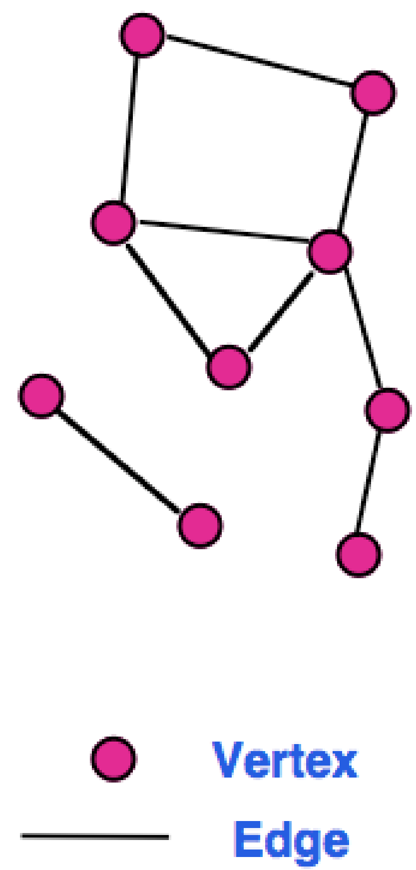
Graphs
Definitions

Examples of Graphs

- Street maps
  - » **Vertices are the intersections**
  - » **Edges are the streets**
- Power line network
  - » **Vertices are the houses & power stations**
  - » **Edges are the power lines**
- WWW pages for the CSE Department
  - » **Vertices are the pages**
  - » **Edges are the links**



Definition

- A graph G contains two components

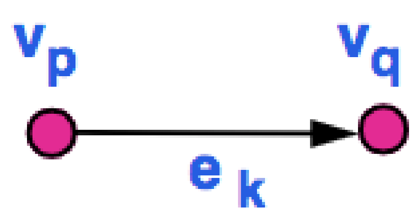
$G = \langle V, E \rangle$	the graph
$V(G)$	set of vertices (or nodes) in the graph
$E(G)$	bag of edges (or arcs) in the graph

- An edge is a pair of vertices with an associated direction – vertices are at the **ends** of an edge

$$e_k = (v_p, v_q) \wedge v_p, v_q \in V$$

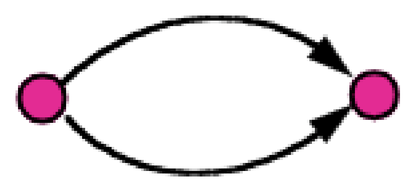
v_p is the **tail** of the edge

v_q is the **head** of the edge

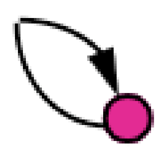


Definition – 2

- Note we have
  - » **Set of vertices**
    - > **Every vertex is distinct**
  - » **Bag of edges**
    - > **Can have more than one edge connecting the same pair of vertices**
      - – **Two or more highways connecting two cities**
      - – **Two or more telephone lines into a house**



- An edge can have both head and tail be the same vertex – a **self-loop**



Directed Graph

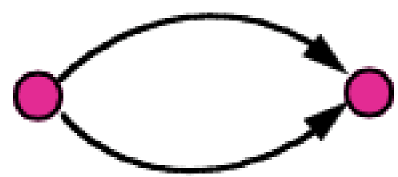
- Also called a **digraph**
- Edges have a direction, a **directed edge**
  - (v1 , v2) are an ordered pair
    - » **Oil pipelines – oil flows in one direction**
    - » **River systems – water flows downhill**
    - » **One way roads**
- Represent edges as arrows in the graph, where v1 is the tail, and v2 is the head

Undirected Graph

- Edges have no direction, an **undirected edge**
  – (v1 , v2) are an unordered pair
  » **Two way streets**
  » **Who danced with whom at a party**
- The graph is considered to have both (v1 , v2) and (v2 , v1) in the graph but only one of the pair is specified as a member of E (G).
- Represent edges with lines – no arrows

Multi-graph

- Multi-graph
  - » **Has more than one edge between one or more pairs of vertices, called parallel edges**
    - > **Highway systems between cities**
    - > **Multiple telephones into a house**
- Edges can be directed or undirected



Simple Graph

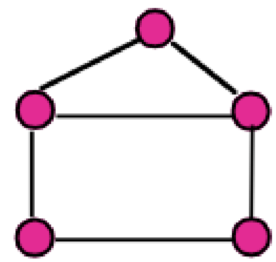
- Simple Graph
  - » **No loops and no parallel edges**
  - » **Edges form a set instead of a bag**
    - > **Simplifies technical details for graph algorithms but does not fundamentally change them**

If a graph has e edges and v vertices then
$e \le v (v - 1) / 2$

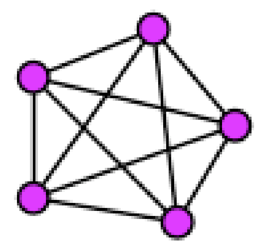
Why?

