Quantum computing lecture notes

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Quantum computing lecture notes (August-October 2021)

Plan for the course:

- QC from the top, comparison to classic, additional resources of QC
- Postulates of QMechanics
- Examples of computing given electron spins as qubits
- Examples of Quantum Algorithms
- Error correction
- On implementation of QComputers
- perspectives of QC

Lecture 1. What is QC, what is quantum computer in general

Classic computers start from A. Turing (math model).

- Turing Machine is a mathematical model of a physical system (a ribbon).
- Any information can be coded in 2-state gates.
- *Bit* is a term for amount of information, but often we can *bit* a gate itself.
- *NOT* and *Controlled-NOT (CNOT)* [quantum] gates are enough to implement any classic program.
- There are also qutrits for 3-state systems. They can sometime be beneficial, but not developed that much.
- First classical computer was developer 1941 ~18000 vacuum tubes, 30 tons. Used relays (before transistors appeared)
- Anything spoken in words can be coded in classical computer... But we need more memory!

Microrelectronics

- Revolution started with semiconducting diodes of $\sim\mu m$ (10^{-4}) size.
- How to create a gate: bomb Si with Fe+ to create ferro-magnetic "lakes" which can have magnetic momenum, controlled by outer force

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- We can have 10^8 gates per cm^2, but at nano scale (10^{-9} linear sizes) world changed. Refer to Feynmans lecture "There's Plenty of Room at the Bottom".

From classic to quantum

- qubit = "quantim bits". We can have macroscopic objects which have quantum properties, e.g. superconductor's conductivity.
- it is hard to address qubits selectively, if they are small. That is why most promising approach is using supercondictors. BUT them we will loose memory capacity!
- For QC it is enough to have ~500 gates to solve any "imaginable" problem. <100 gates are enough to solve almost any.
- This doesn't mean we should stop searching other implementations.
- transition: even if qubit can also be **observed** as a 2-state system, it **has** infinite number of states.

Quantum:

- basic operations are NOT and **C-NOT**: if A then not B.

IN	OUT
00	00
01	01
10	11
11	10

- States - any vector from Hilbert space. E.g. we usually denote

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

- NOT in matrix form:

$$NOT = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} = |1\rangle = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \times |0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

New resources of QC

- 1 qubit $\sim \psi = a|0\rangle + b|1\rangle$, where $|a|^2 + |b|^2 = 1$;
- 2 qubit $\sim \psi = a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle$;
- N qubits - 2^N coefficients
- System can occupy multiple basic states at the same time.
- QC allows to perform logical operations, which cannot be implemented by classic computers, e.g. $\sqrt{NOT}$
- Cross terms: $a * b$ (coherence) increase faster, then just a number of coefficients. They are the major carrier of information in QC.

Problems of quantum computers:

- !! systems are loosing their properties with time.
- quantum coherence disappears with time! Major information is stored not in a population of states (probability to observe a state).
- system evolution needs time. E.g. apping NOT and stopping earlier will not result into "pure NOT". In math this is just a unitary matrix multiplication
- Memory comparison: 48 qubits are "equivalent/comparable" to 10^{14} classic elements ($1cm^2$ with nm scale).
- quantum systems cannot be copied. If we can reproduce a system – we destroy initial state

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How can we build a qubit

- Actually any quantum system. **QD**
- quantum dot. Potential box for an electron. We choose any pair of states of an e^- as a calculation basis, e.g. $n = 0, 1$. If states were equidistant, we could excite it even further ($n = 2$). But as energy level of electron are not equidistant, we are ok.
- We should be sure we address only 2 states and only them. 2 states – is just a metter of convenience, as everything is already binary.
- Using Fullerene with Nitrogen (N) inside was proposed.

Spin - Electron spin! Just an electron has magnetic moment (no wires, no other). $\omega = \gamma B_0$ $|+\frac{1}{2}\rangle => |1\rangle$. Proton also has spin. - Why? 1944 Zavoisky developed EPR to do whatever we need with electron spins. - Irradiated malonic adic. Detach one H^+. You have an electron with $1\mu s$ decoherence time. We can have order of ten operations. Not too much.

Dimond defects - In diamond we replace one C with N. Loss of electron.

Biradicals - CNOT with 2 unpaired electrons. We need intereaction between spins, e.g. inside a molecule.

- concept of coherence is the major concept to learn for the future.

Q&A: - Why do you thing Schrödinger equation will be abandoned? - Because this always happen. We are dissatisfied with some properties of Schrödinger Eq. Scientist are ready to change. Believers are not.

Quantum computing

- Relevant unitary transformation of state qubits. Reversible transitions.
- We search some Hamiltonian to implement desired transition $\Psi(0) \rightarrow QC(U) \rightarrow \Psi(1)$
- Solving **reverse** problem. Construct a Hamiltonial for a desired logical operations.
- in QM any observable has a corresponing operator.

Lecture 2.

Plan - Calculations - Superposition and entanglement - parallelism - Copying of qubits - $\sqrt{NOT}$ - quantum Zeno's paradox

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Computations

- States should correspond to definite values of observable (e.g. enegry level, magentisation)
- This allows direct interpretation of superposition parameters. $\psi = c_1|\psi_1\rangle + c_2|\psi_2\rangle$, where $|c_1|^2 + |c_2|^2 = 1$, where $|c_i|^2$ represents probability to observe state $|\psi_i\rangle$.
- with single experiment we cannot restore the the parameters.
- if we change a basis, for a clean state we will get superposition of the other basis.

Superposition

Independent superposition of 2 states: $|\psi_1\rangle = \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle) = \frac{1}{2}(|0\rangle + |1\rangle)(|0\rangle + |1\rangle)$

Can be represented as a multiplication.

But there are other examples: $|\psi_2\rangle = \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle)$

They cannot be represented as a product!

$|\psi_3\rangle = \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle)$

If state cannot be represented as a product of independent state, these states are called entangled.

Opinion: we should think about something like a "memory" of a quantum system. Qubits could not be in entangled without interacting in the past.

Quantum parallelism

- ability to evaluate a function for all "tapes" of Turing machine. $\frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$ apply some operation $y + f(x) \mod 2$ For many states at one time: $\Psi = \frac{1}{\sqrt{2}}(|0, f(0)\rangle + |1, f(1)\rangle) \to ... \to \frac{1}{\sqrt{2^N}}\Sigma|x\rangle f(x)$

CNOT

In classical computer CNOT makes a copy of a state. For a superposition in created an entangled state!

$\Psi_1 = a|0\rangle + b|1\rangle \to ... \to a|00\rangle + b|11\rangle$

Square root of NOT

$$\sqrt{NOT} = \begin{pmatrix} (1+i)/2 & (1-i)/2 \\ (1-i)/2 & (1+i)/2 \end{pmatrix}$$

$\sqrt{NOT}\sqrt{NOT} = NOT$

Is actially existing operation, which corresponds to an exact action, and the result can be obtained through another calculation basis.

Teleportation

To teleport a qubit STATE: 1. Alice creates a pair of entangled qubits B, C. 2. Entangles with A. 3. Measures the state of one of BC 4. Calls Bob and informs about the state. 5. Bob now knows what to do to proper transformation.

!! Spin depends to creation of molecules. Trees while converting light into molecules create $\frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$ entangled states in a wild.

Quantum Zeno's paradox

To make sure everything is ok, we should control and measure (error correction). But is we measure to often, system slows down. If we control too much, QC will stop!

