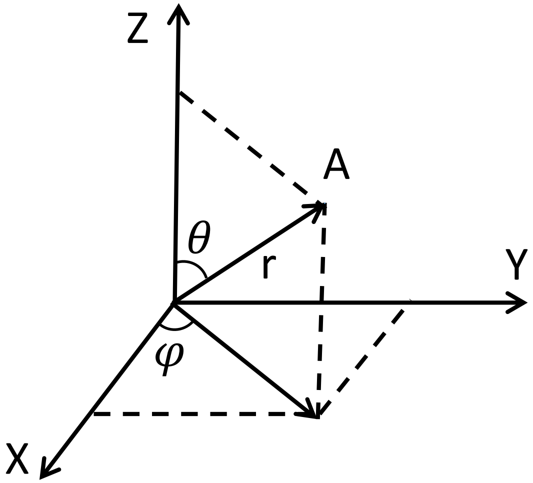
Cartesian and Spherical coordinates

A = (x, y, z) or (θ, φ, r)

$$\begin{cases} x = r \cdot sin(\theta)cos(\varphi) \\ y = r \cdot sin(\theta)sin(\varphi) \\ z = r \cdot cos(\theta) \end{cases}$$



$$\begin{cases} r = \sqrt{x^2 + y^2 + z^2} \\ \theta = cos^{-1}(z/r) \\ \varphi = tan^{-1}(y/x) \end{cases}$$

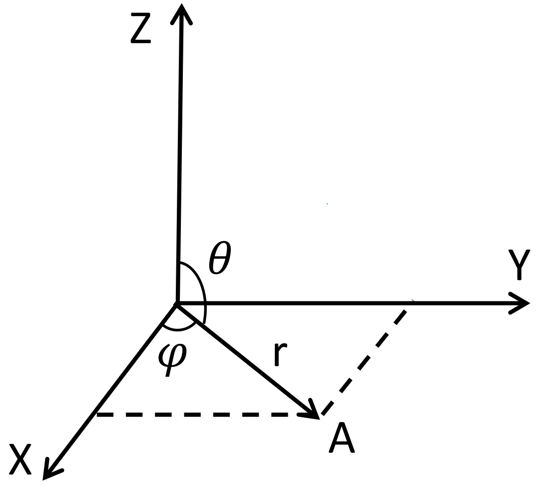
1.

The point A has spherical coordinates θ = π/2, ϕ = π/3, r = 6; the Cartesian coordinates of the point B are x = 5, y = $5\sqrt{3}$, z = 0. Find the distance between the points A and B.

What are the Cartesian coordinates of the point A, if the Spherical coordinates are $\begin{pmatrix} \theta = \pi/2 \\ \varphi = \pi/3 \\ r = 6 \end{pmatrix}$?

Solution

$$\begin{cases} x = r \cdot sin(\theta)cos(\varphi) = 3 \\ y = r \cdot sin(\theta)sin(\varphi) = 3\sqrt{3} \\ z = r \cdot cos(\theta) = 0 \end{cases} \longrightarrow A = (3, 3\sqrt{3}, 0)$$



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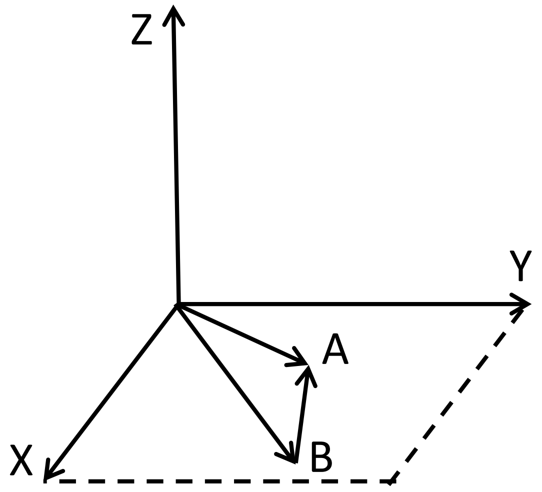


Solution

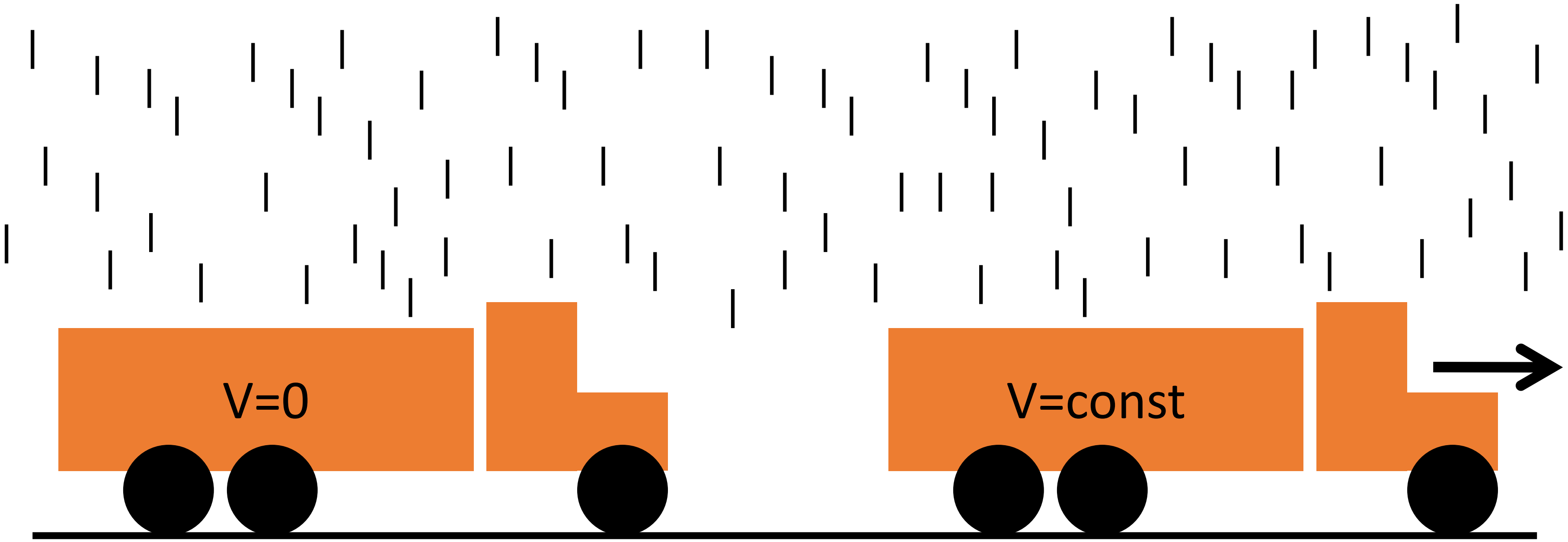
$$A = \left(3, 3\sqrt{3}, 0\right)$$

$$B = \left(5, 5\sqrt{3}, 0\right)$$

$$\left|\vec{B} - \vec{A}\right| = \sqrt{(b_x - a_x)^2 + \left(b_y - a_y\right)^2 + (b_z - a_z)^2} = \sqrt{(5-3)^2 + \left(5\sqrt{3} - 3\sqrt{3}\right)^2 + (0-0)^2} = 4$$

2.

