Calc2 Notes Draft

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Contents

1 Integration By Part **4**

2 Trig **6**
- 2.1 Trigonometric Formulas . . . 6
- 2.2 Application In Integration . . . 6
- 2.3 Inverse Trigonometry . . . 7
- 2.4 Reduction Formula . . . 7
- 2.5 Double Angle Formula . . . 7
- 2.6 Products Of Sin and Cos . . . 7
- 2.7 Sub Into Trig . . . 8

3 Partial Fraction **12**

4 Improper Integral **14**
- 4.1 Normal Infinite Integral . . . 14
- 4.2 Double Infinite Integral . . . 14
- 4.3 Discontinuity . . . 14
- 4.4 Comparison Test . . . 14

5 Application of Integration **16**

6 First Order ODE **17**
- 6.1 Separable ODE . . . 17
- 6.2 Linear ODE . . . 17

7 Second Order Differential equation **19**
- 7.1 Step 1: Find solution to homogeneous equation . . . 19
- 7.2 Step 2: Find solution to non-homogeneous equation . . . 19

8 Applications of Second Order Differential Equations **22**
8.1 Simple Oscillation 22
8.2 Damped Oscillation 22
8.3 Forced Oscillation 22
8.4 Circuit 22

9 Series **23**
9.1 Divergence Test 23
9.2 Geometric Series 23
9.3 p-series 23
9.4 Comparison Test 23
9.5 Integral Test 26
9.6 Alternating Series 27
9.7 Ratio Test 27
9.8 Root Test 28
9.9 Checklist of Tests 28
9.10 Power Series 29
9.11 Term by Term Differentiation and Integration 31
9.12 Taylor Series 31
9.12.1 Function to Taylor Series 31
9.12.2 Visualization of Taylor Series 32
9.13 Estimation using Series 33
9.13.1 Alternating Series 33
9.13.2 Taylor Series 35

10 Parametric Equation **37**
10.1 Eliminate the parameter 37
10.2 Derivative of a parametric equation 37
10.3 Second derivative of a parametric equation 37
10.4 Area of a parametric equation 38
10.5 Length of parametric equation curve 38

11 Polar Form **39**
11.1 Derivative of Polar Form 39
11.2 Switching to coordinates 39
11.3 Switching from coordinates 39
11.4 Polar Equation That You Need To Remember 39
11.4.1 Lines 39
11.4.2 Cardioid 40
11.5 Area of Polar Form 41
11.5.1 Area of one polar equation 41
11.5.2 Area between two polar equations 41
11.6 Curve Length of Polar Form 41

12 Vector **42**
12.1 Magnitude of Vector 42
12.2 Unit vector 42
12.3 Principal/Unit Normal Vector 42

12.4 Binormal Vector 42
12.5 Arc Length 42
12.6 Arc Length Function 42
12.7 Curvature 44
12.8 Acceleration 44

1 Integration By Part

$$\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx \tag{1}$$

Simple Example 1

Let try to integrate:

$$\int (x+1)\cos(x)$$

Let:

$$f(x) = x+1 \qquad g'(x) = cos(x)$$

So that:

$$f'(x) = 1 \qquad g(x) = \sin(x)$$

$$\begin{aligned}\int (x+1)\cos(x) &= (x+1) * \sin(x) - \int 1 * \sin(x)\ dx \\ &= (x+1) * \sin(x) - \int \sin(x)\ dx \\ &= (x+1) * \sin(x) + \cos(x)\end{aligned}$$

Simple Example 2

A weird case that you'll probably learn:

$$\int e^x \sin(x)\ dx$$

Let:

$$f(x) = e^x \qquad g'(x) = \sin(x)$$

So that:

$$f'(x) = e^x \qquad g(x) = -\cos(x)$$

Sub everything into the formula:

$$\int e^x \sin(x)\ dx = e^x * -cos(x) - \int e^x * -\cos(x)\ dx$$
$$\int e^x \sin(x)\ dx = -e^x * \cos(x) + \int e^x * cos(x)\ dx$$

Let:

$$f(x) = e^x \qquad\qquad g'(x) = \cos(x)$$

So that:

$$f'(x) = e^x \qquad\qquad g(x) = \sin(x)$$

$$\begin{aligned}
\int e^x \sin(x)\ dx &= -e^x * \cos(x) + \int e^x * cos(x)\ dx \\
\int e^x \sin(x)\ dx &= -e^x * \cos(x) + \left[e^x * \sin(x) - \int e^x * \sin(x)\ dx\right] \\
\int e^x \sin(x)\ dx &= -e^x * \cos(x) + e^x * \sin(x) - \int e^x * \sin(x)\ dx \\
2\int e^x \sin(x)\ dx &= -e^x * \cos(x) + e^x * \sin(x) \\
\int e^x \sin(x)\ dx &= -\frac{1}{2}e^x * \cos(x) + \frac{1}{2}e^x * \sin(x)
\end{aligned}$$

Theorem 1: LIATE

Here's a quick checklist when picking f(x)

1. L-Logarithmic
2. I-Inverse Trigonometric
3. A-Algebra
4. T-Trigonometric
5. E-Exponential

L is your most preferred f(x) and E is the worst choice.[1]

[1] From my experience, you almost never have to pick L or I. T and E are exchangeable, pick the one that makes solving the question easier

2 Trig

2.1 Trigonometric Formulas

$$sin^2 x = \frac{1 - \cos 2x}{2} \quad (2) \qquad cos^2 x = \frac{1 + \cos 2x}{2} \quad (3)$$

Simple Example 3

Find

$$\int \sin^2(x)dx$$

using trigonometric formulas 2:

$$\begin{aligned}\int \sin^2(x)dx &= \int \frac{1-\cos(2x)}{2}dx \\ &= \frac{1}{2}\int(1-\cos(2x))dx \\ &= \frac{1}{2}\left[\int 1dx - \int \cos(2x)dx\right] \\ &= \frac{1}{2}\left[x - \int \cos(2x)dx\right]\end{aligned}$$

Let $u = 2x$, so that $\frac{du}{dx} = 2$

$$\begin{aligned}\int \cos(2x)dx &= \frac{1}{2}\int \cos(u)du \\ &= \frac{1}{2}\sin(u) \\ &= \frac{1}{2}\sin(2x) + C\end{aligned}$$

Back the the equation above:

$$\begin{aligned}&= \frac{1}{2}\left[x - \int \cos(2x)dx\right] \\ &= \frac{1}{2}\left[x - \frac{1}{2}\sin(2x)\right] + C \\ &= \frac{1}{2}x - \frac{1}{4}\sin(2x) + C\end{aligned}$$

2.2 Application In Integration

$$\int \sin^2 x\ dx = \frac{1}{2}x - \frac{1}{4}\sin 2x + C \tag{4}$$

$$\int \cos^2 x\ dx = \frac{1}{2}x + \frac{1}{4}\sin 2x + C \tag{5}$$

It is strongly recommended that you remember these two equations.

