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<u>Proposition</u> Let (M^{2n+1}, ξ) contact mfd and Σ^{2n} an embedded hypersurface. Let X be a contact VF s.t. $X \pitchfork \Sigma$. Then $\Gamma = \{p \in \Sigma \text{ s.t. } X_p \in \xi_p\}$ is a contact submfd.

<u>Proof</u>

Let α be any contact form for ξ.
Let $f \in C^\infty \Sigma$ given by $f(p) = \alpha_p(X_p)$. Then $\Gamma = f^{-1}0$. We show $df \neq 0$ on Γ. Let $p \in \Gamma$. $\exists \tilde{w} \in \xi_p$ s.t. $d\alpha_p(X_p, w) \neq 0$. For dimension reasons $\exists c \in \mathbb{R}$ $w := \tilde{w} - cX_p \in T\Sigma$. So we have $w \in \xi_p \cap T_p\Sigma$ s.t. $d\alpha_p(w, X_p) \neq 0$. Then $df_p(\tilde{w}) = d(\alpha(X))_p(w) = \mathcal{L}_X\alpha(w) - i_X d\alpha_p(w) = 0$ because $\mathcal{L}_X \alpha = g\alpha$.
Since $w \in T\Sigma$, we have $df \neq 0$. We additionally have $w \in T\Sigma_{|\Gamma} \setminus T\Gamma$, or otherwise said $w \pitchfork \Gamma$.
Now we wish to show $\alpha \wedge (d\alpha)^{n-1}{}_{|T\Gamma} \neq 0$.

Firstly notice that from $df = g\alpha - d\alpha(X, -)$ we get $d\alpha(X, T\Gamma \cap \xi) = 0$. We see $\langle X_p \rangle \oplus \langle w \rangle \oplus T_p\Gamma = T_pM$ and $X_p, w \in \xi_p \Rightarrow \exists r \in T_p\Gamma$ s.t. $\alpha_p(r) \neq 0$. We also see $\dim \xi_p \cap T_p\Gamma = 2n-2$, so take basis $v_1, \dots, v_{2n-2}$ for $\xi_p \cap T_p\Gamma$.
We've seen $d\alpha(X_p, v_i) = 0$. To show that $\alpha \wedge (d\alpha)^{n-1}{}_{|T\Gamma} \neq 0$, we show that $d\alpha$ nondegenerate on $\langle v_1, \dots, v_{2n-2} \rangle$.
Assume $d\alpha(v_1, v_i) = 0 \ \forall i$. Let's look at
$\alpha \wedge (d\alpha)^n (r, X_p, w, v_1, \dots, v_{2n-2}) = \alpha(r)\, d\alpha(X_p, w)\, d\alpha(v_1, -)(d\alpha)^{n-2}(v_2, \dots, v_{2n-2}) = 0$
The first equality happens since only r goes into α, with X_p only w goes. The second equality $\Rightarrow$ no v_i goes with v_1. But this implies $\alpha \wedge (d\alpha)^n = 0$ $\lightning$.