ODVODENIE UNIVERZÁLNEJ SUBSTITÚCIE

a)

$$\boxed{t = \operatorname{tg}\frac{x}{2}}$$

$$\operatorname{arctg} t = \frac{x}{2} \Rightarrow \boxed{x = 2\operatorname{arctg} t}$$

$$\boxed{dx = \frac{2}{1+t^2}\,dt}$$

$$\sin(2x) = 2\sin x\cos x \Rightarrow \sin\left(\cancel{2}\frac{x}{\cancel{2}}\right) = 2\sin\frac{x}{2}\cos\frac{x}{2}$$

$$\cos(2x) = \cos^2 x - \sin^2 x \Rightarrow \cos\left(\cancel{2}\frac{x}{\cancel{2}}\right) = \cos^2\frac{x}{2} - \sin^2\frac{x}{2}$$

$$\boxed{\sin(x)} = \frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{\underbrace{\sin^2\frac{x}{2}+\cos^2\frac{x}{2}}_{1=}} \cdot \underbrace{\frac{\frac{1}{\cos^2\frac{x}{2}}}{\frac{1}{\cos^2\frac{x}{2}}}}_{=1} =$$

$$= 2\,\frac{\overbrace{\frac{\sin\frac{x}{2}\cancel{\cos\frac{x}{2}}}{\cos^{\cancel{2}}\frac{x}{2}}}^{tg\frac{x}{2}}}{\frac{\sin^2\frac{x}{2}+\cos^2\frac{x}{2}}{\cos^2\frac{x}{2}}} = \frac{2\,tg\frac{x}{2}}{tg^2\frac{x}{2}+1} = \boxed{\frac{2t}{t^2+1}}$$

$$\boxed{\cos x} = \frac{\cos^2\frac{x}{2} - \sin^2\frac{x}{2}}{\sin^2\frac{x}{2} + \cos^2\frac{x}{2}} \cdot \frac{\frac{1}{\cos^2\frac{x}{2}}}{\frac{1}{\cos^2\frac{x}{2}}} =$$

$$1 = \sin^2\frac{x}{2} + \cos^2\frac{x}{2}$$

$$= 1$$

$$= \frac{\dfrac{\cos^2\frac{x}{2} - \sin^2\frac{x}{2}}{\cos^2\frac{x}{2}}}{\dfrac{\sin^2\frac{x}{2} + \cos^2\frac{x}{2}}{\cos^2\frac{x}{2}}} = \frac{1 - \operatorname{tg}^2\frac{x}{2}}{\operatorname{tg}^2\frac{x}{2} + 1}$$

$$= \boxed{\frac{1 - t^2}{t^2 + 1}}$$

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$$\boxed{t = \operatorname{tg} x}$$

$$\boxed{\operatorname{arctg} t = x} \Rightarrow \boxed{dx = \frac{1}{1+t^2}\,dt}$$

$$\boxed{\sin^2 x} = \frac{\sin^2 x}{\underbrace{\sin^2 x + \cos^2 x}_{1}} \cdot \underbrace{\frac{\frac{1}{\cos^2 x}}{\frac{1}{\cos^2 x}}}_{=1} =$$

$$= \frac{\frac{\sin^2 x}{\cos^2 x}}{\frac{\sin^2 x + \cos^2 x}{\cos^2 x}} = \frac{\operatorname{tg}^2 x}{\operatorname{tg}^2 x + 1} = \boxed{\frac{t^2}{t^2+1}}$$

$$\boxed{\cos^2 x} = \frac{\cos^2 x}{\sin^2 x + \cos^2 x} \cdot \frac{\frac{1}{\cos^2 x}}{\frac{1}{\cos^2 x}} =$$

$$= \frac{1}{tg^2 x + 1} = \boxed{\frac{1}{t^2 + 1}}$$

PR1

a) $\int \frac{1-\sin x}{1+\cos x}\,dx = \int \frac{1-\frac{2t}{t^2+1}}{1+\frac{1-t^2}{t^2+1}} \frac{2}{1+t^2}\,dt$

