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Reconstructing neighborhood of a convex surface

Let $f_i \in C^\infty(\Sigma^{2n}), \beta_i \in \Omega^1(\Sigma)$ for $i \in \{0,1\}$ such that $f_0^{-1}(0) = f_1^{-1}(0) =: \Gamma$, $d(\frac{1}{f_0}\beta_0) = d(\frac{1}{f_1}\beta_1)$ on $\Sigma_\pm$ are symplectic forms, $\pm f_i > 0$ on $\Sigma_\pm$, $\mathrm{Ker}(\beta_{0|T\Gamma}) = \mathrm{Ker}(\beta_{1|T\Gamma}))$ is a contact structure. We show then that we have contactomorphic neighborhoods of Σ in $(\Sigma \times \mathbb{R}, \mathrm{Ker}(f_i dt + \beta_i))$.

Firstly, since $df_i \neq 0$ at the zero set Γ, we can find g such that $f_0 g = f_1$. Notice the contact structure induced by (β_0, f_0) and $(g\beta_0, gf_0)$ are the same, so replacing f_0, β_0 by $gf_0, g\beta_0$ we can assume that $f_0 = f_1 =: f$. Now we can do linear interpolation between the β_i to get $\beta_s := s\beta_1 + (1-s)\beta_0$. Notice that we still have $d(\frac{1}{f}\beta_0) = d(\frac{1}{f}\beta_s)$ and $\mathrm{Ker}(\beta_{0|T\Gamma}) = \mathrm{Ker}(\beta_{s|T\Gamma})$ for all s. So we have found a 1-parameter family of contact forms from α_0 to α_1 where $\alpha_s = \beta_s + f dt$. Now we apply the Moser's trick. Let X_s be the time dependent vector field with flow θ_s such that $\theta_s^* \alpha_s = \lambda_s \alpha_0$. Differentiating we get

$$di_{X_s}\alpha_s + i_{X_s} d\alpha_s + \dot{\alpha_s} = \mu_s \alpha_s.$$

This we solve as usually. Notice every object of ours is t-invariant, so so is the solution X_s. One can use this to show that θ_1 takes Σ diffeomorphically onto the graph of a function $\Sigma \to \mathbb{R}$. In this way we get a contactomorphism of $(\Sigma \times \mathbb{R}, \alpha_i)$. Keeping this image in mind, we get the following corollary

Corollary

Let (M_i, ξ_i), $i \in \{0,1\}$ be two contact manifolds and f an orientation reversing diffeomorphism from a boundary component Σ_0 of M_0 to a boundary component Σ_1 of M_1. Suppose Σ_i s a convex surface in (M_i, ξ_i). If f is a contactomorphism Γ_{Σ_0} to Γ_{Σ_1} and is a symplectomorphism of the ideal Liouville domains $(\Sigma_i)_\pm$, then one may remove a collar neighborhood of Σ_1 from M_1 and glue it to M_0 so the resulting manifold has a contact structure into which each of the (M_i, ξ_i) contact embed.