2.3.5

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Lemma 2.3.5 (Legendrian realization)

Let $D^n \subseteq (R^{2n}_+, d\lambda_+)$ Lagrangian ($d\lambda|_D = 0$) that has cylindrical end over Legendrian sphere $\Lambda^{n-1} \subset \Gamma$.
Then $\exists$ convex surface $\Sigma' \subset \Sigma \times \mathbb{R}$ s.t.

(1) Σ' graphical over $\Sigma \times \{0\}$, $\Sigma' \pitchfork \partial_t$

(2) $\Sigma' \equiv \Sigma \times \{0\}$ in nbhd of Γ

(3) lift $\tilde{D} \subset \Sigma'$ of D is Legendrian in $\Sigma \times \mathbb{R}$.

Thoughts

- What is the contact form on Γ?

λ here β/f from the notation of previous section.
Let $(R^{2n}, d\lambda)$ be Liouville with Liouville VF $X \pitchfork \partial R$.
Then let's look at $\lambda \wedge (d\lambda)^{n-1}|_{\partial R}$.

$0 \neq i_X (d\lambda)^n_{|\partial R} = n\, i_X d\lambda \wedge (d\lambda)^{n-1}_{|\partial R} = n\, \lambda \wedge (d\lambda)^{n-1}$. ✓

The symplectic structure of the cylindrical end is: $N := ([c, \infty] \times \Gamma, d(e^s \lambda|_\Gamma))$ where $\Gamma = \{\infty\} \times \Gamma$.

- Why is cylindrical end with Legendrian sphere Lagrangian?

Legendrian means $\lambda|_\Lambda = 0$ and Λ is $n-1$ dimensional.
$[c, \infty) \times \Lambda$ is Lagrangian if $d(e^s\lambda)|_{[c,\infty)\times\Lambda} = 0$.

$d(e^s\lambda) = e^s(ds \wedge \lambda + d\lambda)$. Let $v_1, v_2 \in T\Lambda$.
Then $(ds \wedge \lambda)(v_1, v_2) = 0$, $d\lambda(v_1, v_2) = 0$.
If $v \in T\lambda$ then $d\lambda(\partial_s, v) = 0$, $(ds \wedge \lambda)(\partial_s, v) = 0$. ✓

- Why is $[c, \infty) \times \Lambda$ Legendrian in $R_+ \times \mathbb{R}$?

The contact form is $dt + \lambda$, $dt + \lambda|_{[c,\infty)\times\Lambda} = 0$. ✓

<u>Proof</u> Let $\tilde{D}|_U$ the lift of D over subset $U \subset D$.
Set $\tilde{D}|_{D\cap N} := D \cap N$. Let $p \in \partial D'$ where $D' = D \setminus N$.

Proof Let $\tilde{D}|_U$ the lift of D over subset $U \subset D$.
Set $\tilde{D}|_{D \cap N} := D \cap N$. Let $p \in \partial D'$ where $D' = D \setminus N$.
Let $q \in D'$. Let γ path from p to q. Then define
$\tilde{D}|_{\{q\}} = \left(-\int_\gamma \lambda, q\right)$.

• Why is $\int_\gamma \lambda$ well defined, independent of γ?
Let $(X^{2n}, d\lambda)$ symplectic $D^n \subseteq X$ Lagrangian. Let γ loop.
$[0,1] \xrightarrow{\gamma} D$. Then $\int \lambda = 0$? This can be reduced to: given D^n a disk, λ a closed form. Then $\int_\gamma \lambda$ depends on endpts only. True since λ is exact. ✓

• Why $\tilde{D}$ Legendrian? How does a tangent vector in $\tilde{D}$ look like? It is $-\lambda(v)\partial_t + v$ where $v \in TD$.

$$(\lambda + dt)(-\lambda(v)\partial_t + v) = 0 \quad ✓$$

So we have constructed $\tilde{D}$ s.t. (3) holds.
Can we extend the lifting to Σ?
$\tilde{D}$ is created by flow of some VF $g\partial_t$ where $g \in C^\infty D$.
Extend g to Σ. $\square$