2.1.13

Samstag, 2. November 2024 15:20

Exercise 2.1.13 Show that if W is an isotropic, coisotropic or symplectic subspace of a symplectic vector space (V, ω), then any standard basis for (W, ω) extends to a symplectic basis for (V, ω). □

Let $\dim V = 2n$.

Case W symplectic:

By lemma 2.1.1, $(W^\omega, \omega|_{W^\omega})$ symplectic. Hence we can find a symplectic basis for W^ω. The union of any standard basis for W and W^ω is a standard basis for V.

Case W isotropic:

A standard basis for W is of the form $\{a_1, \dots, a_k\}$, $k \le n$ where $\omega(a_i, a_j) = 0 \ \forall i,j$.

Assume we have constructed $\{b_1, \dots, b_\ell\} \in V$ s.t.

$$\left.\begin{array}{l}\{a_1, \dots, a_k, b_1, \dots, b_\ell\} \text{ lin. independent}, \ \ell \in \{0, \dots, k-1\}, \\ \omega(a_i, b_j) = \delta_{ij}, \ \omega(b_i, b_j) = 0, \ \omega(a_i, a_j) = 0.\end{array}\right\} (*)$$

We find $b_{\ell+1}$ s.t. $(*)$ holds.

$E := \langle a_1, \dots, \hat{a}_{\ell+1}, \dots, a_k, b_1, \dots, b_\ell \rangle$ has dimension $k+\ell-1 \le 2n-2 < 2n$ so E^ω has nonzero dimension. Since $a_{\ell+1} \in E^\omega$, $\exists \ b_{\ell+1} \in E^\omega$ s.t. $\omega(a_{\ell+1}, b_{\ell+1}) = 1$.

This way we get symplectic subspace $\langle a_1, \dots, a_k, b_1, \dots, b_k \rangle$. We are finished by previous case.

Case W coisotropic:

We use the same construction as in the previous case.

□