考试课程　　离散数学　　（A 卷）　　2023 年 1 月 4 日

Each problem is of twelve marks except the last one with four marks.

1. Let n be a positive integer. Simplify the sums

   (a) $\displaystyle\sum_{k=1}^{n} k 2^{k-1}$,

   (b) $\displaystyle\sum_{k=0}^{n} \frac{(-1)^k}{k+1}\binom{n}{k}$.

2. Given any five integers, show that three of them can be taken so that their sum is a multiple of 3.

3. Everyday Bob buys either a candy for \$1 or a sundae for \$2. There are two different flavors of candy, but only one kind of sundae. If he has n dollars, in how many ways can he spend the money?

4. Let a, b, c, d be four such integers that $a - c$ divides $ab + cd$. Show that $a - c$ also divides $ad + bc$.

5. Show that a simple graph on n nodes with more than $\binom{n-1}{2}$ edges is connected.

6. A 5×5 square grid is a planar map of a graph G on 36 nodes.

   (a) Has G an Eulerian walk, a Hamilton cycle, and a perfect matching? Are G and its dual both bipartite?

   (b) Has G a Hamilton path from the most left-up node to the most down-right node and why?

7. Given any six points in the plane such that no three of them are on a line, show that there are at least three convex quadrilaterals which four vertices are among those given points.

8. There are eleven girls in a class and everyday only three of them take a walk and walk together. Is it possible to make a plan so that each girl walks with each other exactly once within some consecutive days, and why? What if instead only six of them walk everyday?

9. What is $\gcd(F_{99}, F_{2023})$, where F_n is the Fibonacci number?