\mathcal{T}$. For any $\alpha \in \Phi$ we have the root subgroup $\mathcal{N}_\alpha \subseteq \mathcal{G}$ with Lie algebra $\mathrm{Lie}\,\mathcal{N}_\alpha = (\mathrm{Lie}\,\mathcal{G})_\alpha$. We fix a $\mathbb{Z}_p$-basis $(X_\alpha)_{\alpha\in\Phi}$ of $\mathrm{Lie}\,\mathcal{N}_\alpha$, and note that this choice gives rise to unique isomorphisms of group schemes $x_\alpha \colon \mathbb{G}_a \xrightarrow{\cong} \mathcal{N}_\alpha$ such that $(dx_\alpha)(1) = X_\alpha$. We furthermore fix a basis $\Delta \subseteq \Phi$ of the root system, so we get a decomposition $\Phi = \Phi^+ \cup \Phi^-$ into positive and negative roots. Let $\mathcal{B} = \mathcal{T}\mathcal{N}$ and $\mathcal{B}^+ = \mathcal{T}\mathcal{N}^+$ denote the Borel subgroups of $\mathcal{G}$ corresponding to Φ^- and Φ^+, respectively, with unipotent radicals $\mathcal{N}$ and $\mathcal{N}^+$. Finally let $N = \mathcal{N}(\mathbb{Z}_p)$ and let $\mathfrak{n} = \mathrm{Lie}(\mathcal{N}_{\mathbb{F}_p})$ be the Lie algebra of $\mathcal{N}_{\mathbb{F}_p}$ over $\mathbb{F}_p$.

$\mathbb{Z}$-models. Let $\mathcal{G}_\mathbb{Z}$ be the Chevalley group over $\mathbb{Z}$ corresponding to $\mathcal{G}$ (cf. [Cona, §1]), and consider the subgroups $\mathcal{T}_\mathbb{Z}, \mathcal{B}_\mathbb{Z}, \mathcal{N}_\mathbb{Z}$ corresponding to $\mathcal{T}, \mathcal{B}, \mathcal{N}$. Let furthermore $\mathfrak{n}_\mathbb{Z} = \mathrm{Lie}(\mathcal{N}_\mathbb{Z})$ be the Lie algebra of $\mathcal{N}_\mathbb{Z}$ over $\mathbb{Z}$, and note that $N = \mathcal{N}_\mathbb{Z}(\mathbb{Z}_p)$ and $\mathfrak{n} = \mathfrak{n}_\mathbb{Z} \otimes \mathbb{F}_p$. (Note also that $(\mathcal{G}_\mathbb{Z})_{\mathbb{Z}_p} = \mathcal{G}$, so although we abuse notation a bit here, it wont be a problem.)

Total ordering of Φ^-. For any total ordering of Φ^- the multiplication induces an isomorphism of schemes $\prod_{\alpha\in\Phi^-} \mathcal{N}_\alpha \xrightarrow{\cong} \mathcal{N}$. For convenience we fix a total ordering which has the additional property that $\alpha_1 \geq \alpha_2$ if $\mathrm{ht}(\alpha_1) \leq \mathrm{ht}(\alpha_2)$. All products indexed by Φ^- are meant to be taken according to this ordering. Here we have the height function $\mathrm{ht} \colon \mathbb{Z}[\Delta] \to \mathbb{Z}$ given by

$\sum_{\alpha\in\Delta} m_\alpha \alpha \mapsto \sum_{\alpha\in\Delta} m_\alpha$. In particular, since $\Phi \subseteq \mathbb{Z}[\Delta]$ the height $\mathrm{ht}(\beta)$ of any root $\beta \in \Phi$ is defined.

Coxeter number and p. Let h be the Coxeter number of $\mathcal{G}$ and assume from now on that $p \geq h - 1$.

Weyl group and module. Let $\Phi^\vee$ be the dual root system of Φ and let W be the corresponding Weyl group with length function ℓ on W. Let furthermore $X = X(\mathcal{T}) \cong X(\mathcal{T}_\mathbb{Z})$ be the character group of $\mathcal{T}$, and set

$$X^+ = \{\lambda \in X \mid \langle \lambda, \alpha^\vee \rangle \geq 0 \text{ for all } \alpha \in \Phi^+\}.$$

For any $\lambda \in X^+$, let $V_\mathbb{Z}(\lambda)$ be the Weyl module for $\mathcal{G}_\mathbb{Z}$ over $\mathbb{Z}$ with highest weight λ, and let $V_{\mathbb{F}_p}(\lambda) = V_\mathbb{Z}(\lambda) \otimes_\mathbb{Z} \mathbb{F}_p$.

Lazard theory. We will introduce concepts from Lazard theory in next subsection, but we note now that we will let $\mathfrak{g} = \mathbb{F}_p \otimes_{\mathbb{F}_p[\pi]} \operatorname{gr} N$ be the Lazard Lie algebra corresponding to N.

Cohomology. For any ring R, we denote (using the Chevalley-Eilenberg complex) the Lie algebra cohomology of any R-Lie algebra $\mathfrak{g}$ by $H^\bullet(\mathfrak{g}, \cdot\,)$, while we write $H^\bullet_{\mathrm{dsc}}(G, \cdot\,)$ and $H^\bullet_{\mathrm{cts}}(H, \cdot\,)$ for the discrete (resp. continuous) group cohomology of a topological group G. Later we will introduce filtrations and then gradings on the cohomology, in which case we always use the notation $H^{s,t} = \operatorname{gr}^s H^{s+t}$ for any type of cohomology H.

Spectral sequences. Given a ring R, a cohomological spectral sequence is a choice of $r_0 \in \mathbb{N}$ and a collection of

- R-modules $E_r^{s,t}$ for each $s,t \in \mathbb{Z}$ and all integers $r \geq r_0$
- differentials $d_r^{s,t}\colon E_r^{s,t} \to E_r^{s+r,t+1-r}$ such that $d_r^2 = 0$ and E_{r+1} is isomorphic to the homology of (E_r, d_r), i.e.,
$$E_{r+1}^{s,t} = \frac{\ker(d_r^{s,t}\colon E_r^{s,t} \to E_r^{s+r,t+1-r})}{\operatorname{im}(d_r^{s-r,t+r-1}\colon E_r^{s-r,t+r-1} \to E_r^{s,t})}.$$

For a given r, the collection $(E_r^{s,t}, d_r^{s,t})_{s,t\in\mathbb{Z}}$ is called the r-th page. A spectral sequence *converges* if d_r vanishes on $E_r^{s,t}$ for any s,t when $r \gg 0$. In this case $E_r^{s,t}$ is independent of r for sufficiently large r, we denote it by $E_\infty^{s,t}$ and write

$$E_r^{s,t} \Longrightarrow E_\infty^{s+t}.$$

Also, we say that the spectral sequence collapses at the r'-th page if $E_r = E_\infty$ for all $r \geq r'$, but not for $r < r'$. Finally, when we have terms E_∞^n with a natural filtration $F^\bullet E_\infty^n$ (but no natural double grading), we set $E_\infty^{s,t} = \operatorname{gr}^s E_\infty^{s,t} = F^s E_\infty^{s+t}/F^{s+1}E_\infty^{s+t}$.

Lazard theory

subsec:Laz-theory

In this subsection we will briefly introduce elements of Lazard theory as presented in [Sch].

Let G be any abstract group and let the commutator be normalized to as $[g,h] = ghg^{-1}h^{-1}$.

Definition 1.1. A *p-valuation* ω on G is a real valued function

$$\omega\colon G \setminus \{1\} \to (0,\infty)$$

which, with the convention that $\omega(1) = \infty$, satisfies

(a) $\omega(g) > \frac{1}{p-1}$,

(b) $\omega(g^{-1}h) \geq \min(\omega(g), \omega(h))$,

(c) $\omega([g,h]) \geq \omega(g) + \omega(h)$,

(d) $\omega(g^p) = \omega(g) + 1$

for any $g, h \in G$. ♠

For the rest of this subsection, let (G, ω) be a p-valued group, i.e., a group with a p-valuation.

For any real number $\nu > 0$ put

$$G_\nu \coloneqq \{g \in G : \omega(g) \geq \nu\} \quad \text{and} \quad G_{\nu+} \coloneqq \{g \in G : \omega(g) > \nu\},$$

and note that these are normal subgroups, cf. [Sch, Sect. 23].

The subgroups G_ν form a decreasing exhaustive and separated filtration of G with the additional properties

$$G_\nu = \bigcap_{\nu' < \nu} G_{\nu'} \quad \text{and} \quad [G_\nu, G_{\nu'}] \subseteq G_{\nu+\nu'}.$$

There is a unique (Hausdorff) topological group structure on G for which the G_ν form a fundamental system of open neighborhoods of the identity element. It will be called the *topology defined by* ω. We will assume that G is profinte in the topology defined by ω. Hence $G = \varprojlim_{\nu > 0} G/G_\nu$ as topological groups, and thus G must be a pro-p-group since $\omega(g^p) = \omega(g) + 1$ implies that G/G_ν is a p-group (finite since G_ν is open).

We now form, for each $\nu > 0$, the subquotient group

$$\operatorname{gr}_\nu G \coloneqq G_\nu / G_{\nu+}.$$

It is commutative by (c) and therefore will be denoted additively. We now consider the graded abelian group

$$\operatorname{gr} G := \bigoplus_{\nu>0} \operatorname{gr}_\nu G.$$

An element $\xi \in \operatorname{gr} G$ is called, as usual, homogeneous (of degree ν) if it lies in $\operatorname{gr}_\nu G$. Furthermore, in this case any $g \in G_\nu$ such that $\xi = gG_{\nu+}$ is called a representative of ξ.

Note that $p\xi = 0$ for any homogeneous element $\xi \in \operatorname{gr} G$ since $\omega(g^p) = \omega(g) + 1$. Hence $\operatorname{gr} G$ in fact is an $\mathbb{F}_p$-vector space. Furthermore, by bilinear extension of the map

$$\begin{aligned} \operatorname{gr}_\nu G \times \operatorname{gr}_{\nu'} G &\to \operatorname{gr}_{\nu+\nu'} G \\ (\xi, \eta) &\mapsto [\xi, \eta] := [g, h]G_{\nu+\nu'}+, \end{aligned}$$

for $\nu, \nu' > 0$, we obtain a graded $\mathbb{F}_p$-bilinear map

$$[\,\cdot\,,\,\cdot\,]\colon \operatorname{gr} G \times \operatorname{gr} G \to \operatorname{gr} G$$

which satisfies

$$[\xi, \xi] = 0 \qquad \text{for any } \xi \in \operatorname{gr} G.$$

One can check that $[\,\cdot\,,\,\cdot\,]$ satisfies the Jacobi identity, and thus $\operatorname{gr} G$ is a graded Lie algebra over $\mathbb{F}_p$, cf. [Sch, Sect. 23].

Now, noticing that the map

$$\begin{aligned} \operatorname{gr}_\nu G &\to \operatorname{gr}_{\nu+1} G \\ gG_{\nu+} &\mapsto g^p G_{(\nu+1)+} \end{aligned}$$

is well defined and $\mathbb{F}_p$-linear, by considering for varying ν the direct sum of these maps, we can introduce an $\mathbb{F}_p$-linear map of degree one

$$\pi\colon \operatorname{gr} G \to \operatorname{gr} G.$$

We can and will therefore view $\operatorname{gr} G$ as a graded module over the polynomial ring $\mathbb{F}_p[\pi]$ in one variable over $\mathbb{F}_p$. Furthermore the Lie bracket on $\operatorname{gr} G$ is bilinear for the $\mathbb{F}_p[\pi]$-module structure, i.e., $\operatorname{gr} G$ is a Lie algebra over the ring $\mathbb{F}_p[\pi]$. For more details, we refer to [Sch, Sect. 25].

Definition 1.2. The pair (G, ω) is called of finite rank if $\operatorname{gr} G$ is finitely generated as an $\mathbb{F}_p[\pi]$-module. ♠

Note that G being of finite rank does not depend on the choice of the p-valuation, and assume from now on that (G, ω) is of finite rank. Note that $\operatorname{gr} G$ is finitely generated and torsionfree over the principal ideal domain $\mathbb{F}_p[\pi]$, and thus by the elementary divisor theorem $\operatorname{gr} G$ is free. We call

$$\operatorname{rank}(G, \omega) := \operatorname{rank}_{\mathbb{F}_p[P]} \operatorname{gr} G$$

the *rank* of the pair (G, ω).

For any $g \in G$ note that we then have a group homomorphism

$$\begin{aligned} c\colon \mathbb{Z} &\to G \\ m &\mapsto g^m. \end{aligned}$$

Since G/N, for any $N \lhd G$, is a p-group, we obtain $c^{-1}(N) = p^{a_N}\mathbb{Z}$ for some $a_N \geq 0$. It follows that c extends uniquely to a continuous group homomorphism

$$\tilde{c}\colon \mathbb{Z}_p \to \varprojlim_{N \lhd G} \mathbb{Z}/p^{a_N}\mathbb{Z} \xrightarrow{c} \varprojlim_{N} G/N = G$$

which we always will write as $g^x \coloneqq \tilde{c}(x)$. More generally, for any finitely many elements $g_1, \dots, g_r \in G$, we have the continuous map

$$\begin{aligned} \mathbb{Z}_p^r &\to G \\ (x_1, \dots, x_r) &\mapsto g_1^{x_1} \cdots g_r^{x_r} \end{aligned} \tag{1.1}$$

{eq:ZprtoG}

which depends on the order of the g_i and therefore is not a group homomorphism. However we introduce the following notation, where v_p denotes the usual p-adic valuation on $\mathbb{Q}_p$.

Definition 1.3. The sequence of elements $(g_1, \dots, g_r)$ in G is called an *ordered basis* of (G, ω) if the map (1.1) is a bijection (and hence, by compactness, a homeomorphism) and

$$\omega(g_1^{x_1} \cdots g_r^{x_r}) = \min_{1 \le i \le r} (\omega(g_i) + v(x_i)) \qquad \text{for any } x_1, \dots, x_r \in \mathbb{Z}_p.$$

♠

Definition 1.4. For any $g \in G \setminus \{1\}$, we put $\sigma(g) \coloneqq gG_{\omega(g)+} \in \operatorname{gr} G$. ♠

By [Sch, Remark 26.3], we note that for $g \in G \setminus \{1\}$ and $x \in \mathbb{Z}_p \setminus \{0\}$

$$\omega(g^x) = \omega(g) + v_p(x) \quad \text{and} \quad \sigma(g^x) = \bar{x}\pi^{v_p(x)} . \sigma(g), \tag{1.2}$$

{eq:sigma-gx}

where $\bar{x}$ is the image of $p^{-v_p(x)}x$ in $\mathbb{F}_p$ (i.e., the first non-zero coefficient of $x = \sum_{k=0}^{\infty} a_k p^k$). We note that an ordered basis $(g_1, \dots, g_d)$ of (G, ω) corresponds to an ordered $\mathbb{F}_p[\pi]$-basis $\big(\sigma(g_1), \dots, \sigma(g_d)\big)$ of $\operatorname{gr} G$, cf. [Sch, Prop. 26.5].