Two identical trucks are caught in the rain. First truck is at rest, second one is moving with constant velocity V. Which truck gets more rain water per minute through the open dump truck body?



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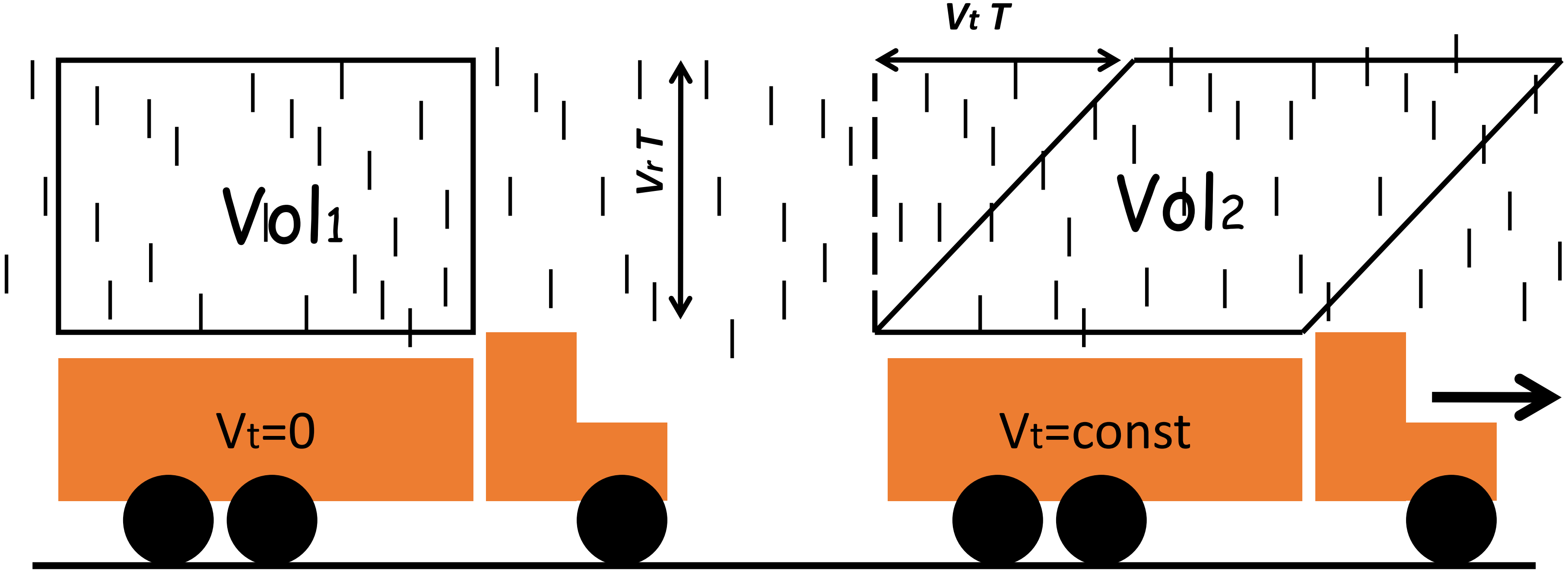
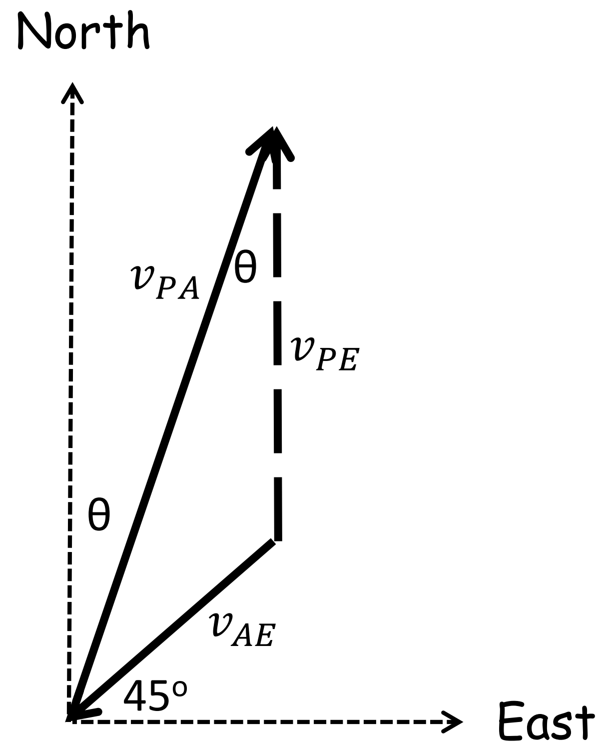
Graphical solution

Let's consider the volumes Vol1 and Vol2 containing rain drops which get into dump truck bodies during time interval T.



3. A pilot with an airspeed (speed with respect to air) of 120 km/h wishes to fly due north. A 40-km/h wind is blowing from the northeast. In what direction should she head, and what will be her ground speed (speed with respect to the ground)?



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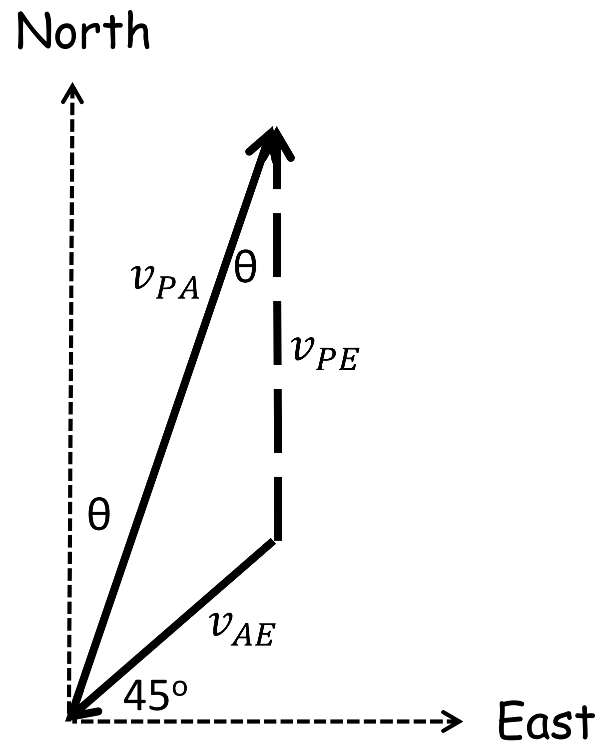
Solution

