定积分的计算例题

这部分例题主要是定积分的换元积分法和分部积分法的应用

注意：即使一个函数存在原函数，则f(x)也未必可积分：例如

$$F(x)=\begin{cases} x^2\sin\frac{1}{x^2}, x\neq 0 \\ 0, x=0 \end{cases}$$

是

$$f(x)=\begin{cases} 2x\sin\frac{1}{x^2}-\frac{2}{x}\cos\frac{1}{x^2}, x\neq 0 \\ 0, x=0 \end{cases}$$

的原函数。但是f在$x=0$无界。因此在任何包含0的区域不可积分。（这属于广义积分。）

二、定积分的换元积分法

定理1：（定积分的换元积分法）

设函数 $f(x)\in C[a,\ b]$,作变换 $x=\varphi(t)$,

满足三个条件：

(1) $\varphi(t)\in C^1[\alpha,\ \beta]$;

(2) $a\le\varphi(t)\le b$;

(3) $\varphi(\alpha)=a,\ \varphi(\beta)=b$,

则有 $\int_a^b f(x)dx=\int_\alpha^\beta f[\varphi(t)]\varphi'(t)dt$

Ex.判断正误

$$\int_{-1}^{1}\frac{1}{1+x^2}dx=\arctan x\Big|_{-1}^{1}=\frac{\pi}{2} \qquad (\sqrt{})$$

$$\int_{-1}^{1}\frac{1}{1+x^2}dx=-\int_{-1}^{1}\frac{1}{1+\frac{1}{x^2}}d\frac{1}{x}=-\arctan\frac{1}{x}\Big|_{-1}^{1}=-\frac{\pi}{2}. \qquad (\times)$$

$$\left(-\arctan\frac{1}{x}\Big|\text{在}x=0\text{处不连续};\ x=\frac{1}{t}\text{在}t=0\text{处不连续}\right)$$

Ex. $f \in C[-a,a]$为偶函数,则$\int_{-a}^{a}\frac{f(x)}{1+e^x}dx=\int_0^a f(x)dx.$

Proof. $$\int_{-a}^{a}\frac{f(x)}{1+e^x}dx=\int_0^a\frac{f(x)}{1+e^x}dx+\int_{-a}^{0}\frac{f(x)}{1+e^x}dx$$

$$=\int_0^a\frac{f(x)}{1+e^x}dx-\int_a^0\frac{1}{1+e^{-t}}f(-t)d(t)\qquad (x=-t)$$

$$=\int_0^a\frac{f(x)}{1+e^x}dx+\int_0^a\frac{e^t}{1+e^t}f(t)dt\qquad (f\text{偶})$$

$$=\int_0^a\frac{f(x)}{1+e^x}dx+\int_0^a\frac{e^x}{1+e^x}f(x)dx=\int_0^a f(x)dx.$$

Ex. $f \in C[0,a]$, $f(x)+f(a-x)\neq 0$, 则 $\int_0^a \frac{f(x)}{f(x)+f(a-x)}dx=\frac{a}{2}$.

Proof. $I=\int_0^a \frac{f(x)}{f(x)+f(a-x)}dx$

(令 $x=a-t$)

$$=-\int_a^0 \frac{f(a-t)}{f(a-t)+f(t)}dt=\int_0^a \frac{f(a-t)}{f(a-t)+f(t)}dt$$

(令 $t=x$)

$$=\int_0^a \frac{f(a-x)}{f(a-x)+f(x)}dx$$

$$2I=\int_0^a\left(\frac{f(x)}{f(x)+f(a-x)}+\frac{f(a-x)}{f(a-x)+f(x)}\right)dx=a.$$ W

Ex. $f \in C[1,+\infty), a>1,$ 则 $\int_1^a f(x^2+\frac{a^2}{x^2})\frac{dx}{x}=\int_1^a f(x+\frac{a^2}{x})\frac{dx}{x}.$

