Brief review of kinematic formulas for one-dimensional motion

- Average vectors of velocity and acceleration of a point:

$$\langle v \rangle = \frac{\Delta x}{\Delta t} \quad , \qquad \langle a \rangle = \frac{\Delta v}{\Delta t} \quad ,$$

where $\Delta \mathbf{r}$ is the displacement vector (an increment of a radius vector).

- Velocity and acceleration of a point:

$$v = \frac{dx}{dt} \quad , \qquad a = \frac{dv}{dt}$$

- Distance covered by a point:

$$s = \int v \, dt,$$

where v is the *modulus* of the velocity vector of a point.

More accurate consideration

$$S(T) = Si + \int_0^T v(t)\,dt$$ The time dependent position (or displacement)

$$v(T) = Vi + \int_0^T a(t)\,dt$$ The time dependent velocity, a(t) is an arbitrary acceleration

$$S(T) = Si + \int_0^T v(t)\,dt = Si + \int_0^T \left(Vi + \int_0^t a(h)\,dh\right) dt = \boxed{Si + \int_0^T Vi\,dt + \int_0^T \left(\int_0^t a(h)\,dh\right) dt}$$

More general expression for S(T), where:

Si – initial position;

Vi – initial velocity;

T – current time;

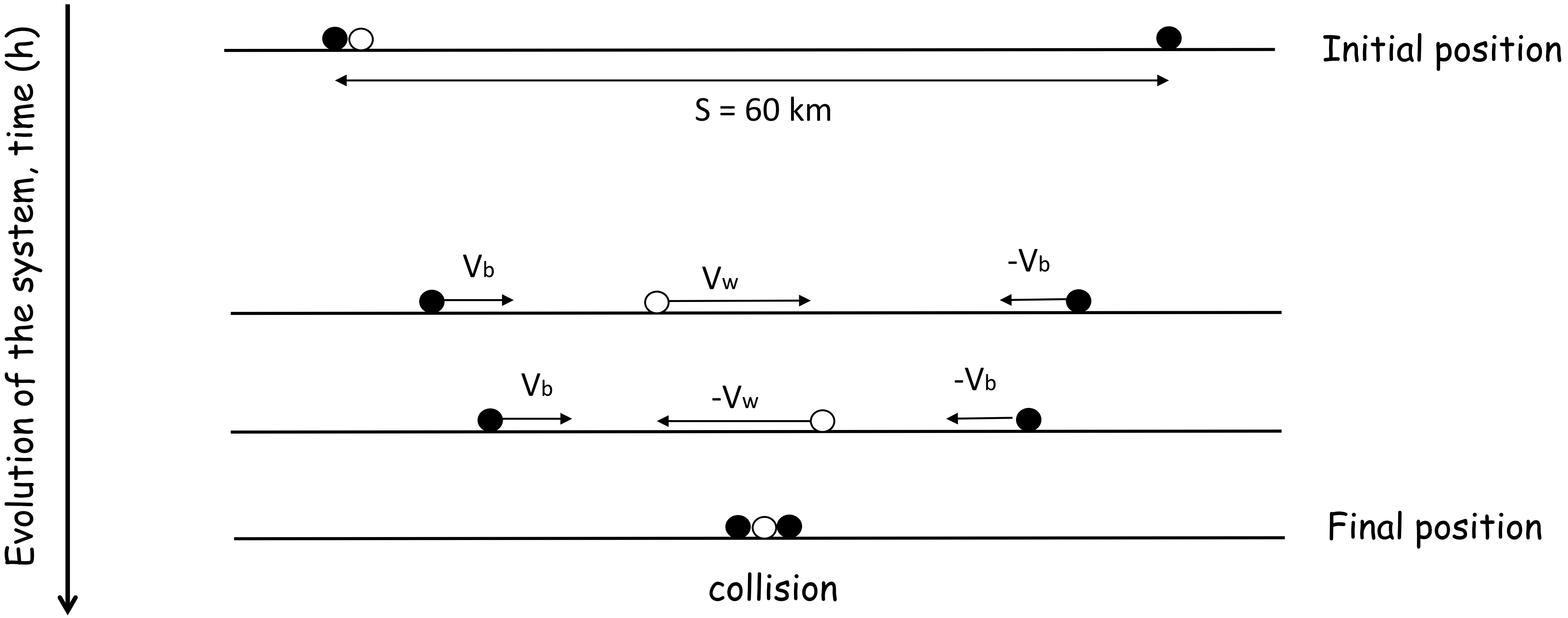
t and h – time variables;

When the acceleration is constant, we get the famous expression for S(T)

$$\left[S(T) = Si + \int_0^T Vi\,dt + \int_0^T \left(\int_0^t a\,dh\right) dt\right] \rightarrow S(T) = \frac{a \cdot T^2}{2} + Vi \cdot T + Si$$

1.

Two ~~trains~~ black material points (MPs), each having a speed of $V_b = 30$ km/h, are headed at each other on the same straight track. A ~~bird~~ white MP that can fly Vw = 60 km/h flies off the front of one ~~train~~ black MP when they are 60 km apart and heads directly for the other ~~train~~ black MP. On reaching the other ~~train~~ black MP, the ~~(crazy) bird~~ white MP flies directly back to the first ~~train~~ black MP, and so forth. What is the total distance the ~~bird~~ white MP travels before the ~~trains~~ MPs collide?



1.

Two black material points (MPs), each having a speed of $V_b = 30$ km/h, are headed at each other on the same straight track. A white MP that can fly $V_w = 60$ km/h flies off the front of one black MP when they are 60 km apart and heads directly for the other black MP. On reaching the other black MP, the white MP flies directly back to the first black MP, and so forth. What is the total distance the white MP travels before the MPs collide?

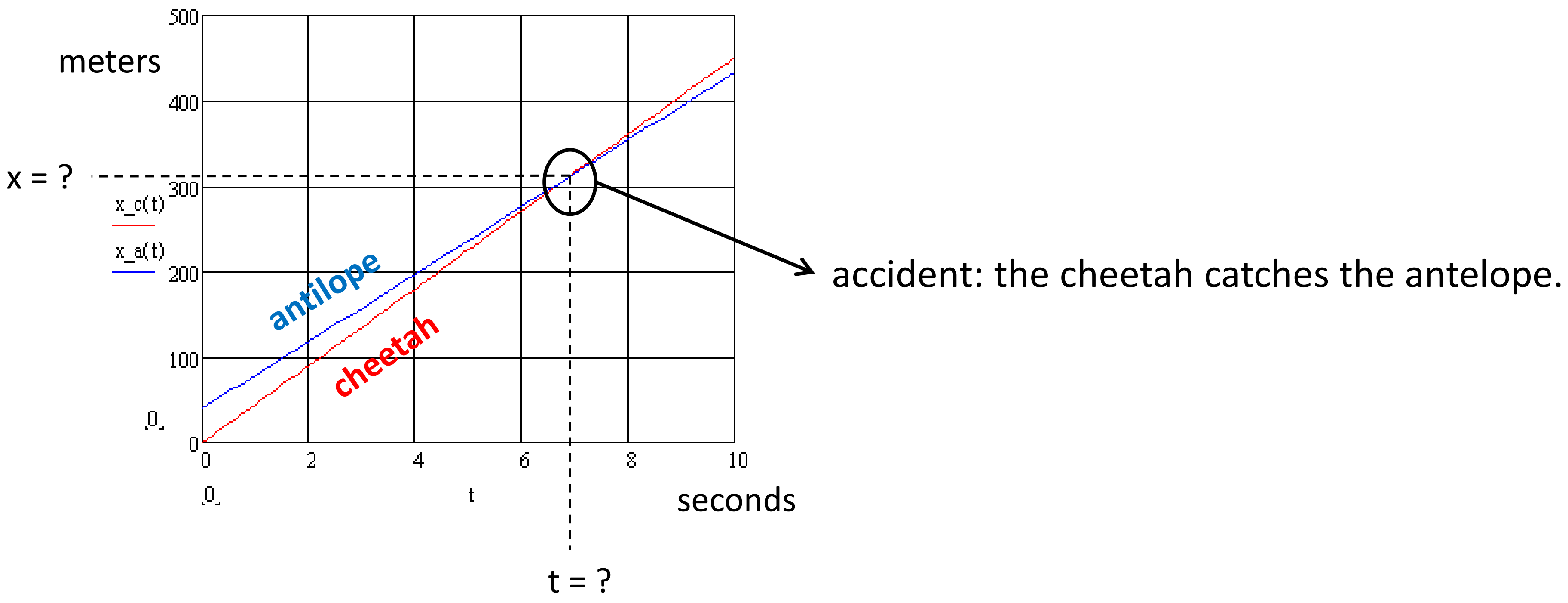
Let's use the fact, that the travel times of MPs (black and white, t_b and t_w) are the same:

