Discrete Mathematics

Lecture 4

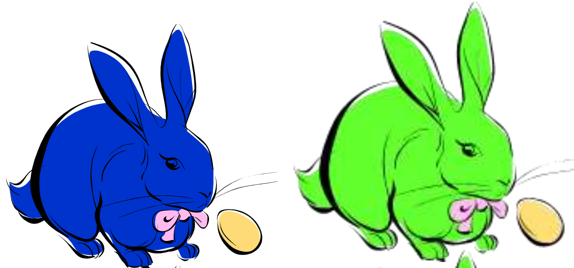
Fibonacci Numbers

Fibonacci问题

Fibonacci数列是递推关系的一典型问题，数列的本身有着许多应用.

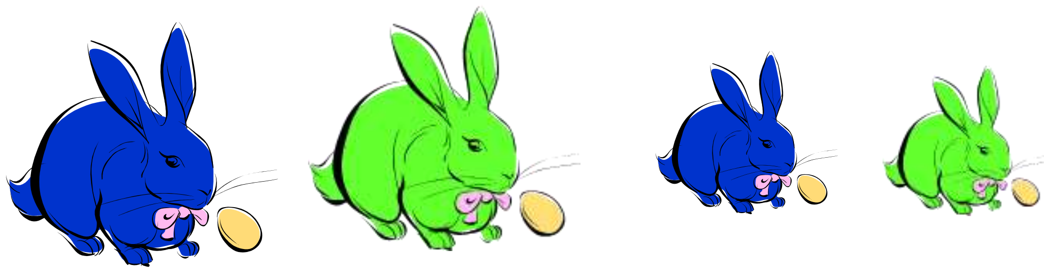
(1) 问题的提出：假定初生的一对雌雄兔子，从出生的第**2**个月之后每个月都可以生出另外一对雌雄兔. 如果第**1**个月只有一对初生的雌雄兔子，问***n***个月之后共有多少对兔子？

1月

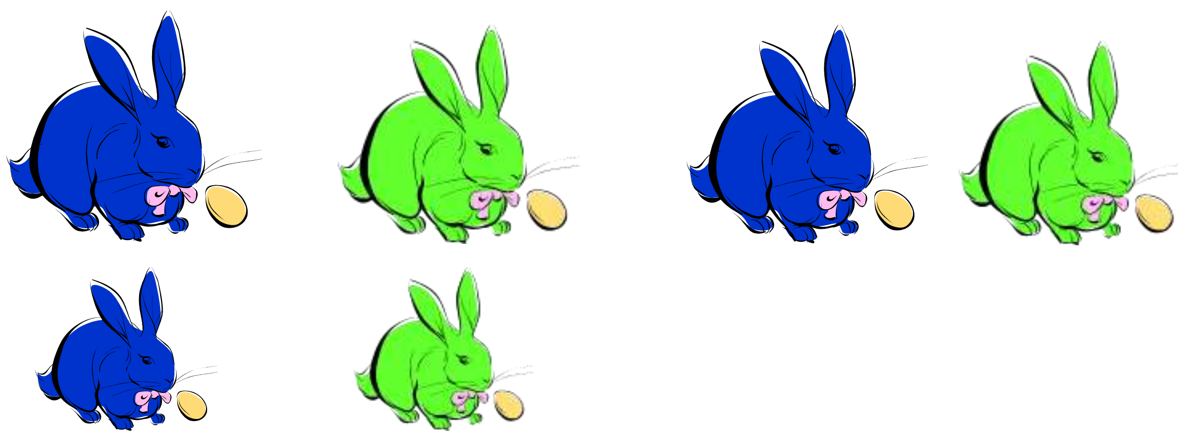


2月

3月



4月



(2) 求递推关系: 设满n个月时兔子对数为F_n,则第n-1个月留下的兔子数目为F_{n-1}对;当月新生兔数目为F_{n-2}对, 即第n-2个月的所有兔子到第n个月都有繁殖能力 ∴ $F_n=F_{n-1}+F_{n-2}$, $F_1=F_2=1$。