Proof. $\int_1^a f(x^2+\frac{a^2}{x^2})\frac{dx}{x}=\frac{1}{2}\int_1^{a^2} f(t+\frac{a^2}{t})\frac{dt}{t}$ $(t=x^2)$

$$=\frac{1}{2}\int_1^a f(t+\frac{a^2}{t})\frac{dt}{t}+\frac{1}{2}\int_a^{a^2} f(t+\frac{a^2}{t})\frac{dt}{t}$$

$$@\frac{1}{2}(I_1+I_2).$$

$$I_2=-\int_a^1 f(s+\frac{a^2}{s})\frac{ds}{s}=\int_1^a f(s+\frac{a^2}{s})\frac{ds}{s}$$ $(s=\frac{a^2}{t})$ W

Ex.(1) $f \in C[a,b]$, 则 $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$;

(2) $I = \int_{\pi/6}^{\pi/3} \frac{\cos^2 x}{x(\pi-2x)}dx = \frac{1}{\pi}\ln 2.$

Proof.(1) $\int_a^b f(a+b-x)dx$

$$\underline{\underline{t=a+b-x}} -\int_b^a f(t)dt = \int_a^b f(t)dt = \int_a^b f(x)dx.$$

(2)利用(1), $I = \int_{\pi/6}^{\pi/3} \frac{\sin^2 x}{x(\pi-2x)}dx = \frac{1}{2}\int_{\pi/6}^{\pi/3} \frac{1}{x(\pi-2x)}dx$

$$= \frac{1}{\pi}\int_{\pi/6}^{\pi/3}\left(\frac{1}{2x}+\frac{1}{\pi-2x}\right)dx = \frac{1}{2\pi}\ln\frac{2x}{\pi-2x}\Bigg|_{\pi/6}^{\pi/3} = \frac{1}{\pi}\ln 2.$$

Ex. $I=\int_0^1 \frac{\ln(1+x)}{1+x^2}dx.$

解: $I=\int_0^{\pi/4}\ln(1+\tan t)dt \qquad (t=\arctan x)$

$$=\int_0^{\pi/4}\ln(\sin t+\cos t)dt-\int_0^{\pi/4}\ln(\cos t)dt=I_1-I_2.$$

$$I_1=\int_0^{\pi/4}\left(\ln\sqrt{2}+\ln\sin(t+\frac{\pi}{4})\right)dt$$

$$=\frac{\pi}{8}\ln 2+\int_0^{\pi/4}\ln\cos(\frac{\pi}{4}-t)dt=\frac{\pi}{8}\ln 2+I_2.$$

$$I=\frac{\pi}{8}\ln 2.$$ W

三、定积分的分部积分法

定理2：（定积分的分部积分法）

设函数$u(x), v(x)$在区间$[a, b]$上有连续的一阶导数$u'(x), v'(x)$，则有分部积分公式

$$\int_a^b u(x)\cdot v'(x)dx$$

$$= u(x)\cdot v(x)\Big|_a^b - \int_a^b v(x)\cdot u'(x)dx$$

Ex. 证明$I_n \triangleq \int_0^{\pi/2} \sin^n x dx = \int_0^{\pi/2} \cos^n x dx$, 并求$I_n$.

Proof. 令$t = \frac{\pi}{2} - x$, 则

$$\int_0^{\pi/2} \sin^n x dx = -\int_{\pi/2}^{0} \sin^n (\frac{\pi}{2} - t) dt = \int_0^{\pi/2} \cos^n t dt.$$

$$I_n = -\int_0^{\pi/2} \sin^{n-1} x d\cos x$$

$$= -\sin^{n-1} x \cos x \Big|_0^{\pi/2} + (n-1)\int_0^{\pi/2} \cos^2 x \sin^{n-2} x dx$$