Lecture 3. Postulates of Quantum Mechanics

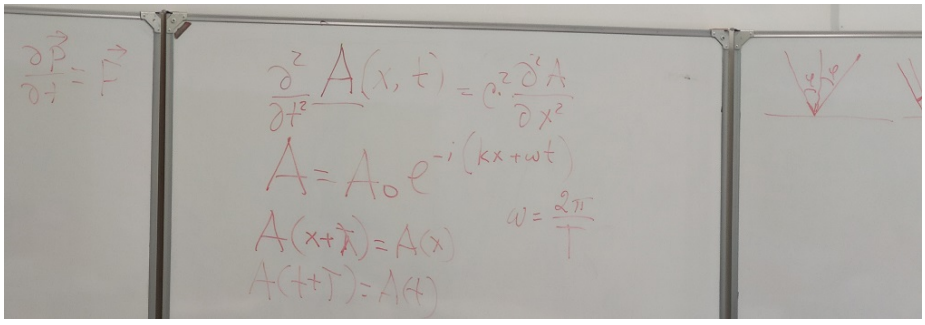
Ideas (recap): - information does not exist without physical implementation. - In QC we should use completely different logic. - In quantum computing $\sqrt{NOT}$ can be measured in experiment!

Agenda: - basics of Quantum Mechanics.

Description on objects in Classic Mechanics is just coordinated and impulse. Far from being full.

Classical mechanics description

- there is radius vector r and momentum p
- Solutions can predict fully the state.



What is light?

Waves or particles? Both. Why light is a particle? Double-slit experiment.

Today we can observe a very small part of light. If we do this with a singe particle, then one quanta of light creates a signal in a single location. No interference effect. Thus it behaves as a particle.

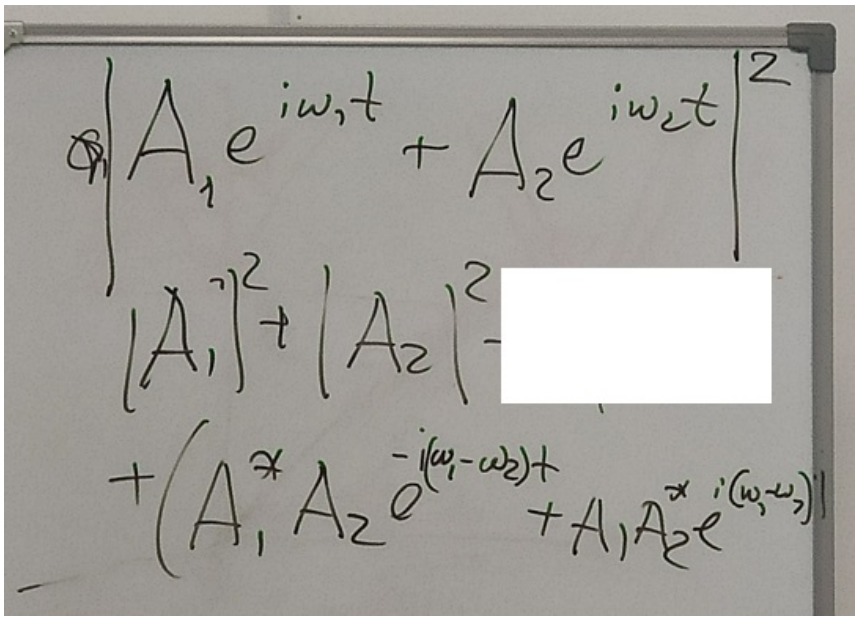
De Broglie - particles should show properties of waves and mass should be connected to wave freq. See this page. $\lambda = h/p = h/(mv) = 10^{-31}cm$ if $m = 1g$ then $v = 10^5 cm/s$, but if $m = 100kg$, then $\lambda = 10^{-36}cm$, that is why we don't feel ourselves as waves :)

Schrodinger classic work "Quantisierung als Eigenwertproblem"

Postulates of QM. Postulate 1

State of a particle is given by a wave function $\psi(r, t)$ (not position and impulse!). Function can be called a "vector of space" (not a typical vector we used to) if it follows some properties. - Wave Function is not observable. Misterious. But any observable should be quadratic functional of WF. - $|\psi(r)|^2$ – is a probabilit to find a particle in r in a single experiment. !! r is radius-vector and t is time. But sometimes there can be time/coordinate-independents functions, thus we sometimes omit.

- Principle of superposition: $\psi = c_1\psi_1 + c_2\psi_2$. This property is known for the waves in classical mechanics, but now we define it for any particle too! In classic superposition in we can observe "something average", but in quantum case we will observe one of 2 states only.



- Wave functions can be defined not only in r (coordinate) space. This can be some other, e.g. p (momentum) of spin.
- 2 non-interacting particles can be separated to a multiplication: $\psi(1,2) = \psi(1)\psi'(2)$ $|\psi(1,2)\rangle = \frac{1}{2}(|0_1 0_2\rangle + |0_1 1_2\rangle + |1_1 0_2\rangle + |1_1 1_2\rangle) = \frac{1}{2}(|0_1\rangle + |1_1\rangle)(|0_2\rangle + |1_2\rangle)$ Subscripts mean that qubits live not in the **same** space, but in **similar** spaces. Also, measurement will give one of allowed states! E.g. if observable is energy, observable will be one of $E_1 + E_1, E_1 + E_2, E_2 + E_1, E_2 + E_2$.

Q: How we distinguish $E_1 + E_2, E_2 + E_1$? A: We observe a superposition $\frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$. Thus we need to run additional experients to collapse. But if we choose different energy levels for different qubits, then we can determine.

- For an interacting particles you cannot represent common wave function as a product. Interaction can be at moment of observation, or any moment at the past. E.g. $|\psi(t)\rangle = c_1(t)|10\rangle - c_2(t)|01\rangle$. This state is called Einstein–Podolsky–Rosen pair.

Alternative description of a state Wave function is determned for for very close systems. We cannot consider models in thermodinamic equilibrium

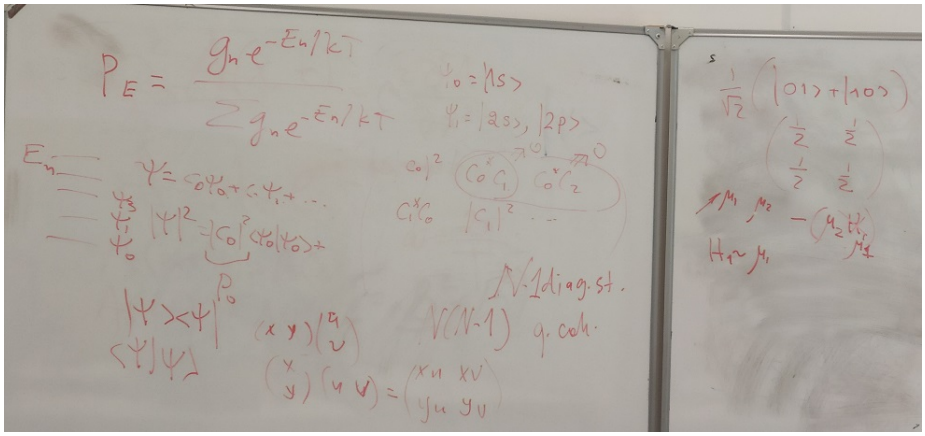
Density matrix In a superposition $c_0\psi_0 + c_1\psi_1 + ...$ coefficient correspond to "contribution". If we build a matrix $2^N \times 2^N$ of coefficients $c_0^* c_0$, $c_0^* c_1$, ..., $c_1^* c_0$,

.... Where N is the number of qubits.

$$\begin{pmatrix} c_0^* c_0 & c_0 c_1 & ... & c_0^* c_{2^N} \\ c_1^* c_0 & c_1^* c_1 & ... & c_1^* c_{2^N} \\ ... & ... & ... & ... \\ c_{2^N}^* c_0 & c_{2^N}^* c_1 & ... & c_{2^N}^* c_{2^N} \end{pmatrix}$$

- diagonal elements $c_i^* c_i = |c_i|^2$ are probabilities to observe in some state.
- non-diagonal characterize quantum coherence. If coherence go goes to 0 quantum computers just becomes classic.
- If we collapse wave function the matrix will be all zeros except single 1 on a diagonal.

