高等微积分

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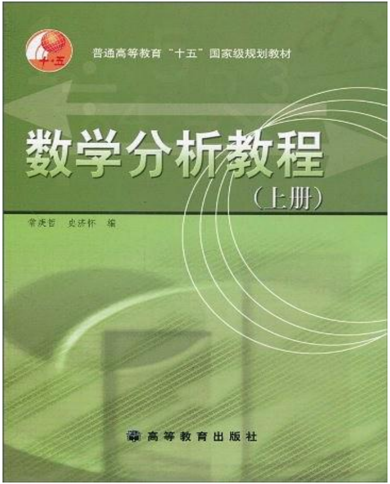
第六章: 求导的逆运算–不定积分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§6.1. 不定积分的概念与积分法

定义 1. 将定义在区间上的函数 f 的原函数的一般表达式称为 f 的不定积分, 记作 $\int f(x)\,\mathrm{d}x$.

评注: (1) $\int f(x)\,\mathrm{d}x$ 是以 x 为自变量的函数.

(2) 若 F, G 均为 f 的原函数, 则 $F' = G'$, 于是存在常数 C 使得 $G - F = C$, 故

$$\int f(x)\,\mathrm{d}x = F(x) + C \text{ (其中 } C \text{ 为常数).}$$

(3) 若 $f \in \mathscr{C}[a,b]$, 则 $\int f(x)\,\mathrm{d}x = \int_a^x f(t)\,\mathrm{d}t + C$.

不定积分与导数、微分的关系

- 若 $\int f(x)\,\mathrm{d}x = F(x) + C$, 则 $F'(x) = f(x)$,

$$\left(\int f(x)\,\mathrm{d}x\right)' = F'(x) = f(x),$$

$$\mathrm{d}F(x) = f(x)\,\mathrm{d}x,\quad \mathrm{d}\left(\int f(x)\,\mathrm{d}x\right) = f(x)\,\mathrm{d}x,$$

$$\int f(x)\,\mathrm{d}x = \int F'(x)\mathrm{d}x = \int \mathrm{d}F(x) = F(x) + C.$$

- **(线性性)** $\forall \alpha, \beta \in \mathbb{R}$, 我们有

$$\int (\alpha f(x) + \beta g(x))\, \mathrm{d}x = \alpha \int f(x)\, \mathrm{d}x + \beta \int g(x)\, \mathrm{d}x.$$

基本的不定积分公式

- $\int \mathrm{d}x = x + C.$
- $\int x^\alpha \,\mathrm{d}x = \frac{x^{\alpha+1}}{\alpha+1} + C \ (\alpha \neq -1),$
  $\int \frac{1}{x} \,\mathrm{d}x = \ln|x| + C.$
- $\int a^x \,\mathrm{d}x = \frac{a^x}{\ln a} + C \ (a > 0,\ a \neq 1),$
  $\int e^x \,\mathrm{d}x = e^x + C.$
- $\int \sin x \,\mathrm{d}x = -\cos x + C,$
  $\int \cos x \,\mathrm{d}x = \sin x + C.$

- $\int \operatorname{sh} x \, \mathrm{d}x = \operatorname{ch} x + C$, $\int \operatorname{ch} x \, \mathrm{d}x = \operatorname{sh} x + C$.
- $\int \sec^2 x \, \mathrm{d}x = \tan x + C$.
- $\int \csc^2 x \, \mathrm{d}x = -\cot x + C$.
- $\int \frac{\mathrm{d}x}{\sqrt{1-x^2}} = \arcsin x + C$.
- $\int \frac{\mathrm{d}x}{1+x^2} = \arctan x + C$.
- $\int \frac{\mathrm{d}x}{\sqrt{x^2+a^2}} = \ln\left|x + \sqrt{x^2 + a^2}\right| + C$.
- $\int \frac{\mathrm{d}x}{\sqrt{x^2-a^2}} = \ln\left|x + \sqrt{x^2 - a^2}\right| + C$.

例 1. 计算 $\int |x-1|\,\mathrm{d}x$.

解: 当 $x \geqslant 1$ 时，我们有

$$\int |x-1|\,\mathrm{d}x = \int (x-1)\,\mathrm{d}x = \frac{1}{2}x^2 - x + C_1.$$

当 $x < 1$ 时，我们则有

$$\int |x-1|\,\mathrm{d}x = \int (1-x)\,\mathrm{d}x = x - \frac{1}{2}x^2 + C_2.$$

由于原函数在点 $x=1$ 处连续，则 $C_1 = 1 + C_2$,

故 $$\int |x-1|\,\mathrm{d}x = \begin{cases} \frac{1}{2}x^2 - x + 1 + C, & \text{若 } x \geqslant 1, \\ x - \frac{1}{2}x^2 + C, & \text{若 } x < 1. \end{cases}$$

例 2. 计算 $\int \frac{2x^2}{1+x^2}\,\mathrm{d}x$.