$$(v_{PE} + v_{AE}\cos 45°)^2 + (v_{AE}\sin 45°)^2 = v_{PA}^2$$

$$(v_{PE} + 40\cos 45°)^2 = (120)^2 - (40\sin 45°)^2$$

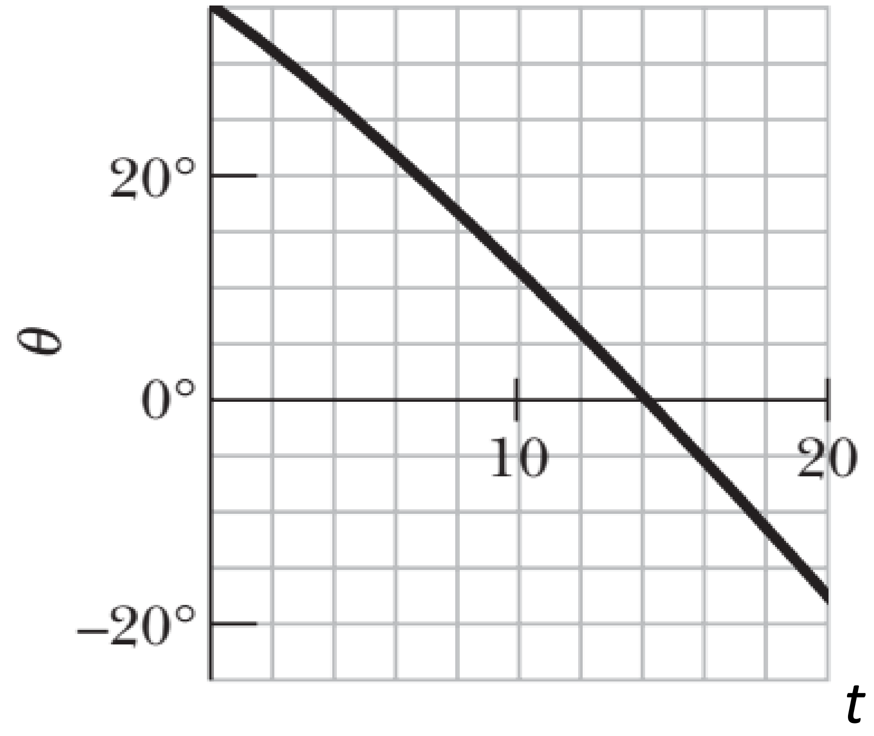
$$v_{PE} = 88.3\,\text{km/h}$$

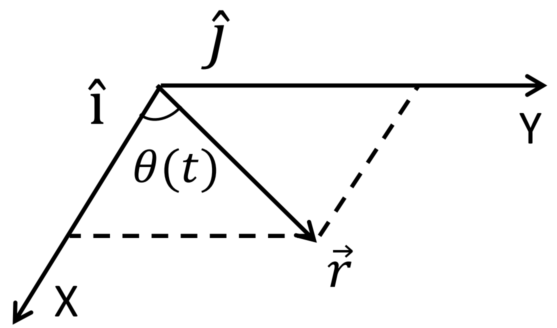
$$\theta = \sin^{-1}\frac{v_{AE}\sin 45°}{v_{PA}} = \sin^{-1}\frac{(40\sin 45°)^2}{120} = 13.6°\,\text{E of N}$$



The plane should head 13.6° east of north.

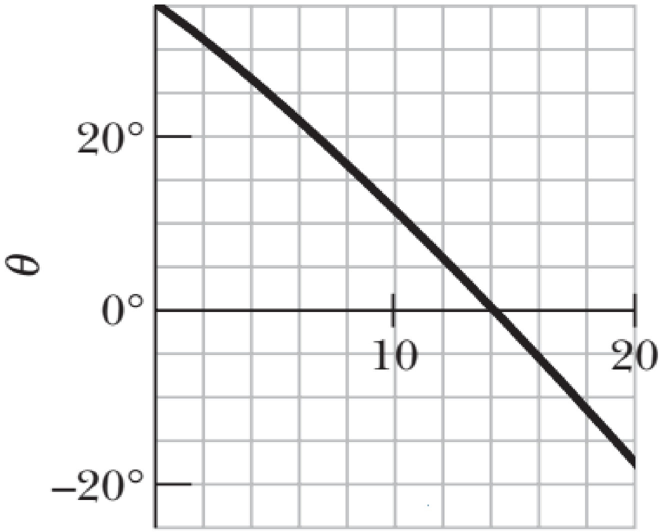
4. The position vector $\vec{r} = 5.00t\hat{\mathrm{i}} + (et + ft^2)\hat{\mathrm{j}}$ locates a particle as a function of time t. Vector $\vec{r}$ is in meters, t is in seconds, and factors e and f are constants. Figure 4-31 gives the angle θ of the particle's direction of travel as a function of t (θ is measured from the positive x direction). What are (a) e and (b) f, including units?





4. From vector representation to independent coordinate functions x(t) and y(t)

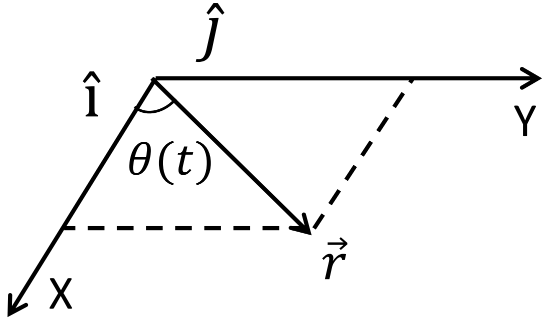
$$\vec{r} = 5.00t\hat{\mathrm{i}} + (et + ft^2)\hat{\mathrm{j}} \quad \longleftrightarrow \quad \begin{cases} x(t) = 5\cdot t \\ y(t) = e\cdot t + f\cdot t^2 \end{cases}$$



Solution

Let's build a function connecting θ and t variables

$$\tan(\theta(t)) = \frac{y(t)}{x(t)} = \frac{\left(e\cdot t + f\cdot t^2\right)}{(5\cdot t)} = \frac{(e + f\cdot t)}{5}$$



Consider two time moments t = 0 and t = 14

t = 0

$$\tan\left(35\cdot\frac{\pi}{180}\right) = \frac{(e + f\cdot 0)}{5}$$

$$0.7 = \frac{e}{5}$$

$$e = 3.5$$

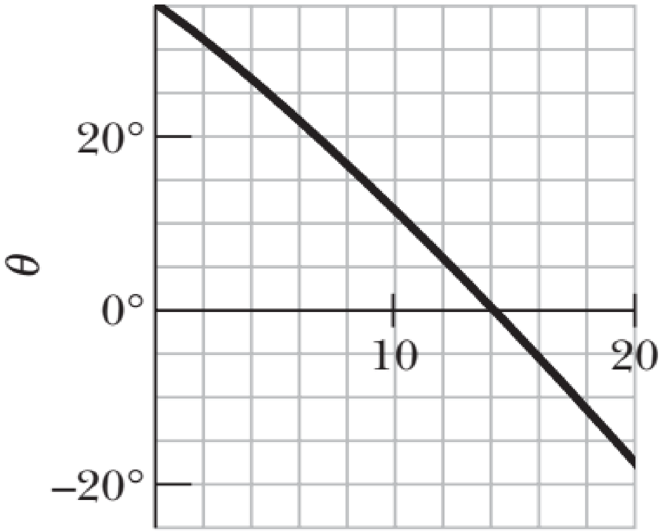
t = 14

$$\tan(0) = \frac{3.5 + f\cdot 14}{5}$$

$$f = -0.25$$

4.

