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Solution for the first mid-term examination.

Task 1.

Initial equation: $12x^2 + 11y^2 + 2001xyy' = 0$

Given homogeneous first-order non-linear ordinary differential equation of 2-nd degree.
Differential form: $(12x^2 + 11y^2)dx + (2001xy)dy = 0$.
Let $P(x, y) = 12x^2 + 11y^2$, $Q(x, y) = 2001xy$.
$P(kx, ky) = k^2P(x, y)$, $Q(kx, ky) = k^2Q(x, y)$ for all $x, y, k \in \mathbb{R}$.

Assume that $x \neq 0$ and $y \neq 0$.
Divide both sides of the initial equation by $2001xy \neq 0$: $\frac{4x}{667y} + \frac{11y}{2001x} + y' = 0$.
Rewrite obtained equation as follows:

$$y' + \frac{11}{2001x}y = -\frac{4x}{667}y^{-1} \quad (1)$$

This is the Bernoulli equation of form $y' + g(x)y = f(x)y^k$ with $g(x) = \frac{11}{2001x}$, $f(x) = -\frac{4x}{667}$ and $k = -1$.

Consider complementary equation: $y_c' + \frac{11}{2001x}y_c = 0$, where y_c is a non-trivial solution of the complementary equation.
Divide both sides by $y_c \neq 0$: $\frac{y_c'}{y_c} + \frac{11}{2001x} = 0$.
Obtained differential equation $\frac{y_c'}{y_c} = -\frac{11}{2001x}$ is separable.
Convert this equation to differential form: $\frac{dy_c}{y_c} = -\frac{11dx}{2001x}$.
In order to solve it integrate both sides: $\int \frac{dy_c}{y_c} = -\frac{11}{2001}\int \frac{dx}{x}$.
Intermediate result: $\ln|y_c| = -\frac{11}{2001}\ln|x|$.
Exponentiate both sides: $e^{\ln y_c} = e^{\ln x^{-\frac{11}{2001}}}$. Here function $x^{-\frac{11}{2001}}$ defined on $x \in (0; +\infty)$.
After simplification, the equation has the following form: $y_c = x^{-\frac{11}{2001}}$.

Let $y = uy_c = ux^{-\frac{11}{2001}}$, where $u = u(x)$.
Find the derivative of y: $y' = (ux^{-\frac{11}{2001}})' = u'x^{-\frac{11}{2001}} - \frac{11u}{2001}x^{-\frac{2012}{2001}}$.
Furthermore, make a substitution $y = ux^{-\frac{11}{2001}}$ in the equation (1):
$u'x^{-\frac{11}{2001}} - \frac{11u}{2001}x^{-\frac{2012}{2001}} + \frac{11u}{2001}x^{-\frac{2012}{2001}} = -\frac{4x}{667}(ux^{-\frac{11}{2001}})^{-1}$.
The second and third terms on left hand side are cancelled out: $u'x^{-\frac{11}{2001}} = -\frac{4x}{667}(ux^{-\frac{11}{2001}})^{-1}$.
Divide both sides of obtained equation by $u^ky_c = u^{-1}x^{-\frac{11}{2001}} \neq 0$.

We get $u'u = -\frac{4}{667}x^{\frac{2023}{2001}}$.
Convert this equation to differential form: $udu = -\frac{4}{667}x^{\frac{2023}{2001}}dx$.
Integrate both sides of equation: $\int udu = -\frac{4}{667}\int x^{\frac{2023}{2001}}dx$.
Intermediate result: $\frac{u^2}{2} = -\frac{4}{667} \cdot \frac{2001}{4024}x^{\frac{4024}{2001}} + C_1$, where constant $C_1 \in \mathbb{R}$.
After simplification, the equation has the following form: $u^2 = -\frac{3}{503}x^{\frac{4024}{2001}} + C_2$, where $C_2 = 2C_1$.
Take the square root of both sides: $u = \pm\sqrt{-\frac{3}{503}x^{\frac{4024}{2001}} + C_2}$.
Finally, make back substitution of u in y:
$y = ux^{-\frac{11}{2001}} = \pm x^{-\frac{11}{2001}}\sqrt{-\frac{3}{503}x^{\frac{4024}{2001}} + C_2} = \pm\sqrt{-\frac{3}{503}x^{\frac{4002}{2001}} + C_2x^{-\frac{22}{2001}}} = \pm\sqrt{\frac{-3x^2+503C_2x^{-\frac{22}{2001}}}{503}} = \pm\sqrt{\frac{-3x^2+C_3x^{-\frac{22}{2001}}}{503}}$, where $C_3 = 503C_2$.

