Ejercicios Mecánica Clásica

- Tomás Rocha
- Óscar López

Punto 1

1. Calcule la trayectoria que da la distancia más corta entre dos puntos sobre la superficie de un cono invertido, con ángulo de vértice α. Use coordenadas cilíndricas.

Tenemos como ligadura a $z = r\cot\alpha$.

Por lo tanto, $q_1 = \theta$, $q_2 = r$

- Hallando la métrica:

$$ds^2 = dx^2 + dy^2 + dz^2 \quad \text{(cartesianas)}$$

$$ds^2 = dr^2 + r^2 d\theta^2 + dz^2 \quad \text{(cilíndricas)}$$

$$\frac{dz}{dr} = \cot\alpha \Rightarrow dz = \cot\alpha\, dr \Rightarrow dz^2 = \cot^2\alpha\, dr^2$$

- Reemplazando:

$$ds^2 = dr^2 + r^2 d\theta^2 + \cot^2\alpha\, dr^2$$

$$ds^2 = (1+\cot^2\alpha)\, dr^2 + d\theta^2$$

$$ds^2 = \csc^2\alpha\, dr^2 + d\theta^2$$

$$I = \int_a^b ds = \int_a^b \sqrt{\csc^2\alpha\, dr^2 + r^2 d\theta^2}$$

$$= \int_a^b \sqrt{\csc^2\alpha + r^2\left(\frac{d\theta}{dr}\right)^2}\, dr$$

$$\Rightarrow f(\theta, \dot{\theta}, r) = \sqrt{\csc^2\alpha + r^2\dot{\theta}^2}$$

• Por la ecuación de Euler:

$$\frac{d}{dr}\left(\frac{\partial f}{\partial \dot{\theta}}\right) - \frac{\partial f}{\partial \theta} = 0 \quad ; \quad \frac{\partial f}{\partial \theta} = 0$$

$$\frac{\partial f}{\partial \dot{\theta}} = \frac{r^2\dot{\theta}}{\sqrt{r^2\dot{\theta}^2 + \csc^2\alpha}} = h \rightsquigarrow cte$$

$$\left(\frac{r^2\dot{\theta}}{\sqrt{r^2\dot{\theta}^2 + \csc^2\alpha}}\right)^2 = h^2$$

$$\frac{r^4\dot{\theta}^2}{r^2\dot{\theta}^2 + \csc^2\alpha} = h^2$$

$$r^4\dot{\theta}^2 = h^2 r^2\dot{\theta}^2 + \csc^2\alpha$$

$$\dot{\theta}^2 = \frac{h^2\csc^2\alpha}{r^2(r^2 - h^2)}$$

$$\dot{\theta} = \frac{d\theta}{dr} = \frac{h\csc^2\alpha}{r\sqrt{r^2 - h^2}}$$

$$\theta = h\csc\alpha \int \frac{dr}{r\sqrt{r^2 - h^2}} = h\csc\alpha\left(\frac{\ln(r^2 - h^2) - \ln x^2}{2h^2} + K\right)$$

$$\Rightarrow \theta = \frac{\ln(r^2 - h^2) - \ln x^2}{2h}\csc\alpha + K$$

Problemas:

- Oscar Danilo López 2230659
- Tomás Santiago Rocha 2230672

2. Calcule el valor mínimo de la integral

$$I = \int_0^1 \left[(y')^2 + 12xy\right] dx$$

donde la función $y(x)$ satisface $y(0) = 0$ y $y(1) = 1$.

$$I = \int_0^1 \left[(y')^2 + 12xy\right] dx \qquad y(0)=0 \quad y(1)=1$$

funcional $\rightarrow$ $f(y', y, x) = (y')^2 + 12xy$

· Aplicamos la ecuación Euler: $\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) - \frac{\partial f}{\partial y} = 0$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left((y')^2 + 12xy\right) \qquad \frac{\partial f}{\partial y'} = \frac{\partial}{\partial y'}\left((y')^2 + 12xy\right)$$

$$\frac{\partial f}{\partial y} = 12x \qquad \frac{\partial f}{\partial y'} = 2y'$$

$$\frac{d}{dx}(2y') - 12x = 0$$

$$2y'' - 12x = 0$$

$$2y'' = 12x$$

$$y'' = 6x$$

· Integramos ambos lados:

$$y' = 3x^2 + C_1$$

· Volvemos a integrar:

$$\boxed{y = x^3 + C_1x + C_2}$$

· Con las condiciones iniciales encontramos el valor de las constantes:

· $y(0) = 0$ $\qquad y(0) = \cancel{(0)^3} + \cancel{C_1(0)} + C_2$

$$y(0) = C_2 \longrightarrow \underline{C_2 = 0}$$

· $y(1) = 1$

$$y(1) = (1)^3 + C_1(1)$$

$$y(1) = 1 + C_1 = 1$$

$$1 + C_1 = 1 \longrightarrow \underline{C_1 = 0}$$

$$\boxed{y(x) = x^3}$$

· Calculamos la integral remplazando la función.