$$t_b = \frac{(S/2)}{V_b} = 1(h) = t_w$$

The total distance for the white MP:

$$S_w = V_w \cdot t_w = 60\ (km)$$

2. A cheetah is the fastest land mammal, and it can run at speeds of about 101 km/h for a period of perhaps 20 s. The next fastest land animal is an antelope, which can run at about 88 km/h for a much longer time. Suppose a cheetah is chasing an antelope, and both are running at top speed, (a) **If the antelope has a 40-m head start, how long will it take the cheetah to catch him, and how far will the cheetah travel in this time?**



2. A cheetah is the fastest land mammal, and it can run at speeds of about 101 km/h for a period of perhaps 20 s. The next fastest land animal is an antelope, which can run at about 88 km/h for a much longer time. Suppose a cheetah is chasing an antelope, and both are running at top speed, (a) **If the antelope has a 40-m head start, how long will it take the cheetah to catch him, and how far will the cheetah travel in this time?**

Solution (a) The speeds are constant, so Eq. 3.10, $x = vt$, applies. Both animals run for the same time, but the cheetah must run 40 m extra. Thus

$$x_C = v_C t = x_A + 40 \qquad (i)$$

and

$$x_A = v_A t \qquad (ii)$$

Substitute *ii* in *i* and solve for *t*:

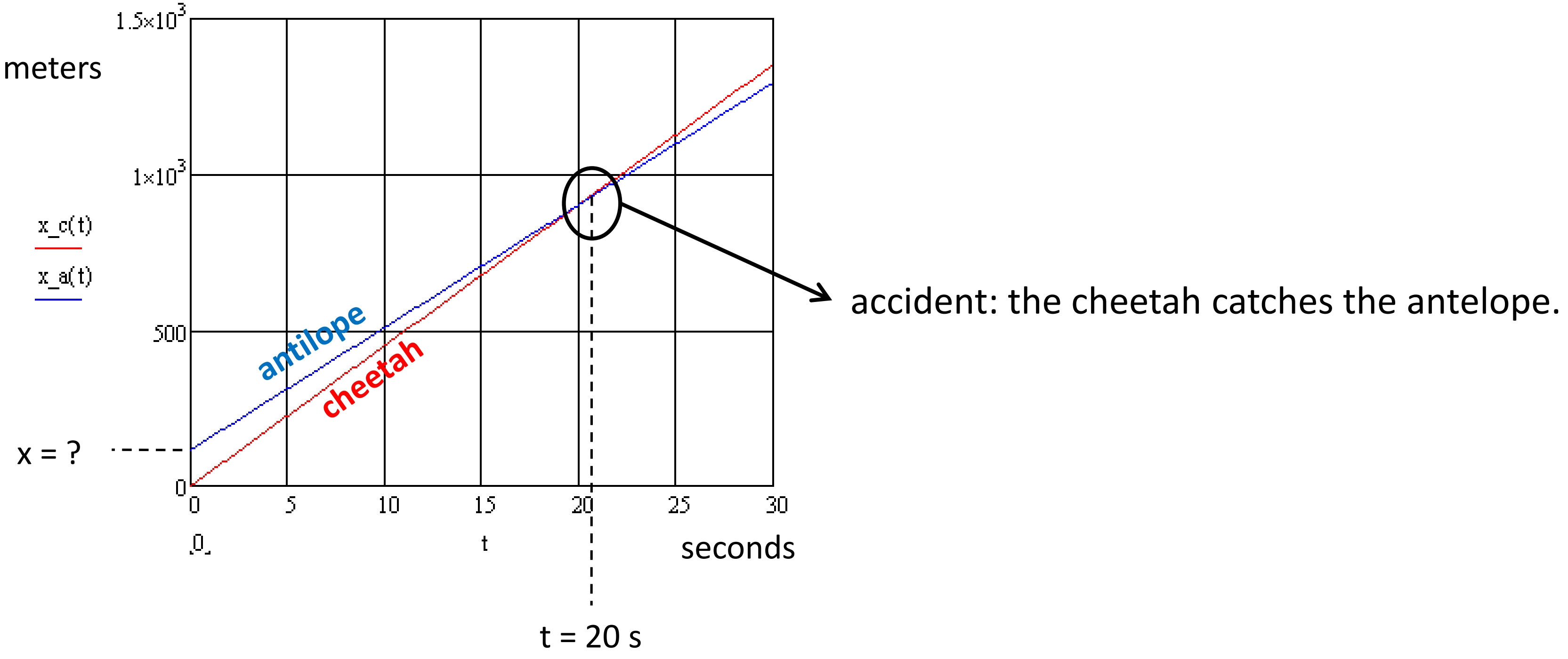
$$v_C t = v_A t + 40 \qquad (v_C - v_A)t = 40 \qquad t = \frac{40}{v_C - v_A}$$

The speeds must be expressed in meters per second, not kilometers per hour.

$$v_C = 101\,\text{km/h} = 28\ \text{m/s}$$

$$v_A = 88\,\text{km/h} = 24.4\ \text{m/s} \qquad t = 11.1\ \text{s}$$

2. A cheetah is the fastest land mammal, and it can run at speeds of about 101 km/h for a period of perhaps 20 s. The next fastest land animal is an antelope, which can run at about 88 km/h for a much longer time. Suppose a cheetah is chasing an antelope, and both are running at top speed. (b) **What is the maximum head start the antelope can have if the cheetah is to catch him within 20 s (at which time the cheetah runs out of breath)?**



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Solution

(b) Let $h =$ head start distance and $t = 20$ s for both animals. If the cheetah is to catch the antelope, then $x_C = x_A + h$.

$$x_C = v_C t \qquad\qquad x_A = v_A t$$

So $\quad v_C t = v_A t + h \qquad h = (v_C - v_A)t = 72\ m$

3. A motorist traveling 31 m/s passes a stationary motorcycle police officer. 2.5 s after the motorist passes, the police officer starts to move and accelerates in pursuit of the speeding motorist. The motorcycle has constant acceleration of 3.6 m/s^2. (a) How fast will the police officer be traveling when he overtakes the car? Draw curves of x versus t for both the motorcycle and the car, taking t = 0 at the moment the car passes the stationary police officer.