解: $\int \frac{2x^2}{1+x^2}\,\mathrm{d}x = 2\int \left(1 - \frac{1}{1+x^2}\right)\mathrm{d}x$

$= 2\int \mathrm{d}x - 2\int \frac{1}{1+x^2}\,\mathrm{d}x$

$= 2x - 2\arctan x + C$.

例 3. 计算 $\int \frac{\mathrm{d}x}{(\cos^2 x)(\sin^2 x)}$.

解: $\int \frac{\mathrm{d}x}{(\cos^2 x)(\sin^2 x)} = \int \frac{\cos^2 x+\sin^2 x}{(\cos^2 x)(\sin^2 x)}\mathrm{d}x$

$= \int \left(\frac{1}{\cos^2 x} + \frac{1}{\sin^2 x}\right)\mathrm{d}x$

$= \tan x - \cot x + C.$

例 4. 计算 $\int \left(3x^2 - \frac{1}{\sqrt[3]{x}} + \frac{1}{x} + \frac{1}{1+x^2}\right) \mathrm{d}x$.

解: $\int \left(3x^2 - \frac{1}{\sqrt[3]{x}} + \frac{1}{x} + \frac{1}{1+x^2}\right) \mathrm{d}x$

$$= \int \mathrm{d}\left(x^3 - \frac{3}{2}x^{\frac{2}{3}} + \ln|x| + \arctan x\right)$$

$$= x^3 - \frac{3}{2}x^{\frac{2}{3}} + \ln|x| + \arctan x + C.$$

例 5. 计算 $\int \frac{\mathrm{d}x}{a^2-x^2}\ (a \neq 0)$.

解: $\int \frac{\mathrm{d}x}{a^2-x^2} = \int \frac{1}{2a}\left(\frac{1}{x+a} - \frac{1}{x-a}\right)\mathrm{d}x$

$= \frac{1}{2a}\big(\ln|x+a| - \ln|x-a|\big) + C$

$= \frac{1}{2a}\ln\left|\frac{x+a}{x-a}\right| + C$.

例 6. 计算 $\int\left(\sin\frac{x}{2}+\cos\frac{x}{2}\right)^2\mathrm{d}x$.

解: $\int\left(\sin\frac{x}{2}+\cos\frac{x}{2}\right)^2\mathrm{d}x=\int(1+\sin x)\mathrm{d}x$

$=x-\cos x+C$.

例 7. 计算 $\int e^{|x|}\mathrm{d}x$.

解: 当 $x \geqslant 0$, $\int e^{|x|}\mathrm{d}x = \int e^{x}\mathrm{d}x = e^{x} + C_1$.

当 $x < 0$ 时, $\int e^{|x|}\mathrm{d}x = \int e^{-x}\mathrm{d}x = -e^{-x} + C_2$.

由于原函数在点 $x = 0$ 连续, 从而 $C_2 = 2 + C_1$.

故 $\int e^{|x|}\mathrm{d}x = \begin{cases} e^{x} + C, & 若\ x \geqslant 0, \\ 2 - e^{-x} + C, & 若\ x < 0. \end{cases}$

§ 6.2. 方法一: 第一换元积分法 (凑微分)

若 $F'(y) = f(y)$, 而 u 为可导函数, 则

$$(F \circ u)'(x) = F'(u(x))u'(x) = f(u(x))u'(x).$$

故 $\int f(u(x))u'(x)\mathrm{d}x = F(u(x)) + C$. 通常也将左式写成 $\int f(u(x))\mathrm{d}u(x)$, 则我们有

$$\begin{aligned}\int f(u(x))u'(x)\mathrm{d}x &= \int f(u(x))\mathrm{d}u(x)\\ &= F(u(x)) + C.\end{aligned}$$

例 8. 计算 $\int 2xe^{x^2}\mathrm{d}x$.

解: $\int 2xe^{x^2}\mathrm{d}x = \int e^{x^2}\mathrm{d}(x^2) = \int \mathrm{d}(e^{x^2}) = e^{x^2} + C$.

例 9. 设 $a \neq 0$. 计算 $\int \frac{\mathrm{d}x}{a^2+x^2}$.

