Physics 1. Mechanics.

Week 2 Motion in 2 and 3 Dimensions

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Objectives

The main objectives of today’s lecture are:

- Study relationships between position, velocity and acceleration in multi-dimensional motion
- Discuss projectile motion
- Study how rabbits run

Question?

Today's question:

You have entered a competition in which the person who throws a ball the farthest wins a chicken dinner. Considering that there is air resistance (air drag),

- what should be the angle θ_0 between the ball's initial speed and the ground so that you can throw it as far as possible?

a) Slightly less than 45 degrees

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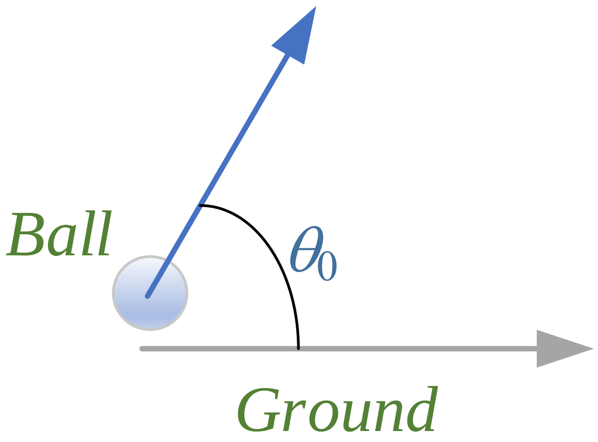

Motion in Two Dimensions

Physics and Mechanics

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Today, we will continue looking at the aspect of physics that analyzes motion, but now the motion can be in two or three dimensions.

Motion in three dimensions is not easy to understand.

- For example, you are probably good at driving a car along a freeway (one-dimensional motion) but would probably have a difficult time in landing an airplane on a runway (three-dimensional motion) without a lot of training.

Image credit: The Telegraph

Position and Displacement

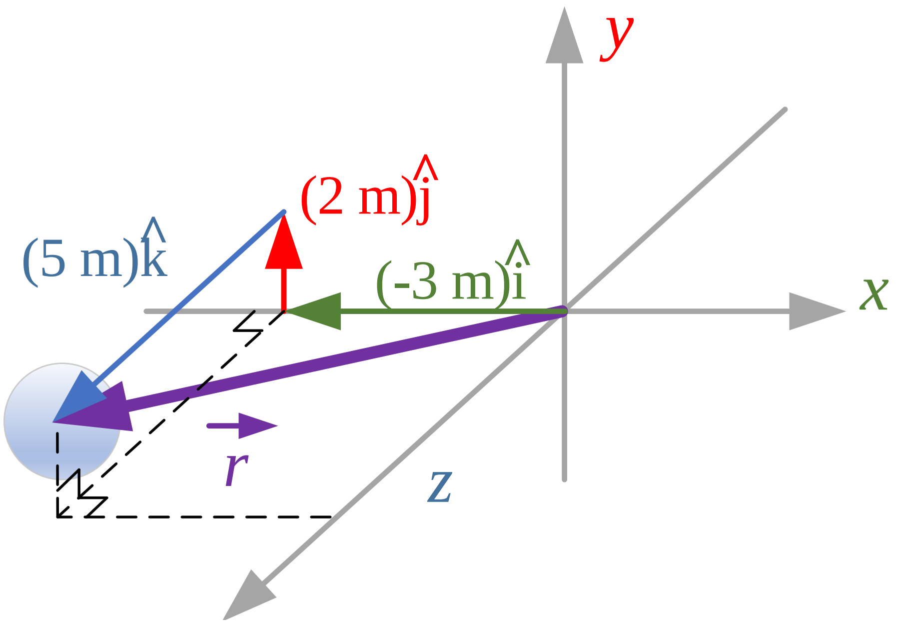
One general way of locating a particle (or particle-like object) is with a position vector $\vec{r}$, which is a vector that extends from a reference point (usually the origin) to the particle.

