Discrete Mathematics

Lecture 5

Combinatorial Probability

The Law of Large Numbers

Toss a coin n times and denote by X the number of heads. We say $X \sim B(n,1/2)$, $P(X=k)=C(n,k)/2^n$ for $k \in [0, n]$.

Theorem 5.3.1(Bernoulli) **For all $\varepsilon>0$, $P(|X/n-0.5|<\varepsilon) \to 1$ as $n \to \infty$.**

Theorem 5.3.2 **If $X \sim B(2n,1/2)$, then for all $t \in [0,n]$, $P(|X-n|>t) \leq \exp\{-t^2/(n+t)\}$.**

Theorem 5.3.1(Bernoulli) For all $\varepsilon>0$, $P(|X/n-0.5|<\varepsilon) \to 1$ as $n\to\infty$.

Theorem 5.3.2 If $X\sim B(2n,1/2)$, then for all $t\in[0,n]$, $P(|X-n|>t) \le \exp\{-t^2/(n+t)\}$.

Theorem 5.3.2⇒Theorem 5.3.1:

For $X\sim B(2n,1/2)$, we need to prove $P(|X/2n-0.5|<\varepsilon) \to 1$ as $n\to\infty$.

$$P(|X/2n-0.5|>\varepsilon)=P(|X-n|>2\varepsilon n)$$
$$\le \exp\{-4\varepsilon^2 n^2/(n+2\varepsilon n)\}\to 0.$$

Theorem 5.3.2 If $X \sim B(2n,1/2)$, then for all $t \in [0,n]$, $P(|X-n|>t) \leq \exp\{-t^2/(n+t)\}$.

Proof. $P(|X-n|>t)=P(X<n-t \text{ or } X>n+t)$

$$=P(X=0)+\cdots+P(X=n-t-1)$$

$$+P(X=n+t+1)+\cdots+P(X=2n)$$

$$=2[P(X=0)+\cdots+P(X=n-t-1)]$$

$$=2[C(2n,0)+\cdots+C(2n,n-t-1)]/2^{2n}$$

$$<C(2n,n-t)/C(2n,n)\leq\exp\{-t^2/(n+t)\}.$$