解: $\int \frac{\mathrm{d}x}{a^2+x^2} = \frac{1}{a}\int \frac{\mathrm{d}(\frac{x}{a})}{1+(\frac{x}{a})^2} = \frac{1}{a}\int \mathrm{d}\Big(\arctan\frac{x}{a}\Big)$

$= \frac{1}{a}\arctan\frac{x}{a} + C$.

例 10. 设 $a > 0$. 计算 $\int \frac{\mathrm{d}x}{\sqrt{a^2-x^2}}$.

解: $\int \frac{\mathrm{d}x}{\sqrt{a^2-x^2}} = \int \frac{\mathrm{d}(\frac{x}{a})}{\sqrt{1-(\frac{x}{a})^2}} = \int \mathrm{d}(\arcsin \frac{x}{a}) = \arcsin \frac{x}{a} + C.$

例 11. 计算 $\int \tan x \, \mathrm{d}x$.

解: $\int \tan x \, \mathrm{d}x = \int \frac{\sin x}{\cos x} \mathrm{d}x = -\int \frac{\mathrm{d}(\cos x)}{\cos x}$

$= -\int \mathrm{d}\big(\ln |\cos x| \big) = -\ln |\cos x| + C.$

例 12. 计算 $\int \cot x \, \mathrm{d}x$.

解: $\int \cot x \, \mathrm{d}x = \int \frac{\cos x \, \mathrm{d}x}{\sin x} = \int \frac{\mathrm{d}(\sin x)}{\sin x} = \ln |\sin x| + C.$

例 13. 计算 $\int \tan^2 x \, \mathrm{d}x$.

解: $\int \tan^2 x \, \mathrm{d}x = \int (\sec^2 x - 1) \mathrm{d}x$

$= \int \mathrm{d}(\tan x - x) = \tan x - x + C$.

例 14. 计算 $\int \frac{\mathrm{d}x}{(1+4x^2)(\arctan 2x+1)^2}$.

解: $\int \frac{\mathrm{d}x}{(1+4x^2)(1+\arctan 2x)^2}$

$$= \frac{1}{2}\int \frac{\mathrm{d}(2x)}{(1+(2x)^2)(1+\arctan 2x)^2} = \frac{1}{2}\int \frac{\mathrm{d}(\arctan 2x)}{(1+\arctan 2x)^2}$$

$$= \frac{1}{2}\int \frac{\mathrm{d}(1+\arctan 2x)}{(1+\arctan 2x)^2} = -\frac{1}{2}\int \mathrm{d}\left(\frac{1}{1+\arctan 2x}\right)$$

$$= -\frac{1}{2(1+\arctan 2x)} + C.$$

例 15. 计算 $\int \frac{\mathrm{d}x}{\sin x}$.

解: $\int \frac{\mathrm{d}x}{\sin x} = \int \frac{\mathrm{d}x}{2\sin\frac{x}{2}\cos\frac{x}{2}} = \int \frac{\mathrm{d}(\frac{x}{2})}{\tan\frac{x}{2}\cos^2\frac{x}{2}}$

$= \int \frac{\mathrm{d}(\tan\frac{x}{2})}{\tan\frac{x}{2}} = \ln\left|\tan\frac{x}{2}\right| + C.$

例 16. 计算 $\int \frac{x}{\sqrt{3+2x-x^2}}\mathrm{d}x$.

解: $\int \frac{x}{\sqrt{3+2x-x^2}}\mathrm{d}x$

$$= \int \left(-\frac{1}{2}\frac{(3+2x-x^2)'}{\sqrt{3+2x-x^2}} + \frac{1}{\sqrt{3+2x-x^2}} \right)\mathrm{d}x$$

$$= -\frac{1}{2}\int \frac{\mathrm{d}(3+2x-x^2)}{\sqrt{3+2x-x^2}} + \int \frac{1}{\sqrt{4-(x-1)^2}}\mathrm{d}x$$

$$= -\sqrt{3+2x-x^2} + \int \frac{1}{\sqrt{1-(\frac{x-1}{2})^2}}\mathrm{d}(\frac{x-1}{2})$$

$$= -\sqrt{3+2x-x^2} + \arcsin\frac{x-1}{2} + C.$$

例 17. 计算 $\int \frac{\mathrm{d}x}{1+e^x}$.

解: $\int \frac{\mathrm{d}x}{1+e^x} = -\int \frac{\mathrm{d}(e^{-x})}{e^{-x}+1} = -\ln(1+e^{-x}) + C$.

例 18. 计算 $\int \sec x \, \mathrm{d}x$.