- According to the notation of this class, vector $\vec{r}$ can be written as

$$\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$$

- The radius vector shown on the right can be written as

$$\vec{r} = (-3\ \mathrm{m})\hat{\imath} + (2\ \mathrm{m})\hat{\jmath} + (5\ \mathrm{m})\hat{k}$$



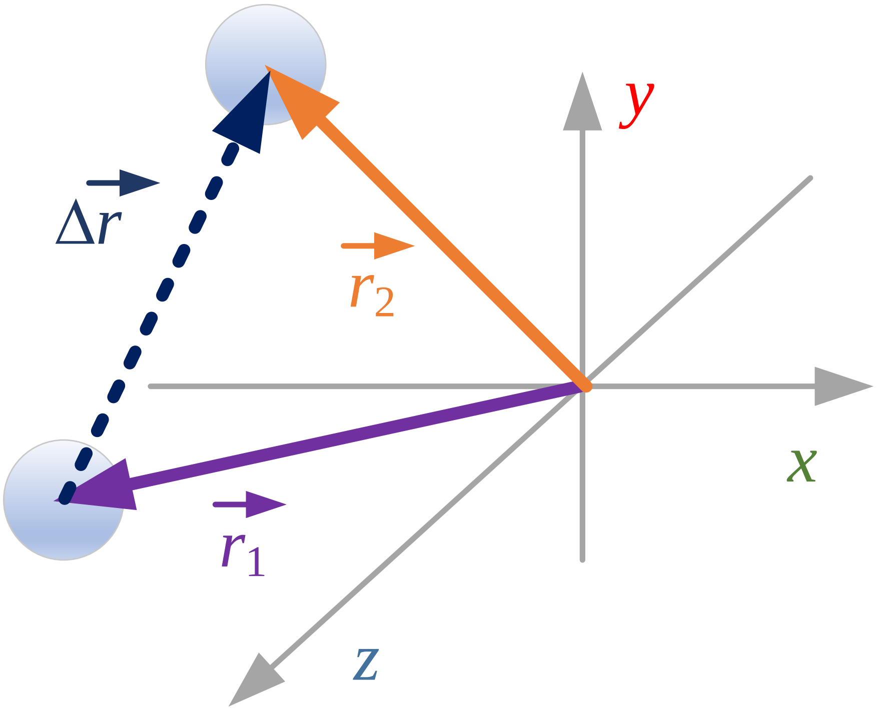
Position and Displacement

As a particle moves, its position vector changes in such a way that the vector always extends to the particle from the reference point (the origin).

- If the position vector changes — say, from $\vec{r}_1$ to $\vec{r}_2$ during a certain time interval — then the particle's displacement $\Delta\vec{r}$ during that time interval is

$$\Delta\vec{r} = \vec{r}_2 - \vec{r}_1$$

$$\Delta\vec{r} = \Delta x\hat{\imath} + \Delta y\hat{\jmath} + \Delta z\hat{k}$$



Average Velocity

If a particle moves through a displacement $\Delta\vec{r}$ in a time interval Δt, then its average velocity $\vec{v}_{avg}$ is

$$\vec{v}_{avg} = \frac{\Delta\vec{r}}{\Delta t}$$

- This tells us that the direction of $\vec{v}_{avg}$ must be the same as that of the displacement $\Delta\vec{r}$
- We can rewrite this equation as

$$\vec{v}_{avg} = \frac{\Delta x\hat{\imath} + \Delta y\hat{\jmath} + \Delta z\hat{k}}{\Delta t} = \frac{\Delta x}{\Delta t}\hat{\imath} + \frac{\Delta y}{\Delta t}\hat{\jmath} + \frac{\Delta z}{\Delta t}\hat{k}$$

- For instance, if a particle moves through displacement $\Delta\vec{r} = (12\text{ m})\hat{\imath} + (3\text{ m})\hat{k}$ in 2 seconds, then its average velocity during that move is

$$\vec{v}_{avg} = \frac{(12\text{ m})\hat{\imath} + (3\text{ m})\hat{k}}{2\text{ s}} = (6\text{ m})\hat{\imath} + (1.5\text{ m})\hat{k}$$

Instantaneous Velocity (1)

When we speak of the velocity of a particle, we usually mean the particle’s instantaneous velocity $\vec{v}$ at some instant. This $\vec{v}$ is the value that $\vec{v}_{avg}$ approaches in the limit as we shrink the time interval Δt to 0 about that instant.