2.3 Inverse Trigonometry

$$\int \frac{1}{\sqrt{a^2 - u^2}}\, du = \sin^{-1}\left(\frac{u}{a}\right) + C \tag{6}$$

$$\int \frac{1}{a^2 + u^2}\, du = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C \tag{7}$$

$$\int \frac{1}{u\sqrt{u^2 - a^2}}\, du = \frac{1}{a}\sec^{-1}\left(\frac{u}{a}\right) + C \tag{8}$$

2.4 Reduction Formula

$$\int \sin^n x\ dx = \frac{1}{n}\cos x \sin^{n-1} x + \frac{n-1}{n}\int \sin^{n-2} x\ dx \tag{9}$$

$$\int \cos^n x\ dx = \frac{1}{n}\sin x \cos^{n-1} x + \frac{n-1}{n}\int \cos^{n-2} x\ dx \tag{10}$$

Simple Example 4

Find

$$\int \sin^4(x) dx$$

using reduction formula
Solution:

$$\begin{aligned}
\int \sin^4(x) &= \frac{1}{4}\cos(x)\sin^{4-1}(x) + \frac{4-1}{4}\int \sin^{4-2}(x) dx \\
&= \frac{1}{4}\cos(x)\sin^3(x) + \frac{3}{4}\int \sin^2(x) dx \\
&= \frac{1}{4}\cos(x)\sin^3(x) + \frac{3}{4} * \left(\frac{1}{2}x - \frac{1}{4}\sin(2x)\right) + C \\
&= \frac{1}{4}\cos(x)\sin^3(x) + \frac{3}{8}x - \frac{3}{16}\sin(2x) + C
\end{aligned}$$

2.5 Double Angle Formula

$$2\sin\theta\cos\theta \;=\; \sin 2\theta \tag{11}$$

2.6 Products Of Sin and Cos

Let A and B represent real number

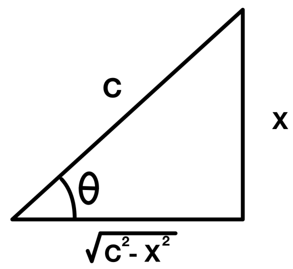
$$\sin(Ax)\cos(Bx) = \frac{1}{2}(\sin(Ax - Bx) + \sin(Ax + B)) \tag{12}$$

$$\sin(Ax)\sin(Bx) = \frac{1}{2}(\cos(Ax - Bx) - \cos(Ax + Bx)) \tag{13}$$

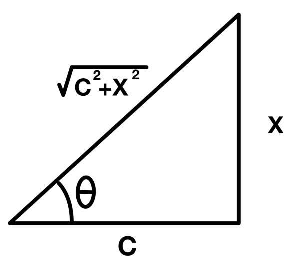
$$\cos(Ax)\cos(Bx) = \frac{1}{2}(\cos(Ax - Bx) + cos(Ax + Bx)) \tag{14}$$

2.7 Sub Into Trig

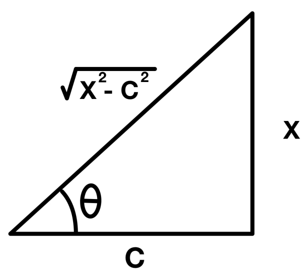
$$\sqrt{c^2 - x^2} \qquad Let\ x = c\sin(\theta) \qquad \sqrt{c^2 - x^2} = c\cos(\theta) \tag{15}$$



$$\sqrt{c^2 + x^2} \qquad Let\ x = c\tan(\theta) \qquad \sqrt{c^2 + x^2} = c\sec(\theta) \tag{16}$$



$$\sqrt{x^2 - c^2} \qquad Let\ x = c\sec(\theta) \qquad \sqrt{x^2 - c^2} = c\tan(\theta) \tag{17}$$



Simple Example 5

Find

$$\int \sqrt{25 - x^2}$$

Let c=5, so that

$$\int \sqrt{25 - x^2} = \int \sqrt{5^2 - x^2}$$

Let $x = 5\sin(\theta)$ and $\sqrt{5^2 - x^2} = 5\cos(\theta)$ so that $\dfrac{dx}{d\theta} = 5\cos(\theta)$ and $\theta = \sin^- 1(\frac{x}{5})$

$$\begin{aligned}
\int \sqrt{5^2 - x^2} dx &= \int 5\cos(\theta) * 5\cos(\theta) d\theta \\
&= 25 \int \cos^2(\theta) \\
&= 25 * \left(\frac{1}{2}\theta + \frac{1}{4}\sin(2\theta) \right) \\
&= 25 * \left(\frac{1}{2}\theta + \frac{1}{4} * 2 * \sin(\theta) * \cos(\theta) \right)
\end{aligned}$$

Look at the triangle from theorem 15, we can see that $\sin(\theta) = \frac{x}{c}$ and $\cos(\theta) = \frac{\sqrt{5^2 - x^2}}{5}$

$$\begin{aligned}
25\left(\frac{1}{2}\theta + \frac{1}{4} * 2 * \sin(x) * \cos(\theta) \right) &= 25\left(\frac{1}{2}\sin^{-1}(x) + \frac{1}{2} * \left(\frac{x}{5}\right) * \left(\frac{\sqrt{5^2 - x^2}}{5} \right) \right) \\
&= 25\left(\frac{1}{2}\sin^{-1}(\frac{x}{5}) + \frac{1}{50} x\sqrt{25 - x^2} \right)
\end{aligned}$$

Challenging Example 1

Find

$$\int x\sqrt{16 - 2x^2} dx$$

Let $c = 4$ and $u = \sqrt{2}x$, so that $\dfrac{du}{dx} = \sqrt{2}$

$$\int x\sqrt{16 - 2x^2} dx = \frac{1}{2}\int u\sqrt{4^2 - u^2} du$$

Let $u = 4\sin(\theta)$ and $\sqrt{4^2 - u^2} = 4\cos(\theta)$
so that $\dfrac{du}{d\theta} = 4\cos(\theta)$

$$\begin{aligned}
\frac{1}{2}\int u\sqrt{4^2 - u^2} du &= \frac{1}{2}\int 4\sin(\theta) * 4\cos(\theta) * 4\cos(\theta) d\theta \\
&= 32 \int \sin(\theta)\cos^2(\theta) d\theta
\end{aligned}$$

Let $v = \cos(\theta)$, so that $\frac{dv}{d\theta} = -\sin(\theta)$

$$\begin{aligned} 32\int \sin(\theta)\cos^2(\theta)d\theta &= -32\int v^2 dv \\ &= -32 * \frac{1}{3}v^3 + C \\ &= -\frac{32}{3}v^3 + C \\ &= -\frac{32}{3}\cos^3(\theta) + C \end{aligned}$$

Looking at the image of equation 15, we observe that

$$\cos(\theta) = \frac{\sqrt{c^2 - x^2}}{c}$$

$$\begin{aligned} -\frac{32}{3}\cos^3(\theta) &= -\frac{32}{3} * \left(\frac{\sqrt{4^2 - u^2}}{4}\right)^3 \\ &= -\frac{32}{3} * \left(\frac{\sqrt{16 - 2x^2}}{4}\right)^3 \\ &= -\frac{1}{6} * \left(\sqrt{16 - 2x^2}\right)^3 \end{aligned}$$

or use u-substitution lol.

