求 $\int_0^{+\infty} (x^{2024}+x^{-2024}) \frac{\ln x\,dx}{1+x^2}$

$x = e^t$

$$\int_0^{+\infty} (x^{2024}+x^{-2024}) \frac{\ln x}{1+x^2}\,dx$$

$$= \int_{-\infty}^{+\infty} (e^{2024t}+e^{-2024t}) \cdot \frac{t}{1+e^{2t}} \cdot e^t\,dt$$

$$= \int_{-\infty}^{+\infty} (e^{2024t}+e^{-2024t}) \frac{t}{e^t+e^{-t}}\,dt$$

被积函数奇函数

$= 0$

3. 求证 $\lim\limits_{n\to\infty} \frac{\ln n}{\ln(1^{2023}+2^{2023}+\cdots+n^{2023})} = \frac{1}{2024}$

$$\lim_{n\to\infty} \frac{\ln(1^{2023}+2^{2023}+\cdots+n^{2023})}{\ln n}$$

$$= \lim_{n\to+\infty} \frac{\ln(1^{2023}+\cdots+n^{2023})-2024\ln n}{\ln n} + 2024$$

$$= \lim_{n\to+\infty} \frac{\ln\left(\frac{1}{n}\left(\left(\frac{1}{n}\right)^{2023}+\cdots+\left(\frac{n}{n}\right)^{2023}\right)\right)}{\ln n} + 2024$$

分子 $\to \ln \int_0^1 x^{2023}\,dx = \ln 2024^{-1}$

分母 $\to +\infty$

$= 2024$

5. $f(x)$可导 $f(0)=0$ $f(1)=1$ → $\int_0^1 |f'(x)+f(x)|\,dx \le e^{-1}$

5. $f(x)$可导, $f(0)=0$, $f(1)=1$ $\Rightarrow$ $\int_0^1 |f'(x)+f(x)|\,dx \leq e^{-1}$

$$\left(e^{-x}\cdot f(x)\right)' = e^{-x}f'(x) + e^{-x}f(x)$$

$$\int_0^1 |f'(x)+f(x)|\,dx \geq \int_0^1 |e^{-x}(f'(x)+f(x))|\,dx$$

$$= \int_0^1 \left|\left(e^{-x}f(x)\right)'\right|\,dx$$

$$\geq \int_0^1 \left(e^{-x}f(x)\right)'\,dx = e^{-x}f(x)\Big|_0^1$$

$$= e^{-1}$$

(1) $f(x)$连续. 试证 $\int_0^{\pi} x f(\sin x)\,dx = \pi\int_0^{\frac{\pi}{2}} f(\cos x)\,dx$.

第(2)：(1)应用.

$$\int_0^{\pi} x f(\sin x)\,dx = \int_0^{\frac{\pi}{2}} x f(\sin x)\,dx + \int_{\frac{\pi}{2}}^{\pi} x f(\sin x)\,dx$$

$$= \int_0^{\frac{\pi}{2}} x f(\sin x)\,dx + \int_{\frac{\pi}{2}}^{0} (\pi - x) f(\sin(\pi-x))\,d(\pi-x)$$

$$= \int_0^{\frac{\pi}{2}} x f(\sin x)\,dx + \int_0^{\frac{\pi}{2}} (\pi - x) f(\sin x)\,dx$$

$$= \pi \int_0^{\frac{\pi}{2}} f(\sin x)\,dx$$

$$= \pi \int_{\frac{\pi}{2}}^{0} f\left(\sin\left(\frac{\pi}{2}-x\right)\right)\,d\left(\frac{\pi}{2}-x\right)$$

$$= \pi \int_0^{\frac{\pi}{2}} f(\cos x)\,dx$$

$f(x) := \cos x - \int_0^x e^{-t} f(x-t)\,dt$

求 $f(x)$.

