Multi-agent safe invariant set.

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Background

There are many smart agents $\{\mathrm{A}_1, \mathrm{A}_2, \mathrm{A}_3 \ldots \mathrm{A}_\mathrm{n}\}$ living in a shared space. Each agent is capable of controlling its own behavior by:

$$\dot{x}_i = f_i(x_i, u_i) \tag{1.1}$$

For easier representation, here also define its integration $F_i: (\mathbb{R}^{n_x}, \mathbb{R}^{n_u}, \mathbb{R}) \to \mathbb{R}^{n_x}$:

$$F_i(x, u, t) \triangleq \begin{cases} x + \int_0^t f_i(F_i(x, u, \tau), u) d\tau & t > 0 \\ x & t = 0 \end{cases} \tag{1.2}$$

For each two of them, certain pair of state is dangerous, known as "crush". We use the following Boolean function to denote that:

$$Crush_i^j(x_i, x_j): (\mathbb{R}^{n_x}, \mathbb{R}^{n_x}) \to \{0,1\} \tag{1.3}$$

Agents are not capable of communicating with each other. However, they can observe the state of any agent in the system.

Given such a system, this article is proposing a strategy for those agents to peacefully live in the space, by applying control supervisor based on safe invariant set.

For each agent, its control u_i is limited by supervision set R_i:

$$u_i \in R_i(x_1, x_2, \ldots x_n) \tag{1.4}$$

In the following pages, the word "regulation" will be used to represent such supervision set.

Principle 1. Non-correlated regulation (NCR)

The author thinks that one agent is never capable of considering all the other agents in a correlated manner at real time. The computational power of a single agent is limited, such that it can only regulate its behavior using following form:

$$R_i(x_1, x_2, \ldots x_n) = \cap_{j=0, j \neq i}^{n} R_i^j(x_i, x_j) \tag{2.1}$$

First, for each other agent A_j, A_i works out a two-body regulation R_i^j considering only the states of A_j and A_i. Then A_i chooses its final regulation to be the intersection of all such two body regulations.

Principle 2. Basic agent right.

There can be many different results of R_i^j considering the abundance of the agents. No matter what they are, the intersection of them in eq (2.1) cannot be empty, or otherwise there would be no choice for $\mathrm{A_i}$ to make.

In fact, each other agent may cast certain regulation upon $\mathrm{A_i}$, but they are not clever enough to think in $\mathrm{A_i}$'s perspective. To achieve the non-emptiness of $\mathrm{A_i}$'s final regulation, one solution is to make the contract of "basic agent right":

For each agent $\mathrm{A_i}$, there exist a control known as the "basic agent right", denoted as . $\mathrm{A_i}$ shall be always allowed to take $\mathrm{u_i} \in B_i(x_i)$ as its next control. On the other hand, $\mathrm{A_i}$'s control needs to be regulated such that it won't harm the basic right of any other agents. ($\mathrm{B_i}$ can be estimated by all agents.)

In the later pages, we will show that the peaceful space can be achieved if every NCR agent observes the principle of basic agent right.

Notion of backup safety.

Given the fact that $\mathrm{A_i}$'s control is regulated by certain strategy $\mathrm{B_i}(x_i)$, there would be an obtainable forward reachable set:

$$FR_i(B_i, x_i, t) \triangleq \{x_f \,|\, A_i\ keep\ use\ u_i \in B_i(x),\ and\ can\ reach\ x_f\ at\ time\ t\} \qquad (3.1)$$

Here the strict definition is not given, since there would be a lot of words.
We further define the collision of two forward reachable set:

$$Crush_i^j(x_i, x_j, B_i, B_j) = \exists\ t \geq 0, s.t.$$

$$\exists\ x_i^c \in FR_i(B_i, x_i, t), x_j^c \in FR_j(B_j, x_j, t), Crush_i^j(x_i^c, x_j^c) \qquad (3.2)$$

The straight forward understanding of eq (3.2) is: if $\mathrm{A_i}$ follow strategy of B_i and A_j follow strategy of B_j, whether it is possible for a future crush to happen.

