Exercise 2.1.2

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Exercise 2.1.2 Let (V, ω) be a symplectic vector space and $\Psi : V \to V$ be a linear map. Prove that Ψ is a linear symplectomorphism if and only if its graph

$$\Gamma_\Psi = \{(v, \Psi v) \mid v \in V\}$$

is a Lagrangian subspace of $V \times V$ with symplectic form

$$(-\omega) \oplus \omega := \mathrm{pr}_2^* \omega - \mathrm{pr}_1^* \omega.$$

Let Ψ be a symplectomorphism.

Then $\omega(\Psi v, \Psi w) = \omega(v, w)$ $\forall v, w \in V$.

$(-\omega \oplus \omega)((v, \Psi v), (w, \Psi w)) = \omega(\Psi v, \Psi w) - \omega(v, w) = 0$

$\forall v, w \implies (-\omega \oplus \omega)|_{\Gamma_\Psi} = 0$.

Since Γ_Ψ has half the dimension of $(V \times V, (-\omega) \oplus \omega)$ and the symplectic form vanishes on it, Γ_Ψ is Lagrangian (Lemma 2.1.1)

Assume Γ_Ψ Lagrangian. Then from above we see $\omega(\Psi v, \Psi w) = \omega(v, w)$ $\forall v, w \in V$. It remains to show Ψ is an isomorphism.

Assume $\Psi(v) = 0$ for some $v \neq 0$.

Then $\exists w \neq 0$ s.t. $\omega(v, w) \neq 0$.

But $0 = \omega(0, \Psi w) = \omega(\Psi v, \Psi w) = \omega(v, w) \neq 0$ ↯.

$\square$