Chapter 3

Vectors

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Scalars and vectors quantities:

- **Scalars and vectors :**

A scalar: is a quantity that has magnitude only**.** **Mass, time, speed, distance, pressure, Temperature and volume** are all examples of scalar quantities .

Note : Magnitude – A numerical value with units.

A vector : is a quantity that has a magnitude and a direction. One example of a vector is velocity. The velocity of an object is determined by the magnitude (speed) and direction of travel. Other examples of vectors are **force, displacement and acceleration .**

Note : Vectors are typically illustrated by drawing an **ARROW above the symbol. The arrow is used to convey direction and magnitude.**

Scalar Quantities and Vector Quantities

Scalar Quantities	Examples	Vector Quantities	Examples
Distance Length without direction	$20\ m$	Displacement Length with direction	$20\ m\ [North]$
Speed How Fast an object moves without direction	$80\ km/h$	Velocity How Fast an object moves with direction	$80\ km/h\ \ [280^{\circ}]$
Mass How much Stuff is IN an object	$56\ kg$	Force $= Mass \times acceleration$	60 N downward
Energy Emits in All directions	$500\ kj$	Weight Force due to gravity	500 N Always downward
Temperature Average Kinetic Energy of an object	25 ℃	Friction Resistance force due to the Surface conditions	15 N Against the direction of motion
Time	$40\ minutes$	Acceleration How Fast Velocity Changes over Time	$6\ \frac{m}{s^2}\ [NW]$

Vectors and Their Components

- **The simplest example is a displacement vector**
- **If a particle changes position from A to B, we represent this by a vector arrow pointing from A to B**

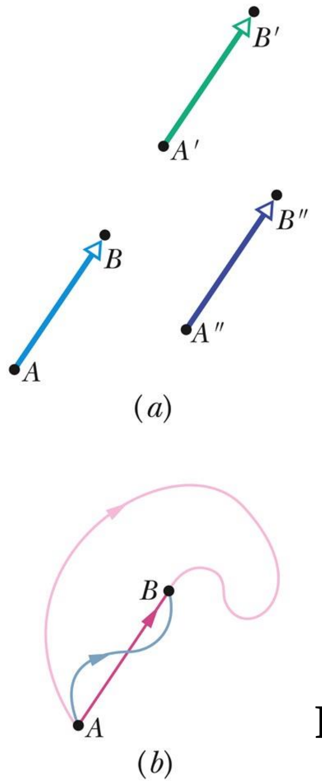

Figure 3-1

- **In (a) we see that all three arrows have the same magnitude and direction: they are identical displacement vectors.**
- **In (b) we see that all three paths correspond to the same displacement vector. The vector tells us nothing about the actual path that was taken between A and B.**

Vectors and Their Components

- The **vector sum**, or **resultant**
  - Is the result of performing vector addition
  - Represents the net displacement of two or more displacement vectors

$$\vec{s} = \vec{a} + \vec{b},$$

Eq. (3-1)

  - Can be added graphically as shown:

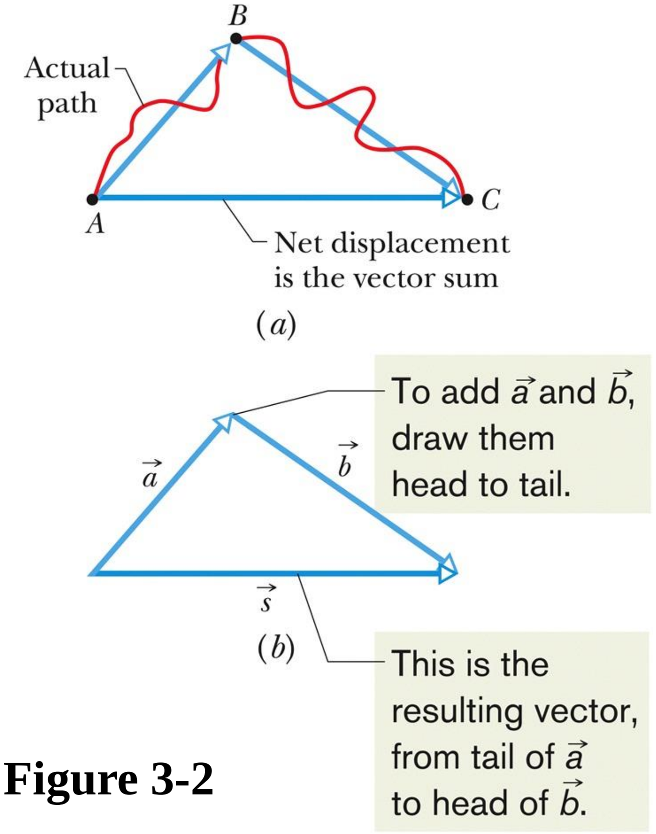

Figure 3-2

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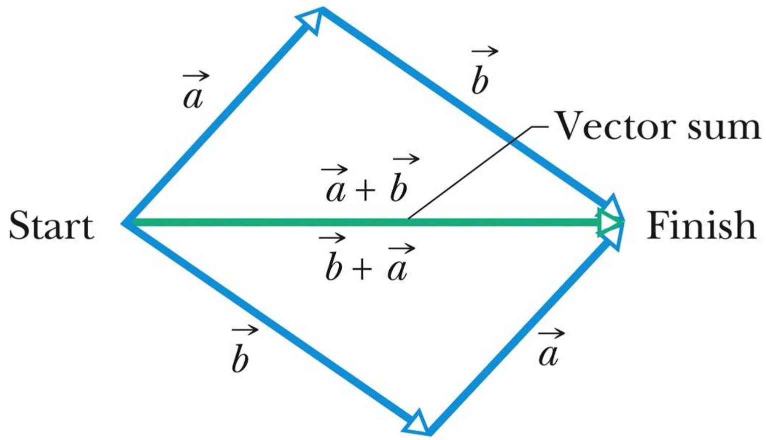
Vectors and Their Components