Finally we let $\mathfrak{g} = \mathbb{F}_p \otimes_{\mathbb{F}_p[\pi]} \operatorname{gr} G = \mathbb{F}_p \otimes_{\mathbb{F}_p[\pi]} \operatorname{gr} G / \pi \operatorname{gr} G$, and note that this is a Lie algebra over $\mathbb{F}_p$ with an $\mathbb{F}_p$-basis of vectors $\xi_i = 1 \otimes \sigma(g_i)$.

Cohomology theories and the spectral sequence

subsec:coh-and-spec-seq

One of the main results we use in this chapter is the spectral sequence introduced in [Sør, §6.1], so in this subsection we aim to introduce the concepts needed to use this spectral sequence. We also look into an important translation between continuous and discrete group cohomology that we will need later.

Let R be a ring and $\mathfrak{g}$ be a R-Lie algebra with R a trivial (left) $\mathfrak{g}$-module. Then we use the cochain complex $C^\bullet(\mathfrak{g}, R) = \mathrm{Hom}_R(\bigwedge^\bullet \mathfrak{g}, R)$, i.e.,

$$0 \longrightarrow R \xrightarrow{\partial_1} \mathrm{Hom}_R(\mathfrak{g}, R) \xrightarrow{\partial_2} \mathrm{Hom}_R\Big(\bigwedge^2 \mathfrak{g}, R\Big) \xrightarrow{\partial_3} \cdots,$$

where the coboundary map ∂_n is given by

$$\partial_n(f)(x_1, \dots, x_n) = \sum_{i<j} (-1)^{i+j} f([x_i, x_j], x_1, \dots, \widehat{x}_i, \dots, \widehat{x}_j, \dots, x_n),$$

where $\widehat{x}_i$ means excluding x_i. For more details we refer to [CE, Thm. 7.1] and note that we are considering the trivial action on R, which simplifies the formula slightly.

Now consider $R = \mathbb{F}_p$ in the following and suppose that $\mathfrak{g} = \mathfrak{g}^0 \oplus \mathfrak{g}^1 \oplus \cdots$ is a graded Lie algebra. Then $\bigwedge^n \mathfrak{g}$ is also graded by letting

$$\mathrm{gr}^j\Big(\bigwedge^n \mathfrak{g}\Big) = \bigoplus_{j_1 + \cdots + j_n = j} \mathfrak{g}^{j_1} \wedge \cdots \wedge \mathfrak{g}^{j_n}.$$

Letting $\mathbb{F}_p$ be a $\mathbb{Z}$-graded (concentrated in degree 0) $\mathfrak{g}$-module, we get a grading

$$\mathrm{Hom}_{\mathbb{F}_p}\Big(\bigwedge^n \mathfrak{g}, \mathbb{F}_p\Big) = \bigoplus_{s \in \mathbb{Z}} \mathrm{Hom}^s_{\mathbb{F}_p}\Big(\bigwedge^n \mathfrak{g}, \mathbb{F}_p\Big)$$

where $\mathrm{Hom}^s_{\mathbb{F}_p}$ denotes the homogeneous $\mathbb{F}_p$-linear maps of degree s, cf. [FF, Lem. 4.2]. One can check that this passes to bigrading of Lie algebra cohomology

$$H^{s,t}(\mathfrak{g}, \mathbb{F}_p) = H^{s+t}\big(\mathrm{gr}^s \mathrm{Hom}_{\mathbb{F}_p}(\bigwedge^\bullet \mathfrak{g}, \mathbb{F}_p)\big).$$

In the spectral sequence described in [Sør, §6.1], we take $r_0 = 1$ (i.e., the spectral sequence start from the first page) and $E_1^{s,t} = H^{s,t}(\mathfrak{g}, \mathbb{F}_p)$, where $\mathfrak{g} = \mathbb{F}_p \otimes \operatorname{gr} G$ indeed is (positively) $\mathbb{Z}$-graded.

Let now G be a topological group and $\mathbb{F}_p$ a G-module. Then we will define two types of group cohommology: continuous and discrete.

Continuous group cohomology $H^n_{\mathrm{cts}}(G, \mathbb{F}_p)$ is the cohomology of the complex $C^\bullet(G, \mathbb{F}_p) = \mathcal{C}(G^\bullet, \mathbb{F}_p)$, i.e.,

$$0 \longrightarrow \mathbb{F}_p \xrightarrow{\partial_1} \mathcal{C}(G, \mathbb{F}_p) \xrightarrow{\partial_2} \mathcal{C}(\mathbb{G}^2, \mathbb{F}_p) \xrightarrow{\partial_3} \mathcal{C}(G^3, \mathbb{F}_p) \xrightarrow{\partial_4} \cdots,$$

where the coboundary map ∂_n is given by

$$\partial_n(f)(g_1, \ldots, g_n) = \sum_{i=1}^{n} (-1)^i f(g_1, \ldots, g_i g_{i+1}, \ldots, g_n),$$

where the n-th term is interpreted as $(-1)^n f(g_1, \ldots, g_{n-1})$, cf. [Sør, §3] and note again that our formula is slightly simpler since we only consider the trivial action on $\mathbb{F}_p$.

Discrete group cohomology $H^n_{\mathrm{dsc}}(G, \mathbb{F}_p)$ is the cohomology of the complex $C^\bullet(G, \mathbb{F}_p) = \operatorname{Hom}_G(\mathbb{Z}[G^\bullet], \mathbb{F}_p)$ as follows. One can check that

$$\cdots \xrightarrow{d_4} \mathbb{Z}[G^3] \xrightarrow{d_3} \mathbb{Z}[G^2] \xrightarrow{d_2} \mathbb{Z}[G] \xrightarrow{d_1} \mathbb{Z} \longrightarrow 0$$

with boundary maps

$$d_n \colon (g_0, g_1, \ldots, g_n) \mapsto \sum_{i=0}^{n} (-1)^i (g_0, \ldots, \widehat{g}_i, \ldots, g_n)$$

is a chain complex, and thus we get a cochain complex $C^\bullet(G, \mathbb{F}_p) = \operatorname{Hom}_G(C_\bullet, \mathbb{F}_p)$,

$$\begin{array}{ccccc} 0 \longrightarrow \operatorname{Hom}_G(\mathbb{Z}, \mathbb{F}_p) & \xrightarrow{\partial_1} & \operatorname{Hom}_G(\mathbb{Z}[G^2], \mathbb{F}_p) & \xrightarrow{\partial_2} & \cdots \\ f & \longmapsto & f \circ d_1 & & \end{array}$$

Note that this discrete cohomology can be viewed as continuous cohomology if we equip G with the discrete topology.

Note that [Sør] gets the spectral sequence we are interested in by using an isomorphism to translate $H^\bullet_{\text{cts}}(G, \mathbb{F}_p)$ to $HH^\bullet(G, \mathbb{F}_p)$ (essentially what's known as Mac Lane isomorphism) and introducing a $\mathbb{Z}$-filtration and grading on $HH^\bullet(G, \mathbb{F}_p)$, which is used in the spectral sequence. We will skip the full details of this translation and just note that we get a $\mathbb{Z}$-filtration and grading on $H^\bullet(G, \mathbb{F}_p)$, which with $k = \mathbb{F}_p$ gives us the following, cf. [Sør, Thm. 5.5–§6.1].

thm:spec-seq **Theorem 1.5.** Let (G, ω) be a p-valuable group and $\mathfrak{g} = \mathbb{F}_p \otimes_{\mathbb{F}_p[\pi]} \operatorname{gr} G$ its Lazard Lie algebra. Then there is a convergent spectral sequence collapsing at a finite stage,

$$E_1^{s,t} = H^{s,t}(\mathfrak{g}, \mathbb{F}_p) \Longrightarrow H^{s+t}(G, \mathbb{F}_p).$$

This means that each sheet E_r has a multiplication $E_r \otimes E_r \to E_r$ compatible with the (s,t)-bigrading and satisfying Leibniz formula. Furthermore $H^*(E_r) \cong E_{r+1}$ as algebras. I.e., the multiplication on E_∞ is compatible with the cup product on $H^*(G, \mathbb{F}_p)$ in the sense that the following diagram commutes.

$$\begin{tikzcd}
E_\infty^{s,n-s} \otimes E_\infty^{s',n'-s'} \arrow[r] \arrow[d, "\cong"'] & E_\infty^{s+s',n+n'-s-s'} \arrow[d, "\cong"] \\
\operatorname{gr}^s H^n(G, \mathbb{F}_p) \otimes \operatorname{gr}^{s'} H^{n'}(G, \mathbb{F}_p) \arrow[r] & \operatorname{gr}^{s+s'} H^{n+n'}(G, \mathbb{F}_p)
\end{tikzcd}$$

♣

Finally we note that [Fer, Thm. 2.10] implies that $H^n_{\text{cts}}(N, \mathbb{F}_p) \cong H^n_{\text{dsc}}(N, \mathbb{F}_p)$ for all n (with $N = \mathcal{N}(\mathbb{Z}_p)$ as above), if we can show that N is a pro-p group which is poly-$\mathbb{Z}_p$ by finite.

Definition 1.6. A group G is poly-$\mathbb{Z}_p$ if it has a normal series

$$G = G_1 \supseteq G_2 \supseteq \cdots \supseteq G_n = 1$$

such that each factor group G_i/G_{i+1} is isomorphic to $\mathbb{Z}_p$.

A group is poly-$\mathbb{Z}_p$ by finite (virtually poly-$\mathbb{Z}_p$) if it contains a poly-$\mathbb{Z}_p$ subgroup of finite index. ♠

Note that [Conb, Prop. 5.1.16(2) and Cor. 5.2.5] (as seen in the proof of [Conb, Cor. 5.2.13] or [Conb, Thm. 5.4.3]) gives us a composition series of $\mathcal{N}$ such that the successive quotients are $\mathbb{G}_a$, which implies that $N = \mathcal{N}(\mathbb{Z}_p)$ is poly-$\mathbb{Z}_p$ by finite since $\mathbb{G}_a(\mathbb{Z}_p) = \mathbb{Z}_p$. Thus, assuming that $\mathcal{N}(\mathbb{Z}_p)$ is a pro-p group, we get that

DK Note: Can probably remove this.

$$H^n_{\mathrm{cts}}(N, \mathbb{F}_p) \cong H^n_{\mathrm{dsc}}(N, \mathbb{F}_p) \qquad \text{for all } n. \tag{1.3}$$

{eq:coh-comp}

Main result

subsec:main-res

We show first that N is p-valuable, which implies by [Sør, §6.1] that we get a convergent multiplicative spectral sequence

DK Note: Rewrite to state Theorem precisely later.

$$E_1^{s,t} = H^{s,t}(\mathfrak{g}, \mathbb{F}_p) \Longrightarrow H^{s+t}(N, \mathbb{F}_p). \tag{1.4}$$

{eq:spec-seq}

We note that $\mathfrak{g} \cong \mathfrak{n}$ and then use ideas of [Gro, §7] to transfer results from [PT] about (the dimension of) $H^n(\mathfrak{n}_{\mathbb{Z}}, \mathbb{F}_p)$ and $H^n(\mathcal{N}_{\mathbb{Z}}(\mathbb{Z}), \mathbb{F}_p)$ to $H^n(\mathfrak{n}, \mathbb{F}_p)$ and $H^n(\mathcal{N}(\mathbb{Z}_p), \mathbb{F}_p)$, giving us that $\sum_{s+t=n} \dim_{\mathbb{F}_p} H^{s,t}(\mathfrak{g}, \mathbb{F}_p) = \dim_{\mathbb{F}_p} H^n(\mathfrak{n}, \mathbb{F}_p) = \dim_{\mathbb{F}_p} H^n(N, \mathbb{F}_p)$. This implies that (1.4) collapses on the first page, and thus $H^{s,n-s}(\mathfrak{n}, \mathbb{F}_p) \cong \mathrm{gr}^s H^n(N, \mathbb{F}_p)$. Noting that $E_\infty^{s,t} = E_1^{s,t}$, we get that the cup product on $E_1^{s,t} = H^{s,t}(\mathfrak{n}, \mathbb{F}_p)$ (from $H^*(\mathfrak{n}, \mathbb{F}_p)$) is compatible with the cup product on $H^*(N, \mathbb{F}_p)$ in the sense that the following diagram commutes.

$$\begin{tikzcd}
H^{s,n-s}(\mathfrak{n}, \mathbb{F}_p) \otimes H^{s',n'-s'}(\mathfrak{n}, \mathbb{F}_p) \arrow[r] \arrow[d, "\cong"'] & H^{s+s',n+n'-s-s'}(\mathfrak{n}, \mathbb{F}_p) \arrow[d, "\cong"] \\
\operatorname{gr}^s H^n(N, \mathbb{F}_p) \otimes \operatorname{gr}^{s'} H^{n'}(N, \mathbb{F}_p) \arrow[r] & \operatorname{gr}^{s+s'} H^{n+n'}(N, \mathbb{F}_p)
\end{tikzcd}$$

1.2 The p-valuation

sec:pval

In this section we will prove that N is p-valuable group, which we will need in multiple arguments later. Note that this section is mainly based on some unpublished notes by Schneider.

Note that as a set N is the direct product $N = \prod_{\alpha \in \Phi^-} x_\alpha(\mathbb{Z}_p)$, which allows us to introduce the function

$$\begin{aligned} \omega \colon N \setminus \{1\} &\to \mathbb{N} \\ \prod_{\alpha \in \Phi^-} x_\alpha(a_\alpha) &\mapsto \min_{\alpha \in \Phi^-} \big(v_p(a_\alpha) - \operatorname{ht}(\alpha)\big), \end{aligned} \tag{1.5}$$ {eq:p-val}

where v_p denotes the usual p-adic valuation on $\mathbb{Z}_p$. Here it is important to note that we write any $g \in N$ uniquely as product

$$g = \prod_{\alpha \in \Phi^-} x_\alpha(a_\alpha)$$

by taking the product following the total ordering $\geq$ of Φ^- defined above. Now, with the convention that $\omega(1) := \infty$, we define the descending sequence of subsets

$$N_m := \{g \in N \mid \omega(g) \geq m\}$$

in N for $m \geq 0$, following the notation used for p-valuable groups. The goal of this section is to show that this ω is a p-valuation by a careful analysis of the sequence of subsets given by N_m.