$$= (n-1)\int_0^{\pi/2} (1-\sin^2 x)\sin^{n-2} x dx$$

$$= (n-1)I_{n-2} - (n-1)I_n.$$

$$I_n = \frac{n-1}{n} I_{n-2}.$$

$$I_0 = \int_0^{\pi/2} dx = \frac{\pi}{2},$$

$$I_1 = \int_0^{\pi/2} \sin x dx = -\cos x \Big|_0^{\pi/2} = 1,$$

$$I_{2n} = \frac{2n-1}{2n} \cdot \frac{2n-3}{2n-2} \cdot \cdots \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \frac{(2n-1)!!}{(2n)!!} \cdot \frac{\pi}{2},$$

$$I_{2n-1} = \frac{2n-2}{2n-1} \cdot \frac{2n-4}{2n-3} \cdot \cdots \cdot \frac{2}{3} \cdot 1 = \frac{(2n-2)!!}{(2n-1)!!}.$$ W

Ex. $f,g\in C[a,b], \int_a^x f(x)dx \ge \int_a^x g(x)dx\ (a\le x\le b),$

$$\int_a^b f(x)dx=\int_a^b g(x)dx, \text{则} \int_a^b xf(x)dx \le \int_a^b xg(x)dx.$$

Proof. 令$F(x)=\int_a^x f(x)dx, G(x)=\int_a^x g(x)dx,$则

$$F(a)=G(a)=0, F(b)=G(b), F(x)\ge G(x)(a\le x\le b).$$

$$\int_a^b x(f(x)-g(x))dx=\int_a^b x\mathrm{d}\left(F(x)-G(x)\right)$$

$$=x(F(x)-G(x))\Big|_a^b-\int_a^b(F(x)-G(x))\mathrm{d}x$$

$$=-\int_a^b(F(x)-G(x))\mathrm{d}x\le 0.$$

Ex. $f\in C^1[a,b],\ f(a)=0$,则

$$\int_a^b f^2(x)dx\le\frac{(b-a)^2}{2}\int_a^b\left(f'(x)\right)^2dx-\frac{1}{2}\int_a^b(x-a)^2\left(f'(x)\right)^2dx.$$

Proof. $f^2(x)=\left(\int_a^x 1\cdot f'(t)dt\right)^2\le(x-a)\int_a^x\left(f'(x)\right)^2dx$

$$\int_a^b f^2(x)dx\le\int_a^b\left(\int_a^x\left(f'(t)\right)^2dt\right)d\frac{(x-a)^2}{2}$$

$$=\frac{(x-a)^2}{2}\int_a^x\left(f'(t)\right)^2dt\Big|_{x=a}^{b}-\int_a^b\frac{(x-a)^2}{2}\left(f'(x)\right)^2dx$$

$$=\frac{(b-a)^2}{2}\int_a^b\left(f'(x)\right)^2dx-\frac{1}{2}\int_a^b(x-a)^2\left(f'(x)\right)^2dx.$$ W

Ex. $f\in C^1[a,b],\ f(a)=f(b)=0,\ \int_a^b f^2(x)dx=1,$ 则

$$\int_a^b \left(f'(x)\right)^2 dx\cdot\int_a^b x^2 f^2(x)dx>\frac{1}{4}.$$

Proof. 若$f'(x)+\lambda xf(x)\equiv 0,$ 则

$$\left(f(x)e^{\frac{1}{2}\lambda x^2}\right)'=(f'(x)+\lambda xf(x))e^{\frac{1}{2}\lambda x^2}\equiv 0,$$

$$f(x)e^{\frac{1}{2}\lambda x^2}\equiv C,\qquad f(x)=Ce^{-\frac{1}{2}\lambda x^2},$$

$f(a)=f(b)=0,$ 则$C=0,\ f(x)\equiv 0,$ 与$\int_a^b f^2(x)dx=1$矛盾.