- Using the language of calculus, we may write $\vec{v}$ as the derivative

$$\vec{v} = \frac{d\vec{r}}{dt}$$

- The direction of $\vec{v}$ approaches the **line tangent to the particle’s path** at a given time.

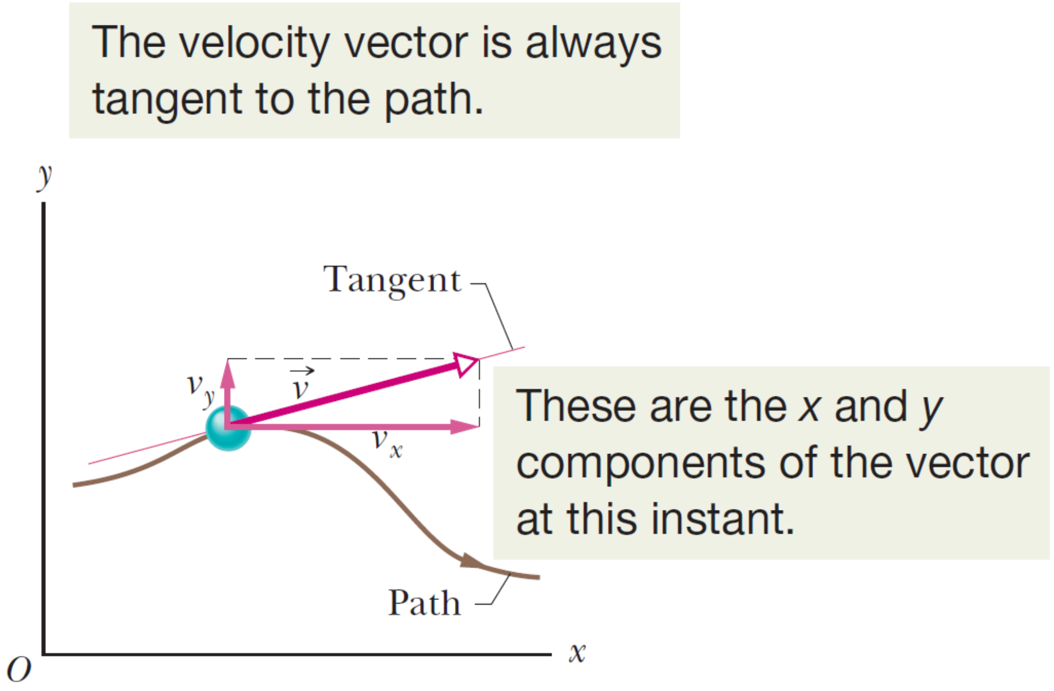
$$\vec{v} = \frac{d}{dt}\left(x\hat{\mathrm{i}} + y\hat{\mathrm{j}} + z\hat{\mathrm{k}}\right) = \frac{dx}{dt}\hat{\mathrm{i}} + \frac{dy}{dt}\hat{\mathrm{j}} + \frac{dz}{dt}\hat{\mathrm{k}} = v_x\hat{\mathrm{i}} + v_y\hat{\mathrm{j}} + v_z\hat{\mathrm{k}}$$

Instantaneous Velocity (2)

Figure here shows a velocity vector $\vec{v}$ and its scalar x and y components.

- Note that $\vec{v}$ is tangent to the particle's path at the particle's position.

- When a velocity vector is drawn, it does *not* extend from one point to another.
- Rather, it shows the instantaneous direction of travel of a particle at the tail, and its length (representing the velocity magnitude) can be drawn to any scale.



Instantaneous Velocity: Exercise (1)

Assume that a **rabbit's** motion can be described by the following equations:

$$x = -0.31t^2 + 7.2t + 28$$

$$y = 0.22t^2 - 9.1t + 30$$

Find the rabbit's velocity at t = 15 s.