We first note that clearly $N_1 = N$, $\bigcap_m N_m = \{1\}$, and

DK Note: Decide whether to use mathmap or not.

$$\begin{aligned} N_m &= \prod_{\alpha\in\Phi^-} x_\alpha(p^{\max(0,m+\mathrm{ht}(\alpha))}\mathbb{Z}_p) \\ &= \prod_{\substack{\alpha\in\Phi^- \\ \mathrm{ht}(\alpha)=-1}} x_\alpha(p^{m-1}\mathbb{Z}_p)\cdots\prod_{\substack{\alpha\in\Phi^- \\ \mathrm{ht}(\alpha)=-(m-1)}} x_\alpha(p\mathbb{Z}_p)\prod_{\substack{\alpha\in\Phi^- \\ \mathrm{ht}(\alpha)\le -m}} x_\alpha(\mathbb{Z}_p). \end{aligned} \tag{1.6}$$

{eq:N_m}

In our analysis of this sequence it will be helpful to introduce the following two other filtrations of N. Firstly we will consider the filtration by congruence subgroups

$$N(m) \coloneqq \ker\bigl(\mathcal{N}(\mathbb{Z}_p) \to \mathcal{N}(\mathbb{Z}/p^m\mathbb{Z})\bigr) = \prod_{\alpha\in\Phi^-} x_\alpha(p^m\mathbb{Z}_p) \tag{1.7}$$

{eq:N-par-m}

for $m \geq 0$. Secondly, using the descending central series of the group $\mathcal{G}(\mathbb{Q}_p)$ defined by $C^1\mathcal{G}(\mathbb{Q}_p) \coloneqq \mathcal{G}(\mathbb{Q}_p)$ and $C^{m+1}\mathcal{G}(\mathbb{Q}_p) \coloneqq [C^m\mathcal{G}(\mathbb{Q}_p), \mathcal{G}(\mathbb{Q}_p)]$, we consider the filtration given by

$$N_{(m)} \coloneqq N \cap C^m\mathcal{G}(\mathbb{Q}_p)$$

for $m \geq 1$. By [BT, Prop. 4.7(iii)] we have that

$$N_{(m)} = \prod_{\substack{\alpha\in\Phi^- \\ \mathrm{ht}(\alpha)\le -m}} x_\alpha(\mathbb{Z}_p), \tag{1.8}$$

{eq:N_par-m}

and we note that the natural map

$$\prod_{\substack{\alpha\in\Phi^- \\ \mathrm{ht}(\alpha)=-m}} x_\alpha(\mathbb{Z}_p) \to N_{(m)}/N_{(m+1)}$$

is an isomorphism of abelian groups, and that all the subgroups $N(m)$ and $N_{(m)}$ are normal in N.

We are now ready to prove the following lemma, which will help us when showing that ω is a p-valuation.

Lemma 1.7.

lem:N_m item:N_m (i) $N_m = \prod_{1\leq i\leq m} N(m-i)\cap N_{(i)}$, for any $m \geq 1$, is a normal subgroup of N which is independent of the choices made.

item:N_mcom (ii) $[N_\ell, N_m] \subseteq N_{\ell+m}$ for any $\ell, m \geq 1$.

(iii) N_m/N_{m+1}, for any $m \geq 1$, is an $\mathbb{F}_p$-vector space of dimension equal to $|\{\alpha \in \Phi^- \mid \mathrm{ht}(\alpha) \geq -m\}|$.

item:g^p (iv) Let $g \in N_m$ for some $m \geq 1$. If $g^p \in N_{m+2}$, then $g \in N_{m+1}$. ♣

Proof. (i) Using (1.7) and (1.8) we note that

$$\prod_{\substack{\alpha\in\Phi^-\\ \mathrm{ht}(\alpha)=-i}} x_\alpha(p^{m-i}\mathbb{Z}_p) \subseteq N(m-i)\cap N_{(i)} \quad \text{and} \quad \prod_{\substack{\alpha\in\Phi^-\\ \mathrm{ht}(\alpha)\leq -m}} x_\alpha(\mathbb{Z}_p) = N(0)\cap N_{(m)}$$

for $1 \leq i < m$, so by (1.6) it's clear that $N_m \subseteq \prod_{1\leq i\leq m} N(m-i)\cap N_{(i)}$. We also note, by (1.7) and (1.8), that

$$\begin{aligned}
&\big(N(m-i)\cap N_{(i)}\big)\big(N(m-i-1)\cap N_{(i+1)}\big)\\
&\subseteq \Big(\prod_{\substack{\alpha\in\Phi^-\\ \mathrm{ht}(\alpha)=-i}} x_\alpha(p^{m-i}\mathbb{Z}_p)\Big)\big(N(m-i-1)\cap N_{(i+1)}\big)
\end{aligned}$$

for any $1 \leq i < m$, so

$$\begin{aligned}
&\prod_{1\leq i\leq m} N(m-i)\cap N_{(i)}\\
&\subseteq \prod_{\substack{\alpha\in\Phi^-\\ \mathrm{ht}(\alpha)=-1}} x_\alpha(p^{m-1}\mathbb{Z}_p)\cdots \prod_{\substack{\alpha\in\Phi^-\\ \mathrm{ht}(\alpha)=-(m-1)}} x_\alpha(p\mathbb{Z}_p)\big(N(0)\cap N_{(m)}\big)\\
&= N_m
\end{aligned}$$

by induction, (1.6) and (1.8). This shows the equality and that N_m is normal clearly follows.

(ii) We first recall the following formulas for commutators

$$[gh,k] = g[h,k]g^{-1}[g,k] \quad \text{and} \quad [g,hk] = [g,h]h[g,k]h^{-1}. \tag{1.9}$$

{eq:comformulas}

Now, using (1.9), (i) and the fact that all the involved subgroups are normal, it's enough to show that

$$[N(\ell) \cap N_{(i)}, N(m) \cap N_{(j)}] \subseteq N(\ell+m) \cap N_{(i+j)}.$$

This further reduces to showing that

$$[N(\ell), N(m)] \subseteq N(\ell+m) \quad \text{and} \quad [N_{(i)}, N_{(j)}] \subseteq N_{(i+j)}.$$

The right inclusion is a well known property of the descending central series, so it follows from our definition of $N_{(m)}$. For the left inclusion it suffices, by (1.7) and (1.9), to show that

$$[x_\alpha(p^\ell \mathbb{Z}_p), x_\beta(p^m \mathbb{Z}_p)] \subseteq N(\ell+m)$$

for any $\alpha, \beta \in \Phi^-$. To show this inclusion we recall Chevalley's commutator formula, cf. [Conb, Prop. 5.1.14],

$$[x_\alpha(a), x_\beta(b)] \in x_{\alpha+\beta}(c_{\alpha,\beta,1,1} ab\mathbb{Z}_p) \prod_{\substack{i,j\geq 1 \\ i+j>2}} x_{i\alpha+j\beta}(c_{\alpha,\beta,i,j} a^i b^j \mathbb{Z}_p),$$

where $c_{\alpha,\beta,i,j} \in \mathbb{Z}_p$ and on the right hand side we use the convention is that $x_{i\alpha+j\beta} \equiv 1$ if $i\alpha + j\beta \notin \Phi$. From (1.7) and Chevalley's commutator formula the inclusion follows.

(iii) We note that

$$N(m-i)\cap N_{(i)} = \prod_{\substack{\alpha\in\Phi^-\\ \mathrm{ht}(\alpha)\leq -i}} x_\alpha(p^{m-i}\mathbb{Z}_p)$$

for $1\leq i\leq m$, so the statement follows from (i) and (ii). DK Note: Write (iii) better.

(iv) For any $1\leq \ell\leq m$ we consider the chain of normal subgroups

$$N_{m+2}(N_m\cap N_{(\ell+1)})\subseteq N_{m+1}(N_m\cap N_{(\ell+1)})\subseteq N_{m+1}(N_m\cap N_{(\ell)})$$

between N_{m+2} and N_m. By (1.9) and an argument like in (ii), we get that

$$[N_{m+1}(N_m\cap N_{(\ell)}), N_{m+1}(N_m\cap N_{(\ell)})]\subseteq N_{m+2}(N_m\cap N_{(\ell+1)}),$$

so the quotient group

$$N_{m+1}(N_m\cap N_{(\ell)})/N_{m+2}(N_m\cap N_{(\ell+1)})$$

is abelian. Now looking carefully at the groups as sets, we see that

$$N_m\cap N_{(\ell)} = \prod_{\substack{\alpha\in\Phi^-\\ \mathrm{ht}(\alpha)\leq -\ell}} x_\alpha(p^{\max(0,m+\mathrm{ht}(\alpha))}\mathbb{Z}_p)$$

and thus (using Chevalley's commutator formula and the fact that $\mathrm{ht}(i\alpha+j\beta)\leq \mathrm{ht}(\alpha+\beta)<\mathrm{ht}(\alpha),\mathrm{ht}(\beta)$ to move the products for the $\mathrm{ht}(\alpha)=-\ell$ term) DK Note: More detail here?

$$\begin{aligned} N_{m+1}(N_m\cap N_{(\ell)}) = &\prod_{\substack{\alpha\in\Phi^-\\ \mathrm{ht}(\alpha)>-\ell}} x_\alpha(p^{\max(0,m+1+\mathrm{ht}(\alpha))}\mathbb{Z}_p)\\ &\cdot\prod_{\substack{\alpha\in\Phi^-\\ \mathrm{ht}(\alpha)=-\ell}} x_\alpha(p^{m-\ell}\mathbb{Z}_p)\\ &\cdot\prod_{\substack{\alpha\in\Phi^-\\ \mathrm{ht}(\alpha)<-\ell}} x_\alpha(p^{\max(0,m+\mathrm{ht}(\alpha))}\mathbb{Z}_p).\end{aligned}$$

Similarly

$$N_{m+2}(N_m \cap N_{(\ell+1)}) = \prod_{\substack{\alpha\in\Phi^- \\ \mathrm{ht}(\alpha)>-\ell}} x_\alpha(p^{\max(0,m+2+\mathrm{ht}(\alpha))}\mathbb{Z}_p)$$
$$\cdot \prod_{\substack{\alpha\in\Phi^- \\ \mathrm{ht}(\alpha)=-\ell}} x_\alpha(p^{m+2-\ell}\mathbb{Z}_p)$$
$$\cdot \prod_{\substack{\alpha\in\Phi^- \\ \mathrm{ht}(\alpha)\leq-(\ell+1)}} x_\alpha(p^{\max(0,m+\mathrm{ht}(\alpha))}\mathbb{Z}_p),$$

and since the quotient group

$$N_{m+1}(N_m \cap N_{(\ell)})/N_{m+2}(N_m \cap N_{(\ell+1)})$$

is abelian, we see that it is isomorphic to

$$\prod_{\substack{\alpha\in\Phi^- \\ \mathrm{ht}(\alpha)>-\ell}} \frac{x_\alpha(p^{\max(0,m+1+\mathrm{ht}(\alpha))}\mathbb{Z}_p)}{x_\alpha(p^{\max(m+2+\mathrm{ht}(\alpha))}\mathbb{Z}_p)} \times \prod_{\substack{\alpha\in\Phi^- \\ \mathrm{ht}(\alpha)=-\ell}} \frac{x_\alpha(p^{m-\ell}\mathbb{Z}_p)}{x_\alpha(p^{m+2-\ell}\mathbb{Z}_p)}.$$

Here the subgroup

$$N_{m+1}(N_m \cap N_{(\ell+1)})/N_{m+2}(N_m \cap N_{(\ell+1)})$$

corresponds to

$$\prod_{\substack{\alpha\in\Phi^- \\ \mathrm{ht}(\alpha)>-\ell}} \frac{x_\alpha(p^{\max(0,m+1+\mathrm{ht}(\alpha))}\mathbb{Z}_p)}{x_\alpha(p^{\max(0,m+2+\mathrm{ht}(\alpha))}\mathbb{Z}_p)} \times \prod_{\substack{\alpha\in\Phi^- \\ \mathrm{ht}(\alpha)=-\ell}} \frac{x_\alpha(p^{m+1-\ell}\mathbb{Z}_p)}{x_\alpha(p^{m+2-\ell}\mathbb{Z}_p)}.$$

It follows that $N_{m+1}(N_m \cap N_{(\ell+1)})/N_{m+2}(N_m \cap N_{(\ell+1)})$ is the p-torsion subgroup of $N_{m+1}(N_m \cap N_{(\ell)})/N_{m+2}(N_m \cap N_{(\ell+1)})$.

Now let $g \in N_m$ for some $m \geq 1$. For $\ell = 1$ we have $g \in N_m = N_{m+1}(N_m \cap N_{(1)})$, since $N_{(1)} = N$, and clearly $g^p \in N_{m+2}(N_m \cap N_{(2)})$ because $g^p \in N_{(2)}$ by

Chevalley's commutator formula and (1.8). Since $N_{m+1}(N_m \cap N_{(2)})/N_{m+2}(N_m \cap N_{(2)})$ is the p-torsion subgroup of $N_{m+1}(N_m \cap N_{(1)})/N_{m+2}(N_m \cap N_{(2)})$, it follows that $g \in N_{m+1}(N_m \cap N_{(2)})$ and thus $g^p \in N_{m+2}(N_m \cap N_{(3)})$ by Chevalley's commutator formula and (1.8). By induction on ℓ, we thus get that $g \in N_{m+1}(N_m \cap N_{(m+1)}) = N_{m+1}$. Here the last equality follows from the fact that $N_{(m+1)} \subseteq N_{m+1}$ by (1.6) and (1.8). □

With this lemma, we are now ready to prove that ω is a p-valuation on N.

Proposition 1.8. The function ω is a p-valuation on N, i.e., it satisfies for any $g, h \in N$:

(a) $\omega(g) > \frac{1}{p-1}$,

(b) $\omega(g^{-1}h) \geq \min(\omega(g), \omega(h))$,

(c) $\omega([g, h]) \geq \omega(g) + \omega(h)$,

(d) $\omega(g^p) = \omega(g) + 1$. ♣

Proof. We note that (a) is obvious by our definition of ω, (c) follows from Lemma 1.7 (ii) and (d) follows from Lemma 1.7 (iv).

It only remains to show (b), which we will do by following the proof idea of [Zab, Lem. 1], i.e., we are going to use triple induction. Here we note that all products $\prod_{\alpha \in \Phi^-} x_\alpha(a_\alpha)$ are in ascending order in Φ^- (so descending in height). For ease of notation, we prove equivalently that $\omega(gh^{-1}) \geq \min(\omega(g), \omega(h))$ for $g, h \in N$.

