<u>Theorem 1.1.8</u> Let ξ_t, $t\in[0,1]$ be a smooth family of contact structures on a $(2n+1)$-dimensional closed manifold M and B a closed contact submanifold for all ξ_t. Then there is an isotopy Ψ_t of M satisfying $(\Psi_t)_*\xi_0 = \xi_1$ that leaves B invariant. If moreover $\xi_t \cap TB = \xi_0 \cap TB$ $\forall t$, then Ψ_t may be chosen to fix B pointwise.

<u>Proof</u> Let α_t be a smooth family of contact forms with $\operatorname{Ker}\alpha_t = \xi_t$. Assume Ψ_t is the time-dependent flow of the time-dependent VF $X_t \in \mathfrak{X}(M)$. Since M compact, the flow will exist for all time.
X_t can be uniquely decomposed $X_t = H_t R_t + Y_t$ where $H_t \in C^\infty M$, R_t the Reeb VF of α_t, $Y_t \in \xi_t$.
$(\Psi_t)_*\xi_0 = \xi_1$ is satisfied if $\exists\ \lambda_t \in C^\infty M$ s.t. $\Psi_t^*\alpha_t = \lambda_t\alpha_0$. Differentiating, we get

$$\Psi_t^*(\mu_t\alpha_t) = \frac{\dot\lambda_t}{\lambda_t}\lambda_t\alpha_0 = \dot\lambda_t\alpha_0 = \Psi_t^*(\dot\alpha_t + dH_t + i_{Y_t}d\alpha_t)$$

where $\mu_t = \left(\frac{d}{dt}\ln(\lambda_t)\right)\circ\Psi_t^{-1}$

Hence we solve $\dot\alpha_t + dH_t + i_{Y_t}d\alpha_t = \mu_t\alpha_t$ for Y_t, μ_t, H_t. Notice, given any H_t, we can uniquely solve for Y_t, μ_t. In the first part we find H_t s.t. Y_t is tangent to B.

Looking at the contact submfd B, we can solve the equation $\dot\alpha_{t|TB} + d\tilde H_t + i_{\tilde Y_t}(d\alpha_{t|TB}) = \tilde\mu_t\alpha_{t|TB}$ by choosing $\tilde H_t \equiv 0$. From this $\tilde Y_t, \tilde\mu_t$ are defined.
Define $\tilde\alpha_t = \dot\alpha_t + i_{\tilde Y_t}(d\alpha_t) - \tilde\mu_t\alpha_t : TM_{|B} \to C^\infty B$.

Define $\tilde{\alpha}_t = \dot{\alpha}_t + \iota_{\tilde{Y}_t}(d\alpha_t) - \tilde{\mu}_t \alpha_t : TM|_B \to C^\infty B$.

<u>Lemma</u> Let B be a closed submfd of M and $\alpha \in \Omega^1 M$ s.t. $\alpha|_{TB} \equiv 0$. Then $\exists f \in C^\infty M$ s.t. $df|_{TM|_B} \equiv \alpha|_{TM|_B}$.

Hence we find H_t s.t. $H_t|_B \equiv 0$, $dH_t|_{TM|_B} = -\tilde{\alpha}_t$. From this we get our solutions Y_t, μ_t and we see that $Y_t \equiv \tilde{Y}_t$ along B, so Y_t tangent to B.

If we moreover assume $\xi_t \cap TB = \xi_0 \cap TB \; \forall t$, then $\alpha_t|_{TB} = \eta_t \alpha_0|_{TB}$ for $\eta_t \in C^\infty B$. Then $\dot{\alpha}_t|_{TB} + \iota_{\tilde{Y}_t}(d\alpha_t|_{TB}) = \tilde{\mu}_t \alpha_t|_{TB}$ has solution $\tilde{\mu}_t = \frac{\dot{\eta}_t}{\eta_t}$, $\tilde{Y}_t = 0$, hence X_t fixes B ($H_t|_B = 0$, $Y_{t|B} = 0 \Rightarrow X_t = 0$). $\square$