- Finding horizontal and vertical velocity components yields

$$v_x = \frac{dx}{dt} = -0.62t + 7.2$$

$$v_y = \frac{dy}{dt} = 0.44t - 9.1$$

Instantaneous Velocity: Exercise (2)

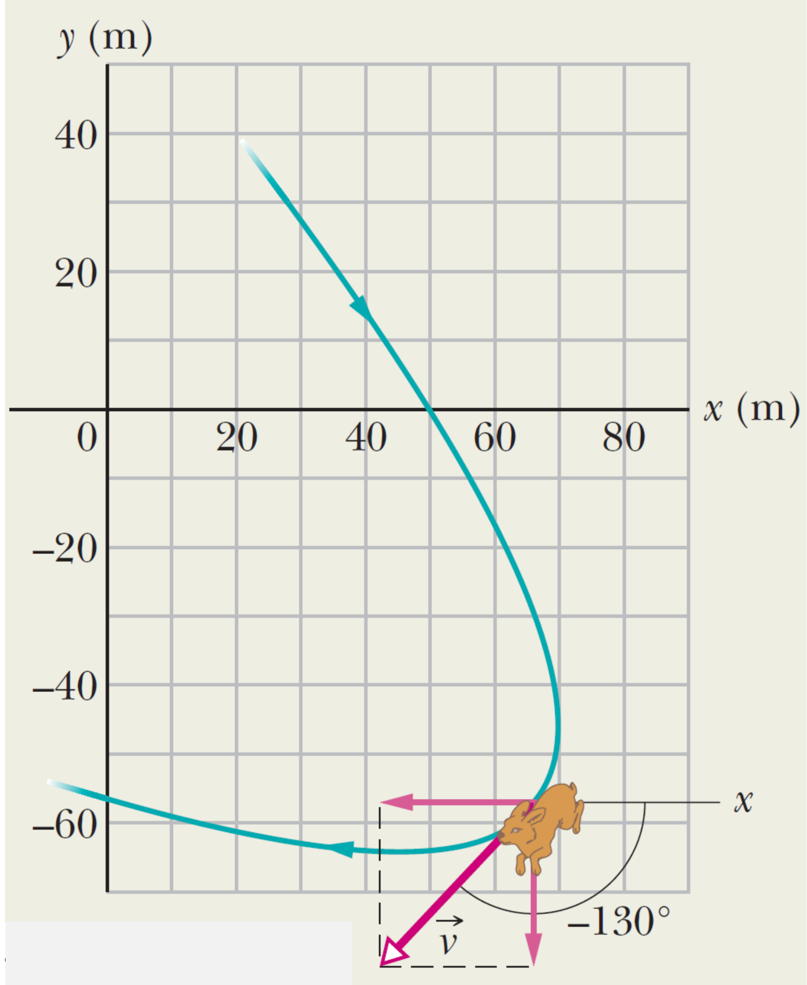
At time t = 15 s,

$$v_x = -2.1 \text{ m/s}$$

$$v_y = -2.5 \text{ m/s}$$

Finally,

$$\vec{v} = (-2.1 \text{ m/s})\hat{\imath} + (-2.5 \text{ m/s})\hat{\jmath}$$



Instantaneous Acceleration

Just like it was the case with motion along a straight line, instantaneous acceleration can be found as

$$\vec{a} = \frac{d\vec{v}}{dt}$$

Thus,

- If the velocity changes in *either* magnitude *or* direction (or both), the particle must have an acceleration.

$$\vec{a} = \frac{d}{dt}\left(v_x\hat{\mathrm{i}} + v_y\hat{\mathrm{j}} + v_z\hat{\mathrm{k}}\right) = \frac{dv_x}{dt}\hat{\mathrm{i}} + \frac{dv_y}{dt}\hat{\mathrm{j}} + \frac{dv_z}{dt}\hat{\mathrm{k}}$$

Projectile Motion

Projectile Motion (1)

We next consider a special case of two-dimensional motion:

A particle moves in a vertical plane with some initial velocity $\vec{v}_0$ but its acceleration is always the free-fall acceleration $\vec{g}$, which is directed downward.

