3. Weinstein handles

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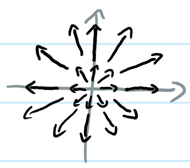
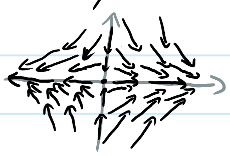
We build new Weinstein domains from old ones by attaching handles.
Consider $(\mathbb{R}^{2n}, \omega = \sum dx^i \wedge dy^i)$.
For $k \in \{0, \dots, n\}$ define

$$X_k := \sum_{j=1}^{n-k} \frac{1}{2}\left(x_j \partial_{x_j} + y_j \partial_{y_j}\right) + \sum_{j=n-k+1}^{n} \left(2x_j \partial_{x_j} - y_j \partial_{y_j}\right)$$

$n=1, k=0$ $\quad$ $n=1, k=1$





"X_k has k hyperbolic-type components in (x_j, y_j) direction for $j \in \{n-k+1, \dots, n\}$."
One can check X_k is a Liouville VF: $d\iota_{X_k}\omega = \omega$ and that X_k is the gradient of the Morse function

$$f_k := \sum_{j=1}^{n-k} \frac{1}{4}\left(x_j^2 + y_j^2\right) + \sum_{j=n-k+1}^{n} \left(x_j^2 - \frac{1}{2} y_j^2\right).$$

Hence f_k is gradient-like for f_k.

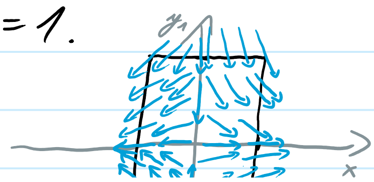
Consider closed unit k disk in $(y_{n-k+1}, \dots, y_n)$ plane. This disk is isotropic. For $\varepsilon > 0$ we consider the following "nbhd":

$$H_k^\varepsilon := \left\{\sum_{j=1}^{n} x_j^2 + \sum_{j=1}^{n-k} y_j^2 \le \varepsilon\right\} \cap \left\{\sum_{j=n-k+1}^{n} y_j^2 \le 1\right\}$$

H_k^ε is diffeomorphic to $D^{2n-k} \times D^k$, so we can view it as a k-handle. One can show

$$X_k\left(\sum_{j=1}^{n} x_j^2 + \sum_{j=1}^{n-k} y_j^2\right) > 0 \text{ and } X_k\left(\sum_{n-k+1}^{n} y_j^2\right) < 0.$$

So X_k flows inwards in the boundary component $D^{2n-k} \times S^{k-1}$ and outwards in $S^{2n-k-1} \times D^k$.
Let's draw the case $n=1, k=1$.



Definition 3.1.

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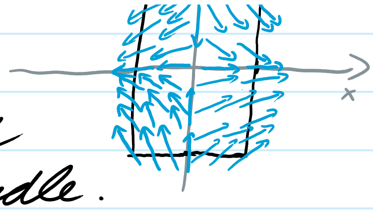

We call H_k^ε the standard $2n$-dimensional Weinstein k-handle. We call the handle critical if $k=n$, otherwise subcritical.

We would like to attach a Weinstein handle to a Liouville domain in a way that the Liouville vector fields glue together nicely to give a Liouville domain. The key to making this work is to use standard nhbd theorems of isotropic submflds.

Denote $\partial_- H_k^\varepsilon := D^{2n-k} \times S^{k-1}$

$\partial_+ H_k^\varepsilon := S^{2n-k-1} \times D^k$.

From chapter 2 we recall $\partial_- H_k^\varepsilon$ is a contact mfd with contact form $i_{X_k}\omega$.

Lemma 3.2 The attaching sphere $S = S^{k-1} = \{0\} \times S^{k-1}$ in $\partial_- H_k^\varepsilon$ is an isotropic submfd.

Theorem 2.5.8 (in Geiges) Let (M_i, ξ_i), $i \in \{0,1\}$ be contact mfds with closed isotropic submfds L_i. Suppose there is an isomorphism of conformal symplectic normal bundles $\Phi : CSN_{M_0} L_0 \to CSN_{M_1} L_1$ that covers a diffeomorphism $\phi : L_0 \to L_1$. Then this diffeo ϕ extends to a contactomorphism of neighborhoods of L_i.

Theorem 3.3 Let W^{2n} be a Liouville domain and S^{k-1} be an isotropic sphere $\subseteq \partial W$ with trivial conformal symplectic normal bundle. The mfd W' obtained by attaching an index k Weinstein handle along bundle. The mfd W' obtained by attaching an index k Weinstein handle along S^{k-1} admits the structure of a Liouville domain. Moreover if W is Weinstein, then W' is as well.