For some two agents $\mathrm{A_i}$, $\mathrm{A_j}$ in the n-agent system, by instinct one necessary condition for "eventually no crush between $\mathrm{A_i}$ and $\mathrm{A_j}$ for all possible trajectories starting from x_i, x_j" is "backup safety":

$$BS_j^i(x_j, x_i) = BS_i^j(x_i, x_j) \triangleq !\,Crush_i^j(x_i, x_j, B_i, B_j) \qquad (3.3)$$

Instinct of necessity: since there are many other agents in the space, it is possible that $\mathrm{A_i}$'s final regulation set get maximumly squeezed by other agents, such that $\forall t > 0$, $\mathrm{A_i}$ has nothing to do but to take the backup choice from B_i. The same goes for $\mathrm{A_j}$. If such situation happens (such situation can always happen no matter how the agents are regulated), eq (3.3) is equivalent to the definition of "eventually no crush between $\mathrm{A_i}$ and $\mathrm{A_j}$, starting from

x_i, x_j". (Elements in B are not allowed to be further regulated according to the description of "basic right")

So the guidance of the design is that eq (3.3) must hold true for all selection of i, j to avoid the risk of collision.

Notion of moral bottom line

To avoid a complicated story, we further limit the dynamic to discrete time axis.

$$x_i(k+1) = F_i(x_i(k), u_i(k), dt) \tag{4.1}$$

As mention before, $\mathrm{A_i}$, $\mathrm{A_j}$ must try to keep eq (5) all the time since from their perspective doing so is necessary for collision prevention.

Suppose $BS_i^j(x_i(k), x_j(k)) = true$ at moment k. To keep such property at $k+1$, A_j must satisfy:

$$R_j^i(x_j, x_i) \subseteq M_j^i(x_j, x_i) \triangleq \{u_j | \forall u_i \in B_i(x_i),\ BS_i^j\left(F_i(x_i, u_i, dt),\ F_j(x_j, u_j, dt)\right)\} \tag{4.2}$$

From the principle 2 basic agent right, A_j must respect the basic right of A_i. It is possible for A_i to choose b_i at k, thus A_j must constraint its action in $M_j(x_j, x_i)$ so as to avoid potential backup safe regression. The constraint of M_j^i, known as A_j's moral bottom line over A_i, must also be enforced to everyone in the space to rule out the risk of collision.

$$BS_i^j(x_i, x_j) \rightarrow M_j^i(x_j, x_i) \neq \emptyset \tag{4.3}$$

Proof of eq (4.3): by showing $B_j(x_j)$ is subset of $M_j^i(x_j, x_i)$.

First order freedom

Knowing that each agent is at least regulated by moral bottom line, there is one design of regulation set that can be sufficient for the purpose of peaceful living:

$$R_i^j(x_i, x_j) \triangleq \{u_i | \ \forall u_j \in\ M_j^i(x_j, x_i),\ BS_i^j\left(F_i(x_i, u_i, dt),\ F_j(x_j, u_j, dt)\right)\} \tag{5.1}$$

It is not hard to prove that:

$$B_i(x_i) \subseteq R_i^j(x_i, x_j) \subseteq M_i^j(x_i, x_j) \tag{5.2}$$

So here such regulation R_i^j observes moral bottom line, while also protects A_i's own basic right.

When A_i regulates its control $u_i \in R_i^j(x_i, x_j)$ defined by (5.1) and A_j observes (4.2), it can be proved that for any step k:

$$BS_i^j(x_i(k), x_j(k)) \rightarrow\ BS_i^j(x_i(k+1), x_j(k+1)) \tag{5.3}$$

The backup safe invariance between A_i and A_j is then achieved.

