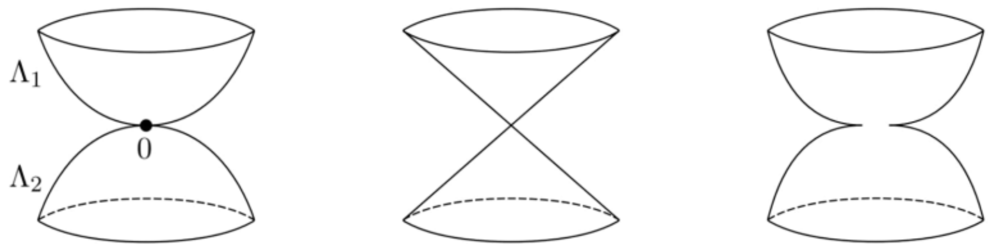
4.2.2

Mittwoch, 23. Oktober 2024 09:03

Lemma 4.2.2. *If U is above Λ, then $U \uplus \Lambda$ is Legendrian isotopic to Λ and $\Lambda \uplus U$ is a stabilization of Λ. If U is below Λ, then $U \uplus \Lambda$ is a stabilization of Λ and $L \uplus U$ is Legendrian isotopic to Λ.*

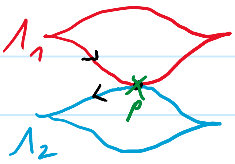
Proof. The assertions of the lemma are most easily verified using the front projection picture of the Legendrian sum (cf. Figure 4.1.1). If U is above Λ, then, by Figure 4.1.1(b), $U \uplus \Lambda$ is obtained from Λ by performing a generalized Reidemeister I move and is therefore Legendrian isotopic to Λ; by Figure 4.1.1(c), $\Lambda \uplus U$ is a stabilization of Λ. The case of U below Λ is analogous. □

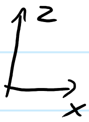


(a) (b) (c)

$\Lambda_1 \uplus \Lambda_2$ $\Lambda_2 \uplus \Lambda_1$

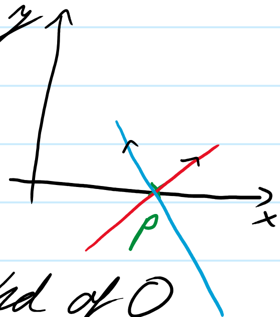
We draw the construction of $\Lambda_1 \uplus \Lambda_2$ in dim 3. Let Λ_1, Λ_2 be Legendrian unknots touching ξ-transversely in a point. (Everything lies in $(\mathbb{R}^3, dz - y\,dx)$)





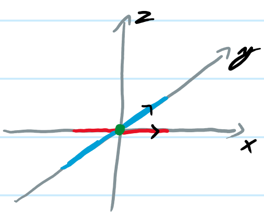
Looking at projection on xy plane it looks like

(We know this due to $y = \frac{z'}{x'}$).

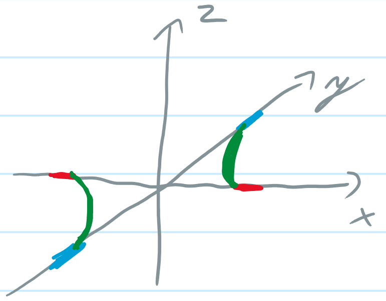


(*)

At p it is contactomorphic to nbhd of 0 in $(\mathbb{R}^3, dz - y\,dx)$ s.t. $\Lambda_1 \cap$ chart will be on x axis, $\Lambda_2 \cap$ chart will be on y axis

