PR 3

a) $$\int \frac{\sqrt[3]{x}}{x+\sqrt[6]{x^5}}\,dx =$$

$t = \sqrt[6]{x} \Rightarrow x = t^6$

$\Downarrow$ $\boxed{dx = 6t^5 dt}$

$\sqrt[3]{x} = t^2 \Rightarrow x^{\frac{1}{3}} \Rightarrow t^{6\cdot\frac{1}{3}} = t^2$

$\sqrt[6]{x^5} = t^5 \Rightarrow x^{\frac{5}{6}} \Rightarrow t^{6\cdot\frac{5}{6}} = t^5$

$$= \int \frac{t^2}{t^6+t^5}\, 6 \cdot t^5 dt = \int \frac{t^2 \cdot 6\cancel{t^5}\, dt}{\cancel{t^5}(t+1)} =$$

$$= 6\int \frac{t^2-1+1}{t+1}\,dt = 6\int \frac{(t-1)\cancel{(t+1)}}{\cancel{t+1}} + \frac{1}{t+1}\,dt$$

$$= 6\int t - 1 + \frac{1}{t+1}\,dt = 6\left[\frac{t^2}{2} - t + \ln|t+1|\right]_t$$

$$= 3t^2 - 6t + 6\ln|t+1| + C \quad ; \quad t = \sqrt[6]{x}$$

$$⑤ \int \frac{1}{x}\sqrt{\frac{1-x}{1+x}}\,dx =$$

$$\boxed{t = \sqrt{\frac{1-x}{1+x}}}$$

$$t^2 = \frac{1-x}{1+x}$$

$$t^2 + xt^2 = 1 - x$$

$$xt^2 + x = 1 - t^2 \quad \Rightarrow \quad x(t^2+1) = 1 - t^2$$

$$\boxed{x = \frac{1-t^2}{t^2+1}}$$

$$\boxed{dx} = \frac{-2t(t^2+1) - (1-t^2)\cdot 2t}{(t^2+1)^2}\,dt =$$

$$= \frac{\cancel{-2t^3} - 2t - 2t + \cancel{2t^3}}{(t^2+1)^2}\,dt = \boxed{\frac{-4t\,dt}{(t^2+1)^2}}$$

$$= \int \frac{1}{\frac{1-t^2}{t^2+1}} \cdot t \cdot \frac{-4t}{(t^2+1)^2}\,dt = \int \frac{\cancel{t^2+1}}{1-t^2} \cdot \frac{(-4t^2)}{(t^2+1)^{\cancel{2}}}\,dt$$

$$= \int \frac{-4t^2}{(1-t^2)(t^2+1)}\,dt = \int \frac{4t^2}{(t^2-1)(t^2+1)}\,dt =$$

$$\boxed{\frac{A}{t-1} + \frac{B}{t+1} + \frac{Ct+D}{t^2+1}} \Rightarrow$$

$$\Rightarrow A(t+1)(t^2+1) + B(t-1)(t^2+1) + (Ct+D)(t^2-1) =$$

$$= A(t^3+t+t^2+1) + B(t^3+t-t^2-1) + Ct^3 - Ct + Dt^2 - D$$

$$\boxed{t=1} \qquad 4A = 4 \Rightarrow \boxed{A=1}$$

$$\boxed{t=-1} \qquad -4B = 4 \Rightarrow \boxed{B=-1}$$

$$\boxed{t=0} \qquad A - B - D = 0$$

$$1 + 1 - D = 0 \Rightarrow \boxed{D=2}$$

$$\boxed{t=2} \qquad 15A + 5B + 8C - 2C + 4D - D = 16$$

$$15 - 5 + 6C + 6 = 16 \Rightarrow \boxed{C=0}$$

$$= \int \frac{1}{t-1} - \frac{1}{t+1} + \frac{2}{t^2+1}\, dt =$$

$$= \ln|t-1| - \ln|t+1| + 2\,\mathrm{arctg}\, t + C =$$

$$= \ln\left|\frac{t-1}{t+1}\right| + 2\,\mathrm{arctg}\, t + C \;; \quad t = \sqrt{\frac{1-x}{1+x}}$$

