2. Weinstein manifolds and open books

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Weinstein & Stein

Definition 2.1 Let $f \in C^\infty M$, $X \in \mathfrak{X}(M)$.
X is **gradient-like** for f if $X(f) > 0$ outside the critical pts. of f.

Definition 2.2 Let (W, ω) symplectic mfd and $f \in C^\infty(W, [0,\infty))$ proper. f is called **ω-convex** if it admits a complete gradient-like Liouville vector field. We call (W,ω) a **Weinstein mfd** if an ω-convex Morse function exists. We call it ~~if~~ **of finite type** if it has only finitely many critical pts.

Definition 2.5 Let (W, J) be an almost complex mfd. We call an $f \in C^\infty W$ **strictly plurisubharmonic** if

$$g(X,Y) := -d(df \circ J)(X, JY)$$

defines a Riemannian metric.

One checks in this case $\omega_f(x,y) := -d(df \circ J)$ defines a symplectic form.

Theorem (Grauert) Every Stein mfd admits a strictly plurisubharmonic function.

We can now see, that strictly plurisubharmonic functions on Stein mfds are ω_f-convex Morse functions:

One can solve (for X) the equation $i_X \omega_f = -df \circ J$ to obtain a Liouville VF that is gradient-like for f, since $Xf = df(X) = -df(J(JX)) = i_X\omega_f(JX) =$
$= \omega_f(X, JX) = g(X,X) \geq 0.$

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Definition A **compact Weinstein mfd** (Σ, ω) is a compact symplectic mfd with boundary K that can be embedded into a Weinstein mfd (W, ω) with an ω-convex function f s.t. $f^{-1}([0,C]) = \Sigma$ and C regular val of f.

Contact open books

Definition 2.7 An **abstract (contact) open book** (Σ, λ, Ψ) consists of a Liouville domain $(\Sigma, d\lambda)$ and a symplectomorphism $\Psi : \Sigma \to \Sigma$ with cpct support s.t. $\Psi^* d\lambda = d\lambda$ and Ψ is id near $\partial\Sigma$.

We now show that an abstract contact open book corresponds to a contact mfd with a supporting open book.

Lemma (Giroux) The symplectomorphism Ψ can be isotoped to a symplectomorphism $\hat{\Psi}$ that is id near the boundary and satisfies $\hat{\Psi}^* \lambda = \lambda - dh$ and h is a negative function.

Proof Let $\mu := \Psi^* \lambda - \lambda \in \Omega^1 \Sigma$. Since $d\lambda$ non-degenerate, we find $Y \in \mathfrak{X}\Sigma$ s.t. $i_Y d\lambda = -\mu$.

$\mathcal{L}_Y d\lambda = d i_Y d\lambda = -d\mu = 0 \Rightarrow$ the flow of Y preserves $d\lambda$ $\quad (Fl_t^{Y\,*} d\lambda = d\lambda)$.

Since $\Psi \equiv id$ near $\partial\Sigma$, we have $\mu \equiv 0, Y \equiv 0$ near $\partial\Sigma$.

Let's inspect $\hat{\Psi} = \Psi \circ Fl_1^Y$. $\hat{\Psi}$ is still id near $\partial\Sigma$ and it's a symplectomorphism:

$\hat{\Psi}^* d\lambda = Fl_1^{Y\,*} \Psi^* d\lambda = Fl_1^{Y\,*} d\lambda = d\lambda$.

$$\hat{\Psi}^* d\lambda = Fl_1^{Y*}\, \Psi^* d\lambda = Fl_1^{Y*} d\lambda = d\lambda .$$

We show $\hat{\Psi}^*\lambda - \lambda$ is exact.

$$\hat{\Psi}^*\lambda - \lambda = (\Psi \circ Fl_1^{Y})^*\lambda - \lambda = Fl_1^{Y*}\Psi^*\lambda - \lambda = Fl_1^{Y*}(\lambda + \mu) - \lambda =$$
$$= \mu + Fl_1^{Y*}\lambda - \lambda$$

$$Fl_1^{Y*}\lambda - \lambda = \int_0^1 \frac{d}{dt} Fl_t^{Y*}\lambda \, dt = \int_0^1 Fl_t^{Y*}(\mathcal{L}_Y \lambda)\, dt =$$
$$= \int_0^1 Fl_t^{Y*}(i_Y d\lambda + d i_Y \lambda)\, dt = -\mu + d\left(\int_0^1 Fl_t^{Y*}(\lambda(Y))\, dt\right).$$