$$I = \int_0^1 \left[(y')^2 + 12xy\right] dx \qquad y' = 3x^2$$

$$\int_0^1 \left[(3x^2)^2 + 12x(x^3)\right] dx$$

$$\int_0^1 \left[9x^4 + 12x^4\right] dx$$

$$\int_0^1 21x^4\, dx = \frac{21}{5}x^5 \Big|_0^1 = \frac{21(1)^5}{5} - \cancel{\frac{21(0)^5}{5}}$$

$$= \frac{21}{5}$$

Respuesta: El valor mínimo de la integral es $\frac{21}{5}$.

3. Encuentre la geodésica (i.e. la trayectoria de menor distancia) entre los puntos $P_1 = (a,0,0)$ y $P_2 = (-a,0,\pi)$ sobre la superficie $x^2+y^2-a^2=0$. Use coordenadas cilíndricas.

x y z

$x^2+y^2=a^2$

Cilíndricas: · $x = r\cos\theta$ · $y = r\sin\theta$

$r^2\cos^2\theta + r^2\sin^2\theta = a^2$

$r^2(\cancel{\cos^2\theta+\sin^2\theta}) = a^2$

$r=a$

Puntos a coordenadas cilíndricas:

P_1: $r=\sqrt{a^2+0}=a$

$\theta=\tan^{-1}(0/a)=0$

$z=0$

$\boxed{P_1(a,0,0)}$

P_2: $r=\sqrt{(-a)^2+0}=a$

$x=r\cos\theta$ → $-a=a\cos\theta$

$\cos\theta=-1$ → $\theta=\pi$

$\boxed{P_2(a,\pi,\pi)}$

$ds=\sqrt{dr^2+r^2d\theta^2+dz^2}$

$r=a \to dr=0$

$ds=\sqrt{a^2d\theta^2+dz^2}$

$S=\int_0^\pi \sqrt{a^2+\left(\frac{dz}{d\theta}\right)^2}\,d\theta$

· $\mathcal{L}=\sqrt{a^2+\dot z^2}$ · Ecuación Euler-Lagrange:

$\frac{d}{d\theta}\left(\frac{\partial \mathcal{L}}{\partial \dot z}\right)-\frac{\partial \mathcal{L}}{\partial z}=0$

$\frac{\partial \mathcal{L}}{\partial z}=0$, $\frac{\partial \mathcal{L}}{\partial\dot z}=\frac{\partial}{\partial \dot z}\left(\sqrt{a^2+\dot z^2}\right)$

$=\frac{1}{\cancel{2}\sqrt{a^2+\dot z^2}}(\cancel{2}\dot z)$

$\frac{\partial \mathcal{L}}{\partial \dot z}=\frac{\dot z}{\sqrt{a^2+\dot z^2}}$

$\frac{\partial \mathcal{L}}{\partial\dot z}=C \longrightarrow \frac{\dot z}{\sqrt{a^2+\dot z^2}}=C$

$\dot z = C\sqrt{a^2+\dot z^2}$

$\dot z^2 = C^2(a^2+z^2)$

$\dot z^2 - c^2\dot z^2 = c^2a^2$

$\dot z^2(1-c^2)=c^2a^2$

$\dot z^2 = \frac{c^2a^2}{(1-c^2)}$

$\dot z = \frac{ca}{\sqrt{1-c^2}}$

· Ahora integro para hallar z

$\boxed{z(\theta)=\frac{Ca}{\sqrt{1-c^2}}\theta + C_2}$

Hallo las constantes con las condiciones iniciales:

$z(0)=0$

$z(0)=\frac{Ca}{\sqrt{1-c^2}}(0)+C_2$

$z(0)=C_2$

$C_2=0$

$z(\theta)=\frac{ca}{\sqrt{1-c^2}}\theta$

$z(\pi)=\pi$

$z(\pi)=\frac{ca}{\sqrt{1-c^2}}\cancel{\pi}=\cancel{\pi}$

$\frac{ca}{\sqrt{1-c^2}}=1$

$c^2a^2=1-c^2$

$c^2a^2+c^2=1$

$c^2(a^2+1)=1$

$c^2=\frac{1}{(a^2+1)}$

$c=\frac{1}{\sqrt{a^2+1}}$

· Remplazo la constante C:

$z(\theta)=\frac{\left(\frac{1}{\sqrt{a^2+1}}\right)a}{\sqrt{1-\frac{1}{(a^2+1)}}}\theta$

$z(\theta)=\frac{(a/\sqrt{a^2+1})}{\sqrt{\frac{a^2}{a^2+1}}}\theta$

$z(\theta)=\frac{a\cancel{\sqrt{a^2+1}}}{\cancel{a\sqrt{a^2+1}}}\theta$

R/ $\boxed{z(\theta)=\theta}$

5. El Lagrangiano de una partícula de masa m es

$$\mathcal{L}=\frac{m^2\dot x^4}{12}+m\dot x^2 f(x)-f^2(x)$$

donde $f(x)$ es una función diferenciable de x. Encuentre la ecuación de movimiento.