$\boxed{t = tg\frac{x}{2}}$

$$= \int \frac{\frac{t^2+1-2t}{t^2+1}}{\frac{\cancel{t^2}+1+1-\cancel{t^2}}{t^2+1}} \cdot \frac{2}{1+t^2}\,dt =$$

$$= \int \frac{(t^2+1-2t)\cancel{(t^2+1)}}{\cancel{(t^2+1)}\cdot \cancel{2}} \cdot \frac{\cancel{2}}{1+t^2}\,dt =$$

$$= \int \frac{t^2+1-2t}{1+t^2}\,dt = \int 1 - \frac{2t}{1+t^2}\,dt =$$

$$= t - \ln|1+t^2| + C = \operatorname{tg}\frac{x}{2} - \ln\left(1+\operatorname{tg}^2\frac{x}{2}\right) + C$$

b) $$\int \frac{\operatorname{tg}x}{1+\operatorname{tg}x}\,dx = \int \frac{t}{1+t}\cdot\frac{dt}{1+t^2} =$$

$\boxed{t = \operatorname{tg}x}$

$$\frac{A}{1+t} + \frac{Bt+C}{1+t^2} =$$

$$= \frac{A + At^2 + Bt + Bt^2 + C + Ct}{(1+t)(1+t^2)} = \frac{t^2(A+B) + t(B+C) + A + C}{(1+t)(1+t^2)}$$

$$A + B = 0 \Rightarrow A = -B \qquad \boxed{A = -\frac{1}{2}}$$

$$\left.\begin{aligned} B + C = 1 &\Rightarrow B + C = 1 \\ A + C = 0 &\Rightarrow -B + C = 0 \end{aligned}\right\} \boxed{C = \frac{1}{2}}$$

$$\boxed{B = \frac{1}{2}}$$

$$= \int \frac{-\frac{1}{2}}{1+t} + \frac{\frac{1}{2}t + \frac{1}{2}}{1+t^2}\, dt = -\frac{1}{2} \ln|1+t| +$$

$$+\frac{1}{2}\frac{1}{2}\int\frac{2t+2\cdot 1}{t^2+1}\,dt = -\frac{1}{2}\ln|1+t| + \frac{1}{4}\ln|t^2+1| +$$

$$+\frac{1}{\cancel{4}}\frac{1}{2}\int\frac{\cancel{2}\,1}{t^2+1}\,dt = -\frac{1}{2}\ln|1+t| + \frac{1}{4}\ln\left(t^2+1\right)$$

$$+\frac{1}{2}\operatorname{arctg} t + C\ ; \qquad t = tg(x)$$

(c) $$\int\frac{1}{\cos x}\,dx = \int\frac{1}{\frac{1-t^2}{\cancel{t^2+1}}}\cdot\frac{2}{\cancel{1+t^2}}\,dt =$$

$$\boxed{t = tg\frac{x}{2}}$$

$$= \int \frac{2}{1-t^2}\,dt \qquad \frac{A}{1-t} + \frac{B}{1+t} = \frac{A(1+t)+B(1-t)}{(1-t)(1+t)}$$

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$$2 = A(1+t) + B(1-t)$$

$t = -1 \Rightarrow 2 = B \cdot 2 \Rightarrow \boxed{B = 1}$

$t = 1 \Rightarrow 2 = 2A \Rightarrow \boxed{A = 1}$

$$A + At + B - Bt = 2 \Rightarrow t(A-B) + A + B = 2$$

$A - B = 0 \Rightarrow A = B$ $\quad \boxed{B = 1}$ $\qquad A + B = 2 \Rightarrow 2A = 2 \Rightarrow \boxed{A = 1}$

$$= -\int \frac{-1}{1-t} + \frac{1}{1+t}\,dt = -\ln|1-t| +$$

$$+ \ln|1+t| + C = \ln\left|\frac{1+\operatorname{tg}\frac{x}{2}}{1-\operatorname{tg}\frac{x}{2}}\right| + C$$