由递推关系式可依次得到

$F_3=F_2+F_1=2$, $F_4=F_3+F_2=3$,

$F_5=F_4+F_3=3+2=5$, …

(3) Fibonacci数列的性质

1. $F_1+F_2+\cdots+F_n=F_{n+2}-1$;
2. $F_1+F_3+F_5+\cdots+F_{2n-1}=F_{2n}$;
3. $F_0-F_1+F_2-F_3+\cdots-F_{2n-1}+F_{2n}=F_{2n-1}-1$;
4. $F_1^2+F_2^2+\cdots+F_n^2=F_nF_{n+1}$;
5. $F_{n-1}F_{n+1}-F_n^2=(-1)^n$;
6. $F_n^2+F_{n-1}^2=F_{2n-1}$;
7. $F_{n+1}F_n+F_nF_{n-1}=F_{2n}$.

1. $F_1+F_2+\cdots+F_n=F_{n+2}-1;$

$$F_n=F_{n+2}-F_{n+1}$$
$$F_{n-1}=F_{n+1}-F_n$$
$$\ldots$$
$$F_1=F_3-F_2$$

1. $F_1+F_2+\cdots+F_n=F_{n+2}-1$;
2. $F_1+F_3+F_5+\cdots+F_{2n-1}=F_{2n}$;

$$F_{2n-1}=F_{2n}-F_{2n-2}$$
$$F_{2n-3}=F_{2n-2}-F_{2n-4}$$
$$\ldots$$
$$F_1=F_2-F_0$$

1. $F_1+F_2+\cdots+F_n=F_{n+2}-1$;
2. $F_1+F_3+F_5+\cdots+F_{2n-1}=F_{2n}$;
3. $F_0-F_1+F_2-F_3+\cdots-F_{2n-1}+F_{2n}=F_{2n-1}-1$;

By (1), $F_1+F_2+\cdots+F_{2n}=F_{2n+2}-1$ (*);
then (3) follows by (*)-2×(2).

1. $F_1+F_2+\cdots+F_n=F_{n+2}-1;$
2. $F_1+F_3+F_5+\cdots+F_{2n-1}=F_{2n};$
3. $F_0-F_1+F_2-F_3+\cdots-F_{2n-1}+F_{2n}=F_{2n-1}-1;$
4. $F_1^2+F_2^2+\cdots+F_n^2=F_nF_{n+1};$

$$F_n^2=F_n(F_{n+1}-F_{n-1})=F_nF_{n+1}-F_nF_{n-1}$$
$$F_{n-1}^2=F_{n-1}F_n-F_{n-1}F_{n-2}$$
$$\ldots$$
$$F_1^2=F_1F_2-F_1F_0$$

1. $F_1+F_2+\cdots+F_n=F_{n+2}-1$;
2. $F_1+F_3+F_5+\cdots+F_{2n-1}=F_{2n}$;
3. $F_0-F_1+F_2-F_3+\cdots-F_{2n-1}+F_{2n}=F_{2n-1}-1$;
4. $F_1^2+F_2^2+\cdots+F_n^2=F_nF_{n+1}$;
5. $F_{n-1}F_{n+1}-F_n^2=(-1)^n$;

   Use induction on *n*.

$$\begin{aligned} F_{n-1}F_{n+1}-F_n^2 &= F_{n-1}(F_{n-1}+F_n)-F_n^2 \\ &= F_{n-1}^2+F_{n-1}F_n-F_n^2 \\ &= F_{n-1}^2+(F_{n-1}-F_n)F_n \\ &= F_{n-1}^2-F_{n-2}F_n=(-1)^n. \end{aligned}$$

- $F_n^2+F_{n-1}^2=F_{2n-1}$;
- $F_{n+1}F_n+F_nF_{n-1}=F_{2n}$.