Planar & Non-planar Graphs

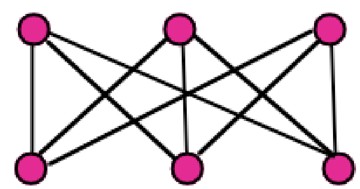
- Planar graph
  - » **Can be drawn on a plane (a sheet of paper) so no edges cross**



- Non-planar graph
  - » **Cannot draw on a plane without edges crossing**
  - » **All non-planar graphs contain one or both of these as sub-graphs**



Complete 5-graph



Houses & Utilities

Adjacent Vertices & Incident Edges

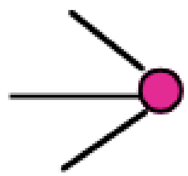
- Two vertices are **adjacent** if the vertices are at the ends of the same edge
  - » **The vertices v_p and v_q are adjacent**
- In directed and undirected graphs
  - » **The edge e_k is incident on the vertices v_p and v_q**

Example edge
$e_k = (v_p , v_q)$

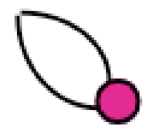
- In a directed graph
  - » **e_k is an outgoing edge from the vertex v_p**
  - » **e_k is an incoming edge to the vertex v_q**

Degree of a Vertex

- In an undirected graph
  - » **Degree is the number of ends of edges incident upon the vertex (number of adjacent vertices)**



degree 3



degree 2

If a graph has e edges then

$$\sum_{v \in G} \deg(v) = 2e$$

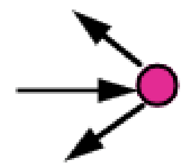
Why?

Degree of a Vertex – 2

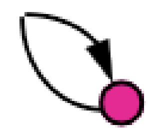
In a directed graph

degree = in-degree + out-degree

- In-degree – the number of times the vertex is at the head of an edge
- Out-degree – the number of times the vertex is at the tail of an edge



degree 3
In-degree 1
out-degree 2



degree 2
In-degree 1
out-degree 1

If a simple graph has e edges and v vertices then
e ≤ v (v – 1)

Why?

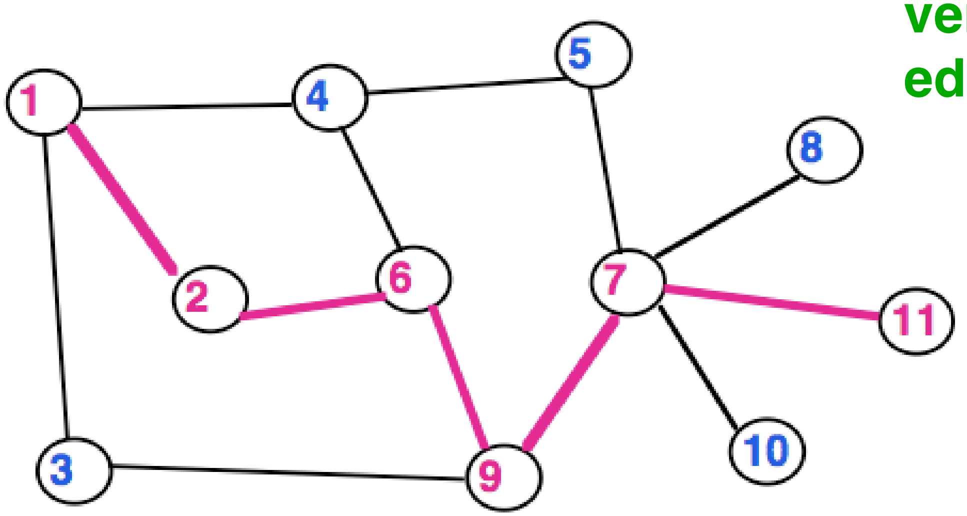
If a graph has e edges then

$$\sum_{v \in G} in\deg(v) = \sum_{v \in G} out\deg(v) = e$$

Why?