- Vector addition is **commutative**
  - We can add vectors in any order

$$\vec{a} + \vec{b} = \vec{b} + \vec{a} \qquad \text{(commutative law).}$$

Eq. (3-2)



You get the same vector result for either order of adding vectors.

Figure (3-3)

Vectors and Their Components

- Vector addition is **associative**
  - We can group vector addition however we like

$$(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c}) \qquad \text{(associative law).}$$

Eq. (3-3)

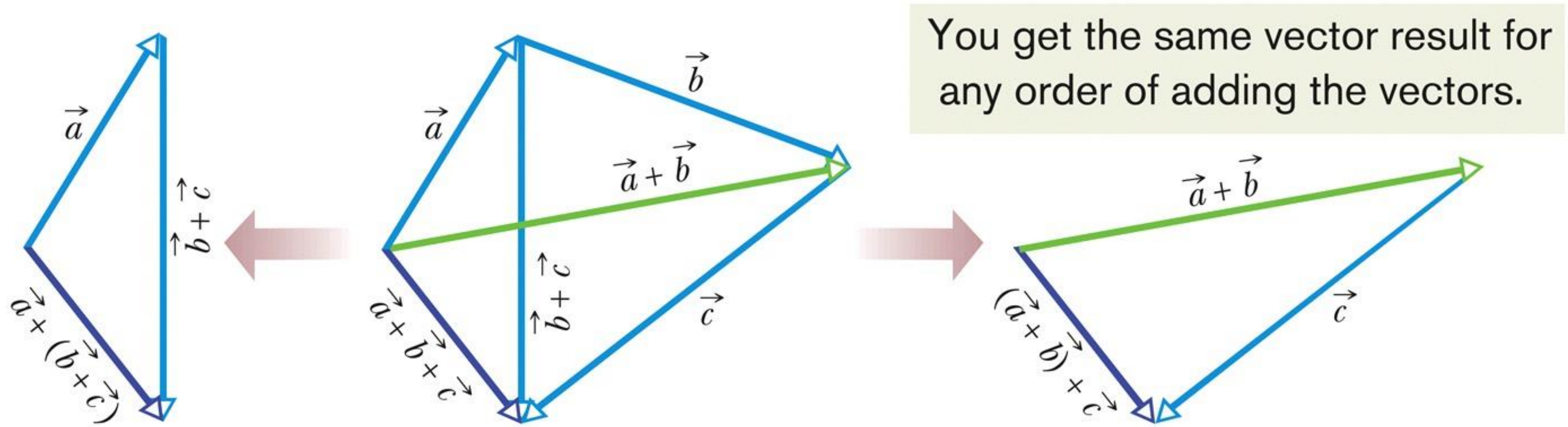


Figure (3-4)

Vectors and Their Components

- A negative sign reverses vector direction

$$\vec{b} + (-\vec{b}) = 0.$$

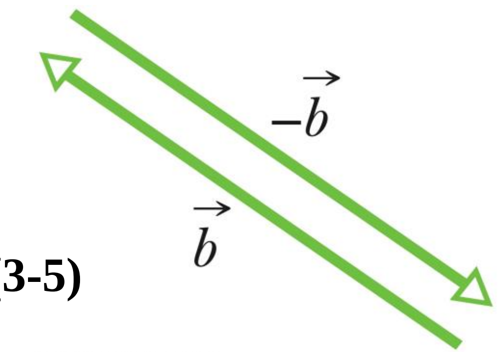


Figure (3-5)

- We use this to define vector subtraction

$$\vec{d} = \vec{a} - \vec{b} = \vec{a} + (-\vec{b})$$

Eq. (3-4)

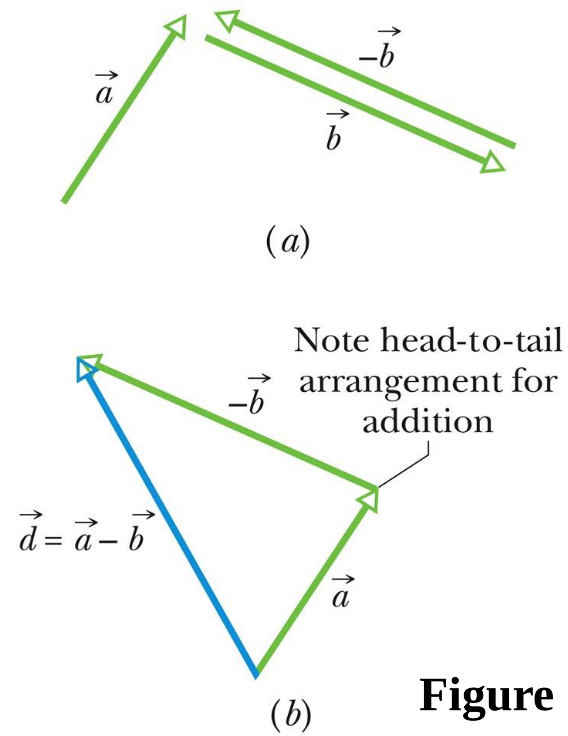


Figure (3-6)