Also use induction on *n*.

$$F_{n+1}F_n+F_nF_{n-1}=(F_n+F_{n-1})F_n+(F_{n-1}+F_{n-2})F_{n-1}$$

$$=F_n^2+F_{n-1}^2+F_nF_{n-1}+F_{n-1}F_{n-2}$$

$$=F_{2n-1}+F_{2n-2}=F_{2n}.$$

- $F_n^2+F_{n-1}^2=F_{2n-1}$;
- $F_{n+1}F_n+F_nF_{n-1}=F_{2n}$.

$$
\begin{aligned}
F_n^2+F_{n-1}^2 &= (F_{n-1}+F_{n-2})^2+F_{n-1}^2 \\
&= F_{n-1}^2+F_{n-2}^2+2F_{n-1}F_{n-2}+F_{n-1}^2 \\
&= F_{2n-3}+F_{n-1}(F_{n-2}+F_{n-1})+F_{n-1}F_{n-2} \\
&= F_{2n-3}+F_nF_{n-1}+F_{n-1}F_{n-2} \\
&= F_{2n-3}+F_{2n-2}=F_{2n-1}.
\end{aligned}
$$

n	$F_n^2+F_{n-1}^2=F_{2n-1}$	$F_{n+1}F_n+F_nF_{n-1}=F_{2n}$
1	√	√
2	√	√
3	√	

n	$F_n^2+F_{n-1}^2=F_{2n-1}$	$F_{n+1}F_n+F_nF_{n-1}=F_{2n}$
1	√	√
2	√	√
3	√	√
4	√	√
5	√	√
6	√	√
…	…	…

Modified Fibonacci

$E_0=A$, $E_1=B$, $E_{n+1}=E_n+E_{n-1}$.

$E_2=A+B$, $E_3=B+(A+B)=A+2B$,

$E_4=A+B+(A+2B)=2A+3B$,

$E_5=A+2B+(2A+3B)=3A+5B$,

$E_6=2A+3B+(3A+5B)=5A+8B$,

$E_7=3A+5B+(5A+8B)=8A+13B$, …

$\Rightarrow E_n=F_{n-1}A+F_nB$,

$E_n = F_{n-1}A + F_nB$.

If $A = F_a$ and $B = F_{a+1}$, then

$$F_{a+b+1}\ (=E_{b+1}) = F_{a+1}F_{b+1} + F_aF_b.$$

Corollary

- $F_n^2 + F_{n-1}^2 = F_{2n-1}$;
- $F_{n+1}F_n + F_nF_{n-1} = F_{2n}$.

$F_n = ?$

- $F_1 = F_2 < F_3 < F_4 < \cdots < F_n < \cdots$
- $2F_{n-2} < F_{n-2} + F_{n-1} = F_n < 2F_{n-1}$,
- $2^{n/2} < F_n < 2^n$ for $n > 6$.

So the Fibonacci numbers grow exponentially,

but how exactly?

1/1=1, 2/1=2, 3/2=1.5, 5/3=1.666666667,

8/5=1.600000000, 13/8=1.625000000,

21/13=1.615384615, 34/21=1.619047619,

55/34=1.617647059, 89/55=1.618181818,

144/89=1.617977528, 233/144=1.618055556,

377/233=1.618025751, …

Let $G_n = cq^n$ satisfy $G_{n+1} = G_n + G_{n-1}$.

