第三次作业

1. (1) $P_k = \dfrac{366^k - \frac{366!}{(366-k)!}}{366^k}$

(2) 证.

$$1-P_k = \frac{\frac{366!}{(366-k)!}}{366^k} = \frac{366}{366}\cdot\frac{365}{366}\cdot\cdots\frac{366-k+1}{366}$$

欲证 $e^{-\frac{(k-1)k}{2(n-k+1)}} < 1-P_k < e^{-\frac{(k-1)k}{2n}}$

$$\Leftarrow \quad \frac{(k-1)k}{2n} < \ln\frac{366}{366-k+1} + \ln\frac{366}{366-k+2} + \cdots + \ln\frac{366}{366} < \frac{(k-1)k}{2(n-k+1)}$$

对于 $\ln x$ 而言有 $\frac{x-1}{x} < \ln x < x-1 \ (x \neq 1)$

则 $\ln\frac{366}{366-k+1} + \cdots + \ln\frac{366}{366} < \frac{k-1}{n-k+1} + \frac{k-2}{n-k+2} + \cdots + \frac{0}{n} < \frac{1}{n-k+1}(1+2+\cdots+k-1)$

$\ln\frac{n}{n-k+1} + \cdots + \ln\frac{n}{n} > \frac{k-1}{n} + \frac{k-2}{n} + \cdots + \frac{0}{n} = \frac{k(k-1)}{2n} \qquad = \frac{k(k-1)}{2(n-k+1)}$

故原式得证

(3) 令 $P_k > \frac{1}{2} \Rightarrow \quad 1-P_k < \frac{1}{2}$

$\Rightarrow$

2. 解: (1) $\dfrac{C_{2m}^{m}}{C_{2m}^{m-t}} = \dfrac{m+t}{m} \cdot \dfrac{m+t-1}{m-1} + \cdots + \dfrac{m+1}{m-t+1}$

$$= \left(1+\frac{t}{m}\right)\cdot\left(1+\frac{t}{m-1}\right)\cdots\left(1+\frac{t}{m-t+1}\right)$$

由于 $\frac{x}{1+x} < \ln(1+x) \le x$

$$\ln\left(\frac{C_{2m}^{m}}{C_{2m}^{m-t}}\right) \ge \frac{t}{t+m} + \frac{t}{t+m-1} + \cdots + \frac{t}{t+m-t+1} \ge \frac{t^2}{t+m} \ge \frac{\ln^2 c + m\ln c + 2\ln c\sqrt{m\ln c}}{\ln c + m + \sqrt{m\ln c}}$$

$$= \ln c + \frac{\ln c\sqrt{m\ln c}}{\ln c + m + \sqrt{m\ln c}}$$

$$\ge \ln c$$

即 $\dfrac{C_{2m}^{m}}{C_{2m}^{m-t}} \ge C$

\#

(2) $\ln\left(\dfrac{C_{2m}^{m}}{C_{2m}^{m-t}}\right) \leq \dfrac{t}{m} + \dfrac{t}{m-1} + \cdots + \dfrac{t}{m-t+1} \leq \dfrac{t^2}{m-t+1} \leq \dfrac{m\ln c + \ln^2 c - 2\ln c\sqrt{m\ln c}}{m + \ln c - \sqrt{m\ln c} + 1} \leq \ln c$

则 $\dfrac{C_{2m}^{m}}{C_{2m}^{m-t}} \leq c$ #

3、(1) Markov取等条件为 $x=\lambda$或0.

Chebyshev取等条件为 X恒为一定值 或 $|X-E(x)|=\lambda$.

4. pf: 设每个$w\in\Omega$, $E_1 \cdots E_n$中恰好有$d(w)$个E_i包含w.

则 $\sum_{i=1}^{n} P(E_i) = \sum_{i=1}^{n} \sum_{w\in\Omega, w\in E_i} p_w = \sum_{w\in\Omega} d(w) p_w$

$\xlongequal[\text{分类}]{\text{按}d(w)\text{与}m\text{大小}} \sum_{\substack{w\in\Omega \\ d(w)\geq m}} d(w) p_w + \sum_{\substack{w\in\Omega \\ d(w)<m}} d(w) p_w$

$\geq \sum_{w\in\Omega, d(w)\geq m} d(w) p_w \geq m \sum_{w\in\Omega, d(w)\geq m} p_w = m P(F)$

则 $P(F) \leq \dfrac{\sum_{i=1}^{n} P(E_i)}{m}$ #

5. (1) pf: 记 $A=\{w \mid X(w) > \frac{\mu}{2}\}$, $B=\Omega\backslash A$. 则有

$E[X] = \sum_{w\in A} X(w) p_w + \sum_{w\in B} X(w) p_w \leq E(Y) + \frac{\mu}{2} P(B) \leq E[Y] + \frac{\mu}{2} = E[Y] + \frac{E[X]}{2}$

故 $E[Y] \geq \frac{1}{2} E[X]$

(2) pf: $E[Y] = \sum_{X(w)>\frac{\mu}{2}} X(w) P(x) \leq M \sum_{X(w)>\frac{\mu}{2}} P(x) = M P(X>\frac{\mu}{2})$

有 $E[Y] \geq \frac{1}{2} E[X]$

则 $\frac{1}{2} E[X] \leq M P(X>\frac{\mu}{2})$

即 $P(X>\frac{\mu}{2}) \geq \frac{1}{2M} E[X]$