At first by induction on the number of non-zero coordinates among $(a_\beta)_{\beta \in \Phi^-}$ in $\prod_{\beta \in \Phi^-} x_\beta(a_\beta)$ we are reduced to the case where h is of the form $h = x_\beta(a_\beta)$

for some $\beta \in \Phi^-$ and $a_\beta \in \mathbb{Z}_p$. To see this let $h \in N \setminus \{1\}$ and write $h = \prod_{\beta\in\Phi^-} x_\beta(a_\beta)$ in our unique way (according to the ordering of Φ^-), and let α be the smallest element of Φ^- for which $a_\alpha \neq 0$ so that $h = x_\alpha(a_\alpha) \cdot h'$. Then $gh^{-1} = g(h')^{-1} \cdot x_\alpha(a_\alpha)^{-1}$ and thus strong induction will imply that

$$\begin{aligned}\omega(gh^{-1}) &\geq \min\bigl(\omega(g(h')^{-1}), v(a_\alpha) - \mathrm{ht}(\alpha)\bigr)\\ &\geq \min\bigl(\omega(g), \omega(h'), v(a_\alpha) - \mathrm{ht}(\alpha)\bigr) = \min\bigl(\omega(g), \omega(h)\bigr).\end{aligned}$$

Fix $h = x_\beta(a_\beta)$ and let now g be of the form $g = \prod_{k=1}^r x_{\alpha_k}(a_{\alpha_k})$ with $\alpha_1 < \alpha_2 < \cdots < \alpha_r$ in Φ^-. If $\beta > \alpha_r$, then $gh^{-1} = \prod_{k=1}^{r-1} x_{\alpha_k}(a_{\alpha_k}) \cdot x_{\alpha_r}(a_{\alpha_r}) x_\beta(-a_\beta)$, so (b) is clearly true if $\beta > \alpha_1$ (by the definition of ω), and if $\beta = \alpha_r$, then $x_{\alpha_r}(a_{\alpha_r}) x_\beta(-a_\beta) = x_\beta(a_{\alpha_r} - a_\beta)$ and (b) follows from $v_p(a-b) \geq \min(v_p(a), v_p(b))$ for $a, b \in \mathbb{Z}_p$.

On the other hand, if $\beta < \alpha_r$, then we write

$$\begin{aligned}gh^{-1} &= \prod_{k=1}^{r} x_{\alpha_k}(a_{\alpha_k}) \cdot x_\beta(-a_\beta)\\ &= \prod_{k=1}^{r-1} x_{\alpha_k}(a_{\alpha_k}) \cdot x_\beta(-a_\beta) \cdot x_{\alpha_r}(a_{\alpha_r}) \cdot [x_{\alpha_r}(-a_{\alpha_r}), x_\beta(a_\beta)].\end{aligned}$$

Now we use descending induction on β in the chosen ordering of Φ^- and suppose that the statement (b) is true for any g and any h' of the form $h' = x_{\beta'}(a_{\beta'})$ with $\beta' > \beta$. Note that the base case is trivial and recall that Φ^- is finite and totally ordered. Note furthermore that Chevalley's commutator formula gives us

$$[x_{\alpha'}(a_{\alpha'}), x_{\beta'}(a_{\beta'})] = \prod_{\substack{i\alpha'+j\beta'\in\Phi^- \\ i,j>0}} x_{i\alpha'+j\beta'}(c_{\alpha',\beta',i,j} a_{\alpha'}^i a_{\beta'}^j) \qquad (1.10)$$

{eq:Chevalley}

for any $\alpha', \beta' \in \Phi^-$, where $c_{\alpha',\beta',i,j} \in \mathbb{Z}_p$. Also, we have $\mathrm{ht}(i\alpha' + j\beta') \le \mathrm{ht}(\alpha' + \beta') < \mathrm{ht}(\alpha'), \mathrm{ht}(\beta')$, so we can apply the induction hypothesis for $x_{\alpha_r}(a_{\alpha_r})$ and each $x_{i\alpha_r + j\beta}(c_{\alpha_r,\beta,i,j}(-a_{\alpha_r})^i a_\beta^j)$ in $[x_{\alpha_r}(-a_{\alpha_r}, x_\beta(a_\beta))]$, since $\alpha_r > \beta$ and all terms on the right side of (1.10) are larger than β (and α_r) in the ordering of Φ^-. We thus obtain

$$\begin{aligned} \omega(gh^{-1}) \ge \min\Big(&\min_{\substack{i\alpha_r + j\beta \in \Phi^- \\ i,j>0}} \omega(x_{i\alpha_r+j\beta}(c_{\alpha_r,\beta,i,j}(-a_{\alpha_r})^i a_\beta^j)), \\ &\omega(x_{\alpha_r}(a_{\alpha_r})), \omega\Big(\prod_{k=1}^{r-1} x_{\alpha_k}(a_{\alpha_k}) \cdot x_\beta(-a_\beta)\Big)\Big). \end{aligned} \tag{1.11}$$

{eq:omega-par-ginvh}

Now, for $i, j > 0$ with $i\alpha' + j\beta' \in \Phi^-$,

$$\begin{aligned} \omega(x_{i\alpha'+j\beta'}(c_{\alpha',\beta',i,j} a_{\alpha'}^i a_{\beta'}^j)) &= v_p(c_{\alpha',\beta',i,j} a_{\alpha'}^i a_{\beta'}^j) - \mathrm{ht}(i\alpha' + j\beta') \\ &\ge v_p(c_{\alpha',\beta',i,j}) + v_p(a_{\alpha'}^i) + v_p(a_{\beta'}^j) - \mathrm{ht}(\alpha' + \beta') \\ &\ge v_p(a_{\alpha'}) - \mathrm{ht}(\alpha') + v_p(a_{\beta'}) - \mathrm{ht}(\beta') \\ &= \omega(x_{\alpha'}(a_{\alpha'})) + \omega(x_{\beta'}(a_{\beta'})) \\ &\ge \min\big(\omega(x_{\alpha'}(a_{\alpha'})), \omega(x_{\beta'}(a_{\beta'}))\big). \end{aligned} \tag{1.12}$$

{eq:omega-par-Chev}

So taking $\alpha' = \alpha_r$ and $\beta' = \beta$ and using (1.12) in (1.11), we get that

$$\omega(gh^{-1}) \ge \min\Big(\omega(x_{\alpha_r}(a_{\alpha_r})), \omega(x_\beta(a_\beta)), \omega\Big(\prod_{k=1}^{r-1} x_{\alpha_k}(a_{\alpha_k}) \cdot x_\beta(-a_\beta)\Big)\Big). \tag{1.13}$$

{eq:omega-par-ginvh-2

Finally induction on r will imply that

$$\begin{aligned} \omega\Big(\prod_{k=1}^{r-1} x_{\alpha_k}(a_{\alpha_k}) \cdot x_\beta(-a_\beta)\Big) &\ge \min\Big(\omega\Big(\prod_{k=1}^{r-1} x_{\alpha_k}(a_{\alpha_k})\Big), \omega(x_\beta(a_\beta))\Big) \\ &= \min\big(\min_{1\le k\le r-1} \omega(x_{\alpha_k}(a_{\alpha_k})), \omega(x_\beta(a_\beta))\big), \end{aligned}$$

which by (1.13) implies that

$$\omega(gh^{-1}) \geq \min\bigl(\min_{1\leq k\leq r} \omega(x_{\alpha_k}(a_{\alpha_k})), \omega(x_\beta(a_\beta))\bigr)$$
$$= \min\bigl(\omega(g), \omega(h)\bigr),$$

thus finishing the proof. $\square$

We have now shown that $N = \mathcal{N}(\mathbb{Z}_p)$ is a p-valuable group with the p-valuation ω introduced in (1.5), which is the main result of this section. Before continuing, we will clarify what this means based on Lazard theory as described in Section 1.1.

We note that

$$\operatorname{gr} N := \bigoplus_{m\geq 1} N_m/N_{m+1}$$

is a graded $\mathbb{F}_p$-vector space, and recall the following well known result, cf. [Laz] or [Sch, Sect. 25].

Proposition 1.9. $\operatorname{gr} N$ is a Lie algebra over the polynomial ring $\mathbb{F}_p[\pi]$ in one variable π where

$$[gN_{\ell+1}, hN_{m+1}] := [g,h]N_{\ell+m+1} \quad \text{and} \quad \pi(gN_{m+1}) := g^p N_{m+2},$$

and as an $\mathbb{F}_p[\pi]$-module $\operatorname{gr} N$ is free of rank $|\Phi^-|$. ♣

1.3 Spectral sequence and cohomology

sec:specsec

Recall that $N = \mathcal{N}(\mathbb{Z}_p)$, $\mathfrak{g} = \mathbb{F}_p \otimes_{\mathbb{F}_p[\pi]} \operatorname{gr} G$ and $\mathfrak{n} = \operatorname{Lie}(\mathcal{N}_{\mathbb{F}_p})$. In this section we will first look at the spectral sequence from [Sør] (cf. Theorem 1.5), i.e.,

$$E_1^{s,t} = H^{s,t}(\mathfrak{g}, \mathbb{F}_p) \Longrightarrow H^{s+t}_{\text{cts}}(N, \mathbb{F}_p),$$

and note that we can work with the left side using that $H^{s,t}(\mathfrak{g},\mathbb{F}_p) \cong H^{s,t}(\mathfrak{n},\mathbb{F}_p)$ and for the right side $H^{s+t}_{\mathrm{cts}}(N,\mathbb{F}_p) \cong H^{s+t}_{\mathrm{dsc}}(N,\mathbb{F}_p)$. Afterwards, we will use results from [PT] to argue that the spectral sequence collapses on the first page.

We will start by showing that $\mathfrak{g} \cong \mathfrak{n}$, for which we will need the following lemma.

Lemma 1.10. $\operatorname{gr} N \cong \mathbb{F}_p[\pi] \otimes_{\mathbb{F}_p} \mathfrak{n}$ as graded Lie algebras (where π has degree 1). ♣

Proof. We first note that the elements X_α, where X_α is our fixed $\mathbb{Z}_p$-basis of $\operatorname{Lie}\mathcal{N}_\alpha$, reduce modulo p to an $\mathbb{F}_p$-basis $\{\overline{X}_\alpha\}_{\alpha\in\Phi^-}$ of $\mathfrak{n}$. On the other hand all

$$\sigma\big(x_\alpha(1)\big) \in \operatorname{gr}_{-\operatorname{ht}(\alpha)} N,$$

with $x_\alpha(1) \in N_{-\operatorname{ht}(\alpha)}$, form an $\mathbb{F}_p[\pi]$-basis of $\operatorname{gr} N$, cf. [Sch] Proposition 26.5. Hence the map

$$\begin{aligned} \mathbb{F}_p[\pi] \otimes_{\mathbb{F}_p} \mathfrak{n} &\to \operatorname{gr} N \\ f \otimes \overline{X}_\alpha &\mapsto f\,.\,\sigma\big(x_\alpha(1)\big) \end{aligned}$$

is an isomorphism of graded modules. Chevalley's commutator formula says that there are p-adic integers $c_{\alpha,\beta}$ such that $[X_\alpha, X_\beta] = c_{\alpha,\beta}X_{\alpha+\beta}$ and DK Note: clarify

$$[x_\alpha(1), x_\beta(1)] \in x_{\alpha+\beta}(c_{\alpha,\beta})N_{-\operatorname{ht}(\alpha)-\operatorname{ht}(\beta)+1} = x_{\alpha+\beta}(1)^{c_{\alpha,\beta}}N_{-\operatorname{ht}(\alpha)-\operatorname{ht}(\beta)+1},$$

where $X_{\alpha+\beta} = 0$ and $x_{\alpha+\beta} \equiv 1$ if $\alpha+\beta \notin \Phi$. This implies that the image of the above map is a Lie subalgebra, and thus that the map is an isomorphism of Lie algebras. $\square$

Now $\operatorname{gr} N \cong \mathbb{F}_p[\pi] \otimes_{\mathbb{F}_p} \mathfrak{n}$ implies that $\mathfrak{g} \cong \mathbb{F}_p \otimes_{\mathbb{F}_p[\pi]} \mathbb{F}_p[\pi] \otimes_{\mathbb{F}_p} \mathfrak{n} \cong \mathfrak{n}$, where both $\mathfrak{g}$ and $\mathfrak{n}$ is graded by the height function. From this it clearly follows that $H^{s,t}(\mathfrak{g}, \mathbb{F}_p) \cong H^{s,t}(\mathfrak{n}, \mathbb{F}_p)$. Note that this can also be seen directly by looking at the Chevalley constants. Finally, since we proved in the previous section that N is a pro-p group, we get (as noted in (1.3)) that $H^n_{\mathrm{cts}}(N, \mathbb{F}_p) \cong H^n_{\mathrm{dsc}}(N, \mathbb{F}_p)$ for all n.

By [PT, §2.10] (using that $p \geq h-1$) and the Universal Coefficient Theorem (as used in [PT, §3.8]), we get a $\mathbb{F}_p$-vector space isomorphism

$$H^n(\mathfrak{n}_{\mathbb{Z}}, \mathbb{F}_p) = H^n(\mathfrak{n}_{\mathbb{Z}}, V_{\mathbb{F}_p}(0)) \cong \bigoplus_{\substack{w \in W \\ \ell(w)=n}} V_{\mathbb{F}_p}(w \cdot 0),$$

where $V_{\mathbb{F}_p}(0) = \mathbb{F}_p$ with the trivial action (concentrated in degree 0). Similarly, by the corollary in **??**§3.8]PT, we have a $\mathbb{F}_p$-vector space isomorphism

$$\operatorname{gr} H^n_{\mathrm{dsc}}(\mathcal{N}_{\mathbb{Z}}(\mathbb{Z}), \mathbb{F}_p) = \operatorname{gr} H^n_{\mathrm{dsc}}(\mathcal{N}_{\mathbb{Z}}(\mathbb{Z}), V_{\mathbb{F}_p}(0)) \cong \bigoplus_{\substack{w \in W \\ \ell(w)=n}} V_{\mathbb{F}_p}(w \cdot 0).$$

Here the grading on cohomology won't be important, since we just need that

$$\dim_{\mathbb{F}_p} H^n(\mathfrak{n}_{\mathbb{Z}}, \mathbb{F}_p) = \dim_{\mathbb{F}_p} H^n_{\mathrm{dsc}}(\mathcal{N}_{\mathbb{Z}}(\mathbb{Z}), \mathbb{F}_p). \tag{1.14}$$ {eq:PT-dims}

We now equip $\mathcal{N}_{\mathbb{Z}}(\mathbb{Z})$ with the discrete topology and claim that

$$H^n_{\mathrm{dsc}}(\mathcal{N}_{\mathbb{Z}}(\mathbb{Z}), \mathbb{F}_p) = H^n_{\mathrm{cts}}(\mathcal{N}_{\mathbb{Z}}(\mathbb{Z}), \mathbb{F}_p) \cong H^n_{\mathrm{cts}}(\mathcal{N}(\mathbb{Z}_p), \mathbb{F}_p).$$

Here the first equality is clear since $\mathcal{N}_{\mathbb{Z}}(\mathbb{Z})$ is equipped with the discrete topology. To see the isomorphism, first note that $\mathbb{Z}$ is a discrete group, $\mathbb{Z}_p$ is a profinite group, and the homomorphism $\mathbb{Z} \to \mathbb{Z}_p$ has dense image in $\mathbb{Z}_p$. So we have homomorphisms

$$H^n_{\mathrm{cts}}(\mathbb{Z}_p, \mathbb{F}_p) \to H^n_{\mathrm{cts}}(\mathbb{Z}, \mathbb{F}_p)$$

for all $n \geq 0$ from [Ser, Sect. I §2.6]. Now both $H^0_{\text{cts}}(\mathbb{Z}, \,\cdot\,)$ and $H^0_{\text{cts}}(\mathbb{Z}_p, \,\cdot\,)$ are the functor of taking invariant, both $H^1_{\text{cts}}(\mathbb{Z}, \,\cdot\,)$ and $H^1_{\text{cts}}(\mathbb{Z}_p, \,\cdot\,)$ are the functor of taking "coinvariants", and all $H^n(\mathbb{Z}, \,\cdot\,)$ and $H^n(\mathbb{Z}_p, \,\cdot\,)$ vanish for $n \geq 2$, so $\mathbb{Z}$ is "good" in the sense of [Ser, Section I §2.6 Exercise 2]. Thus [Ser, Section I §2.6 Exercise 2(d)] implies that the homomorphisms

DK Note: Rewrite without "coinvariants"

$$H^n_{\text{cts}}(\mathcal{N}(\mathbb{Z}_p), \mathbb{F}_p) \to H^n_{\text{cts}}(\mathcal{N}(\mathbb{Z}), \mathbb{F}_p) \qquad n \geq 0,$$

DK Note: Rewrite this more like in the introduction.

induced by the homomorphism $\mathcal{N}(\mathbb{Z}) \to \mathcal{N}(\mathbb{Z}_p)$, are all isomorphisms.