Challenging Example 2

Find

$$\int \frac{x^2}{(4 - x^2)^{\frac{3}{2}}} dx$$

Let $x = 2\sin(\theta)$ and $\sqrt{4 - x^2} = 2\cos(\theta)$
so that $\theta = \sin^{-1}(x)$, $\dfrac{dx}{d\theta} = 2\cos(\theta)$

$$\begin{aligned}
\int \frac{x^2}{(4-x^2)^{\frac{3}{2}}}dx &= \int \frac{4\sin^2(\theta)}{8\cos^3(\theta)} * 2\cos(\theta)d\theta \\
&= \int \frac{8\sin^2(\theta)\cos(\theta)}{8\cos^3(\theta)}d\theta \\
&= \int \frac{\sin^2(\theta)}{\cos^2(\theta)}d\theta \\
&= \int \tan^2(\theta)d\theta \\
&= \int \left(\sec^2(\theta) - 1\right)d\theta \\
&= \tan(\theta) - \theta + C
\end{aligned}$$

Take a look back into the triangle in equation 15, we notice that $\tan(\theta) = \dfrac{x}{\sqrt{4-x^2}}$

$$\tan(\theta) - \theta + C = \tan(\theta) = \frac{x}{\sqrt{4-x^2}} - \sin^{-1}\left(\frac{x}{2}\right) + C$$

3 Partial Fraction

Checklist:

1. Check for long division
2. Factor
3. Write out all variables...
4. Cross multiply
5. Isolate for x, $x^2 \ldots$
6. Linear algebra
7. Sub everything back in

Simple Example 6

Find:

$$\int \frac{x+5}{x^2+3x+2}$$

1. Check for long division:

   We can see that no long division is required.

2. Factor:
   $$x^2+3x+2=(x+1)(x+2)$$

3. Write out all variables
   $$\frac{x+5}{(x+1)(x+2)}=\frac{A}{x+1}+\frac{B}{x+2}$$

4. Cross multiply
   $$x+5=A(x+2)+B(x+1)$$

5. Isolate for x, $x^2 \ldots$
   $$x+5=Ax+2A+Bx+B$$
   $$x+5=x(A+B)+2A+B$$

6. Linear Algebra
   $$=\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 2 & 1 & -5 \end{array}\right]$$
   $$=\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & -1 & -7 \end{array}\right]$$
   $$=\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 1 & 7 \end{array}\right]$$
   $$=\left[\begin{array}{cc|c} 1 & 0 & -6 \\ 0 & 1 & 7 \end{array}\right]$$

   A=-6 B=7

7. Sub everything back in

$$\begin{aligned}
\int \frac{x+5}{(x+1)(x+2)} &= \int \frac{A}{x+1} + \int \frac{B}{x+2} \\
&= \int \frac{-6}{x+1} + \int \frac{7}{x+2} \\
&= -6 \int \frac{1}{x+1} + 7 \int \frac{1}{x+2} \\
&= -6 \ln{(x+1)} + 7 \ln{(x+2)} + C
\end{aligned}$$

4 Improper Integral

4.1 Normal Infinite Integral

Formula:

$$\int_c^\infty f(x) = \lim_{t\to-\infty} \int_c^t f(x) \tag{18}$$

$$\int_{-\infty}^c f(x) = \lim_{t\to-\infty} \int_t^c f(x) \tag{19}$$

c can be any constant.

Simple Example 7

Find:

$$\begin{aligned}
\int_1^\infty \frac{1}{x^2} &= \lim_{t\to\infty} \int_1^t \frac{1}{x^2}\,dx \\
&= \lim_{t\to\infty} \frac{1}{x^2}\Big|_1^t \\
&= \lim_{t\to\infty} \left(\frac{1}{t^2} + \frac{1}{1}\right) \\
&= \frac{1}{\infty} + \frac{1}{1} \\
&= 0 + 1 \\
&= 1
\end{aligned}$$

4.2 Double Infinite Integral

Formula:

$$\int_{-\infty}^\infty f(x) = \lim_{t\to\infty} \int_c^t f(x) + \lim_{t\to-\infty} \int_t^c f(x) \tag{20}$$

c can be any constant.

4.3 Discontinuity

Given f(x) is continuous at all points between a and b except at x=c

$$\int_a^b f(x)\,dx = \lim_{t\to c^+} \int_t^b f(x) + \lim_{t\to c^-} \int_a^t f(x)\,dx \tag{21}$$

4.4 Comparison Test

Theorem 2

If $f_1(x)$ is always bigger than $f_2(x)$, then if $f_2(x)$ diverges, $f_1(x)$ must diverge.
If $f_2(x)$ is always smaller than $f_1(x)$, then if $f_1(x)$ converge, $f_2(x)$ must converge.

Simple Example 8

In figure 1 blue line represents $\dfrac{1}{\sqrt{x}}$ and red line represents $\dfrac{1}{x}$. Since $\dfrac{1}{x}$ diverges[3] $\dfrac{1}{\sqrt{x}}$ has to diverge.

Figure 1

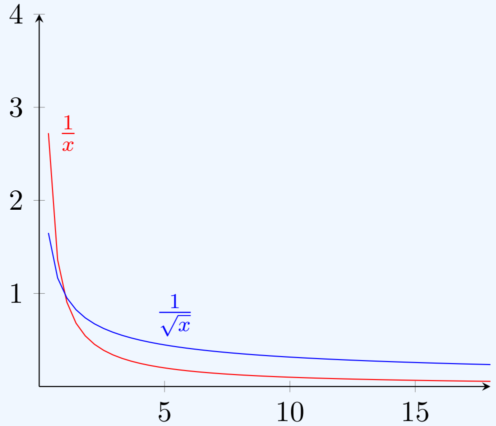
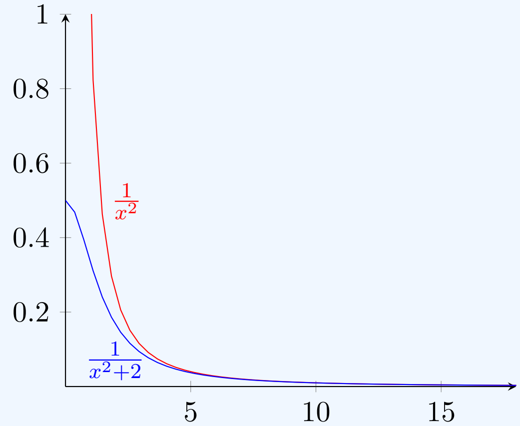


Figure 2



While in figure 2 blue line represents $\dfrac{1}{x^2+2}$ and red line represents $\dfrac{1}{x^2}$. Since $\dfrac{1}{x^2}$ converges[1] $\dfrac{1}{x^2+2}$ has to converge.