$\mathcal{L}=\frac{m^2\dot x^4}{12}+m\dot x^2f(x)-f^2(x)$

Ecuación Euler Lagrange:

$\frac{d}{dt}\left[\frac{\partial \mathcal{L}}{\partial\dot x}\right]-\frac{\partial \mathcal{L}}{\partial x}=0$

$\frac{\partial \mathcal{L}}{\partial x}=\frac{\partial}{\partial x}\left[\frac{m^2\dot x^4}{12}+m\dot x^2f(x)-f^2(x)\right]$

$=m\dot x^2f'(x)-2f(x)f'(x)$.

$\frac{\partial \mathcal{L}}{\partial \dot x}=\frac{\partial}{\partial\dot x}\left[\frac{m^2\dot x^4}{12}+m\dot x^2f(x)-f^2(x)\right]$

$=\frac{1}{3}m^2\dot x^3+2m\dot x f(x)$

$\frac{d}{dt}\left[\frac{1}{3}m^2\dot x^3+2m\dot x f(x)\right]-m\dot x^2 f'(x)+2f(x)f'(x)=0$

$= \cancel{3}\dot x^2\ddot x + 2m\ddot x f(x)+2m\dot x^2 f'(x)$

$-3\dot x^2\ddot x$

$\frac{d}{dt}(f(x))=\frac{df(x)}{dx}\cdot\frac{\partial x}{\partial t}=f'(x)\dot x$

$m^2\dot x^2\ddot x+2m\ddot x f(x)\dot x+2m\dot x^2 f'(x)-m\dot x^2 f'(x)+2f(x)f'(x)=0$

R/ $\boxed{m^2\dot x^2\ddot x+2m\ddot x f(x)\dot x+m\dot x^2f'(x)+2f(x)f'(x)=0}$

Punto 4

4. Un cuerpo se deja caer desde una altura h y alcanza el suelo en un tiempo T. La ecuación de movimiento concebiblemente podría tener cualquiera de las formas

$$y = h - g_1 t, \quad y = h - \frac{1}{2} g_2 t^2, \quad y = h - \frac{1}{4} g_3 t^3$$

donde g_1, g_2, g_3 son constantes apropiadas. Demuestre que la forma correcta es aquella que produce el mínimo valor de la acción.

- **Primera ecuación:**

$$y = h - g_1 t \quad , \quad \dot{y} = -g_1$$

$$\mathcal{L} = \frac{1}{2} m \dot{y}^2 - m g y$$

$$\mathcal{L} = \frac{1}{2} m g_1^2 - m g (h - g_1 t)$$

$$S_1 = \int_0^T \left[\frac{1}{2} m g_1^2 - m g (h - g_1 t) \right] dt$$

$$S_1 = \frac{1}{2} m g_1^2 t - m g \left(h t - \frac{1}{2} g_1 t^2 \right) \Big|_0^T$$

$$S_1 = \frac{1}{2} m g_1^2 T - m g \left(h T - \frac{1}{2} g_1 T^2 \right)$$

- **Segunda ecuación:**

$$y = h - \frac{1}{2} g_2 t^2 \quad , \quad \dot{y} = -g_2 t$$

$$\mathcal{L} = \frac{1}{2} m g_2^2 t^2 - m g \left(h - \frac{1}{2} g_2 t^2 \right)$$

$$S_2 = \int_0^T \left[\frac{1}{2} m g_2^2 t^2 - mg\left(h - \frac{1}{2} g_2 t^2\right) \right] dt$$

$$S_2 = \frac{1}{6} m g_2^2 t^3 - mg\left(ht - \frac{1}{6} g_2 t^3\right) \Big|_0^T$$

$$S_2 = \frac{1}{6} m g_2^2 T^3 - mg\left(hT - \frac{1}{6} g_2 T^3\right)$$

- Tercera Ecuación:

$$y = h - \frac{1}{4} g_3 t^3 \quad , \quad \dot{y} = -\frac{3}{4} g_3 t^2$$

$$\mathcal{L} = \frac{9}{32} m g_3^2 t^4 - mg\left(h - \frac{1}{4} g_3 t^3\right)$$

$$S_3 = \int_0^T \left[\frac{9}{32} m g_3^2 t^4 - mg\left(h - \frac{1}{4} g_3 t^3\right) \right] dt$$

$$S_3 = \frac{9}{160} m g_3 t^5 - mg(ht - \frac{1}{16} g_3 t^4 \Big|_0^T$$

$$S_3 = \frac{9}{160} m g_3 T^5 - mg\left(hT - \frac{1}{16} g_3 T^4\right)$$

→ Se hallaron los valores de S_1, S_2 y S_3 mediante métodos computacionales y condiciones iniciales iguales.

