2.1.15

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Exercise 2.1.15 Recall that $\Omega(V)$ denotes the space of all symplectic forms on the vector space V. Consider the (covariant) action of the general linear group $\mathrm{GL}(2n,\mathbb{R})$ on $\Omega(V)$ via $\mathrm{GL}(2n,\mathbb{R}) \times \Omega(V) \to \Omega(V) : (\Psi,\omega) \mapsto (\Psi^{-1})^*\omega$ and show that $\Omega(V)$ is homeomorphic to the homogeneous space $\mathrm{GL}(2n,\mathbb{R})/\mathrm{Sp}(2n)$. □

We show $\frac{GL(2n)}{Sp(2n)}$ is homeomorphic to $\Omega(\mathbb{R}^{2n})$.

Identify $\Omega(\mathbb{R}^{2n})$ with the skew symmetric invertable matrices. Then $J_0 = \begin{pmatrix} 0 & -\mathbb{1}_n \\ \mathbb{1}_n & 0 \end{pmatrix}$ is the standard symplectic form.

Consider
$$\begin{aligned} GL(2n) &\longrightarrow \Omega(\mathbb{R}^{2n}). \\ A &\longmapsto A^T J_0 A \end{aligned}$$

This is surjective because of theorem 2.1.3.
It is obviously continuous.
Passing to quotient, we get
$$\frac{GL}{Sp} \longrightarrow \Omega$$
continuous and bijective.
We show $GL \to \Omega$ is open, hence $GL/_{Sp} \to \Omega$ will be a homeom.

Since GL open subset of $\mathbb{R}^{n^2}$ and Ω of $\mathbb{R}^{\frac{n(n-1)}{2}}$, we show the image of the tangent map is surjective at every point.
The vector in direction B at point A gets sent to $B^T J_0 A + A^T J_0 B$.
To show surjectivity, we take C skew symmetric.
Then take $B := \frac{1}{2} J_0^{-1} (A^{-1})^T C$. □

If the last part is unclear, check the proof that $O(n)$ is a Lie group.