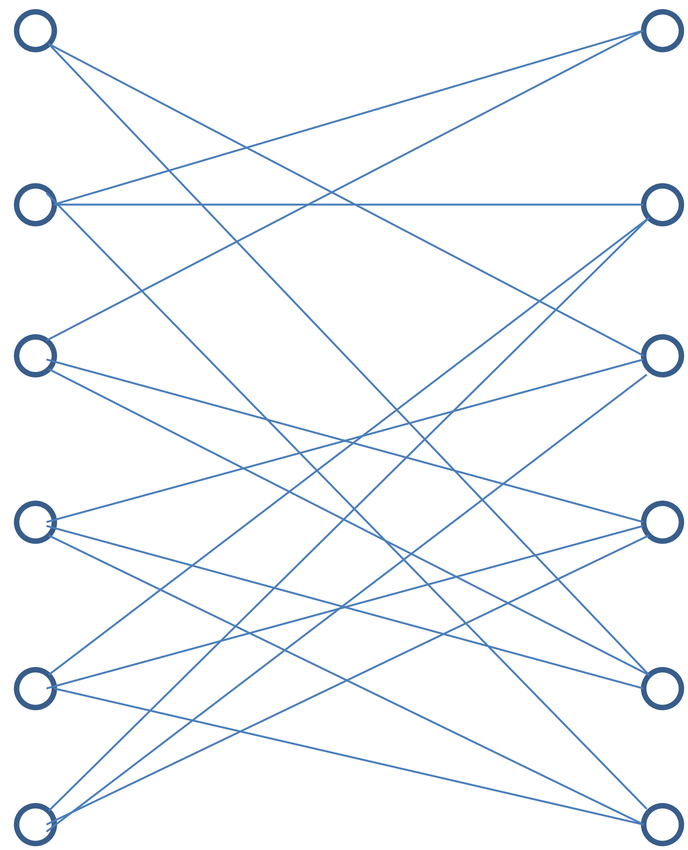
# Discrete Mathematics

## Lecture 10

## *Matching in Graphs*

# 10.1-2 A Dancing Problem

A graph is **bipartite** if its nodes can be partitioned into two classes, say *A* and *B*, such that every edge connects a node in *A* to a node in *B*.



# 10.1-2 A Dancing Problem

Is a star bipartite?

Yes, it is.

Is a path bipartite?

Yes, it is.

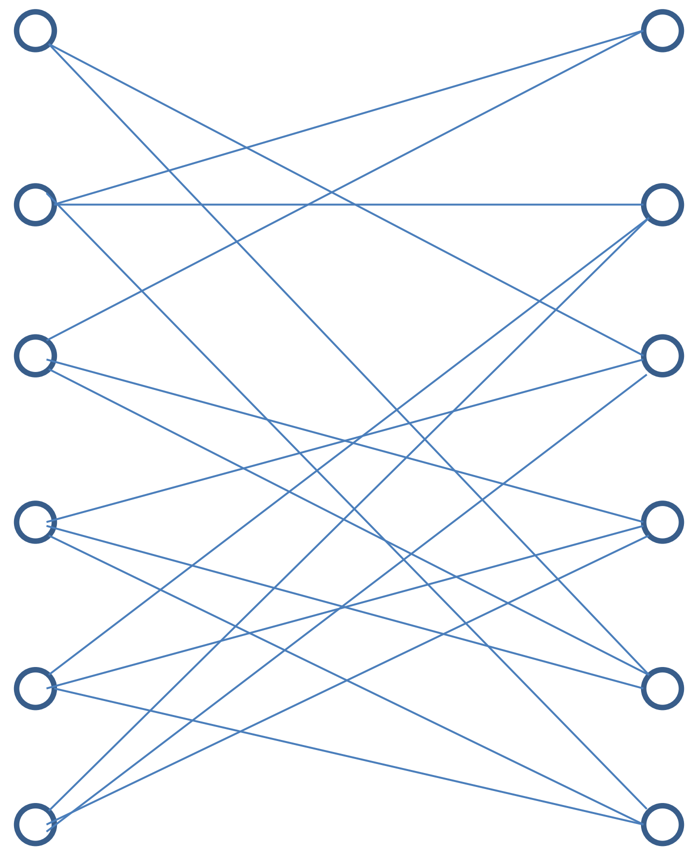
Is a tree bipartite?

Yes, it is, and why?

**How to test whether a graph is bipartite?**

# 10.1-2 A Dancing Problem

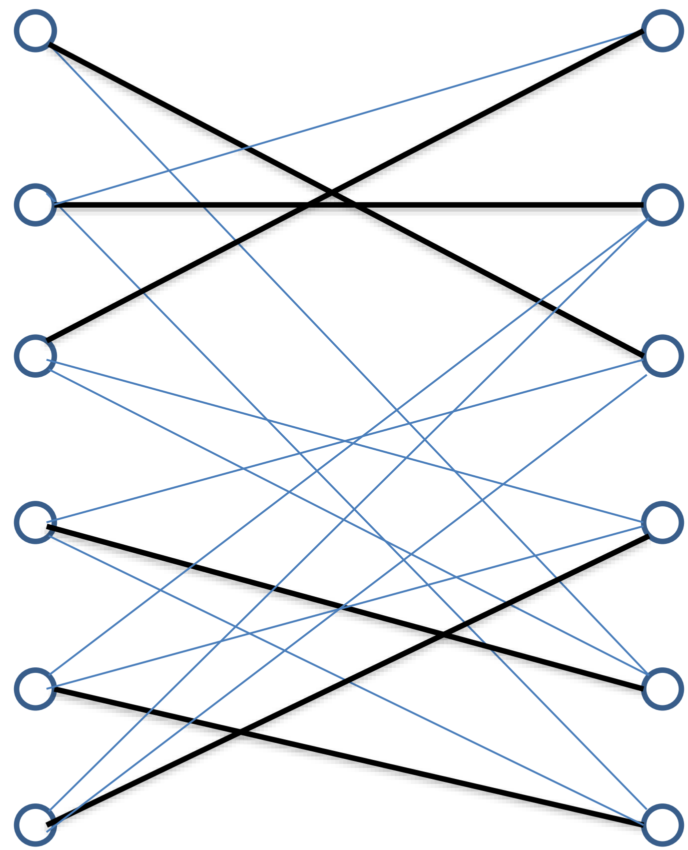
A graph is **bipartite** if its nodes can be partitioned into two classes, say *A* and *B*, such that every edge connects a node in *A* to a node in *B*.



# 10.1-2 A Dancing Problem

A **matching** is a set of edges that have no endpoint in common.

A **matching** is **perfect** if it covers all nodes.



## Theorem 10.1.1 (König's Theorem)

**Every bipartite $d$-regular graph with $d>0$ contains a perfect matching.**

**Def. Given a graph $G=(V, E)$ and $v \in V$, the neighborhood of $v$ is denoted by $N_G(v)$. For $S \subset V$, define the neighborhood of $S$ by**

$$N_G(S) := \bigcup_{v \in S} N_G(v).$$

# 10.3 The Main Theorem

**Theorem 10.3.1 (The Marriage Theorem)**

**A bipartite graph has a perfect matching iff $|A|=|B|$ and for any subset $S \subset A$,**

$$|S| \leq |N(S)|.$$

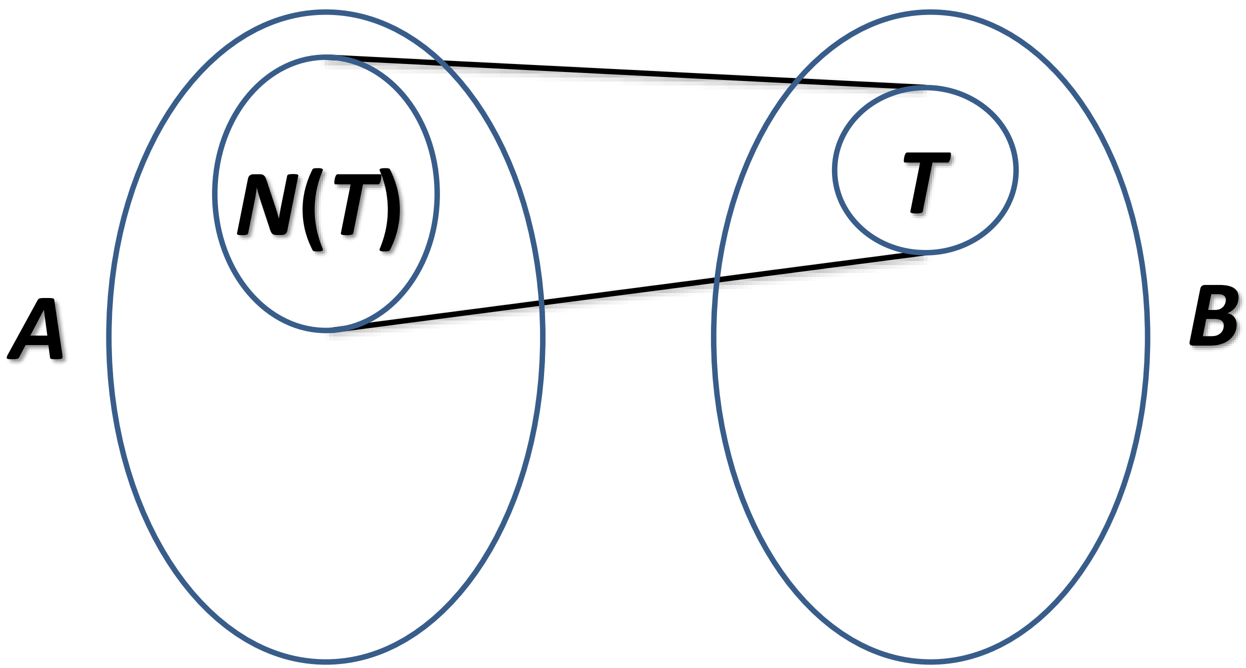
**Note. If we interchange "$A$" and "$B$", then perfect matchings remain perfect matchings. But what happens to the condition stated in the theorem?**