- Such a particle is called a projectile (meaning that it is projected or launched)
- In our analysis today, we will assume that air has no effect on the projectile.

Richard Megna/Fundamental Photographs

A tennis ball bouncing off a hard surface.
Between impacts, the ball has projectile motion

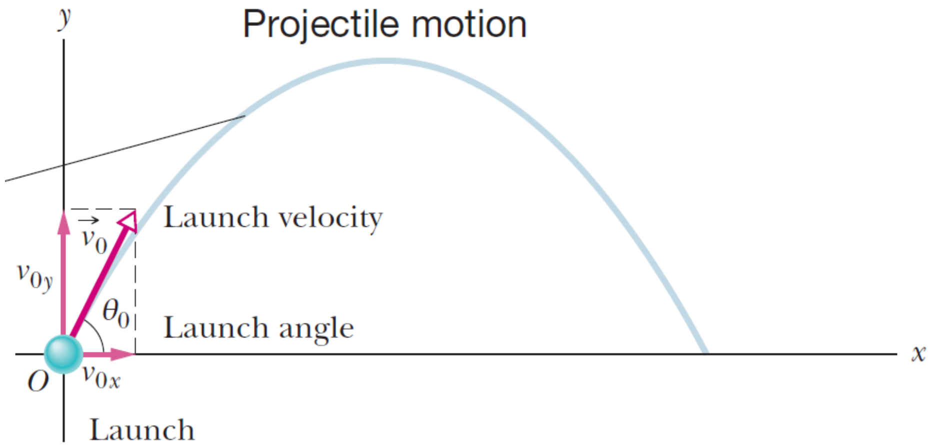
Projectile Motion (2)

Assume that a projectile is launched with an initial velocity $\vec{v}_0$ that can be written as

$$\vec{v}_0 = v_{0x}\hat{\imath} + v_{0y}\hat{\jmath}$$

The components v_{0x} and v_{0y} can then be found if we know the angle θ_0 between $\vec{v}_0$ and the positive x direction:

$$v_{0x} = v_0 \cos\theta_0, \qquad v_{0y} = v_0 \sin\theta_0$$



- During its two-dimensional motion, the projectile's position vector $\vec{r}$ and velocity vector $\vec{v}$ change continuously, but its
- acceleration vector a is constant and *always* directed vertically downward.
- The projectile has *no* horizontal acceleration.

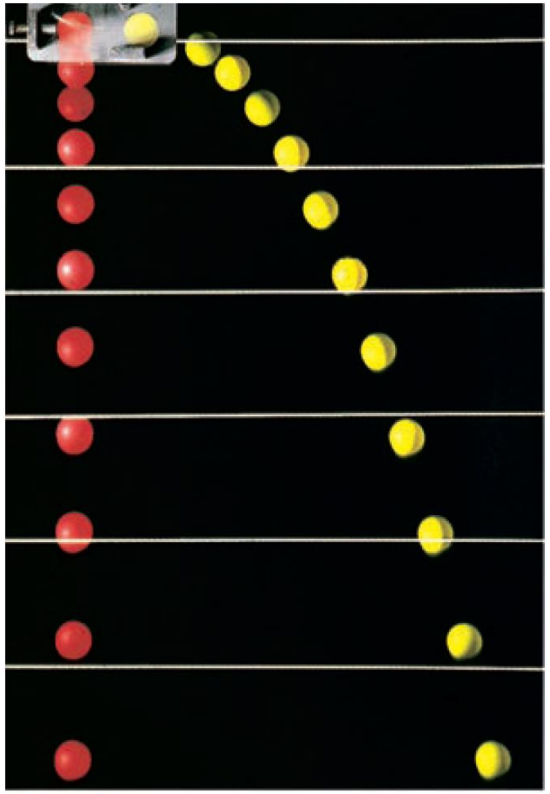
Projectile Motion (3)

What do experiments tell us about projectile motion?

- In projectile motion, the horizontal motion and the vertical motion are independent of each other; that is, neither motion affects the other.

This feature allows us to break up a problem involving two-dimensional motion into two separate and easier one-dimensional problems:

- one for the horizontal motion (with *zero acceleration*) and
- one for the vertical motion (with *constant downward acceleration*).