$f(0) = 1$

$$f'(x) = \sin x - e^{-x} - \int_0^x e^{-t} f'(x-t)\,dt$$

$$= \sin x - e^{x} - \int_0^x e^{-t}\, d f(x-t)$$

$$= \sin x - e^{x} - e^{-y} f(x-y)\Big|_0^x + \int_0^x f(x-t)\, d e^{-t}$$

$$= \sin x - e^{-x} - e^{-x} - f(x) - \int_0^x e^{-t} f(x-t)\,dt$$

$$= \sin x - 2e^{-x} + \cos x.$$

$$\Rightarrow f(x) = -\cos x + 2e^{-x} + \sin x + C$$

$$f(0) = -1 + 2 + 0 + C = 1 \Rightarrow C = 0$$

$$\Rightarrow f(x) = -\cos x + 2e^{-x} + \sin x$$

9. $I_n := \int_0^{\frac{\pi}{2}} \frac{\sin^2(nt)}{\sin t}\, dt$.

求: $\lim\limits_{n\to\infty} \frac{I_n}{\ln n}$

$$I_n = \int_0^{\frac{\pi}{2}} \frac{\sin^2(nt)}{\sin t}\, dt$$

$\forall$ 固定 n, $\frac{\sin^2(nt)}{\sin nt}$ 在 0 附近 $= O(t)$

$\Rightarrow$ I_n 可积

注意到: $\sin t + \sin 3t + \cdots + \sin\big((2n-1)t\big) = \frac{\sin^2 nt}{\sin t}$

事实上: $\sin t\,\big(\sin t + \sin 3t + \cdots + \sin((2n-1)t)\big)$

$$= \frac{1}{2}\Big[\big(\cos(0) - \cos 2t\big) + \big(\cos(2t) - \cos(4t)\big) + \cdots + \big(\cos(2(n-1)t) - \cos(2nt)\big)\Big]$$

$$= \frac{1}{2}\big(\cos(0) - \cos(2nt)\big) = \sin^2(nt)$$

$$\Rightarrow I_n = \int_0^{\frac{\pi}{2}} \left(\sum_{k=1}^{n} \sin((2k-1)t)\right) dt$$

$$= \sum_{k=1}^{n} \frac{1}{2k-1}$$

$$\lim_{n\to+\infty} \frac{I_n}{\ln n} = \lim_{n\to+\infty} \frac{\frac{1}{2n+1}}{\ln(n+1) - \ln n} = \frac{1}{2}$$

$f(x)\downarrow,\quad 0<a\le b \xrightarrow{?} b\int_0^a f \le a\int_0^b f$

$$\frac{1}{a}\int_0^a f(x)\,dx-\frac{1}{b}\int_0^b f(x)\,dx$$

$$=\int_0^1 f(at)\,dt-\int_0^1 f(bt)\,dt$$

$$=\int_0^1 \big(f(at)-f(bt)\big)\,dt\ge 0$$

8. $f(x)$ 二阶可导，$f(0)=f(1)=0 \xrightarrow{?} \left|\int_0^1 f(x)\,dx\right| \le \frac{1}{12}\max|f''(x)|$

$$\int_0^1 f(x)\,dx=-\int_0^1 x f'(x)\,dx=-\int_0^1\left(x-\frac{1}{2}\right)f'(x)\,dx$$

$$=-\frac{1}{2}\int_0^1 f'(x)\,d(x^2-x)=-\frac{1}{2}(x^2-x)f'(x)+\frac{1}{2}\int_0^1(x^2-x)f''(x)\,dx$$

$$\Rightarrow 2\left|\int_0^1 f(x)\,dx\right|\le\left|\int_0^1(x^2-x)f''(x)\,dx\right|$$

$$\le\max|f''(x)|\cdot\int_0^1(x-x^2)\,dx$$

$$=\max|f''(x)|\cdot\underbrace{\left(\frac{1}{2}x^2-\frac{1}{3}x^3\right)\Big|_0^1=\frac{1}{6}}$$

$$\Rightarrow\left|\int_0^1 f(x)\,dx\right|\le\frac{1}{12}\max|f''(x)|$$