$R_i(x_1, x_2, \dots x_n)$, as the intersection of R_i^j, is always respecting other's basic right, while protecting A_i's own basic right, using (5.2), and can be proved to be not empty if $BS_i^j(x_i, x_j)$. Together with (5.3), the design is done.

Multiple possibilities of basic right

In practice, it is hard for one to tell the basic right of an unfamiliar agent. For A_i the choice of B_j varies, depending upon the background of A_j, or culture, ideology… However, the existence and unique-ness of B_j still stands. A_i may anticipate a list of possibilities for $B_j \in G_j^i = \{B_j^{i1}, B_j^{i2}, \dots . B_j^{im}\}$, while not knowing what exactly B_j is.

The moral bottom line becomes the intersection of enumerating through the other agent's basic right list:

$$M_j^i(B_2, B_1, x_j, x_i) \triangleq \{u_j | \forall u_i \in B_1(x_i), !\,Crush_i^j(F_i(x_i, u_i, dt), F_j(x_j, u_j, dt), B_1, B_2)\} \qquad (6.1)$$

$$R_j^i(x_j, x_i) \subseteq M_j^i(x_j, x_i) = \cap_{B_1 \in G_i^j,\ (skip\ unsafe\ B_1, B_j)} M_j^i(B_j, B_1, x_j, x_i) \qquad (6.2)$$

Some anticipation B_1 can lead to empty result of $M_j^i(B_j, B_1, x_j, x_i)$, which shall be skipped in the calculation of moral bottom line. (B_1 can be ruled out by assuming the current situation is backup safe).

With A_j observes (6.2), the action set of A_i is then defined as:

$$R_i^j(B, x_i, x_j) \triangleq \{u_i | \forall u_j \in M_j^i(B, B_i, x_j, x_i),\ !\,Crush_i^j(F_i(x_i, u_i, dt), F_j(x_j, u_j, dt), B_i, B)\}$$

(6.3)

$$R_i^j(x_i, x_j) \triangleq \cap_{B \in G_j^i,\ (skip\ unsafe\ B_i, B)} R_i^j(B, x_i, x_j) \qquad (6.4)$$

It is not hard to show that basic right is protected:

$$B_i(x_i) \subseteq R_i^j(x_i, x_j)$$

For the observation of moral bottom line:

$$R_i^j(x_i, x_j) \subseteq M_i^j(x_i, x_j)$$

For each single enumeration of B, $R_i^j(B, x_i, x_j) \subseteq M_i^j(B_i, B, x_i, x_j)$ can be proved. The intersection operation will simply keep that property.

If $B_j \in G_j^i$ and $B_i \in G_i^j$, we have:

$$R_i^j(x_i, x_j) \subseteq R_i^j(B_j, x_i, x_j)$$

$$M_j^i(x_j, x_i) \subseteq M_j^i(B_j, B_i, x_j, x_i)$$

The safe invariance property in eq (5.3) can be proved.

General right system and right graph analysis.

In practice, the right of an agent A_i is normally more than what is referred as the "basic right" B_i. Generally speaking, the right of an agent grows as it levels up in social status. We use a double parameter b_i to represent the right level of an agent: $u_i \in K_i(b_i,\ x_i)$ is always allowed when A_i 's right level equals b_i (and $u_i \notin K_i(b_i,\ x_i)$ is not always allowed). K_i is monotonically non-decreasing over b input:

$$\forall i, x_i,\ b_1 < b_2 \rightarrow K_i(b_1,\ x_i) \subseteq K_i(b_2,\ x_i) \qquad (7.1)$$

Given the right level and state of A_i and A_j, the invariance item is defined as:

$$I_j^i(b_j, b_i, x_j, x_i) = I_i^j(b_i, b_j, x_i, x_j) \triangleq\ !Crush_i^j(x_i, x_j, K_i(b_i, x_i), K_j(b_j, x_j)) \qquad (7.2)$$

known as the **invariance indicator function**. For the same states, a point (b_i, b_j) is invariant true indicates its left bottom side region is invariant true:

$$\forall b_1 \leq b_i, \forall b_2 \leq b_j, I_i^j(x_i, x_j, b_i, b_j) \rightarrow I_i^j(x_i, x_j, b_1, b_2) \qquad (7.3)$$

Take the crossing problem in autonomous driving as example, let b_i, b_j be the max acceleration of the vehicles:

$$K_i(b_2,\ x_i) = \{u_i | u_i.acc < b_2\} \qquad (7.4)$$

And place their states at the deep-colored boxes with certain initial speed in figure (7.1). Their invariance chart would be like that shown on the left side of figure (7.1).