Opinion: we can include different time-dependent relaxation (e.g. interaction with thermostat).



Postulate 2

Observables of Classic Mechanics are represented by Hermitian operators $H^* = H$

Example: Coordinate and it's operator $x \to x$

Impulse and it's operator in classical form $p_x = -i\hbar \frac{\partial}{\partial x}$

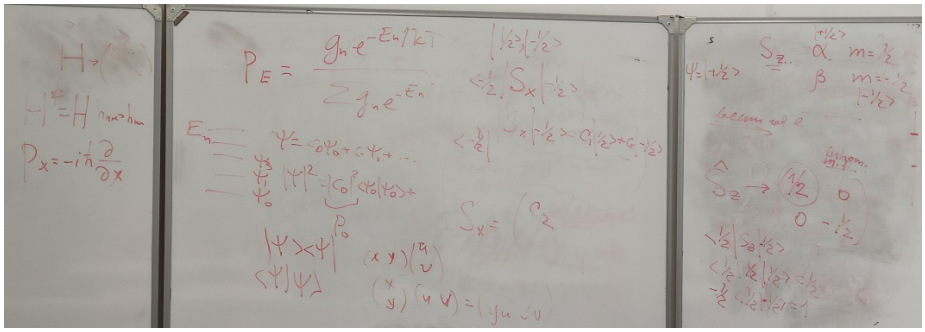
Energy operator (Hamiltonian) – here we consider only kinetic energy:

$E_k = \frac{1}{2} m v^2 = \frac{1}{2} p v = \frac{p^2}{2m}$

$H = \frac{p_x^2}{2m} \to -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2}$

Properties of: - there are actions which does nothing with a state of the system (see eigenvectors and eigenvalues). Eigenfunction. $\hat{H}\psi_n = E_n \psi_n$. - Eigenvalues of Hermitian operators are all real numbers. Otherwise observables would

become complex, which is physically strange. - Not required, but usually we use eigenfunctions to present a spin.



Pauli matrices

Electron spins (these are state, not numbers!) $\alpha = |+\frac{1}{2}\rangle$ $\beta = |-\frac{1}{2}\rangle$

S_x – observable (spin projection to x), which should be from $\{-\frac{1}{2}, \frac{1}{2}\}$. Thus, applying operator of a spin:

$$S_x|-\tfrac{1}{2}\rangle = S_x \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

See quadratic form

$$\langle -\tfrac{1}{2}|S_x|-\tfrac{1}{2}\rangle = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Square root of NOT

If we apply not, then we just transfer from one state to another. Square root creates a superposition.

$$\sqrt{NOT} = \begin{pmatrix} (1+i)/2 & (1-i)/2 \\ (1-i)/2 & (1+i)/2 \end{pmatrix} = \frac{1}{2(1-i)} \begin{pmatrix} (1+i)(1-i) & (1-i)(1-i) \\ (1-i)(1-i) & (1+i)(1-i) \end{pmatrix}$$

$$= \frac{1}{2(1-i)} \begin{pmatrix} 1+i^2 & 1+i^2-2i \\ 1+i^2-2i & 1+i^2 \end{pmatrix} = \frac{1}{2(1-i)} \begin{pmatrix} 0 & -2i \\ -2i & 0 \end{pmatrix} = -\frac{i}{1-i} S_x$$

From eigenstate of one observable S_z we go to eigenstate of another observable S_x.

Lecture 3.

Start from refresher of Postulate 1.

$\psi = a\psi_1 + b\psi_2$ can be expressed in matrix form

$$\rho = \begin{pmatrix} |a|^2 & a^*b \\ ab^* & |b|^2 \end{pmatrix}$$

$\rho_{kn} = \langle\psi_k|\rho|\psi_n\rangle, n = 1, 2$

Any matrix elements of density matrix can be observed, they are all observables.

Any observable is has an operator, this operator has to be Hermitian (H transposed and complex conjugate = H)

Eigenvalues can be measured in experiment.

Postulate 2 Observables of classic mechanics are Hermitian operators

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Posutulate 3.

Every observable in quantum mechanics is represented by an operator which is used to obtain physical information about the observable from the state function. For an observable that is represented in classical physics by a function $Q(x, p)$, the corresponding operator is $Q(\hat{x}, \hat{p})$

$\hat{H}\psi(r, t) = i\hbar\frac{\partial\psi}{\partial t}$

In isolated system H does not change in time.

$\psi(t) = e^{-iHt/\hbar}\psi(0)$

H - operator of energy, kinetic (of impulse p_x) + potential (of position in a field x)

$E = \frac{p_x^2}{2m} + V(x) = -\frac{\hbar^2\partial^2}{2m\partial x^2} + V(x)$

If we are in one of eigenstate, with probability 1 we will measure eigen-energy E_n.

$H\psi_n = E_n\psi_n$

In superposition we will obtain E_n with probability $p_n = |\langle\psi|\psi_n\rangle|^2$

$\langle\psi|\psi_n\rangle = \int dv\psi^*\psi_n$

!!$\langle a|b\rangle$ – bra-ket notation means the scalar product.

This can be understood as a projection of one vector a (state) onto another b (e.g. base vector).

Creation of pure state

pure quantum states correspond to vectors in a Hilbert space, while each observable quantity (such as the energy or momentum of a particle) is associated with a mathematical operator.

We believe we can run an experiment without dsiturbing the system (in classic). In Quantum – differerent.

Suppose we have a partice in superposition. $\psi = \sum_n c_n \psi_n$

Suppose ψ_n are eigenvectors of Hamiltonian H.

When we measure energy of thea particle, we will obtain only ONE, k-th of the eigenvalues of H $E = E_k$, and wave function will **collapse** to ψ_n.

Desription of state ensemble

Suppose qubits never interacted.

$\psi_1 = a_1|0_1\rangle + b_1|1_1\rangle$ – particle 1 $\psi_2 = a_2|0_2\rangle + b_2|1_2\rangle$ – particle 2

Together they are

$\psi = \psi_1\psi_2 = a_1a_2|0_10_2\rangle + a_1b_2|0_11_2\rangle + b_1a_2|1_10_2\rangle + b_1b_2|0_11_2\rangle$

$CNOT\psi = a_1a_2|0_10_2\rangle + a_1b_2|0_11_2\rangle + b_1a_2|1_11_2\rangle + b_1b_2|0_10_2\rangle$

Now we cannot represent this as a product of $\psi_1'\psi_2'$.

Now we want to learn in which state is the first qubit?

Solution can be done using density matrix. $|1\rangle = |00\rangle, |2\rangle = |01\rangle, |3\rangle = |10\rangle, |4\rangle = |11\rangle$

$\psi = a_1|1\rangle + a_2|2\rangle + a_3|3\rangle + a_4|4\rangle$

in matrix form: $\rho = \begin{pmatrix} |a_1|^2 & a_1^*a_2 & a_1^*a_3 & a_1^*a_4 \\ a_2^*a_1 & |a_2|^2 & a_2^*a_3 & a_2^*a_4 \\ a_3^*a_1 & a_3^*a_2 & |a_3|^2 & a_3^*a_4 \\ a_4^*a_1 & a_4^*a_2 & a_4^*a_3 & |a_4|^2 \end{pmatrix}$

Summing the 2 terms give the probability irrespective to the other qubit. $\sigma(1) = Tr_2\rho(1,2)$ and $\sigma(2) = Tr_1\rho(1,2)$

$\sigma_{00} = \rho_{11} + \rho_{22}$

Non-diagonal elements: $\sigma_{01} = \rho_{13} + \rho_{24}$

Transition from AB system (ρ) to A (σ) we **loose information about quantum coherence** (0-, and 2-quantum coherence).