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c) $\displaystyle\int \frac{\sqrt{1+x}}{x}\, dx = \left|\begin{array}{l} t = \sqrt{1+x} \\ t^2 = 1+x \Rightarrow x = t^2 - 1 \\ \qquad\qquad\qquad dx = 2t\, dt \end{array}\right.$

$$= \int \frac{t}{t^2-1}\, 2t\, dt = 2\int \frac{t^2-1+1}{t^2-1}\, dt = 2\int 1 + \frac{1}{t^2-1}\, dt$$

$$= 2\left[t + \frac{1}{2} \ln \left| \frac{t-1}{t+1} \right| + C \right] = 2\sqrt{1+x} + \ln \left| \frac{\sqrt{1+x}-1}{\sqrt{1+x}+1} \right|$$

$$+ C$$

d) $$\int \frac{\sqrt{2x+3}+x}{\sqrt{2x+3}-x}\, dx = \left| \begin{array}{l} \boxed{t = \sqrt{2x+3}} \\ t^2 = 2x+3 \\ \boxed{x = \frac{t^2-3}{2}} \end{array} \right.$$

$$dx = \frac{1}{2} 2t\,dt \Rightarrow \boxed{dx = t\,dt}$$

$$= \int \frac{t + \frac{t^2-3}{2}}{t - \frac{t^2-3}{2}}\, t\,dt = \int \frac{\frac{2t + t^2 - 3}{\cancel{2}}}{\frac{2t - t^2 + 3}{\cancel{2}}}\, t\,dt =$$

$$= \int \frac{2t^2 + t^3 - 3t}{2t - t^2 + 3}\,dt = \int \frac{t^3 + 2t^2 - 3t}{-t^2 + 2t + 3}\,dt$$

$$(t^3+2t^2-3t):(-t^2+2t+3) = -t - 4 + \frac{8t+12}{-t^2+2t+3}$$

$$\begin{array}{l} -(+t^3-2t^2-3t) \\ \hline \quad 4t^2 \\ \quad -(+4t^2-8t-12) \\ \hline \quad\quad 8t+12 \end{array}$$

$$\Rightarrow -(t^2-2t-3) =$$

$$= -(t-3)(t+1)$$

$$= \int -t - 4 + \frac{-8t-12}{(t-3)(t+1)}\,dt = -\frac{t^2}{2} - 4t + \int \frac{-8t-12}{(t-3)(t+1)}\,dt$$

$$\frac{A}{t-3}+\frac{B}{t+1}=\frac{At+A+Bt-3B}{(t-3)(t+1)}$$

$$\left.\begin{aligned} A+B&=-8 \quad (-1)\\ A-3B&=-12 \end{aligned}\right\} +$$

$$-4B=-4 \quad \Rightarrow \quad A=-8-B$$

$$\boxed{B=1} \qquad \boxed{A=-8-1=-9}$$

$$= -\frac{t^2}{2} - 4t + \int \frac{-9}{t-3} + \frac{1}{t+1} \, dt =$$

$$= -\frac{t^2}{2} - 4t - 9\ln|t-3| + \ln|t+1| + C$$

$$t = \sqrt{2x+3}$$

PR 5 a) $\int \frac{1}{\sqrt{8-6x-9x^2}} dx = \int \frac{1}{\sqrt{8-9x^2-6x}} dx =$

$$= \int \frac{1}{\sqrt{8 - 9\left(x^2 + \frac{6}{9}x\right)}}\,dx = \int \frac{1}{\sqrt{8 - 9\left(x^2 + \frac{2}{3}x\right)}}\,dx =$$

$$= \int \frac{1}{\sqrt{8 - 9\left[\left(x + \frac{1}{3}\right)^2 - \frac{1}{9}\right]}}\,dx = \int \frac{1}{\sqrt{8 - 9\left(x + \frac{1}{3}\right)^2 + 1}}\,dx$$

$$= \int \frac{1}{\sqrt{9 - 9\left(x + \frac{1}{3}\right)^2}}\,dx = \frac{1}{3}\int \frac{1}{\sqrt{1 - \left(x + \frac{1}{3}\right)^2}}\,dx$$

$$= \underline{\underline{\frac{1}{3}\arcsin\left(x + \frac{1}{3}\right) + C}}$$

b) $\displaystyle\int \frac{3x+2}{\sqrt{x^2+x+3}}\,dx =$

$$t = -x + \sqrt{x^2+x+3}$$

$$t + x = \sqrt{x^2+x+3}$$

$$t^2 + 2\cdot t x + \cancel{x^2} = \cancel{x^2} + x + 3$$

$$2tx - x = 3 - t^2$$

$$\boxed{x = \frac{3-t^2}{2t-1}}$$

$$\Rightarrow \boxed{dx} = \frac{-2t(2t-1) - (3-t^2)\,2}{(2t-1)^2}\,dt$$

$$= \frac{-4t^2+2t-6+2t^2}{(2t-1)^2}\,dt =$$

$$= \boxed{\frac{-2t^2+2t-6}{(2t-1)^2}\,dt}$$

$$= -\int \frac{3\left(\frac{3-t^2}{2t-1}\right) + 2}{t + \frac{3-t^2}{2t-1}} \cdot \frac{-2t^2+2t-6}{(2t-1)^2}\,dt =$$