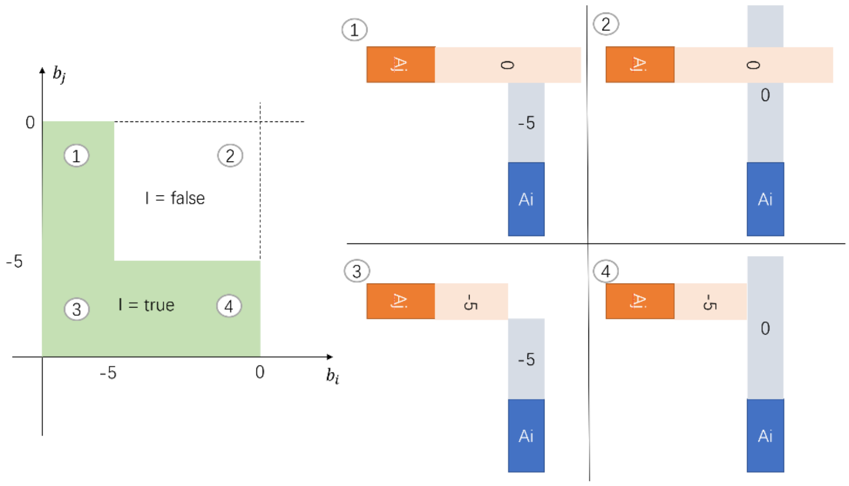

Figure 7.1. Crossing Problem Invariance chart.

Explanation: There are 4 typical cases:

1. b_j is at keep running, b_i is at braking. ($b_j = 0,\quad b_i = -5$)
2. b_j is at keep running, b_i is at keep running. ($b_j = 0,\quad b_i = 0$)
3. b_j is at braking, b_i is at braking. ($b_j = -5,\quad b_i = -5$)
4. b_j is at braking, b_i is at keep running. ($b_j = -5,\quad b_i = 0$)

1,3 and 4 are invariant true and 2 is invariant false. If both vehicles are always allowed to keep running, they would possibly collide.

The invariance indicator graph generally says, int the current situation there is no way such that both vehicles can enjoy the right of "keep running". To the contrast, either to let A_i yield and keep A_j's keeping running right, or to let A_j yield and keep A_i's keeping running right are acceptable invariance plans. The final result of each agent right is jointly decided by both agents.

We define the **moral function** of as follows:

$$M_i^j(b_i, b_j, x_i, x_j) \triangleq \{u_i | \forall u_j \in K_j(b_j, x_j), I_i^j\left(b_i, b_j, F_i(x_i, u_i, dt), F_j(x_j, u_j, dt)\right)\} \qquad (7.5)$$

If A_i 's next step action u_i is inside $M_i^j(b_i, b_j, x_i, x_j)$, we say A_i guards the morality of (b_i, b_j). The physical meaning of u_i guarding (b_i, b_j) is that, if A_j choose to exercise its right $u_j \in K_j(b_j, x_j)$, the invariance of (b_i, b_j) can be preserved from current frame to next frame. (Invariant true indicates moral function is non-empty for (b_i, b_j)). It is not hard to show that:

$$u_i \in M_i^j(b_i, b_j, x_i, x_j) \rightarrow u_i \in M_i^j(b_1, b_2, x_i, x_j), \forall b_1 < b_i, b_2 < b_j \qquad (7.6)$$

