



$$\frac{1}{2} \int_0^{+\infty} (x^{2024} + x^{-2024}) \frac{\ln x dx}{1+x^2}$$

$$x = e^t$$

$$\int_0^{+\infty} (x^{2024} + x^{-2024}) \frac{\ln x}{1+x^2} dx$$

$$= \int_{-\infty}^{+\infty} (e^{2024t} + e^{-2024t}) \cdot \frac{t}{1+e^{2t}} \cdot e^t dt$$

$$= \int_{-\infty}^{+\infty} (e^{2024t} + e^{-2024t}) \frac{t}{e^t + e^{-t}} dt$$

被积函数奇函数

$$= 0$$

$$3. \lim_{n \rightarrow \infty} \frac{\ln n}{\ln(1^{2023} + 2^{2023} + \dots + n^{2023})} = \frac{1}{2024}$$

$$\lim_{n \rightarrow \infty} \frac{\ln(1^{2023} + 2^{2023} + \dots + n^{2023})}{\ln n}$$

$$= \lim_{n \rightarrow +\infty} \frac{\ln(1^{2023} + \dots + n^{2023}) - 2024 \ln n}{\ln n} + 2024$$

$$= \lim_{n \rightarrow +\infty} \frac{\ln\left(\frac{1}{n} \left(\left(\frac{1}{n}\right)^{2023} + \dots + \left(\frac{n}{n}\right)^{2023}\right)\right)}{\ln n} + 2024$$

$$\text{分子} \rightarrow \ln \int_0^1 x^{2023} dx = \ln 2024^{-1}$$

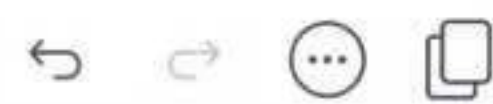
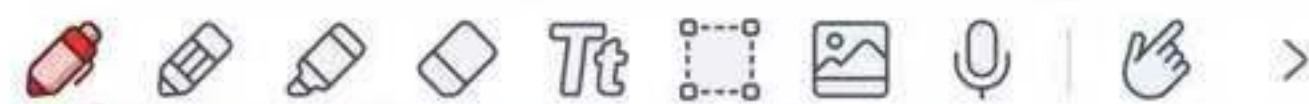
$$\text{分母} \rightarrow +\infty$$

$$= 2024$$

$$\lim_{n \rightarrow \infty} f_n(x) = 0, f_1(x) = 1 \rightarrow \int_0^1 |f_1(x) - f_n(x)| dx \leq e^{-1}$$



024



$$5. f(x) \text{ 连续, } f(0) = 0, f(1) = 1 \rightarrow \int_0^1 |f'(x) + f(x)| dx \leq e^{-1}$$

$$(e^{-x} \cdot f(x))' = e^{-x} f'(x) + e^{-x} f(x)$$

$$\begin{aligned} \int_0^1 |f'(x) + f(x)| dx &\geq \int_0^1 |e^{-x} (f'(x) + f(x))| dx \\ &= \int_0^1 |(e^{-x} f(x))'| dx \\ &\geq \int_0^1 (e^{-x} f(x))' dx = e^{-x} f(x) \Big|_0^1 \\ &= e^{-1} \end{aligned}$$

$$1) f(x) \text{ 连续, } \int_0^\pi x f(\sin x) dx = \pi \int_0^{\frac{\pi}{2}} f(\cos x) dx.$$

证 (2) : (1) 证法.

$$\begin{aligned} \int_0^\pi x f(\sin x) dx &= \int_0^{\frac{\pi}{2}} x f(\sin x) dx \\ &\quad + \int_{\frac{\pi}{2}}^\pi x f(\sin x) dx \end{aligned}$$

$$= \int_0^{\frac{\pi}{2}} x f(\sin x) dx + \int_{\frac{\pi}{2}}^0 (\pi - x) f(\sin(\pi - x)) d(\pi - x)$$

$$= \int_0^{\frac{\pi}{2}} x f(\sin x) dx + \int_0^{\frac{\pi}{2}} (\pi - x) f(\sin x) dx$$