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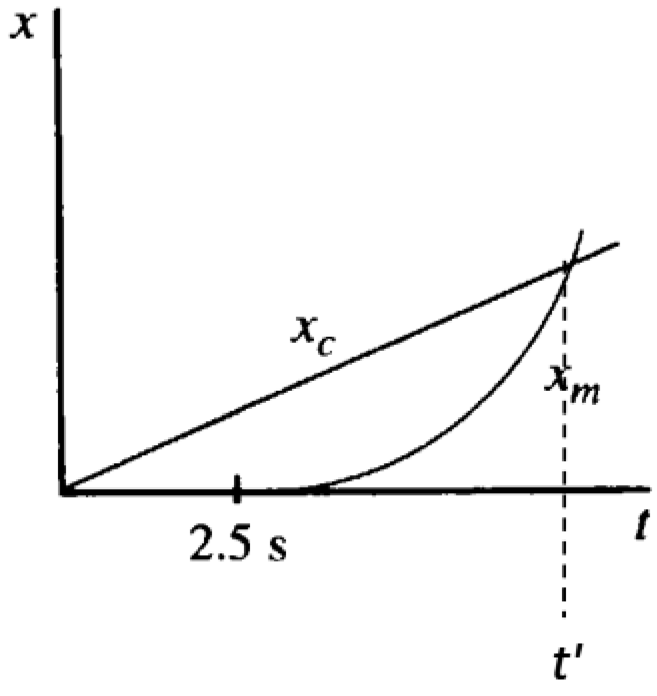
Solution

The car has constant velocity and travels a distance x_c in time t:

$$x_c = v_c t$$

The motorcycle starts from rest ($v_0 = 0$) and moves a distance x_m in time $t - 2.5$ with constant acceleration:

$$x_m(t) = \frac{1}{2} a(t - 2.5)^2$$



These curves are sketched here. When the motorcycle overtakes the car, both will have traveled the same distance. Thus

$$\frac{1}{2} a(t' - 2.5)^2 = v_c t'$$

$$t' = \frac{42 \pm \sqrt{(42)^2 - 4(1.8)(11.25)}}{(2)(1.8)}$$

$t' = 0.27$ s or 23 s

The motorcycle did not start until $t' = 2.5$ s, so the solution we want is $t' = 23$ s.

$$v_m = v_0 + a(t' - 2.5s) = 73.8\ m/s$$

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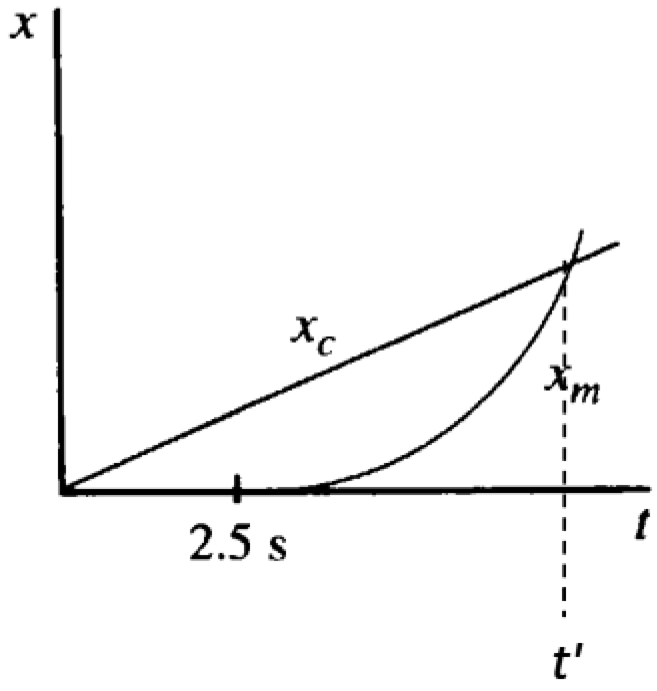

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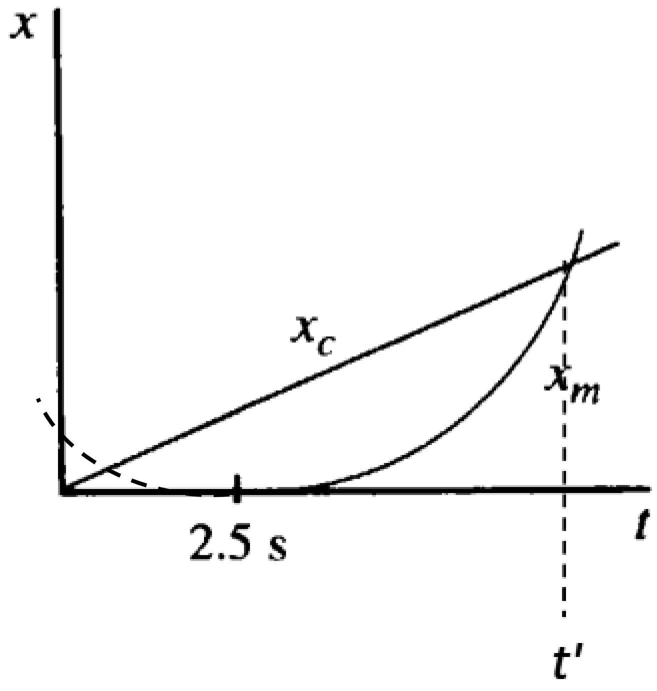

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$$v_m = v_0 + a(t' - 2.5s) = 73.8\ m/s$$

3.

$Vc := 31$

$a := 3.6$

Solution 1

$$\frac{1}{2}\cdot a\cdot(t-2.5)^2 = Vc\cdot t$$

$$\begin{bmatrix} \dfrac{0.5\cdot\left[2.0\cdot Vc + 5.0\cdot a - 2.0\cdot\left(5.0\cdot Vc\cdot a + Vc^{2.0}\right)^{0.5}\right]}{a^{1.0}} \\ \dfrac{0.5\cdot\left[2.0\cdot Vc + 5.0\cdot a + 2.0\cdot\left(5.0\cdot Vc\cdot a + Vc^{2.0}\right)^{0.5}\right]}{a^{1.0}} \end{bmatrix} = \begin{pmatrix} 0.285 \\ 21.937 \end{pmatrix}$$