Vectors and Their Components

- These rules hold for all vectors, whether they represent displacement, velocity, etc.
- Only vectors of the same kind can be added
  - (*distance*) + (*distance*) makes sense
  - (*distance*) + (*velocity*) does not

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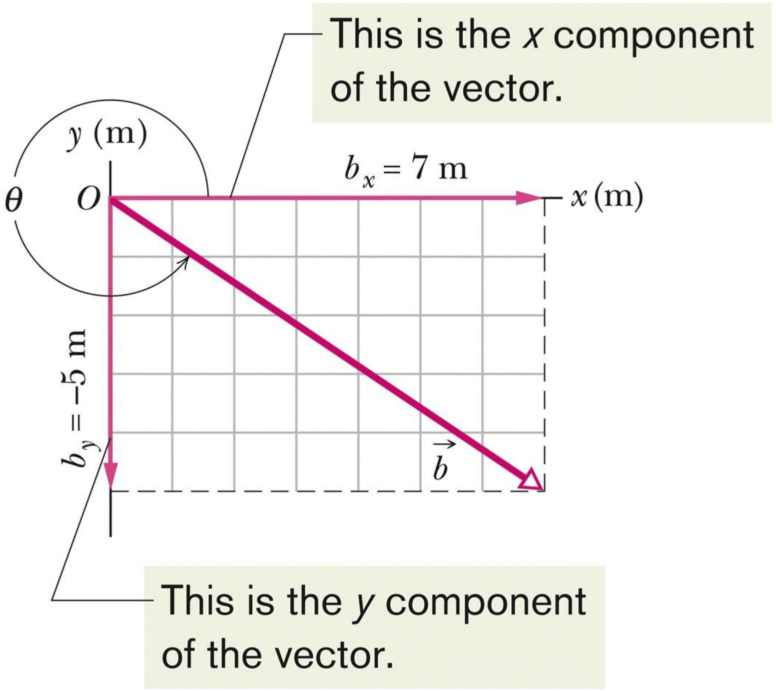
Vectors and Their Components

- Rather than using a graphical method, vectors can be added by **components**
    - A component is the projection of a vector on an axis
- The process of finding components is called **resolving the vector**
- The components of a vector can be positive or negative.
- They are unchanged if the vector is shifted in any direction (but not rotated).



Vectors and Their Components

- Components in two dimensions can be found by:

$$a_x = a\cos\theta \quad \text{and} \quad a_y = a\sin\theta,$$

Eq. (3-5)

- Where θ is the angle the vector makes with the positive x axis, and a is the vector length
- The length and angle can also be found if the components are known

$$a = \sqrt{a_x^2 + a_y^2} \qquad \text{and} \qquad \tan\theta = \frac{a_y}{a_x}$$

Eq. (3-6)

- Therefore, components fully define a vector

2D

Vectors and Their Components

- In the three dimensional case we need more components to specify a vector
  - (a, θ, φ) or (a_x, a_y, a_z)

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Vectors and Their Components

- Angles may be measured in degrees or radians
- Recall that a full circle is 360˚, or 2π rad

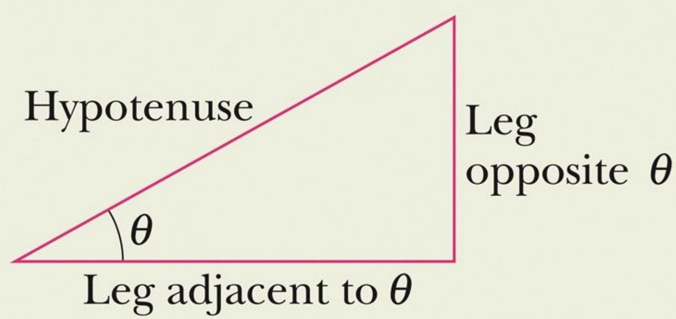
$$40^\circ \frac{2\pi \text{ rad}}{360^\circ} = 0.70 \text{ rad.}$$

- Know the three basic trigonometric functions

$$\sin\theta = \frac{\text{leg opposite } \theta}{\text{hypotenuse}}$$

$$\cos\theta = \frac{\text{leg adjacent to } \theta}{\text{hypotenuse}}$$

$$\tan\theta = \frac{\text{leg opposite } \theta}{\text{leg adjacent to } \theta}$$



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Unit Vectors, Adding Vectors by Components

- A **unit vector**
  - Has magnitude 1
  - Has a particular direction
  - Lacks both dimension and unit
  - Is labeled with a hat: ^
- We use a **right-handed coordinate system**
  - Remains right-handed when rotated

$$\vec{a} = a_x\hat{\mathrm{i}} + a_y\hat{\mathrm{j}} \quad \textbf{Eq. (3-7)}$$

$$\vec{b} = b_x\hat{\mathrm{i}} + b_y\hat{\mathrm{j}}. \quad \textbf{Eq. (3-8)}$$