Path

- Can be one of
  - » Sequence of edges – counting rails
    - > Edges are head to tail in the sequence
  - » Sequence of adjacent vertices – counting fence posts
    - > Edges between adjacent vertices are head to tail



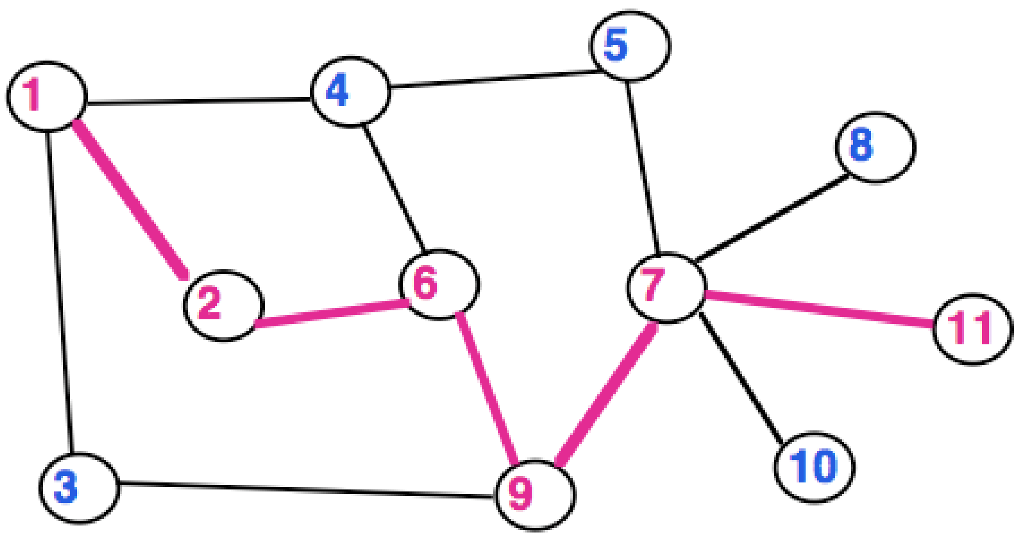
Path from vertex 1 to 11

vertices < 1 , 2 , 6 , 9, 7 , 11 >

edges < (1 , 2)
, (2 , 6)
, (6 , 9)
, (9 , 7)
, (7 , 11)
>

Simple Path

- No vertex occurs more than once in a path



Path from vertex 1 to 11

vertices < 1 , 2 , 6 , 9, 7 , 11 >

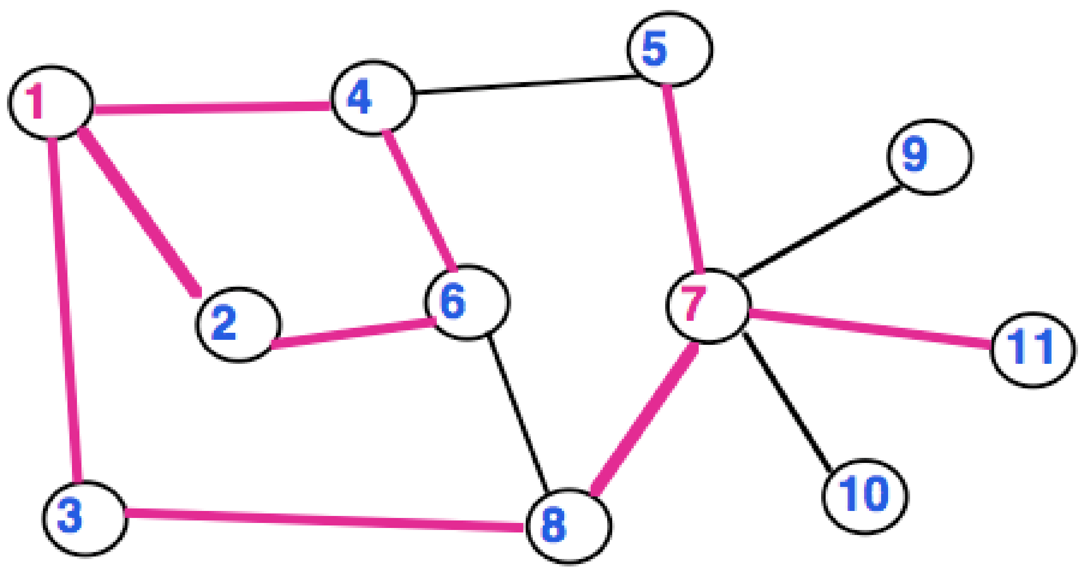
edges < (1 , 2)
, (2 , 6)
, (6 , 9)
, (9 , 7)
, (7 , 11)
>