[1] Because of p-series

5 Application of Integration

1. Method of disk(x axis):
$$\int_a^b \pi(f(x))^2 \, dx \tag{22}$$

2. Method of Washer(x axis):
$$\pi \int_a^b (f(x))^2 - (g(x))^2 \, dx \tag{23}$$

3. Method of Shell(y axis):
$$2\pi \int_a^b x f(x) \, dx \tag{24}$$

4. Arc Length:
$$L = \int_a^b \sqrt{1 + (f'(x))^2} \, dx \tag{25}$$

5. Surface Area:
$$A = 2\pi \int_a^b f(x)\sqrt{1 + (f'(x))^2} \, dx \tag{26}$$

6. Work:
$$W = \int_a^b f(x) \, dx \tag{27}$$
In equation 27 f(x) represent force applied, x is distance travelled.

7. Mass:
$$M = \int_a^b p(x) \, dx \tag{28}$$
In equation 28 p(x) represent density, x is distance travelled.

8. Moments:
$$M_x = p \int_a^b \frac{1}{2} \left((f(x))^2 - (g(x))^2\right) \tag{29}$$
$$M_y = p \int_a^b x \left(f(x) - g(x)\right) \, dx \tag{30}$$
p represent density.

9. Center of Mass
$$\bar{x} = \frac{1}{A} \int_a^b x(f(x) - g(x)) \, dx \tag{31}$$
$$\bar{y} = \frac{1}{A} \int_a^b \frac{1}{2} \left((f(x))^2 - (g(x))^2\right) \, dx \tag{32}$$
A is the area between two functions.

6 First Order ODE

6.1 Separable ODE

Checklist:

1. Seperate it into $f(y)g(x)$
$$\frac{dy}{dx} = f(y) * g(x) \tag{33}$$
2. Move $f(y)$ to other side:
$$\frac{1}{f(y)}\ dy = g(x)\ dx \tag{34}$$
3. Integrate both sides:
$$\int \frac{1}{f(y)}\ dy = \int g(x)\ dx \tag{35}$$
4. Isolate for y if possible

Simple Example 9

Find:
$$\frac{dy}{dx} = \frac{x}{e^y}, \quad y(2) = 3$$
Solution:
$$\begin{aligned} e^y\ dy &= x\ dx \\ \int e^y\ dy &= \int x\ dx \\ e^y + C_1 &= \frac{1}{2}x^2 + C_2 \\ e^y &= \frac{1}{2}x^2 + C \\ y &= C\ln\left(\frac{1}{2}x^2\right) \end{aligned}$$
Using initial condition:
$$\begin{aligned} 3 &= C\ln\left(\frac{1}{2} * 2^2\right) \\ 3 &= C\ln(2) \\ C &= \frac{3}{\ln(2)} \end{aligned}$$

6.2 Linear ODE

1. Put the given equation into standard form:
$$\frac{dy}{dx} + f(x)y = g(x) \tag{36}$$

2. Find the integrating factor
$$I(x) = e^{\int f(x)\ dx} \tag{37}$$

3. Multiply both sides by integrating factor (Math magic)
$$I(x) * \left(\frac{dy}{dx} + f(x) * y\right) = I(x) * g(x)$$
$$\frac{dy}{dx}\left(I(x) * y\right) = I(x) * g(x)$$

4. Integrate both sides
$$I(x) * y = \int I(x) * g(x) \tag{38}$$

5. Find C

7 Second Order Differential equation

7.1 Step 1: Find solution to homogeneous equation

You are given:

$$ay'' + by' + c = f(x)$$

Find the complementary equation:

$$a\lambda^2 + b\lambda + c = 0$$

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Three types of solutions:

1. Two different real solutions (λ_1, λ_2)

$$y = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x} \tag{39}$$

2. One real solution (λ)

$$y = C_1 e^{\lambda x} + C_2 x e^{\lambda x} \tag{40}$$

3. Two different imagery solutions $\left(\alpha = -\frac{b}{2a}, \beta = \frac{\sqrt{4ac - b^2}}{2a}\right)$

$$y = e^{\alpha x}\left(C_1 \cos(\beta x) + C_2 \sin(\beta x)\right) \tag{41}$$

7.2 Step 2: Find solution to non-homogeneous equation

Skip this step if f(x) equals 0.

1. Make an educated guess[4]:

f(x)	Educated Guess
c_1x+c_2	Ax+B
c_1x^2+c_2x+c_3	$Ax^2 + Bx + C$
$c_1 e^{\lambda x}$	$Ae^{\lambda x}$
$c_1 e^{\alpha x}\cos\beta x + c_2 e^{\alpha x}\sin\beta x$	$Ae^{\alpha x}\cos\beta x + Be^{\alpha x}\sin\beta x$
$\left(c_1x^2 + c_2x + c_3\right)e^{\lambda x}$	$\left(Ax^2 + Bx + C\right)e^{\lambda x}$
$(c_1x + c_2)\cos(\lambda x)$	$(Ax + B)\cos(\lambda x) + (Cx + D)\sin(\lambda x)$
$(c_1x + c_2)\sin(\lambda x)$	$(Ax + B)\cos(\lambda x) + (Cx + D)\sin(\lambda x)$
$\left(c_1x^2 + x_2x + c_3\right)e^{\lambda x}\cos(\beta x)$	$(Ax^2 + Bx + C)e^{\lambda x}\cos(\beta x) + (Dx^2 + Ex + F)e^{\lambda x}\sin(\beta x)$
$\left(c_1x^2 + x_2x + c_3\right)e^{\lambda x}\sin(\beta x)$	$(Ax^2 + Bx + C)e^{\lambda x}\cos(\beta x) + (Dx^2 + Ex + F)e^{\lambda x}\sin(\beta x)$

Simplified version:

f(x)	Educated Guess
Ploynomial	Same order polynomial
$e^{\lambda x}$	$Ae^{\lambda x}$
$\sin(\lambda x)\ and/or\ \cos(\lambda x)$	$A\sin(\lambda x) + B\cos(\lambda x)$

[4]Must be same λx in f(x) and guess

2. Let g(x) represent your educated guess, find g'(x) and g"(x):

3. Sub your guess back into the ODE For example:

$$y'' + 2y' + 3y = 10e^{2x}$$

   In this case, g(x)=Ae^{2x}, g'(x)=$2Ae^{2x}$, g"(x)=$4Ae^{2x}$:

$$\begin{aligned} g''(x) + 2g'(x) + 3g(x) &= 10e^{2x} \\ 4Ae^{2x} + 2 * 2Ae^{2x} + 3 * Ae^{2x} &= 10e^{2x} \end{aligned}$$

4. Find undetermined coefficient
   Simple example:

$$\begin{aligned} 4A + 4A + 3 * A &= 10 \\ A &= \frac{10}{11} \end{aligned}$$

5. Find the particular solution: Sub the coefficients into your guess

$$g(x) = \frac{10}{11}e^{2x}$$

6. The general solution is combination of homogeneous solution and a particular solution:

$$y_{general} = y_{homogeneous} + y_{particular} \tag{42}$$

Challenging Example 3

Find:

$$2y'' + 13y' + 6y = -13te^{4t}$$

1. Find the solution for homo-generous equation

   Complementary equation

$$\begin{aligned} 2\lambda^2 + 13\lambda + 6 &= 0 \\ (2\lambda + 1)(\lambda + 6) &= 0 \\ \lambda_1 = -\frac{1}{2} \quad \lambda_2 &= -6 \end{aligned}$$

