3. Contact surgery and symplectic handle attachment

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3.1 "Flat" Weinstein model for contact surgery

Let $(M, \xi = \ker \alpha)$ be a contact mfd and suppose that S is an isotropic k-sphere with trivial conformal symplectic normal bundle, trivialized by the framing ε. Using a nbhd thm we find a strict contact embedding

$$\Psi : (\upsilon(S), \alpha) \longrightarrow \left(\mathbb{R} \times T^*S^k \times \mathbb{R}^{2(n-k)}, dz + p\,dq + \tfrac{1}{2}(x\,dy - y\,dx)\right)$$

A priori we get only that a nbhd of S is contactomorphic to a nbhd of $(0, S^k, 0)$ in $\mathbb{R} \times T^*S^k \times \mathbb{R}^{2(n-k)}$. We can enlarge this second nbhd by composing with the contactomorphism

$$\begin{aligned} \varphi_C : \mathbb{R} \times T^*S^k \times \mathbb{R}^{2(n-k)} &\longrightarrow \mathbb{R} \times T^*S^k \times \mathbb{R}^{2(n-k)} \\ (z; q, p; x, y) &\longmapsto (Cz; q, Cp; \sqrt{C}\,x, \sqrt{C}\,y) \end{aligned}$$

Now consider the following model for contact surgery & symplectic handle attachment.
Consider the symplectic mfd $(\mathbb{R}^{2n+2}, \omega_0)$.
We use coordinates $(x, y; z, w)$ where there are $n-k$ pairs of (x,y) coordinates and $k+1$ pairs of (z,w) coordinates. The symplectic form is then given by $\omega_0 = dx \wedge dy + dz \wedge dw$.
Note the VF $X = \frac{1}{2}(x\partial_x + y\partial_y) + 2z\partial_z - w\partial_w$ is Liouville. Now consider

$$S_{-1} := \{(x, y, z, w) \mid |w|^2 = 1\}.$$

X is transverse to this submfd and induces the contact form $\alpha = \frac{1}{2}(x\,dy - y\,dx) + 2z\,dw + w\,dz$

Λ is transverse to this subbundle and induces the contact form $\alpha = \frac{1}{2}(xdy - ydx) + 2zdw + wdz$.

We see the sphere $\{(x,y,z,w) \mid x=y=0, z=0, |w|^2=1\} \cong S^k$ describes an isotropic sphere in S_{-1} with trivial conformal symplectic normal bundle.
We think of S_{-1} as a nbhd of the isotropic sphere S before surgery. That is, S_{-1} is a standard nbhd of an isotropic sphere of dim k with trivial normal bundle, since we have the following contactomorphism:

$$\Psi_k : \mathbb{R} \times T^*S^k \times \mathbb{R}^{2(n-k)} \longrightarrow S_{-1}$$
$$(z, q, p, x, y) \to (x, y, zq + p, q)$$

Here we regard T^*S^k as the submfd of $\mathbb{R}^{2(k+1)}$ by using coordinates $(q,p) \in \mathbb{R}^{2(k+1)}$ where $|q|^2 = 1$, $q \cdot p = 0$.
Note that Ψ_k is a strict contactomorphism.

Combining all three maps, we get a contactomorphism
$\underset{\underset{M}{\in}}{\vartheta(S)} \longrightarrow S_{-1}$ $\quad \phi_c = \Psi_k \circ \varphi_c \circ \Psi$

This is not a strict contactomorphism, but the following lemma makes gluing possible.

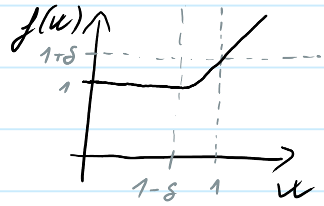
Lemma 3.3 For $i \in \{0,1\}$ let (M_i, α_i) be a contact type hypersurface in the symplectic mfd (W_i, ω_i) wrt the Liouville VF Y_i. Suppose $\varphi : (M_0, \alpha_0) \to (M_1, \alpha_1)$ is a contactomorphism s.t. $\varphi^* \alpha_1 = C\alpha_0$ for some constant C.
Then φ extends to a symplectomorphism between nbhds of M_0 and M_1 by sending flowlines of Y_0 to flow lines of Y_1.

Attaching a symplectic handle

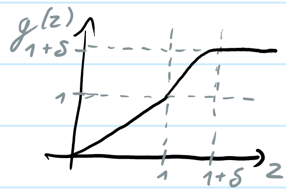
We now define a symplectic handle. The contactomorphism Φ_c identifies the nbhd $\nu_M(S)$ with a nbhd of the isotropic sphere in S_{-1}.

We define the profile for the handle. Fix $\delta > 0$, which will serve as smoothing parameter. Choose smooth $f, g : \mathbb{R}_{\geq 0} \to \mathbb{R}$ s.t.

