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In Geiges

Lemma 5.2.4 Let M_i, $i \in \{0,1\}$ be compact hypersurfaces of contact type in symplectic mfds (W_i, ω_i) with corresponding Liouville VFs Y_i. Let there be a strict contactomorphism $\phi: (M_0, i_{Y_0}\omega_0) \to (M_1, i_{Y_1}\omega_1)$. Then ϕ can be extended to a symplectomorphism of neighborhoods of M_0 and M_1.

Proof Define $\varphi: (-\varepsilon, \varepsilon) \times M \longrightarrow W$

$$(t, x) \mapsto Fl_t^Y x$$

This is a diffeo onto it's image for small enough ε.

$\varphi^*\omega$ is a symplectic form on $(-\varepsilon,\varepsilon)\times M$.

$$\mathcal{L}_{\partial_t}(\varphi^*\omega) = d i_{\partial_t}(\varphi^*\omega) = d\varphi^* i_Y \omega = \varphi^* d i_Y \omega = \varphi^*\omega$$

$$\frac{d}{dt}\Big|_{t=t_0} Fl_t^{\partial_t *}(\varphi^*\omega) = Fl_{t_0}^{\partial_t *}(\mathcal{L}_{\partial_t}\varphi^*\omega) = Fl_{t_0}^{\partial_t *}(\varphi^*\omega)$$

So $t \mapsto (Fl_t^{\partial_t *}\varphi^*\omega)(t_0, x)$ is a curve in a vector space where it's derivative is itself $\Rightarrow$

$$Fl_t^{\partial_t *}\varphi^*\omega = e^t(\varphi^*\omega) \Rightarrow$$

$$(\varphi^*\omega)(t,x)(a_1\partial_t + v_1, a_2\partial_2 + v_2) = e^t(\varphi^*\omega)(0,x)(a_1\partial_t + v_1, a_2\partial_t + v_2),$$

since $T_0\mathbb{R} \times T_x M \xrightarrow{TFl_t^{\partial_t}} T_t\mathbb{R} \times T_x M$ is just "id" $\times$ id

Hence $(\varphi^*\omega)(t,x) = e^t(\varphi^*\omega)(0,x)$.

$$(\varphi^*\omega)(t,x) = (d i_{\partial_t}\varphi^*\omega)(t,x) = (d(i_{\partial_t} e^t \varphi^*\omega))(0,x) = d(e^t\varphi^* i_Y\omega)(0,X).$$

So we see, a nbhd of M is symplectomorphic to the symplectization of $(M, i_Y\omega)$. Hence when the M_i are are strictly contactomorphic, then also their symplectizations are symplectomorphic. $\square$