2 Definitions and Examples

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Definition 2.1 A Liouville form on a symplectic mfd (W, ω) is a 1-form λ such that $\omega = d\lambda$. The VF $X \in \mathfrak{X}W$ s.t. $\iota_X \omega = \lambda$ is the Liouville VF.

Observe $\mathcal{L}_X \omega = \omega \Rightarrow \phi_t^* \omega = e^t \omega$.
"the symplectic form expands along the flow of X".
An important note is: all such mfds W have boundary:

$$0 < \int_W \omega^n = \int_W d(\lambda \wedge \omega^{n-1}) = \int_{\partial W} \lambda \wedge \omega^{n-1} \Rightarrow \partial W \neq \emptyset.$$

Definition 2.2 A Liouville domain is a compact symplectic mfd $(W, d\lambda, X)$ with boundary such that the Liouville VF X points transversally out of ∂W.

In this case $(\partial W, \lambda|_{\partial W})$ is a contact mfd:

$$\lambda \wedge (d\lambda)^{n-1} = (\iota_X d\lambda) \wedge (d\lambda)^{n-1} = \frac{1}{n} \iota_X (d\lambda)^n$$

$\Rightarrow$ Liouville domain is an exact symplectic filling of $(\partial W, \lambda)$.

Definition 2.3 A Weinstein domain $(W, d\lambda = \omega, X, f)$ is a Liouville domain (W, ω, X) with a Morse function $f: W \to \mathbb{R}$ such that f locally constant on ∂W s.t. X is gradient like for f i.e.

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$$X(f) \geq \delta (|X|^2 + |df|^2)$$

for some Riemannian metric on W and some $\delta > 0$.

Using Cauchy-Schwarz

$$\delta |df|^2, \delta |X|^2 \leq \delta(|X|^2 + |df|^2) \leq |X(f)| \leq |df||X|$$

$$\Rightarrow \quad \delta |X| \leq |df| \leq \frac{1}{\delta} |X|$$

From this we see $X_p = 0 \iff df_p = 0$, that is X is 0 exactly at the critical pts of f. One can check that λ is a contact form on each regular level set of f, so a Weinstein domain is a symplectic mfd which "decomposes" into layers of contact mflds.

Example (very important for contact topologists)
Let $\Sigma^{2n} \subset (M^{2n+1}, \xi = \operatorname{Ker} \alpha)$ be a closed oriented convex hypersurface. Then Σ has a nbhd contactomorphic to $(\mathbb{R}_t \times \Sigma, \operatorname{Ker}(f dt + \beta))$ where $f \in C^\infty \Sigma$, $\beta \in \Omega^1 \Sigma$ (by following the flow of the contact VF). The contact condition $\alpha \wedge (d\alpha)^n > 0$ on Σ implies that

$$\Theta := (d\beta)^{n-1} \wedge (f d\beta + n \beta \wedge df)$$

is a volume form on Σ. Define the characteristic foliation of Σ to be directed by the unique $Y \in \mathfrak{X}\Sigma$ s.t. $i_Y \Theta = \beta \wedge (d\beta)^{n-1}$. Take $R_+ := \{x \in \Sigma \text{ s.t. } fx > 0\}$. On R_+ the contact condition implies $(d\beta')^n > 0$ where $\beta' := \beta/f$. Hence β' is a Liouville form on R_+. Let X be the Liouville VF. One can show $X = nfY$. Consider $R_+^\varepsilon = \{f \geq \varepsilon > 0\}$.

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$X = \nabla f\, Y$. Consider $R_+^\varepsilon = \{f \geq \varepsilon > 0\}$.
One can show Y is outwardly transverse to $\partial \overline{R_+} = \{f = 0\} =: \Gamma$. So for small ε X will be outwardly transverse to ∂R_+^ε.
Hence R_+^ε is a Liouville domain.

One can show not all Liouville domains are Weinstein:

<u>Lemma</u> Let $(W^{2n}, \omega = d\lambda, X, f)$ be a Weinstein domain. The index of each critical point of f is $\leq n$.

<u>Proposition</u> For $n \geq 2$, ∂W is connected.

So if we construct a Liouville domain with disconnected ∂, we have won. Such object indeed exists.