$\square$

We can now define $A_{(\Sigma,\phi)} := \dfrac{\Sigma \times \mathbb{R}}{(x,t) \sim (\Psi(x), t + h(x))}$

This mapping torus carries the contact form $\alpha := \lambda + dt$. Since $\Psi \equiv id$ near $\partial\Sigma$, a nbhd of the bndry looks like $(-\frac{1}{2}, 0] \times \partial\Sigma \times S^1$ with the contact form $\alpha = e^r \lambda_{|\partial\Sigma} + dt$.
Denote the annulus $\{z \in \mathbb{C} \mid r < |z| < R\}$ by $A(r,R)$.
We can glue the mapping torus $A_{(\Sigma,\Psi)}$ along its boundary to $B_\Sigma := \partial\Sigma \times D^2$
using the map ϕ_{glue}

$$\partial\Sigma \times A(\tfrac{1}{2}, 1) \longrightarrow (-\tfrac{1}{2}, 0] \times \partial\Sigma \times S^1$$
$$(x, re^{i\varphi}) \longmapsto (\tfrac{1}{2} - r, x, \varphi)$$

Pulling back the form α by ϕ_{glue} we obtain
$$e^{\frac{1}{2}-r}\lambda_{|\partial\Sigma} + d\varphi \text{ on } \partial\Sigma \times A(\tfrac{1}{2}, 1)$$
which can be extended to a contact form
$\beta = h_1(r)\lambda_{|\partial\Sigma} + h_2(r)\, d\varphi$ on B_Σ.
We call $M := A_{(\Sigma,\Psi)} \cup_{\phi_{glue}} B_\Sigma$ an **abstract open book for M**. We call the resulting contact mfd a **contact open book** and denote it by $OB(\Sigma, \lambda; \Psi)$.

Definition 2.14 A (**concrete**) open book

Definition 2.14 A (concrete) open book on M is a pair (B, ϑ) where

(i) B is a codim-2 submfd of M with trivial normal bundle

(ii) $\vartheta : M \setminus B \to S^1$ endows $M \setminus B$ with the structure of a fiber bundle over S^1 s.t. ϑ gives the angular coordinate of the D^2-factor of a nbhd $B \times D^2$ of B.

We call B the binding, closure of a fiber of ϑ a page.

We can now relate abstract & concrete OBs. Let $OB(\Sigma, \lambda; \Psi)$ be the mfd stemming from an abstract OB. Away from the set B_Σ it has the structure of a fibration over S^1. Disregarding the contact structure we can rescale h to the constant function -2π, so we may put

$$\vartheta : OB(\Sigma, \Psi) - \partial\Sigma = A_{(\Sigma,\Psi)} \to S^1 = \mathbb{R}/2\pi\mathbb{Z}$$
$$[x,t] \mapsto [t]$$

Choose a Riemannian metric s.t. the fibers of ϑ are orthogonal to the VF ∂_φ on $\nu_M(B) \setminus B \cong B \times (D^2 \setminus \{0\})$ where D^2 has polar coordinates (r, φ). Extend this metric and define a connection by the rules:

- the vertical spaces Vert are tangent spaces to the fibers $\vartheta^{-1}([t])$
- the horizontal spaces Hor are the orthogonal complement wrt the metric.

To define the smooth monodromy we consider the loop $t \mapsto e^{it}$ in S^1. We lift the tangent VF to this loop, given by ∂_φ, to the horizontal

the loop $e^{i\theta}$ in S^1. We lift the tangent VF to this loop, given by ∂_φ, to the horizontal VF X_M on M. Note this VF $= \partial_\varphi$ near binding B. The smooth monodromy of the open book is the time 2π flow of this VF.
In other words the map Ψ is the monodromy. On the mapping torus we use the obvious lift to find $[x, 0] \mapsto [x, 2\pi] = [\Psi x, 0]$.

