1. 证：$n=1$时，$(x_1+\cdots+x_k)^n = x_1+\cdots+x_k = \sum_{i=1}^{k} x_i$ 成立，符合题意

当n满足 $(x_1+\cdots+x_k)^n = \sum_{\alpha_1,\cdots,\alpha_k\in\mathbb{Z}\geq 0,\alpha_1+\cdots+\alpha_k=n}\binom{n}{\alpha_1,\cdots,\alpha_k}x_1^{\alpha_1}\cdots x_k^{\alpha_k}$

则 $(x_1+\cdots+x_k)^{n+1} = \sum_{i=1}^{k} x_i \sum_{\alpha_1\cdots\alpha_k\in\mathbb{Z}\geq 0,\alpha_1+\cdots+\alpha_k=n}\binom{n}{\alpha_1,\cdots,\alpha_k}x_1^{\alpha_1}\cdots x_k^{\alpha_k}$

$$= \sum_{\alpha_1\cdots\alpha_k\in\mathbb{Z}\geq 0,\alpha_1+\cdots+\alpha_k=n}\binom{n}{\alpha_1,\cdots,\alpha_k}x_1^{\alpha_1+1}\cdots x_k^{\alpha_k}+\cdots+\sum_{\alpha_1\cdots\alpha_k\in\mathbb{Z}\geq 0,\alpha_1+\cdots+\alpha_k=n}\binom{n}{\alpha_1,\alpha_2,\cdots,\alpha_k}x_1^{\alpha_1}\cdots x_k^{\alpha_k+1}$$

令 $n+1=P$。

$$=\sum_{i=1}^{k}\sum_{\lambda_1\cdots\lambda_k\in\mathbb{Z}\geq 0,\lambda_i\geq 1,\lambda_1+\lambda_2+\cdots+\lambda_k=n+1}\binom{n}{\lambda_1,\lambda_2\cdots,\lambda_i-1,\cdots,\lambda_k}x_1^{\lambda_1}\cdots x_i^{\lambda_i}\cdots x_k^{\lambda_k}$$

$$=\sum_{\lambda_1\cdots\lambda_k\in\mathbb{Z}\geq 0,\lambda_1+\cdots+\lambda_k=n+1}\ \sum_{i=1,\lambda_i\geq 1}^{k}\binom{n}{\lambda_1,\cdots,\lambda_i-1,\cdots,\lambda_k}x_1^{\lambda_1}\cdots x_i^{\lambda_i}\cdots x_k^{\lambda_k}$$

$$=\sum_{\lambda_1\cdots\lambda_k\in\mathbb{Z}\geq 0,\lambda_1+\cdots+\lambda_k=n+1}\binom{n+1}{\lambda_1,\cdots,\lambda_k}x_1^{\lambda_1}\cdots x_k^{\lambda_k}$$

也即 $n+1$也成立. 得证

2. 设 $A_i=\{x \mid 1\leq x\leq n$ 且 x是p_i倍数$\}$，则有 $|A_i|=[\frac{n}{p_i}]$

对于 $|A_{i_1}\cap A_{i_2}\cap\cdots\cap A_{i_r}| = [\frac{n}{p_{i_1}p_{i_2}\cdots p_{i_r}}]$

则 $|A_1\cup A_2\cup A_3\cup\cdots\cup A_k| = \sum_{r=1}^{k}(-1)^{r-1}\sum_{1\leq i_1<\cdots<i_r\leq k}|A_{i_1}\cap\cdots\cap A_{i_r}| = \sum_{r=1}^{k}(-1)^{r-1}\sum_{1\leq i_1<\cdots<i_r\leq k}[\frac{n}{p_{i_1}p_{i_2}\cdots p_{i_r}}]$

故都互素的个数为 $n-\sum_{r=1}^{k}(-1)^{r-1}\sum_{1\leq i_1<\cdots<i_r\leq k}[\frac{n}{p_{i_1}p_{i_2}\cdots p_{i_r}}]$

3. (1) Pf: $n=1$时，$P(A_1)=P(A_1)$ 显然成立，$n=2$时，$P(A_1\cup A_2)=P(A_1)+P(A_2)-P(A_1\cap A_2)$，也成立

设n时成立，下证$n+1$也成立.