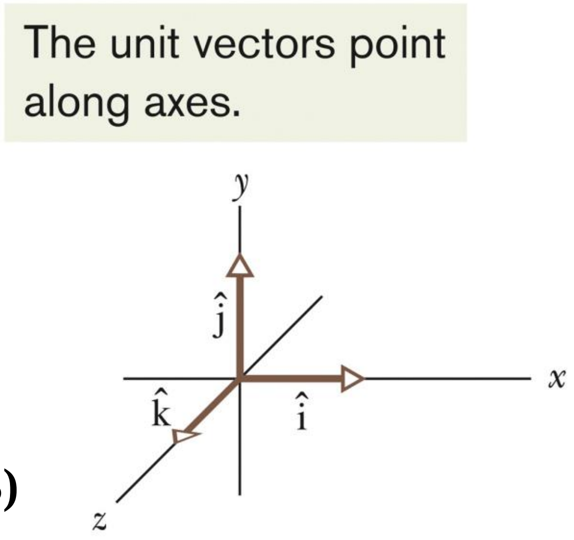


Figure (3-13)

Unit Vectors, Adding Vectors by Components

- The quantities $a_x\mathbf{i}$ and $a_y\mathbf{j}$ are **vector components**

$$\vec{a} = a_x\hat{\mathrm{i}} + a_y\hat{\mathrm{j}} \qquad \text{Eq. (3-7)}$$

$$\vec{b} = b_x\hat{\mathrm{i}} + b_y\hat{\mathrm{j}}. \qquad \text{Eq. (3-8)}$$

- The quantities a_x and a_y alone are **scalar components**
  - Or just "components" as before
- Vectors can be added using components

Eq. (3-9) $$\vec{r} = \vec{a} + \vec{b}, \longrightarrow$$

$$r_x = a_x + b_x \qquad \text{Eq. (3-10)}$$

$$r_y = a_y + b_y \qquad \text{Eq. (3-11)}$$

$$r_z = a_z + b_z. \qquad \text{Eq. (3-12)}$$

Unit Vectors, Adding Vectors by Components

- To subtract two vectors, **we subtract components**

$$d_x = a_x - b_x, \quad d_y = a_y - b_y, \quad \text{and} \quad d_z = a_z - b_z,$$

Eq. (3-13)

$$\vec{d} = d_x\hat{\mathrm{i}} + d_y\hat{\mathrm{j}} + d_z\hat{\mathrm{k}}.$$

Unit Vectors, Adding Vectors by Components

- Vectors are independent of the coordinate system used to measure them
- We can rotate the coordinate system, without rotating the vector, and the vector remains the same
- All such coordinate systems are equally valid

$$a = \sqrt{a_x^2 + a_y^2} = \sqrt{a_x'^2 + a_y'^2}$$ **Eq. (3-14)**

$$\theta = \theta' + \phi.$$ **Eq. (3-15)**

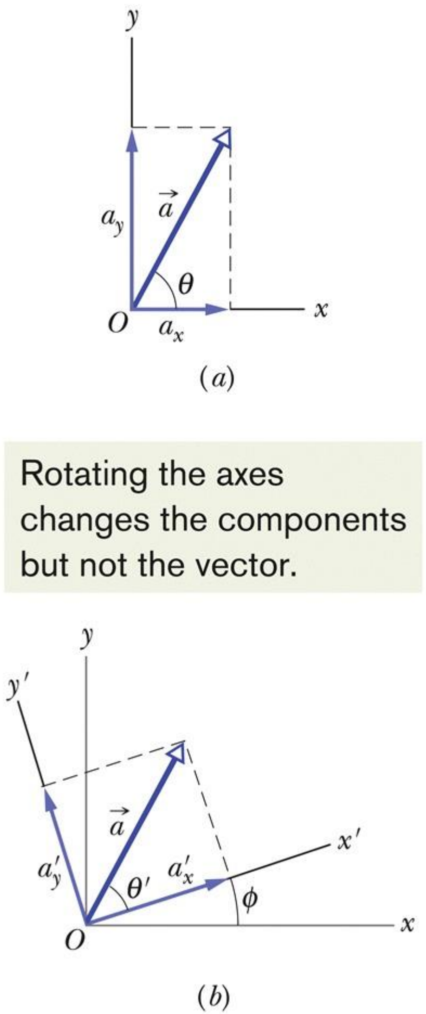


Figure (3-15)

Example 3: if $\vec{A} = 5\hat{\imath} + 3\hat{\jmath} - 6\hat{k}$ $\quad \vec{B} = 8\hat{\imath} + \hat{\jmath} - 4\hat{k}$, Find

1. $\vec{A} + \vec{B} = \quad 13\hat{\imath} + 4\hat{\jmath} - 10\hat{k}$
2. $\vec{A} - \vec{B} = \quad -3\hat{\imath} + 2\hat{\jmath} - 2\hat{k}$
3. $\vec{C}$ where $\vec{C} = 2A - 3\vec{B} = \quad -14\hat{\imath} + 3\hat{\jmath}$
4. The Magnitude and direction for $\vec{C}$.

