高等微积分

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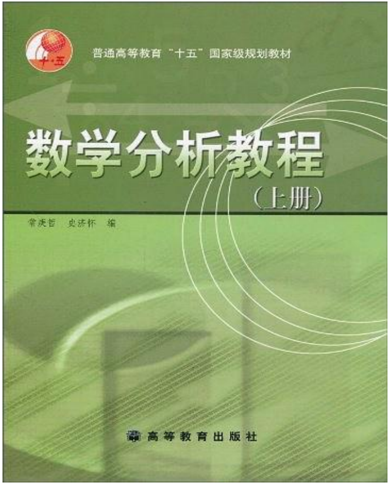
第六章: 求导的逆运算–不定积分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小结: 不定积分

- 将定义在区间上的函数 f 的原函数的一般表达式称为 f 的不定积分, 记作 $\int f(x)\,\mathrm{d}x$. 这是一个以 x 为自变量的函数.
- 若 $f \in \mathscr{C}[a,b]$, 则 $\int f(x)\mathrm{d}x = \int_a^x f(t)\mathrm{d}t + C$.

不定积分与导数、微分的关系

- 若 $\int f(x)\,\mathrm{d}x = F(x) + C$, 则 $F'(x) = f(x)$,

$$\left(\int f(x)\,\mathrm{d}x\right)' = F'(x) = f(x),$$

$$\mathrm{d}F(x) = f(x)\,\mathrm{d}x,\quad \mathrm{d}\left(\int f(x)\,\mathrm{d}x\right) = f(x)\,\mathrm{d}x,$$

$$\int f(x)\,\mathrm{d}x = \int F'(x)\mathrm{d}x = \int \mathrm{d}F(x) = F(x) + C.$$

- (线性性) $\forall \alpha, \beta \in \mathbb{R}$, 我们有

$$\int (\alpha f(x) + \beta g(x))\,\mathrm{d}x = \alpha \int f(x)\,\mathrm{d}x + \beta \int g(x)\,\mathrm{d}x.$$

基本的不定积分公式

- $\int \mathrm{d}x = x + C.$
- $\int x^\alpha \,\mathrm{d}x = \frac{x^{\alpha+1}}{\alpha+1} + C \ (\alpha \neq -1),$
  $\int \frac{1}{x} \,\mathrm{d}x = \ln|x| + C.$

- $\int a^x \mathrm{d}x = \frac{a^x}{\ln a} + C \ (a > 0,\ a \neq 1)$,
  $\int e^x \mathrm{d}x = e^x + C$.
- $\int \sin x \, \mathrm{d}x = -\cos x + C$,
  $\int \cos x \, \mathrm{d}x = \sin x + C$.

- $\int \operatorname{sh} x \, \mathrm{d}x = \operatorname{ch} x + C$, $\int \operatorname{ch} x \, \mathrm{d}x = \operatorname{sh} x + C$.
- $\int \sec^2 x \, \mathrm{d}x = \tan x + C$.
- $\int \csc^2 x \, \mathrm{d}x = -\cot x + C$.
- $\int \frac{\mathrm{d}x}{\sqrt{1-x^2}} = \arcsin x + C$.

- $\int \frac{\mathrm{d}x}{1+x^2} = \arctan x + C.$
- $\int \frac{\mathrm{d}x}{\sqrt{x^2+a^2}} = \ln \left|x + \sqrt{x^2 + a^2}\right| + C.$
- $\int \frac{\mathrm{d}x}{\sqrt{x^2-a^2}} = \ln \left|x + \sqrt{x^2 - a^2}\right| + C.$

回顾：求不定积分的基本方法

- 第一换元积分法 (凑微分): 若 $F'(y) = f(y)$, 则 $\int f(u(x))u'(x)\mathrm{d}x = \int f(u(x))\mathrm{d}u(x) = F(u(x)) + C$.

回顾：求不定积分的基本方法

- 第二换元积分法：如果 $f(x(t))x'(t) = F'(t)$, 则

$$\int f(x)\,\mathrm{d}x \overset{x=x(t)}{=} \int f(x(t))x'(t)\,\mathrm{d}t$$
$$= F(t) + C \overset{t=t(x)}{=} F(t(x)) + C.$$

- $\int \sec x \, \mathrm{d}x = \ln|\sec x + \tan x| + C.$
- $\int \csc x \, \mathrm{d}x = \ln|\csc x - \cot x| + C.$

回顾：求不定积分的基本方法

下面假设 $a>0$.

