Discrete Mathematics

Lecture 3

贾宽三角

3.1 The Binomial Theorem

$(x+y)^2=x^2+2xy+y^2,$

$(x+y)^3=(x+y)(x+y)^2=(x+y)(x^2+2xy+y^2)$

$=x^3+3x^2y+3xy^2+y^3,$

$(x+y)^4=(x+y)(x+y)^3=x^4+4x^3y+6x^2y^2+4xy^3+y^4,$

$(x+y)^5=(x+y)(x+y)(x+y)(x+y)(x+y)$

$=C(5,0)x^5+C(5,1)x^4y+\cdots+C(5,5)y^5.$

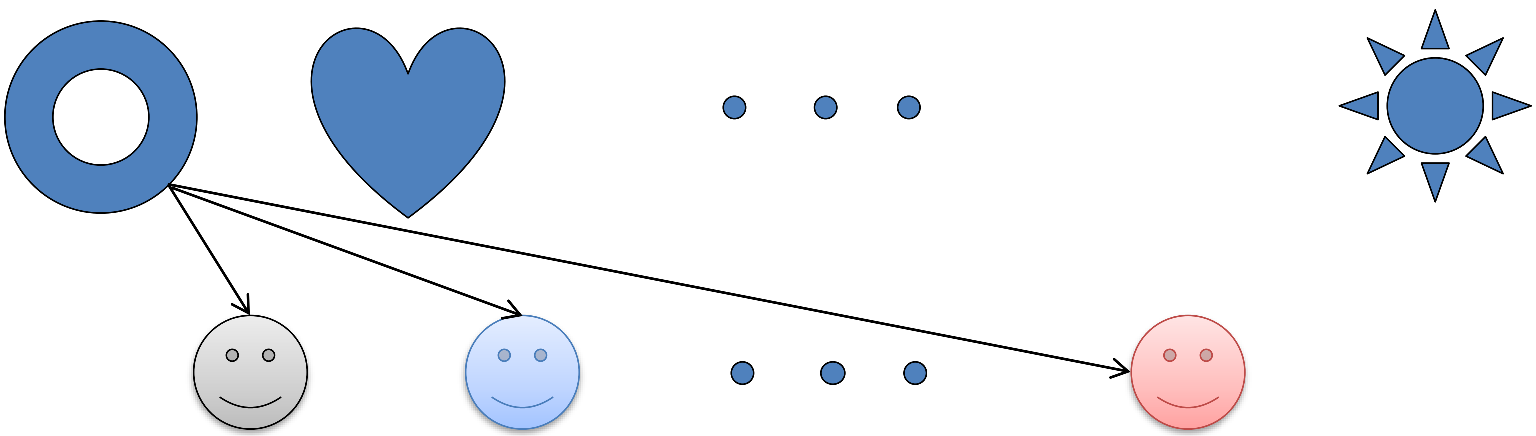
Theorem 3.1.1 (The Binomial Theorem)

$(x+y)^n = C(n,0)x^n + C(n,1)x^{n-1}y + \cdots + C(n,n)y^n$.

Corollary $2^n = C(n,0) + C(n,1) + \cdots + C(n,n)$.

3.2-3 Distributing Presents & Anagrams

Suppose we have *n* different presents.
We want to distribute to *k* children.
How many ways can this be done?



The answer is k^n.

3.2-3 Distributing Presents & Anagrams

If the ith child should get n_i presents, then how many ways can these presents be distributed? $\dfrac{n!}{n_1!\,n_2!\cdots n_k!}$.

Corollary $$\sum_{n_1+n_2+\cdots+n_k=n} \frac{n!}{n_1!\,n_2!\cdots n_k!} = k^n.$$

The Multinomial Theorem

$$\left(\sum_{i=1}^{k} x_i\right)^n = n! \sum_{n_1+n_2+\cdots+n_k=n} \prod_{i=1}^{k} x_i^{n_i} \Big/ n_i!$$

3.2-3 Distributing Presents & Anagrams

If the ith child should get n_i presents, then how many ways can these presents be distributed?