Richard Megna/Fundamental Photographs

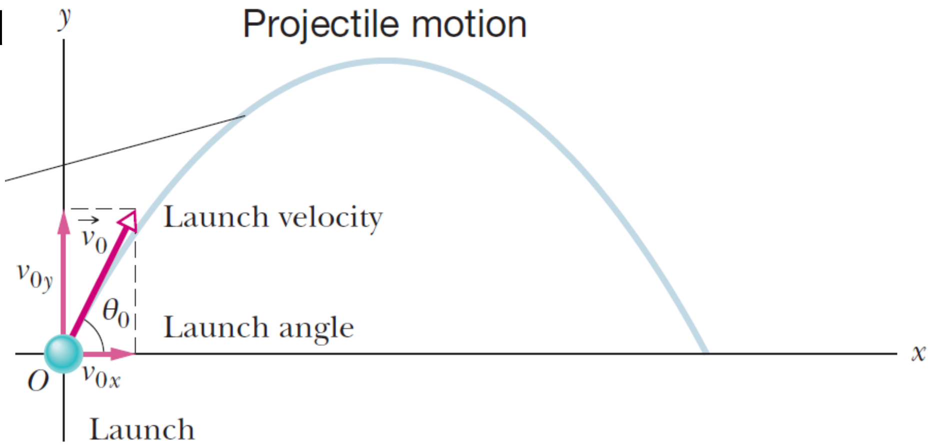
One ball is released from rest at the same instant that another ball is shot horizontally to the right. Their vertical motions are identical.

Projectile Motion (4): Horizontal Speed

$$x = v_{0x}t + {}^{1}/_{2}\, a_x t^2 + x_0$$

Now we are ready to analyze projectile motion, horizontally and vertically.

- We start with the horizontal motion. Because there is *no acceleration* in the horizontal direction, the horizontal component v_x of the projectile's velocity remains unchanged from its initial value v_{0x} throughout the motion.



- At any time t, the projectile's horizontal displacement $x - x_0$ from an initial position x_0 is given by equation with a = 0:

$$x - x_0 = v_{0x} \cdot t = (v_0 \cos\theta_0)t$$

Projectile Motion (5): Vertical Speed

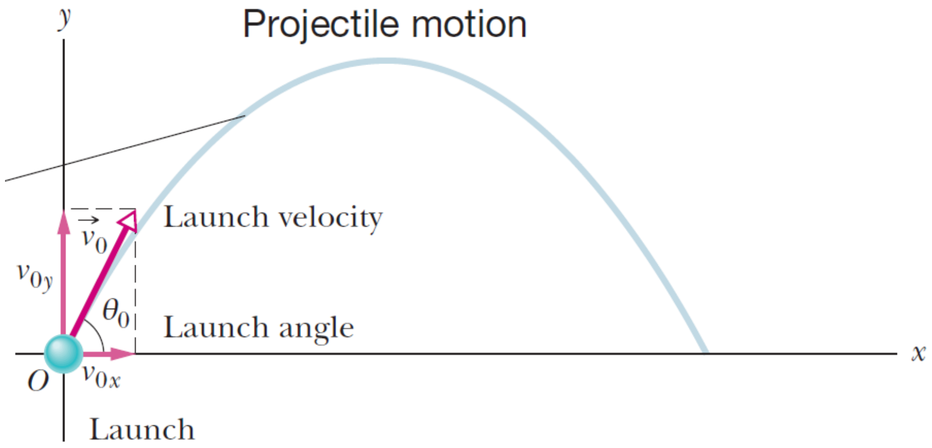
$$y = v_{0y}t + {}^{1}/_{2}\, a_y t^2 + y_0$$

The vertical motion is the motion we discussed for a particle in free fall. Most important is that the acceleration is constant.