**It remains valid. For $T \subset B$, clearly**

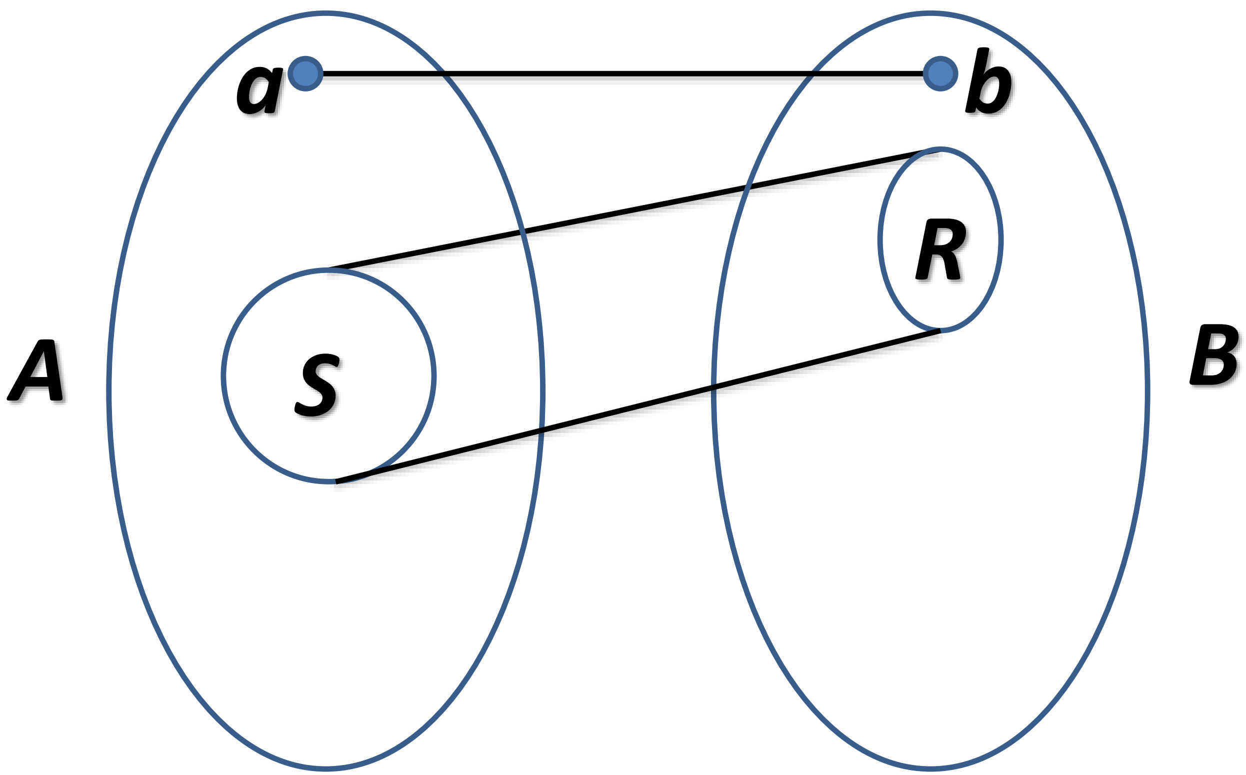
**$N(A \setminus N(T)) \subset B \setminus T$,**

**$|A| - |N(T)| = |A \setminus N(T)| \le |B \setminus T| = |B| - |T|$,**

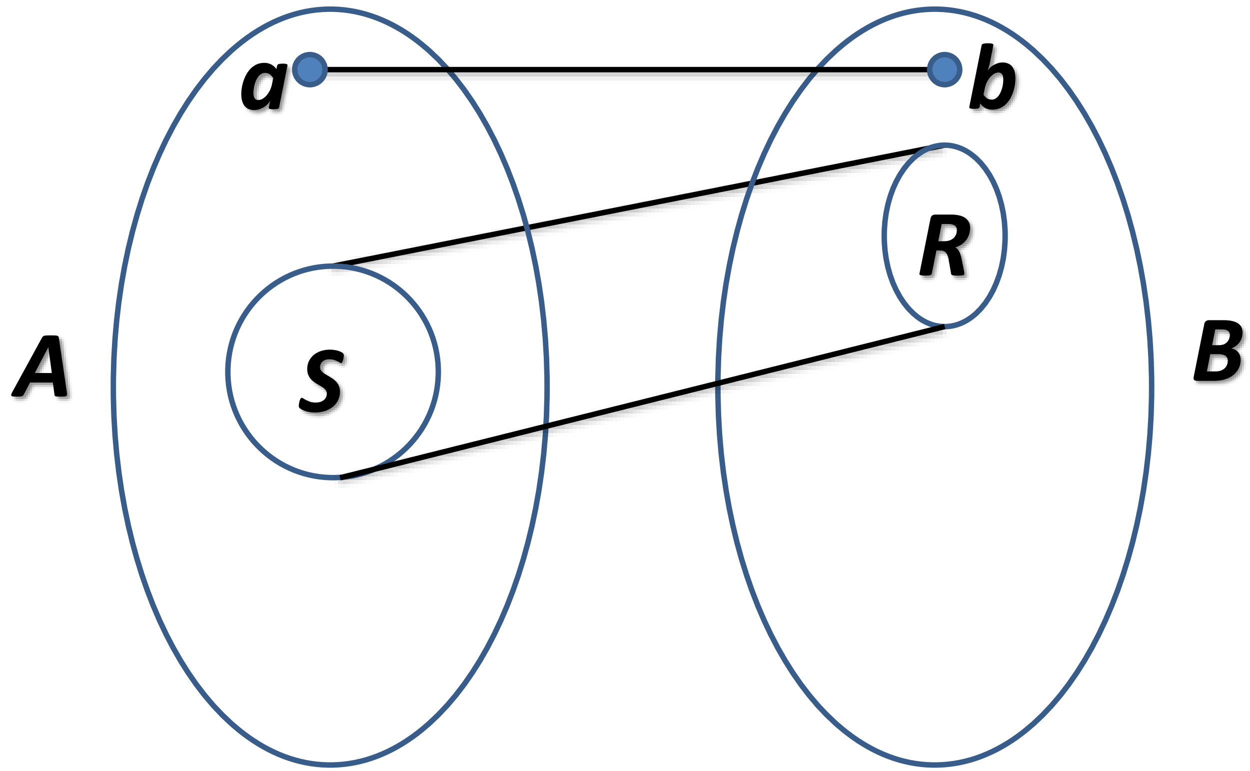
**$\Rightarrow |T| \le |N(T)|$.**



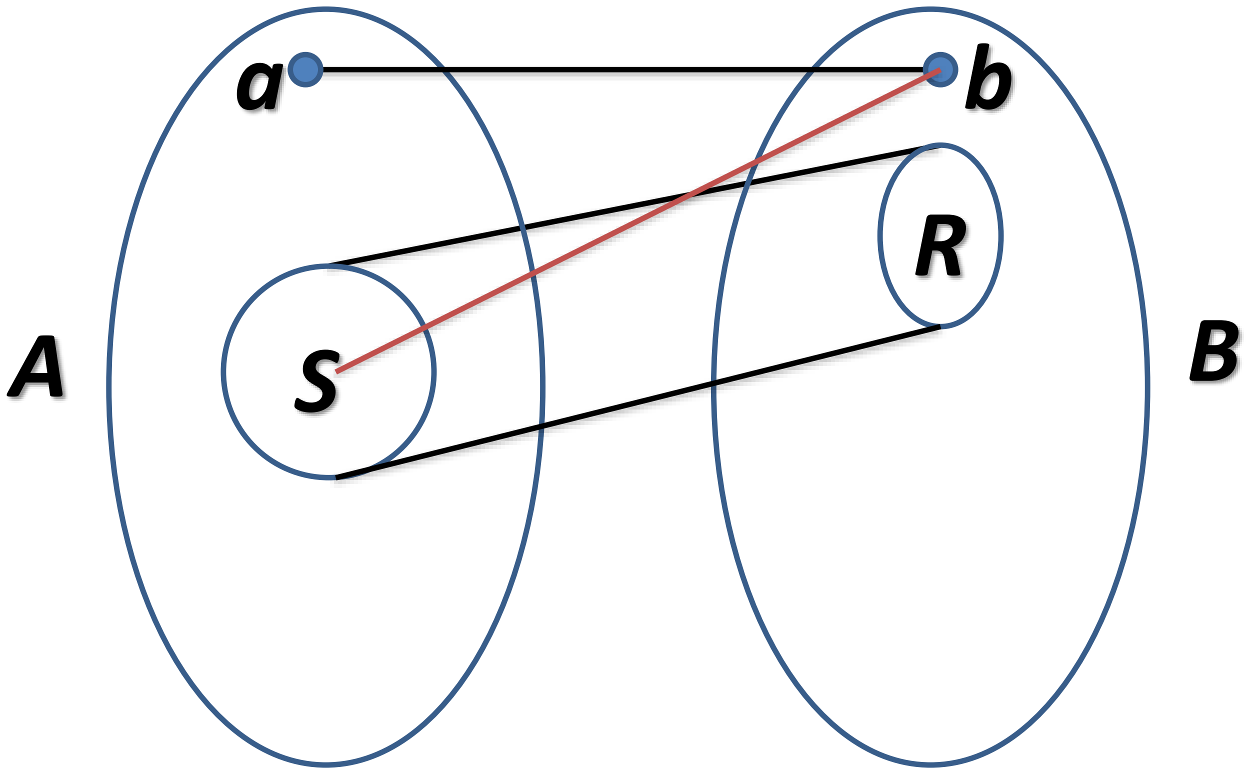
**Proof of The Marriage Theorem.**
**Use induction on the number of nodes of the bipartite graph *G*.**



**Let $H := G - a - b$. If for all $S \subset A \backslash \{a\}$ and $R = N_H(S)$, $|S| \leq |R|$, then by induction, $H$ has a perfect matching and so has $G$.**



**If $\exists S \subset A \setminus \{a\}$ so that $|S| > |R|$, then let $T = R \cup \{b\}$. We have $T = N(S)$ & $|S| = |T|$.**



If $\exists S \subset A \backslash \{a\}$ so that $|S|>|R|$, then let $T=R \cup \{b\}$. We have $T=N(S)$ & $|S|=|T|$.