$$\vec{r} = 5.00t\hat{\mathrm{i}} + (et + ft^2)\hat{\mathrm{j}} \quad \longleftrightarrow \quad \begin{cases} x(t) = 5\cdot t \\ yt(t) = e\cdot t + f\cdot t^2 \end{cases}$$

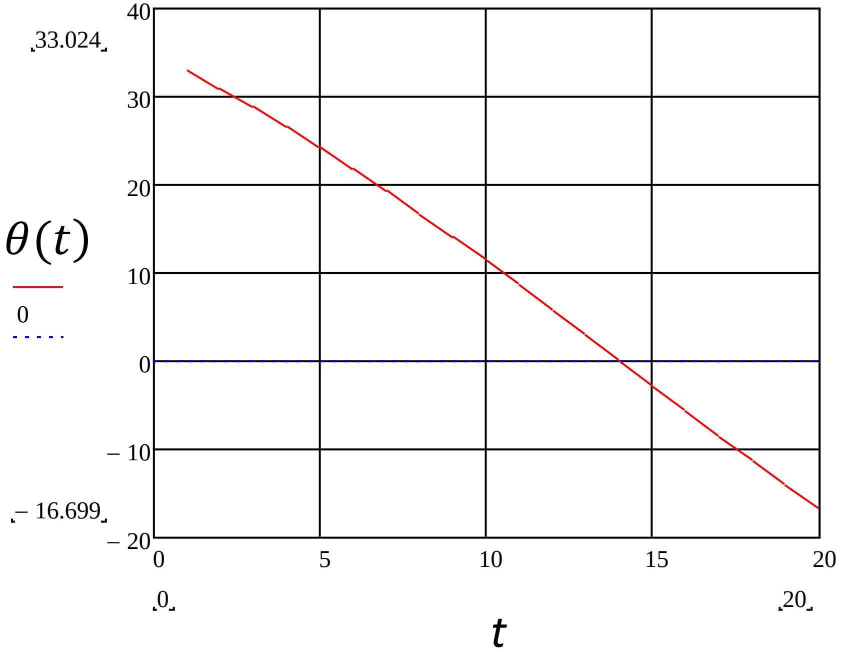


Solution

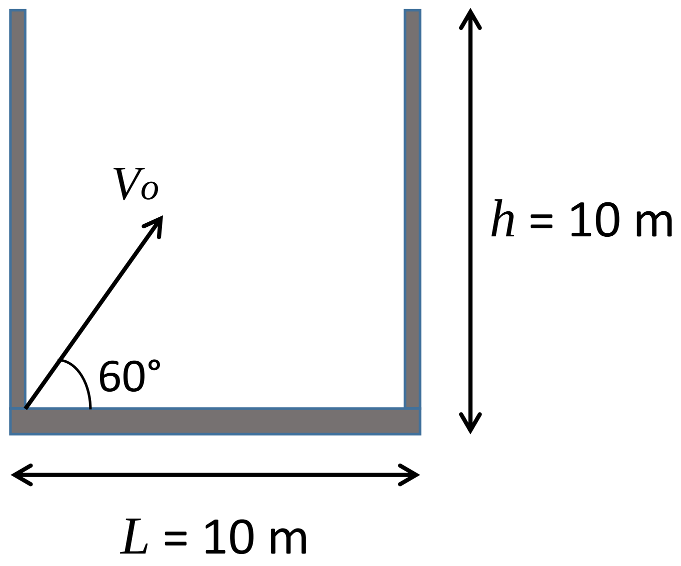
$$\tan(\theta(t)) = \frac{y(t)}{x(t)} = \frac{\left(e\cdot t + f\cdot t^2\right)}{(5\cdot t)} = \frac{(e + f\cdot t)}{5} \qquad e = 3.5 \qquad f = -0.25$$

Checking the solution by building the graph θ(t)

$$\theta(t) := \operatorname{atan}\left[\frac{\left(3.5\cdot t + -0.25\cdot t^2\right)}{5\cdot t}\right]$$



5. The ball is thrown from the bottom of a rectangular pit with an initial velocity V_o = 20 m/s which direction makes an angle of α=60 ° to the horizon line. The pit depth is h = 10 m, the distance from the throwing point to the pit wall is L = 10 m. Will the ball leave the pit?



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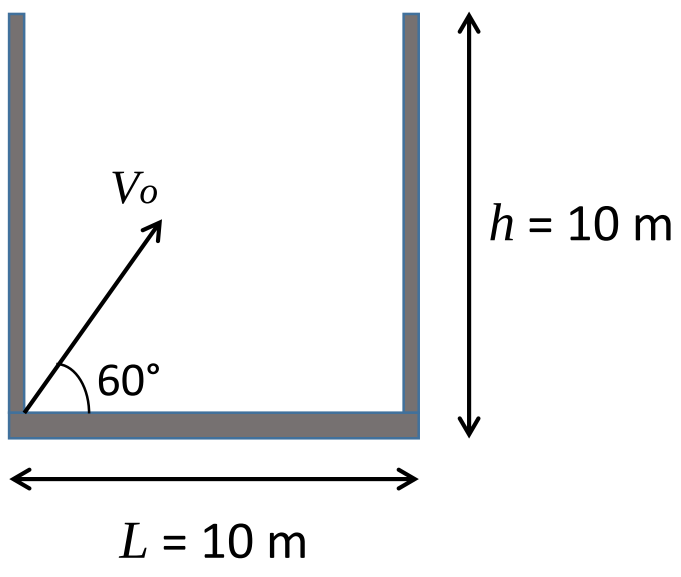

Solution

Transform from 2-dimensional motion to 2 × 1-dimensional motions

$$V_x = V_0 \cdot \cos(\alpha)$$

$$V_y = V_0 \cdot \sin(\alpha)$$

time of flying to the wall

$$t = \frac{L}{V_x} = \frac{L}{\left(V_0 \cdot \cos(\alpha)\right)}$$

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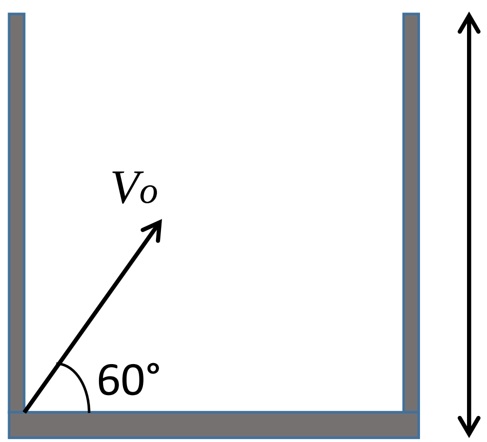