Non-simple Path

- At least one vertex occurs more than once in a path

Path from vertex 1 to 11

vertices < 1 , 2 , 6 , 4 , 1 , 3 , 8 , 7 , 5 , 7 , 11 >

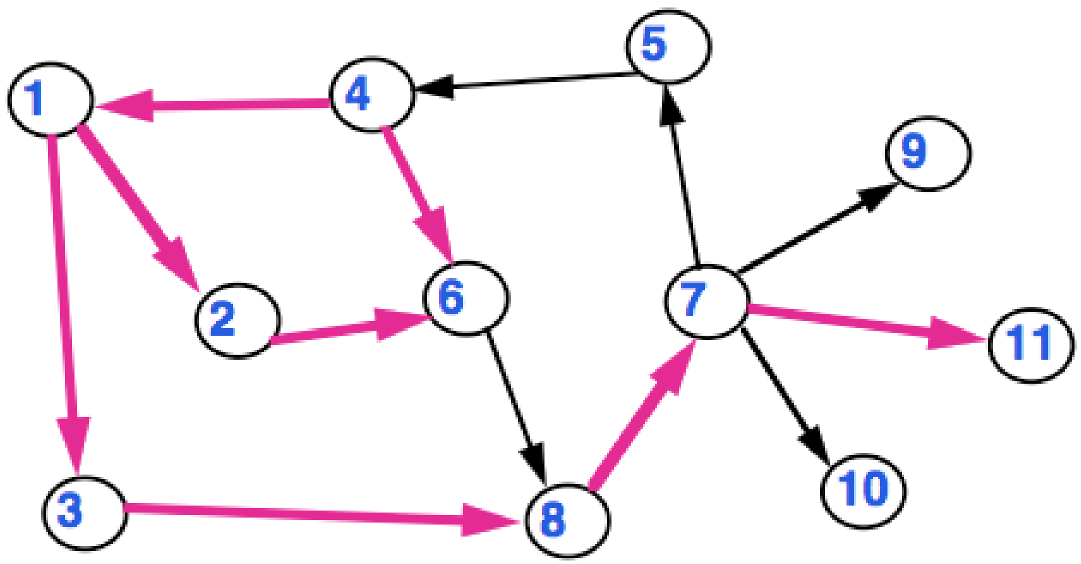


Directed Path

- In a directed graph a path follows the edge directions – follow the arrows

Path from vertex 1 to 11

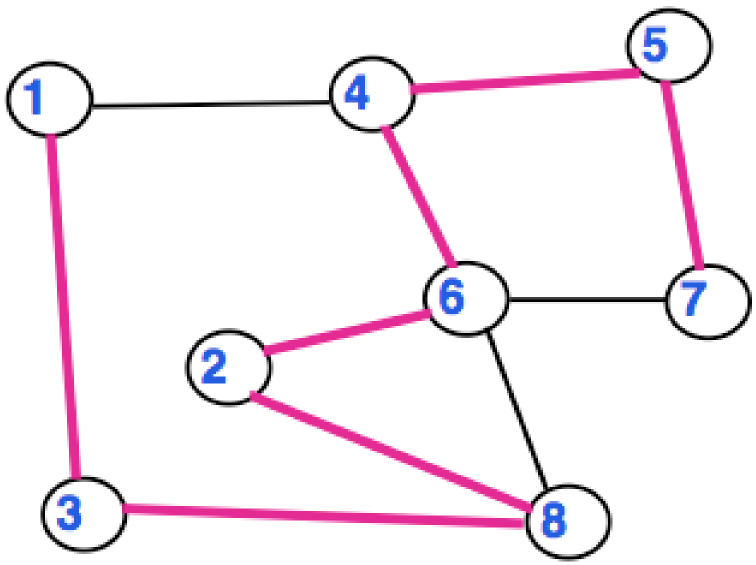
vertices < 1 , 2 , 6 , 4 , 1 , 3 , 8 , 7 , 11 >



Hamiltonian Path

- A path that visits all the vertices exactly once
  - » **Can be directed or undirected**