- 若含 $\sqrt{a^2-x^2}$, 作变换 $x=a\sin t$ $(|t|\leqslant\frac{\pi}{2})$.
- 若含 $\sqrt{x^2+a^2}$, 作变换 $x=a\tan t$ $(|t|<\frac{\pi}{2})$.
- 若含 $\sqrt{x^2-a^2}$, 要分情况讨论: 当 $x>a$ 时, 定义 $x=a\sec t$ $(0\leqslant t<\frac{\pi}{2})$; 而当 $x<-a$ 时, 定义 $x=-u$ 或 $x=-a\sec t$ $(0\leqslant t<\frac{\pi}{2})$.

§6.2. 方法三：分部积分法

设函数 u, v 均为一阶连续可导. 由于

$$\mathrm{d}(uv) = u\,\mathrm{d}v + v\,\mathrm{d}u,$$

则 $u\,\mathrm{d}v = \mathrm{d}(uv) - v\,\mathrm{d}u$, 于是

$$\int u\,\mathrm{d}v = uv - \int v\,\mathrm{d}u.$$

例 26. 计算 $\int x \cos x \, \mathrm{d}x$.

解: $\int x \cos x \, \mathrm{d}x = \int x \, \mathrm{d}(\sin x)$

$= x \sin x - \int \sin x \, \mathrm{d}x = x \sin x + \cos x + C$.

例 27. 计算 $\int \ln x \,\mathrm{d}x$.

解: $\int \ln x \,\mathrm{d}x = x \ln x - \int x \mathrm{d}(\ln x)$

$= x \ln x - \int x \cdot \frac{1}{x} \,\mathrm{d}x = x \ln x - x + C$.

例 28. 计算 $\int \arcsin x \, \mathrm{d}x$.

解: $\int \arcsin x \, \mathrm{d}x = x \arcsin x - \int x \, \mathrm{d}(\arcsin x)$

$= x \arcsin x - \int \frac{x}{\sqrt{1-x^2}} \, \mathrm{d}x = x \arcsin x - \int \frac{\mathrm{d}(x^2)}{2\sqrt{1-x^2}}$

$= x \arcsin x + \int \frac{\mathrm{d}(1-x^2)}{2\sqrt{1-x^2}} = x \arcsin x + \int \mathrm{d}\sqrt{1-x^2}$

$= x \arcsin x + \sqrt{1-x^2} + C.$

例 29. 计算 $\int xe^x \mathrm{d}x$.

解: $\int xe^x \mathrm{d}x = \int x \,\mathrm{d}(e^x) = xe^x - \int e^x \mathrm{d}x = xe^x - e^x + C$.

例 30. 计算 $\int 3x^2 \arctan x \,\mathrm{d}x$.

解: $\int 3x^2 \arctan x \,\mathrm{d}x = \int \arctan x \,\mathrm{d}(x^3)$

$= x^3 \arctan x - \int x^3 \,\mathrm{d}(\arctan x)$

$= x^3 \arctan x - \int \frac{x^3}{1+x^2} \,\mathrm{d}x$

$= x^3 \arctan x - \int \left(x - \frac{x}{1+x^2}\right) \mathrm{d}x$

$= x^3 \arctan x - \frac{1}{2}x^2 + \frac{1}{2} \int \frac{\mathrm{d}(1+x^2)}{1+x^2}$

$= x^3 \arctan x - \frac{1}{2}x^2 + \frac{1}{2} \ln(1+x^2) + C$.

例 31. 计算 $\int x\ln^2 x\,\mathrm{d}x$.

解: $\int x\ln^2 x\,\mathrm{d}x = \int \ln^2 x\,\mathrm{d}(\frac{x^2}{2})$

$$= \frac{x^2}{2}\ln^2 x - \int \frac{x^2}{2}\,\mathrm{d}(\ln^2 x)$$

$$= \frac{1}{2}x^2\ln^2 x - \int \frac{x^2}{2}\cdot\frac{2\ln x}{x}\,\mathrm{d}x$$

$$= \frac{1}{2}x^2\ln^2 x - \int x\ln x\,\mathrm{d}x$$

$$= \frac{1}{2}x^2\ln^2 x - \int \ln x\,\mathrm{d}(\frac{x^2}{2})$$

$$= \frac{1}{2}x^2\ln^2 x - \frac{1}{2}x^2\ln x + \int \frac{x^2}{2}\cdot\frac{1}{x}\,\mathrm{d}x$$

$$= \frac{1}{2}x^2\ln^2 x - \frac{1}{2}x^2\ln x + \frac{x^2}{4} + C.$$

例 32. 计算 $\int \sqrt{a^2 - x^2}\,\mathrm{d}x\ (a > 0)$.