Let $G_1:=G[S \cup T]$ the subgraph induced by $S \cup T$, and similarly $G_2:=G[(A \backslash S) \cup (B \backslash T)]$.

**What is the subgraph $G_1$?**

**Well**, $G_1=(V_1,E_1)$ where $V_1=S \cup T$ and $E_1=\{e \in E \mid e \cap S \neq \emptyset, e \cap T \neq \emptyset\}$.

So $G_1$ is bipartite with $|S|=|T|$.

Has it a perfect matching?

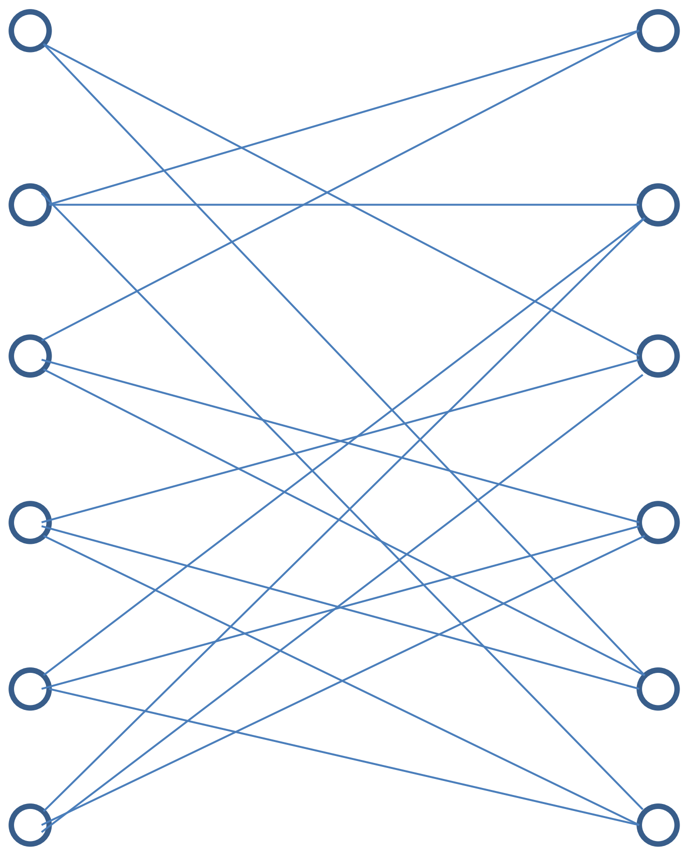
If $\exists S \subset A \setminus \{a\}$ so that $|S|>|R|$, then let $T=R \cup \{b\}$. We have $T=N(S)$ & $|S|=|T|$.

Let $G_1:=G[S \cup T]$ the subgraph induced by $S \cup T$, and similarly $G_2:=G[(A \setminus S) \cup (B \setminus T)]$.

Then $G_1$ has a perfect matching by induction, since for all $X \subset S \subset A$, we have $|X| \leq |N(X)| = |N_{G1}(X)|$.

Similarly, $G_2$ has a perfect matching since for all $Y \subset B \setminus T \subset B$, we also have $|Y| \leq |N(Y)| = |N_{G2}(Y)|$.

# 10.4 How to Find a Perfect Matching



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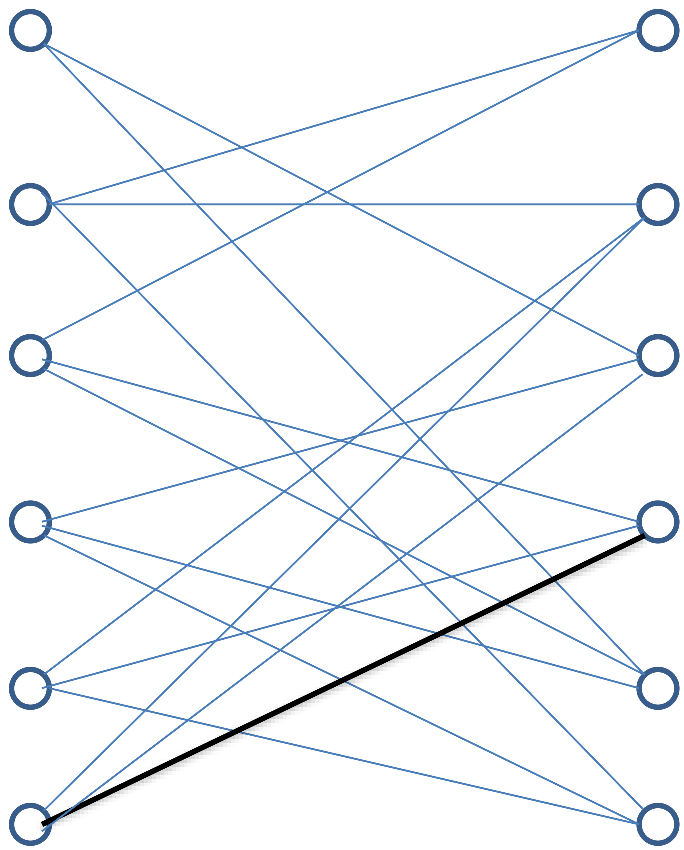

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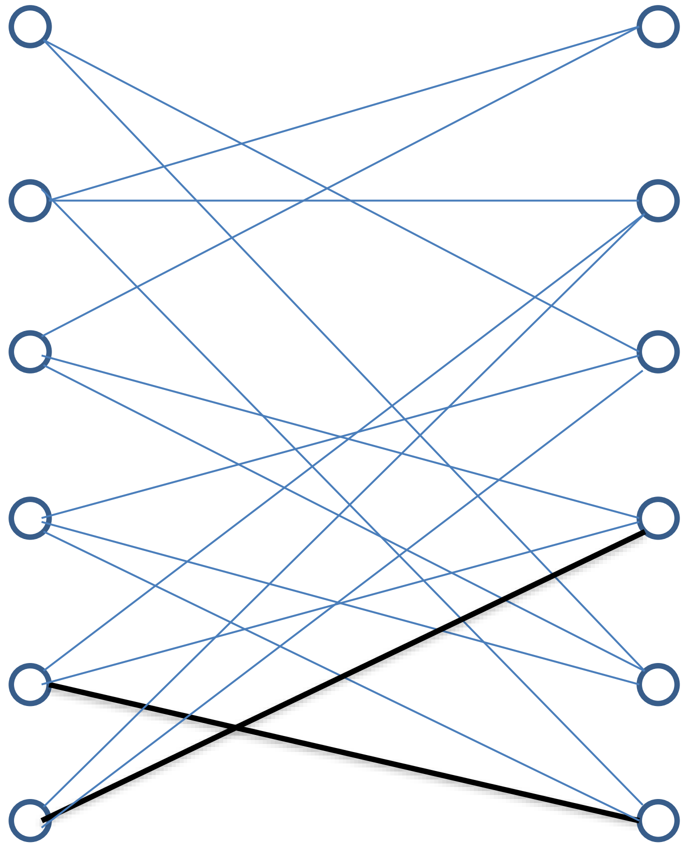

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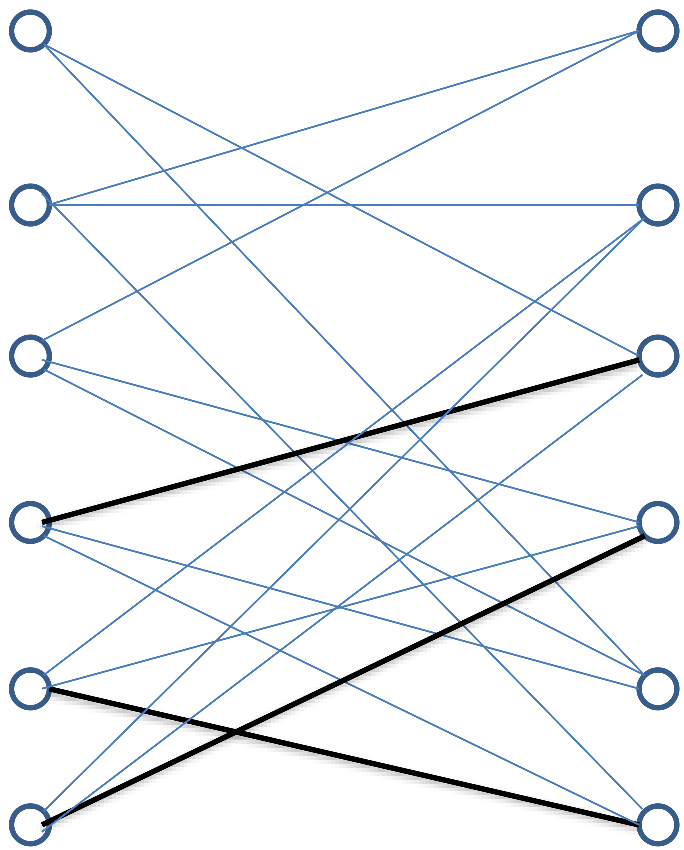