ρ express transition. Term ρ_{14} two-quantum coherence: both qubits become excited. Term ρ_{23} zero-quantum coherence: one looses energy, another obtains.

In the unusual world strange properties are declared. But then these properties are supported. The paradigm shift. New generations challenge it, but the idea survive. ref

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Two-state systems

Circularly polarized photons, topological isolators, superconductive units can serve as qubits.

Electron spins $\pm\frac{1}{2}$ as qubits in atoms with one unpaired electron: $H, Li, Na, K, ...$ Also organic free radicals CH_3-, CH_3-CH_2. Also nuclear spin $I = \pm\frac{1}{2}$

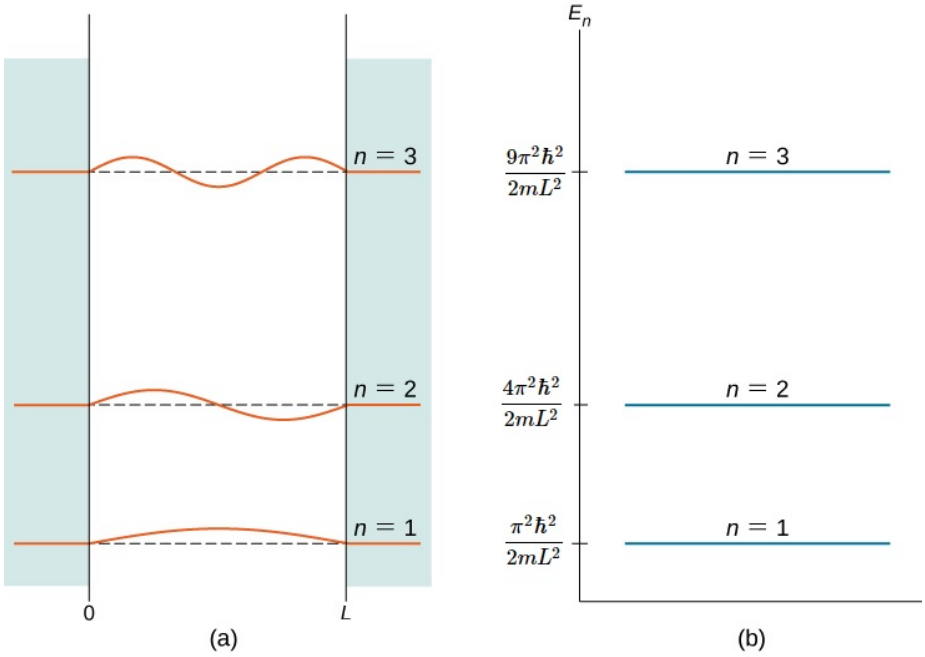
In Russia photon is considered the most promising candidates.

In principle, hidrogen has two 2-level systems: electon and nuclear spin.

Electron in 2 potential boxes

$E_n = \frac{\pi^2\hbar^2}{2ma}n^2$

$\psi_n = \sqrt{\frac{2}{a}}\sin(\frac{\pi b}{a}x)$



Along with mass and charge electron has inherent angular moment, called spin.

Magnetic moment $\mu = -2\mu_B S$ See Bohr magneton.

$H = -\mu B$ – scalar product of moment in external magnetic field.

Suppose $OZ||B$, Then $B = (0, 0, B)^T$

Thus, $H = -2\mu_B BS$

From experiments we know S_z has 2 values $m = \pm\frac{1}{2}$

Q: why half? A: we have 2 values. In classic mechanics spin appears as an additive to normal angular moment (from motion in space). BUT normal orbital magnetic moment ... $1s$ in hidrogen has 0 moment. angular momentum $\overrightarrow{L} = \overrightarrow{r} \times \overrightarrow{p}$. There are $2L + 1$ projections: $-L, -L + 1, ...0, ..., L$. There is also a spin with 2 projections. It has the same properties as momentum. We want them to be $-s, -s + 1, .. + s$. We want the difference to be 1, and there is no 0.

All 2-level systems create isomorphic groups. Any discoveries for one 2-level system can be used for the others. We can "define" a "spin" for any other system, thus we can proceed with spin as general approach.

Let's suggest we have a system with 2 projections.

$S_z|m\rangle = m|m\rangle$

$|-1/2\rangle = |0\rangle, |+1/2\rangle = |1\rangle$

$S_z = \begin{pmatrix} -1/2 & 0 \\ 0 & 1/2 \end{pmatrix}$ $S_x = \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix}$, $B = (B, 0, 0)^T$

This is an evolition operator:

$\psi(t) = e^{-iHt/\hbar}\psi(0)$.

If we consider $H = \hbar\omega S_z$, so we need to find **evolution operator**. It can be reduces to a very simple formula:

$L = e^{-i\omega t S_z} = ... = cos(\omega t/2)U - i2sin(\omega t/2)S_z$

!! $U = I$ – identity operator. !! $e^{ix} = 1 - ix - \frac{x^2}{2!} + ...$

$e^{-i\omega t S_z}(a|0\rangle + b|1\rangle) = ae^{i\omega t/2}|0\rangle + be^{-i\omega t/2}|1\rangle$

Evolution operator acts on $|0\rangle$. This eigenfunction of Hamiltonian. Eigenvalue is $-1/2$, thus we have first term. The same for the second term. Probability to observe doesn't change (absolute values of exponent are 1), but we have phase change, coherence change, non-diagonal elements change.

When we apply magnetic field along X-axis.

$e^{-i\omega t S_x} = cos(\omega t/2)U - i2sin(\omega t/2)S_x$ $e^{-i\omega t S_x}|0\rangle = \cos(\omega t/2)|0\rangle - i\sin(\omega t/2)|1\rangle$

$e^{-i\omega t S_x}|1\rangle = \cos(\omega t/2)|1\rangle - i\sin(\omega t/2)|0\rangle$

When $\omega t = \pi$, then $e^{-i\pi S_x}$ gives the operator NOT.

When $\omega t = \pi/2$, then $e^{-i(\pi/2)S_x}$ gives the operator $\sqrt{NOT}$.

Just applying constant magnetic field during definite time, we can convert 0 to 1 and vice versa.

Math description of 2-state system

We have to know, which property we use to differentiate the states. 1. We choose observable to measure. E.g. $H\psi_n = E_n\psi$ 2. The set of eignestates $\{\psi_n\}$ we chose basis states $\langle\psi_n|\psi_k\rangle = \delta_{nk}$ – delta-function $n == k \to 1$ 3. $\psi = \sum c_n\psi_n$, $c_n = \langle\psi_n|\psi\rangle$ 4. For qubits. We have 3 independent parameters for a qubit. - either $c_1 = |c_1|e^{-i\phi}, c_2 = |c_2|e^{-i\theta}, |c_1|^2 + |c_2|^2 = 1$ - or density matrix (4 elements, but with constraints): $\sigma_{21} = \sigma_{12}^*$ and $\sigma_{11} + \sigma_{22} = 1$

Thus we can express any 2-state system with 3 independent parameters, one of which can be $n = \sigma_{11} - \sigma_{22}$ – population difference of states 1 and 2.