- f increasing on $(1-\delta, \infty)$
- $f(u) = 1$ for $u \in [0, 1-\delta]$
- $f(u) = u + \delta$ for $u > 1 - \delta/2$



- g increasing on $(0, 1+\delta)$
- $g(z) = z$ for $z < 1$
- $g(u) = 1+\delta$ for $u > 1+\delta$

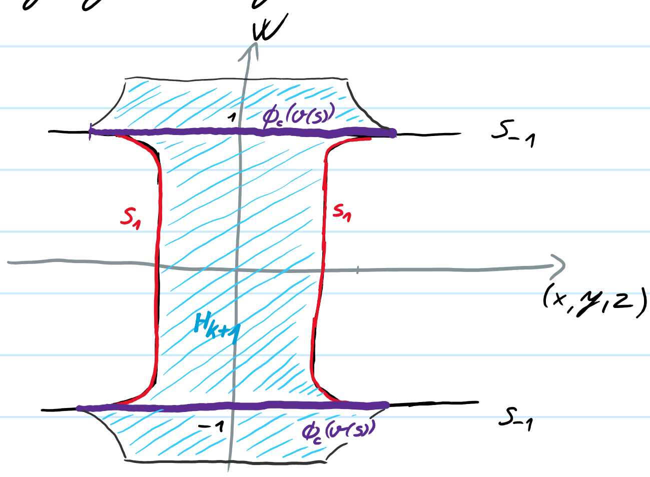


Define $F(x,y,z,w) := -f(w^2) + g(x^2+y^2+z^2) : \mathbb{R}^{2n+2} \to \mathbb{R}$

Define a hypersurface $S_1 := \{(x,y,z,w) \mid F(x,y,z,w) = 0\}$

This hypersurface is of contact type, because the Liouville VF X is transverse to S_1 ($X(F) > 0$). (we are in $(\mathbb{R}^{2(n+1)}, \omega_0)$).

The hypersurface S_1 is meant to describe the result of the surgery along S.



In order to describe the surgery we shall use handle attachment along a symplectic mfd (W, ω) with contact type boundary M. Define the symplectic handle as follows: the handle H_{k+1} consists of those points $p \in (\mathbb{R}^{2n+2}, \omega_0)$ s.t. one of following hold:

as follows: the handle H_{k+1} consists of those points $p \in (\mathbb{R}^{2n+2}, \omega_0)$ s.t. one of following hold:

- $\exists t \in [0,1]$ s.t. $Fl_t^X(p) \in \phi_c(\vartheta(S))$

(This is the gluing part of the symplectic handle)

- $\exists t_1 \leq 0 \leq t_2$ s.t. $Fl_{t_1}^X(p) \in \phi_c(\vartheta(S))$, $Fl_{t_2}^X(p) \in S_1$
- p is the unique critical point of $X \Rightarrow p = 0$.

(H_{k+1} is blue in picture above)

Now we attach this symplectic handle H_{k+1} to (W, ω). A nbhd of the boundary is symplectomorphic to $([-1,0] \times M, d(e^t \alpha))$; call this symplectomorphism $\psi_\partial : \vartheta_W(M) \longrightarrow [-1,0] \times M$. Restricting, one gets the symplectomorphism $\vartheta_W(\vartheta_M(S)) \longrightarrow [-1,0] \times \vartheta_M(S)$. We can compose this symplectomorphism with the map

$$\tilde{\phi}_c : [-1,0] \times \vartheta_M(S) \longrightarrow H_{k+1} .$$
$$(t, p) \longmapsto Fl_t^X(\phi_c(p))$$

This is also a symplectomorphism. Now attach the symplectic handle $\tilde{W} := W \cup H_{k+1} / \sim$ where we glue $x \in \vartheta_W(\vartheta_M(S)) \subseteq W$ to $y \in H_{k+1}$ iff $\tilde{\phi}_c \circ \psi_\partial(x) = y$. By lemma 3.3 the resulting mfd $\tilde{W}$ is symplectic and its boundary is a contact mfd that is diffeomorphic to the surgered mfd $\widetilde{(M,\xi)}_{S,\varepsilon}$ obtained by performing surgery on M along the isotropic submfd S with framing ε.

Definition 3.6 The above attaching procedure is called symplectic handle attachment along S at the convex end of W^{2n+2}, which is a $2n+1$ dimensional contact mfd. We call the dimension subcritical if $\dim S < n$, critical if $\dim S = n$. The induced operation on the convex end is called contact surgery along S. The contact surgery on the convex end is called contact surgery along S. The contact surgery is subcritical if $\dim S < n$, and critical or Legendrian if $\dim S = n$.

Remark Since we attach a symplectic handle to a cobordism by gluing flowlines of the respective Liouville VF, we get that the new mfd is also Liouville.

Summarised:

Proposition 3.8 Let (W, ω) be a symplectic cobordism. Suppose that $i: S \to \partial W$ is an embedded isotropic k-sphere in the convex end of W whose conformal symplectic normal bundle is trivialized by ε.
Then we can attach a handle H_{k+1} to W along S with framing ε to obtain a symplectic cobordism $(\tilde{W}, \tilde{\omega})$. Furthermore if (W, ω) admits an ω-convex function f, then f can be extended to an $\tilde{\omega}$-convex function $\tilde{f}$ on $\tilde{W}$ s.t. $\tilde{f}$ has only one additional critical point.