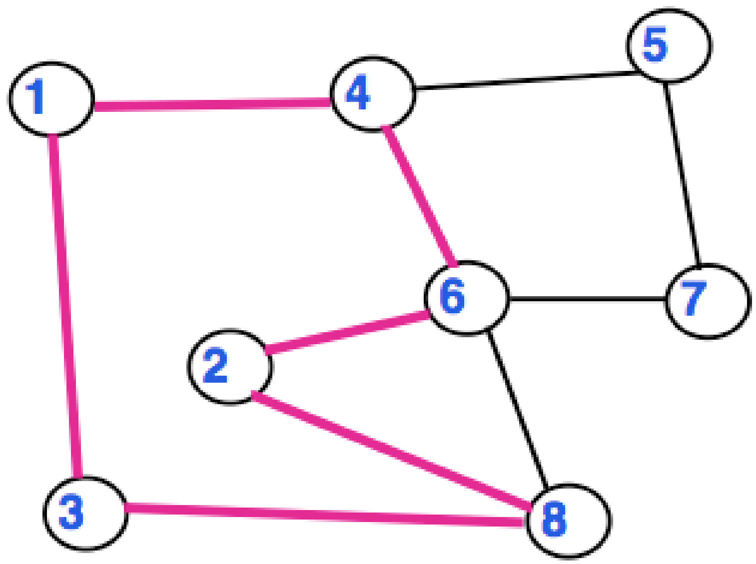
vertices < 1 , 3 , 8 , 2 , 6 , 4 , 5 , 7 >



Cycle

- A path that starts and ends with the vertex
  - » **Can be simple or non-simple, and directed or undirected**

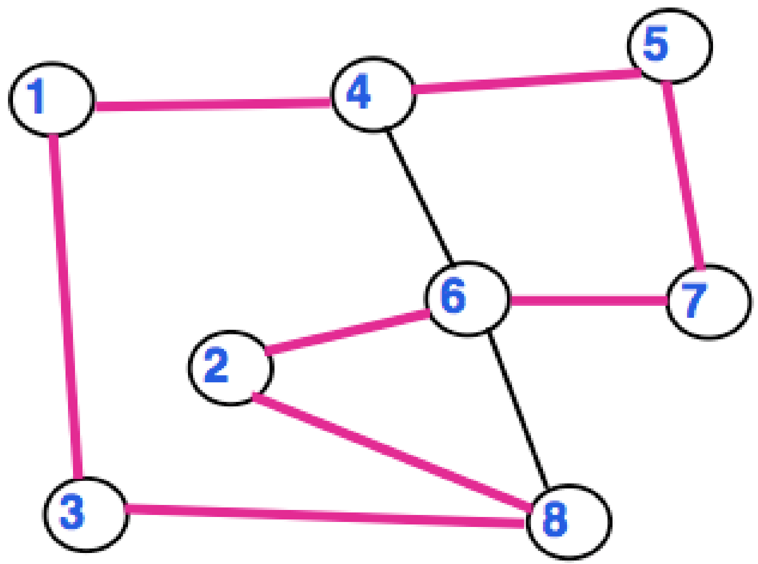
vertices < 1 , 3 , 8 , 2 , 6 , 4 , 1 >



Hamiltonian Cycle

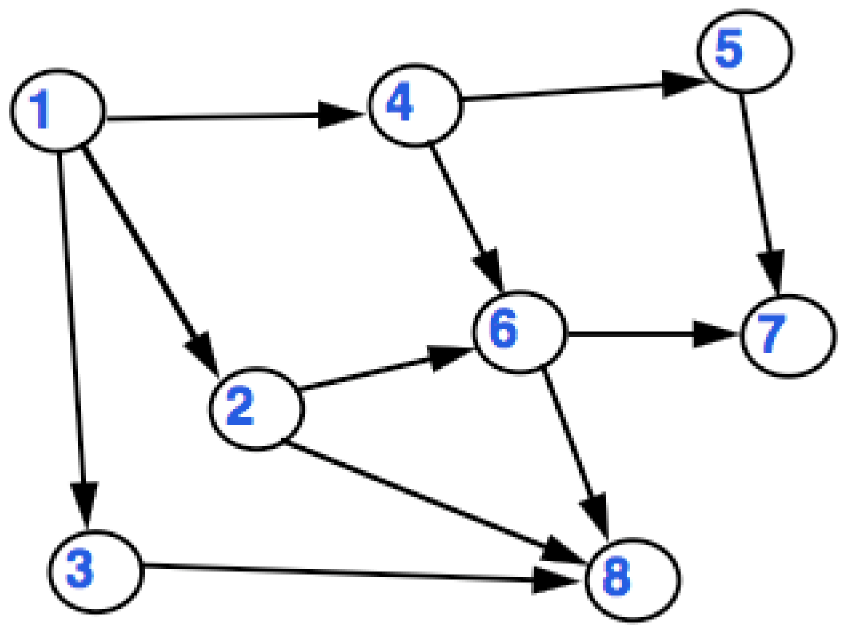
- A path that visits all the vertices exactly once, except for the first vertex, which is also the last vertex
  - » **Can be directed or undirected**