Solution 2

$$T = t - 2.5$$

$$\frac{1}{2}\cdot a\cdot(T)^2 = Vc\cdot(T+2.5)$$

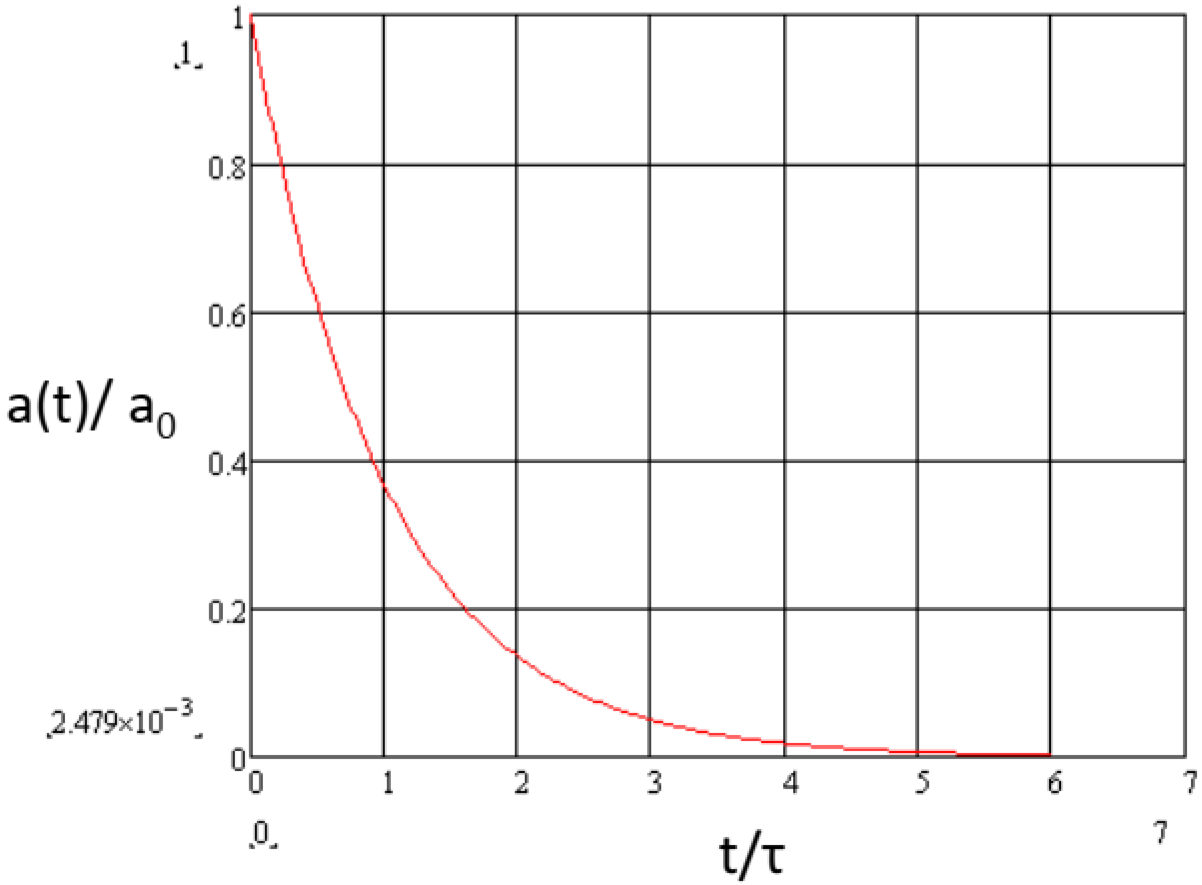
$$\begin{bmatrix} \dfrac{Vc + \left(5.0\cdot Vc\cdot a + Vc^{2.0}\right)^{0.5}}{a^{1.0}} \\ \dfrac{Vc - 1.0\cdot\left(5.0\cdot Vc\cdot a + Vc^{2.0}\right)^{0.5}}{a^{1.0}} \end{bmatrix} = \begin{pmatrix} 19.437 \\ -2.215 \end{pmatrix}$$

$$t = T + 2.5$$

$$t := 19.437 + 2.5 \rightarrow 21.937$$

4.

Find the maximum speed v_{max} and the dependence of the distance on time $x(t)$ in the time interval $[0,\infty]$ in the case of the non-constant acceleration $a(t) = a_0 \exp(-t/\tau)$. Assume that the initial velocity is zero.



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Find the maximum speed v_{max} and the dependence of the distance on time x(t) in the time interval [0,∞] in the case of the non-constant acceleration a(t) = a_0 exp(-t/τ). Assume that the initial velocity is zero.

$$a(t) = a\_0 \cdot e^{\frac{-t}{\tau}}$$

$$\frac{dv}{dt} = a\_0 \cdot e^{\frac{-t}{\tau}} \quad \Rightarrow \quad v(t) = \left(\int_0^t a\_0 \cdot e^{\frac{-h}{\tau}} \, dh \right) \rightarrow \boxed{v(t) = -a\_0 \cdot \tau \cdot \left(e^{-\frac{t}{\tau}} - 1 \right)}$$

$$v(t) = \frac{dx}{dt} \quad \Rightarrow \quad x(t) = \int_0^t a\_0 \cdot \tau \cdot \left(1 - e^{-\frac{h}{\tau}} \right) dh \rightarrow \boxed{x(t) = a\_0 \cdot \tau \cdot t + a\_0 \cdot \tau^2 \cdot \left(e^{-\frac{t}{\tau}} - 1 \right)}$$

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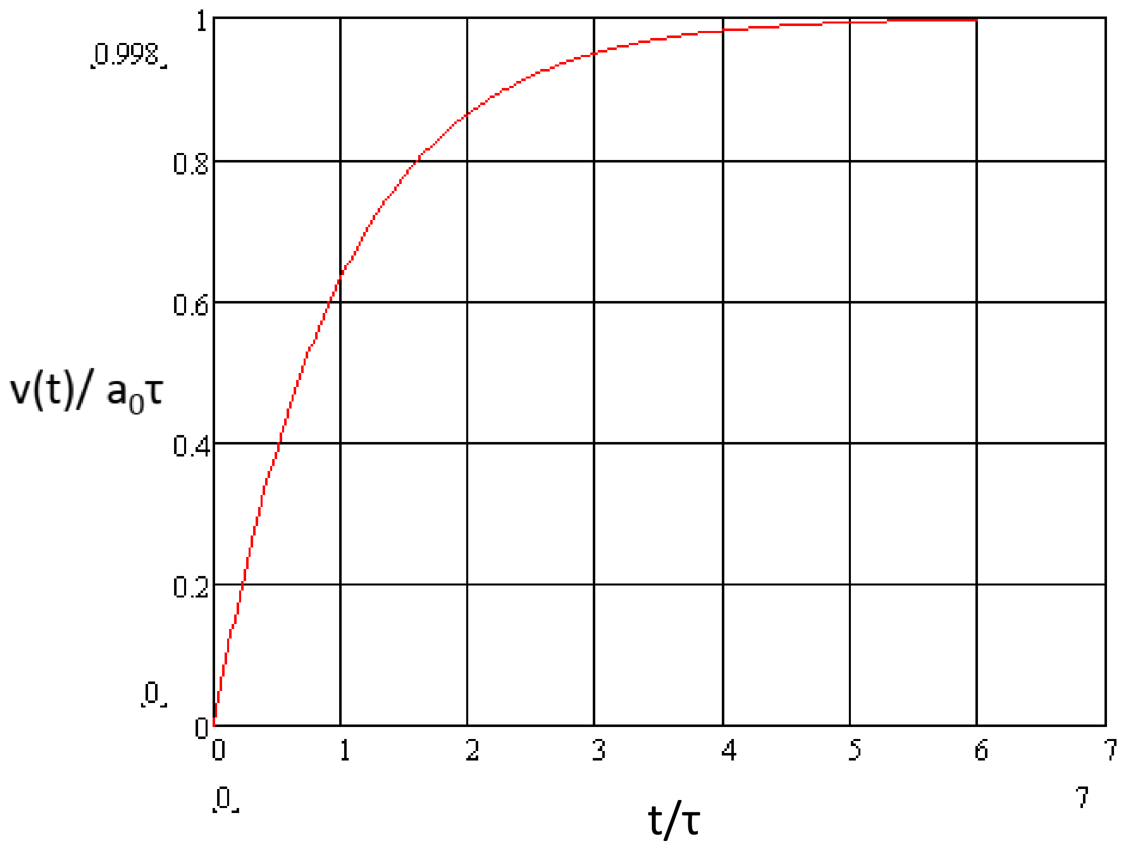
$$v(t) = a\_0 \cdot \tau \cdot \left(1 - e^{-\frac{t}{\tau}}\right)$$