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1. All observables are presented as 2x2 Hermitians
2. Any Unitary operator Q can be represented in a unique way $Q = q_0 I + q_x\sigma_x + q_y\sigma_y + q_z\sigma_z$ in a basis of Identity matrix + Pauli matrices.
3. When spin s=1/2 is chosen as qubit, then $\sigma_z = 2S_z$ $H = 2\mu_B(SB)$. Support $OZ||B$, then $B = (0, 0, B)^T$, $H = 2\mu_B B S_z$

We can use S_i instead of σ_i as they coinside with coefficient 1/2.

Lecture 4. Using e^- spins to implement logic operations

1. Any 2-level quantum system can solve as a qubit, physically they might be different, evolutions are done diferently, but mathematically they are similar (equivalent isomorphic).
2. We have naturally 2-level systems – particles with spin $\frac{1}{2}$.

To build a quantum computer this is enough to have NOT and $CNOT$.

Evolution of a system can be interpreted as realization of certain logic operations.

$\psi(t) = c_o(t)|0\rangle + c_1(t)|1\rangle$

$\psi(t) = e^{iHt/\hbar}\psi(0)$. – at one moment t_0 we can observe (during evolution) $|c_1(t_0)|^2 = 1$, which can be considered as $NOT|0\rangle$. This creates opportunities as well as problems.

There are no "ready" qubits in nature. There are no constant Hamiltonians. We search for dynamic implementations.

- required 1-qubit operations: NOT, Hadamard H, Phase shift
- required 2-qubit: $CNOT$

If relaxation is fast, we cannot state "we implemented NOT". To serve as qubit values, relaxation shouls be 10^{3-4} times longer, then the process of operation itself.

One qubit logic operators.

σ_x is exactly NOT implementation.

$NOT = \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

Problem is to find a corresponding Hamiltonian.

Representing Hamiltonian in matrix basis

Matrix can be represented in a basis of I and Pauli matrices.

$H = c_0 I + \sum_{i=x,y,z} c_i \sigma_i = c_0 I + 2\sum_{i=x,y,z} c_i S_i$

$H = c_o I + 2\epsilon(nS)$

$nS = \frac{1}{\epsilon}\sum_{i=x,y,z} c_i S_i$

$\epsilon = \sqrt{(|c_x|^2 + |c_y|^2 + |c_z|^2)}$

Simplification

Exponential operator can be reduces to a very nice form due to $e^{-i\phi} = \cos\phi - i\sin\phi$ followed by:

$e^{-i\phi S_z} = cos(\phi/2) * I - i2S_z \sin(\phi/2)$

Thus we can rewrite:

$L = e^{-iHt/\hbar} = e^{2i\epsilon t/\hbar} = cos(\frac{\epsilon t}{2}) * I - 2i \sin(\frac{\epsilon t}{2}) * (nS)$

What if we apply magnetic field with one projection

If we only have S_x (magnetic field $\perp$ to quantization axis Z):

$B = \{B, 0, 0\}$

$H = 2\mu_B B S_x$

Rabi frequency

$\omega = 2\frac{\mu_B B}{\hbar}$

$H = \hbar\omega S_x$

How to catch the moment to build NOT gate

if $\epsilon t = \pi$, then **for this particular time**:

$L_\pi = e^{-i\pi S_x} = -2S_x = -NOT$

$L_\pi(a|0\rangle + b|1\rangle) = (-1)(b|0\rangle + a|1\rangle)$

in magnetic field ~1 Guass for electron this time is ~$10ns$ with relaxation of microseconds. Inside fullerene relaxation time is up to milliseconds, for the nuclear spin relaxation time is hours, but operator may take milliseconds (can work for demo only).

Square root of NOT

If time is twice as small $\epsilon t = \pi/2$:

$L_{\pi/2} = -i2Sx = (1+i)\sqrt{(NOT)}$ – ideal operator to produce random numbers, since $\sqrt{NOT}|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)$

Hadamard

$B = \{B, 0, B\}$ – external field has both components. Vector precese around middle position.

$H = \omega(S_x + S_z)$

$L_{\omega t} = \sqrt{\pi/2} = e^{-i\omega t(S_x+S_z)} = = \frac{i}{\sqrt{2}} \begin{pmatrix} sin(\frac{\omega t}{\sqrt{2}}) & -sin(\frac{\omega t}{\sqrt{2}}) \\ -sin(\frac{\omega t}{\sqrt{2}}) & (-i)cos(\frac{\omega t}{\sqrt{2}}) - sin(\frac{\omega t}{\sqrt{2}}) \end{pmatrix}$

$L = -iH = -i * \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ where H stands for Hadamard operator.

CNOT

$$CNOT = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$CNOT(c_0|00\rangle + c_1|01\rangle + c_2|10\rangle + c_3|11\rangle) = = c_0|00\rangle + c_1|01\rangle + c_3|10\rangle + c_2|11\rangle$

No independent implementation

It is evident, we cannot implement $CNOT$ without considering interation of two qubits.

if

$H = H_1 + H_2$

then $L = e^{-iHt/\hbar} = L_1 * L_2$ $\psi = \psi_1\psi_2$ since $[H_1, H_2]$ (they commutate)

Impementation (result)

We will look in a form of stroboscopic form (time dependent, as we did for NOT)

$H = \omega s S_z + \omega f F_z + K S_z F_z$

by [J. A. Jones, M. Mosca, 1998] for nuclear spins

and

$H = \omega a S_z + \omega b F_z + A S_z F_z + B(S_x F_x + S_y F_y)$

by [Volkov, Salikhov, 2011] for electron spins

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whiteboard

Let $\hbar = 1$ $H = \omega_s S_z + \omega_z F_z + j S_z F_z$

$L = e^{-iHt}$

$F_z, S_z|0\rangle = \frac{1}{2}|0\rangle$ $F_z, S_z|00\rangle = \frac{1}{4}|00\rangle$

We should force spins with this Hamiltonian interfere using pulses of magnetic field.

Formula explanation: We insert $L^{-1}L = I$ - 2 pulses rotating in opposite direction. Thus we obtain in the formula $LHL^{-1} = \hat{H}$ twice in a row ($\hat{H}^2$), which can be considered as free evolution under new Hamiltonian (we "edited" Hamiltonian).

Q&A: electrons should not be closer then $1nm$, but not too far (magnetic fields should overlap to some extent). E.g., in H_2 (hydrogen) 2 electrons with opposite spins are too close, that they are magnetically neutral and impossible to manipulate with magnetic impulses.

Simplest basis to represent matrix is 4x4 matrix with only 1 non-zero elements. There will be 16 of them.

Notation for solution:

$I = U_0$

$$I_x = \begin{pmatrix} 0 & 1/2 & 0 & 0 \\ 1/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/2 \\ 0 & 0 & 1/2 & 0 \end{pmatrix}$$

$$S_z = \begin{pmatrix} 1/2 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \\ 0 & 0 & -1/2 & 0 \\ 0 & 0 & 0 & -1/2 \end{pmatrix}$$

Thus we can construct: $E_{1+} = I/2 + S_z = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

$$E_{1-} = I/2 - S_z = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

They are interesting. Squares are equal to itself. And they are orthogonal (we use it later).

