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Lemma If S is an isotropic $(k-1)$-sphere in a contact manifold (M^{2n-1}, ξ) with trivial conformal symplectic normal bundle $\mathrm{CSN}(S)$, then in any open set containing S there is a standard neighborhood N with convex boundary $\partial N = S^{k-1} \times S^{2n-k-1}$ contactomorphic to an ϵ-neighborhood N_ϵ of the zero section Z in $J^1(S^{k-1}) \oplus \mathbb{R}^{2(n-k)}$. Here the contact form is $dz - \lambda + \sum x_i dy_i - y_i dx_i$. Moreover, under this contactomorphism, the dividing set on the boundary is $\Gamma_{\partial N} = \partial N_\epsilon \cap \{z = 0\} \cong S^{k-1} \times S^{2n-k-2}$ and $(\partial N)_\pm = (\partial N_\epsilon \cap \{\pm z > 0\})$ with ideal Liouville forms given by $d(\alpha/z)$.

Proof We already know that some neighborhoods of S and Z are contactomorphic, since both have trivial CSN. Now if we are given an open neighborhood of S, since $S \cong Z$ are compact, we can indeed take ϵ small enough such that N_ϵ fits into this open neighborhood. One can check that the vector field given by $\sum \tilde{\varphi}_i \partial_{\tilde{\varphi}_i}$ is a well defined vector field on the cotangent space of any manifold. Define

$$v := z\partial_z + \sum \tilde{\varphi}_i \partial_{\tilde{\varphi}_i} + \frac{1}{2} \sum_{i-1}^{n-k} x_i \partial_{x_i} + y_i \partial_{y_i}.$$

One can check that v is a contact vector field. One can identify $T^*S^{k-1} \times \mathbb{R}$ with $S^{k-1} \times \mathbb{R}^k$ to see that a neighborhood of our sphere is diffeomorphic to $S^{k-1} \times \mathbb{R}^{2n-k}$ and $\partial N_\epsilon = S^{k-1} \times S^{2n-k-1}$. Next, one computes that $i_v \alpha = z$. One can also show that the dividing curve is diffeomorphic to $S^{k-1} \times S^{2n-k-2}$.