The magnitude of $\vec{C}$ is :

$$\left|\vec{C}\right| = \sqrt{(C_x)^2 + (C_y)^2 + (C_z)^2} = \sqrt{(-14)^2 + (3)^2 + (0)^2} =$$

$$\sqrt{205} = 14.32$$

The angle (direction) that $\vec{C}$ makes with the x-axis is :

$$\theta = \tan^{-1}\left[\frac{C_y}{C_x}\right] = \tan^{-1}\left[\frac{3}{-14}\right] = H.W.^{\circ}$$

Multiplying Vectors

- Multiplying a vector ***z*** by a scalar *c*
  - Results in a new vector
  - Its magnitude is the magnitude of vector ***z*** times $|c|$
  - Its direction is the same as vector ***z***, or opposite if *c* is negative
  - To achieve this, we can simply multiply each of the components of vector ***z*** by *c*
- To divide a vector by a scalar we multiply by 1/*c*

Example : Multiply vector ***z*** by 5

- $\boldsymbol{z} = -3\,\mathbf{i} + 5\,\mathbf{j}$
- $5\,\boldsymbol{z} = -15\,\mathbf{i} + 25\,\mathbf{j}$

Multiplying Vectors

- Multiplying two vectors: the **scalar product**
  - Also called the **dot product**
  - Results in a scalar, where *a* and *b* are magnitudes and φ is the angle between the directions of the two vectors:

$$\vec{a} \cdot \vec{b} = ab \cos \phi,$$

Eq. (3-20)

- The commutative law applies, and we can do the dot product in component form

$$\vec{a} \cdot \vec{b} = (a_x\hat{\mathrm{i}} + a_y\hat{\mathrm{j}} + a_z\hat{\mathrm{k}}) \cdot (b_x\hat{\mathrm{i}} + b_y\hat{\mathrm{j}} + b_z\hat{\mathrm{k}}),$$

$$\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}.$$

$$\vec{a} \cdot \vec{b} = a_x b_x + a_y b_y + a_z b_z.$$

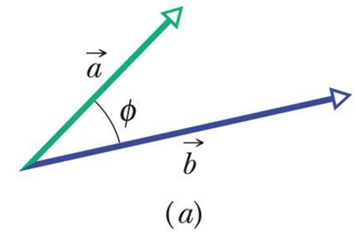
Multiplying Vectors

- A dot product is: the product of the magnitude of one vector times the scalar component of the other vector in the direction of the first vector

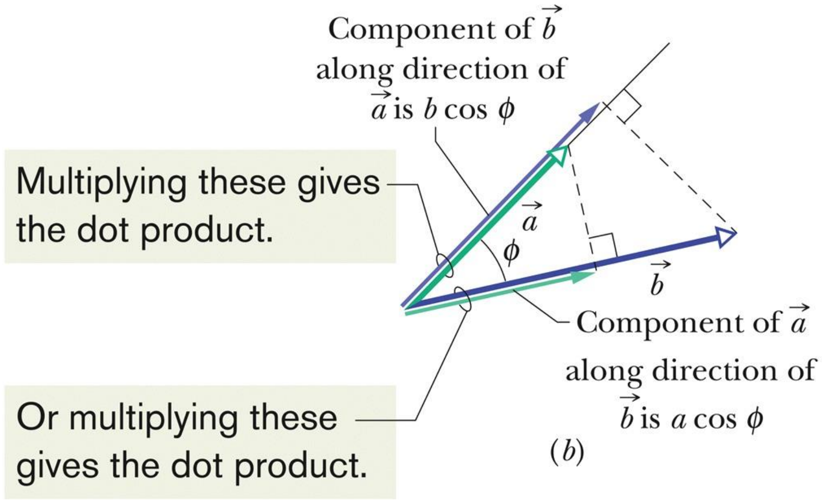
$$\vec{a}\cdot\vec{b} = (a\cos\phi)(b) = (a)(b\cos\phi).$$

Eq. (3-21)

Figure (3-18)



- Either projection of one vector onto the other can be used
- To multiply a vector by the projection, multiply the magnitudes



Multiplying Vectors

If the angle ϕ between two vectors is $0°$, the component of one vector along the other is maximum, and so also is the dot product of the vectors. If, instead, ϕ is $90°$, the component of one vector along the other is zero, and so is the dot product.

Dot product

Example : two vectors are defined as :

$\vec{A} = 3\hat{\imath} - 4\hat{\jmath} + 4\hat{k}$ *and* $\vec{B} = 2\hat{\imath} + 3\hat{\jmath} - 7\hat{k}$

Find the following :

1) $\vec{A}.\vec{B}$

2) The angle θ between $\vec{A}$ *and B* .

Solution :

$$\vec{A}.\vec{B} = A_xB_x + A_yB_y + A_zB_z = \left(3\hat{\imath} - 4\hat{\jmath} + 4\hat{k}\right).\left(2\hat{\imath} + 3\hat{\jmath} - 7\hat{k}\right) = -34$$

$$\boldsymbol{\theta} = \mathbf{cos}^{-1}\left[\frac{\vec{\mathbf{A}}.\vec{\mathbf{B}}}{|\vec{\mathbf{A}}|.|\vec{\mathbf{B}}|}\right]$$

$$|\vec{A}| = \sqrt{(A_x)^2 + (A_y)^2 + (A_z)^2} = \sqrt{(3)^2 + (-4)^2 + (4)^2} = 6.4$$

$$|\vec{B}| = \sqrt{(B_x)^2 + (B_y)^2 + (B_z)^2} = \sqrt{(2)^2 + (3)^2 + (7)^2} = 7.9$$