$$\frac{n!}{n_1!n_2!\cdots n_k!}.$$

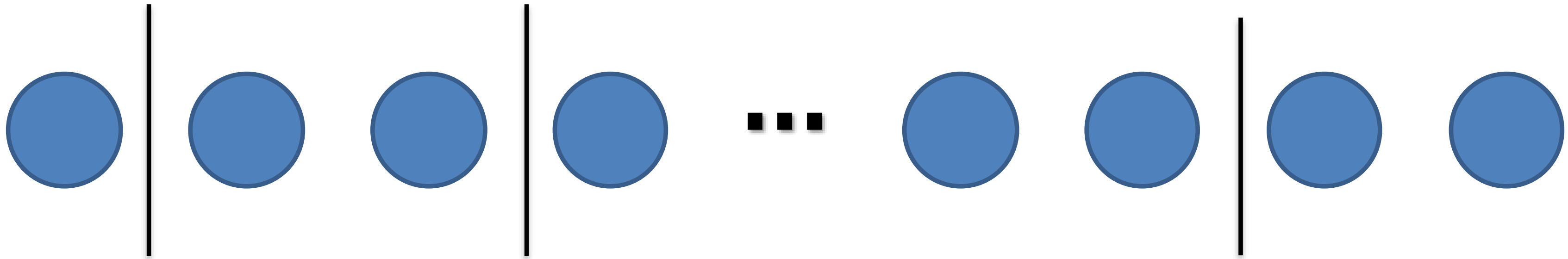
Corollary

$$\sum_{n_1+n_2+\cdots+n_k=n} \frac{n!}{n_1!n_2!\cdots n_k!} = k^n.$$

How many terms are in the summation?

3.4 Distributing Money

Theorem 3.4.1 **The number of ways to distribute n identical pennies to k children so that each child get at least one is $C(n-1, k-1)$.**



Theorem 3.4.2 **The number of ways to distribute n identical pennies to k children is $C(n+k-1, k-1)$.**

允许重复的组合

1. 取n个相同的硬币分给k个孩子是允许重复的组合的典型问题.
2. 在k个不同的元素中取n个进行组合,且允许重复,则组合数为$C(n+k-1, n)$.
3. 即: n个无区别的球放进k个有标志的盒子里,每盒放的球可多于一个,则共有$C(n+k-1,n)$种方案.

证明：只要证允许重复的组合与从$n+k-1$个不同的元素中取n个作不重复的组合一一对应，就得证。

假设k个不同元素为$1,2,\ldots,k$。从中取n个作允许重复的组合（$a_1, a_2, \ldots, a_n$）。不失一般性设 $a_1 \le a_2 \le \cdots \le a_n$。

这个组合自然地对应到一不重复的组合

$$(a_1, a_2+1, \ldots, a_i+i-1, \ldots, a_n+n-1)。$$

注意到这相当于从$n+k-1$个不同的元素中取n个不重复的组合，故为$C(n+k-1, n)$。

3.5 贾宪三角

C(0,0)

C(1,0) C(1,1)

C(2,0) C(2,1) C(2,2)

C(3,0) C(3,1) C(3,2) C(3,3)

C(4,0) C(4,1) C(4,2) C(4,3) C(4,4)

C(5,0) C(5,1) C(5,2) C(5,3) C(5,4) C(5,5)

C(6,0) C(6,1) C(6,2) C(6,3) C(6,4) C(6,5) C(6,6)

…

3.5 贾宪三角

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

3.6 贾宪三角等式

(1) $C(n,k)=C(n,n-k)$;

(2) $C(n,k)=C(n-1,k-1)+C(n-1,k)$;

(3) $C(n,0)+C(n,1)+\cdots+C(n,n)=2^n$;

(4) $C(n,0)-C(n,1)+C(n,2)-\cdots+(-1)^nC(n,n)=0$;

(5) $C(n+m,k)=C(n,0)C(m,k)+C(n,1)C(m,k-1)$

$+\cdots+ C(n,k)C(m,0)$, $k\leq\min(m,n)$.

(6) $C(m+n,m)=C(m,0)C(n,0)+ C(m,1)C(n,1)$

$+\cdots+ C(m,m)C(n,m)$, $m\leq n$.

(7) $C(n+k+1,k)=C(n+k,k)+C(n+k-1,k-1)+$

$C(n+k-2,k-2)+\cdots+ C(n+1,1)+C(n,0)$.

(8) $C(n,k)C(k,r)=C(n,r)C(n-r,k-r)$, $(k\geq r)$.

3.7 鸟瞰贾宪三角

贾宪三角除了对称性，另一重要性质为每一行的值从一开始单调增加到中点，然后单调减少到一。

$C(n,k)$? $C(n,k+1)$,

$$\frac{n(n-1)\cdots(n-k+1)}{k!} \; ? \; \frac{n(n-1)\cdots(n-k)}{(k+1)!},$$

$$1 \; ? \; \frac{n-k}{k+1}, \qquad k \; ? \; \frac{n-1}{2},$$

C(n,k) ? C(n,k+1)⇔k ? (n-1)/2,

So if k<(n-1)/2, then C(n,k)<C(n,k+1);

If k=(n-1)/2, then C(n,k)=C(n,k+1);

If k>(n-1)/2, then C(n,k)>C(n,k+1).