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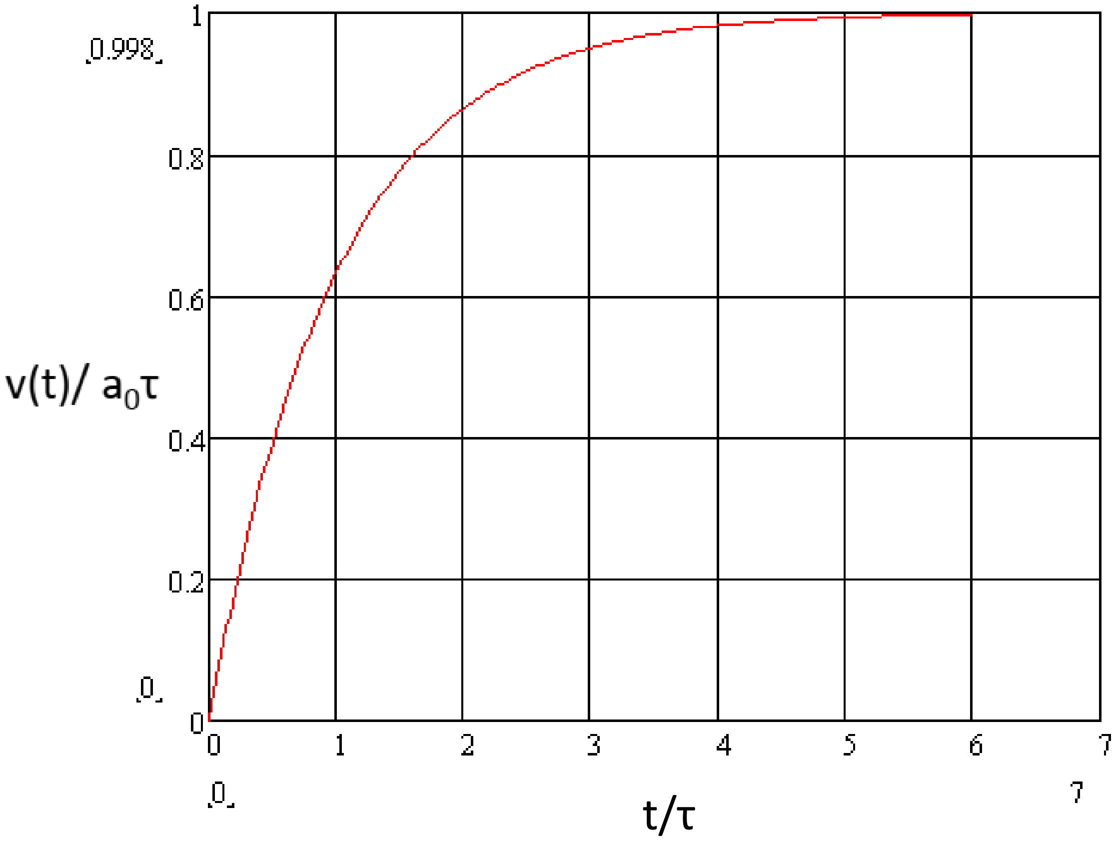


$$V\_max = v(t)\Big|_{t>\tau} = a\_0\cdot\tau\cdot\left(1 - \cancel{e^{-\frac{t}{\tau}}}\right) = a\_0\cdot\tau$$

$$x(t)\Big|_{t>\tau} = a\_0\cdot\tau\cdot t + a\_0\cdot\tau^{2}\cdot\left(\cancel{e^{-\frac{t}{\tau}}} - 1\right) = a\_0\cdot\tau\cdot(t - \cancel{\tau}) = a\_0\cdot\tau\cdot t = V\_max\cdot t$$

5. A point moves rectilinearly in one direction. Fig. 1.1 shows

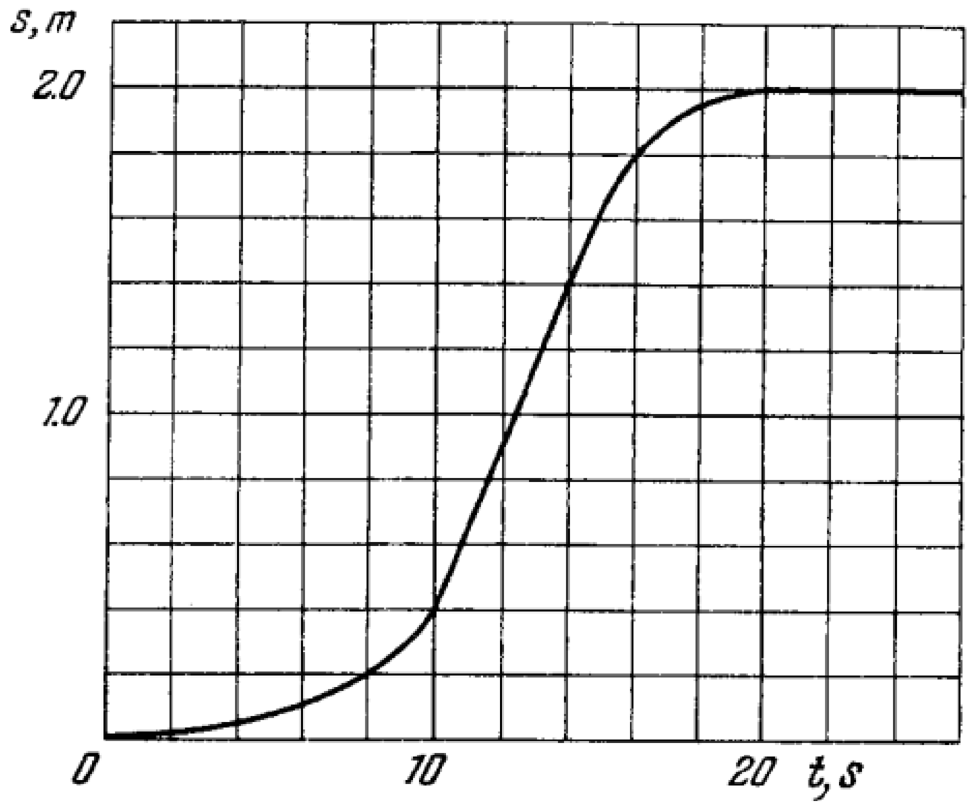


Fig. 1.1.

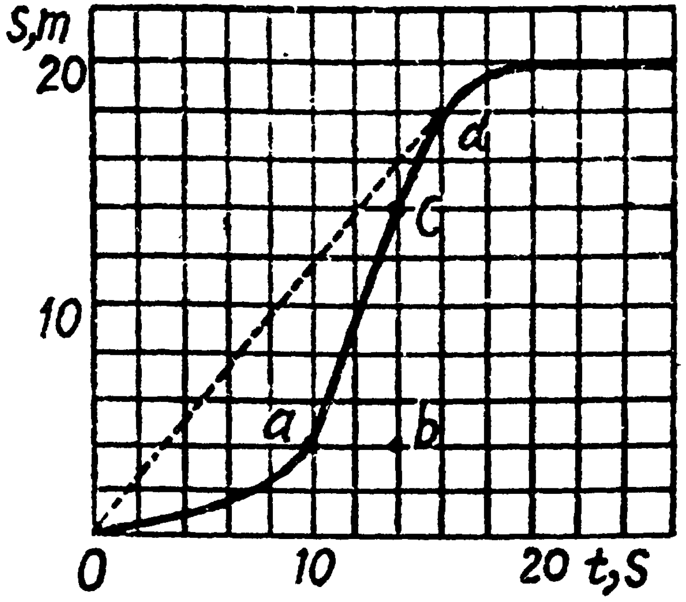
the distance s traversed by the point as a function of the time t. Using the plot find:

(a) the average velocity of the point during the time of motion;

(b) the maximum velocity;

(c) the time moment t_0 at which the instantaneous velocity is equal to the mean velocity averaged over the first t_0 seconds.