The monotonicity of eq (7.6) and eq (7.3) indicates that we can describe the set of invariance and moral using a border function:

$$DM_{i\,\{x_i, x_j\}}^j(u_i,\ b_i) \triangleq \max(b_j)\ subject\ to\ u_i \in M_i^j(b_i, b_j, x_i, x_j) \qquad (7.7)$$

$$DI_{i\,\{x_i, x_j\}}^j(\ b_i) \triangleq \max(b_j)\ subject\ to\ I_i^j(b_i, b_j, x_i, x_j) \qquad (7.8)$$

Known as the invariance border and moral border.

Still using the crossing problem as example, the two border functions are displayed in figure (7.2)

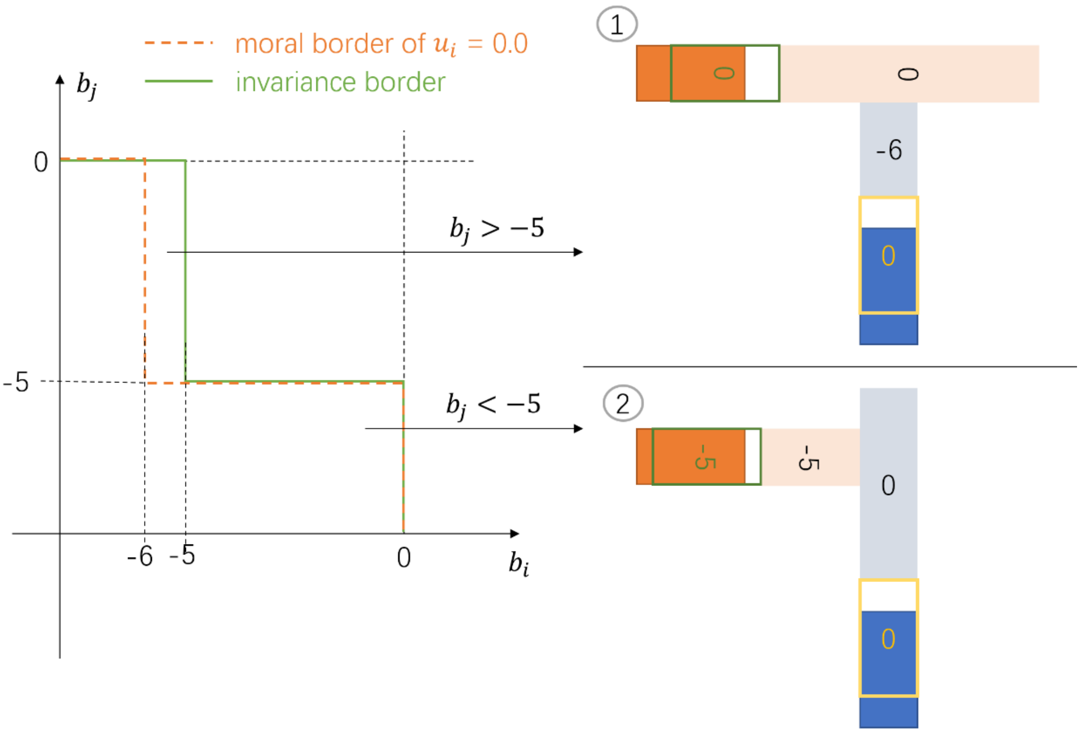

Figure 7.2 Border function of moral and invariance.

Explanation: The invariance border function (in green solid line) is simply the boundary of the invariant true region in figure (7.1). The moral border function of A_i, taking $u_i = 0$ as example, is displayed in orange dashed line in figure (7.2). Generally speaking, there are two conditions after A_i chooses $u_i = 0$:

1. If the invariance plan allows $b_j > -5$, as shown by the top right, the old invariance region of case 1 in figure (7.1) would be squeezed, since after keep running ($u_i = 0$) for a while, A_i would have to brake more than -5 to keep the invariance true.
2. If the invariance plan does not allow $b_j > -5$, as shown by the bottom right, there won't be anything bad to come since $u_i = 0$ is always allowed by the invariance plan for A_i.