So the angle between the vectors $\vec{A}$ and $\vec{B}$ is

$$\boldsymbol{\theta} = \mathbf{cos}^{-1}\left[\frac{\vec{\mathbf{A}}.\vec{\mathbf{B}}}{|\vec{\mathbf{A}}|.|\vec{\mathbf{B}}|}\right] = \mathbf{cos}^{-1}\left[\frac{\mathbf{-34}}{\mathbf{6.4 \times 7.9}}\right] = \mathbf{132}^{\circ}$$

Multiplying Vectors (Vector product or cross product)

- Multiplying two vectors: the **vector product**
  - The **cross product** of two vectors with magnitudes *a* & *b*, separated by angle *φ*, produces a vector with magnitude:

$$c = ab \sin \phi, \qquad \text{Eq. (3-24)}$$

  - And a direction perpendicular to both original vectors
- Direction is determined by the **right-hand rule**
- Place vectors tail-to-tail, sweep fingers from the first to the second, and thumb points in the direction of the resultant vector

If $\vec{a}$ and $\vec{b}$ are parallel or antiparallel, $\vec{a} \times \vec{b} = 0$. The magnitude of $\vec{a} \times \vec{b}$, which can be written as $|\vec{a} \times \vec{b}|$, is maximum when $\vec{a}$ and $\vec{b}$ are perpendicular to each other.

Cross product of Unit Vectors :

$$\hat{\imath}\cancel{.}\times\hat{\imath} = \hat{\jmath}\cancel{.}\times\hat{\jmath} = \hat{k}.\hat{k} = 0$$

$$\hat{\boldsymbol{\imath}}\cancel{.}\times\hat{\boldsymbol{\jmath}} = \widehat{\boldsymbol{k}} \qquad \theta = 90^\circ \qquad\qquad \hat{\boldsymbol{\jmath}}\cancel{.}\times\widehat{\boldsymbol{k}} = \hat{\boldsymbol{\imath}} \qquad\qquad \widehat{\boldsymbol{k}}\cancel{.}\times\hat{\boldsymbol{\imath}} = \hat{\boldsymbol{\jmath}}$$

$$\hat{\boldsymbol{\jmath}}\cancel{.}\times\hat{\boldsymbol{\imath}} = -\widehat{\boldsymbol{k}} \qquad \theta = 270^\circ \qquad\qquad \widehat{\boldsymbol{k}}\cancel{.}\times\hat{\boldsymbol{\jmath}} = -\hat{\boldsymbol{\imath}} \qquad\qquad \hat{\boldsymbol{\imath}}\cancel{.}\times\widehat{\boldsymbol{k}} = -\hat{\boldsymbol{\jmath}}$$

$$\boldsymbol{A} \times \boldsymbol{B} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \boldsymbol{A_x} & \boldsymbol{A_y} & \boldsymbol{A_z} \\ \boldsymbol{B_x} & \boldsymbol{B_y} & \boldsymbol{B_z} \end{vmatrix} = \begin{vmatrix} \boldsymbol{A_y} & \boldsymbol{A_z} \\ \boldsymbol{B_y} & \boldsymbol{B_z} \end{vmatrix} \hat{\imath} - \begin{vmatrix} \boldsymbol{A_x} & \boldsymbol{A_z} \\ \boldsymbol{B_x} & \boldsymbol{B_z} \end{vmatrix} \hat{\jmath} + \begin{vmatrix} \boldsymbol{A_x} & \boldsymbol{A_y} \\ \boldsymbol{B_x} & \boldsymbol{B_y} \end{vmatrix} \hat{k}$$