   Solution for homo-generous equation:

$$y = C_1 e^{-\frac{1}{2}t} + C_2 e^{-6t}$$

2. Make an educated guess

   Let $g(x) =$

$$(At + B)e^{4t}$$

   so that $g'(x) =$

$$\begin{gathered} (4At + 4B)e^{4t} + Ae^{4t} \\ 4Ate^{4t} + 4Be^{4t} + Ae^{4t} \end{gathered}$$

   and $g''(x) =$

$$\begin{gathered} (16At + 16B)e^{4t} + 4Ae^{4t} + 4Ae^{4t} \\ 16Ate^{4t} + 16Be^{4t} + 8Ae^{4t} \end{gathered}$$

   Sub everything back into the equation:

$$\begin{aligned} 16Ate^{4t} + 16Be^{4t} + 8Ae^{4t} + 6(4Ate^{4t} + 4Be^{4t} + Ae^{4t}) + 5((At + B)e^{4t}) &= -13te^{4t} \\ 16Ate^{4t} + 16Be^{4t} + 8Ae^{4t} + 24Ate^{4t} + 24Be^{4t} + 6Ae^{4t} + 5Ate^{4t} + 5Be^{4t} &= -13te^{4t} \\ 45Ate^{4t} + 45Be^{4t} + 14Ae^{4t} &= -13te^{4t} \\ 45At + 45B + 14A &= -13t \end{aligned}$$

   Using some simple linear algebra:

$$\begin{aligned} &= \left[\begin{array}{cc|c} 45 & 0 & -13 \\ 14 & 45 & 0 \end{array}\right] \\ &= \left[\begin{array}{cc|c} 1 & 0 & -\frac{13}{45} \\ 14 & 45 & 0 \end{array}\right] \\ &= \left[\begin{array}{cc|c} 1 & 0 & -\frac{13}{45} \\ 0 & 45 & \frac{182}{45} \end{array}\right] \\ &= \left[\begin{array}{cc|c} 1 & 0 & -\frac{13}{45} \\ 0 & 1 & \frac{182}{2025} \end{array}\right] \end{aligned}$$

$g(x) =$

$$-\frac{13}{45}te^{4t} + \frac{182}{2025}e^{4t}$$

3. The general solution to the question is:

$$y = C_1e^{-\frac{1}{2}t} + C_2e^{-6t} - \frac{13}{45}te^{4t} + \frac{182}{2025}e^{4t}$$

8 Applications of Second Order Differential Equations

8.1 Simple Oscillation

$$my'' + ky = 0 \tag{43}$$

8.2 Damped Oscillation

$$my'' + by' + ky = 0 \tag{44}$$

8.3 Forced Oscillation

$$my'' + by' + ky = F(t) \tag{45}$$

8.4 Circuit

$$Lq'' + Rq' + \frac{1}{C}q = V(t) \tag{46}$$

9 Series

9.1 Divergence Test

If the series is:

$$\sum_{n=0}^{\infty} a_n$$

if

$$\lim_{n\to\infty} a_n \neq 0$$

then the series diverges.

9.2 Geometric Series

Geometry series takes the form of:

$$\sum_{n=0}^{\infty} ar^n \tag{47}$$

The series only converges to

$$\frac{a}{1-r} \tag{48}$$

if and only if $|r| < 1$

9.3 p-series

p-series takes the form of

$$\frac{1}{n^p} \tag{49}$$

p-series only converges when $p > 1$

9.4 Comparison Test

Theorem 3

If there are two series

$$\sum_{n\to\infty}^{\infty} a_n \tag{50}$$

$$\sum_{n\to\infty}^{\infty} b_n \tag{51}$$

and

$$a_n \text{ is always bigger than or equal to } b_n\ as\ n \to \infty$$

or

$$b_n \text{ is always smaller than or equal to } a_n\ as\ n \to \infty$$

if $\sum_{n\to\infty}^{\infty} a_n$ converges then $\sum_{n\to\infty}^{\infty} b_n$ must also converge.

if $\sum_{n\to\infty}^{\infty} b_n$ diverges then $\sum_{n\to\infty}^{\infty} a_n$ must also diverge. Its really similar to the comparison test of infinite integral.

Theorem 4

The bigger the denominator gets, the smaller the value become

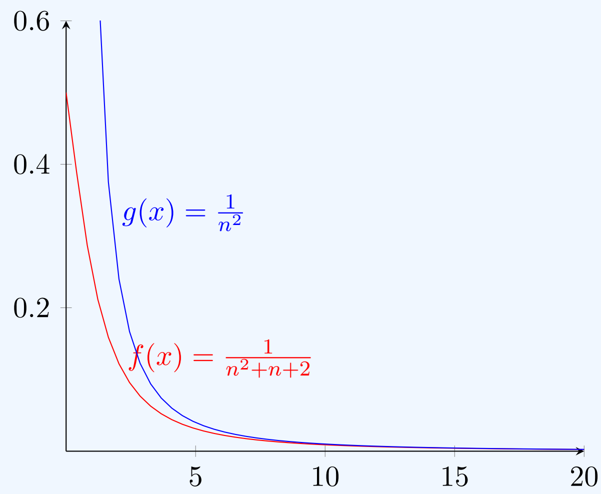
Simple Example 10

Find if

$$\frac{1}{x^2+x+2}$$

will converge.

1. Think of a similar sequence that might help It takes a lot of practice.
   In this case we will use $\frac{1}{x^2}$ for converge
2. Make sure it is actually smaller/bigger than the series of interest.
   We need $\frac{1}{x^2}$ to be smaller than $\frac{1}{x^2+x+2}$



   In this case, $\frac{1}{n^2}$ is always bigger than $\frac{1}{n^2+n+2}$
3. Since $\frac{1}{x^2}$ converges$_{p-series}$, $\frac{1}{x^2+x+2}$ must also converge.

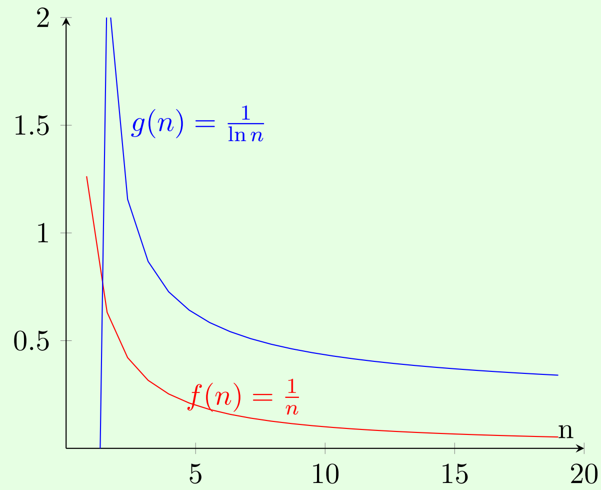
Challenging Example 4

Find if

$$\sum_{n=0}^{\infty} \frac{1}{\ln n}$$

converges.