d) $$\int \frac{1}{\sin x - \cos x}\,dx = \int \frac{1}{\frac{2t}{1+t^2} - \frac{1-t^2}{1+t^2}} \cdot \frac{2}{1+t^2}\,dt$$

$$\boxed{t = \operatorname{tg}\frac{x}{2}}$$

$=$

$$\int \frac{1}{\frac{2t-1+t^2}{\cancel{1+t^2}}} \cdot \frac{2}{\cancel{1+t^2}}\,dt = \int \frac{2}{t^2+2t-1}\,dt =$$

$$D = b^2 - 4ac = 4 - 4(1)\cdot(-1) = \underline{\underline{8}}$$

$$t_{1,2} = \frac{-2 \pm 2\sqrt{2}}{2} = \underline{\underline{-1 \pm \sqrt{2}}}$$

$$= \int \frac{2}{(t+1-\sqrt{2})(t+1+\sqrt{2})}\,dt =$$

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$$\frac{A}{t+1+\sqrt{2}} + \frac{B}{t+1-\sqrt{2}} = \frac{At + A - \sqrt{2}A + Bt + B + \sqrt{2}B}{(t+1+\sqrt{2})(t+1-\sqrt{2})}$$

$$\Rightarrow \quad A + B = 0 \quad \Rightarrow \boxed{A = -B} \quad \Rightarrow \boxed{A = -\frac{\sqrt{2}}{2}}$$

$$A - \sqrt{2}A + B + \sqrt{2}B = 2$$

$$\cancel{-B} + \sqrt{2}B + \cancel{B} + \sqrt{2}B = 2$$

$$\cancel{2}\sqrt{2}B = \cancel{2}\,1$$

$$\boxed{B = \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}}$$

$$= \int \frac{-\frac{\sqrt{2}}{2}}{t+1+\sqrt{2}} + \frac{\frac{\sqrt{2}}{2}}{t+1-\sqrt{2}} \, dt =$$

$$= -\frac{\sqrt{2}}{2} \ln|t+1+\sqrt{2}| + \frac{\sqrt{2}}{2} \ln|t+1-\sqrt{2}| + c$$

$$= \frac{\sqrt{2}}{2} \ln\left|\frac{t+1-\sqrt{2}}{t+1+\sqrt{2}}\right| + c \;; \quad t = \operatorname{tg}\frac{x}{2}$$

PR 2 a)

$$\int \sin^3 x \cos^2 x \, dx = \int \sin x \cdot (1-\cos^2 x) \cos^2 x \, dx$$

$\sin x \cdot \sin^2 x$ $\quad$ $=-dt$

$t = \cos x$

$dt = -\sin x \, dx$

$$= -\int (1-t^2) t^2 \, dt = -\int t^2 - t^4 \, dt =$$

$$= -\frac{t^3}{3} + \frac{t^5}{5} + C = -\frac{\cos^3 x}{3} + \frac{\cos^5 x}{5} + C$$

(b) $\int \frac{1}{\cos x}\,dx = \int \frac{\cos x}{\cos^2 x}\,dx = \int \frac{\cos x}{1-\sin^2 x}\,dx$

$$= \left| \begin{array}{c} t = \sin x \\ dt = \cos x\,dx \end{array} \right| = \int \frac{dt}{1-t^2} \;\ldots$$

(c) $\int \sin \overset{\alpha}{3x} \cos \overset{\beta}{2x}\,dx$

$\alpha + \beta = 5x$

$\alpha - \beta = x$

$$= \int \frac{1}{2}\left(\sin(x) + \sin(5x)\right)dx = \underline{-\frac{1}{2}\cos(x) - \frac{1}{10}\cos(5x) + c}$$

d) $\int \cos(2x)\cos(3x)\,dx = \int \frac{1}{2}\left(\cos(x) + \cos(5x)\right)dx =$

$= \frac{1}{2}\sin(x) + \frac{1}{10}\sin(5x) + C$

e) $\int \sin(2x)\cdot\sin(3x)\,dx = \int \frac{1}{2}\left(\cos(x) - \cos(5x)\right)dx$

$= \frac{1}{2}\sin(x) - \frac{1}{10}\sin(5x) + C$