Multiplying Vectors

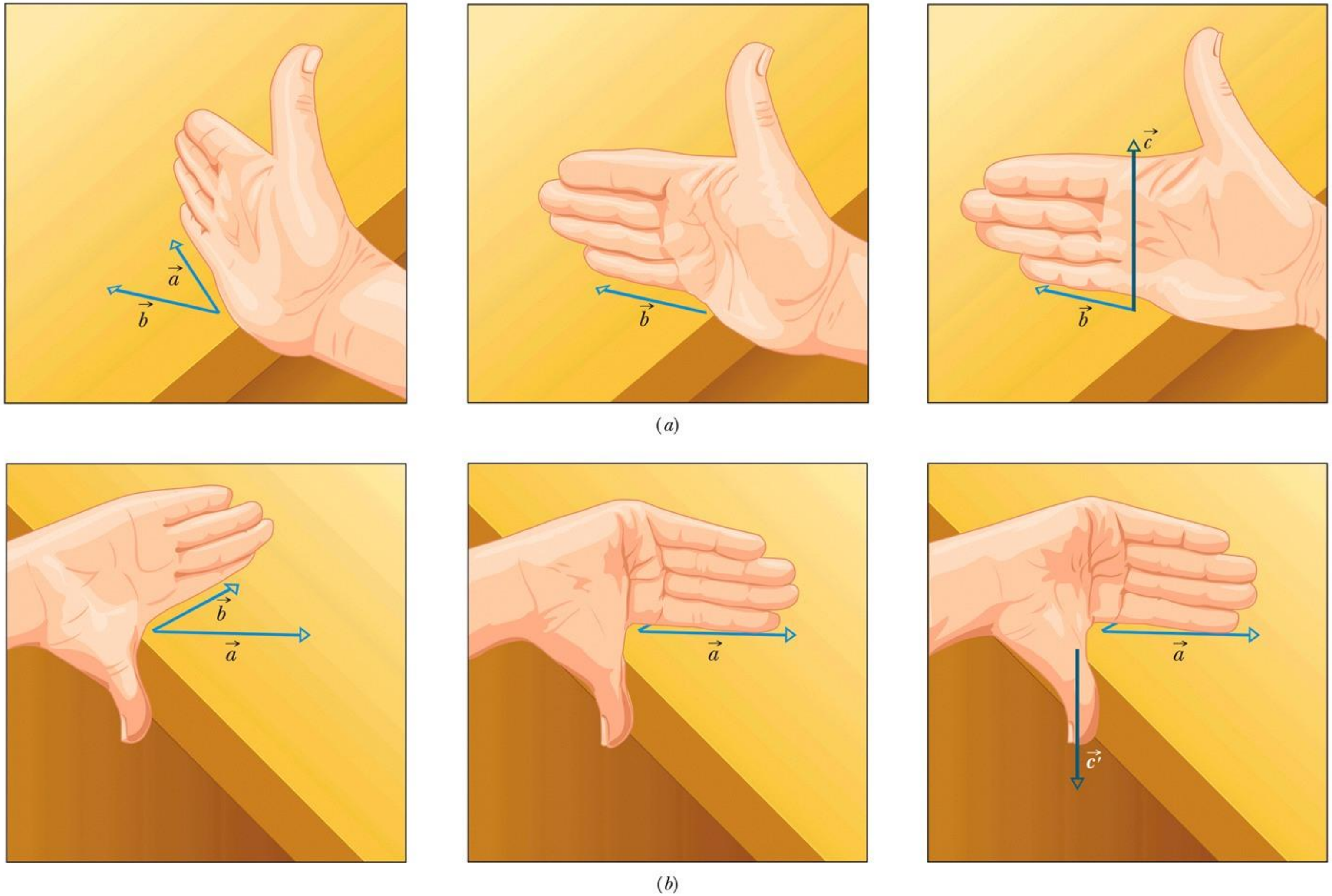


Figure (3-19)

The upper shows vector *a* cross vector *b*, the lower shows vector *b* cross vector *a*

Multiplying Vectors

- **The cross product is not commutative**

$$\vec{b} \times \vec{a} = -(\vec{a} \times \vec{b}).$$ **Eq. (3-25)**

- **To evaluate, we distribute over components:**

$$\vec{a} \times \vec{b} = (a_x\hat{\mathrm{i}} + a_y\hat{\mathrm{j}} + a_z\hat{\mathrm{k}}) \times (b_x\hat{\mathrm{i}} + b_y\hat{\mathrm{j}} + b_z\hat{\mathrm{k}}),$$ **Eq. (3-26)**

$$a_x\hat{\mathrm{i}} \times b_x\hat{\mathrm{i}} = a_x b_x(\hat{\mathrm{i}} \times \hat{\mathrm{i}}) = 0,$$

$$a_x\hat{\mathrm{i}} \times b_y\hat{\mathrm{j}} = a_x b_y(\hat{\mathrm{i}} \times \hat{\mathrm{j}}) = a_x b_y\hat{\mathrm{k}}.$$

- **Therefore, by expanding (3-26):**

$$\vec{a} \times \vec{b} = (a_y b_z - b_y a_z)\hat{\mathrm{i}} + (a_z b_x - b_z a_x)\hat{\mathrm{j}} + (a_x b_y - b_x a_y)\hat{\mathrm{k}}.$$

Eq. (3-27)

Example : three vectors are defined as :

$\vec{A} = 3\hat{\imath} - 4\hat{\jmath} + 4\hat{k}$, $\vec{B} = 2\hat{\imath} + 3\hat{\jmath} - 7\hat{k}$ *and* $\vec{C} = -4\hat{\imath} + 2\hat{\jmath} + 5\hat{k}$

Determine :

1) $\vec{A} \times \vec{B}$

2) $\vec{A} \times B\ .\vec{C}$

Solution :

1) $\vec{A} \times \vec{B}$

$$\vec{\mathrm{A}} \times \vec{\mathrm{B}} = \begin{vmatrix} \hat{\mathrm{i}} & \hat{\mathrm{j}} & \hat{\mathrm{k}} \\ 3 & -4 & 4 \\ 2 & 3 & -7 \end{vmatrix} = 16\hat{\mathrm{i}} + 29\hat{\mathrm{j}} + 17\hat{\mathrm{k}}$$

2) $(\vec{A} \times \vec{B}).\vec{C} = (16\hat{\mathrm{i}} + 29\hat{\mathrm{j}} + 17\hat{\mathrm{k}}).(-4\hat{\imath} + 2\hat{\jmath} + 5\hat{k})$