vertices < 1 , 3 , 8 , 2 , 6 , 7 , 5 , 4 , 1 >



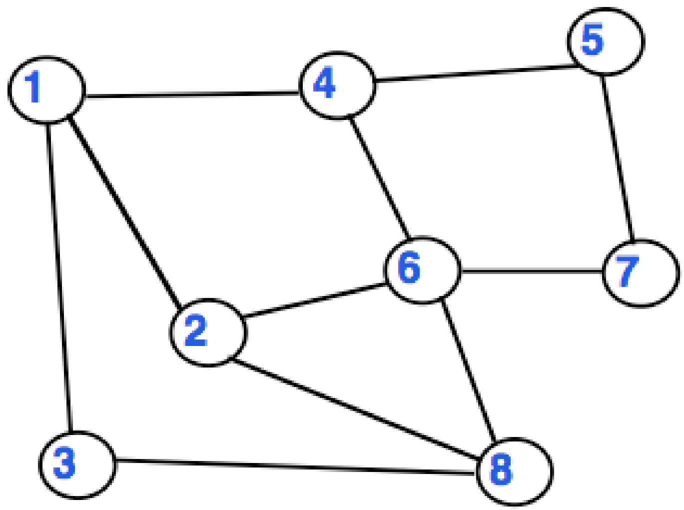
Acyclic Graph

- A directed graph with no cycles or self-loops
  - » **A tree is an acyclic graph**

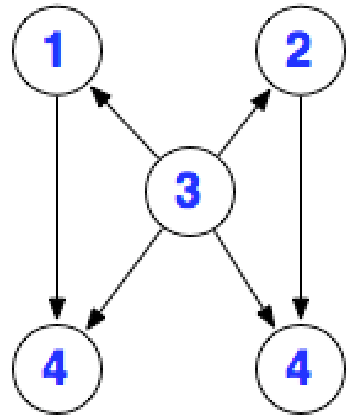


Connected Graph

- For any two vertices there exists an undirected path that joins them
  - » **For a directed graph, consider all edges to be undirected**



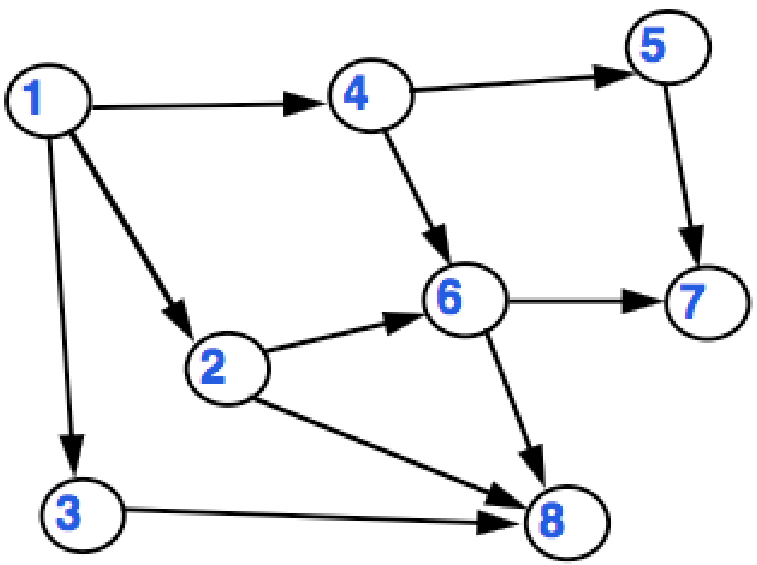
Connected undirected graph



Connected directed graph

Strongly Connected Graph

- For every pair of vertices in a directed graph there exists a directed path between them