Symplectic handle cancellation

Lemma 3.11 (Cancellation theorem)
Let (W, ω) be a symplectic mfd and f an ω-convex function. Let p, q be non-degenerate critical pts of f and $d \in (f(p), f(q))$.
Suppose
- $\mathrm{index}_q(f) = \mathrm{index}_p(f) + 1$
- The spheres $S_q^- := W^s(q) \cap \{x \mid f(x) = d\}$
$S_p^+ := W^u(p) \cap \{x \mid f(x) = d\}$

$$S^+_p := W^{u}(p) \cap \{x \mid f(x) = d\}$$

intersect transversely in one pt.

Then the critical pts can be cancelled by a J-convex deformation of f in a nbhd of $[f(p), f(q)]$.

We now describe a situation in which handle cancellation occurs.

Let (W_1^{2n}, ω) be a symplectic mfd s.t. $M_1 \subset \partial W_1$ is a convex end. Choose an ω-convex function f_1 near the convex end and let X_1 be the associated Liouville VF.

Now suppose that $\Sigma_1 \subset M_1^{2n-1}$ is an isotropic $(n-2)$ sphere with a trivialization ε of its conformal symplectic normal bundle. Suppose Σ_1 bounds a Legendrian $(n-1)$-disk D_1 in M_1.

Now form the symplectic mfd (W_2, ω_2) by attaching a symplectic $n-1$ handle along Σ_1,

$$(W_2, \omega_2) = W_1 \cup_{\Sigma_1, \varepsilon} H_{n-1}$$

The ω_1-convex function f_1 extends to the ω_2-convex function f_2. This f_2 has one additional critical pt corresponding to the middle of the handle - denote it by p.

Note that the convex end M_1 is surgered into a new convex end $M_2 \subset \partial W_2$. This convex end comes with a Legendrian $(n-1)$-sphere Σ_2 which is formed as follows:

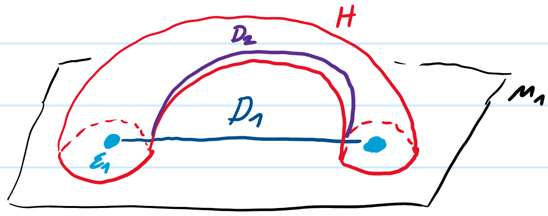
First observe that there is a parallel copy D_2 of the core of H_{n-1} which is a Legendrian $(n-1)$-disk. Using the flat Weinstein model we put

$\varphi : D^{n-1} \longrightarrow H_{n-1}$, $w \longmapsto (1, 0; 0, w)$.

One checks $\alpha|_{D_2} = 0$ where $D_2 = \varphi(D^{n-1})$.

$\varphi: D^{n-1} \to H_{n-1}$, $w \mapsto (1,0;0,w)$.
One checks $\alpha|D_2 = 0$ where $D_2 = \varphi(D^{n-1})$.
After smoothing, we can glue D_1 to D_2 to get the Legendrian sphere Σ_2.

Observe Σ_2 intersects the belt sphere of H_{n-1} transversely in one point $\varphi(0)$.



Since Σ_2 is Legendrian, the conformal symplectic normal bundle is trivial, so we can form W_3 by critical n-handle attachement along Σ_2 without reference to framing. $(W_3, \omega_3) := W_2 \cup_{\Sigma_2} H_n$.
As before we can extend the ω_2-convex function f_2 to an ω_3-convex function f_3 on W_3. Denote the additional critical pt of f_3 by q. We shall denote the gradient-like Liouville VF on W_3 by X_3.
The convex end M_2 is surgered yielding the contact mfd M_3.
Now intersect a level set $\{f_3 = d\}$ with d between $f_3(p)$ and $f_3(q)$, with the mfds $W^s q$ and $W^u p$ to form the spheres $S^- q$ and $S^+ p$.
These spheres intersect transversely in one point, as we can see the unique flow line of the Liouville VF X_3 from p to q.

This means Lemma 3.11 applies, so we can deform f_3 to another ω_3 convex function g_3 s.t. $g_3(x) = f_3(x)$ on sublevel sets $\{f_3 < c = f_1(p) - \delta\}$. In particular on such sublevel sets $g_3 \equiv f_1$. Furthermore g_3 has no critical pts whenever $g_3(x) \geq c$. This means sublevel sets $g_3 \equiv f_1$. Furthermore g_3 has no critical pts whenever $g_3(x) \geq c$. This means $\{g_3(x) \geq c\}$ is a symplectization. So we conclude that the completion of W_1 (the mfd we get from W_1 by attaching the positive end of a symplectization) is symplectom. to the completion of W_3. Summarised:

<u>Lemma 3.14</u> (Handle cancellation in successive handle attachment)
Let (W_1, ω_1) and (W_3, ω_3) be the symplectic mfds formed as above by successive handle attachment. Then the completions of (W_1, ω_1), (W_3, ω_3) are symplectomorphic. In particular M_1 and M_3 are contactomorphic.