Proceeding in a similar fashion, we write

$$y - y_0 = v_{0y} \cdot t - {}^{1}/_{2}\, g t^2 = \quad (v_0 \sin\theta_0)t - {}^{1}/_{2}\, g t^2$$

- Q: How to find vertical speed? $v_y = v_0 \sin\theta_0 - gt$



The vertical velocity component behaves just as for a ball thrown vertically upward:

1. Velocity is directed upward initially, and its magnitude steadily decreases to zero, *which marks the <u>maximum height</u> of the path.*
2. Vertical velocity component then reverses direction, and its magnitude is growing.

Projectile Motion: Equation of the Path

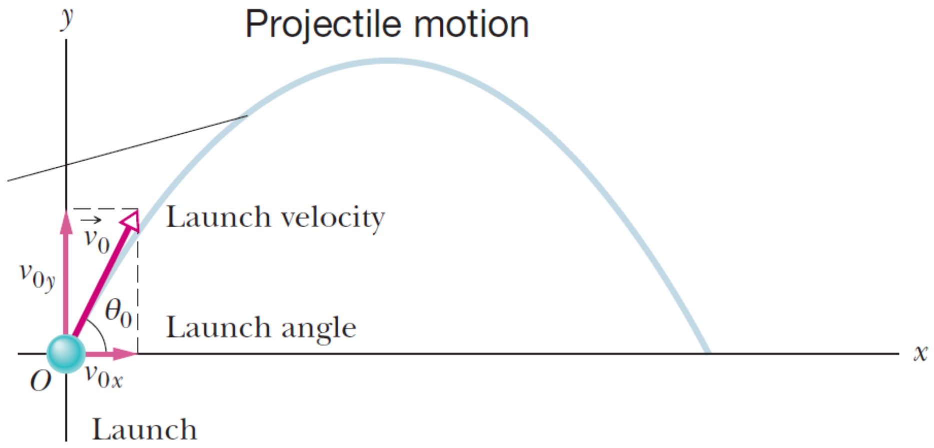
We can find the equation of the projectile's path (its **trajectory**) by eliminating time t between the two equations:

$$x - x_0 = (v_0 \cos\theta_0)t$$

$$y - y_0 = (v_0 \sin\theta_0)t - {}^1/_2\, g t^2$$

We obtain the following (for zero initial conditions):

$$y = (\tan\theta_0)x - \frac{g x^2}{2(v_0\cos\theta_0)^2}$$



- Because g, θ_0, and v_0 are constants, this equation is of the form $y = ax + bx^2$, in which a and b are constants.
- This is the equation of a parabola, so the path is *parabolic*.

Projectile Motion: Horizontal Range

$$x - x_0 = (v_0 \cos\theta_0)t$$
$$y - y_0 = (v_0 \sin\theta_0)t - {}^1\!/_2\, gt^2$$

The horizontal range R of the projectile is the horizontal distance the projectile has traveled when it returns to its initial height (the height at which it is launched).

- To find range R, let us put $x - x_0 = R$ and $y - y_0 = 0$, obtaining

$$R = (v_0 \cos\theta_0)t$$
$$0 = (v_0 \sin\theta_0)t - {}^1\!/_2\, gt^2$$

- Eliminating t between these two equations yields:

$$R = \frac{2{v_0}^2}{g} \sin\theta_0 \cos\theta_0 = \frac{{v_0}^2}{g} \sin 2\theta_0$$

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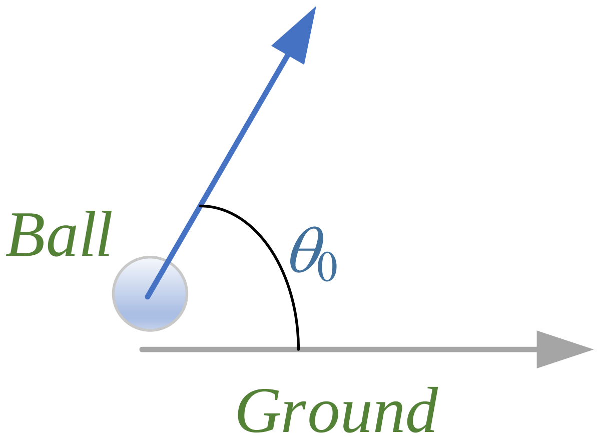

Thank you for your attention!

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