1. We can use $\frac{1}{n}$ to compare
2. $\frac{1}{n}$ is always going to be smaller than $\frac{1}{\ln n}$



$\frac{1}{\ln n}$ is always bigger than $\frac{1}{n}$ in the longer run.

3. $\frac{1}{n}$ diverges$_{p} - series$, so $\frac{1}{\ln n}$ has to diverge.

Challenging Example 5

Find if

$$\sum_{n=0}^{\infty} \frac{1}{0.789^n + 0.789^{n-1} + 0.789^{n-2} + \dots}$$

converges

1. Compare it to the equation:
$$\frac{1}{0.789^n}$$

2. Using theorem 2 stated above, we can conclude $\frac{1}{0.789^n+0.789^{n-1}+0.789^{n-2}+\dots}$ is always smaller than $\frac{1}{0.789^n}$ in the long run.

3. $\displaystyle\sum_{n=0}^{\infty} \frac{1}{0.789^n}$ can be written in the form of a geometric series:

$$\sum_{n=0}^{\infty} \frac{1}{0.789^n} = \sum_{n=0}^{\infty} 1 * \frac{1}{0.789^n}{}_{Hint: \sum_{n=0}^{\infty} a * r^n}$$
$$= \sum_{n=0}^{\infty} 1 * \left(\frac{1}{0.789}\right)^n$$

In this case

$$a = 1, \quad r = \frac{1}{0.789}$$

$|r|$ is smaller than 1, subbing variables into $\dfrac{a}{1-r}$

$$\sum_{n=0}^{\infty} \frac{1}{0.789^n} = \frac{1}{1-0.789} \approx 4.73$$

4. Since $\sum_{n=0}^{\infty} \frac{1}{0.789^n}$ converges, $\sum_{n=0}^{\infty} \frac{1}{0.789^n+0.789^{n-1}+0.789^{n-2}+...}$ must also converge

9.5 Integral Test

Given $f(n) = a_n$ for all integer values of n and f(x) is a continuous and decreasing function.

Theorem 5

if $\displaystyle\int_c^{\infty} f(x)$ converges then $\displaystyle\sum_{n=0}^{\infty} a_n$ will converge.

if $\displaystyle\int_c^{\infty} f(x)$ diverges then $\displaystyle\sum_{n=0}^{\infty} a_n$ will diverge.

c can be any finite integer

Simple Example 11

Find if

$$\frac{1}{n^2}$$

converges?

We can take a look at the integral:

$$\begin{aligned}
\int_1^{\infty} \frac{1}{n^2} dn &= \lim_{t\to\infty} \int_0^t \frac{1}{n^2} dn \\
&= \lim_{t\to\infty} -\left.\frac{1}{n}\right|_1^t \\
&= \lim_{t\to\infty} -\left(\frac{1}{t} - \frac{1}{1}\right) \\
&= -(0-1) \\
&= 1
\end{aligned}$$

Since the integral $\displaystyle\int_1^{\infty} \frac{1}{n^2}$ converges, the series will converge.

The challenge to this kind of questions is to solve the integral, not apply integral test.

9.6 Alternating Series

Theorem 6

If a series takes one of these forms:

$$\sum_{n=0}^{\infty}(-1)^n a_n \quad (52) \qquad \sum_{n=0}^{\infty}(-1)^{n+1} a_n \quad (53)$$

and a_n is a constantly decreasing towards 0.
The series is going to converge.

In simpler words, the requirements for an alternating series are:

1. Constantly alternating between positive and negative
2. $\lim_{n\to\infty} = 0$
3. $a_n \geq a_{n+1}$

Simple Example 12

Will

$$\sum_{n=0}^{\infty}\frac{(-1)^n n}{n^3}$$

converge?
Solution: We will check if it satisfies the conditions of an alternating series.

1. The function is going to switch between positive and negative
2. $\lim_{n\to\infty}\frac{n}{n^3} = 0$
3. $\frac{1}{(n+1)^2} \leq \frac{1}{n^2}$

The series qualifies for all three conditions, therefore it is an alternating series and must converge.

9.7 Ratio Test

Theorem 7: Absolute Convergence

When $\sum_{n=0}^{\infty}|a_n|$ converges, $\sum_{n=0}^{\infty}a_n$ must also converge.

Theorem 8: Conditional Convergence

When $\sum_{n=0}^{\infty}a_n$ converges, $\sum_{n=0}^{\infty}|a_n|$ might not converge.

Theorem 9

Given a function in the form:

$$\sum_{n=0}^{\infty} a_n$$

Find

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| \tag{54}$$

There are three possible outcomes

1. The limit is smaller than $1_{including\ 0}$, the series is absolute convergent.
2. The limit is equal to 1, the test is inconclusive.
3. the limit is greater than $1_{including\ \infty}$, the series diverges

Ratio test is really helpful when you have exponents or factorial

9.8 Root Test

This test is really similar to ratio test.

Theorem 10

Given a function in the form:

$$\sum_{n=0}^{\infty} a_n$$

Find

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} \tag{55}$$

There are three possible outcomes

1. The limit is smaller than $1_{including\ 0}$, the series is absolute convergent.
2. The limit is equal to 1, the test is inconclusive.
3. the limit is greater than $1_{including\ \infty}$, the series diverges

9.9 Checklist of Tests

1. Comparison Test (p-series/geometric series)
2. Alternating Series Test ($(-1)^n$)
3. Ratio Test (exponents or factorial)
4. Integral Test (decreasing, positive and continuous)
5. Divergent Test (Last resort)

9.10 Power Series

Theorem 11

A power series centered at x_0 is a series takes the form:

$$\sum_{n=0}^{\infty} c_n(x-x_0)^n \tag{56}$$

Unlike before, power series has a variable. x can be any value and it will influence whether the power series converges.

Simple Example 13

We can take a look at

$$\sum_{n=0}^{\infty} \frac{(x)^n}{2^n}$$

If we let x=5

$$\sum_{n=0}^{\infty} \frac{5^n}{2^n} = \sum_{n=0}^{\infty} \left(\frac{5}{2}\right)^n$$

It becomes a geometric series. Since $|r|$ is bigger than 1, the series diverges.

If we let x=1

$$\sum_{n=0}^{\infty} \frac{1^n}{2^n} = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n$$

In this case $|r|$ is smaller than 1, the series converges.
Note that not all power series are this simple!

Theorem 12

There are three possible cases of x that will convergence the series:

1. Only converge at x_0
2. Converges within an interval
3. Converges for all x

It should be noted the series will always converge at x_0.

When you are finding interval of convergence for some series, always try ratio test.