贾宪三角的第*n*行的最大值有多大?

$$2^n/(n+1)<C(n,n/2)<2^n.$$

$$\binom{n}{n/2} = \frac{n!}{(n/2)!(n/2)!},$$

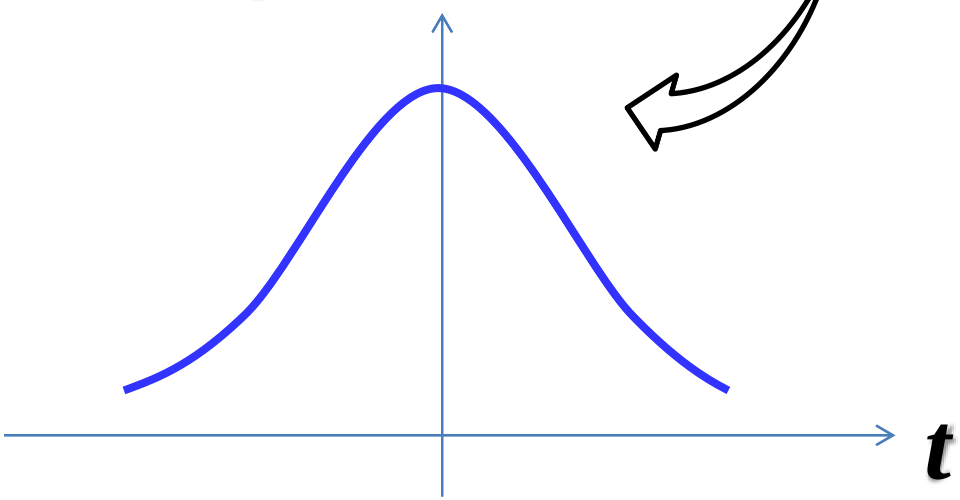
$$n! \approx \sqrt{2\pi n}(n/e)^n, \quad (n/2)! \approx \sqrt{\pi n}[n/(2e)]^{n/2}.$$

$$\binom{n}{n/2} \approx \frac{\sqrt{2\pi n}\,(n/e)^n}{\pi n[n/(2e)]^n} = \sqrt{\frac{2}{\pi n}}\,2^n,$$

3.8 鹰瞰贾宪三角

$$\binom{2n}{n-t}\Big/\binom{2n}{n} \approx \exp\{-t^2/n\},$$

where the graph of the right hand side is the famous **Gauβ curve**.



$$\exp\{-t^2/(n-t+1)\} \le \binom{2n}{n-t}\Big/\binom{2n}{n} \le \exp\{-t^2/(n+t)\}.$$

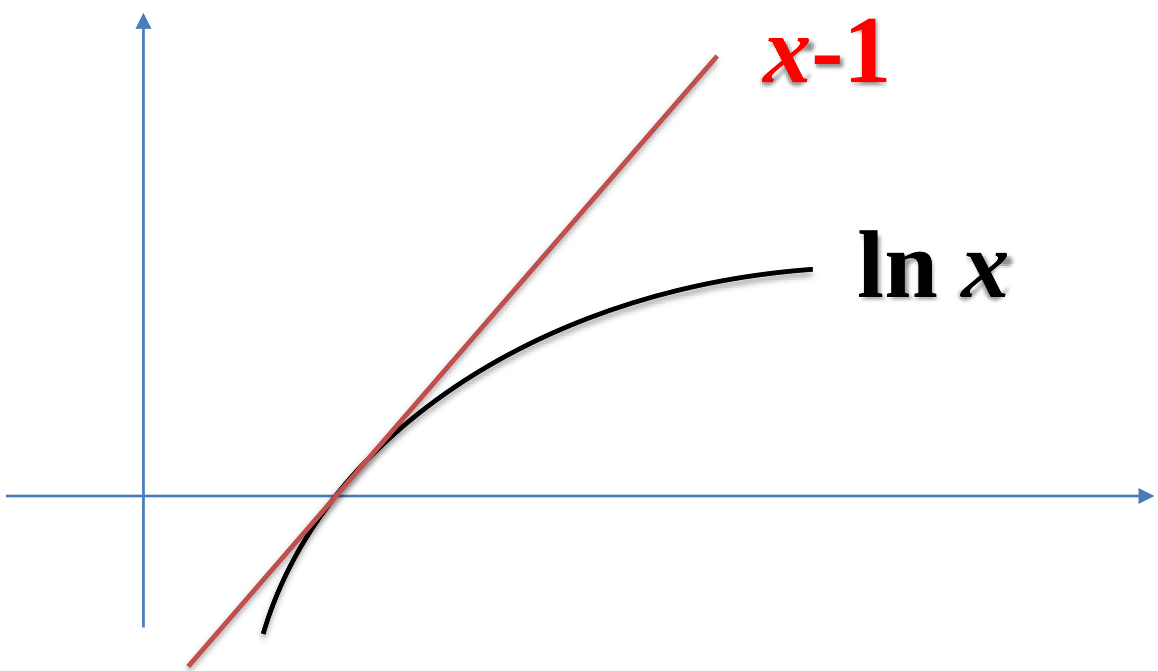
$$\binom{2n}{n-t}\Big/\binom{2n}{n} = \frac{(2n)!}{(n-t)!(n+t)!}\Big/\frac{(2n)!}{n!n!}$$