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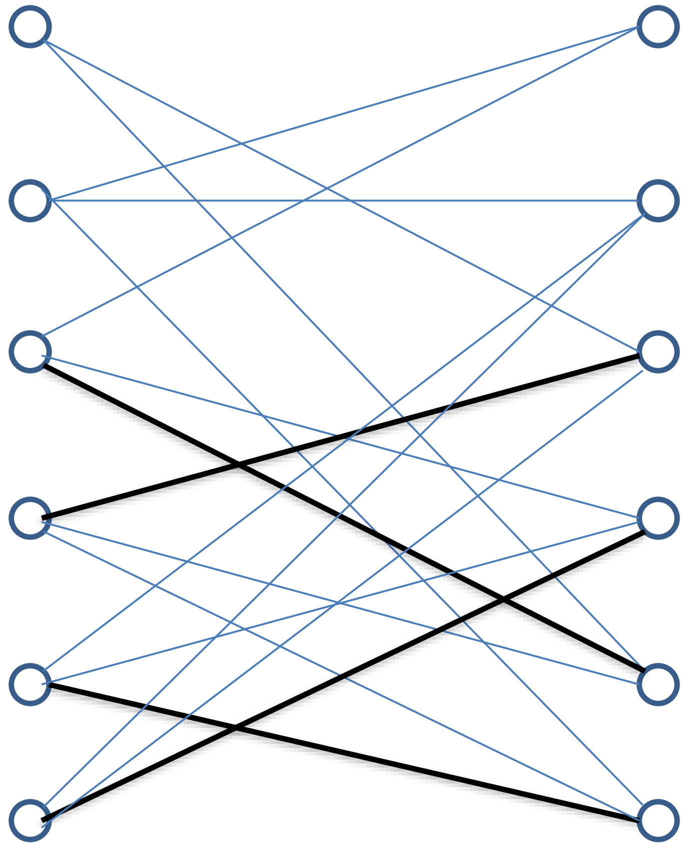

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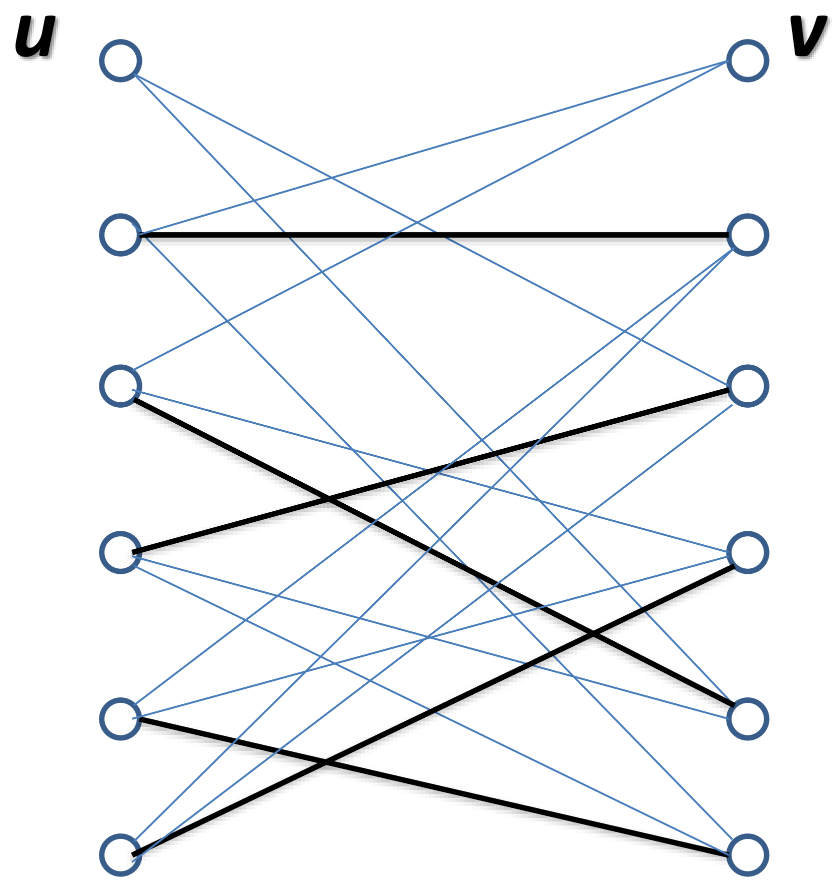

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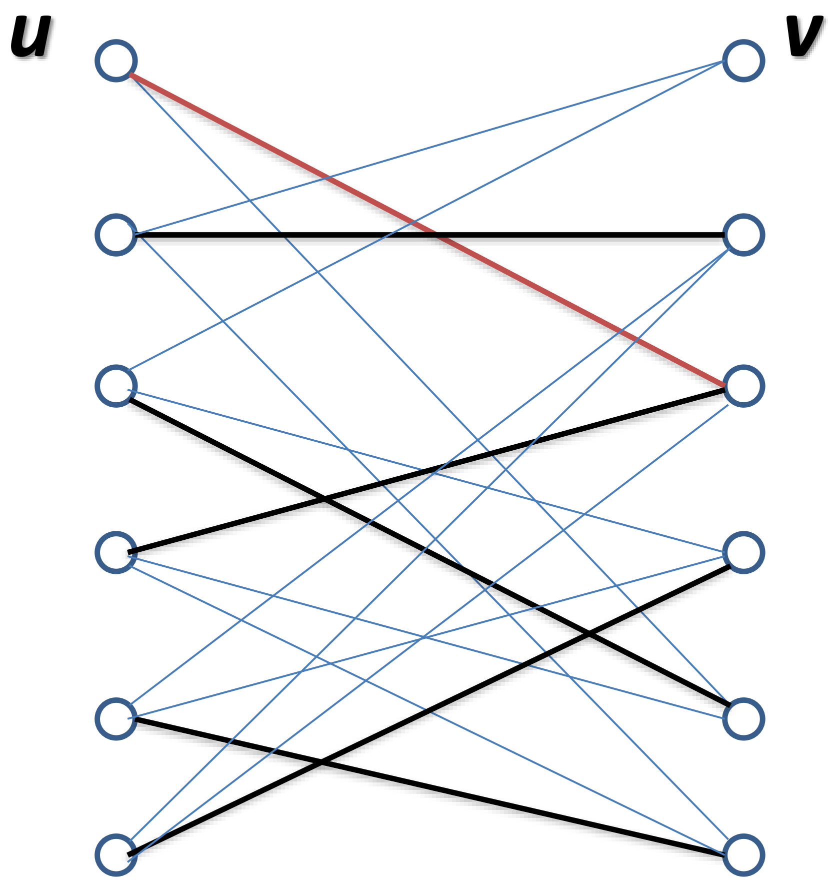

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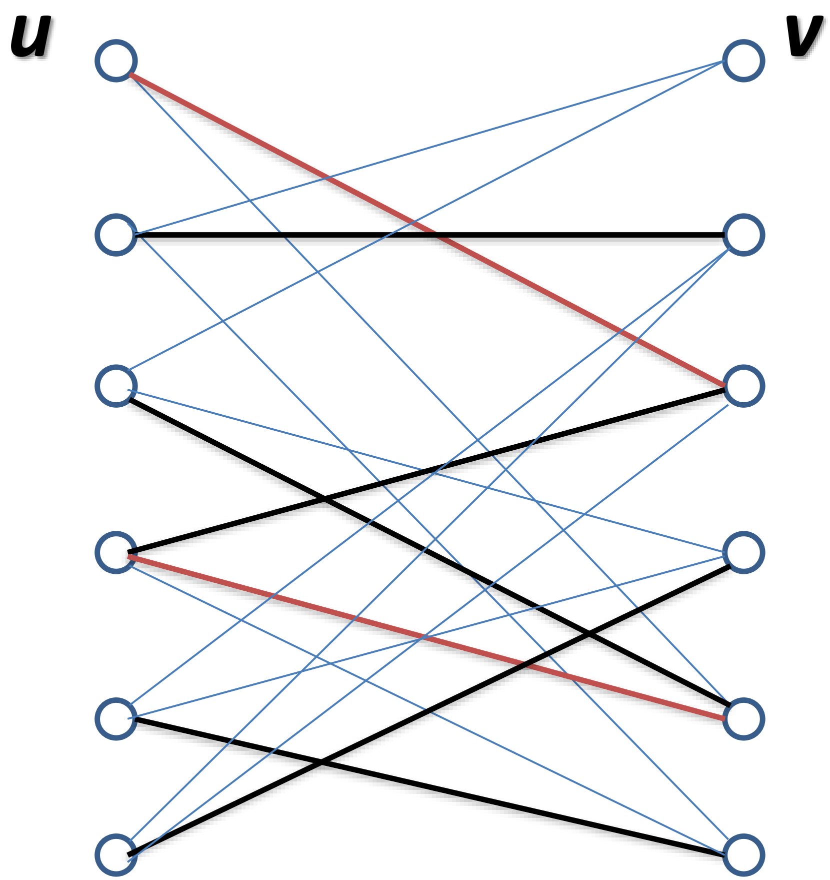