5.



(a) the average velocity of the point during the time of motion;

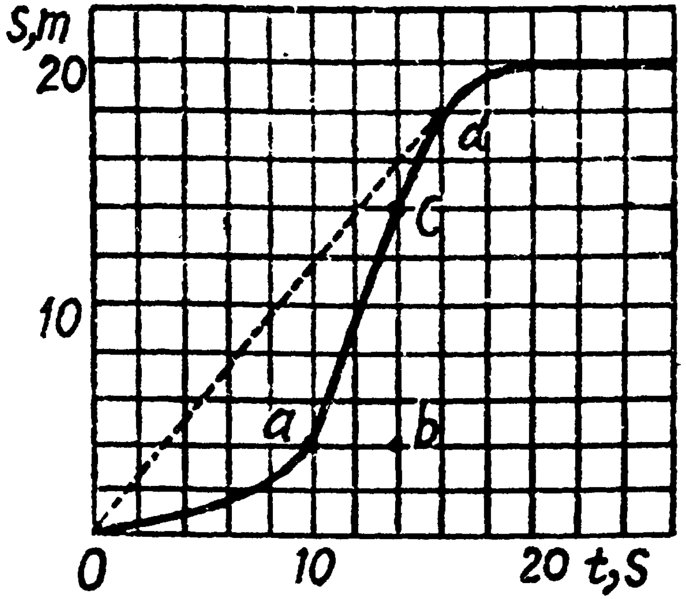
Solution

Sought average velocity

$$\langle v\rangle = \frac{s}{t} = \frac{200\text{ cm}}{20\text{s}} = 10\text{ cm/s}$$

5.

(b) the maximum velocity;



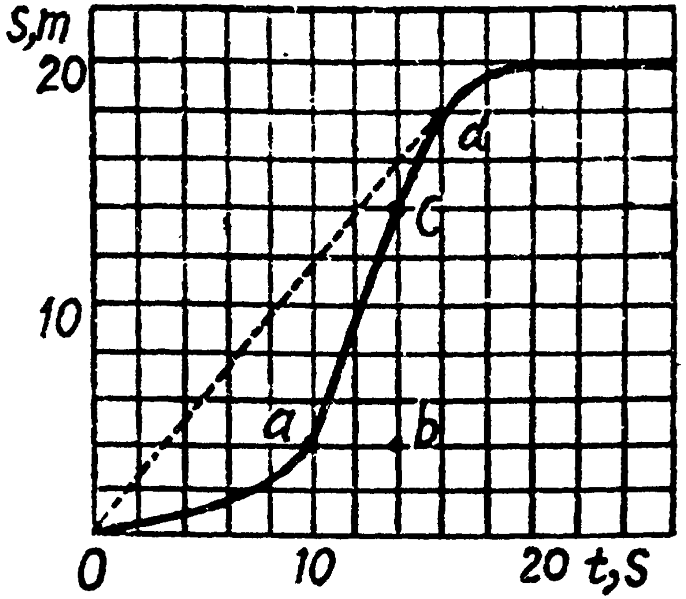
Solution

(b) For the maximum velocity, $\frac{ds}{dt}$ should be maximum. From the figure $\frac{ds}{dt}$ is maximum for all points on the line *ac*, thus the sought maximum velocity becomes average velocity for the line *ac* and is equal to :

$$\frac{bc}{ab} = \frac{100 \text{ cm}}{4\text{s}} = 25 \text{ cm/s}$$

5.

(c) the time moment t_0 at which the instantaneous velocity is equal to the mean velocity averaged over the first t_0 seconds.



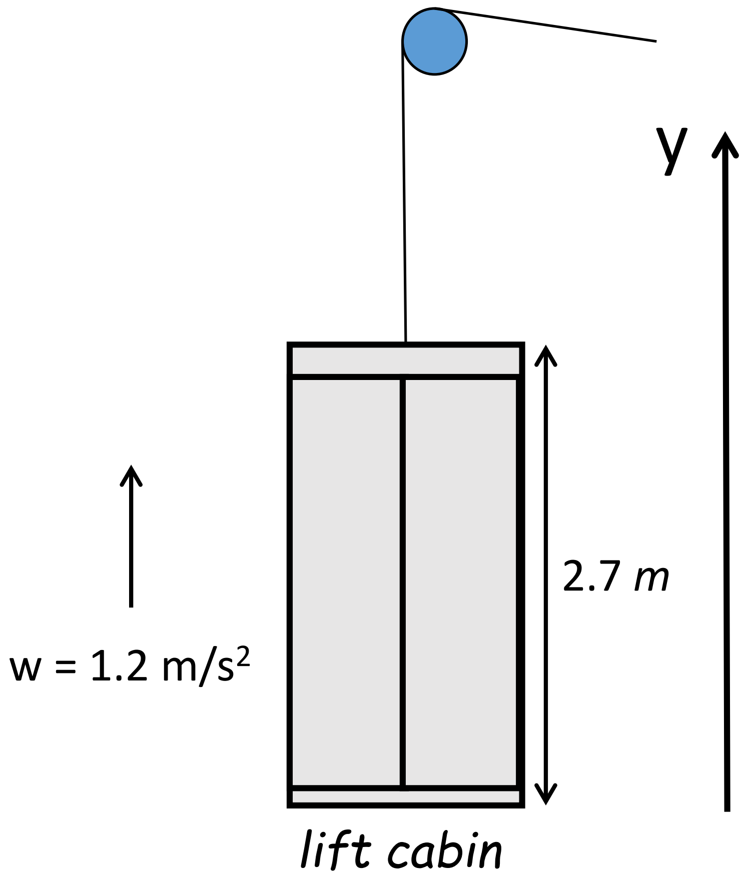
Solution

Time t_0 should be such that corresponding to it the slope $\frac{ds}{dt}$ should pass through the point O (origin), to satisfy the relationship $\frac{ds}{dt} = \frac{s}{t_0}$. From figure the tangent at point d passes through the origin and thus corresponding time $t = t_0 = 16$ s.

6. An elevator car whose floor-to-ceiling distance is equal to 2.7 m starts ascending with constant acceleration 1.2 m/s^2; 2.0 s after the start a bolt begins falling from the ceiling of the car. Find:

(a) the bolt's free fall time;

(b) the displacement and the distance covered by the bolt during the free fall in the reference frame fixed to the elevator shaft.



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(a) the bolt's free fall time;

(b) the displacement and the distance covered by the bolt during the free fall in the reference frame fixed to the elevator shaft.

Solution

(a) One good way to solve the problem is to work in the elevator's frame having the observer at its bottom (Fig.).