If we are in the first subspace E_{1+} – nothing happens. Otherwise in E_{1-} subspace $NOT = \alpha I_x$ flip happens: $(2I_xE_{1-})$

$CNOT = E_{1+} + 2I_xE_{1-} =$

$= E_{1+}E_{1+} + 2I_xE_{1-}E_{1-} =$

added 0

$= E_{1+}E_{1+} + 2I_xE_{1-}E_{1-} + iE_{1+}E_{1-} =$

added one more 0

$= E_{1+}E_{1+} + (-i^2)2I_xE_{1-}E_{1-} + iE_{1+}E_{1-} - 2iI_xE_{1+}E_{1-} =$

$= E_{1+}(E_{1+} + iE_{1-}) - i2I_xE_{1-}(E_{1+} + iE_{1-}) = = (E_{1+} - i2I_xE_{1-})(E_{1+} + iE_{1-})$ (+)

knowing that:

$e^{GE_{1+}} = I + GE_{1+} + \frac{1}{2!}G^2E_{1+} + \frac{1}{3!}G^3E_{1+}$

add 0:

$e^{GE_{1+}} = I + (E_{1+} - E_{1-}) + GE_{1+} + \frac{1}{2!}G^2E_{1+} + \frac{1}{3!}G^3E_{1+} + ... =$

$= E_{1+}(1 + G + \frac{G^2}{2} + ...) = E_{1+}e^G$

and

$i = e^{i\pi/2}$

Then (+) becomes

$= (e^{-2I_xE_{1-}})(e^{i\frac{\pi}{2}E_{1-}})$

This is a product of 2 exponents. This is a form of solution of Schrödinger equation.

How to make our Hamiltonial $Le^{-\alpha S_zI_z}L^{-1}$ to function as I_xS_z?

we can rotate a spin 90^0 around Y.

$e^{i\pi/2y}e^{-\alpha S_zI_z}e^{-i\pi/2y}$.

Note: System will work as quantum computer, if the fidelity is more then 80%.

Lecture 5. QFT, quantum teleportation, and Quantum Zeno's paradox

Notes: - systems evolve without control, as they are implemented as physical objects. - superposition is rather trivial. But observables interesting things. We will not see superposed values, but only 2 allowed states. - But if we do multiple experiments – we (using Ergodic hypothesis) can observe how values converge to. - when we speak about $|0\rangle$ and $|1\rangle$ – they are not abstract. They are eigenstates

of some observable. - when we use different observables, they have different bases. We usially take spin projection, but we can take Zeeman effect $H = -\vec{\mu}\vec{B}$. - Any operator space and be represented using set of 2×2 matrices - Identity and Pauli matrices. - in fluctuating systems, if vector precesses (rotates) around axis, we can use Rabi frequency to understand, **when** e.g. NOT occures. Thus, we manipulate with short pulses. - Errors occur when we are not exact in pulse lengthes. Longer the sequence of pulses, bigger is the accumulated error. - In classic systems we assume measurements does not distub the system. Error correction – if wrong, then flip. Measurement in Q systems is different. Thus, error correction is more complicated. - Any program can be reduced to a set 1- or 2-qubit operations. - Quantum probabilisting vs classic: each time we perform operation, classic computer performs an operation and "reduces" the state. In quantum systems until reduction we evolve by all possible trajectories.

Quantum Fourier Transform

Some vector in the basis: $Y_k = \frac{1}{\sqrt{2^N}} \sum_m X_m e^{\frac{i2\pi mk}{2^n}}$ m - means time dependence k - which state we have.

Thus, Fourier transform – is an operator.

$UQFT = \frac{1}{\sqrt{2^N}} \sum_{m=0}^{n-1} \sum_{k=0}^{n-1} e^{\frac{i2\pi mk}{2^n}} |k\rangle\langle m|$

$|m\rangle = \frac{1}{\sqrt{2^N}} \sum_m e^{\frac{i2\pi mk}{2^n}} |k\rangle$

for n = 1 this is just Hadamard gate :)

for n=2 we have the following:

$|m\rangle = \frac{1}{2}(|0\rangle + e^{i\pi m}|1\rangle)(|0\rangle + e^{i\pi m/2}|1\rangle)$

SHOW THIS IS TRUE – possible exam question

$(|0\rangle + e^{i\phi}|1\rangle)$ is already familiar - Hadamard + phase shift.

Quantum teleportation

Suggested in 1993. Teleportation: take and object, describe it fully, send information and then restore.

QT – the state of one qubit we reproduce in another state. The state of original qubit is lost.

Alice wants to send a qubit state to Bob. Alice has a qubit with **unknown** state (if known - this means she measured it!). Alice also prepares a pair of highly entangled state (Bell state) $|\psi_{BC}\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$ – singlet state.

$|\psi_{BC}\rangle \neq |\psi_B\rangle|\psi_C\rangle$

After this pair is shared between Alice and Bob.

Let us consider aggregate state of the system.

$|\psi_{ABC}\rangle = \frac{1}{2}(a|0\rangle + b|1\rangle)(|01\rangle - |10\rangle) = \frac{1}{2}(a|001\rangle - a|010\rangle + b|101\rangle - b|110\rangle)$.

Total wave function in the basis of Bell states:

$|\psi_{ABC}\rangle = |\psi_A\rangle|\psi_{BC}\rangle = c_1|\psi_A s\rangle + c_2|\psi_A T_1\rangle + c_3|\psi_A T_{-1}\rangle + c_4|\psi_A T_0\rangle$.

Alice wants to know in which states are her qubits? She measures in a basis of Bell states. T stands for Triplet states.

For example, multiply by $(|0_A 1_B\rangle - |1_A 0_B\rangle)$. Why? We want to measure (project) AB onto a single Bell state (for clarification read below). This is not the same as two measurements! And we get:

$\langle\psi_{AB}|\psi_{ABC}\rangle = -\frac{1}{\sqrt{2}}(a|0\rangle + b|1\rangle)$

$-a|0_C\rangle - b|1_C\rangle = e^{i\pi}(a|0_C\rangle + b|1_C\rangle)$.

Wow! Qubit C is in a superposition with the coefficient of A. If we multiply with other Bell states, we get either swapped a,b or sign changed. This can be easily fixes with X and Z gates if Bob knows exact observed state of Alice's pair.

Alice calls Bob and tells information is in a definite state (2 bits).

!! Sound like we are missing A and B qubits entanglement? **For details look here**

This thing is tricky. The protocol doesn't specify **how to measure a pair of qubits in the basis of Bell states**. In other implementation to avoid this operation, A and B are entangled, thus we can be SURE that measured A and B states does not depend on this unknows operation, we can do the same with regular measurements.

Quantum Zeno's paradox

Consequence of quantum objects participating in computations. Measurement of qubits leads to decoherence. Decoherence affects the qubit dynamics. Frequent control (e.g. in error correction) reduces the rate of quantum computations.

$|0\rangle$. Start with state. Apply magnetic field. Magnetic moment will rotate (from the pole). Now $a|0\rangle + b|1\rangle$. The whole process of continued transition with $prob(t) = |b|^2$ form an oscillating function of probability of NOT. If at some time we check that there is "ok", we immediatly transfer the system to one of the states. Thus we "steal" some time from computations. The more frequently we check, the slower is the operator.

Lecture 6. Error correction

Errors come form interation with environment. How environment can affect, which types of error can occur, how to correct such errors?

Two major sources - dissipation and decoherence.

Formally thiere should be interation, this disturbance can be implemented as an unitary operation, which can be written in Pauli basis $V = c_0 I + c_x \sigma_x + c_y \sigma_y + c_z \sigma_z$.

σ_x has the shape of $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ in the basis of σ_z. This acts as NOT, thus disturbance of this type of environment interation can bring an error ($|0\rangle \to |1\rangle$).

Also, if splitting of 2 levels fluctuate with time, this with time can bring coherence to 0.

How σ_z changes the phase of coherence?

$\sigma_z \psi = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ -b \end{pmatrix}$

$\psi = a|0\rangle + b|1\rangle \to^{\sigma_z} \to a|0\rangle + e^{i\pi} b|1\rangle$.

Operations which can be described as σ_z work as phase change.

How σ_y changes?