$$cq^{n+1} = cq^n + cq^{n-1},$$

$$q^2 = q + 1.$$

$\Rightarrow q_1 = (1+\sqrt{5})/2 \approx 1.618034$ (golden ratio),

$q_2 = (1-\sqrt{5})/2 \approx -0.618034.$

Let $F_n = Aq_1^n + Bq_2^n$. Then

$F_0 = A + B = 0,$

$F_1 = Aq_1 + Bq_2 = 1.$

$\Rightarrow A = 1/\sqrt{5}$, $B = -1/\sqrt{5}$. $\Rightarrow$

$$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$$

$$\approx \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n.$$

$$\frac{F_n}{F_{n-1}} \approx \frac{1+\sqrt{5}}{2} \approx 1.618.$$

$$F_n \equiv \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$$

$$\approx \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n.$$

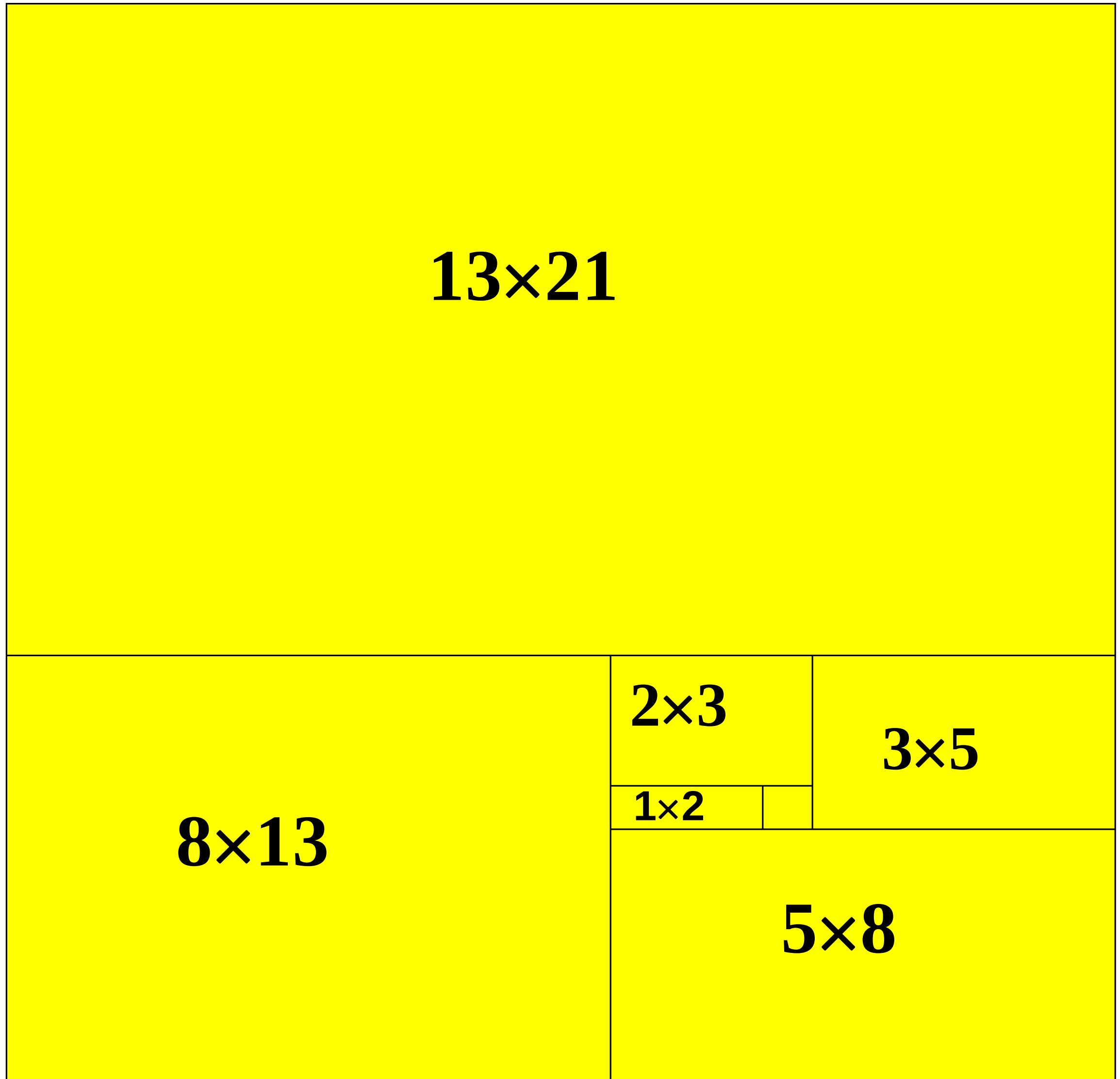
Fibonacci数列还有下面有趣的性质：

(a) Fibonacci数列可以作为表示任意正整数N的“**基**”：

$$N = \sum_{i=2}^{n} a_i F_i,$$

其中a_i=0, 1，而且$a_i a_{i+1}$=0。但是n的值不太容易决定，例如: 11=F_6+F_4=8+3，用**Fibonacci** 数列表示为一个5位数: **10100**

(b) 有所谓的**Fibonacci**方形，即边长为F_n的正方形，可以分解为若干形如$F_{i+1}\times F_i$的**Fibonacci**矩形的“和”。后面是边长为$F_8=21$的正方形的分解。


13×21
2×3
3×5
1×2
8×13
5×8

References

Arthur Benjamin: The magic of Fibonacci numbers #TED : http://www.ted.com/talks/arthur_benjamin_the_magic_of_fibonacci_numbers