$$= \pi \int_0^{\frac{\pi}{2}} f(\sin x) dx$$

$$= \pi \int_{\frac{\pi}{2}}^0 f(\sin(\frac{\pi}{2} - x)) d(\frac{\pi}{2} - x)$$

$$= \pi \int_0^{\frac{\pi}{2}} f(\cos x) dx$$



$$f(x) := \cos x - \int_0^x e^{-t} f(x-t) dt$$

$$\text{求 } f(x).$$

$$f(0) = 1$$

$$f'(x) = \sin x - e^{-x} - \int_0^x e^{-t} f'(x-t) dt$$

$$= \sin x - e^{-x} - \int_0^x e^{-t} d f(x-t)$$

$$= \sin x - e^{-x} - e^{-y} f(x-y) \Big|_0^x + \int_0^x f(x-t) d e^{-t}$$

$$= \sin x - e^{-x} - e^{-x} - f(x) - \int_0^x e^{-t} f(x-t) dt$$

$$= \sin x - 2e^{-x} + \cos x.$$

$$\Rightarrow f(x) = -\cos x + 2e^{-x} + \sin x + C$$

$$f(0) = -1 + 2 + 0 + C = 1 \Rightarrow C = 0$$

$$\Rightarrow f(x) = -\cos x + 2e^{-x} + \sin x$$





9.  $I_n := \int_0^{\frac{\pi}{2}} \frac{\sin^2(nt)}{\sin t} dt.$

证:  $\lim_{n \rightarrow \infty} \frac{I_n}{\ln n}$

$$I_n = \int_0^{\frac{\pi}{2}} \frac{\sin^2(nt)}{\sin t} dt$$

$\forall$  固定  $n$ ,  $\frac{\sin^2(nt)}{\sin t}$  在 0 附近  $= O(t)$

$\Rightarrow I_n$  可积

注意到:  $\sin t + \sin 3t + \dots + \sin((2n-1)t) = \frac{\sin^2 nt}{\sin t}$

事实上:  $\sin t (\sin t + \sin 3t + \dots + \sin((2n-1)t))$

$$= \frac{1}{2} \left[ \cos(0) - \cos(2t) + (\cos(2t) - \cos(4t)) \right. \\ \left. + \dots + (\cos(2(n-1)t) - \cos(2nt)) \right]$$

$$= \frac{1}{2} (\cos(0) - \cos(2nt)) = \sin^2(nt)$$

$$\Rightarrow I_n = \int_0^{\frac{\pi}{2}} \left( \sum_{k=1}^n \sin((2k-1)t) \right) dt$$

$$= \sum_{k=1}^n \frac{1}{2k-1}$$

$$\lim_{n \rightarrow \infty} \frac{I_n}{\ln n} = \lim_{n \rightarrow \infty} \frac{\frac{1}{2n+1}}{\ln(n+1) - \ln n} = \frac{1}{2}$$



$$f(x) \downarrow, 0 < a \leq b \rightarrow b \int_0^a f \leq a \int_0^b f$$

$$\begin{aligned} & \frac{1}{a} \int_0^a f(x) dx - \frac{1}{b} \int_0^b f(x) dx \\ &= \int_0^1 f(at) dt - \int_0^1 f(bt) dt \\ &= \int_0^1 (f(at) - f(bt)) dt \geq 0 \end{aligned}$$

$$8. f(0) = f(1) = 0, f'(0) = f'(1) = 0 \rightarrow \left| \int_0^1 f(x) dx \right| \leq \frac{1}{12} \max |f''(x)|$$

$$\begin{aligned} \int_0^1 f(x) dx &= - \int_0^1 x f'(x) dx = - \int_0^1 (x - \frac{1}{2}) f'(x) dx \\ &= \frac{1}{2} \int_0^1 f'(x) d(x^2 - x) = \frac{1}{2} (x^2 - x) f'(x) + \frac{1}{2} \int_0^1 (x^2 - x) f''(x) dx \end{aligned}$$

$$\Rightarrow 2 \left| \int_0^1 f(x) dx \right| \leq \left| \int_0^1 (x^2 - x) f''(x) dx \right|$$

$$\leq \max |f''(x)| \cdot \int_0^1 (x - x^2) dx$$

$$= \max |f''(x)| \cdot \left( \frac{1}{2} x^2 - \frac{1}{3} x^3 \right) \Big|_0^1 = \frac{1}{6}$$

$$\Rightarrow \left| \int_0^1 f(x) dx \right| \leq \frac{1}{12} \max |f''(x)|$$