Hence

$$\dim_{\mathbb{F}_p} H^n(\mathfrak{n}_{\mathbb{Z}}, \mathbb{F}_p) = \dim_{\mathbb{F}_p} H^n_{\text{dsc}}(\mathcal{N}_{\mathbb{Z}}(\mathbb{Z}), \mathbb{F}_p) = \dim_{\mathbb{F}_p} H^n_{\text{cts}}(\mathcal{N}(\mathbb{Z}_p), \mathbb{F}_p).$$

Now $\mathfrak{n} = \mathfrak{n}_{\mathbb{Z}} \otimes \mathbb{F}_p$, and $H^n(\mathfrak{g}, \mathbb{F}_p) \cong H^n(\mathfrak{n}, \mathbb{F}_p)$ (since $\mathfrak{g} \cong \mathfrak{n}$) is the cohomology of the complex

$$C^\bullet(\mathfrak{n}, \mathbb{F}_p) = \operatorname{Hom}_{\mathbb{F}_p}\Big(\bigwedge^{\bullet} \mathfrak{n}, \mathbb{F}_p\Big)$$

while $H^n(\mathfrak{n}_{\mathbb{Z}}, \mathbb{F}_p)$ is the homology of the complex

$$C^\bullet(\mathfrak{n}_{\mathbb{Z}}, \mathbb{F}_p) = \operatorname{Hom}_{\mathbb{F}_p}\Big(\bigwedge^{\bullet} \mathfrak{n}_{\mathbb{Z}}, \mathbb{F}_p\Big).$$

Here $\bigwedge^\bullet \mathfrak{n}_{\mathbb{Z}}$ is a free $\mathbb{Z}$-module and $(\bigwedge^\bullet \mathfrak{n}_{\mathbb{Z}}) \otimes \mathbb{F}_p \cong \bigwedge^\bullet(\mathfrak{n}_{\mathbb{Z}} \otimes \mathbb{F}_p) \cong \bigwedge^\bullet \mathfrak{n}$, so we have natural isomorphisms

$$\operatorname{Hom}_{\mathbb{F}_p}\Big(\bigwedge^{\bullet} \mathfrak{n}_{\mathbb{Z}}, \mathbb{F}_p\Big) \cong \operatorname{Hom}_{\mathbb{F}_p}\Big(\Big(\bigwedge^{\bullet} \mathfrak{n}_{\mathbb{Z}}\Big) \otimes \mathbb{F}_p, \mathbb{F}_p\Big) \cong \operatorname{Hom}_{\mathbb{F}_p}\Big(\bigwedge^{\bullet} \mathfrak{n}, \mathbb{F}_p\Big).$$

These isomorphisms are clearly compatible with the differentials, so $C^\bullet(\mathfrak{n}, \mathbb{F}_p) \cong C^\bullet(\mathfrak{n}_{\mathbb{Z}}, \mathbb{F}_p)$, and thus $H^n(\mathfrak{n}, \mathbb{F}_p) \cong H^n(\mathfrak{n}_{\mathbb{Z}}, \mathbb{F}_p)$. Hence

$$\dim_{\mathbb{F}_p} H^n(\mathfrak{n}, \mathbb{F}_p) = \dim_{\mathbb{F}_p} H^n(\mathfrak{n}_{\mathbb{Z}}, \mathbb{F}_p) = \dim_{\mathbb{F}_p} H^n(\mathcal{N}(\mathbb{Z}_p), \mathbb{F}_p).$$

Now $\dim_{\mathbb{F}_p} H^n(\mathfrak{n}, \mathbb{F}_p) = \dim^n_{\mathbb{F}_p}(\mathfrak{g}, \mathbb{F}_p)$ and $N = \mathcal{N}(\mathbb{Z}_p)$ implies that

$$\sum_{s+t=n} \dim_{\mathbb{F}_p} H^{s,t}(\mathfrak{g}, \mathbb{F}_p) = \dim_{\mathbb{F}_p} H^n(\mathfrak{g}, \mathbb{F}_p) = \dim_{\mathbb{F}_p} H^n(N, \mathbb{F}_p),$$

so the multiplicative spectral sequence

$$E_1^{s,t} = H^{s,t}(\mathfrak{g}, \mathbb{F}_p) \Longrightarrow H^{s+t}(N, \mathbb{F}_p)$$

collapses on the first page, since the dimension of $E_r^{s,t}$ is non-increasing as r increases. Since the spectral sequence collapses on the first page, we get that $E_1^{s,t} = E_\infty^{s,t}$, so

$$\operatorname{gr}^s H^n(N, \mathbb{F}_p) \cong H^n(\mathfrak{g}, \mathbb{F}_p) \cong H^n(\mathfrak{n}, \mathbb{F}_p),$$

giving us a good description of $H^n(\mathcal{N}(\mathbb{Z}_p), \mathbb{F}_p)$. Furthermore, we can describe the cup product, by calculating it in $H^*(\mathfrak{g}, \mathbb{F}_p)$ or $H^*(\mathfrak{n}, \mathbb{F}_p)$, cf. Theorem 1.5 for the details.

DK Note: Rewrite theorem nicely here.

1.4 Example: $N \subseteq \mathrm{SL}_3(\mathbb{Z}_p)$

sec:ex-N-in-SL3

In the case of $\mathcal{G} = \mathrm{SL}_3$ (in this case $h = 4$, so $p \geq 3$), we can take $\mathcal{T}$ to be the diagonal matrices in SL_3 ($\det = 1$), $\mathcal{B}$ upper triangular matrices in SL_3 and

$$\mathcal{N} = \left\{ \begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix} \right\} \subseteq \mathrm{SL}_n .$$

Furthermore we can take $\Phi^- = \{\alpha_1, \alpha_2, \alpha_3 = \alpha_1 + \alpha_2\}$ with

$$X_{\alpha_1} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad x_{\alpha_1}(A)(a) = \begin{pmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$X_{\alpha_2} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad x_{\alpha_2}(A)(a) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & a \\ 0 & 0 & 1 \end{pmatrix},$$

$$X_{\alpha_3} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad x_{\alpha_3}(A)(a) = \begin{pmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

for $\mathbb{Z}_p$-algebra A and $a \in A$. Here $\mathrm{ht}(\alpha_1) = \mathrm{ht}(\alpha_2) = -1$ and $\mathrm{ht}(\alpha_3) = -2$, and explicit calculations show that, in $N = \mathcal{N}(\mathbb{Z}_p)$, $g_1 = x_{\alpha_1}(1), g_2 = x_{\alpha_2}(1), g_3 = x_{\alpha_3}(1)$ is an ordered basis of (N, ω). Thus (cf. [Sch, Prop. 26.5]) $\sigma(g_1), \sigma(g_2), \sigma(g_3)$ is a basis of the $\mathbb{F}_p[\pi]$-module $\mathrm{gr}\, N$, and ξ_1, ξ_2, ξ_3 is a basis of $\mathfrak{g} = \mathbb{F}_p \otimes_{\mathbb{F}_p[\pi]} \mathrm{gr}\, N$, where $\xi_i = 1 \otimes \sigma(g_i)$. Furthermore $\mathfrak{g} = \mathfrak{g}^1 \oplus \mathfrak{g}^2$, where $\mathfrak{g}^1 = \mathrm{span}(\xi_1, \xi_2)$ and $\mathfrak{g}^2 = \mathrm{span}(\xi_3)$.

DK Note: Maybe show the calculations.

The only non-trivial commutator among the g_i's is $[g_1, g_2] = x_{\alpha_3}(-1)$, which implies (cf. [Sch, Rem. 26.3]) that $\sigma([g_1, g_2]) = -\sigma(g_3)$ and thus $[\xi_1, \xi_2] = -\xi_3$. So $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}^2$.

Now $H^1(\mathfrak{g}, \mathbb{F}_p) = \mathrm{Hom}_k(\mathfrak{g}/[\mathfrak{g},\mathfrak{g}], \mathbb{F}_p) = H^{-1,2}(\mathfrak{g}, \mathbb{F}_p)$, and, since $\bigwedge^3 \mathfrak{g} = \mathfrak{g}^1 \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^2$ is degree 4, $H^3(\mathfrak{g}, \mathbb{F}_p) = H^{-4,7}(\mathfrak{g}, \mathbb{F}_p)$. And a version of Poincaré duality (cf. [Fuk]) gives us that $H^1 \times H^2 \to H^3$ with $H^{-1,2} \times H^{s,t} \to H^{-4,7}$ only works for $(s,t) = (-3, 5)$, so $H^2(\mathfrak{g}, \mathbb{F}_p) = H^{-3,5}(\mathfrak{g}, \mathbb{F}_p)$. This gives us a description of $H^*(N, \mathbb{F}_p)$, and we note (either by explicit calculation or by considering properties of the wedge product) that the only non-trivial cup product is $H^1(N, \mathbb{F}_p) \times H^2(N, \mathbb{F}_p) \to H^3(N, \mathbb{F}_p)$.

DK Note: Write more details here.

Chapter 2

Cohomology of Iwahori Subgroups

cha:cohiwagps

2.1 Intoduction

sec:cohiwagps

2.2 $I \subseteq \mathrm{SL}_2(\mathbb{Z}_p)$

sec:Iwa-SL2

$$I = \begin{pmatrix} 1 + p\mathbb{Z}_p & \mathbb{Z}_p \\ p\mathbb{Z}_p & 1 + p\mathbb{Z}_p \end{pmatrix} \subseteq \mathrm{SL}_2(\mathbb{Z}_p).$$

Obvious try (using that $(1+p)^{\mathbb{Z}_p} = 1 + p\mathbb{Z}_p$):

$$g_1' = \begin{pmatrix} 1 & 0 \\ p & 1 \end{pmatrix}, \qquad g_2' = \begin{pmatrix} 1+p & 0 \\ 0 & (1+p)^{-1} \end{pmatrix}, \qquad g_3' = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Better:

$$g_1 = \begin{pmatrix} 1 & 0 \\ p & 1 \end{pmatrix}, \qquad g_2 = \begin{pmatrix} \exp(p) & 0 \\ 0 & \exp(-p) \end{pmatrix}, \qquad g_3 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \tag{2.1}$$

{eq:gis-SL2}

For $g = (a_{ij})$

$$\omega(g) := \min\bigl(v_p(a_{11}-1), \tfrac{1}{2} + v_p(a_{12}), -\tfrac{1}{2} + v_p(a_{21}), v_p(a_{22}-1)\bigr).$$

$$g_1^{x_1} g_2^{x_2} g_3^{x_3} = \begin{pmatrix} \exp(px_2) & x_3 \exp(px_2) \\ px_1 \exp(px_2) & px_1 x_3 \exp(px_2) + \exp(-px_2) \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}. \tag{2.2}$$

{eq:gixi-SL2}

$g_{ij} = [g_i, g_j]$

In the following we use that $\frac{1}{p-1} = 1 + p + p^2 + \cdots$ and $\log(1-p) = -p - \frac{p^2}{2} - \frac{p^3}{3} - \cdots$.

$g_{12} = \begin{pmatrix} 1 & 0 \\ p\bigl(1-\exp(-2p)\bigr) & \end{pmatrix}$**:** Comparing g_{12} with (2.2), we see that $x_2 = x_3 = 0$. This leaves $a_{21} = px_1 = p\bigl(1-\exp(-2p)\bigr) = 2p^2 + O(p^3)$, which implies that $x_1 = 2p + O(p^2)$. Hence $\sigma(g_{12}) = 2\pi \,.\, \sigma(g_1)$, which implies that $\xi_{12} = 0$.

$g_{13} = \begin{pmatrix} 1-p & p \\ -p^2 & 1+p+p^2 \end{pmatrix}$**:** Comparing g_{13} with (2.2), we see that

$$\begin{aligned} a_{11} &= \exp(px_2) = 1-p, \\ a_{12} &= x_3 \exp(px_2) = x_3(1-p) = p, \\ a_{21} &= px_1 \exp(px_2) = px_1(1-p) = -p^2, \end{aligned}$$

and thus

$$x_2 = \frac{1}{p}\log(1-p) = \frac{1}{p}\big((-p) + O(p^2)\big) = -1 + O(p),$$
$$x_3 = \frac{p}{1-p} = p + O(p^2),$$
$$x_1 = \frac{-p^2}{p(1-p)} = -p + O(p^2).$$

Hence $\sigma(g_{13}) = -\pi \,.\, \sigma(g_1) - \sigma(g_2) - \pi \,.\, \sigma(g_3)$, which implies that $\xi_{13} = -\xi_2$.