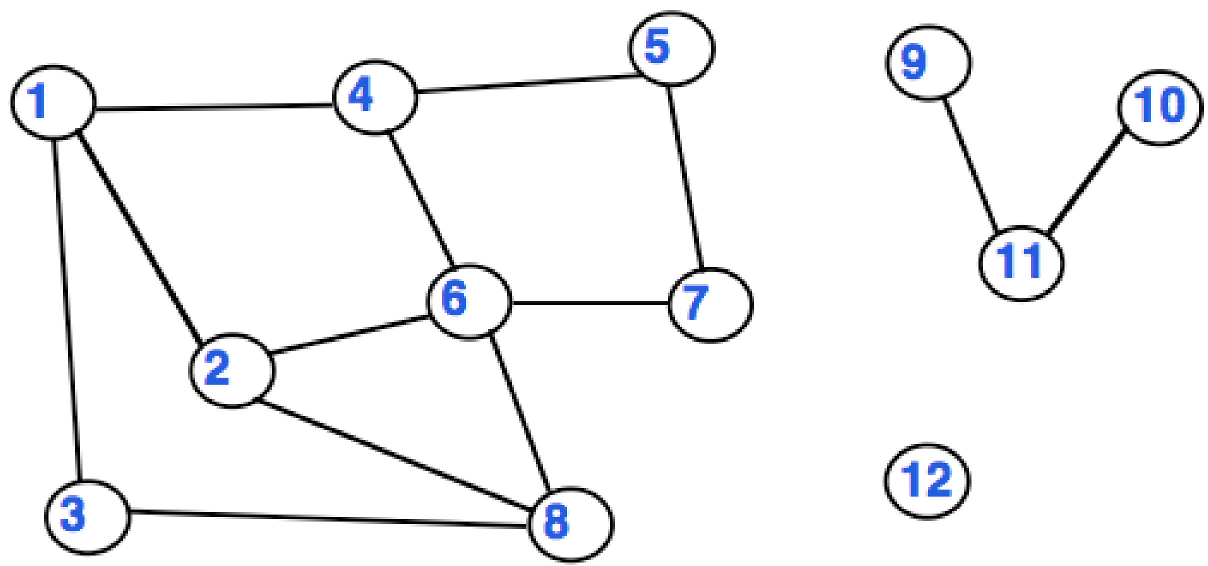


Not strongly connected
– no path from 8 to 6

Connected
– path from 6 to 8

Connected Components

- A connected component of a graph is a the largest subgraph such that it is a connected graph
  - » **The graph has 3 connected components**