解: $\int \sec x \, \mathrm{d}x = \int \frac{\mathrm{d}x}{\cos x} = \int \frac{\mathrm{d}(x+\frac{\pi}{2})}{\sin(x+\frac{\pi}{2})}$

$= \ln \left| \csc(x + \frac{\pi}{2}) - \cot(x + \frac{\pi}{2}) \right| + C$

$= \ln \left| \sec x + \tan x \right| + C.$

§6.2 方法二：　第二换元积分法

$$\int f(x)\,\mathrm{d}x \overset{x=x(t)}{=} \int f(x(t))x'(t)\,\mathrm{d}t$$

$$= F(t) + C \overset{t=t(x)}{=} F(t(x)) + C.$$

例 19. 计算 $\int \frac{\sin\sqrt{x}}{\sqrt{x}}\mathrm{d}x$.

解: $\int \frac{\sin\sqrt{x}}{\sqrt{x}}\mathrm{d}x \overset{t=\sqrt{x}}{=} \int \frac{\sin t}{t}\mathrm{d}(t^2) = \int \frac{\sin t}{t} \cdot (2t)\,\mathrm{d}t$

$= \int 2\sin t\,\mathrm{d}t = -2\cos t + C$

$= -2\cos\sqrt{x} + C.$

例 20. 计算 $\int \frac{\mathrm{d}x}{1+\sqrt[3]{1+x}}$.

解: $\int \frac{\mathrm{d}x}{1+\sqrt[3]{1+x}} \overset{u=\sqrt[3]{1+x}}{=\!=} \int \frac{\mathrm{d}(u^3-1)}{1+u}$

$= \int \frac{3u^2\,\mathrm{d}u}{1+u} = 3\int \left(u - 1 + \frac{1}{1+u}\right)\mathrm{d}u$

$= 3\left(\frac{1}{2}u^2 - u + \ln|1+u|\right) + C$

$= 3\left(\frac{1}{2}(1+x)^{\frac{2}{3}} - \sqrt[3]{1+x} + \ln\left|1+\sqrt[3]{1+x}\right|\right) + C.$

例 21. 计算 $\int \frac{x-2}{1-\sqrt{x+1}}\mathrm{d}x$.

解: $\int \frac{x-2}{1-\sqrt{x+1}}\mathrm{d}x \overset{u=\sqrt{x+1}}{=\!=} \int \frac{u^2-3}{1-u}\mathrm{d}(u^2-1)$

$= \int \left(-2u^2-2u+4-\frac{4}{1-u}\right)\mathrm{d}u$

$= -\frac{2}{3}u^3-u^2+4u+4\ln|1-u|+C_1$

$= -\frac{2}{3}(x-5)\sqrt{x+1}-x+4\ln\left|1-\sqrt{1+x}\right|+C.$

注: 当被积函数中含有 $\sqrt{a^2 - x^2}$, $\sqrt{x^2 \pm a^2}$ 时, 常用三角函数代换法.

例 22. 计算 $\int \sqrt{a^2 - x^2}\, dx \ (a > 0)$.

解: $\int \sqrt{a^2 - x^2}\, dx$

$$\overset{x=a\sin t}{\underset{|t| \leqslant \frac{\pi}{2}}{=\!=}} \int \sqrt{a^2 - a^2 \sin^2 t}\, d(a \sin t)$$

$$= \int (a\cos t) \cdot (a \cos t)\, dt = a^2 \int \cos^2 t\, dt$$

$$= a^2 \int \frac{1+\cos 2t}{2}\, dt = \frac{a^2}{2}\left(t + \frac{\sin 2t}{2}\right) + C$$

$$= \frac{a^2}{2}\left(t + \sin t \cos t\right) + C$$

$$= \frac{a^2}{2} \arcsin \frac{x}{a} + \frac{x}{2}\sqrt{a^2 - x^2} + C.$$

例 23. 计算 $\int \frac{\mathrm{d}x}{\sqrt{x^2+a^2}}$ $(a > 0)$.