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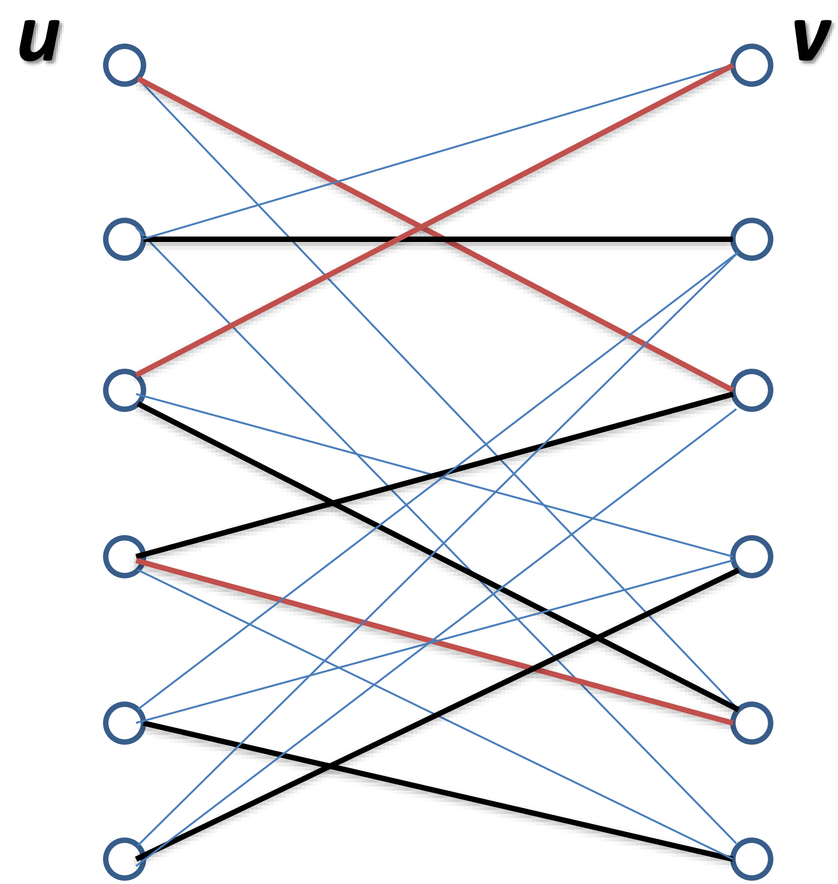
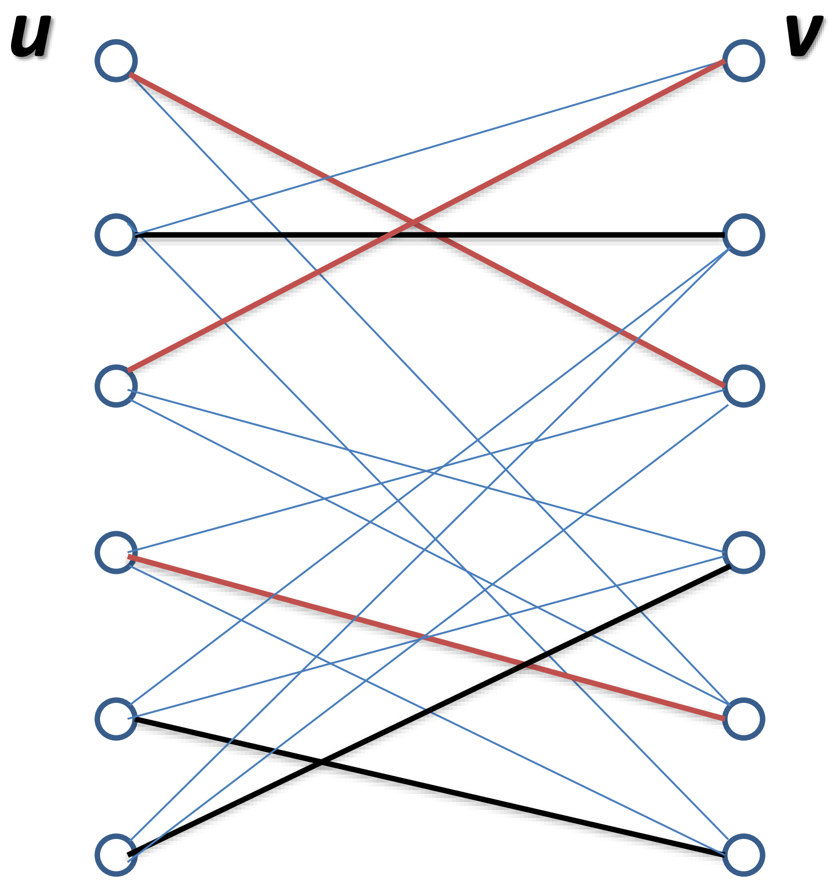



**An augmenting path**

# 10.4 How to Find a Perfect Matching



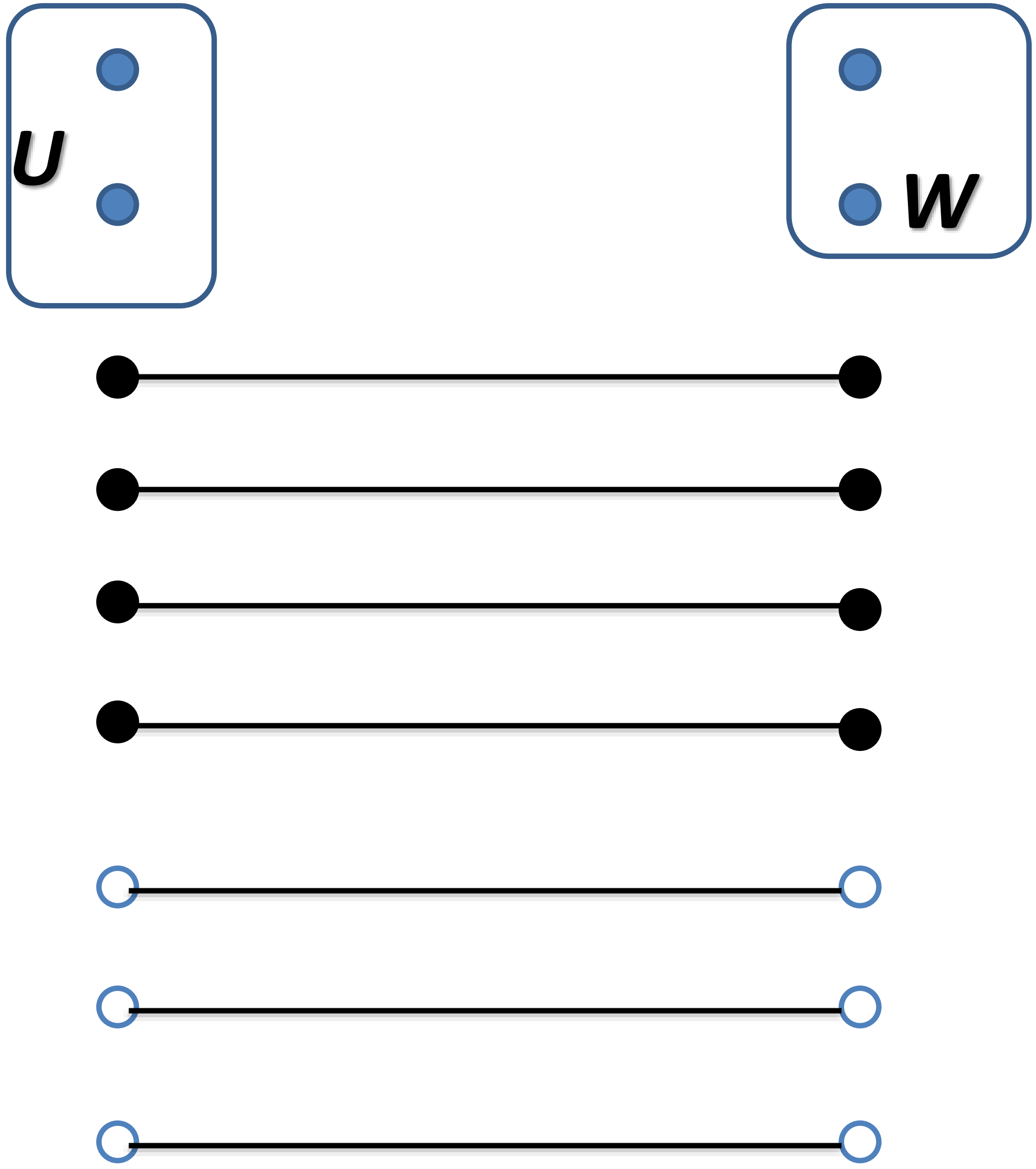
**See FIGURE 10.7 in Page 175.**

**Let *U* be the set of unmatched nodes in *A*.**

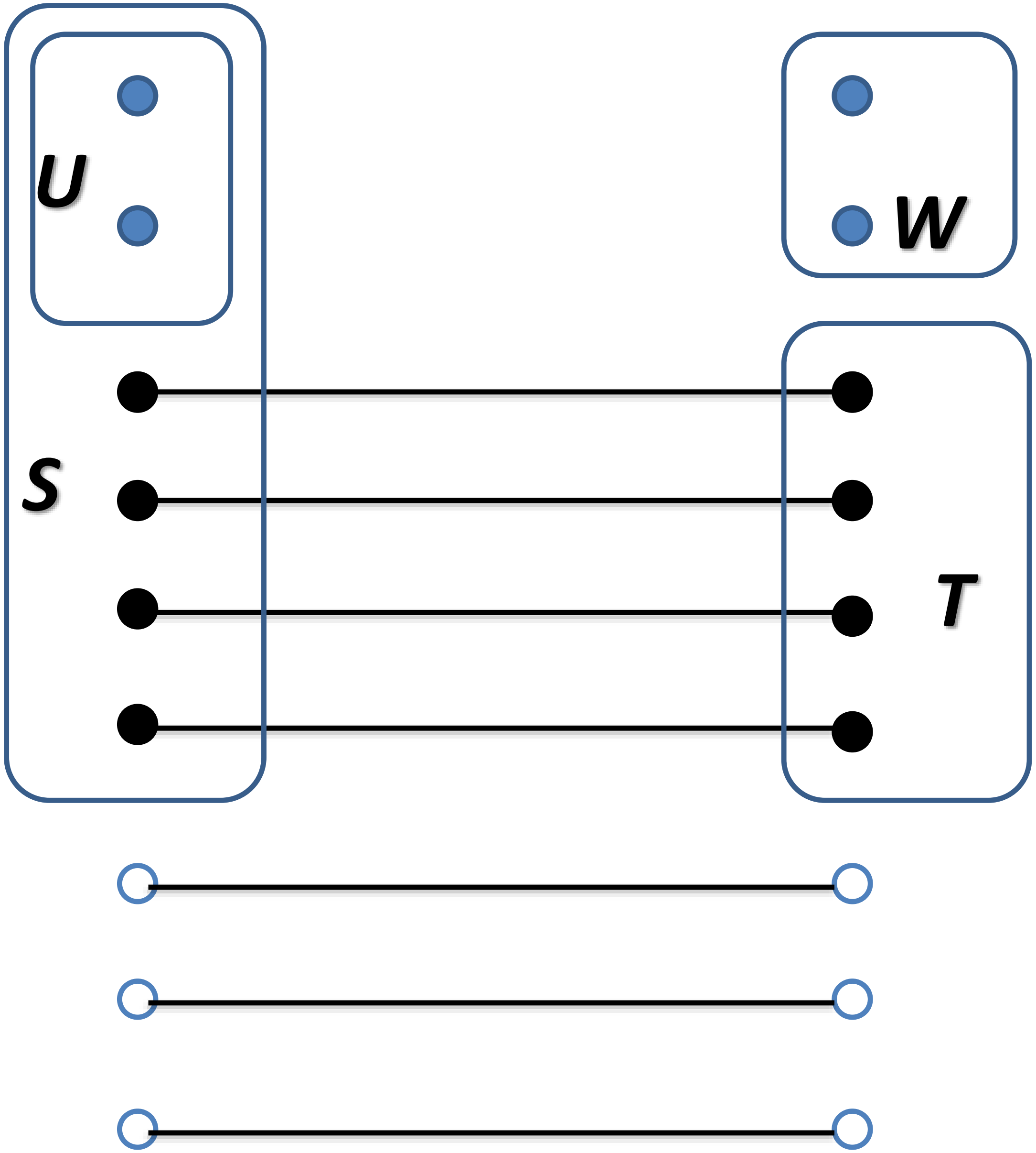
**Let *W* be the set of unmatched nodes in *B*.**