令 $A_1=B_1$, $A_2=B_2$, $\cdots$, $A_{n-1}=B_{n-1}$, $A_n\cup A_{n+1}=B_n$

则 $P(A_1\cup\cdots\cup A_n\cup A_{n+1}) = \sum_{k=1}^{n}(-1)^{k-1}\sum_{1\leq i_1<\cdots<i_k\leq n}P(B_{i_1}\cap\cdots\cap B_{i_k})$

$$=\sum_{k=1}^{n-1}(-1)^{k-1}\sum_{1\leq i_1<\cdots<i_k<n}P(A_{i_1}\cap\cdots\cap A_{i_k})+\sum_{k=1}^{n}(-1)^{k-1}\sum_{1\leq i_1<\cdots<i_{k-1}<n,\ i_k=n}P(A_{i_1}\cap\cdots\cap A_{i_{k-1}}\cap(A_n\cup A_{n+1}))$$

$$=\sum_{k=1}^{n-1}(-1)^{k-1}\sum_{1\leq i_1<\cdots<i_k<n}P(A_{i_1}\cap\cdots\cap A_{i_k})+\sum_{k=1}^{n}(-1)^{k-1}\Big[\sum_{1\leq i_1<\cdots<i_{k-1}<n}P(A_{i_1}\cap\cdots\cap A_{i_{k-1}}\cap A_n)+\sum_{1\leq i_1<\cdots<i_{k-1}<n}P(A_{i_1}\cap\cdots\cap A_{i_{k-1}}\cap A_{n+1})$$

$$-\sum_{1\leq i_1<\cdots<i_{k-1}<n}P(A_{i_1}\cap\cdots\cap A_n\cap A_{n+1})\Big]$$

$n+1$ 版本 $RHS=\sum_{k=1}^{n+1}(-1)^{k-1}\sum_{1\le i_1<\cdots<i_k\le n+1}P(A_{i_1}\cap\cdots\cap A_{i_k})$

$$=\sum_{k=1}^{n-1}(-1)^{k-1}\sum_{1\le i_1<\cdots<i_k<n}P(A_{i_1}\cap\cdots\cap A_{i_k})+\sum_{k=1}^{n}(-1)^{k-1}\left[\sum_{1\le i_1<\cdots<i_{k-1}<n}P(A_{i_1}\cap\cdots\cap A_{i_{k-1}}\cap A_n)+\sum_{1\le i_1<\cdots<i_{k-1}<n}P(A_{i_1}\cap\cdots\cap A_{n+1})\right]$$

$$+(-1)^{k}\sum_{1\le i_1<\cdots<i_{k-1}<n}P(A_{i_1}\cap\cdots\cap A_{i_{k-1}}\cap A_n\cap A_{n+1})$$

$=$上式 RHS. 得证

(2) pf: 下证: $\sum_{i_1<\cdots<i_k}P(A_{i_1}\cap\cdots\cap A_{i_k})\le\sum_{i_1<i_2<\cdots<i_k<i_{k+1}}P(A_{i_1}\cap\cdots\cap A_{i_k}\cap A_{i_{k+1}})$

$$RHS-LHS=\sum_{k=1}^{n+1}(-1)^{k-1}\sum_{i_1<\cdots<i_k}P(A_{i_1}\cap\cdots\cap A_{i_{k+1}})-\sum_{k=1}^{n}(-1)^{k-1}\sum_{i_1<\cdots<i_k}P(A_{i_1}\cap\cdots\cap A_{i_k})$$

$$=\sum_{k=1}^{n}(-1)^{k-1}\sum_{i_1<\cdots<i_{k-1}}P(A_{i_1}\cap\cdots\cap A_{i_{k-1}}\cap A_{i_{k+1}})\ge 0$$

故 $RHS\ge LHS$. 以下先证 r 为奇的式子

则 若 n 为奇数, $r=n$ 时, 显然成立.

