线性代数小测验-I

考试课程　　线性代数　　　　　　　2013 年 10 月 28 日

姓名____________　　　　学号____________

一、 (20分) 求解下列方程组 $\begin{cases} x-y+z+w &= 5 \\ y-z+2w &= 8 \\ 2x-y-3z+4w &= 18. \end{cases}$

二、 (20分) 设A是一个$m\times n$阶矩阵，$N(A)$是A的零空间(null space). 证明：

(a) 若$A^TA=0$,则$A=0$.

(b) $N(A)=N(A^TA)$.

三、 (10分) 假设A是一个4阶矩阵，B是一个4×3的矩阵, C是一个3×4的矩阵满足$A=BC$. 证明A是不可逆的(not invertible). 反之，若A是一个4阶不可逆矩阵，则存在一个4×3的矩阵B和一个3×4的矩阵C使得$A=BC$.

四、 (10分)是否存在3阶矩阵A满足A的列空间(column space)$C(A)$和零空间(Null space)$N(A)$重合, 即$C(A)=N(A)$. 如果A是一个6阶矩阵呢？如果存在，举例说明。否则，解释原因。

五、 (10分) 设$A=I_3-2\alpha\alpha^T$，其中$\alpha=(x_1,x_2,x_3)^T$，且$\alpha^T\alpha=1$，证明A可逆并求A的逆. 令$\alpha=(0,0,1)^T$,定义一个映射$f:\mathbb{R}^3\to\mathbb{R}^3$满足：$\forall x\in\mathbb{R}^3, f(x)=Ax$. 试解释这个映射的几何含义。

六、 (10分) 设$A=\begin{pmatrix} 1 & 1 & 1 \\ 1 & 3 & 1 \end{pmatrix}$.

(a) 求所有3×2的矩阵X使得$AX=0$.

(b) 找一个3×2矩阵X_0，满足$AX_0=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

(c) 求所有3×2的矩阵X使得$AX=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

七、 (10分) 求矩阵$A=\begin{pmatrix} 0 & 0 & 1 & 2 \\ 0 & 0 & 3 & 0 \\ 1 & -1 & 0 & 1 \\ 2 & 0 & -1 & 3 \end{pmatrix}$的$PLU$分解。

八、 (10分) 设$A=\begin{pmatrix} 0 & 1 & 2 & 2 \\ 0 & 3 & 8 & 7 \\ 0 & 0 & 4 & 2 \end{pmatrix}$. 求分块矩阵(block matrix)$B=\begin{pmatrix} A & A \\ A & A \end{pmatrix}$的简化行阶梯型(reduced row echelon form).

一、The reduced row echelon form

$$(A|b) = \begin{pmatrix} 1 & 0 & 0 & 3 & 13 \\ 0 & 1 & 0 & 2 & 8 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 13 \\ 8 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} -3 \\ -2 \\ 0 \\ 1 \end{pmatrix}.$$

二、(a) Assume that $A = (a_{ij})_{m\times n}$ and $A^T A = (b_{ij})_{n\times n}$. Then by definition, $b_{ii} = a_{1i}^2 + \cdots + a_{ni}^2$. Hence, $A^T A = 0$ implies $A = 0$. (b)It is clear that $N(A) \subseteq N(A^T A)$. Conversely, if $A^T Ax = 0$, then $x^T A^T Ax = (Ax)^T (Ax) = 0$, by (a), $Ax = 0$.

三、$N(C) \subseteq N(A)$. Since $rank(C) \le 3$, $N(C) \ne \{0\}$.

四、$n = 6$, $A = \begin{pmatrix} 0 & I_3 \\ 0 & 0 \end{pmatrix}$

五、$A^2 = (I_n - 2\alpha\alpha^T)(I_n - 2\alpha\alpha^T) = I_n^2 - 4\alpha\alpha^T + 4\alpha(\alpha^T\alpha)\alpha^T = I_n$. A: reflection matrix with respect to the mirror $\alpha^T x = 0$. If β is parallel to α, then $A\beta = -\beta$. If β is perpendicular to α (i.e., β lies in the plane with normal vector α), then $A\beta = \beta$.

六、(a)Each column of X is the linear combination of vectors in $N(A)$.（详细答案见下页）

七、The answer is not unique.

$$P = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 3 & 1 \end{pmatrix}, U = \begin{pmatrix} 1 & -1 & 0 & 1 \\ 0 & 2 & -1 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & -6 \end{pmatrix}.$$

八、

$$RREF(B) = \begin{pmatrix} RREF(A) & RREF(A) \\ 0 & 0 \end{pmatrix}$$

(a) Let $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ be a column of X. Then x, y and z satisfy

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0.$$

We apply elimination to get

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

from which we deduce that $y = 0$, and $x = -z$. So each column of X is a multiple of the vector $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$. Since there are two columns of X, X can be written as

$$\begin{bmatrix} a & b \\ 0 & 0 \\ -a & -b \end{bmatrix}.$$

The basis for this space of matrices is given by

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & -1 \end{bmatrix}.$$

(b) We first solve

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Elimination gives We apply elimination to get

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

From which we see that we can take $y = -1/2, x = 3/2, z = 0$. This will be the first column of X.

We now solve

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Elimination gives We apply elimination to get

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

And we can take $y = 1/2, x = -1/2, z = 0$. This is the second column of X.

So one possible solution for X is

$$\begin{bmatrix} 3/2 & -1/2 \\ -1/2 & 1/2 \\ 0 & 0 \end{bmatrix}.$$

(c) The set of complete solutions is given by

$$X_{\text{particular}} + X_{\text{special}} = \begin{bmatrix} 3/2 & -1/2 \\ -1/2 & 1/2 \\ 0 & 0 \end{bmatrix} + a \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ -1 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & -1 \end{bmatrix}.$$