A_j is required to observe certain moral bottom line, such that for the basic right level b_j^0, b_i^0, the control u_j must guard it. Using this fact, the **freedom function** and its border function is defined as follows:

$$R_i^j(b_i^0, b_j^0, b_i, b_j, x_i, x_j) \triangleq \{u_i | \forall u_j \in M_j^i(b_j^0, b_i^0, x_j, x_i), I_i^j\left(b_i, b_j, F_i(x_i, u_i, dt), F_j(x_j, u_j, dt)\right)\}$$

(7.9)

$$DR^j_{i\{b_i^0, b_j^0, x_i, x_j\}}(u_i,\ b_i) \triangleq \max(b_j)\ subject\ to\ u_i \in R_i^j(b_i^0, b_j^0, b_i, b_j, x_i, x_j) \qquad (7.10)$$

If the current state satisfies basic right invariance, it can be proved that the freedom function is a subset of the moral function:

$$I_i^j(b_i^0, b_j^0, x_i, x_j) \to K_j(b_j^0, x_j) \subseteq M_j^i(b_j^0, b_i^0, x_j, x_i)$$

$$\to R_i^j(b_i^0, b_j^0, b_i, b_j, x_i, x_j) \subseteq M_i^j(b_i, b_j, x_i, x_j)$$

(7.11)

It is also easy to show that freedom function of A_i is not empty by showing that $K_i\left(b_i^0, x_i\right)$ is a subset of it.

$$I_i^j\left(b_i^0, b_j^0, x_i, x_j\right) \rightarrow K_i\left(b_i^0, x_i\right) \subseteq R_i^j\left(b_i^0, b_j^0, b_i^0, b_j^0, x_i, x_j\right) \tag{7.12}$$

Generally speaking, the freedom function is there to be the safe guard of the minimum invariance $I_i^j\left(b_i^0, b_j^0, x_i, x_j\right)$, whereas the moral function is there to be the guide towards maximized invariance. An agent A_i 's full mission for crush avoidance can be summarized as follows:

$$Find\ \ u_i,\ b_1, b_2, \ldots b_n,$$

$$to\ maximize:\quad G_i\begin{pmatrix} x_1, x_2, \quad \ldots x_n, \\ u_i, \\ b_1, b_2, \ldots b_n \end{pmatrix} \triangleq k_i\ H_{self}(b_i, u_i) + \sum_{j \neq i} k_j\ H_{other}(b_j)$$

$$subject\ to:\qquad \begin{cases} u_i \in M_i^j\left(b_i, b_j, x_i, x_j\right), \forall j \neq i \\ u_i \in R_i^j\ \left(b_i^0, b_j^0, b_i^0, b_j^0, x_i, x_j\right), \forall j \neq i \end{cases} \tag{7.13}$$

$$(Then\ apply\ u_i)$$

Where:
$H_{self}: (\mathbb{R}, \mathbb{R}^{n_u}) \rightarrow \mathbb{R}$ evaluates the overall benefit of self enjoying right level b_i after one step plus self doing control u_i in this step.
$H_{other}: (\mathbb{R}, \mathbb{R}^{n_u}) \rightarrow \mathbb{R}$ evaluates the overall benefit of other enjoying right level b_j after one step.
$k_i \in \mathbb{R}$: weight factor of each agent. For those agents with larger k, their right is more respected by A_i.

H_{self}, H_{other} and k_i are known as perspective parameters, which may vary among agents. The more they are alike, the faster the invariance true region grows along time. In such case agents' perspective and behavior are united.

If however, perspective parameters don't agree among agents, we still have the contract of basic agent right $b_1^0, b_2^0, \ldots\ b_n^0$, which helps to guarantee a minimum level of safe invariance, the world stay peace.