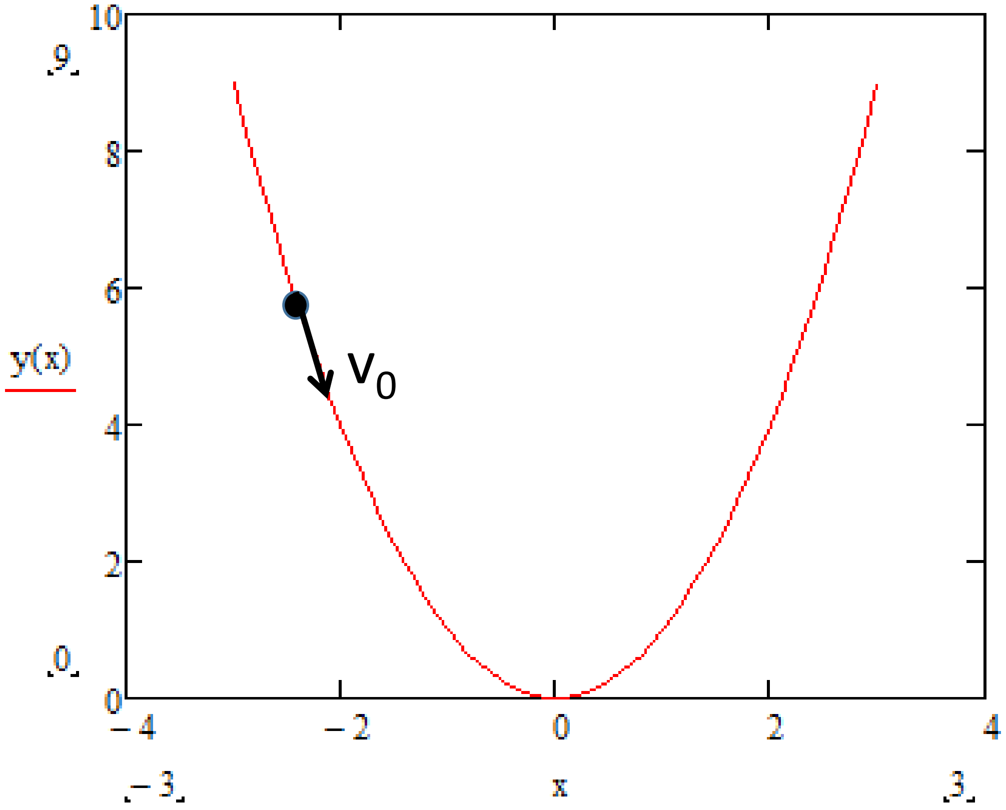
Solution

Elevation at moment $t = \frac{L}{V_x} = \frac{L}{\left(V_0 \cdot \cos(\alpha)\right)}$

$$H = V_y \cdot t - \frac{g \cdot t^2}{2} = \frac{L \cdot \sin(\alpha)}{\cos(\alpha)} - \frac{g \cdot L^2}{2 \cdot \left(V_0 \cdot \cos(\alpha)\right)^2} = 12.4 \text{ (m)}$$

$$H > h$$

6. The material point moves in plane with a constant magnitude of the velocity v_0. Its trajectory is given by equation $y\ (x) = c\ x^2$. Find at the point $x = 0$ the acceleration of the material point ~~and the trajectory radius~~.



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<u>Solution</u>

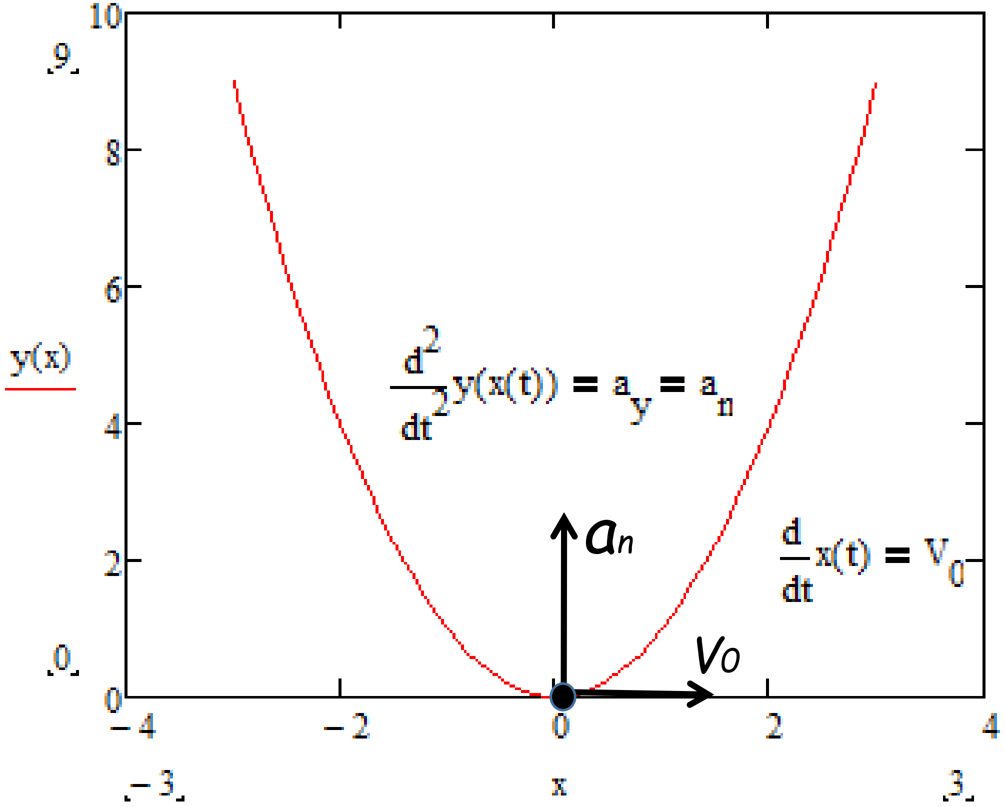
x-motion: velocity and acceleration

$$\frac{d}{dt}x(t) = V_0$$

$$\frac{d^2}{dt^2}x(t) = 0$$

y-motion: acceleration

$$\frac{d^2}{dt^2}y(x(t)) = a_y = a_n$$



6. The material point moves in plane with a constant magnitude of the velocity v_0. Its trajectory is given by equation y (x) = c x^2. Find at the point x = 0 the acceleration of the material point ~~and the trajectory radius~~.

