# 2.3.1

Donnerstag, 17. Oktober 2024 08:49

**Lemma 2.3.1** Let $\Sigma$ be closed convex hypersurface in $(M^{2n+1}, \ker\alpha = \xi)$ with transverse contact VF $v \in \mathfrak{X}M$. Then $\overline{R_\pm}$ are ideal compactifications of Liouville mfds.

**Proof** We know in a nbhd $\Sigma \times \mathbb{R}$ $\alpha = \beta + f dt$ where $f \in C^\infty \Sigma$, $\beta \in \Omega^1 \Sigma$. $\alpha \wedge (d\alpha)^n \neq 0 \Rightarrow$
$\Rightarrow (\beta + f dt) \wedge (d\beta + df \wedge dt)^n = f dt \wedge (d\beta)^n + n \beta \wedge (d\beta)^{n-1} \wedge df \wedge dt \neq 0$
$\Rightarrow \Theta := (d\beta)^{n-1} \wedge (f d\beta + n \beta \wedge df) \neq 0$ on $\Sigma$.
Let $Y \in \mathfrak{X}(\Sigma)$ s.t. $i_Y \Theta = \beta \wedge (d\beta)^{n-1}$. (Exists for dimension reasons). On $R_+$ we scale $\alpha$ and look at $(R_+ \times \mathbb{R}, \ker \frac{\alpha}{f})$. So we have contact form $\beta' + dt$ where $\beta' = \beta/f$. The contact condition implies $(d\beta')^n \neq 0$ on $\Sigma$. Hence $\beta'$ is a Liouville form on $R_+$. One shows that $X = nfY$ is the Liouville VF.
We show $Y \pitchfork \Gamma$. Let $p \in \Gamma$.
$\Theta(p) = (d\beta)^{n-1} \wedge n \beta \wedge df = -(n\, df) \wedge i_Y \Theta$
$0 \neq i_Y \Theta = -nY(f) \cdot i_Y \Theta \Rightarrow -nY(f) = 1.$ ✓
Since $X \parallel Y$, we have $X \pitchfork \partial R_\pm^\varepsilon$ where $R_\pm^\varepsilon = \{x : \pm f x \geq \varepsilon\}$. So the main message is that $\alpha|_\Sigma$ is not the Liouville 1-form, but $\alpha'$ is.

**Claim** $\beta(p) \neq 0 \Rightarrow (\beta \wedge (d\beta)^{n-1})(p) \neq 0$
Assume $\beta(p) \neq 0$, $\beta \wedge (d\beta)^{n-1}(p) = 0$
We know $(d\beta)^n \neq 0$.
$\ker \beta(p)$ has dim $2n-1$. So $(d\beta)^{n-1}$ has rank $2n-2$ on $\ker \beta(p)$.
$\Rightarrow \beta \wedge (d\beta)^{n-1}(p) \neq 0$.

**Claim** $Y \in \ker \beta$.

$0 = i_Y i_Y \theta = i_Y \, \beta \wedge (d\beta)^{n-1} =$
$\beta(Y)(d\beta)^{n-1} - (n-1)\beta \wedge i_Y d\beta \wedge (d\beta)^{n-2}.$
Take $\wedge \beta$ to get $0 = \beta(Y)\beta \wedge (d\beta)^{n-1}.$

From $Y \in \ker(\beta)$, $\ker(i_Y \theta = \beta \wedge (d\beta)^{n-1})$ we get $Y \in \ker(d\beta)$.

So one automatically gets $\mathcal{L}_{nY} \beta = 0$
Now use diffeomorphism $(-\varepsilon,\varepsilon) \times \Gamma \to \Sigma$
$\tau, x \mapsto Fl_\tau^{nY}(x)$
to identify nbhd. The VF $\partial_\tau \mapsto -nY$. $\beta$ is $\tau$ invariant.
Now we just want to know $nY(f) = -1 \Rightarrow \partial_\tau(f) = 1$
So in these charts $f = \tau$ and $\beta$ is $\tau$ invariant.
$\beta = \beta_0 \in \Omega^1 \Gamma$
So on nbhd $\Gamma \times [0,\varepsilon)$, the Liouville VF is just $X = fnY = -\tau \partial_\tau$. This implies that $\Gamma$ is an ideal boundary of $R_+$:
$$((0,\varepsilon)_\tau \times \Gamma, d(\tfrac{\beta_0}{\tau})) \longrightarrow (\Gamma \times (c,\infty)_s, d(e^s \beta_0))$$
$$t, x \longmapsto (x, \ln \tfrac{1}{\tau})$$
is a symplectomorphism. $\square$