$$= \frac{n!n!}{(n-t)!(n+t)!}$$

$$= \frac{n(n-1)\cdots(n-t+1)}{(n+t)(n+t-1)\cdots(n+1)},$$

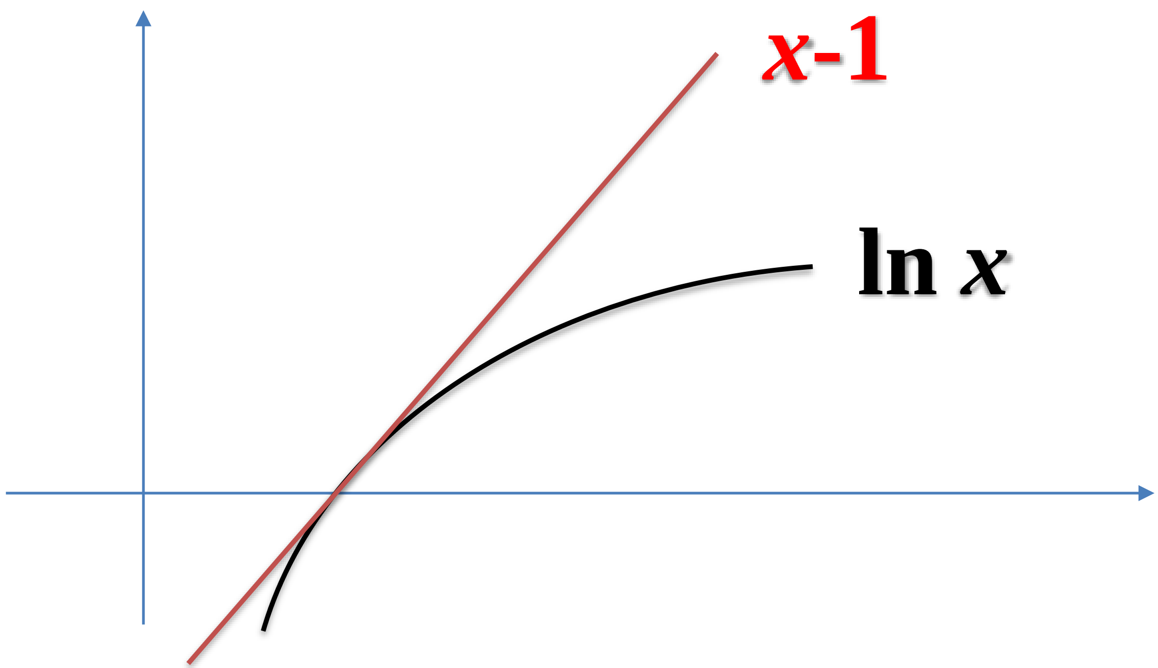
$$\binom{2n}{n-t}\Big/\binom{2n}{n}=\frac{n(n-1)\cdots(n-t+1)}{(n+t)(n+t-1)\cdots(n+1)},$$

$$\ln\left[\binom{2n}{n-t}\Big/\binom{2n}{n}\right]=\ln\left(\frac{n}{n+t}\right)+\ln\left(\frac{n-1}{n+t-1}\right)+\cdots+\ln\left(\frac{n-t+1}{n+1}\right),$$



$$\ln\left(\frac{n-k}{n+t-k}\right) \le \frac{n-k}{n+t-k} - 1 = -\frac{t}{n+t-k},$$

$$\ln\left[\binom{2n}{n-t}\bigg/\binom{2n}{n}\right] = \ln\left(\frac{n}{n+t}\right) + \ln\left(\frac{n-1}{n+t-1}\right) + \cdots + \ln\left(\frac{n-t+1}{n+1}\right),$$



$$\ln\left(\frac{n-k}{n+t-k}\right) \le \frac{n-k}{n+t-k} - 1 = -\frac{t}{n+t-k},$$

$$\ln\left[\binom{2n}{n-t} \Big/ \binom{2n}{n}\right] = \ln\left(\frac{n}{n+t}\right) + \ln\left(\frac{n-1}{n+t-1}\right) + \cdots + \ln\left(\frac{n-t+1}{n+1}\right),$$