$$\mathcal{L} = \frac{1}{2} g_{ab}(q_c)\, \dot{q}^a \dot{q}^b = \frac{1}{2} g_{cb}(q_c)\, \dot{q}^c \dot{q}^b$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}^a}\right) - \frac{\partial \mathcal{L}}{\partial q^a} = 0 \qquad (*)$$

$$\frac{\partial \mathcal{L}}{\partial \dot{q}^a} = \frac{1}{2} g_{cb}(q_c)\left(\frac{\partial \dot{q}^c}{\partial \dot{q}^a}\dot{q}^b + \dot{q}^c \frac{\partial \dot{q}^b}{\partial \dot{q}^a}\right)$$

$$= \frac{1}{2} g_{cb}(q_c)\left(\delta^c_a \dot{q}^b + \dot{q}^c \delta^b_a\right)$$

$$= \frac{1}{2} g_{cb}(q_c)\, \delta^c_a \dot{q}^b + \frac{1}{2} g_{cb}(q_c)\, \delta^b_a \dot{q}^c$$

$$= \frac{1}{2} g_{ab}(q_c)\, \dot{q}^b + \frac{1}{2} g_{ca}(q_c)\, \dot{q}^c \qquad ; \; g_{ca} = g_{ac}$$

$$= \frac{1}{2} g_{ab}(q_c)\, \dot{q}^b + \frac{1}{2} g_{ab}(q_c)\, \dot{q}^b$$

$$\frac{\partial \mathcal{L}}{\partial \dot{q}^a} = g_{ab}(q_c)\, \dot{q}^b$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}^a}\right) = \frac{d}{dt}\left(g_{ab}(q_c)\, \dot{q}^b\right)$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}^a}\right) = \frac{\partial g_{ab}}{\partial q^c}\dot{q}^c \dot{q}^b + g_{ab}(q_c)\, \ddot{q}^b$$

Ahora,

$$\frac{\partial \mathcal{L}}{\partial q^a} = \frac{1}{2}\frac{\partial g_{ab}}{\partial q^a}\dot{q}^a \dot{q}^b = \frac{1}{2}\frac{\partial g_{cb}}{\partial q^a}\dot{q}^c \dot{q}^b$$

Reemplazando en $(*)$,

$$\frac{\partial g_{ab}}{\partial q^c}\dot{q}^c \dot{q}^b + g_{ab}(q_c)\, \ddot{q}^b - \frac{1}{2}\frac{\partial g_{cb}}{\partial q^a}\dot{q}^c \dot{q}^b = 0$$

$$g_{ab}(q_c)\, \ddot{q}^b + \left(\frac{\partial g_{ab}}{\partial q^c} - \frac{1}{2}\frac{\partial g_{cb}}{\partial q^a}\right)\dot{q}^c \dot{q}^b = 0$$

Multiplico por g^{ad}

$$\underbrace{g^{ad} g_{ab}}\ddot{q}^b + g^{ad}\left(\frac{\partial g_{ab}}{\partial q^c} - \frac{1}{2}\frac{\partial g_{bc}}{\partial q^a}\right)\dot{q}^c \dot{q}^b = 0$$

$$(g^{da} g_{ab})^T = (\delta^d{}_b)^T = \delta^b{}_d$$

$$\ddot{q}^d + \frac{1}{2} g^{ad}\left(2\frac{\partial g_{ab}}{\partial q^c} - \frac{\partial g_{bc}}{\partial q^a}\right)\dot{q}^c \dot{q}^b = 0$$

$$\hookrightarrow 2\frac{\partial g_{ab}}{\partial q^c} = \frac{\partial g_{ab}}{\partial q^c} + \frac{\partial g_{ab}}{\partial q^c} = \frac{\partial g_{ab}}{\partial q^c} + \frac{\partial g_{ca}}{\partial q^b}$$

$$\Rightarrow \ddot{q}^d + \frac{1}{2} g^{ad}\left(\frac{\partial g_{ab}}{\partial q^c} + \frac{\partial g_{ca}}{\partial q^b} - \frac{\partial g_{bc}}{\partial q^a}\right)\dot{q}^c \dot{q}^b = 0$$

$$\Rightarrow \boxed{\ddot{q}^d + \Gamma^a_{bc}\,\dot{q}^b \dot{q}^c = 0}$$