Suppose M is oriented with open book (B, ϑ). Since S^1 is oriented, each page inherits an orientation. If this orientation of a page Σ matches the orientation as a symplectic mfd, we call a symplectic form ω on Σ positive.
We shall orient B as the boundary of a page. If this orientation matches the one coming from a contact form α, then we say that α induces a positive contact structure.

<u>Definition 2.16</u> A positive contact structure ξ on an oriented mfd M is said to be carried by an open book (B, ϑ) if ξ admits a contact form satisfying:

(i) α induces a positive contact structure on B

(ii) $d\alpha$ induces a positive symplectic structure on each fiber of ϑ.

Such α is said to be adapted to (B, ϑ).

<u>Lemma 2.17</u> Suppose B is a connected contact submfd of a contact mfd (M, ξ). A contact form α for (M, ξ) is adapted to

contact submfd of a contact mfd (M, ξ).
A contact form α for (M,ξ) is adapted to an OB (B, ϑ) iff the Reeb VF R_α is positively transverse to the fibers of ϑ (i.e. $R_\alpha(\vartheta) > 0$).

<u>Proof</u> Let α be adapted. Let Σ be a page and $x \in \Sigma$. At x we find $v_1, \dots, v_{2n} \in T_x\Sigma$ s.t. $(d\alpha)^n(v_1, \dots, v_{2n}) > 0$. Hence $\alpha \wedge (d\alpha)^n (R, v_1, \dots, v_{2n}) > 0$. Since α adapted, $d\alpha$ is positive and this symplectic orientation matches the induced bundle orientation. Hence $v_1, \dots, v_{2n}$ orients every page s.t.
$\{S^1 \text{ direction}, v_1, \dots, v_{2n}\}$ is oriented for M.
Hence S^1, R_α point the same direction. (S^1 is by definition positively transverse to the fibers).
Conversely if R_α positively transverse to the fibers of ϑ, then $i_{R_\alpha}(\alpha \wedge (d\alpha)^n) > 0$ on each Σ. So $d\alpha$ positive symplectic. Note that

$$\int_{\partial\Sigma} \alpha \wedge (d\alpha)^{n-1} = \int_\Sigma d\alpha \wedge (d\alpha)^{n-1} = \int_\Sigma (d\alpha)^n > 0$$

Since B connected, we must have $\alpha \wedge (d\alpha)^{n-1} > 0$. $\square$

<u>Proposition 2.18</u> An abstract contact open book $OB(\Sigma, \Psi)$ admits a natural open book carrying the contact structure ξ.

<u>Proof</u> We define the binding of $OB(\Sigma, \Psi)$ to be $B := \partial\Sigma \times \{0\}$. The map ϑ is defined by

$$\underset{\in \partial\Sigma \times D^2}{(p; r, \varphi)} \longmapsto \varphi \quad \text{and} \quad \underset{\in A}{x} \longmapsto \underset{\in S^1}{\rho(x)}.$$

The Reeb VF of $OB(\Sigma, \Psi)$ is ∂_φ, so it is

$\partial\tilde{\Sigma}\times D^2$ $\tilde{A}$ $\tilde{S}^1$

The Reeb VF of $OB(\Sigma,\Psi)$ is ∂_φ, so it is positively transverse to all pages. $\square$

We now look at the converse.

<u>Proposition 2.19</u> Suppose (B,ϑ) is an OB on M carrying the contact structure ξ. Then there is a Liouville domain $(\tilde{\Sigma},\omega=d\lambda)$ and a symplectomorphism $\Psi:\tilde{\Sigma}\to\tilde{\Sigma}$ which is the identity near the boundary and M is contactomorphic to $OB(\tilde{\Sigma},\Psi)$. This Ψ represents the symplectic monodromy, which is well-defined up to symplectic isotopy rel. boundary.