Let us denote the separation between floor and celing by $h = 2{\cdot}7$ m. and the acceleration of the elevator by $w = 1{\cdot}2\ \mathrm{m/s^2}$

From the kinematical formula

$$y = y_0 + v_{0y}\,t + \frac{1}{2} w_y\, t^2 \qquad (1)$$

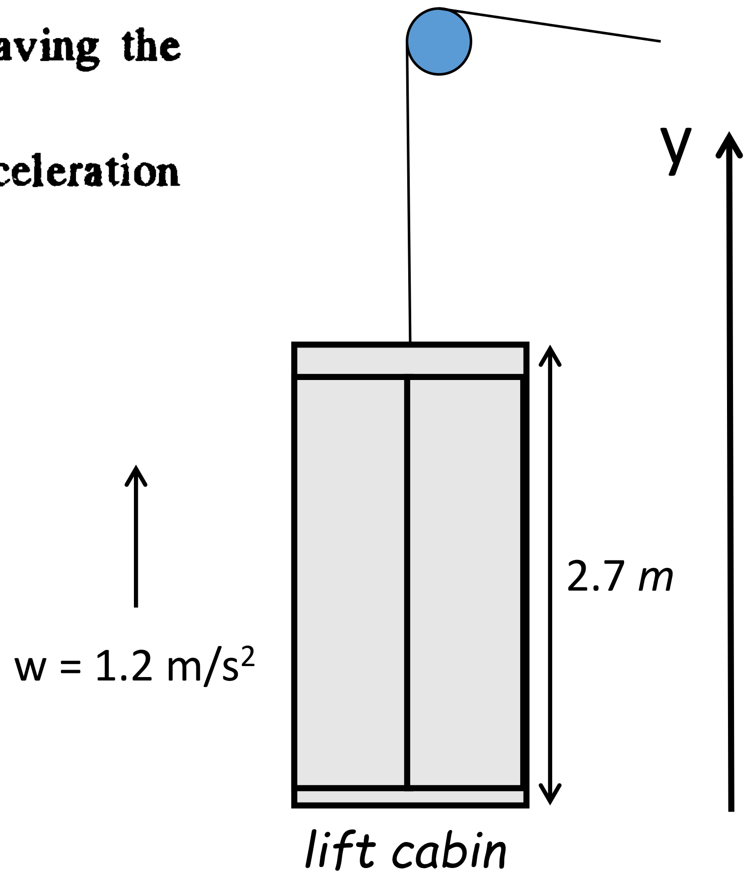
Here $y = 0,\ y_0 = +h,\ v_{0y} = 0$

and $w_y = w_{\text{bolt}\,(y)} - w_{\text{ele}\,(y)}$

$$= (-g) - (w) = -(g + w)$$

So, $$0 = h + \frac{1}{2}\{-(g + w)\}\, t^2$$

or, $$t = \sqrt{\frac{2h}{g + w}} = 0.7\ s$$



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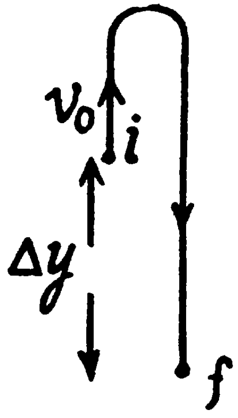
(a) the bolt's free fall time;
(b) the displacement and the distance covered by the bolt during the free fall in the reference frame fixed to the elevator shaft.

Solution

(b) At the moment the bolt loses contact with the elevator, it has already aquired the velocity equal to elevator, given by :

$$v_0 = (1 \cdot 2)(2) = 2.4 \text{ m/s}$$

In the reference frame attached with the elevator shaft (ground) and pointing the y-axis upward, we have for the displacement of the bolt,



$$\Delta y = v_{0y}\, t + \frac{1}{2} w_y\, t^2$$

$$= v_0\, t + \frac{1}{2}(-g)\, t^2$$

or, $\Delta y = -0.7$ m

Hence the bolt comes down or displaces downward relative to the point, when it loses contact with the elevator by the amount 0·7 m (Fig.).
Obviously the total distance covered by the bolt during its free fall time

$$s = |\Delta y| + 2\left(\frac{v_0^2}{2g}\right) = 1.3 \text{ m}$$

7.

The position of a particle moving along an x axis is given by $x(t) = 12t^2 - 2t^3$, where x is in meters and t is in seconds. Determine (a) the position, (b) the velocity, and (c) the acceleration of the particle at $t = 3.0$ s. (d) What is the maximum positive coordinate reached by the particle and (e) at what time is it reached? (f) What is the maximum positive velocity reached by the particle and (g) at what time is it reached? (h) What is the acceleration of the particle at the instant the particle is not moving (other than at $t = 0$)? (i) Determine the average velocity of the particle between $t = 0$ and $t = 3$ s.

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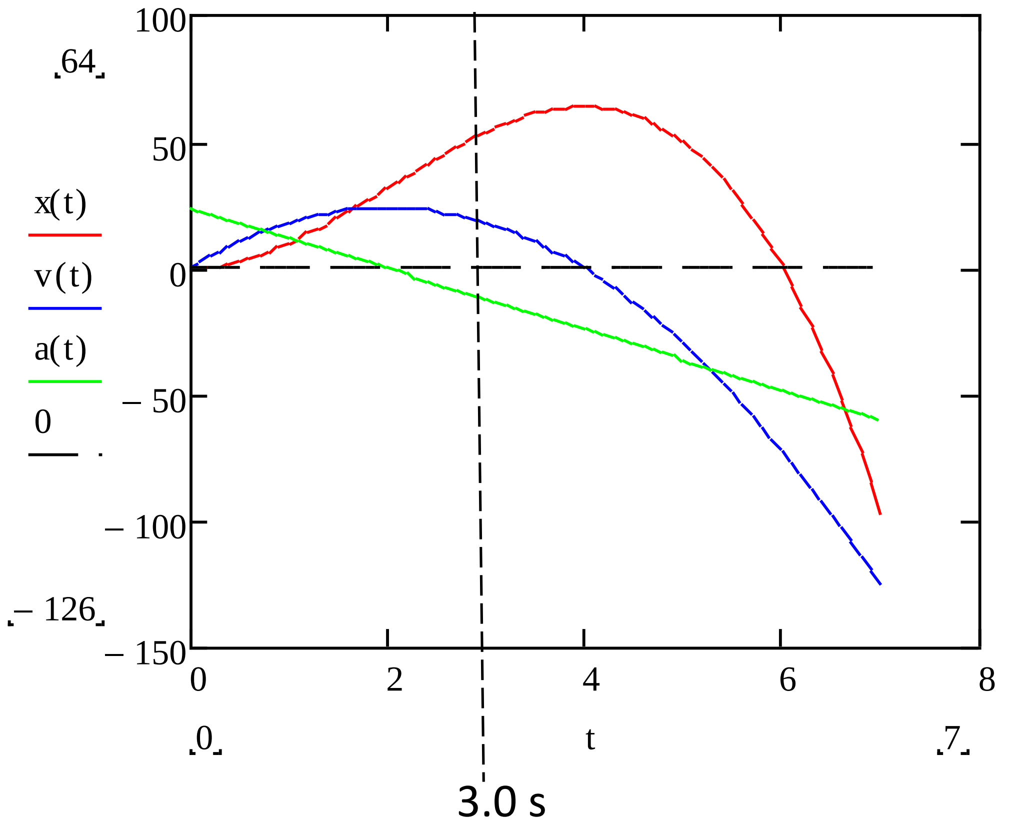
The position of a particle moving along an *x* axis is given by $x(t) = 12t^2 - 2t^3$, where *x* is in meters and *t* is in seconds. Determine (a) the position, (b) the velocity, and (c) the acceleration of the particle at $t = 3.0$ s. (d) What is the maximum positive coordinate reached by the particle and (e) at what time is it reached? (f) What is the maximum positive velocity reached by the particle and (g) at what time is it reached? (h) What is the acceleration of the particle at the instant the particle is not moving (other than at $t = 0$)? (i) Determine the average velocity of the particle between $t = 0$ and $t = 3$ s.

