Output Analysis for Simulation Model

Output Analysis

- **Output analysis** is the analysis of data generated by a simulation run to predict or compare the performance of two (2) or more system designs. In *stochastic simulations* – a simulation of a system that has variables that can change randomly with individual probability – multiple runs are always necessary.
- Output analysis is performed since the output data from a simulation exhibits random variability when random or pseudorandom generators are in use.
- Output analysis can determine the estimate of the mean and variance of random variables or the number of observations required to achieve a desired estimate precision.
- The output data of simulation models are used to help evaluate *performance measures* – reflect the characteristics of the stochastic process being modeled.
- The basic output data from a simulation run is called the *trace*.
- The most common measures used to represent the results of simulation runs are the following (Garrido, 2009):
  - The *mean*, also known as the *average*, gives the fundamental characteristics of the output data
  - The *standard deviation*, to summarize the variability of the output data
  - The *frequency plot*, also known as the *histogram*, represents the distribution of the output data

  Note: The mean and the standard deviation can be computed by accumulating values of the random variable while the simulation run is carried out.

In output simulation for simulation models, there are two (2) methods to statistically analyze the collected simulation output, which are *point estimation* and *interval estimation*. In most cases, both methods are needed for a meaningful data interpretation.

Point Estimation

- **Point estimators** are functions that are used to find an approximate value of a population parameter from random samples in the population. It produces a single value.
- The following are the main characteristics of point estimators (Corporate Finance Institute®, n.d.):
  - **Bias** – This pertains to the difference between the expected value of the estimator and the value of the parameter being estimated. If the estimated value and the value of the parameter being estimated are equal, then the estimator is considered as *unbiased*.
  - **Consistency** – This states how close is the point estimator to the value of the parameter as it increases in size. The point estimator requires a large sample size for it to be more consistent and accurate.
  - **Efficiency** – The efficiency of the estimator depends on the distribution of the population. The most efficient point estimator is the one with the smallest variance of all the unbiased and consistent estimators.
- Point estimation for discrete-time data: $[Y_1, Y_2, \ldots, Y_n]$ with ordinary mean θ (theta)

$$\hat{\theta} = \frac{1}{n}\sum_{i=1}^{n} Y_i$$

  - It is biased if:

$$E(\hat{\theta}) \neq \theta$$

  - It is unbiased if the expected value of θ is: (this condition is desired)

$$E(\hat{\theta}) = \theta$$

  Where:

  Y – is a specific value relative to a simulation run
  n – is the number of simulation run(s)
  E – is a specific event that stops the simulation

- Point estimation for continuous-time data: $\{Y(t), 0 \leq t \leq T_E\}$ with time weighted mean ϕ (phi)

$$\hat{\phi} = \frac{1}{T_E}\int_0^{T_E} Y(t)dt$$

  - The point estimator is generally biased.
  - An unbiased or low-bias estimator is still desired.

  Where:

  T_E – is the period of time or duration of the simulation run
  Y(t) – is a function representing replication(s), relative to time *(t)*

Interval Estimation

- In interval estimation, the desired parameter to be estimated lies between the two (2) values, *l* and *u*, that define the interval with a given probability. The interval [*l*, *u]* is called the confidence interval.
- **Confidence interval** is the statistical tool mainly used in output analysis. It displays the probability that a parameter will fall between a pair of values around the mean (Hayes, 2021).
- Confidence interval is an interval estimate for a parameter that specifies the level of confidence that provides a way of quantifying imprecision. Its goal is to form an interval with endpoints determined by the samples, that will contain or "cover" the target parameter with pre-specified (high) probability called the *confidence level*.
- The **Central Limit Theorem (CLT)** states that as the number of samples ($n \geq 30$) increases, the distribution of the mean will be approximately normal. The following must be observed in applying CLT (Garrido, 2009):
  - Each single run of a stochastic simulation model can be considered as a single sample.
  - Each independent model replication, where replications are performed using random numbers, produces another sample point.
- The mathematical representation of confidence interval is (Banks, Carson, Nelson & Nicol, n.d.):

$$\overline{Y}_{..} \pm t_{\alpha/2,R-1} \frac{S}{\sqrt{R}}$$

Where:

$\overline{Y}$	: sample mean
$\pm\, t_{\alpha/2,R-1}$	: the t-multiplier
$\frac{S}{\sqrt{R}}$	: standard error

- **Prediction interval** is a measure of risk or uncertainty. It is a range of values that is likely to contain the value of a single new observation given a specific setting or initial condition.
- Prediction intervals predict the spread for individual observations rather than the mean. The interval in this estimation method is wider than confidence interval due to the uncertainty involved in predicting a single response.
- The mathematical representation of prediction interval is (Banks, Carson, Nelson & Nicol, n.d.):

$$Y_{..} \pm t_{\alpha/2,R-1} S\sqrt{1+\frac{1}{R}}$$

Where:

Y	: sample estimate or predicted value
$\pm\, t_{\alpha/2,R-1}$	: the t-multiplier
$S\sqrt{1+\frac{1}{R}}$	: standard error of prediction

Types of Simulation with Respect to Output Analysis

The type of simulation, either terminating or non-terminating, greatly depends on the following:

- The objective(s) of the modeling and simulation study.
- The nature of the system.

Terminating Simulation

- A terminating simulation is carried out to study the behavior of a system over a particular time interval.
- The simulation starts and ends at a defined state or time.
- Majority of service systems are modeled as terminating systems.
- The analysis of terminating simulations involves multiple runs using different seeds for the random or pseudorandom number generations.
- The data is gathered for successive time intervals during the simulation period (Garrido, 2009).

  Examples:
  1. A car manufacturer receives a contract to produce 120 cars, which must be delivered within 18 months. The company would like to simulate various manufacturing configurations to see which can meet the delivery request at the least cost.
  2. A shop that sells a single product would like to decide how many items they should have in the inventory for the next 120 months. Given some initial inventory data, the objective is to determine the monthly order to minimize the expected averaging cost per month in the inventory system.

- The following are some possible statistical analysis formula, applicable for terminating simulations (Noche, 2014):
  - Unbiased estimator for the population mean (μ)

$$\overline{Y}(n) = \frac{\sum_{i=1}^{n} Y_i}{n}$$

  - Sample variance

$$S^2(n) = \frac{\sum_{i=1}^{n} [Y_i - \overline{Y}(n)]^2}{n-1}$$

  - Confidence interval for the population mean (μ)

$$\overline{Y}(n) \pm t_{n-1,1-\alpha/2} \frac{S(n)}{\sqrt{n}}$$

Non-Terminating Simulation

- A non-terminating simulation is carried out to study the steady-state and/or long-term average behavior of a system.
- The long-term average behavior of the system is analyzed by calculating the adequate length of the simulation period.
- The simulation runs are performed to gather data for the statistical analysis of the steady-state behavior of the system.
- The simulation runs begin with a warm-up state, also known as *transient state* – the output process for the initial condition at a discrete time, and gradually moves to a *steady state* – shows approximately the same distribution of random variables from a specific point (Garrido, 2009).
  Examples:
  1. Telecommunications systems
  2. Assembly lines that often halt operations
  3. Emergency rooms in hospitals

- The following are some possible statistical analysis method, applicable for non-terminating simulations:
  - Welch Method
  - Replication-Deletion Approach
  - Batch Mean Method

References:
Chaturvedi, D. (2010). *Modeling and simulation of systems using MATLAB and Simulink*. CRC Press – Taylor & Francis Group, LLC
Corporate Finance Institute® (n.d.). *Point estimators*. Retrieved on April 13, 2021 from https://corporatefinanceinstitute.com/resources/knowledge/other/point-estimators/
Banks, Carson, Nelson & Nicol. (n.d.). *Chapter 11: Output analysis for a single model*. Retrieved April 10, 2021 from https://cs.wmich.edu/~alfuqaha/Spring09/cs6910/lectures/Chapter11.pdf
Noche, B. (2014, November 26). *Output analysis for simulation model*. Retrieved on April 10, 2021 from https://www.uni-due.de/imperia/md/content/tul/download/ews2014_2015_sl02_im_output_analysis.pdf
Garrido, J. (2009). *Object oriented simulation a modeling and programming perspective*. Springer Science+Business Media, LLC
Hayes, A. (2021, March 31). *Confidence interval*. Retrieved on April 11, 2021 from https://www.investopedia.com/terms/c/confidenceinterval.asp