Let's determine the constraints on x. One constraint was found during complementary equation solving, i.e., $x > 0$. Also, the numerator should be greater or equal than zero: $-3x^2 + C_3x^{-\frac{22}{2001}} \geqslant 0$. The solution of this inequality is $x \leqslant (\frac{C_3}{3})^{\frac{2001}{4024}}$. Therefore, $0 < x \leqslant (\frac{C_3}{3})^{\frac{2001}{4024}}$, where $C_3 > 0$.

Consider our assumptions $x \neq 0$ and $y \neq 0$.
Substitute them in the initial equation:
for $x = 0$ we get $12 \cdot 0^2 + 11y^2 + 2001 \cdot 0yy' = 0$. It yields us $11y^2 = 0$, so $x = 0$ is not a trivial solution.
for $y = 0$ we get $12x^2 + 11 \cdot 0^2 + 2001x \cdot 0 \cdot 0 = 0$. It yields us $12x^2 = 0$, so $y = 0$ is not a trivial solution.

Answer: given equation does not have a trivial solution and has the most general non-trivial solution for the initial equation:
$y = \pm\sqrt{\frac{-3x^2+C_3x^{-\frac{22}{2001}}}{503}}$ on $\{x|\ x \in \mathbb{R}\ \wedge\ C_3 \in \mathbb{R}\ \wedge\ C_3 > 0\ \wedge\ (0 < x \leqslant (\frac{C_3}{3})^{\frac{2001}{4024}})\}$.

Task 2.
Initial equation: $dy - y(12 + xy^{11})dx = 0$

Given non-homogeneous first-order non-linear ordinary differential equation.
Let $P(x, y) = -y(12 + xy^{11})$, $Q(x, y) = 1$.
$P(kx, ky) \neq k^nP(x, y)$, $Q(kx, ky) \neq k^nQ(x, y)$ for all $x, y, k \in \mathbb{R}$ and $n \in \mathbb{N}$.

Firstly, let's convert equation to explicit form: $\frac{dy}{dx} - y(12 + xy^{11}) = 0$.
Open the parenthesis and rewrite obtained equation as follows:

$$y' - 12y = xy^{12} \quad (2)$$

This equation is the Bernoulli equation of form $y' + g(x)y = f(x)y^k$ with $g(x) = -12$, $f(x) = x$, $k = 12$.

Consider complementary equation: $y_c' - 12y_c = 0$, where y_c is a non-trivial solution of the complementary equation.
Divide both sides by $y_c \neq 0$: $\frac{y_c'}{y_c} - 12 = 0$.
Obtained differential equation $\frac{y_c'}{y_c} = 12$ is separable.
Convert this equation to differential form: $\frac{dy_c}{y_c} = 12dx$.
In order to solve it firstly integrate both sides: $\int \frac{dy_c}{y_c} = 12 \int dx$.
Intermediate result: $\ln|y_c| = 12x$.
Exponentiate both sides: $e^{\ln y_c} = e^{12x}$.
After simplification, the equation has the following form: $y_c = e^{12x}$.