下有若 $r=m$ 成立, 则 $r=m-2$ 也成立

显然 $[r=m$ 式子$]-[r=m-2$ 式子$]=\sum_{i_1<\cdots<i_m}P(A_{i_1}\cap\cdots\cap A_{i_m})-\sum_{i_1<\cdots<i_{m-1}}P(A_{i_1}\cap\cdots\cap A_{i_{m-1}})\ge 0$

则得证

若 n 为偶数, $r=n-1$ 时, $P(A_1\cup\cdots\cup A_n)-[r=n-1$ 式子$]\le 0$

同 $r=m$ 式子, $r=n-1$ 成立时, 则 $r=n-3$ 成立

故综上, 得证原命题.

再证 r 为偶的式子

若 n 为奇数. $r=n-1$ 时, 同上, $P[A_1\cup\cdots\cup A_n]-[r=n-1$ 式子$]\le 0$

而 $[r=m$ 式子$]\ge[r=m-2$ 式子$]$. 则全成立

若 n 为偶数 $r=n$ 时, $P(A_1\cup\cdots\cup A_n)=[r=n$ 式子$]$

而 $[r=m$ 式子$]\ge[r=m-2$ 式子$]$ 则全成立

故 r 为偶式子成立.

4.(1) $L_{n+2}=L_{n+1}+L_n$

特征根方程为 $x^2=x+1$

$\Rightarrow x_1=\frac{1-\sqrt5}{2},\ x_2=\frac{1+\sqrt5}{2}$

则通项公式 $L_n=Ax_1^n+Bx_2^n$

$$\begin{cases} L_1=1=Ax_1+Bx_2 \\ L_2=3=Ax_1^2+Bx_2^2 \end{cases} \Rightarrow \begin{cases} A=1 \\ B=1 \end{cases}$$

则 $L_n=(\frac{1-\sqrt5}{2})^n+(\frac{1+\sqrt5}{2})^n$

(2) pf：由题 $F_n=\frac{1}{\sqrt5}\left[(\frac{1+\sqrt5}{2})^n-(\frac{1-\sqrt5}{2})^n\right]$

对于式1： $LHS=\frac{2}{\sqrt5}\left[(\frac{1+\sqrt5}{2})^{k+n}-(\frac{1-\sqrt5}{2})^{k+n}\right]$

$$RHS=\frac{1}{\sqrt5}\left[(\frac{1+\sqrt5}{2})^k-(\frac{1-\sqrt5}{2})^k\right]\left[(\frac{1-\sqrt5}{2})^n+(\frac{1+\sqrt5}{2})^n\right]+\frac{1}{\sqrt5}\left[(\frac{1+\sqrt5}{2})^n-(\frac{1-\sqrt5}{2})^n\right]\left[(\frac{1-\sqrt5}{2})^k+(\frac{1+\sqrt5}{2})^k\right]$$

$=\frac{2}{\sqrt5}\left[(\frac{1+\sqrt5}{2})^{k+n}-(\frac{1-\sqrt5}{2})^{k+n}\right]=LHS$ 成立

对于式2： $LHS=2\left[(\frac{1-\sqrt5}{2})^{k+n}+(\frac{1+\sqrt5}{2})^{k+n}\right]$

$$RHS=5\cdot\frac{1}{5}\left[(\frac{1+\sqrt5}{2})^n-(\frac{1-\sqrt5}{2})^n\right]\left[(\frac{1+\sqrt5}{2})^k-(\frac{1-\sqrt5}{2})^k\right]+\left[(\frac{1-\sqrt5}{2})^n+(\frac{1+\sqrt5}{2})^n\right]\left[(\frac{1-\sqrt5}{2})^k+(\frac{1+\sqrt5}{2})^k\right]$$