Simple Example 14

Find the interval of convergence for

$$\sum_{n=0}^{\infty} \frac{x^n}{n * 3^n}$$

Solution:

$$\lim_{n\to\infty}\left|\frac{\frac{x^{n+1}}{(n+1)3^{n+1}}}{\frac{x^n}{n*3^n}}\right| = \lim_{n\to\infty}\left|\frac{x*n}{(n+1)3}\right|$$
$$= \lim_{n\to\infty}\left|\frac{x}{3}\frac{n}{n+1}\right|$$
$$= \left|\frac{x}{3}\right|$$

$\left|\frac{x}{3}\right|$ has to be smaller than 1 for the series will converge.

$$\left|\frac{x}{3}\right| < 1$$
$$-1 < \frac{x}{3} < 1$$
$$-3 < x < 3$$

Now that we know the radius of convergence, we need to find if the boundaries converge.

When x=-3:

$$= \sum_{n=0}^{\infty}\frac{(-3)^n}{n*3^n}$$
$$= \sum_{n=0}^{\infty}\frac{(-1)^n 3^n}{n*3^n}$$
$$= \sum_{n=0}^{\infty}(-1)^n\frac{1}{n}$$

Alternating series, the series converges when x=-3

When x=3:

$$= \sum_{n=0}^{\infty}\frac{(3)^n}{n*3^n}$$
$$= \sum_{n=0}^{\infty}\frac{1}{n}$$

p-series, the series diverges at x=3

The interval of convergence is $[-3 < x < 3)$

Geometric series can be considered as a power series centered at 0 and all constant $c_n = a$.

$$\sum_{n=0}^{\infty} ar^n = \sum_{n=0}^{\infty} c_n x^n$$

Simple Example 15

Express

$$f(x) = \frac{3}{1+x^3}$$

as a series.

Solution:

$$f(x) = \frac{3}{1+x^3} = \frac{3}{1-(-x^3)}$$
$$= \sum_{n=0}^{\infty} 3(-x^3)^n$$

9.11 Term by Term Differentiation and Integration

Theorem 13

$$f(x) = \sum_{n=0}^{\infty} c_n(x-x_0)^n \tag{57}$$

$$f'(x) = \sum_{n=1}^{\infty} c_n * n * (x-x_0)^{n-1} \tag{58}$$

$$\int f(x)dx = \sum_{n=1}^{\infty} c_n \frac{(x-x_0)^{n+1}}{n+1} + C \tag{59}$$

P.S. $f(0) = C$ and remember n=1 for differentiation of series.

9.12 Taylor Series

9.12.1 Function to Taylor Series

Theorem 14

Taylor series of f(x) is:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^n(x_0)}{n!}(x-x_0)^n \tag{60}$$

Simple Example 16

$$f(x) = \sin(x)$$

Find f(x) represented using power series centered at $x_0 = \dfrac{\pi}{2}$

$$\begin{aligned} f(x) &= \sin(x) \\ f'(x) &= \cos(x) \\ f''(x) &= -\sin(x) \\ f^3(x) &= -\cos(x) \\ f^4(x) &= \sin(x) \end{aligned} \qquad \begin{aligned} f(\tfrac{\pi}{2}) &= 1 \\ f'(\tfrac{\pi}{2}) &= 0 \\ f''(\tfrac{\pi}{2}) &= -1 \\ f^3(\tfrac{\pi}{2}) &= 0 \\ f^4(\tfrac{\pi}{2}) &= 1 \end{aligned}$$

$$
\begin{aligned}
f(x) = &\sin\left(\frac{\pi}{2}\right) \\
&+ \frac{\cos(\frac{\pi}{2})}{1!}\left(x-\frac{\pi}{2}\right) \\
&+ \frac{-\sin\left(\frac{\pi}{2}\right)}{2!}\left(x-\frac{\pi}{2}\right)^2 \\
&+ \frac{-\cos\left(\frac{\pi}{2}\right)}{3!}\left(x-\frac{\pi}{2}\right)^3 \\
&+ \frac{\sin(x)}{4!}\left(x-\frac{\pi}{2}\right)^4 \dots \\
f(x) = &1 + 0 - \frac{1}{2}\left(x-\frac{\pi}{2}\right)^2 + \frac{1}{24}\left(x-\frac{\pi}{2}\right)^4 \dots
\end{aligned}
$$

By looking at the pattern we observe that

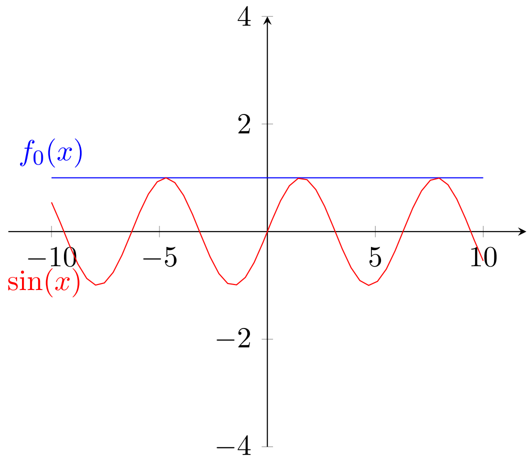
$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\left(x-\frac{\pi}{2}\right)^{2n}$$

9.12.2 Visualization of Taylor Series

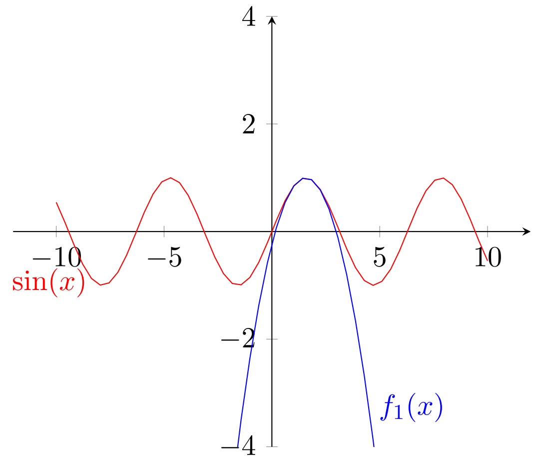
You don't need to know this for the course, but its pretty fun.

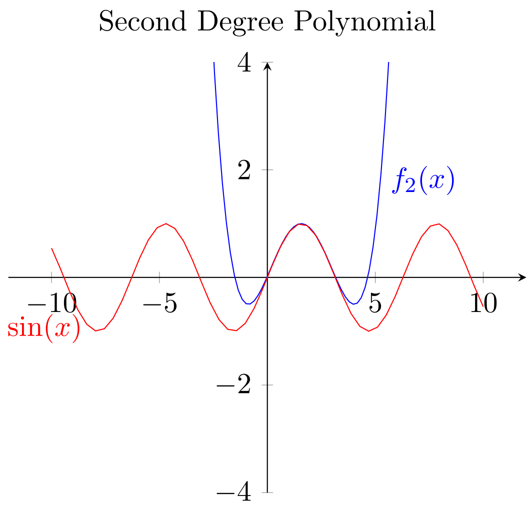
$$
\begin{aligned}
f_0(x) &= 1 \\
f_1(x) &= 1 - \frac{1}{2}\left(x-\frac{\pi}{x}\right)^2 \\
f_2(x) &= 1 - \frac{1}{2}\left(x-\frac{\pi}{x}\right)^2 + \frac{1}{24}\left(x-\frac{\pi}{2}\right)^4 \\
f_3(x) &= 1 - \frac{1}{2}\left(x-\frac{\pi}{x}\right)^2 + \frac{1}{24}\left(x-\frac{\pi}{2}\right)^4 - \frac{1}{720}\left(x-\frac{\pi}{2}\right)^6 \\
f_4(x) &= 1 - \frac{1}{2}\left(x-\frac{\pi}{x}\right)^2 + \frac{1}{24}\left(x-\frac{\pi}{2}\right)^4 - \frac{1}{720}\left(x-\frac{\pi}{2}\right)^6 + \frac{1}{40320}\left(x-\frac{\pi}{2}\right)^8
\end{aligned}
$$