Solution (a), (b), (c)

$$x(t) := 12 \cdot t^2 - 2 \cdot t^3 \qquad x(3) = 54 \text{ m}$$

$$v(t) := \frac{d}{dt} x(t) \rightarrow 24 \cdot t - 6 \cdot t^2 \qquad v(3) = 18 \text{ m/s}$$

$$a(t) := \frac{d^2}{dt^2} x(t) \rightarrow 24 - 12 \cdot t \qquad a(3) = -12 \text{ m/s}^2$$



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Solution (f), (g)

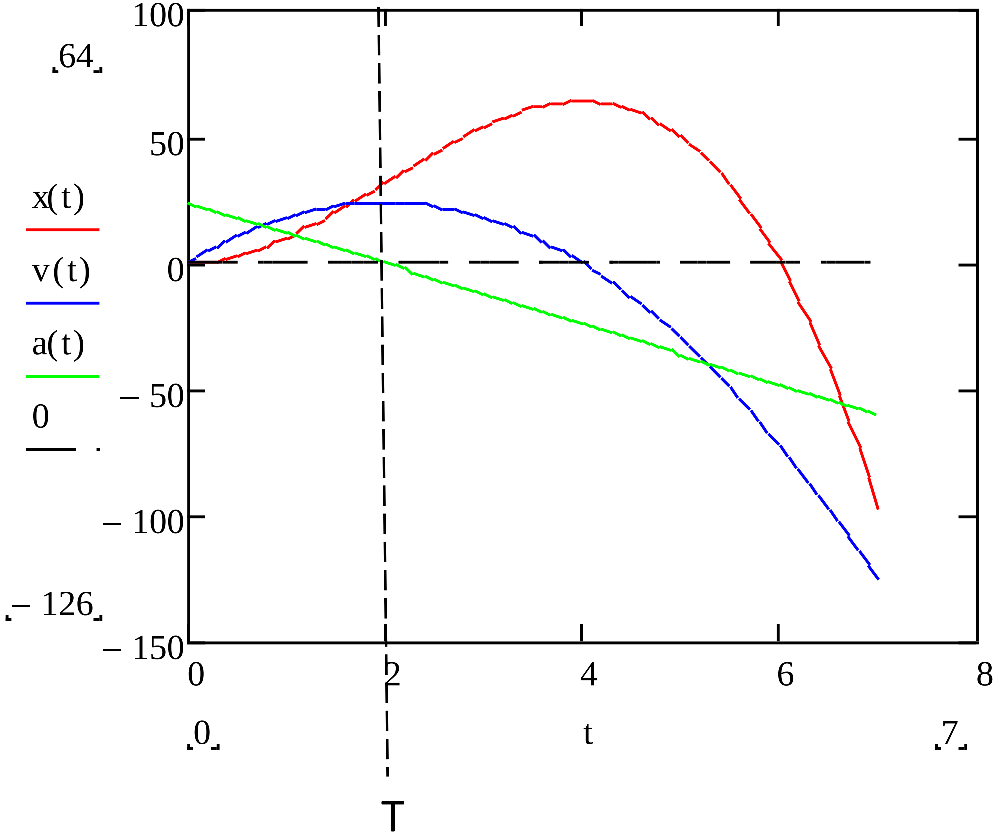
The velocity is positive and maximum when the acceleration is zero a(t) = 0

$$a(t) := \frac{d^2}{dt^2}x(t) \rightarrow 24 - 12\cdot t$$

$$24 - 12\cdot t = 0$$

$$T = 2 \text{ s}$$

$$v(2) = 24 \text{ m/s}$$



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Solution (d), (e), (h)

The particle is not moving when the velocity is zero v(T) = 0. At the same time the particle has maximum positive coordinate.

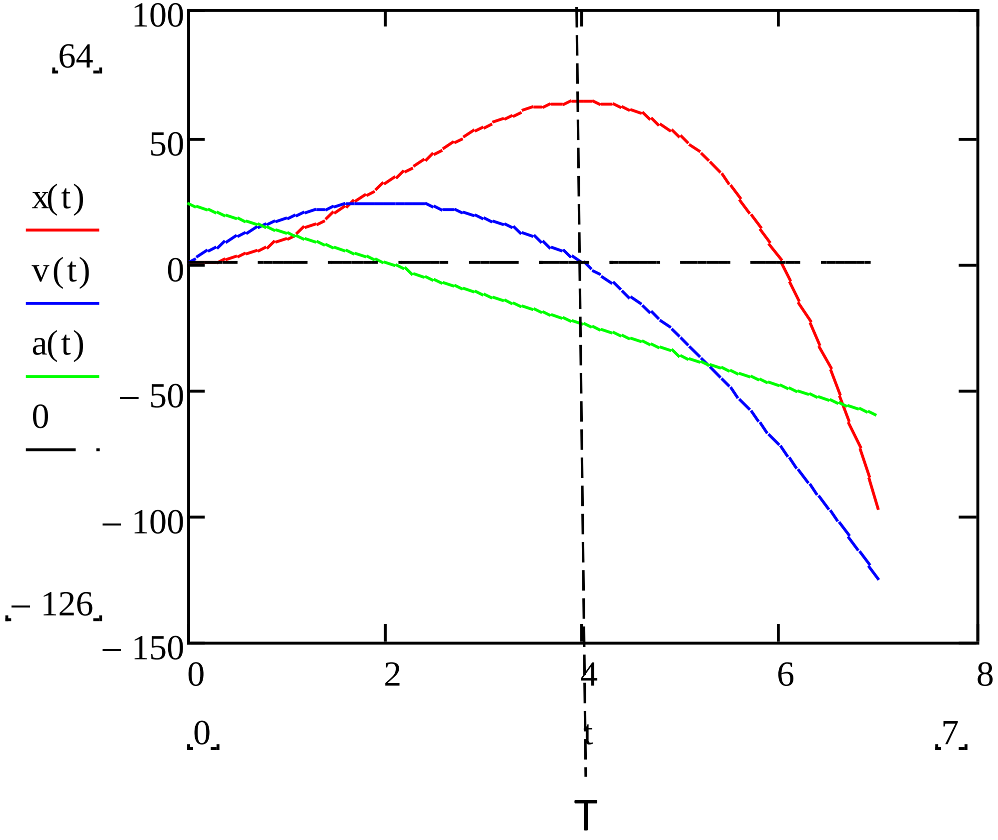
$$v(t) := \frac{d}{dt}x(t) \rightarrow 24\cdot t - 6\cdot t^2$$

$$24 - 6\cdot T = 0$$

$$T = 4\ \text{s}$$

$$x(4) = 64\ \text{m}$$

$$a(4) = -24\ \text{m/s}^2$$



7.

The position of a particle moving along an x axis is given by $x(t) = 12t^2 - 2t^3$, where x is in meters and t is in seconds. Determine (a) the position, (b) the velocity, and (c) the acceleration of the particle at $t = 3.0$ s. (d) What is the maximum positive coordinate reached by the particle and (e) at what time is it reached? (f) What is the maximum positive velocity reached by the particle and (g) at what time is it reached? (h) What is the acceleration of the particle at the instant the particle is not moving (other than at $t = 0$)? (i) Determine the average velocity of the particle between $t = 0$ and $t = 3$ s.

Solution (i)

The average velocity of the particle

$$x(t) := 12 \cdot t^2 - 2 \cdot t^3$$

$$V\_avg = \frac{x(3) - x(0)}{3 - 0} \rightarrow V\_avg = 18 \ \text{m/s}$$