解: $\int \frac{\mathrm{d}x}{\sqrt{x^2+a^2}} \overset{x=a\tan t}{\underset{|t|<\frac{\pi}{2}}{=}} \int \frac{\mathrm{d}(a\tan t)}{\sqrt{a^2\tan^2 t+a^2}}$

$= \int \frac{\cos t\,\mathrm{d}t}{\cos^2 t} = \int \frac{\mathrm{d}t}{\cos t}$

$= \ln|\sec t + \tan t| + C_1$ (见例 18)

$= \ln|\frac{1}{a}\sqrt{x^2+a^2} + \frac{x}{a}| + C_1$

$= \ln|x + \sqrt{x^2+a^2}| + C.$

例 24. 计算 $\int \frac{1-x+x^2}{\sqrt{1+x-x^2}}\,\mathrm{d}x.$

解: $\int \frac{1-x+x^2}{\sqrt{1+x-x^2}}\,\mathrm{d}x = \int \frac{1-x+x^2}{\sqrt{(\frac{\sqrt{5}}{2})^2-(x-\frac{1}{2})^2}}\,\mathrm{d}x$

$$\underset{|t|\leqslant\frac{\pi}{2}}{\overset{x=\frac{1}{2}+\frac{\sqrt{5}}{2}\sin t}{=\!=}} \int \frac{1-(\frac{1}{2}+\frac{\sqrt{5}}{2}\sin t)+(\frac{1}{2}+\frac{\sqrt{5}}{2}\sin t)^2}{\sqrt{(\frac{\sqrt{5}}{2})^2-(\frac{\sqrt{5}}{2}\sin t)^2}}\,\mathrm{d}\Big(\frac{1}{2}+\frac{\sqrt{5}}{2}\sin t\Big)$$

$$= \int (\tfrac{3}{4}+\tfrac{5}{4}\sin^2 t)\,\mathrm{d}t = \int (\tfrac{3}{4}+\tfrac{5}{4}\tfrac{1-\cos 2t}{2})\,\mathrm{d}t$$

$$= \frac{11}{8}t - \frac{5}{16}\sin 2t + C = \frac{11}{8}t - \frac{5}{8}\sin t\cos t + C$$

$$= \frac{11}{8}\arcsin\frac{2}{\sqrt{5}}(x-\tfrac{1}{2}) - \frac{5}{8}\cdot\frac{2}{\sqrt{5}}(x-\tfrac{1}{2})\cdot\sqrt{1-(\tfrac{2}{\sqrt{5}}(x-\tfrac{1}{2}))^2}+C$$

$$= \frac{11}{8}\arcsin\frac{2}{\sqrt{5}}(x-\tfrac{1}{2}) - \tfrac{1}{2}(x-\tfrac{1}{2})\sqrt{1+x-x^2}+C.$$

例 25. 计算 $\int \frac{\mathrm{d}x}{\sqrt{x^2-a^2}}\ (a>0)$.

解: 被积函数的定义域为 $(-\infty,-a)\cup(a,+\infty)$.

当 $x>a$ 时, 考虑变换 $x=a\sec t\ (0<t<\frac{\pi}{2})$,

则 $\mathrm{d}x=a\frac{\sin t}{\cos^2 t}\mathrm{d}t$, 由此可得

$$\int \frac{\mathrm{d}x}{\sqrt{x^2-a^2}}=\int \frac{1}{a\tan t}\cdot a\frac{\sin t}{\cos^2 t}\mathrm{d}t=\int \frac{\mathrm{d}t}{\cos t}$$

$$=\ln|\sec t+\tan t|+C_1=\ln\left|\frac{x}{a}+\frac{\sqrt{x^2-a^2}}{a}\right|+C_1$$

$$=\ln\left|x+\sqrt{x^2-a^2}\right|+C_1'.$$

当 $x < -a$ 时，考虑变换 $u = -x$，则有

$$\int \frac{\mathrm{d}x}{\sqrt{x^2 - a^2}} = -\int \frac{\mathrm{d}u}{\sqrt{u^2 - a^2}} = -\ln\left|u + \sqrt{u^2 - a^2}\right| + C_2$$

$$= \ln\left|\frac{1}{-x + \sqrt{x^2 - a^2}}\right| + C_2 = \ln\left|x + \sqrt{x^2 - a^2}\right| + C_2'.$$

于是我们有

$$\int \frac{\mathrm{d}x}{\sqrt{x^2 - a^2}} = \begin{cases} \ln\left|x + \sqrt{x^2 - a^2}\right| + C_1', & \text{若 } x > a, \\ \\ \ln\left|x + \sqrt{x^2 - a^2}\right| + C_2', & \text{若 } x < -a. \end{cases}$$

因为原函数的定义域由两个不相交的区间组成, 故常数 C_1' 和 C_2' 可以不同, 但计算不定积分的目的只是为了得到一个原函数, 因此人们通常将上式合并成一个统一的表达式:

$$\int \frac{\mathrm{d}x}{\sqrt{x^2 - a^2}} = \ln \left| x + \sqrt{x^2 - a^2} \right| + C.$$

同学们辛苦了!