$$\frac{d^2}{dt^2}y(x(t)) = \frac{d}{dt}\left(2\cdot c\cdot x(t)\cdot V_0\right) = 2\cdot c\cdot V_0\cdot\left(\frac{d}{dt}x(t)\right) = 2\cdot c\cdot\left(V_0\right)^2$$

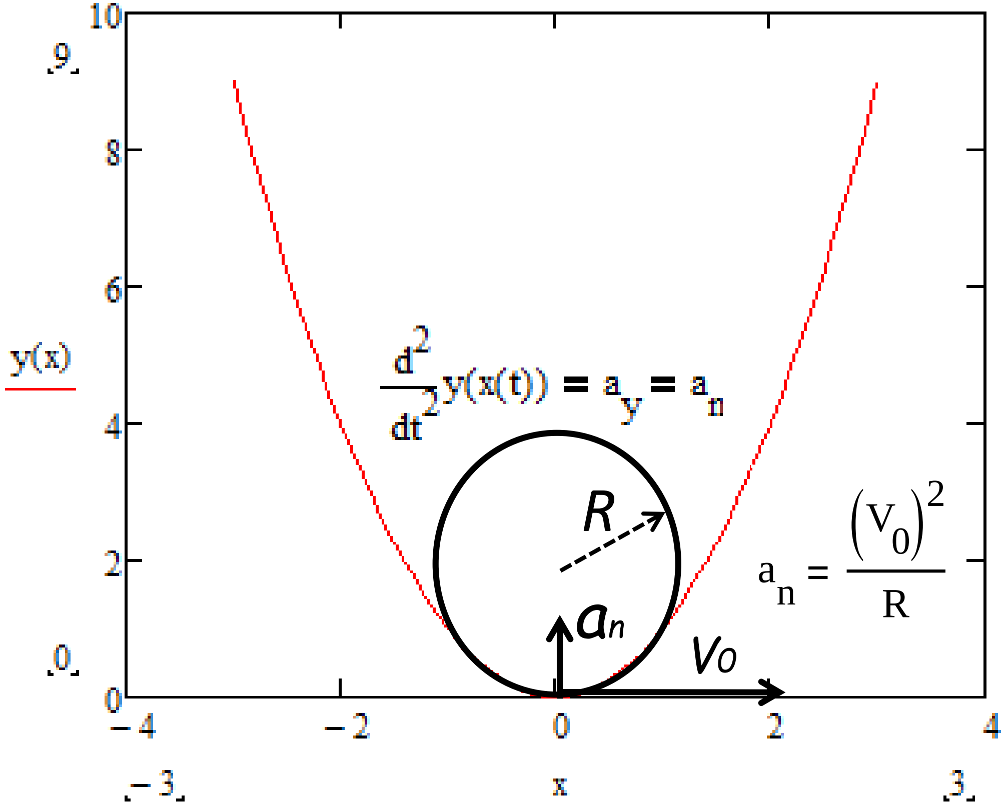
Physical solution

Use as a fact of circular motion. The normal acceleration is

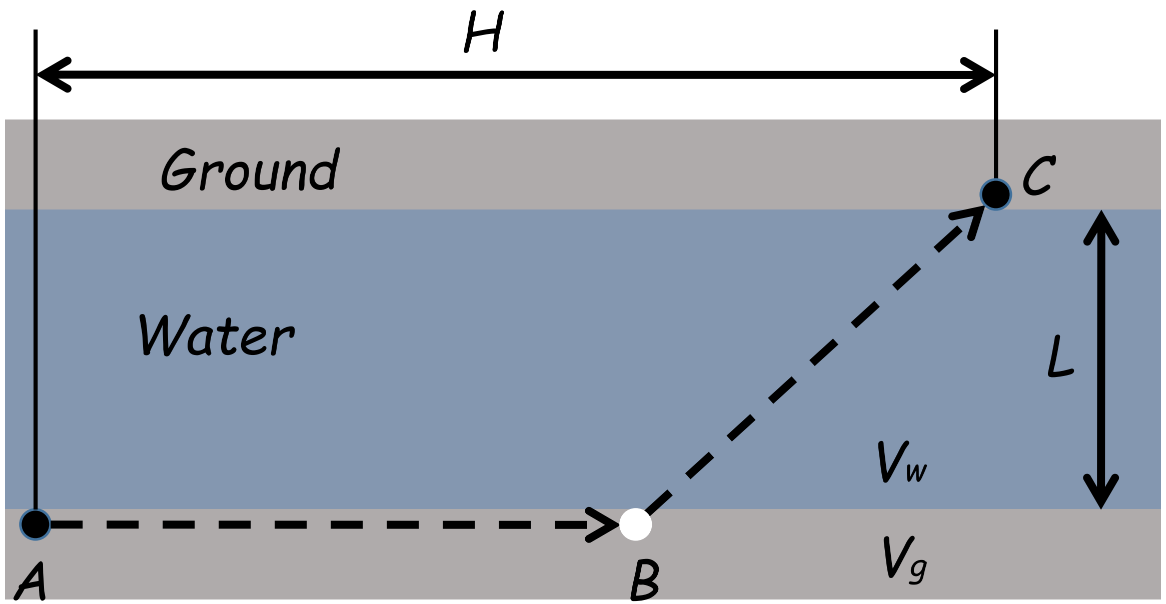
$\boxed{a_n = \frac{\left(V_0\right)^2}{R}}$, where R is the trajectory radius.