<u>Lemma 2.20</u> Suppose (M^{2n+1},ξ) is a compact cooriented contact mfd with supporting open book (B,ϑ). Assume α contact form for ξ adapted to (B,ϑ). Then, after an isotopy through adapted contact forms, there is an embedding $B\times D^2 \xrightarrow{j} \nu_M B$ such that $j|_{B\times\{0\}}=id$ and $j^*\alpha = h_1(r)\alpha_B + h_2(r)\,d\varphi$ where h_i satisfy

(i) $h_2(r)=r^2$ near $r=0$, $h_1(r)>0$

(ii) $h_1^{n-1}(h_1h_2'-h_2h_1')>0$ for $r>0$

(iii) $h_1'<0$ for $r>0$

<u>Proof of 2.19</u> Choose a C^∞ increasing $f:[0,r_0]\to\mathbb{R}$ s.t. $f(r)=r$ near 0 and $f(r)=c$ constant near r_0. $M\setminus B$ fibers over S^1 with fiber Σ so we have inclusion maps $j_\varphi:\Sigma\to\vartheta^{-1}\varphi$.

Symplectic structure on fibers We claim $(\Sigma, j_\varphi^* d(\frac{1}{f(r)^2}\alpha))$ is exact symplectic. For $r \geq r_0$ f is constant, so it follows from the assumption of adaptedness. For small r, using the lemma, $d(h_1/f^2 \, \alpha_B)$ is a symplectic form, since h_1/f^2 decreasing.

Symplectic connection We define the connection by the following rule:

- the vertical bundle Vert is given by the tangent spaces to the fiber.
- the horizontal bundle Hor is given by

$$Hor = \{ v \in T(M \setminus B) \mid i_v d(\tfrac{1}{f^2}\alpha) = 0 \}$$

Consider the loop $t \mapsto e^{it} \in S^1$ and lift the tangent VF using this symplectic connection to get a horizontal VF X_h. The lemma tells us that $h_2 r = r^2$ near $r = 0$, which implies that $X_h = \partial_\varphi$ near the binding B. Hence the 2π flow is defined and equal to the identity near the boundary. The computation

$\frac{d}{dt} Fl_t^{X_h *} d(\frac{1}{f^2}\alpha) = 0$ shows the flow preserves the symplectic structure. This shows the 2π-flow is a symplectomorphism of $(\Sigma, j_\varphi^* d(\frac{1}{f^2}\alpha))$.

The fiber Σ is Liouville We want to show the Liouville VF X is complete.
X is given by $i_X j^* d(\frac{1}{f^2}\alpha) = j^*(\frac{1}{f^2}\alpha)$.
For $0 < r \leq r_0$ we obtain

$$X = \frac{f(r)^2 h_1(r)}{h_1'(r) f(r)^2 - 2 f(r) f'(r) h_1(r)} \partial_r = -\frac{r}{2}(1 + o(1)) \partial_r$$

We see that $dr(X) < 0$ so the Liouville VF is outward pointing at the subdomains we see that $dr(X) < 0$ so the Liouville VF is outward pointing at the subdomains

$$\Sigma_{r_1} = \{x \in \Sigma \mid r(x) \geq r_1\}.$$

Hence Σ_{r_1} is a Liouville domain. From the above formula we see that the flow of X exists for all time, hence Σ is is symplectomorphic to the completion of Σ_{r_1}. This implies that the symplectomorphism type of Σ doesn't depend on the choices made in the construction.

The symplectic monodromy is well defined Different horizontal lifts X_h corresponding to different choices of h_1, h_2 and f give rise to symplectomorphisms that are symplectically isotopic rel boundary. To see this, note that two choices can be linearily interpolated giving a 1-parameter family of horizontal vector fields X_h^s $s \in [0,1]$. Here it still holds $X_h^s = \partial_\varphi$ near B, so we get a 1-parameter family of symplectomorphisms ψ_s for (Σ, ω_s). By Moser stability we find symplectomorphisms $\varphi_s : (\Sigma, \omega_s) \to (\Sigma, \omega_0)$. Hence the maps $\varphi_s \psi_s \varphi_s^{-1}$ form the desired symplectic isotopy.