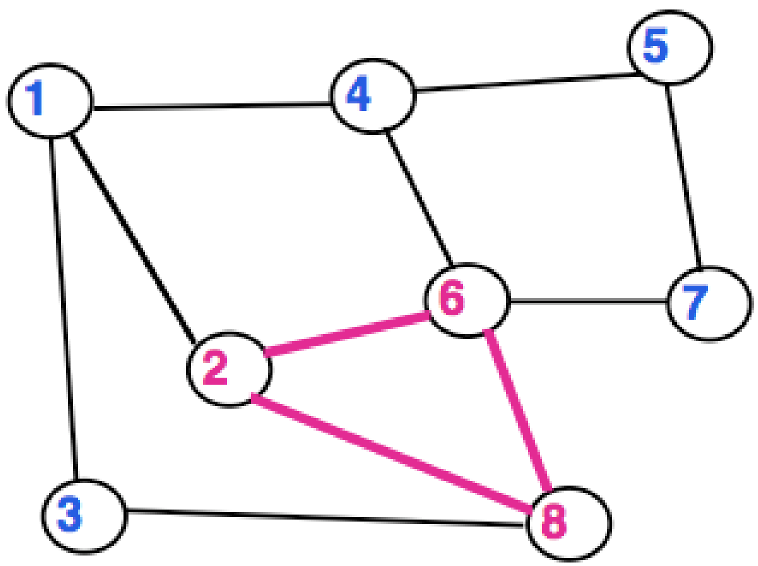
Subgraph

- A subgraph S of a graph G has the following components

  $V(S) \subseteq V(G)$

  $E(S) \subseteq E(G)$

  $\forall e : E(S) \bullet head(e) \in V(S) \wedge tail(e) \in V(S)$



Subgraph

Vertices $\{ 2 , 6 , 8 \}$

Edges $[(2, 6)$
$, (6, 8)$
$, (2, 8)$
$]$

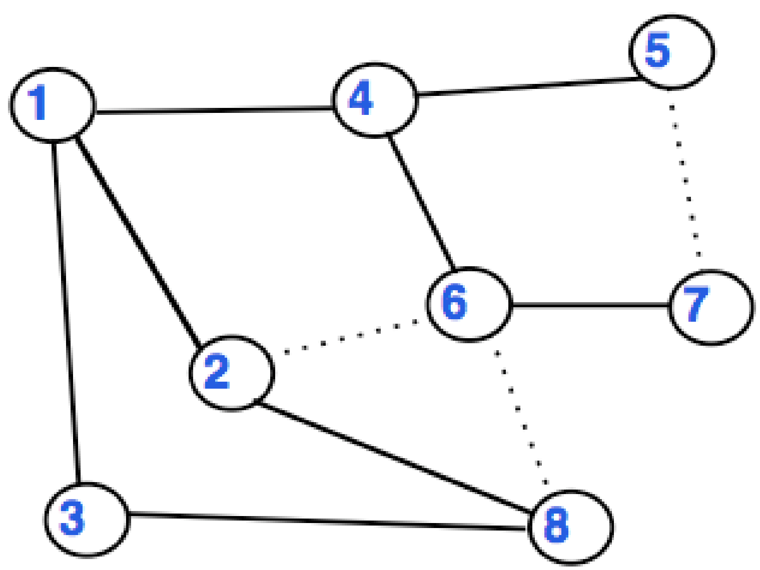
Spanning Subgraph

- Subgraph S of graph G contains all the vertices of G
    - > **Only has meaning if G is a connected component**

    » **Missing one or more edges**

$$V(S) = V(G)$$

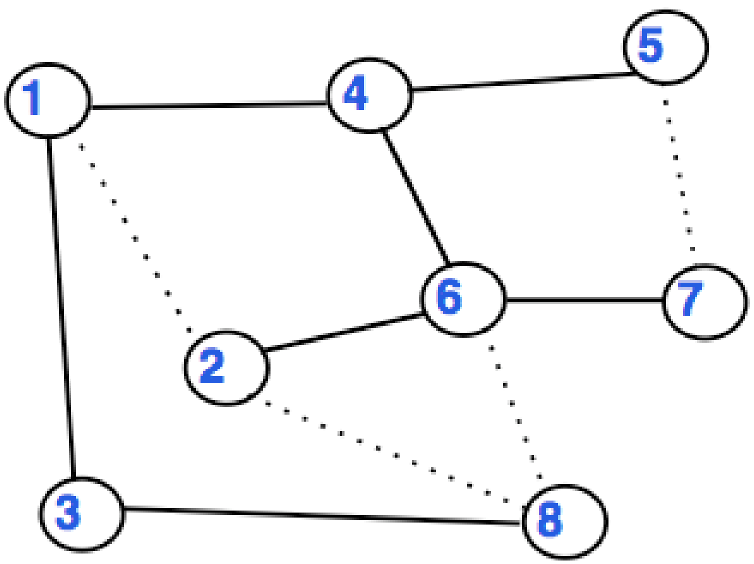
$$E(S) \subset E(G)$$



Spanning subgraph – removed edges (5 , 7), (6 , 8) and (2 , 6)

Spanning Tree

- Spanning subgraph such that the graph is a tree
  - » **In a graph it is called a free tree – no vertex is distinguished as the root**
  - » **The tree structure is called a rooted tree**

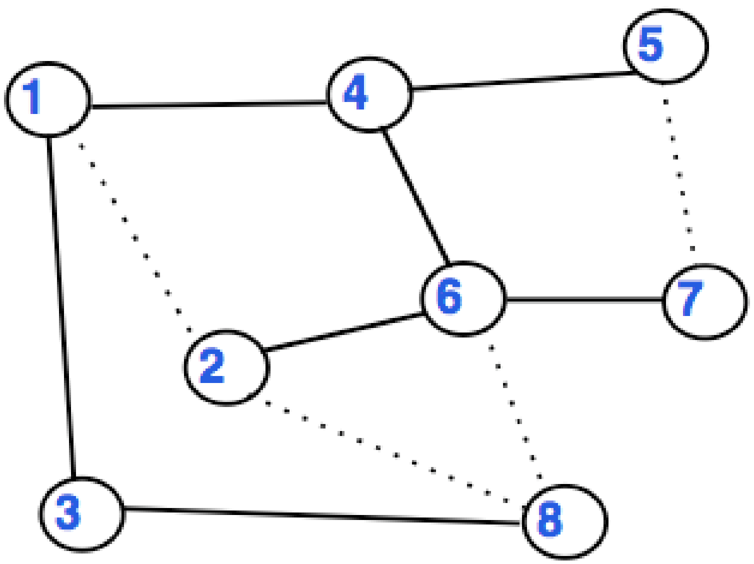


**Spanning tree
– removed edges
(5 , 7) , (6 , 8)
(1 , 2) and (2 , 8)**

Remove enough edges to break all cycles but leave the graph connected

Forest

- An undirected graph without cycles is called a **forest**
  - » **Any vertex could be considered to be the root of a tree**



Example
– the graph consisting only of the solid lines