$$2\cdot c\cdot\left(V_0\right)^2 = \frac{\left(V_0\right)^2}{R}$$

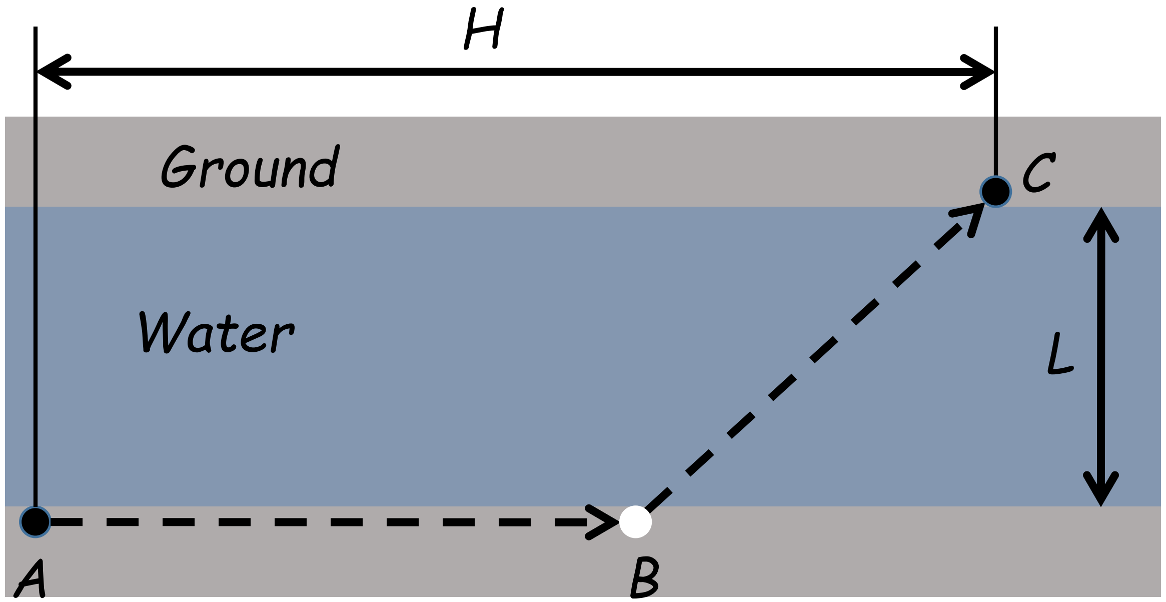
$$R = \frac{1}{2\cdot c}$$



7. A traveler has to get at the final C position from the initial A position via the intermediate B point. His walking speed is 7 km/h and swimming speed is 2 km/h. Find the distance AB, which allows fastest travel from A to C. Take into account H = 200 m, L = 45 m.



7.



Solution 1

Let's construct a function t(AB)

$$t(AB) = \frac{AB}{V_g} + \frac{\sqrt{(H - AB)^2 + L^2}}{V_w}$$

Finding extremum

$$\frac{d}{d(AB)}t(AB) = \frac{1}{V_g} + \frac{AB - H}{V_w \cdot \sqrt{(H - AB)^2 + L^2}}$$

$$\frac{1}{V_g} + \frac{AB - H}{V_w \cdot \sqrt{(H - AB)^2 + L^2}} = 0$$

$$AB = H - \frac{L \cdot V_w}{\sqrt{\left(V_g\right)^2 - \left(V_w\right)^2}} = 187 \text{ m}$$

7.

Solution 2

Speed of BC-changing (shortening)

$$V_{BC} = V_g \cdot \cos(\alpha)$$

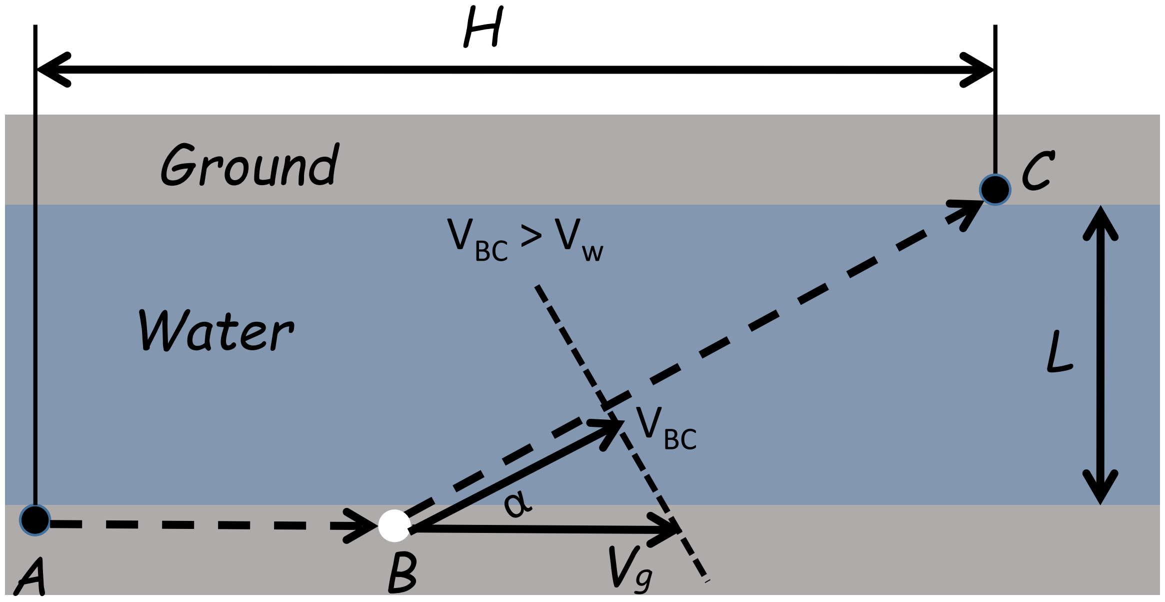
When should the traveler go to water?