Let $y = uy_c = ue^{12x}$, where $u = u(x)$.
Assume $u \neq 0$.
Find the derivative of y: $y' = (ue^{12x})' = u'e^{12x} + 12ue^{12x}$.
Furthermore, make a substitution $y = ue^{12x}$ in the equation (2):
$u'e^{12x} + 12ue^{12x} - 12ue^{12x} = x(ue^{12x})^{12}$.
The second and third terms on left hand side are cancelled out: $u'e^{12x} = x(ue^{12x})^{12}$.
Divide both sides of obtained equation by $u^k y_c = u^{12}e^{12x} \neq 0$: $\frac{u'}{u^{12}} = xe^{132x}$.
Convert this equation to differential form: $\frac{du}{u^{12}} = xe^{132x}dx$.
Integrate both sides of equation: $\int \frac{du}{u^{12}} = \int xe^{132x}dx$.
Intermediate result: $-\frac{u^{-11}}{11} = \frac{xe^{132x}}{132} - \frac{e^{132x}}{17424} + C_1$, where constant $C_1 \in \mathbb{R}$.
Multiply both sides of the equation by -11: $u^{-11} = -\frac{xe^{132x}}{12} + \frac{e^{132x}}{1584} + C_2$, where $C_2 = -11C_1$.
Raise both sides to the power of $-\frac{1}{11}$: $u = \sqrt[11]{\frac{1}{-\frac{xe^{132x}}{12} + \frac{e^{132x}}{1584} + C_2}}$.
Finally, make back substitution of $u(x)$ in y: $y = ue^{12x} = e^{12x}\sqrt[11]{\frac{1}{-\frac{xe^{132x}}{12} + \frac{e^{132x}}{1584} + C_2}} = \sqrt[11]{\frac{e^{132x}}{-\frac{xe^{132x}}{12} + \frac{e^{132x}}{1584} + C_2}} = \sqrt[11]{\frac{1584e^{132x}}{-132xe^{132x} + e^{132x} + 1584C_2}} = \sqrt[11]{\frac{1584}{-132x + 1 + C_3e^{-132x}}}$,where $C_3 = 1584C_2$.

Let's determine the constraints on x. The denominator should be non-zero: $-132x + 1 + C_3e^{-132x} \neq 0$.

Consider our assumption $u \neq 0$:

if $u = 0$, then $y = 0 \cdot e^{12x} = 0$. Substitute $y = 0$ in the initial equation: $0 - 0 \cdot (12 + x \cdot 0^{11})dx = 0$. It yields us $0 = 0$, so $y = 0$ is the trivial solution.

Answer: given equation has trivial solution $y \equiv 0$ on $\mathbb{R}$ and the general solution $y = \sqrt[11]{\frac{1584}{-132x+1+C_3e^{-132x}}}$ on $\{x|\ x \in \mathbb{R}\ \wedge C_3 \in \mathbb{R} \wedge (-132x + 1 + C_3e^{-132x}) \neq 0\}$.

Task 3.

Given homogeneous second-order linear ordinary differential equation $12y'' + 11y = 0$. We have to prove or refute that $y = C_1 \sin(\sqrt{\frac{11}{12}}x) + C_2 \cos(\sqrt{\frac{11}{12}}x)$ is the most general solution of the given equation.

Primarily, let's prove that $y = C_1 \sin(\sqrt{\frac{11}{12}}x) + C_2 \cos(\sqrt{\frac{11}{12}}x)$ is the solution of the given equation.
Assume that $y = C_1 \sin(\sqrt{\frac{11}{12}}x) + C_2 \cos(\sqrt{\frac{11}{12}}x)$ is the solution of the given equation. Then its first and second order derivatives will be defined as follows:
$y' = \sqrt{\frac{11}{12}}C_1 \cos(\sqrt{\frac{11}{12}}x) - \sqrt{\frac{11}{12}}C_2 \sin(\sqrt{\frac{11}{12}}x)$.
$y'' = -\frac{11}{12}C_1 \sin(\sqrt{\frac{11}{12}}x) - \frac{11}{12}C_2 \cos(\sqrt{\frac{11}{12}}x)$.
Substitute y, y', and y'' back in source equation:
$12(-\frac{11}{12}C_1 \sin(\sqrt{\frac{11}{12}}x) - \frac{11}{12}C_2 \cos(\sqrt{\frac{11}{12}}x)) + 11(C_1 \sin(\sqrt{\frac{11}{12}}x) + C_2 \cos(\sqrt{\frac{11}{12}}x)) = 0$
$-11C_1 \sin(\sqrt{\frac{11}{12}}x) - 11C_2 \cos(\sqrt{\frac{11}{12}}x) + 11C_1 \sin(\sqrt{\frac{11}{12}}x) + 11C_2 \cos(\sqrt{\frac{11}{12}}x) = 0$
All terms of left hand side are cancelled out and the equation takes the form $0 = 0$. That is true. Therefore, our assumption is true and we proved that $y = C_1 \sin(\sqrt{\frac{11}{12}}x) + C_2 \cos(\sqrt{\frac{11}{12}}x)$ is the solution for $12y'' + y = 0$ by construction.

Unfortunately, I didn't find the way to prove that $y = C_1 \sin(\sqrt{\frac{11}{12}}x) + C_2 \cos(\sqrt{\frac{11}{12}}x)$ is the most general solution of the given equation.