$g_{23} = \begin{pmatrix} 1 & \exp(2p) - 1 \\ 0 & 1 \end{pmatrix}$**:** Comparing g_{23} with (2.2), we see that $x_1 = x_2 = 0$. This leaves $a_{12} = x_3 = \exp(2p) - 1 = 2p + O(p^2)$. Hence $\sigma(g_{23}) = 2\pi \,.\, \sigma(g_3)$, which implies that $\xi_{23} = 0$.

$$\sigma(g_{12}) = 2\pi \,.\, \sigma(g_1),$$
$$\sigma(g_{13}) = \pi \,.\, \sigma(g_1) + (p-1)\sigma(g_2) + \pi \,.\, \sigma(g_3),$$
$$\sigma(g_{23}) = \pi \,.\, \sigma(g_3).$$

So with $\xi_i = 1 \otimes \sigma(g_i)$:

$$[\xi_1, \xi_2] = 0, \qquad [\xi_1, \xi_3] = -\xi_2, \qquad [\xi_2, \xi_3] = 0.$$

2.3 $I \subseteq \mathrm{GL}_2(\mathbb{Z}_p)$

sec:Iwa-GL2

$$\begin{aligned} g_1 &= \begin{pmatrix} 1 & 0 \\ p & 1 \end{pmatrix}, \qquad g_2 = \begin{pmatrix} \exp(p) & 0 \\ 0 & \exp(-p) \end{pmatrix}, \\ g_3 &= \begin{pmatrix} \exp(p) & 0 \\ 0 & \exp(p) \end{pmatrix}, \qquad g_4 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \end{aligned} \tag{2.3}$$

{eq:gis-GL2}

$$\begin{aligned} &g_1^{x_1} g_2^{x_2} g_3^{x_3} g_4^{x_4} \\ &= \begin{pmatrix} \exp\bigl(p(x_2+x_3)\bigr) & \exp\bigl(p(x_2+x_3)\bigr)x_4 \\ px_1\exp\bigl(p(x_2+x_3)\bigr) & \exp\bigl(p(x_2+x_3)\bigr)px_1x_4 + \exp\bigl(p(x_3-x_2)\bigr) \end{pmatrix} \\ &= \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}. \end{aligned} \tag{2.4}$$

$g_{ij} = [g_i, g_j]$

$$\begin{aligned} \sigma(g_{12}) &= (p-2)\pi \,.\, \sigma(g_1), \\ \sigma(g_{14}) &= (p-1)\pi \,.\, \sigma(g_1) + (p-1)\sigma(g_2) + \pi \,.\, \sigma(g_3), \\ \sigma(g_{24}) &= (p-2)\pi \,.\, \sigma(g_3), \\ \sigma(g_{13}) &= \sigma(g_{23}) = \sigma(g_{24}) = 0. \end{aligned}$$

So with $\xi_i = 1 \otimes \sigma(g_i)$:

$$[\xi_1, \xi_4] = -\xi_2$$

is the only non-zero commutator.

2.4 $I \subseteq \mathrm{SL}_3(\mathbb{Z}_p)$

sec:Iwa-SL3

To make the notation easier to read for the bigger matrices, we will write any zeros as blank space in matrices in this section.

$$g_1 = \begin{pmatrix} 1 & & \\ & 1 & \\ p & & 1 \end{pmatrix}, \quad g_2 = \begin{pmatrix} 1 & & \\ p & 1 & \\ & & 1 \end{pmatrix}, \quad g_3 = \begin{pmatrix} 1 & & \\ & 1 & \\ & p & 1 \end{pmatrix},$$
$$g_4 = \begin{pmatrix} \exp(p) & & \\ & \exp(-p) & \\ & & 1 \end{pmatrix}, \quad g_5 = \begin{pmatrix} 1 & & \\ & \exp(p) & \\ & & \exp(-p) \end{pmatrix}, \tag{2.5}$$
$$g_6 = \begin{pmatrix} 1 & & \\ & 1 & 1 \\ & & 1 \end{pmatrix}, \quad g_7 = \begin{pmatrix} 1 & 1 & \\ & 1 & \\ & & 1 \end{pmatrix}, \quad g_8 = \begin{pmatrix} 1 & & 1 \\ & 1 & \\ & & 1 \end{pmatrix}.$$

{eq:gis-SL3}

$$g_1^{x_1} g_2^{x_2} g_3^{x_3} g_4^{x_4} g_5^{x_5} g_6^{x_6} g_7^{x_7} g_8^{x_8} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

where

$$
\begin{aligned}
a_{11} &= \exp(px_4),\\
a_{12} &= x_7 \exp(px_4),\\
a_{13} &= x_8 \exp(px_4),\\
a_{21} &= px_2 \exp(px_4),\\
a_{22} &= px_2x_7 \exp(px_4) + \exp\big(p(x_5 - x_4)\big),\\
a_{23} &= px_2x_8 \exp(px_4) + x_6 \exp\big(p(x_5 - x_4)\big),\\
a_{31} &= px_1 \exp(px_4),\\
a_{32} &= px_1x_7 \exp(px_4) + px_3 \exp\big(p(x_5 - x_4)\big),\\
a_{33} &= px_1x_8 \exp(px_4) + px_3x_6 \exp\big(p(x_5 - x_4)\big) + \exp(-px_5).
\end{aligned}
\tag{2.6}
$$

{eq:gixi-SL3}

Non-identiy $[g_i, g_j]$

subsec:non-id-gij-SL3

$g_{ij} = [g_i, g_j]$

Except in the first case, we will note that $x_i \in p\mathbb{Z}_p$ implies that the coefficient on ξ_k in ξ_{ij} is zero.

DK Note: Introduce $O(p^k)$ notation.

Note that we repeatedly use that $-1 = (p-1) + (p-1)p + (p-1)p^2 + \cdots$ in $\mathbb{Z}_p$ and $-1 = p - 1$ in $\mathbb{F}_p$.

$g_{14} = \begin{pmatrix} & 1 & \\ & & 1 \\ p\big(1-\exp(-p)\big) & & 1 \end{pmatrix}$**:** Comparing g_{14} with (2.6), we see that $x_2 = x_4 = x_7 = x_8 = 0$, and thus also $x_3 = x_5 = x_6 = 0$. This leaves $a_{31} = px_1 = p\big(1 - \exp(-p)\big) = p^2 + O(p^3)$, which implies that $x_1 = p + O(p^2)$. Hence $\sigma(g_{14}) = \pi \,.\, \sigma(g_1)$, which implies that $\xi_{14} = 0$.

$g_{15} = \begin{pmatrix} 1 & & \\ & 1 & \\ p\big(1-\exp(-p)\big) & & 1 \end{pmatrix}$**:** Since $g_{15} = g_{14}$, the above calculation shows that $\xi_{15} = 0$.

$g_{16} = \begin{pmatrix} 1 & & \\ -p & 1 & \\ & & 1 \end{pmatrix}$**:** Comparing g_{16} with (2.6), we see that $x_1 = x_4 = x_7 = x_8 = 0$, and thus also $x_3 = x_5 = x_6 = 0$. This leaves $a_{21} = px_2 = -p$, which implies that $x_2 = -1$. Hence $\sigma(g_{16}) = -\sigma(g_2)$, which implies that $\xi_{16} = -\xi_2$.

$g_{17} = \begin{pmatrix} 1 & & \\ & 1 & \\ & p & 1 \end{pmatrix}$**:** Comparing g_{17} with (2.6), we see that $x_1 = x_2 = x_4 = x_7 = x_8 = 0$, and thus also $x_5 = x_6 = 0$. This leaves $a_{32} = px_3 = p$, which implies that $x_3 = 1$. Hence $\sigma(g_{17}) = \sigma(g_3)$, which implies that $\xi_{17} = \xi_3$.

$g_{18} = \begin{pmatrix} 1-p & & p \\ & 1 & \\ -p^2 & & 1+p+p^2 \end{pmatrix}$**:** Comparing g_{18} with (2.6), we see that $x_2 = x_7 = 0$, and thus also $x_3 = x_6 = 0$ and $x_4 = x_5$. Using

$$a_{11} = \exp(px_4) = 1 - p,$$

$$a_{13} = x_8 \exp(px_4) = x_8(1-p) = p,$$

$$a_{31} = px_1 \exp(px_4) = px_1(1-p) = -p^2,$$

we get that

$$x_4 = \frac{1}{p}\log(1-p) = \frac{1}{p}\big((-p) + O(p^2)\big) = -1 + O(p),$$
$$x_8 = \frac{p}{1-p} = p + O(p^2),$$
$$x_1 = \frac{-p^2}{p(1-p)} = -p + O(p^2).$$

Hence $\sigma(g_{18}) = -\pi \,.\, \sigma(g_1) - \sigma(g_4) - \sigma(g_5) + \pi \,.\, \sigma(g_8)$, which implies that $\xi_{18} = -(\xi_4 + \xi_5)$.

$g_{23} = \begin{pmatrix} 1 & & \\ & 1 & \\ -p^2 & & 1 \end{pmatrix}$**:** Comparing g_{23} with (2.6), we see that $x_2 = x_4 = x_7 = x_8 = 0$, and thus also $x_3 = x_5 = x_6 = 0$. This leaves $a_{31} = px_1 = -p^2$, which implies that $x_1 = -p$. Hence $\sigma(g_{23}) = -\pi \,.\, \sigma(g_1)$, which implies that $\xi_{23} = 0$.

$g_{24} = \begin{pmatrix} & 1 & \\ p\big(1-\exp(-2p)\big) & 1 & \\ & & 1 \end{pmatrix}$**:** Comparing g_{24} with (2.6), we see that $x_1 = x_4 = x_7 = x_8 = 0$, and thus also $x_3 = x_5 = x_6 = 0$. This leaves $a_{21} = px_2 = p\big(1-\exp(-2p)\big) = p\big(1-\big(1+(-2p)+O(p^2)\big)\big) = 2p^2+O(p^3)$, which implies that $x_2 = 2p + O(p^2)$. Hence $\sigma(g_{24}) = 2\pi \,.\, \sigma(g_1)$, which implies that $\xi_{24} = 0$.

$g_{25} = \begin{pmatrix} & 1 & \\ p\big(1-\exp(p)\big) & 1 & \\ & & 1 \end{pmatrix}$**:** Except a factor -2 in the exponential, which clearly doesn't change the final result, we have the same calculation as for g_{24}. Thus $\xi_{25} = 0$.

$g_{27} = \begin{pmatrix} 1-p & p & \\ -p^2 & 1+p+p^2 & \\ & & 1 \end{pmatrix}$: Comparing g_{27} with (2.6), we see that $x_1 = x_8 = 0$, and thus also $x_3 = x_6 = 0$, so $x_5 = 0$. Using

$$a_{11} = \exp(px_4) = 1 - p,$$

$$a_{12} = x_7 \exp(px_4) = x_8(1-p) = p,$$

$$a_{21} = px_2 \exp(px_4) = px_2(1-p) = -p^2,$$

we get that

$$\begin{aligned} x_4 &= \frac{1}{p}\log(1-p) = \frac{1}{p}\big((-p) + O(p^2)\big) = -1 + O(p), \\ x_7 &= \frac{p}{1-p} = p + O(p^2), \\ x_2 &= \frac{-p^2}{p(1-p)} = -p + O(p^2). \end{aligned}$$

Hence $\sigma(g_{27}) = -\pi\,.\,\sigma(g_2) - \sigma(g_4) + \pi\,.\,\sigma(g_7)$, which implies that $\xi_{27} = -\xi_4$.

$g_{28} = \begin{pmatrix} 1 & & \\ & 1 & p \\ & & 1 \end{pmatrix}$: Comparing g_{28} with (2.6), we see that $x_1 = x_2 = x_4 = x_7 = x_8 = 0$, and thus also $x_3 = x_5 = 0$. This leaves $a_{23} = x_6 = p$. Hence $\sigma(g_{28}) = \pi\,.\,\sigma(g_6)$, which implies that $\xi_{28} = 0$.

$g_{34} = \begin{pmatrix} 1 & & \\ & 1 & \\ & p\big(1-\exp(p)\big) & 1 \end{pmatrix}$: Comparing g_{34} with (2.6), we see that $x_1 = x_2 = x_4 = x_7 = x_8 = 0$, and thus also $x_5 = x_6 = 0$. This leaves

$a_{32} = px_3 = p\bigl(1 - \exp(p)\bigr) = p\bigl(1 - \bigl(1 + p + O(p^2)\bigr)\bigr) = -p^2 + O(p^3),$ which implies that $x_3 = -p + O(p^2)$. Hence $\sigma(g_{34}) = -\pi \,.\, \sigma(g_3)$, which implies that $\xi_{34} = 0$.

$g_{35} = \begin{pmatrix} 1 & & \\ & 1 & \\ & p\bigl(1-\exp(-2p)\bigr) & 1 \end{pmatrix}$: Except a factor -2 in the exponential, which clearly doesn't change the final result, we have the same calculation as for g_{34}. Thus $\xi_{35} = 0$.

$g_{36} = \begin{pmatrix} 1 & & \\ & 1-p & p \\ & -p^2 & 1+p+p^2 \end{pmatrix}$: Comparing g_{36} with (2.6), we see that $x_1 = x_2 = x_4 = x_7 = x_8 = 0$. Using

$$\begin{aligned} a_{22} &= \exp(px_5) = 1 - p, \\ a_{23} &= x_6 \exp(px_5) = x_6(1-p) = p, \\ a_{32} &= px_3 \exp(px_5) = px_3(1-p) = -p^2, \end{aligned}$$

we get that

$$\begin{aligned} x_5 &= \frac{1}{p}\log(1-p) = \frac{1}{p}\bigl((-p) + O(p^2)\bigr) = -1 + O(p), \\ x_6 &= \frac{p}{1-p} = p + O(p^2), \\ x_3 &= \frac{-p^2}{p(1-p)} = -p + O(p^2). \end{aligned}$$

Hence $\sigma(g_{36}) = -\pi \,.\, \sigma(g_3) - \sigma(g_5) + \pi \,.\, \sigma(g_6)$, which implies that $\xi_{36} = -\xi_5$.

$g_{38} = \begin{pmatrix} 1 & -p & \\ & 1 & \\ & & 1 \end{pmatrix}$: Comparing g_{38} with (2.6), we see that $x_1 = x_2 = x_4 = x_8 = 0$, and thus also $x_3 = x_5 = x_6 = 0$. This leaves $a_{12} = x_7 = -p$. Hence $\sigma(g_{38}) = -\pi \,.\, \sigma(g_3)$, which implies that $\xi_{38} = 0$.

$g_{46} = \begin{pmatrix} 1 & & \\ & 1 & \exp(-p) - 1 \\ & & 1 \end{pmatrix}$: Comparing g_{46} with (2.6), we see that $x_1 = x_2 = x_4 = x_7 = x_8 = 0$, and thus also $x_3 = x_5 = 0$. This leaves $a_{23} = x_6 = \exp(-p) - 1 = -p + O(p^2)$. Hence $\sigma(g_{46}) = -\pi \,.\, \sigma(g_6)$, which implies that $\xi_{46} = 0$.

$g_{47} = \begin{pmatrix} 1 & \exp(2p) - 1 & \\ & 1 & \\ & & 1 \end{pmatrix}$: Comparing g_{47} with (2.6), we see that $x_1 = x_2 = x_4 = x_8 = 0$, and thus also $x_3 = x_5 = x_6 = 0$. This leaves $a_{12} = x_7 = \exp(2p) - 1 = 2p + O(p^2)$. Hence $\sigma(g_{47}) = 2\pi \,.\, \sigma(g_7)$, which implies that $\xi_{47} = 0$.

$g_{48} = \begin{pmatrix} 1 & & \exp(p) - 1 \\ & 1 & \\ & & 1 \end{pmatrix}$: Comparing g_{48} with (2.6), we see that $x_1 = x_2 = x_4 = x_7 = 0$, and thus also $x_3 = x_5 = x_6 = 0$. This leaves $a_{13} = x_8 = \exp(p) - 1 = p + O(p^2)$. Hence $\sigma(g_{48}) = \pi \,.\, \sigma(g_8)$, which implies that $\xi_{48} = 0$.