解: $\int \sqrt{a^2 - x^2}\,\mathrm{d}x = x\sqrt{a^2 - x^2} - \int x\,\mathrm{d}(\sqrt{a^2 - x^2})$

$= x\sqrt{a^2 - x^2} + \int \frac{x^2}{\sqrt{a^2 - x^2}}\,\mathrm{d}x$

$= x\sqrt{a^2 - x^2} + \int \left(\frac{a^2}{\sqrt{a^2 - x^2}} - \sqrt{a^2 - x^2}\right)\mathrm{d}x$

$= x\sqrt{a^2 - x^2} + a^2 \int \frac{\mathrm{d}(\frac{x}{a})}{\sqrt{1 - (\frac{x}{a})^2}} - \int \sqrt{a^2 - x^2}\,\mathrm{d}x$

$= x\sqrt{a^2 - x^2} + a^2 \arcsin\frac{x}{a} - \int \sqrt{a^2 - x^2}\,\mathrm{d}x.$

由此我们立刻可得

$$\int \sqrt{a^2 - x^2}\,\mathrm{d}x = \frac{1}{2}\left(x\sqrt{a^2 - x^2} + a^2 \arcsin\frac{x}{a}\right) + C.$$

例 33. 计算 $\int \sqrt{x^2+a^2}\,\mathrm{d}x \ (a>0)$.

解: $\int \sqrt{x^2+a^2}\,\mathrm{d}x = x\sqrt{x^2+a^2} - \int \frac{x^2\,\mathrm{d}x}{\sqrt{x^2+a^2}}$

$= x\sqrt{x^2+a^2} - \int \left(\sqrt{x^2+a^2} - \frac{a^2}{\sqrt{x^2+a^2}}\right)\mathrm{d}x$

$= x\sqrt{x^2+a^2} + a^2 \ln\left|x+\sqrt{x^2+a^2}\right| - \int \sqrt{x^2+a^2}\,\mathrm{d}x,$

由此立刻可得

$$\int \sqrt{x^2+a^2}\,\mathrm{d}x = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\ln\left|x+\sqrt{x^2+a^2}\right| + C.$$

例 34. 设 $m \in \mathbb{N}^*$. 求 $I_m = \int \frac{\mathrm{d}x}{(x^2+a^2)^m}$ $(a > 0)$.

解: $I_m = \frac{x}{(x^2+a^2)^m} - \int x \,\mathrm{d}\frac{1}{(x^2+a^2)^m}$

$$= \frac{x}{(x^2+a^2)^m} + 2m \int \frac{x^2 \,\mathrm{d}x}{(x^2+a^2)^{m+1}}$$

$$= \frac{x}{(x^2+a^2)^m} + 2m \int \frac{\mathrm{d}x}{(x^2+a^2)^m} - 2ma^2 \int \frac{\mathrm{d}x}{(x^2+a^2)^{m+1}}$$

$$= \frac{x}{(x^2+a^2)^m} + 2mI_m - 2ma^2 I_{m+1}.$$

于是 $I_{m+1} = \frac{x}{2a^2m(x^2+a^2)^m} + \frac{2m-1}{2a^2m} I_m$. 注意到

$$I_1 = \int \frac{\mathrm{d}x}{x^2+a^2} = \frac{1}{a} \arctan \frac{x}{a} + C,$$

由此可得 I_m 的一般表达式.

例 35. 计算 $\int e^{ax}\cos bx\,\mathrm{d}x,\ \int e^{ax}\sin bx\,\mathrm{d}x\ (ab\neq 0)$.

解: 利用分部积分可得

$$\begin{aligned}\int e^{ax}\cos bx\,\mathrm{d}x &= \int e^{ax}\,\mathrm{d}\Big(\frac{1}{b}\sin bx\Big)\\ &= \frac{1}{b}e^{ax}\sin bx-\frac{a}{b}\int e^{ax}\sin bx\,\mathrm{d}x,\\ \int e^{ax}\sin bx\,\mathrm{d}x &= \int e^{ax}\,\mathrm{d}\Big(-\frac{1}{b}\cos bx\Big)\\ &= -\frac{1}{b}e^{ax}\cos bx+\frac{a}{b}\int e^{ax}\cos bx\,\mathrm{d}x,\end{aligned}$$

由此立刻可得

$$\int e^{ax}\cos bx\,\mathrm{d}x = \frac{e^{ax}}{a^2+b^2}(a\cos bx+b\sin bx)+C,$$

同学们辛苦了!