故$\forall\lambda\in\text{¡}\ ,\int_a^b\left(f'(x)+\lambda xf(x)\right)^2 dx>0,$ 即

$$\int_a^b \left(f'(x)\right)^2 dx + 2\lambda\int_a^b xf(x)f'(x)dx + \lambda^2\int_a^b x^2 f^2(x)dx > 0.$$

于是$\int_a^b \left(f'(x)\right)^2 dx \cdot \int_a^b x^2 f^2(x)dx > \left(\int_a^b xf(x)f'(x)dx\right)^2.$

而$\int_a^b xf(x)f'(x)dx = \frac{1}{2}\int_a^b x\,\mathrm{d}f^2(x)$

$$= \frac{1}{2}xf^2(x)\Big|_a^b - \frac{1}{2}\int_a^b f^2(x)\,\mathrm{d}x = -\frac{1}{2}$$

故$\int_a^b \left(f'(x)\right)^2 dx \cdot \int_a^b x^2 f^2(x)dx > \frac{1}{4}$.W

Ex. $\left(\dfrac{2n-1}{e}\right)^{\frac{2n-1}{2}} < 1\cdot 3\cdot 5\mathrm{L}\ (2n-1) < \left(\dfrac{2n+1}{e}\right)^{\frac{2n+1}{2}}.$

Proof. $S_n\ @1\cdot 3\cdot 5\mathrm{L}\ (2n-1),$

$$2\ln S_n = 2\sum_{k=2}^{n}\ln(2k-1) < \sum_{k=2}^{n}\int_{2k-1}^{2k+1}\ln x dx$$

$$= \int_3^{2n+1}\ln x dx = x\ln x\Big|_3^{2n+1} - \int_3^{2n+1} x\cdot\frac{1}{x}dx$$

$$= (x\ln x - x)\Big|_3^{2n+1} = x\ln\frac{x}{e}\Bigg|_3^{2n+1} < (2n+1)\ln\frac{2n+1}{e}.$$

同理, $2\ln S_n > \sum_{k=2}^{n}\int_{2k-3}^{2k-1}\ln x dx = \int_1^{2n-1}\ln x dx = (2n-1)\ln\dfrac{2n-1}{e} + 1.$W

Thm.(带积分余项的Taylor公式) $f \in C^{n+1}[a,b], x_0 \in [a,b]$,

则$\forall x \in [a,b]$,有

$$f(x)=\sum_{k=0}^{n}\frac{f^{(k)}(x_0)}{k!}(x-x_0)^k+\frac{1}{n!}\int_{x_0}^{x}(x-t)^n f^{(n+1)}(t)dt.$$

Proof. $n=0$时, 即Newton-Leibniz公式.

假设$n=m-1$时,定理成立, 即

$$f(x)=\sum_{k=0}^{m-1}\frac{f^{(k)}(x_0)}{k!}(x-x_0)^k+\frac{1}{(m-1)!}\int_{x_0}^{x}(x-t)^{m-1} f^{(m)}(t)dt.$$

对余项分部积分,得

$$\frac{1}{(m-1)!}\int_{x_0}^{x}(x-t)^{m-1}f^{(m)}(t)dt=\frac{-1}{m!}\int_{x_0}^{x}f^{(m)}(t)d(x-t)^m$$

$$=-\frac{1}{m!}f^{(m)}(t)(x-t)^m\Big|_{t=x_0}^{x}+\frac{1}{m!}\int_{x_0}^{x}(x-t)^m f^{(m+1)}(t)dt$$

$$=\frac{1}{m!}f^{(m)}(x_0)(x-x_0)^m+\frac{1}{m!}\int_{x_0}^{x}(x-t)^m f^{(m+1)}(t)dt.$$

故 $f(x)=\sum_{k=0}^{m}\frac{f^{(k)}(x_0)}{k!}(x-x_0)^k+\frac{1}{m!}\int_{x_0}^{x}(x-t)^m f^{(m+1)}(t)dt,$

即$n=m$时,定理成立.W