$g_{56} = \begin{pmatrix} 1 & & \\ & 1 & \exp(2p)-1 \\ & & 1 \end{pmatrix}$: Except a factor -2 in the exponential, which clearly doesn't change the final result, we have the same calculation as for g_{46}. Thus $\xi_{56} = 0$.

$g_{57} = \begin{pmatrix} 1 & \exp(-p)-1 & \\ & 1 & \\ & & 1 \end{pmatrix}$: Except a factor -2 in the exponential, which clearly doesn't change the final result, we have the same calculation as for g_{47}. Thus $\xi_{57} = 0$.

$g_{58} = \begin{pmatrix} 1 & & \exp(p)-1 \\ & 1 & \\ & & 1 \end{pmatrix}$: Since $g_{58} = g_{48}$, the above calculation shows that $\xi_{58} = 0$.

$g_{67} = \begin{pmatrix} 1 & & -1 \\ & 1 & \\ & & 1 \end{pmatrix}$: Comparing g_{67} with (2.6), we see that $x_1 = x_2 = x_4 = x_7 = 0$, and thus also $x_3 = x_5 = x_6 = 0$. This leaves $a_{13} = x_8 = -1$. Hence $\sigma(g_{67}) = -\sigma(g_8)$, which implies that $\xi_{67} = -\xi_8$.

The non-zero commutators are:

$$\begin{aligned} &[\xi_1, \xi_6] = -\xi_2, \quad [\xi_1, \xi_7] = \xi_3, \qquad [\xi_1, \xi_8] = -(\xi_4 + \xi_5), \\ &[\xi_2, \xi_7] = -\xi_4, \quad [\xi_3, \xi_6] = -\xi_5, \quad [\xi_6, \xi_7] = -\xi_8. \end{aligned} \tag{2.7}$$ {eq:xi_ij-SL3}

$$\mathfrak{g} = k \otimes_{\mathbb{F}_p[\pi]} \operatorname{gr} I = \operatorname{span}\{\xi_1, \dots, \xi_8\} = \mathfrak{g}_{\frac{1}{3}} \oplus \mathfrak{g}_{\frac{2}{3}} \oplus \mathfrak{g}_1 = \mathfrak{g}^1 \oplus \mathfrak{g}^2 \oplus \mathfrak{g}^3.$$

$$[\mathfrak{g}^i, \mathfrak{g}^j] = \begin{cases} \mathfrak{g}^2 & \text{if } i = j = 1, \\ \mathfrak{g}^3 & \text{if } (i,j) \in \{(1,2),(2,1)\}, \\ 0 & \text{otherwise.} \end{cases} \tag{2.8}$$ {eq:5}

$$\mathrm{gr}^j\Big(\bigwedge^n \mathfrak{g}\Big) = \bigoplus_{j_1+\cdots+j_n=j} \mathfrak{g}^{j_1} \wedge \cdots \wedge \mathfrak{g}^{j_n}.$$

$n \geq 9$:

$$\mathrm{gr}^j\Big(\bigwedge^n \mathfrak{g}\Big) = 0 \text{ for all } j.$$

$n = 8$:

$$\mathrm{gr}^j\Big(\bigwedge^8 \mathfrak{g}\Big) = \begin{cases} \mathfrak{g}^1 \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^3 \wedge \mathfrak{g}^3 & j = 15, \\ 0 & \text{otherwise.} \end{cases}$$

$n = 7$:

$$\mathrm{gr}^j\Big(\bigwedge^7 \mathfrak{g}\Big) = \begin{cases} \mathfrak{g}^1 \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^3 \wedge \mathfrak{g}^3 & j = 14, \\ \mathfrak{g}^1 \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^3 \wedge \mathfrak{g}^3 & j = 13, \\ \mathfrak{g}^1 \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^3 & j = 12, \\ 0 & \text{otherwise.} \end{cases}$$

$n = 6:$

$$\mathrm{gr}^j\Big(\bigwedge^6 \mathfrak{g}\Big) = \begin{cases} \mathfrak{g}^1 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^3 \wedge \mathfrak{g}^3 & j = 13, \\ \mathfrak{g}^1 \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^3 \wedge \mathfrak{g}^3 & j = 12, \\ \begin{array}{l} \mathfrak{g}^1 \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^3 \wedge \mathfrak{g}^3 \\ \oplus\, \mathfrak{g}^1 \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^3 \end{array} & j = 11, \\ \mathfrak{g}^1 \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^3 & j = 10, \\ \mathfrak{g}^1 \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^2 & j = 9, \\ 0 & \text{otherwise.} \end{cases}$$

$n = 5:$

$$\mathrm{gr}^j\Big(\bigwedge^5 \mathfrak{g}\Big) = \begin{cases} \mathfrak{g}^2 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^3 \wedge \mathfrak{g}^3 & j = 12, \\ \mathfrak{g}^1 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^3 \wedge \mathfrak{g}^3 & j = 11, \\ \begin{array}{l} \mathfrak{g}^1 \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^3 \wedge \mathfrak{g}^3 \\ \oplus\, \mathfrak{g}^1 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^3 \end{array} & j = 10, \\ \begin{array}{l} \mathfrak{g}^1 \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^3 \wedge \mathfrak{g}^3 \\ \oplus\, \mathfrak{g}^1 \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^3 \end{array} & j = 9, \\ \begin{array}{l} \mathfrak{g}^1 \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^3 \\ \oplus\, \mathfrak{g}^1 \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^2 \end{array} & j = 8, \\ \mathfrak{g}^1 \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^2 & j = 7, \\ 0 & \text{otherwise.} \end{cases}$$

$n = 4:$

$$\mathrm{gr}^j\Big(\bigwedge^4 \mathfrak{g}\Big) = \begin{cases} \mathfrak{g}^2 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^3 \wedge \mathfrak{g}^3 & j = 10, \\ \mathfrak{g}^1 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^3 \wedge \mathfrak{g}^3 & \\ \quad \oplus \mathfrak{g}^2 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^3 & j = 9, \\ \mathfrak{g}^1 \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^3 \wedge \mathfrak{g}^3 & \\ \quad \oplus \mathfrak{g}^1 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^3 & j = 8, \\ \mathfrak{g}^1 \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^3 & \\ \quad \oplus \mathfrak{g}^1 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^2 & j = 7, \\ \mathfrak{g}^1 \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^3 & \\ \quad \oplus \mathfrak{g}^1 \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^2 & j = 6, \\ \mathfrak{g}^1 \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^2 & j = 5, \\ 0 & \text{otherwise.} \end{cases}$$

$n = 3:$

$$\mathrm{gr}^j\Big(\bigwedge^3 \mathfrak{g}\Big) = \begin{cases} \mathfrak{g}^2 \wedge \mathfrak{g}^3 \wedge \mathfrak{g}^3 & j = 8, \\ \mathfrak{g}^1 \wedge \mathfrak{g}^3 \wedge \mathfrak{g}^3 \\ \quad \oplus \mathfrak{g}^2 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^3 & j = 7, \\ \mathfrak{g}^1 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^3 \\ \quad \oplus \mathfrak{g}^2 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^2 & j = 6, \\ \mathfrak{g}^1 \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^3 \\ \quad \oplus \mathfrak{g}^1 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^2 & j = 5, \\ \mathfrak{g}^1 \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^2 & j = 4, \\ \mathfrak{g}^1 \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^1 & j = 3, \\ 0 & \text{otherwise.} \end{cases}$$

$n = 2:$

$$\mathrm{gr}^j\Big(\bigwedge^2 \mathfrak{g}\Big) = \begin{cases} \mathfrak{g}^3 \wedge \mathfrak{g}^3 & j = 6, \\ \mathfrak{g}^2 \wedge \mathfrak{g}^3 & j = 5, \\ \mathfrak{g}^1 \wedge \mathfrak{g}^3 \\ \quad \oplus \mathfrak{g}^2 \wedge \mathfrak{g}^2 & j = 4, \\ \mathfrak{g}^1 \wedge \mathfrak{g}^2 & j = 3, \\ \mathfrak{g}^1 \wedge \mathfrak{g}^1 & j = 2, \\ 0 & \text{otherwise.} \end{cases}$$

$n = 1:$

$$\mathrm{gr}^j(\mathfrak{g}) = \begin{cases} \mathfrak{g}^3 & j = 3, \\ \mathfrak{g}^2 & j = 2, \\ \mathfrak{g}^1 & j = 1, \\ 0 & \text{otherwise.} \end{cases}$$

$n = 0:$

$$\mathrm{gr}^j(k) = \begin{cases} k & j = 0, \\ 0 & \text{otherwise.} \end{cases}$$

$n \backslash j$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	1															
1		3	3	2												
2			3	9	9	6	1									
3				1	9	15	19	9	3							
4						3	11	21	21	11	3					
5								3	9	19	15	9	1			
6										1	6	9	9	3		
7													2	3	3	
8																1

Table 2.1: Dimensions of $\mathrm{gr}^j\left(\bigwedge^n \mathfrak{g}\right)$. `tab:graded-dims-SL3`

$$\mathrm{Hom}_k\Big(\bigwedge^n \mathfrak{g}, k\Big) = \bigoplus_{s \in \mathbb{Z}} \mathrm{Hom}_k^s\Big(\bigwedge^n \mathfrak{g}, k\Big)$$

With $j = -s$, we get the same table for dimensions of the graded hom-spaces.

Note that when finding cohomology, we only need to consider $H^{s,t} = H^{s,n-s}$ for the non-zero entries of Table 2.1.

We repeatedly use that, if

$$d \overset{\mathsf{SNF}}{\sim} \mathrm{SNF}^{n,m}(a_1, \ldots, a_r, 0, \ldots, 0)$$

with $a_1, \ldots, a_r$ non-zero (in $\mathbb{F}_p$), then

$$\dim \ker(d) = m - r,$$
$$\dim \operatorname{im}(d) = r,$$
$$\dim \operatorname{coker}(d) = n - r.$$

gr^0 :

$$0 \longrightarrow k \longrightarrow 0$$

$$0 \longleftarrow \mathrm{Hom}_k^0(k,k) \longleftarrow 0$$

So $H^0 = H^{0,0}$ with $\dim H^{0,0} = 1$.

gr^1 :

$$0 \longrightarrow \mathfrak{g}^1 \longrightarrow 0$$

$$0 \longleftarrow \mathrm{Hom}_k^{-1}(\mathfrak{g},k) \longleftarrow 0$$

So $\dim H^{-1,2} = 3$ by Table 2.1.

gr^2 :

$$0 \longrightarrow \mathfrak{g}^1 \wedge \mathfrak{g}^1 \xrightarrow{\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}} \mathfrak{g}^2 \longrightarrow 0$$

$$\begin{aligned}
&\mathfrak{g}^1 \wedge \mathfrak{g}^1 \to \mathfrak{g}^2 \\
&\xi_1 \wedge \xi_6 \mapsto -[\xi_1, \xi_6] = \xi_2 \\
&\xi_1 \wedge \xi_7 \mapsto -[\xi_1, \xi_7] = -\xi_3 \\
&\xi_6 \wedge \xi_7 \mapsto -[\xi_6, \xi_7] = \xi_8.
\end{aligned}$$

$$0 \longleftarrow \mathrm{Hom}_k^{-2}\left(\bigwedge^2 \mathfrak{g}, k\right) \xleftarrow{\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}} \mathrm{Hom}_k^{-2}(\mathfrak{g}, k) \longleftarrow 0$$

$$d = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \overset{\mathsf{SNF}}{\sim} \mathrm{SNF}^{3\times 3}(1, -1, 1).$$

So

$$\begin{aligned}
\dim H^{-2,3} &= \dim \ker(d) = 0, \\
\dim H^{-2,4} &= \dim \operatorname{coker}(d) = 0.
\end{aligned}$$

gr^3 :

$$0 \longrightarrow \mathfrak{g}^1 \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^1 \xrightarrow{\begin{pmatrix} 0 & 0 & -1 & 0 & -1 & 0 & -1 & 0 & 0 \end{pmatrix}^{\top}} \mathfrak{g}^1 \wedge \mathfrak{g}^2 \xrightarrow[\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \end{pmatrix}]{} \mathfrak{g}^3 \longrightarrow 0$$

$$
0 \leftarrow \operatorname{Hom}_k^{-3}\left(\bigwedge^3 \mathfrak{g}, k\right) \xleftarrow{\begin{pmatrix} 0 & 0 & -1 & 0 & -1 & 0 & -1 & 0 & 0 \end{pmatrix}} \operatorname{Hom}_k^{-3}\left(\bigwedge^2 \mathfrak{g}, k\right) \xleftarrow[\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \end{pmatrix}^\top]{} \operatorname{Hom}_k^{-3}(\mathfrak{g}, k) \leftarrow 0
$$

$$
d_1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \\ 0 & 0 \\ 0 & -1 \\ 0 & 0 \\ 0 & -1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \overset{\mathsf{SNF}}{\sim} \mathrm{SNF}^{9\times 2}(1, -1),
$$

$$
d_2 = \begin{pmatrix} 0 & 0 & -1 & 0 & -1 & 0 & -1 & 0 & 0 \end{pmatrix} \overset{\mathsf{SNF}}{\sim} \mathrm{SNF}^{1\times 9}(-1).
$$

So

$$
\begin{aligned}
\dim H^{-3,4} &= \dim \ker(d_1) = 2 - 2 = 0, \\
\dim H^{-3,5} &= \dim \frac{\ker(d_2)}{\operatorname{im}(d_1)} = (9-1) - 2 = 6, \\
\dim H^{-3,6} &= \dim \operatorname{coker}(d_2) = 1 - 1 = 0.
\end{aligned}
$$

gr^4 :

$$
0 \longrightarrow \mathfrak{g}^1 \wedge \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^2 \xrightarrow{d^\top} \begin{matrix} \mathfrak{g}^1 \wedge \mathfrak{g}^3 \\ \oplus\, \mathfrak{g}^2 \wedge \mathfrak{g}^2 \end{matrix} \longrightarrow 0
$$

$$0 \longleftarrow \mathrm{Hom}_k^{-4}\big(\textstyle\bigwedge^3 \mathfrak{g}, k\big) \xleftarrow{d} \mathrm{Hom}_k^{-4}\big(\textstyle\bigwedge^2 \mathfrak{g}, k\big) \longleftarrow 0$$

$$d \overset{\mathsf{SNF}}{\sim} \mathrm{SNF}^{9\times 9}(1,1,1,-1,1,-1,0,0,0)$$

So

$$\dim H^{-4,6} = \dim\ker(d) = 9-6 = 3,$$
$$\dim H^{-4,7} = \dim\operatorname{coker}(d) = 9-6 = 3.$$

gr^5 :

$$0 \longrightarrow \mathfrak{g}^1 \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^2 \xrightarrow{d_2^\top} \begin{matrix} \mathfrak{g}^1 \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^3 \\ \oplus\, \mathfrak{g}^1 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^2 \end{matrix} \xrightarrow{d_1^\top} \mathfrak{g}^2 \wedge \mathfrak{g}^3 \longrightarrow 0$$

$$0 \leftarrow \mathrm{Hom}_k^{-5}\big(\textstyle\bigwedge^4 \mathfrak{g}, k\big) \overset{d_2}{\leftarrow} \mathrm{Hom}_k^{-5}\big(\textstyle\bigwedge^3 \mathfrak{g}, k\big) \overset{d_1}{\leftarrow} \mathrm{Hom}_k^{-5}\big(\textstyle\bigwedge^2 \mathfrak{g}, k\big) \leftarrow 0$$

$$d_1 \overset{\mathsf{SNF}}{\sim} \mathrm{SNF}^{15\times 6}(1,1,-1,-1,1,1),$$
$$d_2 \overset{\mathsf{SNF}}{\sim} \mathrm{SNF}^{3\times 15}(-1,1,1).$$

