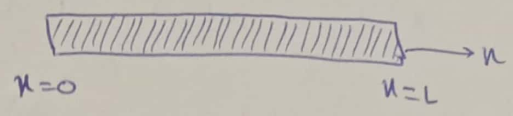
Heat Equation

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}$$



$u(x,t) \rightarrow$ Temperature of Bar

Boundary conditions

End of the bar are at zero Temperature

$$u(0,t) = u(L,t) = 0$$

We will use seperation of variables technique to make this PDE to ODEs for simple solution.

For this, we assume

$$u(x,t) = X(x)\,T(t)$$

Put this into the main equation

$$X(x)\frac{dT(t)}{dt} = \alpha \frac{d^2X(x)}{dx^2} T(t)$$

$$\boxed{X\dot{T} = \alpha X'' T}$$

where, $\dot{T} = \frac{dT}{dt}$, $X'' = \frac{d^2X}{dx^2}$

Divide it with $X(x)T(t)$

$$\frac{\cancel{X}\dot{T}}{\cancel{X}T} = \frac{\alpha X''\cancel{T}}{X\cancel{T}}$$

$$\frac{\dot{T}}{T} = \alpha\frac{X''}{X}$$

Separate out the ODEs as,

$$\frac{X''}{X} = \gamma \Rightarrow \frac{\dot{T}}{T} = \alpha\gamma$$

$$\boxed{X'' = \gamma X} \qquad \boxed{\dot{T} = \alpha\gamma T}$$

Now, we can solve these ODEs.

Boundary conditions will be

$$u(0,t) \Rightarrow X(0)T(t) = 0$$

$$\boxed{X(0) = 0}$$

$$u(L,t) \Rightarrow X(L)T(t) = 0$$

$$\boxed{X(L) = 0}$$

Case - I $(\gamma = 0)$

$$X'' = 0 \Rightarrow X(x) = Ax + B$$

$x=0: \quad X(0) \Rightarrow B=0$

$X(x) = Ax$

$x=L: \quad X(L) = A \cdot L = 0 \Rightarrow A=0$

$X(x) = 0$??

Not a valid case.

Case - II ($\gamma > 0$)

Let $\gamma = k^2 > 0$

$X'' = k^2 X \Rightarrow X(x) = Ae^{kx} + Be^{-kx}$

$x=0: \quad X(0) = A+B = 0 \Rightarrow A = -B$

$X(x) = A(e^{kx} - e^{-kx})$

$x=L: \quad X(L) = A(e^{kL} - e^{-kL}) = 0$

$k>0 \quad L>0 \quad$ so, $A=0$

$X(x) = 0$??

Not a valid case.

Case - III ($\gamma < 0$)

Let $\gamma = -k^2 < 0$

$X'' = -k^2 X \Rightarrow X(x) = A\cos(kx) + B\sin(kx)$

$x=0: \quad X(0) = A = 0 \Rightarrow X(x) = B\sin(kx)$

$x=L: \quad X(L) = B\sin(kL) = 0 \Rightarrow \sin(kL) = 0$

$\therefore kL = n\pi$

$k = n\pi/L \quad$ n is an integer

Valid case.

$$\boxed{X(x) = B\sin\left(\frac{n\pi}{L}x\right)}$$

It satisfies our Boundary conditions.

$\dot{T} = \gamma\alpha T(t)$

$\dot{T} = -k^2\alpha T(t)$

$T(t) = c\, e^{-k^2\alpha t} \qquad \because k = \frac{n\pi}{L}$

$$\boxed{T(t) = c\, e^{-\alpha \frac{n^2\pi^2}{L^2} t}}$$

$$U(x,t) = X(x)\,T(t)$$

$$\boxed{U(x,t) = B \sin\left(\frac{n\pi}{L}x\right) c\, e^{-\alpha \frac{n^2\pi^2}{L^2} t}}$$

It is true for all values of n so will be true for sum of n.

$$\boxed{U(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L}x\right) e^{-\alpha \frac{n^2\pi^2}{L^2} t}} \quad -\alpha$$

This gives us the General solution with Boundary Conditions.

To Get B_n, we will use the initial conditions:

Let $U(x,0) = f(x) \longrightarrow$ Some initial Profile

$$f(x) = U(x,0) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L}x\right) \cancel{e^{-\alpha \frac{n^2\pi^2}{L^2} t}} \to 0$$

$$= \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L}x\right) \quad -①$$

we will solve this fourier series to get B_n.

we know that

$$\int_0^L \sin\left(\frac{n\pi}{L}x\right) \sin\left(\frac{m\pi}{L}x\right) dx = \begin{cases} 0 & \text{if } n \neq m \\ \frac{L}{2} & \text{if } n = m \end{cases} \Rightarrow \frac{L}{2}\delta_{mn}$$

So, multiply $\int_0^L \sin\left(\frac{m\pi}{L}x\right) dx$ on both sides of ①

$$\int_0^L f(x) \sin\left(\frac{m\pi}{L}x\right) dx = \int_0^L \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L}x\right) \sin\left(\frac{m\pi}{L}x\right) dx$$

$$= \sum_{n=1}^{\infty} B_n \cdot \frac{L}{2} \delta_{mn}$$

0 for all $m \neq n$

$$= \frac{L}{2} B_m$$

$$\boxed{B_m = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{m\pi}{L}x\right) dx}$$

Insert into α to get the solution.

~~Insert it into ① to Get~~

$$\cancel{u(x,t) = \sum_{n=1}^{\infty} \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)}$$

For initial condition: $u(x,0) = T_0 \Leftarrow f(x)$

Boundary conditions: $u(0,t) = u(L,t) = 0$

$u(x,t) \to 0$ as $t \to \infty$

~~Bar at Temp T_0 is allowed to cool so that $u(x,t)$ approaches to zero~~

$$B_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)dx$$

$$= \frac{2T_0}{L}\int_0^L \sin\left(\frac{n\pi}{L}x\right)dx$$

let $\frac{\pi}{L}x = \theta$

$dx = d\theta$

at $x=0$, $\theta=0$

at $x=L$, $\theta=\pi$

$$= \frac{2T_0}{\pi}\int_0^\pi \sin n\theta \, d\theta$$

$$= \frac{2T_0}{\pi}\left[-\frac{\cos n\theta}{n}\right]_0^\pi$$

if $n=2m$ is even, $\to = 0$

if $n = 2m-1$ is odd, $\to \frac{2}{2m-1}$

$$u(x,t) = \sum_{m=1}^{\infty} \frac{2}{2m-1} \cdot \frac{2T_0}{\pi} e^{-\left(\frac{(2m-1)c\pi}{L}\right)^2 t} \sin (2m-1)\frac{\pi}{L}x$$