$$= \int \frac{9-3t^2+4t-2}{\cancel{(2t-1)}} \cdot \frac{\cancel{2t-1}}{2t^2-t+3-t^2} \cdot \frac{(-2t^2+2t-6)}{(2t-1)^2}\,dt$$

$$= \int \frac{(7-3t^2+4t)(-2)\cancel{(t^2-t+3)}}{\cancel{(t^2-t+3)}(2t-1)^2}\,dt =$$

$$= -2\int \frac{-3t^2+4t+7}{4t^2-4t+1}\,dt = -2\int -\frac{3}{4} + \frac{t+\frac{31}{4}}{4t^2-4t+1}\,dt$$

$$(-3t^2+4t+7) : (4t^2-4t+1) = -\frac{3}{4}$$

$$-\left(-3t^2+3t-\frac{3}{4}\right)$$

$$\overline{\qquad t+\frac{3}{4}+7 = t+\frac{31}{4}}$$

$$= \frac{3}{2}t - \frac{2}{8}\int \frac{8\left(t + \frac{31}{4}\right)}{4t^2 - 4t + 1}\,dt = \frac{3}{2}t - \frac{1}{4}\int \frac{8t + 62}{4t^2 - 4t + 1}\,dt$$

$$= \frac{3}{2}t - \frac{1}{4}\int \frac{8t - 4 + 66}{4t^2 - 4t + 1}\,dt = \frac{3}{2}t - \frac{1}{4}\ln\left|4t^2 - 4t + 1\right|$$

$$- \frac{1}{\cancel{4}\,2}\int \frac{\cancel{66}\;33}{(2t-1)^2}\,ds = \frac{3}{2}t - \frac{1}{4}\ln\left|4t^2 - 4t + 1\right| - \frac{33}{2}\int \frac{1}{(2t-1)^2}\,dt$$

$$s = 2t - 1$$

$$ds = 2dt \Rightarrow dt = \frac{ds}{2} \qquad \Rightarrow \int \frac{1}{s^2}\,\frac{ds}{2} = \frac{1}{2}\,\frac{s^{-1}}{-1} = -\frac{1}{2s}$$

$$= \frac{3}{2} t - \frac{1}{4} \ln |4t^2 + 4t + 1| + \frac{33}{4} \cdot \frac{1}{(2t-1)} + C;$$

$$t = -x + \sqrt{x^2 + x + 3}$$

ⓒ $\displaystyle\int \frac{1}{x - \sqrt{x^2 - x + 1}} \, dx \Rightarrow$

$$t = -x + \sqrt{x^2 - x + 1}$$

$$t + x = \sqrt{x^2 - x + 1}$$

$$t^2 + 2tx + \cancel{x^2} = \cancel{x^2} - x + 1$$

$$2tx + x = 1 - t^2$$

$$\boxed{x = \frac{1 - t^2}{2t + 1}}$$

$$dx = \frac{-2t(2t+1) - (1-t^2)\cdot 2}{(2t+1)^2}\,dt =$$

$$= \frac{-4t^2 - 2t - 2 + 2t^2}{(2t+1)^2}\,dt = \frac{-2t^2 - 2t - 2}{(2t+1)^2}\,dt$$

$$= \int \frac{1}{t} \cdot \frac{(-2)(t^2+t+1)}{(2t+1)^2}\,dt \Rightarrow \frac{A}{t} + \frac{B}{2t+1} + \frac{C}{(2t+1)^2}$$

$$\Rightarrow A(4t^2+4t+1) + B(2t^2+t) + Ct = 2t^2+2t+2$$

$$t^2(4A+2B) + t(4A+B+C) + A = 2t^2+2t+2$$

$\boxed{A = 2}$ $\qquad 4A + 2B = 2 \Rightarrow 2B = 2 - 8 = -6$

$$\boxed{B = -3}$$

$$4A + B + C = 2$$

$$C = 2 - 8 + 3 \Rightarrow \boxed{C = -3}$$

$$\Rightarrow \int \frac{2}{t} - \frac{3}{2t+1} - \frac{3}{(2t+1)^2}\, dt = \underline{\underline{2\ln|t| - \frac{3}{2}\ln|2t+1|}}$$

$$\underline{\underline{+ \frac{3}{2}\frac{1}{|2t+1|} + C}}; \quad t = -x + \sqrt{x^2 - x + 3}$$