**Any augmenting path must have an odd number of edges, hence it must connect a node in *U* to a node in *W*.**

**A path is almost augmenting if it starts at a node in *U*, ends at a node in *A*, and every second edge of it belongs to *M*.**


U
W

- **Start with $S$:=$U$.**
- **At any stage, the set $S$ will consist of nodes we already know are reachable by some almost augmenting path.**
- **Denote by $T$ the set of nodes in $B$ that are matched with nodes in $S$.**
- **$|S|=|T|+|U|$.**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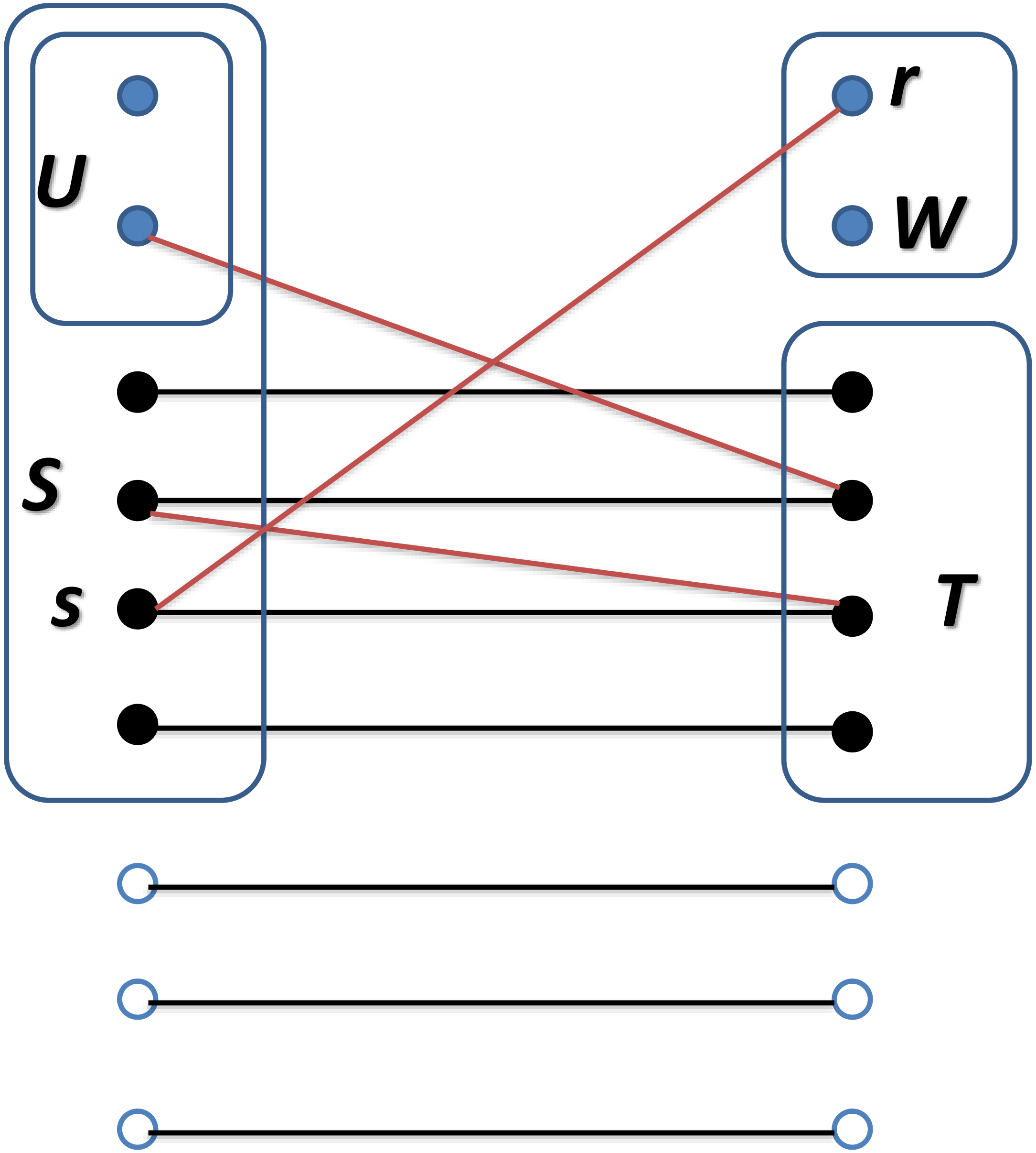
- **Let $sr \in E \backslash M$, $s \in S$, $r \in B \backslash T$.**
- **Let $Q$ be an almost augmenting path starting at some node $u \in U$ and ending at $s$.**

➢ **If $r$ is unmatched ($r \in W$), then by appending the edge $sr$ to $Q$ we get an augmenting path. So we can increase the size of $M$.**



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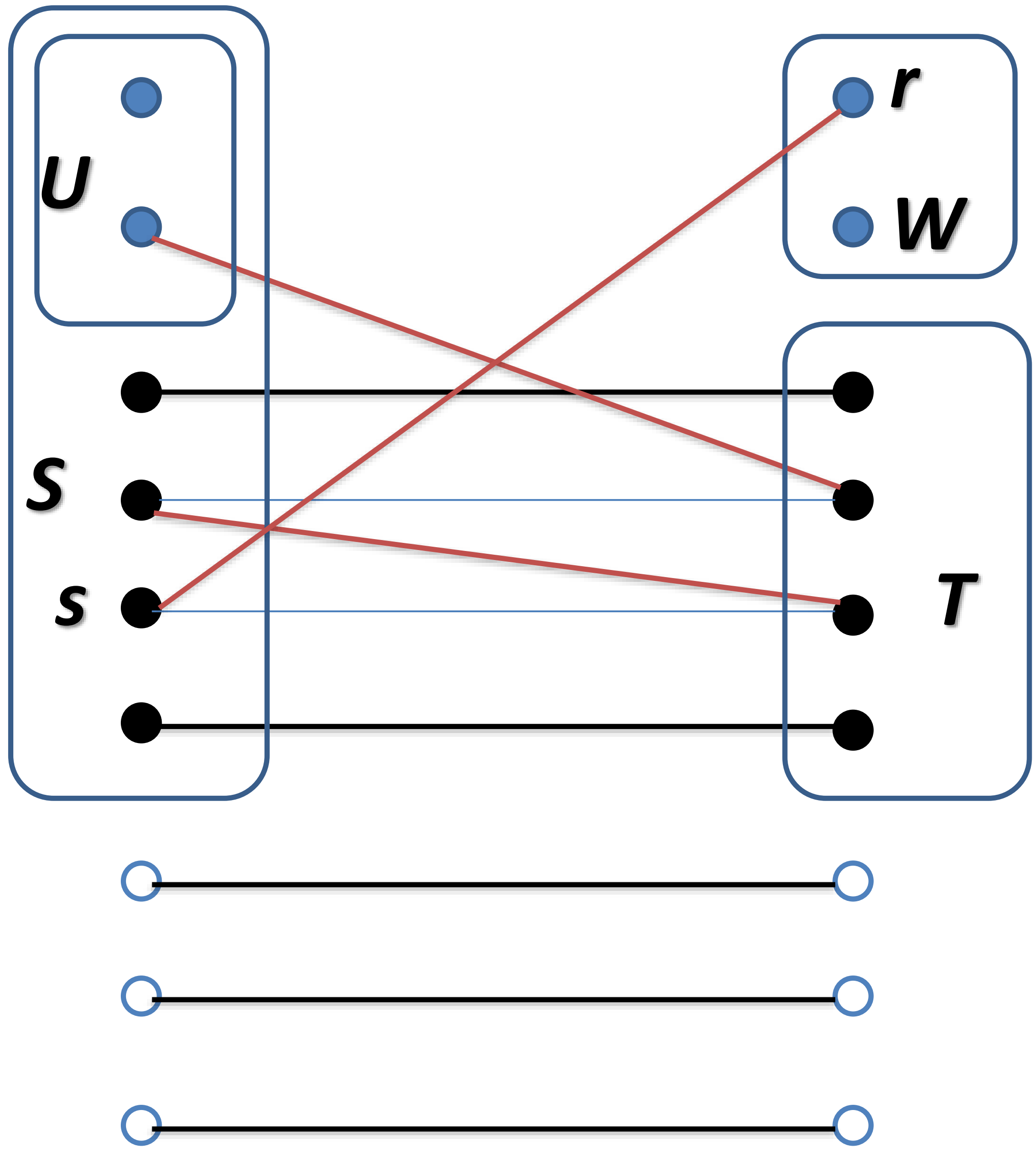
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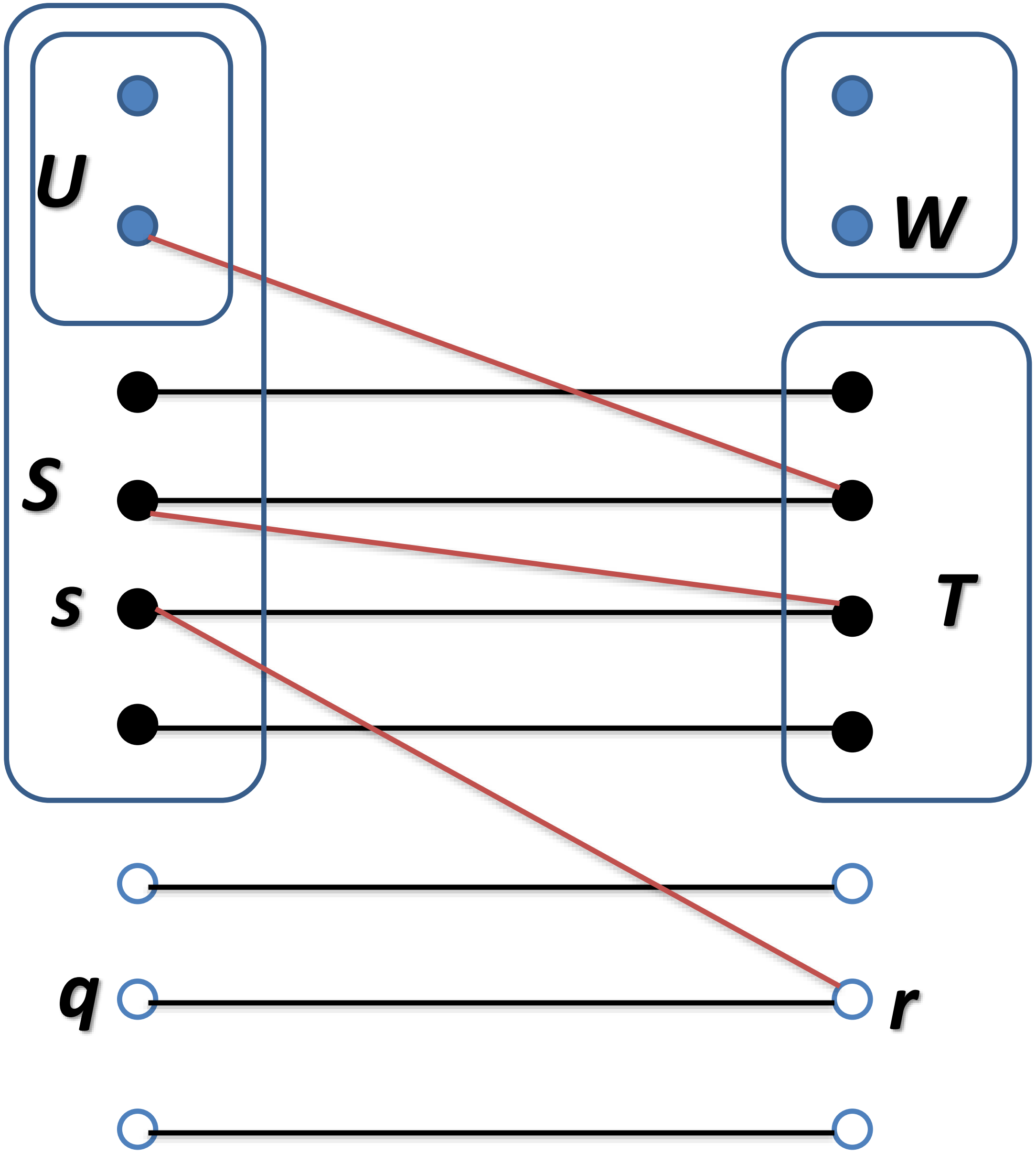
- Let $sr \in E \backslash M$, $s \in S$, $r \in B \backslash T$.
- Let $Q$ be an almost augmenting path starting at some node $u \in U$ and ending at $s$.