$= 16(-4) + 29(2) + 17(5) = 79$

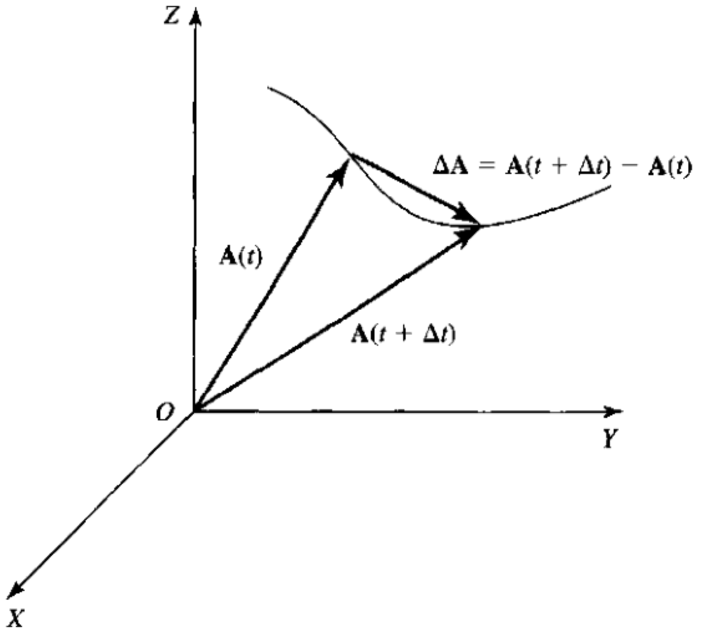
Vector Calculus

Differentiation of Vectors

$$\mathbf{A} = \mathbf{A}(t) = A_x(t)\hat{\mathbf{i}} + A_y(t)\hat{\mathbf{j}} + A_z(t)\hat{\mathbf{k}}$$

The derivative of **A** with respect to *t* is defined in a manner similar to the derivative of a scalar function. That is,

$$\frac{d\mathbf{A}}{dt} = \lim_{\Delta t \to 0} \frac{\mathbf{A}(t + \Delta t) - \mathbf{A}(t)}{\Delta t}$$



$$\frac{d\mathbf{A}}{dt} = \left(\frac{dA_x}{dt}, \frac{dA_y}{d}, \frac{dA_z}{dt}\right) = \frac{dA_x}{dt}\hat{\mathbf{i}} + \frac{dA_y}{dt}\hat{\mathbf{j}} + \frac{dA_z}{dt}\hat{\mathbf{k}}$$

$$\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$$ Position vec.

$$\mathbf{v} = \dot{\mathbf{r}} = \dot{x}\hat{\mathbf{i}} + \dot{y}\hat{\mathbf{j}} + \dot{z}\hat{\mathbf{k}}$$ Veloc.

$$\mathbf{a} = \ddot{\mathbf{r}} = \dot{\mathbf{v}} = \ddot{x}\hat{\mathbf{i}} + \ddot{y}\hat{\mathbf{j}} + \ddot{z}\hat{\mathbf{k}}$$ acc.

Vector Calculus

Note that the magnitudes of the velocity and the acceleration are

$$v = |\mathbf{v}| = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}$$

$$a = |\mathbf{a}| = \sqrt{\ddot{x}^2 + \ddot{y}^2 + \ddot{z}^2}$$

Thus the differentiation of a vector follows the same procedure used in the differentiation of a scalar function. We can extend these rules to the following particular cases:

$$\frac{d}{ds}(\mathbf{A} \pm \mathbf{B}) = \frac{d\mathbf{A}}{ds} \pm \frac{d\mathbf{B}}{ds}$$

$$\frac{d}{ds}[f(s)\mathbf{A}(s)] = \frac{df}{ds}\mathbf{A} + f\frac{d\mathbf{A}}{ds}$$

$$\frac{d}{ds}(\mathbf{A} \cdot \mathbf{B}) = \frac{d\mathbf{A}}{ds} \cdot \mathbf{B} + \mathbf{A} \cdot \frac{d\mathbf{B}}{ds}$$

$$\frac{d}{ds}(\mathbf{A} \times \mathbf{B}) = \frac{d\mathbf{A}}{ds} \times \mathbf{B} + \mathbf{A} \times \frac{d\mathbf{B}}{ds}$$

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Summary

Scalars and Vectors

- Scalars have magnitude only
- Vectors have magnitude and direction
- Both have units!

Adding Geometrically

- Obeys commutative and associative laws

$$\vec{a} + \vec{b} = \vec{b} + \vec{a}$$ Eq. (3-2)

$$(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c}).$$ Eq. (3-3)

Vector Components

- Given by

$$a_x = a\cos\theta \quad \text{and} \quad a_y = a\sin\theta,$$ Eq. (3-5)

- Related back by

$$a = \sqrt{a_x^2 + a_y^2} \quad \text{and} \quad \tan\theta = \frac{a_y}{a_x}$$ Eq. (3-6)

Unit Vector Notation

- We can write vectors in terms of unit vectors

$$\vec{a} = a_x\hat{\mathrm{i}} + a_y\hat{\mathrm{j}} + a_z\hat{\mathrm{k}},$$ Eq. (3-7)

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Summary

Adding by Components

- **Add component-by-component**

$$r_x = a_x + b_x$$

$$r_y = a_y + b_y$$

Eqs. (3-10) - (3-12) $$r_z = a_z + b_z.$$

Scalar Times a Vector

- **Product is a new vector**
- **Magnitude is multiplied by scalar**
- **Direction is same or opposite**

Scalar Product

- **Dot product**

$$\vec{a} \cdot \vec{b} = ab \cos \phi,$$ **Eq. (3-20)**

$$\vec{a} \cdot \vec{b} = (a_x\hat{\mathrm{i}} + a_y\hat{\mathrm{j}} + a_z\hat{\mathrm{k}}) \cdot (b_x\hat{\mathrm{i}} + b_y\hat{\mathrm{j}} + b_z\hat{\mathrm{k}}),$$

Eq. (3-22)

Cross Product

- **Produces a new vector in perpendicular direction**
- **Direction determined by right-hand rule**

$$c = ab \sin \phi,$$ **Eq. (3-24)**