So

$$\dim H^{-5,7} = \dim\ker(d_1) = 6-6 = 0,$$
$$\dim H^{-5,8} = \dim\frac{\ker(d_2)}{\mathrm{im}(d_1)} = (15-3)-6 = 6,$$
$$\dim H^{-5,9} = \dim\operatorname{coker}(d_2) = 3-3 = 0.$$

gr^6 :

$$0 \longrightarrow \begin{array}{c} \mathfrak{g}^1 \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^3 \\ \oplus\, \mathfrak{g}^1 \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^2 \end{array} \xrightarrow{d_2^\top} \begin{array}{c} \mathfrak{g}^1 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^3 \\ \oplus\, \mathfrak{g}^2 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^2 \end{array} \xrightarrow{d_1^\top} \mathfrak{g}^3 \wedge \mathfrak{g}^3 \longrightarrow 0$$

$$0 \leftarrow \mathrm{Hom}_k^{-6}\big(\textstyle\bigwedge^4 \mathfrak{g}, k\big) \overset{d_2}{\leftarrow} \mathrm{Hom}_k^{-6}\big(\textstyle\bigwedge^3 \mathfrak{g}, k\big) \overset{d_1}{\leftarrow} \mathrm{Hom}_k^{-6}\big(\textstyle\bigwedge^2 \mathfrak{g}, k\big) \leftarrow 0$$

$$\begin{aligned} d_1 &\overset{\mathsf{SNF}}{\sim} \mathrm{SNF}^{19\times 1}(-1), \\ d_2 &\overset{\mathsf{SNF}}{\sim} \mathrm{SNF}^{11\times 19}(-1,1,-1,1,-1,-1,-1,1,1,1,-2). \end{aligned}$$

So

$$\begin{aligned} \dim H^{-6,8} &= \dim\ker(d_1) = 1-1 = 0, \\ \dim H^{-6,9} &= \dim \frac{\ker(d_2)}{\mathrm{im}(d_1)} = (19-11)-1 = 7, \\ \dim H^{-6,10} &= \dim\mathrm{coker}(d_2) = 11-11 = 0. \end{aligned}$$

gr^7 :

$$0 \to \mathfrak{g}^1 \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^2 \overset{d_2^\top}{\to} \begin{array}{c} \mathfrak{g}^1 \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^3 \\ \oplus\, \mathfrak{g}^1 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^2 \end{array} \overset{d_1^\top}{\to} \begin{array}{c} \mathfrak{g}^1 \wedge \mathfrak{g}^3 \wedge \mathfrak{g}^3 \\ \oplus\, \mathfrak{g}^2 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^3 \end{array} \to 0$$

$$0 \leftarrow \mathrm{Hom}_k^{-7}\big(\textstyle\bigwedge^5 \mathfrak{g}, k\big) \overset{d_2}{\leftarrow} \mathrm{Hom}_k^{-7}\big(\textstyle\bigwedge^4 \mathfrak{g}, k\big) \overset{d_1}{\leftarrow} \mathrm{Hom}_k^{-7}\big(\textstyle\bigwedge^3 \mathfrak{g}, k\big) \leftarrow 0$$

$$\begin{aligned} d_1 &\overset{\mathsf{SNF}}{\sim} \mathrm{SNF}^{21\times 9}(-1,-1,-1,1,1,1,1,-1,1), \\ d_2 &\overset{\mathsf{SNF}}{\sim} \mathrm{SNF}^{3\times 21}(1,1,-1). \end{aligned}$$

So

$$\dim H^{-7,10} = \dim \ker(d_1) = 9 - 9 = 0,$$
$$\dim H^{-7,11} = \dim \frac{\ker(d_2)}{\mathrm{im}(d_1)} = (21 - 3) - 9 = 9,$$
$$\dim H^{-7,12} = \dim \mathrm{coker}(d_2) = 3 - 3 = 0.$$

The following calculations are not necessary, since we can get the results using a version of Poincaré duality for Lie algebra cohomology, but we keep the sketch work to make it clear that nothing goes wrong.

gr^8 :

$$d_1 \overset{\mathsf{SNF}}{\sim} \mathrm{SNF}^{21\times 3}(1, -1, 1),$$
$$d_2 \overset{\mathsf{SNF}}{\sim} \mathrm{SNF}^{9\times 21}(-1, -1, -1, 1, 1, -1, -1, 1, -1).$$

So

$$\dim H^{-8,11} = \dim \ker(d_1) = 3 - 3 = 0,$$
$$\dim H^{-8,12} = \dim \frac{\ker(d_2)}{\mathrm{im}(d_1)} = (21 - 9) - 3 = 9,$$
$$\dim H^{-8,13} = \dim \mathrm{coker}(d_2) = 9 - 9 = 0.$$

gr^9 :

$$d_1 \overset{\mathsf{SNF}}{\sim} \mathrm{SNF}^{19\times 11}(-1, -1, 1, -1, 1, -1, -1, -1, -1, 1, -1),$$
$$d_2 \overset{\mathsf{SNF}}{\sim} \mathrm{SNF}^{1\times 19}(-1).$$

So

$$\dim H^{-9,13} = \dim \ker(d_1) = 11 - 11 = 0,$$
$$\dim H^{-9,14} = \dim \frac{\ker(d_2)}{\operatorname{im}(d_1)} = (19 - 1) - 11 = 7,$$
$$\dim H^{-9,15} = \dim \operatorname{coker}(d_2) = 1 - 1 = 0.$$

gr^{10} :

$$d_1 \overset{\mathsf{SNF}}{\sim} \mathrm{SNF}^{15\times 3}(1, 1, -1),$$
$$d_2 \overset{\mathsf{SNF}}{\sim} \mathrm{SNF}^{6\times 15}(-1, 1, 1, -1, 1, 1).$$

So

$$\dim H^{-10,14} = \dim \ker(d_1) = 3 - 3 = 0,$$
$$\dim H^{-10,15} = \dim \frac{\ker(d_2)}{\operatorname{im}(d_1)} = (15 - 6) - 3 = 6,$$
$$\dim H^{-10,16} = \dim \operatorname{coker}(d_2) = 6 - 6 = 0.$$

gr^{11} :

$$d \overset{\mathsf{SNF}}{\sim} \mathrm{SNF}^{9\times 9}(1, 1, -1, -1, -1, -1, 0, 0, 0).$$

So

$$\dim H^{-11,16} = \dim \ker(d) = 9 - 6 = 3,$$
$$\dim H^{-11,17} = \dim \operatorname{coker}(d) = 9 - 6 = 3.$$

gr^{12} :

$$d_1 \overset{\mathsf{SNF}}{\sim} \mathrm{SNF}^{9\times 1}(1),$$
$$d_2 \overset{\mathsf{SNF}}{\sim} \mathrm{SNF}^{2\times 9}(1,-1).$$

So

$$\dim H^{-12,17} = \dim \ker(d_1) = 1-1 = 0,$$
$$\dim H^{-12,18} = \dim \frac{\ker(d_2)}{\mathrm{im}(d_1)} = (9-2)-1 = 6,$$
$$\dim H^{-12,19} = \dim \mathrm{coker}(d_2) = 2-2 = 0.$$

gr^{13} :

$$d \overset{\mathsf{SNF}}{\sim} \mathrm{SNF}^{3\times 3}(-1,1,-1).$$

So

$$\dim H^{-13,19} = \dim \ker(d) = 3-3 = 0,$$
$$\dim H^{-13,20} = \dim \mathrm{coker}(d) = 3-3 = 0.$$

gr^{14} :

$$0 \longrightarrow \mathfrak{g}^1 \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^3 \wedge \mathfrak{g}^3 \longrightarrow 0$$

$$0 \longleftarrow \mathrm{Hom}_k^{-14}\big(\textstyle\bigwedge^7 \mathfrak{g}, k\big) \longleftarrow 0$$

So $\dim H^{-14,21} = 3$ by Table 2.1.

gr^{15} :

$$0 \longrightarrow \mathfrak{g}^1 \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^3 \wedge \mathfrak{g}^3 \longrightarrow 0$$

$$0 \longleftarrow \operatorname{Hom}_k^{-15}\big(\textstyle\bigwedge^8 \mathfrak{g}, k\big) \longleftarrow 0$$

So $H^8 = H^{-15,23}$ with $\dim H^{-15,23} = 1$ by Table 2.1.

Altogether:

$$\begin{aligned}
H^0 &= H^{0,0},\\
H^1 &= H^{-1,2},\\
H^2 &= H^{-3,5} \oplus H^{-4,6},\\
H^3 &= H^{-4,7} \oplus H^{-5,8} \oplus H^{-6,9},\\
H^4 &= H^{-7,11} \oplus H^{-8,12},\\
H^5 &= H^{-9,14} \oplus H^{-10,15} \oplus H^{-11,16},\\
H^6 &= H^{-11,17} \oplus H^{-12,18},\\
H^7 &= H^{-14,21},\\
H^8 &= H^{-15,23}
\end{aligned}$$

and we have the following table: Thus

$t \backslash s$	0	-1	-2	-3	-4	-5	-6	-7	-8	-9	-10	-11	-12	-13	-14	-15
0	1															
1																
2		3														
3																
4																
5				6												
6					3											
7					3											
8						6										
9							7									
10																
11								9								
12									9							
13																
14										7						
15											6					
16												3				
17												3				
18													6			
19																
20																
21															3	
22																
23																1

Table 2.2: Dimensions of $E_1^{s,t} = H^{s,t} = \mathrm{gr}^s H^{s+t}(\mathfrak{g}, k)$. `tab:graded-coh-dims-SI`

$$\dim H^i = \begin{cases} 1 & i = 0, \\ 3 & i = 1, \\ 9 & i = 2, \\ 16 & i = 3, \\ 18 & i = 4, \\ 16 & i = 5, \\ 9 & i = 6, \\ 3 & i = 7, \\ 1 & i = 8. \end{cases}$$

2.5 $I \subseteq \mathrm{GL}_3(\mathbb{Z}_p)$

sec:Iwa-GL3

$$g_1 = \begin{pmatrix} 1 & & \\ & 1 & \\ p & & 1 \end{pmatrix}, \quad g_2 = \begin{pmatrix} 1 & & \\ p & 1 & \\ & & 1 \end{pmatrix}, \quad g_3 = \begin{pmatrix} 1 & & \\ & 1 & \\ & p & 1 \end{pmatrix},$$
$$g_4 = \begin{pmatrix} \exp(p) & & \\ & \exp(-p) & \\ & & 1 \end{pmatrix}, \quad g_5 = \begin{pmatrix} 1 & & \\ & \exp(p) & \\ & & \exp(-p) \end{pmatrix},$$
$$g_6 = \begin{pmatrix} \exp(p) & & \\ & \exp(p) & \\ & & \exp(p) \end{pmatrix},$$
$$g_7 = \begin{pmatrix} 1 & & \\ & 1 & 1 \\ & & 1 \end{pmatrix}, \quad g_8 = \begin{pmatrix} 1 & 1 & \\ & 1 & \\ & & 1 \end{pmatrix}, \quad g_9 = \begin{pmatrix} 1 & & 1 \\ & 1 & \\ & & 1 \end{pmatrix}. \tag{2.9}$$

{eq:gis-GL3}

$t \backslash s$	0	−1	−2	−3	−4	−5	−6	−7	−8	−9	−10	−11	−12	−13	−14	−15	−16	−17	−18
0	1																		
1																			
2		3																	
3																			
4				1															
5				6															
6					6														
7					3														
8						6													
9							13												
10								3											
11								12											
12									15										
13										7									
14										7									
15											15								
16												12							
17												3							
18													13						
19														6					
20															3				
21															6				
22																6			
23																1			
24																			
25																		3	
26																			
27																			1

Table 2.3: Dimensions of $E_1^{s,t} = H^{s,t} = \mathrm{gr}^s H^{s+t}(\mathfrak{g}, k)$. tab:graded-coh-dims-GI

Chapter 3

List-Decodable Mean Estimation and Clustering

cha:robstat

3.1 Introduction

sec:robstat-intro

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List of Symbols

Cohomology of Unipotent groups

$H^\bullet_{\mathrm{cts}}(H,\,\cdot\,)$	the continuous group cohomology of a topological group G 4
$H^\bullet_{\mathrm{dsc}}(G,\,\cdot\,)$	the discrete group cohomology of a topological group H 4
$H^{s,t}$	$= \mathrm{gr}^s H^{s+t}$ for some cohomology H 4
$H^{s,t}(\mathfrak{g}, \mathbb{F}_p)$	$= H^{s+t}\big(\mathrm{gr}^s \mathrm{Hom}_{\mathbb{F}_p}(\bigwedge^\bullet \mathfrak{g}, \mathbb{F}_p)\big)$ 11
$\mathcal{N}$ / $\mathcal{N}^+$	the unipotent radical of $\mathcal{B}$ / $\mathcal{B}^+$ 3
$\omega\colon G \setminus \{1\} \to (0, \infty)$	a p-valuation on G 5
p	a prime, $p \geq h - 1$, where h is the Coxeter number of $\mathcal{G}$ 4
Φ	$= \Phi(\mathcal{G}, \mathcal{T})$, the root system of $\mathcal{G}$ with respect to $\mathcal{T}$ 3
Φ^+ / Φ^-	the positive/negative roots in Φ with respect to Δ 3
$\Phi^\vee$	the dual root system of Φ 4
$\pi\colon\ \mathrm{gr}\, G \to \mathrm{gr}\, G$	the direct sum of the maps $gG_{\nu+} \mapsto g^p G_{(\nu+1)+}$ 8
$\mathrm{rank}(G, \omega)$	$:= \mathrm{rank}_{\mathbb{F}_p[P]} \mathrm{gr}\, G$ the rank of the pair (G, ω) 8
$\mathcal{T}$	a (fixed) split maximal torus of $\mathcal{G}$ 3
$V_{\mathbb{F}_p}(\lambda)$	$= V_{\mathbb{Z}}(\lambda) \otimes_{\mathbb{Z}} \mathbb{F}_p$ 4
$V_{\mathbb{Z}}(\lambda)$	the Weyl module for $\mathcal{G}_{\mathbb{Z}}$ over $\mathbb{Z}$ with highest weight λ 4

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