$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

$\sigma_y(a|0\rangle + b|1\rangle) = -ib|0\rangle + ia|1\rangle = -i(b|0\rangle - a|1\rangle) = -i(b|0\rangle + e^{i\pi} a|1\rangle)$

It gives the flip of amplitudes **and** phase shift. This is natural, as $\sigma_x \sigma_z = i \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = i * \sigma_y$

Any relaxation can be produces with flip (σ_x) and phase change (σ_z).

Spontaneous change

These transition are connected with change of energy (as we define qubit levels as 2 energetic levels). There can be spontaneous transition with probability $\sim\omega^3$. In case of spins, $\omega \approx 10^{10}$. $p = |\frac{V_{int}}{\hbar}|^2 * \tau_c$, there $\tau_c = \frac{1}{t}$, where t is characteristic time of change.

If process of decoherence is faster than computing, we can only have probabilistic classic computer.

Why density matrix is better for descrition?

$\rho = |\psi\rangle\langle\psi| = \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix} = \begin{pmatrix} |a|^2 & ab^* \\ a^*b & |b|^2 \end{pmatrix}$

Density matrix elements are observable and can be measured. It allows to describe irreversible relaxation processes. ρ_{12} can dissappear, but ρ_{11}, ρ_{22} stay unchanged, and in wave function form we never nothice that.

$ab^* = \rho_{12}$ $\langle\psi|S_x|\psi\rangle = Re(ab^*) = Re[\rho_{12}]$ $\langle\psi|S_y|\psi\rangle = Im(ab^*) = Im[\rho_{12}]$

Ref: Mesoscopic systems – systems of the edge of classic (macro) and quantum systems, where one cannot reject quntum effects

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We cannot average wave functions of 2 systems (why? e.g. for multiple experiments with the same process, or ergodic process is average over time). If we multiply a state by $e^{i\pi} = -1$ (this means nothing for a particle, but phase change). Average of original and changed state gives 0, but in fact density matrix doesn't change. Density matrix gives ability to provide description avareged over system ensemble.

$\frac{\partial\rho}{\partial t} = -\frac{i}{\hbar}[H, \rho] + (\frac{\partial\rho}{\partial t})_{relax}$,

where

$(\frac{\partial\rho}{\partial t})_{relax} = -\frac{\rho}{\tau}$

With characherisic time τ we can include relaxation process in equation.

How decoherence can happen

Electron spin is a qubit. $H = -\vec{\mu}B_o = +\hbar\omega_0\hat{S}_z$

$H = \hbar\omega S_z$

$S_z = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Let's apply this Hamiltonian over time.

$\psi_{t=0} = a|0\rangle + b|1\rangle \rightarrow^{H(t)} \rightarrow e^{-i\hat{H}t/\hbar}[a|0\rangle + b|1\rangle] =$

$= ae^{-i\omega_0 t/2}|0\rangle + be^{+i\omega_0 t/2}|1\rangle = e^{-i\omega_0 t/2}(|0\rangle + e^{i\omega_0 t}b|1\rangle)$

What do we see? Phase of coherence changed. Phase changed.

What if $\vec{B}(t)$ is changing in time, preserving direction? It may fluctuate due to electric circuit, producing the field. This leads to fluctuating enery level splitting: $\Delta E = E_0 - E_1$.

Then we get solution of Shr. eq. $e^{-i\int_0^t \omega(t)dt}$, which for constant ω converges to what we already shown. We can also write $\omega = \omega_{average} + x(t)$

Suppose $x(t)$ is normal. Any sum of centered normal processes is also normal. Thus we can express error as.

$< e^{i\int_0^t x(t)dt} >= e^{-<x^2>\frac{t^2}{2}}$

Phenomenologically we can describe decoherence might be described as

$$\rho = \begin{pmatrix} \rho_{11} & \rho_{12}k(t) \\ \rho_{21}k(t) & \rho_{22} \end{pmatrix}$$

where $k(t)$ decreases from 1 to 0 with time. It might have such forms as:

$k(t) = e^{\frac{-t}{T2}}$, $k(t) = e^{-gt^2}$, $k(t) = e^{-c\sqrt{t}}$.

Higher is the order of coherence, usually faster is the decoherence process.

If we identify the error, we can produce correcting operation. E.g. additionally applying σ_x

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To make corrections we should know what happened and what to do.

Instead of qubits we use codewords: $\psi_0 = a|000\rangle + b|111\rangle$. We add 2 more qubits. I want to check if qubit is in a proper state. We convert second and 3rd spins using $CNOT$ operator twice: $CNOT(1,2)$ and $CNOT(1,3)$.

Measurements of additional states can help us to identify types of disturbance, thus identifying reverse procedures.

E.g. some noise produced signle flip ($\sigma_x = NOT$) of one of qubits in $\psi_0 = a|000\rangle + b|111\rangle$ state.

$\psi_1 = \sigma_{1x}\psi = a|100\rangle + b|011\rangle$

$\psi_2 = \sigma_{2x}\psi = a|010\rangle + b|101\rangle$

$\psi_3 = \sigma_{3x}\psi = a|001\rangle + b|110\rangle$

even with 2 flips.

$\psi_3 = \sigma_{1x}\sigma_{2x}\psi = a|110\rangle + b|001\rangle$

We don't investigate state of 1st! Prohibited.

In experiment we can observe (with sigle error!) one of $|00\rangle, |01\rangle, |10\rangle, |11\rangle$.

$|00\rangle, |11\rangle$ say "there was no single-flip error in 2,3 qubits". If we are sure the error occurs, then it happened for the first.

We can also use other observables, e.g. $|+\rangle, |-\rangle$. These are eigenfunctions of S_x with eigenvalues $\pm\frac{1}{2}$

For this we additionally apply H gate to all 3 qubits:

$\psi_0 = a|{+}{+}{+}\rangle + b|{-}{-}{-}\rangle$.

Strategy of error correction.

- inspect and correct a logical qubit
- you encode the state of a logical qubit into some (up to hundreds) of qubits. This form a codeword.
- We should decide from physics which types of disturbances may occur, and how they project into errors (spectral diffusions, magnetic field fluctuations, ...).
- We measure states of all encoding qubits (except the logic one). Suppose we have: $\psi = c_0|00011\rangle + c_1|10011\rangle$. We observe 2 qubits converted. We will apply $\sigma_{4x}\sigma_{5x}$ for fix.

Note: no one is discussing that qubits can affect each other, which is additional source of mistake.

Lecture 7. Quantum cryptography. Quantum key distribution

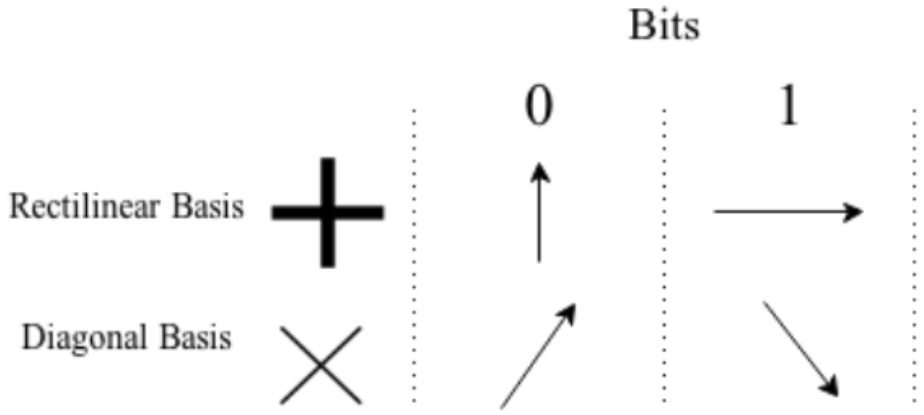
Plan

- secret keys
- why quanutum key distribution
- protocol of quantum key distribution
- Physical implementation of quantum computers.