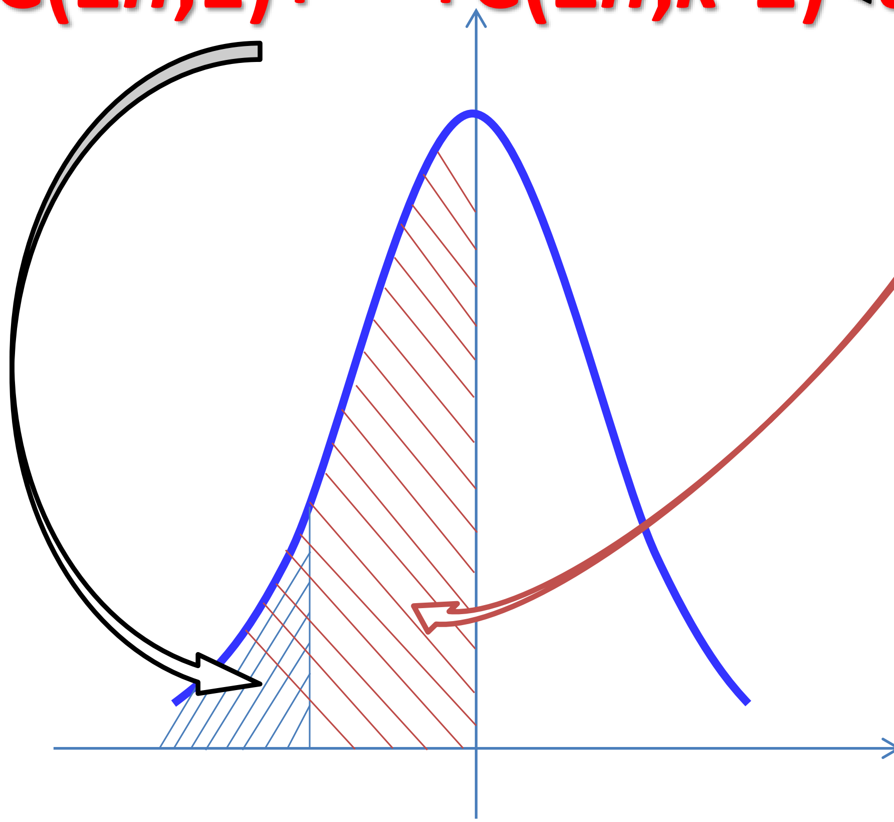
$$\le -\frac{t}{n+t} - \frac{t}{n+t-1} - \cdots - \frac{t}{n+1}$$

$$\le -\frac{t^2}{n+t}.$$

Lemma 3.8.2

Let $0 \le k \le n$ and $c = C(2n,k)/C(2n,n)$. Then

$C(2n,0)+C(2n,1)+\cdots+C(2n,k-1) < c2^{2n-1}$.



Lemma 3.8.2

Let $0 \le k \le n$ and $c = C(2n,k)/C(2n,n)$. Then

$C(2n,0)+C(2n,1)+\cdots+C(2n,k-1) < c2^{2n-1}$.

Proof. Note that $C(2n,k) = cC(2n,n)$.

$\Rightarrow C(2n,k-1) < cC(2n,n-1)$,

$\Rightarrow C(2n,k-i) < cC(2n,n-i)$ for every $i>0$.

Summing up gives

$C(2n,0)+C(2n,1)+\cdots+C(2n,k-1) < c2^{2n-1}$.

The Law of Large Numbers

Toss a coin n times and denote by X the number of heads. We say $X \sim B(n,1/2)$, $P(X=k)=C(n,k)/2^n$ for $k \in [0, n]$.

Theorem 5.3.1(Bernoulli) **For all $\varepsilon>0$, $P(|X/n-0.5|<\varepsilon) \to 1$ as $n\to\infty$.**

Theorem 5.3.2 **If $X \sim B(2n,1/2)$, then for all $t \in [0,n]$, $P(|X-n|>t) \leq \exp\{-t^2/(n+t)\}$.**