$=2\left[(\frac{1-\sqrt5}{2})^{k+n}+(\frac{1+\sqrt5}{2})^{k+n}\right]=LHS$ 得证

对于式3： $LHS=(\frac{1-\sqrt5}{2})^{4k}+(\frac{1+\sqrt5}{2})^{4k}=(\frac{7-3\sqrt5}{2})^k+(\frac{7+3\sqrt5}{2})^k$

$RHS=\left[(\frac{3-\sqrt5}{2})^k+(\frac{3+\sqrt5}{2})^k\right]^2-2=(\frac{7-3\sqrt5}{2})^k+(\frac{7+3\sqrt5}{2})^k+2-2=LHS$ 得证

对于式4： $LHS=\frac{3-\sqrt5}{2}\cdot(\frac{7-3\sqrt5}{2})^k+\frac{3+\sqrt5}{2}(\frac{7+3\sqrt5}{2})^k$

$RHS=\left[\frac{1-\sqrt5}{2}(\frac{3-\sqrt5}{2})^k+\frac{1+\sqrt5}{2}(\frac{3+\sqrt5}{2})^k\right]^2+2$

$=\frac{3-\sqrt5}{2}(\frac{7-3\sqrt5}{2})^k+\frac{3+\sqrt5}{2}(\frac{7+3\sqrt5}{2})^k-2+2=LHS$ 得证

5. 由题意，$[n]$ 对应个数为 a_n

则对于 $[n+1]$ 而言，若无 $n+1$，则为 a_n

若有 $n+1$，则为 a_{n-1}

则 $a_{n+1}=a_n+a_{n-1}$. $a_0=1,\ a_1=2,\ a_2=3$

则特征方程根为 $x_1=\frac{1-\sqrt5}{2},\ x_2=\frac{1+\sqrt5}{2}$. 设 $a_n=A(\frac{1-\sqrt5}{2})^n+B(\frac{1+\sqrt5}{2})^n$

则 $\begin{cases} a_0=A+B \\ a_1=Ax_1+Bx_2 \end{cases} \Rightarrow a_n=\frac{5-3\sqrt5}{10}(\frac{1-\sqrt5}{2})^n+\frac{5+3\sqrt5}{10}(\frac{1+\sqrt5}{2})^n$

(2) $[n]$对应 b_n

对于$[n+1]$，若有 n 与 $n+1$，则为 b_{n-2}

若有$n+1$无n，则为 b_{n-1}

若无$n+1$，则为 b_n

故 $b_{n+1}=b_n+b_{n-1}+b_{n-2}$

(3) $[n]$对应 c_n

对于$[n+1]$，若有 $n+1$，则为 a_{n-2}

若无$n+1$，则为 a_n

则 $c_{n+1}=a_{n-2}+a_n$

$$
\begin{aligned}
\text{则 } c_n=a_{n-3}+a_{n-1} &= \frac{5-\sqrt{5}}{10}\cdot\left(\frac{2}{1-\sqrt{5}}\right)^3\cdot\left(\frac{1-\sqrt{5}}{2}\right)^n+\frac{5+\sqrt{5}}{10}\cdot\left(\frac{2}{1+\sqrt{5}}\right)^3\left(\frac{1+\sqrt{5}}{2}\right)^n+\frac{5-\sqrt{5}}{10}\cdot\frac{2}{1-\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n+\frac{5+\sqrt{5}}{10}\cdot\frac{2}{1+\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n \\
&= -\frac{3+\sqrt{5}}{2\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n+\frac{3-\sqrt{5}}{2\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n-\frac{\sqrt{5}}{5}\left(\frac{1-\sqrt{5}}{2}\right)^n+\frac{\sqrt{5}}{5}\left(\frac{1+\sqrt{5}}{2}\right)^n \\
&= \left(\frac{1+\sqrt{5}}{2}\right)^{n-1}-\left(\frac{1-\sqrt{5}}{2}\right)^{n-1}
\end{aligned}
$$