Alice and Bob exchanges, Eva is eavesdropping.

Simple key - length is the same as the length of the message.

Basis states - rectilinear and diagonal polarizations



$|0_r\rangle, |1_r\rangle, |0_d\rangle, |1_d\rangle$

1. States of photons can be encoded with **either polarization basis randomly**.
2. Polarization of photons is easily controlled/manipulated.

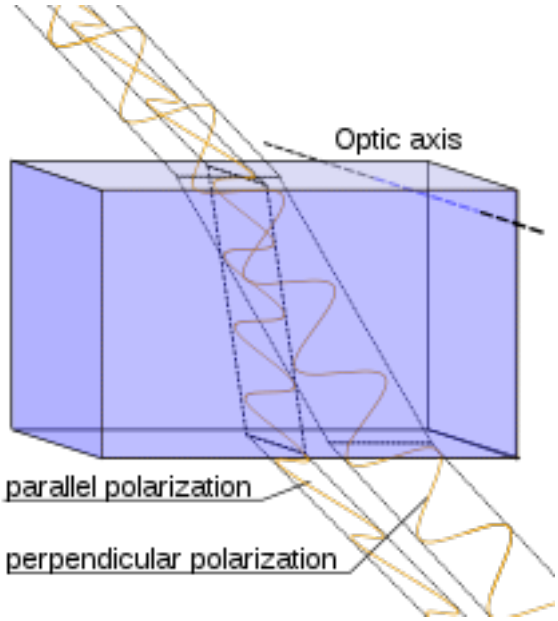
Light polarization

Some medium (e.g. salt $NaCl$, longer in 1 direction) make light to develop differently if it travels in different directions inside the crystal. And the speed of light is different. This is called anisotropy.

What if photon falls with some angle?

Birefringence

Depending on polarization some materials have different refraction angles. This is called birefringence. Video demo.



This leads that for some polarized photons can be detected by a detector, but othes - not.

We can chose an observable, for which vertical polarization is $|1\rangle$ and hoizontal is $|0\rangle$ (rectilinear). But we can also choose diagonal basis ($\frac{\pi}{4}$ is $|0_d\rangle$ and $\frac{3\pi}{4}$ as $|1_d\rangle$). This can be achieved by $Rot(\pi/4)$ gate.

$P_r = |0\rangle\langle 0| - |1\rangle\langle 1| = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. This is σ_z Pauli matrix. Eigenvalues are 1 and -1. Eigenvectors are $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Transition from rectilinear (+) basis to diagonal (x) is given by $\frac{\pi}{4}$ rotation.

$|0\rangle_d = \frac{1}{\sqrt{2}}(|0\rangle_r + |1\rangle_r)$

$|1\rangle_d = \frac{1}{\sqrt{2}}(|0\rangle_r - |1\rangle_r)$

$P_d = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_d = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}_r$

Commutator?

$[P_r, R_d]_r = P_r P - d - P_d P_r \neq 0$

Means we cannot change the order of operators.

We address cryptography with 2 types of polarizers.

Alice will randomly choose the type of polarizers (given some generator). She knows.

Bob chooses the polarizer also. He knows his own basis, but he has no idea about Alice's choise. Also, what is $|0\rangle$ and $|1\rangle$ is Alice's basis.

Protocol

We want to send some 0-1's, and only Alice and Bob knows the sequence.

```
Alice (random basis choice):
+ + x + x x x x + x + + +
1 1 0 1 0 0 0 0 1 1 0 1 1

Bob (random basis choice):
+ x x + + x x x + + x + +
1 1 0 1 1 0 0 0 1 1 1 1 0

Sifted key (where basis match):
1   0 1   0 0 0 1     1 0

Testing (removed from final key):
    ? ?     ? ?       ?

Final key:
1         0     1       0
```

1. Alice sends randomly chosen polarized photons.
2. Bob measures photons in **randomly** chosen basis.
3. A and B shared polarization information (+, x) for the whole set, or just a subset.
4. This is how they choose where they used the same type of polarizer ("sifting"). BUT there might be a mistake, because Eva (eavesdropper) measures and re-issues a photon.
5. For a subset of photons A tells B **polarization orientation**, but doesn't tell what those bit values were.
6. Bob compares his polarization orientations with those of Alice. Then Bobs finds matching cases. It can be an evidence of eavesdropping. Testing qubits are not included in a key.

Physical implementation of quantum computers with photons

Requirements: 1. 2-level system – polarization (**solvable**) 2. Long enough decoherence time (**solvable**). Still there are external things which can change polarization (e.g. nuclear explosion). Thus we use optical fibers. If we are at the edge of good and bad time – we are in the area of mesoscopic systems. 3. Able to measure a state of a qubit, manipulate the state selectively (**solvable**) 4. Implement 1- and 2-qubit gates, e.g. NOT and $CNOT$. (**problem**)

Realization of $CNOT$ is not straightforward. We need interaction, but we know that photons do not mutually interact.

For 2 photons we can for add a beam of **atoms**. E.g. one photon polarizes the atom, and the atom will dictate the behavior of the second.

On density matrix versus wave function (eigenvector)

In classic computers nobody cared about physical systems implementing the gates. For quantum computing this is of extreme importance. Otherwise this means we exclude quantum coherence.

Wave function, full system description, but **not observable**: $|\psi\rangle = a|0\rangle + b|1\rangle$

For ensembles we cannot add/average wave functions, only products. We cannot use our intuitive way.

Density matrix: $\rho = |\psi\rangle\langle\psi| = (a|0\rangle + b|1\rangle)(a^*\langle 0| + b^*\langle 1|)$

$$\rho = \begin{pmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{pmatrix}$$

Density matrix is always given is some basis. **Diagonal numbers are always real numbers**, as they express level populations. Non-diagonal – contribution of different basis states.

in 0-1-basis:

$\rho_{00} = \langle 0|\rho|0\rangle = aa^*$ $\rho_{01} = \langle 0|\rho|1\rangle = ab^*$ $\rho_{10} = \langle 1|\rho|0\rangle = ba^*$ $\rho_{11} = \langle 1|\rho|1\rangle = bb^*$

Example. When 2 atoms in a molecule electron occupies 2 orbital of 2 atoms. In co-valent connection product non-diagonal elements is highly close to the valency.

For density matrix we have equivalent (to Schrödinger eq.) equation:

$\frac{\partial \rho}{\partial t} = -\frac{i}{\hbar}[H, \rho]$.

Advantages and disadvantages of density matrix

1. Can be obtained using wave function.
2. Number of equations increased (wave function - 2 numbers, matrix - 4).
3. Can be averaged in ensembles. This allows to work with **macroscopic** systems.
4. We know, how we can treat a part A of a big system AB. If we are not going to discuss all parameters (e.g. spins only). B might be a thermostat. The recipe – convolution. For B subsystem keep only populations, no coherence (Tr_B). In total: $\rho_A = Tr_B \rho_{AB}$. We loose a lot of coherence values and neglect coupling of A and AB. If B is thermostat, them trace is justified good with thermal equilibrium density matrix of B is reduced to diagonal, diagonal elements are given by Gibbs population of energy level.
5. Density matrix allows to describe the relaxation processes like decoherence of qubits. In this case the evolution of the density matrix is **not** described by unitary transformation. As a result of this non-unitary evolution of qubits, quantum gates loose their reversible evolution in time.

Question for self test.

Suppose you have 2-qubit system. There are 2 computation bases of 2 qubits (different), and here is the state:

$\psi = a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle$.

1. Find $\rho = |\psi\rangle\langle\psi|$
2. And then find ρ_A using $\rho_A = Tr_B \rho_{AB}$.