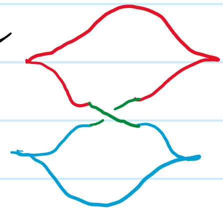


The construction changes this to



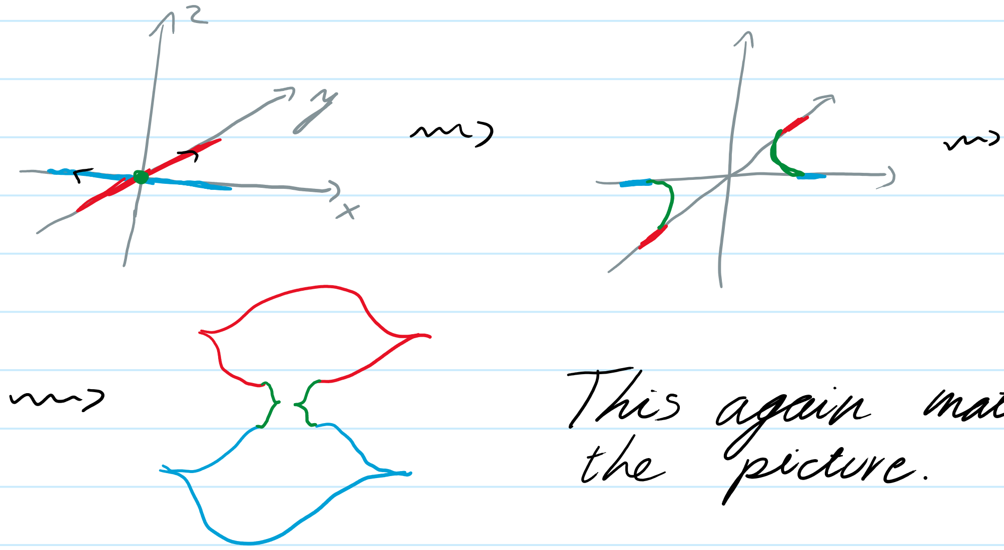
Now we have to orient.
M oriented, ξ cooriented
at p Λ_1, Λ_2 ξ-transverse.
If we orient Λ_1, we get
orientation automatically on Λ_2.

This changes the front projection to



which is what we expected.

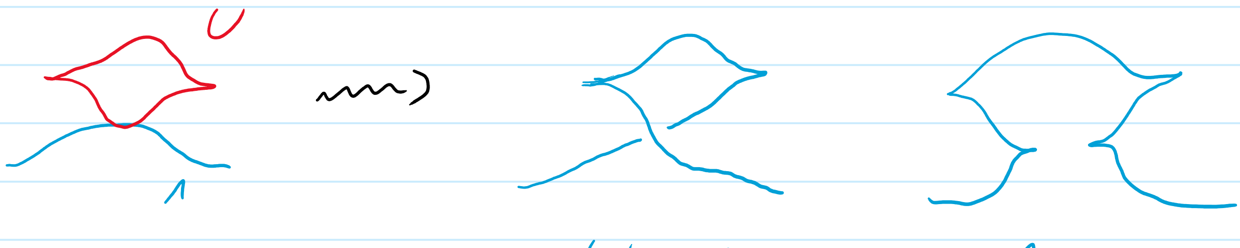
Now if we do $\Lambda_2 \oplus \Lambda_1$ we have



This again matches the picture.

Note if we change orientation of Λ_1, then Λ_2 also changes orientation and the picture remains the same.

In the case $M = \mathbb{R}^3$, the Reeb VF is ∂_z, so above means "more positive in z coordinate".
Let Λ Legendrian knot, U unknot, U above Λ.



$U \oplus \Lambda$ (Legendrian isotopic to Λ)

$\Lambda \oplus U$ Λ stabilized

$U \uplus \Lambda$ $\qquad\qquad$ $\Lambda \uplus U$

(Legendrian isotopic to Λ) $\qquad$ Λ stabilized twice

If Λ above U then things change since in (*) we assumed who is above, but it works out. In $U \uplus \Lambda$ we get right side turned upside down, in $\Lambda \uplus U$ we get left side turned upside down.

One can now think, if this is a sensible proof for higher dimensions if we replace $x, y \in \mathbb{R}$ by $\mathbf{x} = (x_1, \dots, x_n)$, $\mathbf{y} = (y_1, \dots, y_n)$