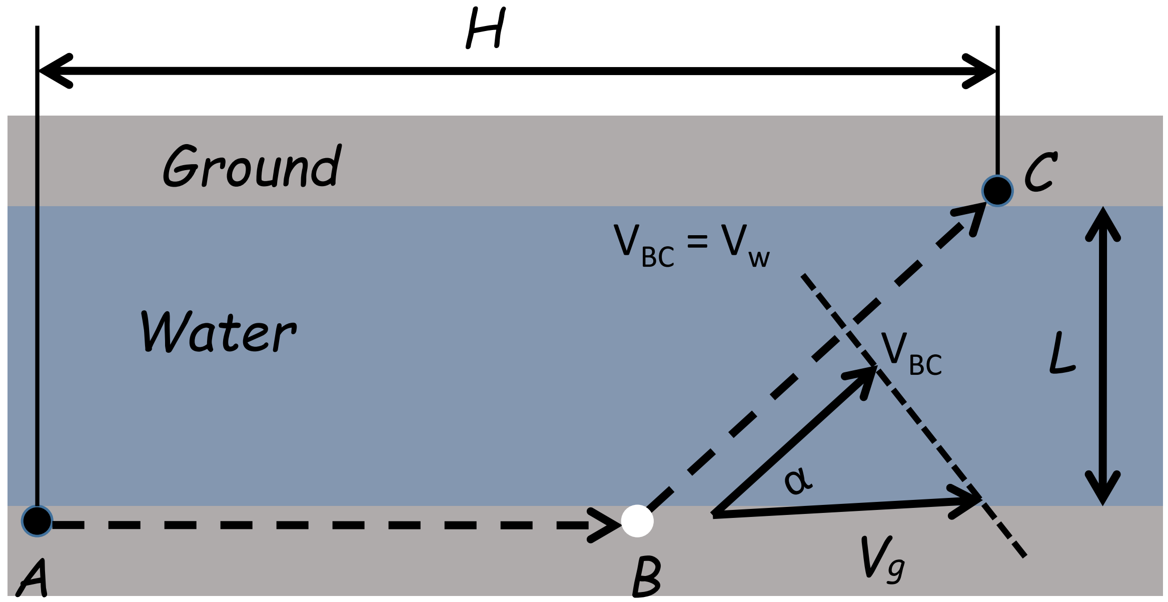
$$V_g \cdot \cos(\alpha) = V_w$$

$$\cos(\alpha) = \frac{H - AB}{\sqrt{L^2 + (H - AB)^2}}$$

$$V_g \cdot \frac{H - AB}{\sqrt{L^2 + (H - AB)^2}} = V_w$$

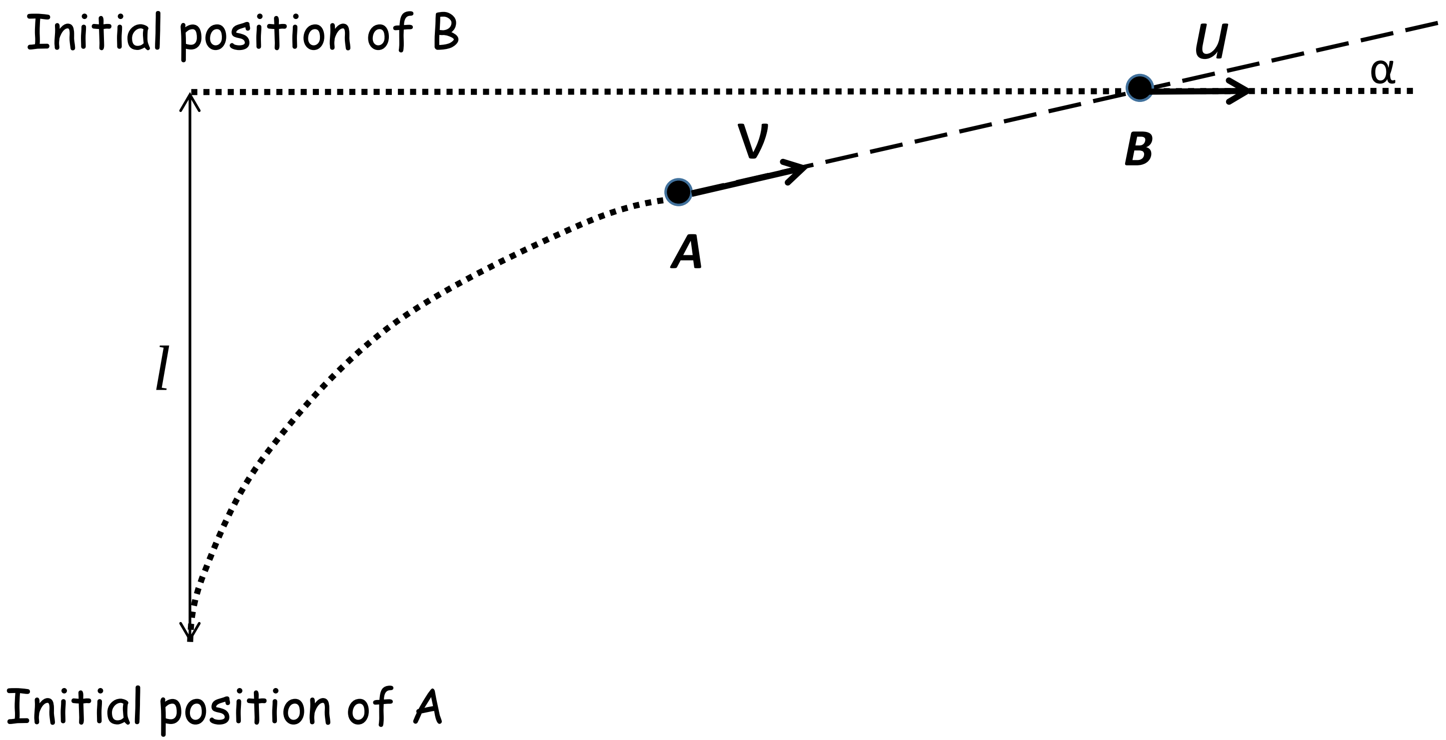
$$AB = H - \frac{L \cdot V_w}{\sqrt{(V_g)^2 - (V_w)^2}} = 187 \text{ m}$$



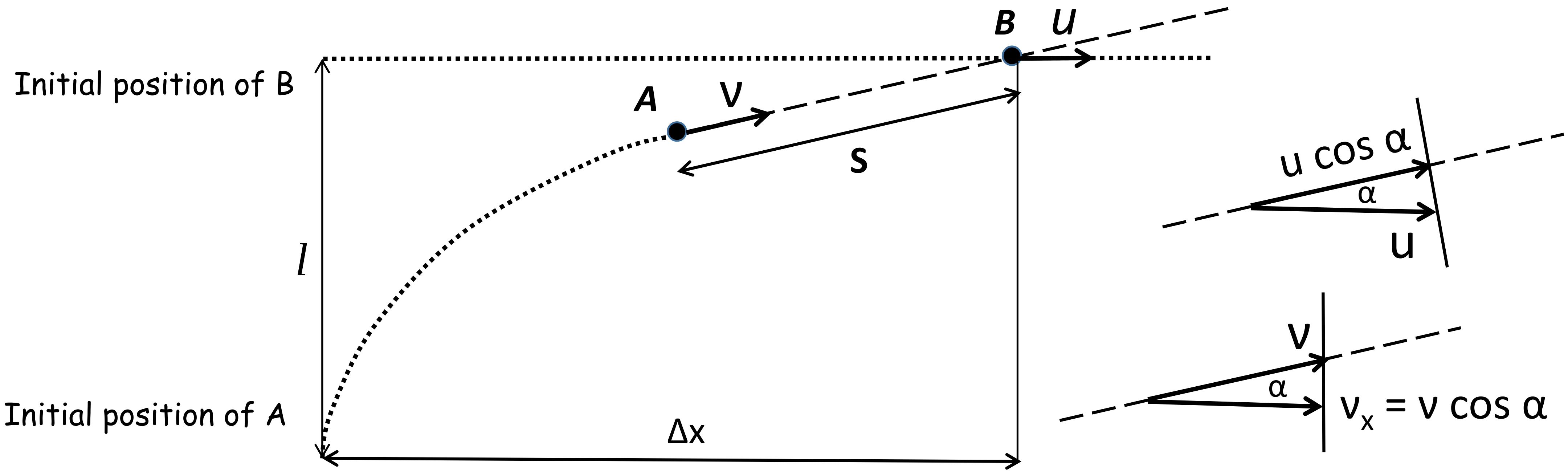


8.

Point A moves uniformly with velocity v so that the vector $\mathbf{v}$ is continually "aimed" at point B which in its turn moves rectilinearly and uniformly with velocity $u < v$. At the initial moment of time $\mathbf{v} \perp \mathbf{u}$ and the points are separated by a distance l. How soon will the points converge?



8.



If A and B are separated by the distance s at this moment, then the points converge or point A approaches B with velocity $\frac{-ds}{dt} = v - u\cos\alpha$ where angle α varies with time.

$$-\int_l^0 ds = \int_0^T (v - u\cos\alpha)\,dt,$$

(where T is the sought time.)

$$l = \int_0^T (v - u\cos\alpha)\,dt$$

$$\Delta x = \int v_x\,dt \qquad uT = \int_0^T v\cos\alpha\,dt$$

$$T = \frac{ul}{v^2 - u^2}$$

Is this answer correct?