➢ If $r$ is matched with a node $q \in A$, then we can append the edges $sr$ and $rq$ to $Q$ to get an almost augmenting path from $U$ to $q$. So we can add $q$ to $S$.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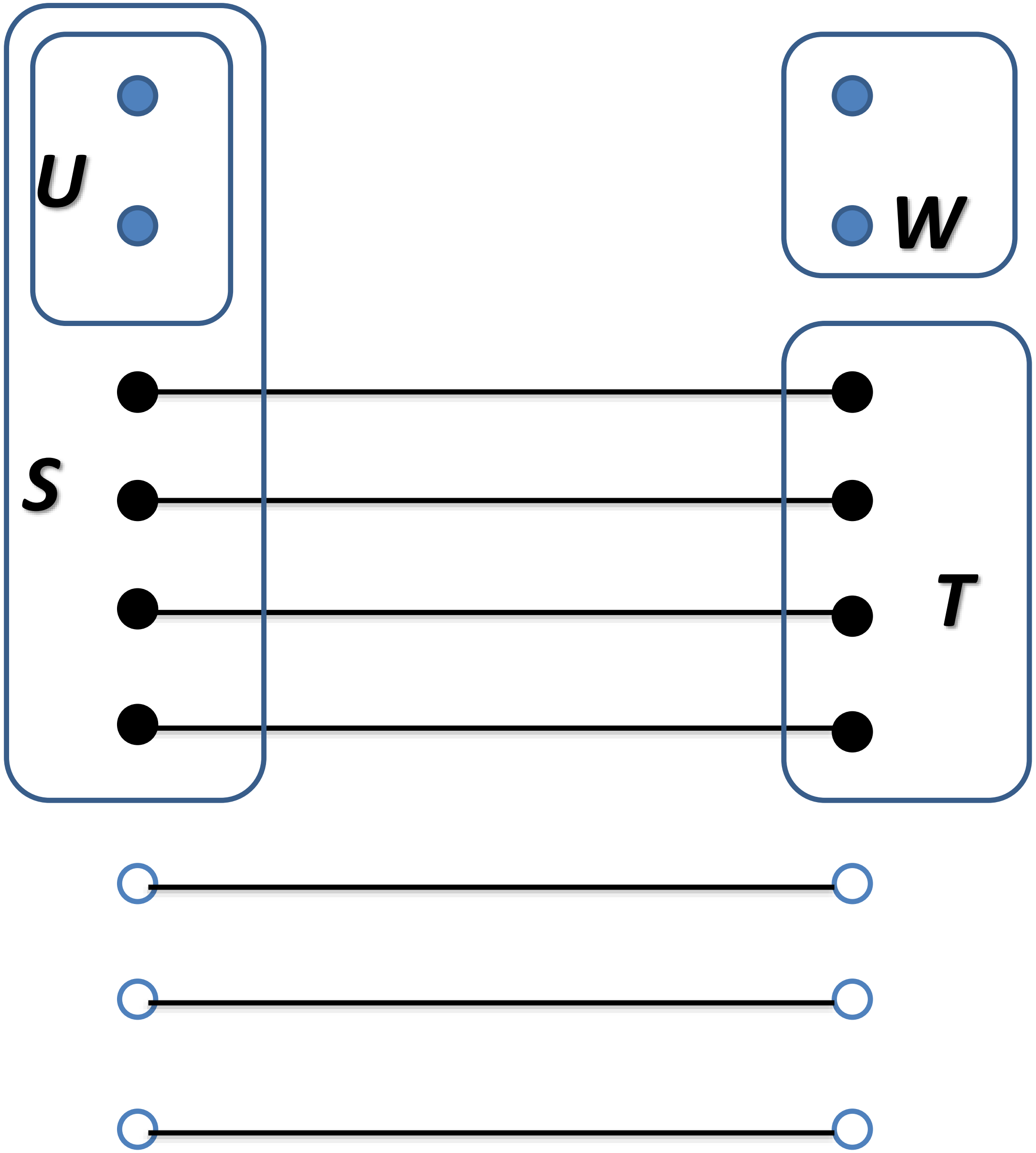
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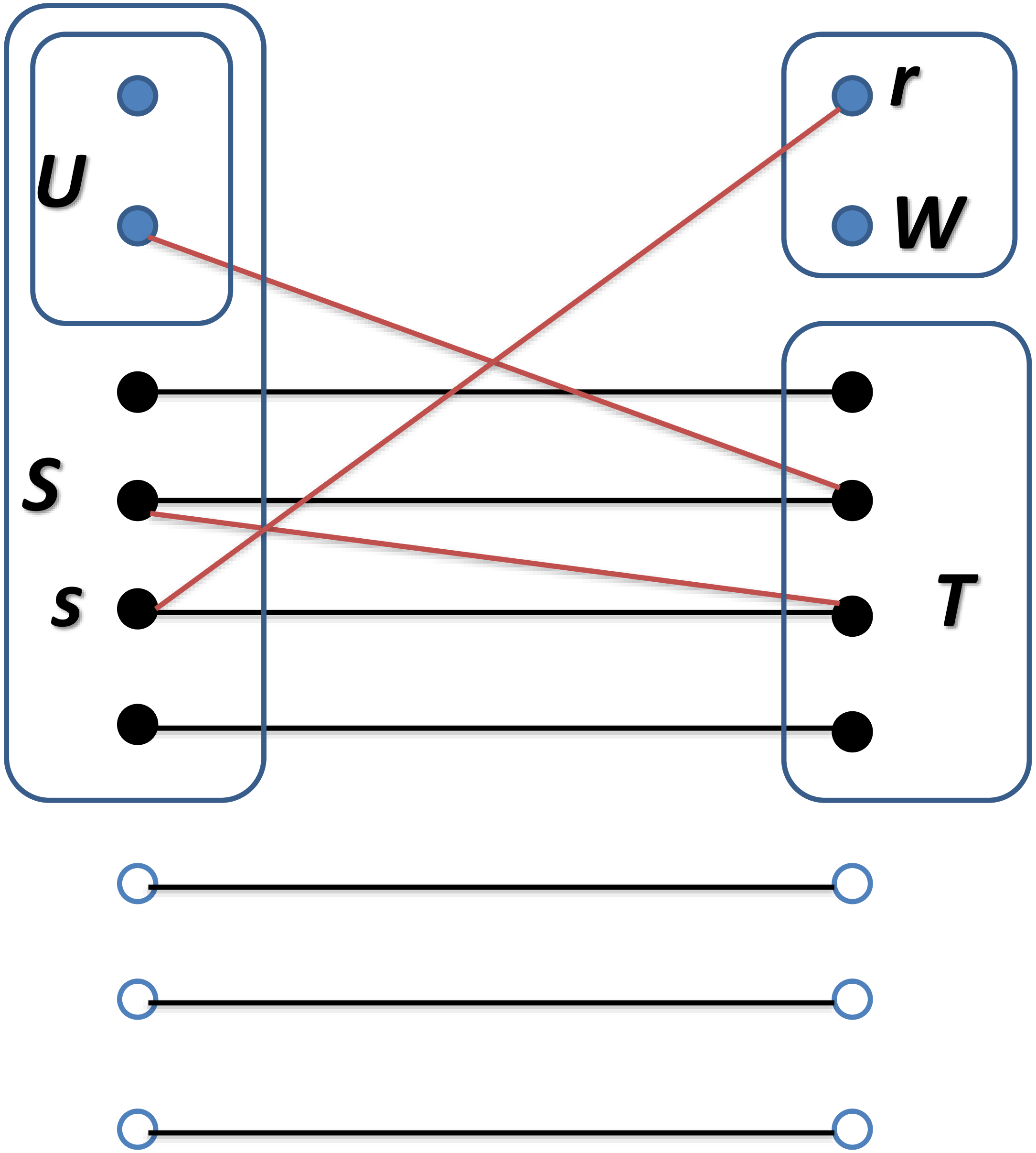
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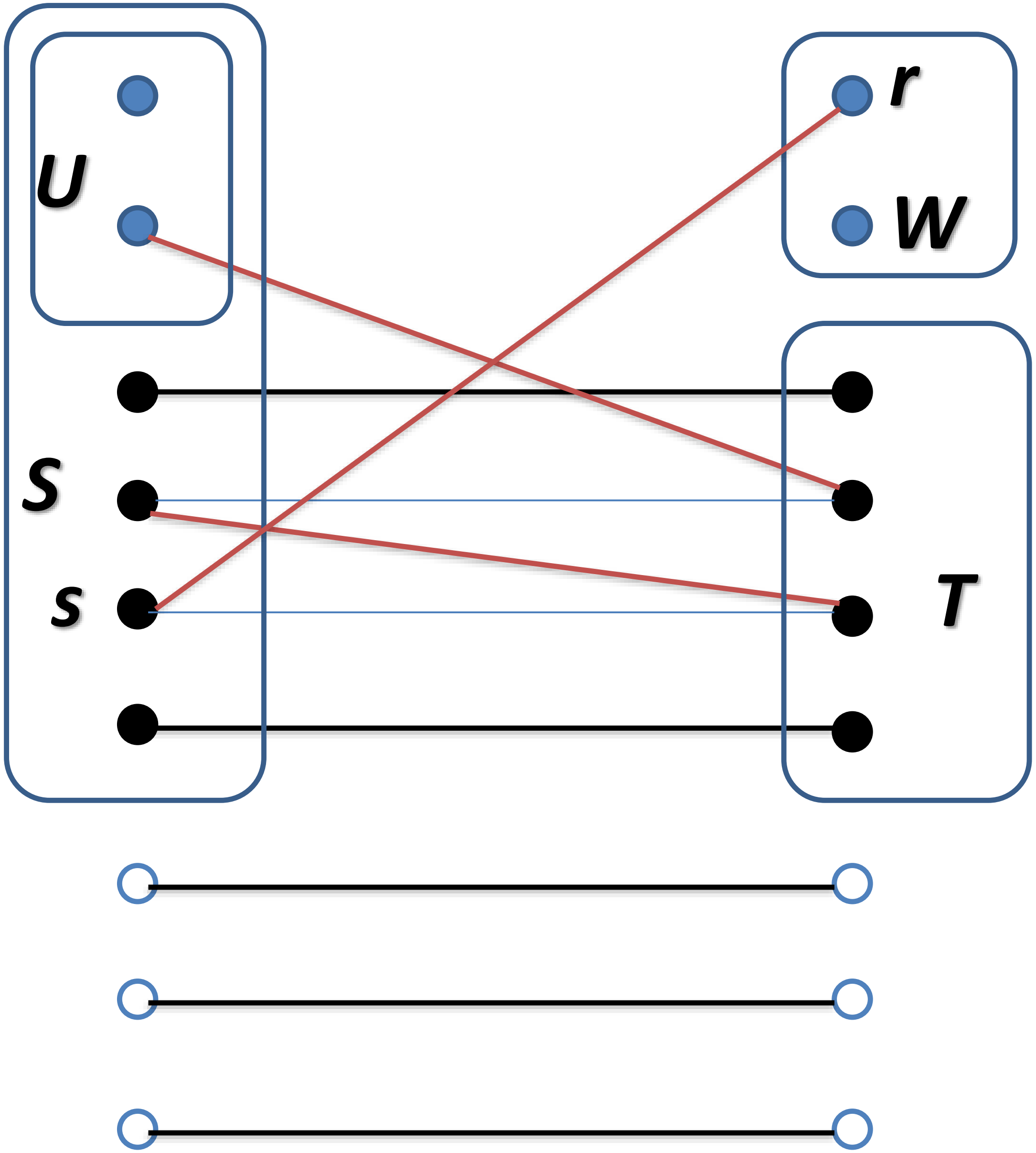
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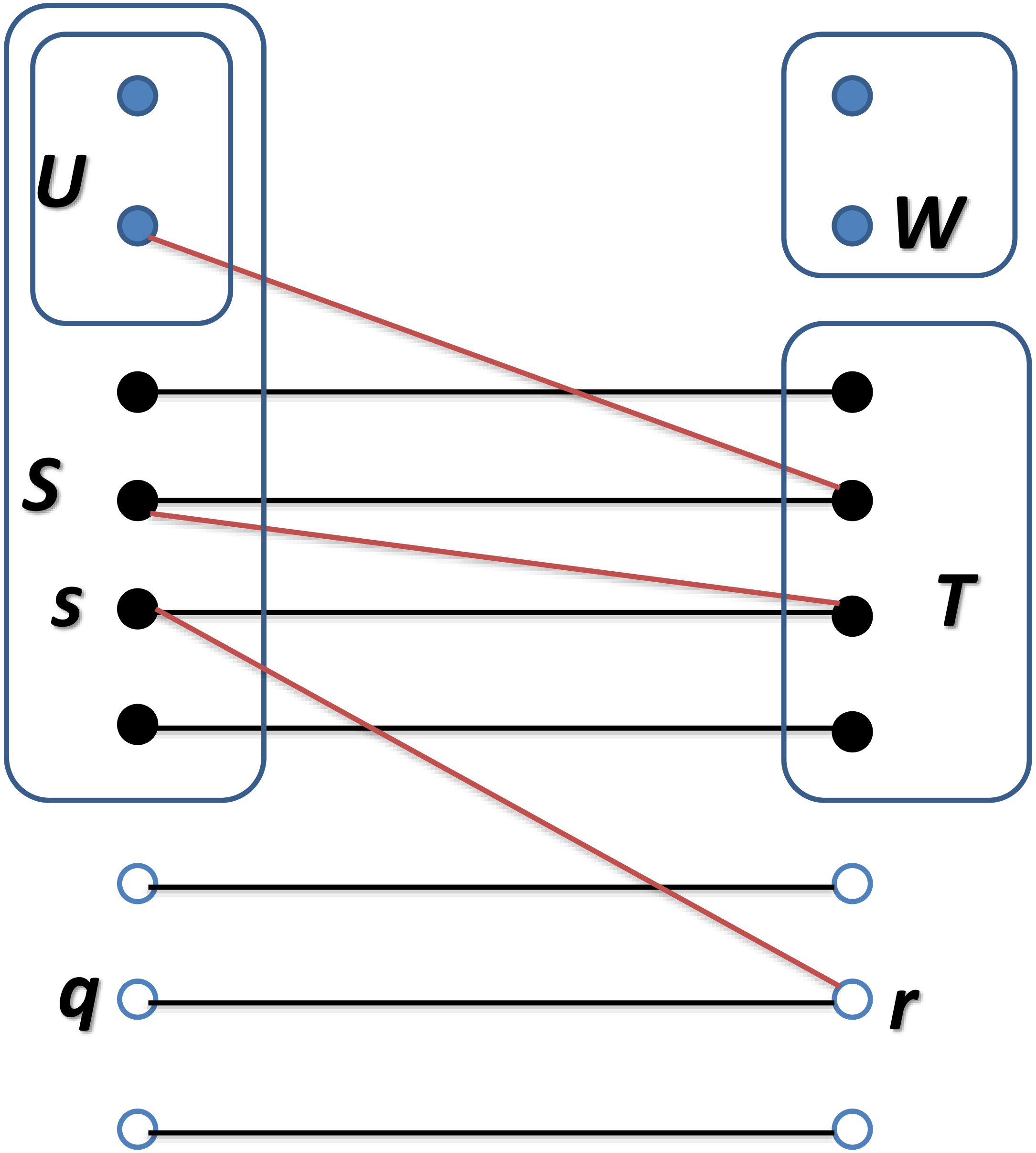
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- **So if we find an edge connecting a node in *S* to a node not in *T*, we can increase either the size of *M* or the set *S*.**
- **Finally we must encounter a situation where either *M* is a perfect matching, or *M* is not perfect, but no edge connects *S* to any node outside *T*, i.e.**

$$T=N(S)\ \&\ |T|=|S|-|U|<|S|.$$

**This implies no perfect matching at all in the graph.**

# Berge-Fulkerson Conjecture (1971)

**Every bridgeless cubic graph admits a double cover by six perfect matchings.**

**Note. Such a graph has a perfect matching by the Petersen theorem (1891).**