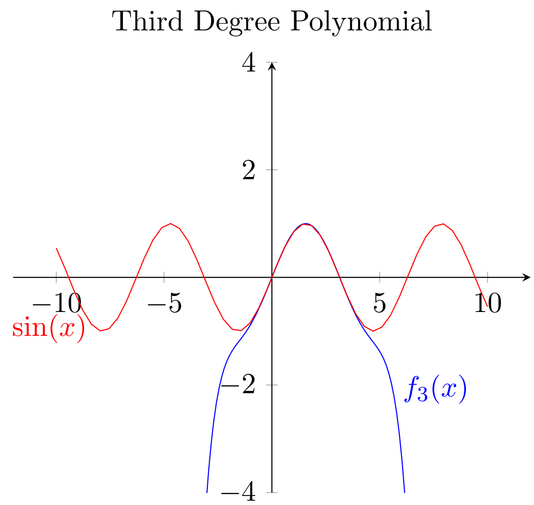
Zero Degree Polynomial

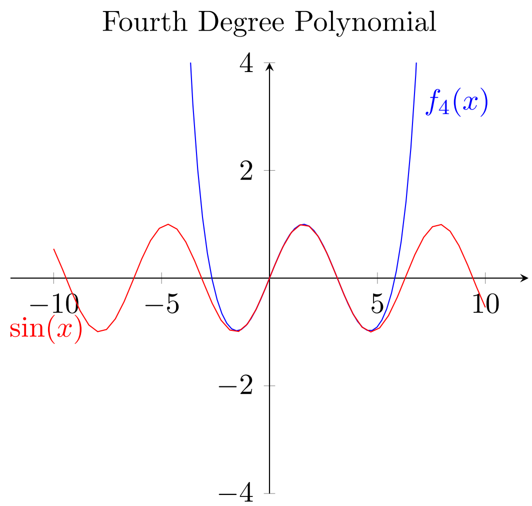


First Degree Polynomial









As the number of Taylor polynomial terms increases, the series approaches the original function.

Theorem 15

Maclaurin Series are Taylor Series centered at x=0

$$\sum_{n=0}^{\infty} \frac{f^n(0)}{n!} x^n \tag{61}$$

9.13 Estimation using Series

9.13.1 Alternating Series

Let's do another visualization:

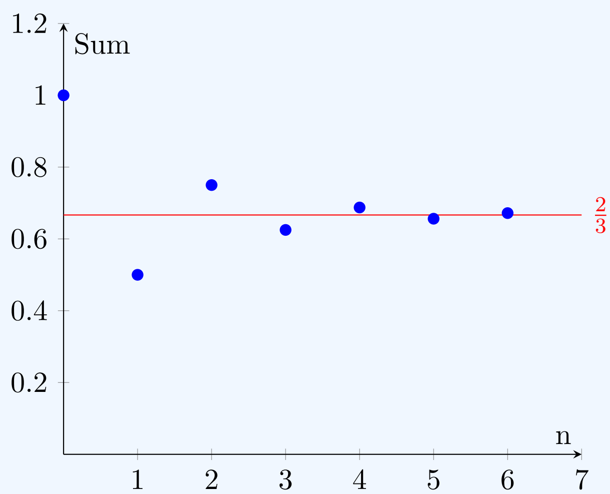
Simple Example 17

Let S_N denote the sum of N terms of the alternating series. We will try to approximate:

$$\sum_{n=0}^{\infty}(-1)^n\frac{1}{2^n}$$

$$\begin{aligned}
S_0 &= 1\\
S_1 &= 1-\frac{1}{2}\\
S_2 &= 1-\frac{1}{2}+\frac{1}{4}\\
S_3 &= 1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}\\
S_4 &= 1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\frac{1}{16}\\
S_5 &= 1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\frac{1}{16}-\frac{1}{32}\\
S_6 &= 1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\frac{1}{16}-\frac{1}{32}+\frac{1}{64}
\end{aligned}$$

Visualization of Alternating Series



As you can verify, the series sums to $\frac{2}{3}$.

Theorem 16

Let S represent the sum of an alternating series.

$$\sum_{n=0}^{\infty}(-1)^n a_n$$

Let R_N represent the remainder of the series so that $R_N = S - S_N$.

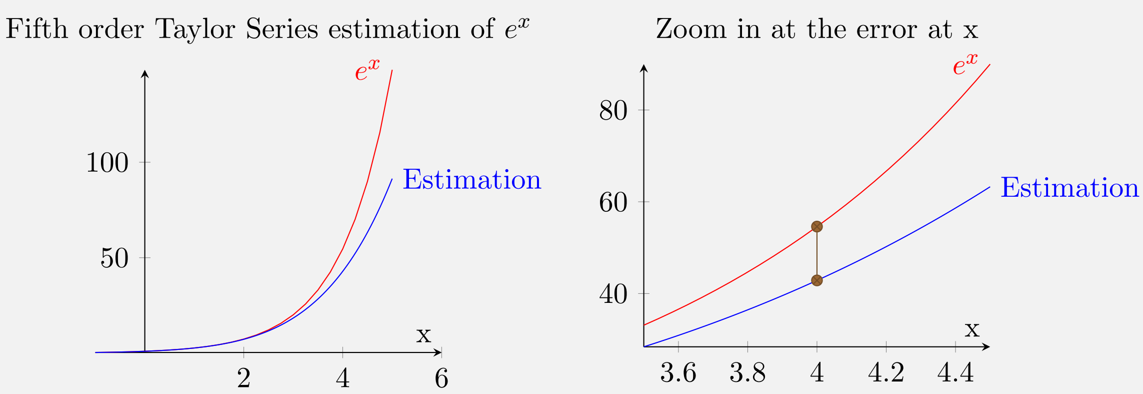
$$|R_N| \leq a_{n+1} \tag{62}$$

In simpler words, the distance from S_N to series value is always smaller than distance to S_{N+1}.

9.13.2 Taylor Series

Theorem 17

Let S denote the value of f(x) at x(Top dot).



Let S_N represent the sum of the series evaluated at x for N terms.(Bottom dot)
Let R_N represent the error such that $R_n = S - S_N$(The line connecting two dots).

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x - x_0|^{n+1} \tag{63}$$

Where M is the maximum value of f'(x) between x and x_0

Proof. Bruh, just trust me. □

Theorem 18: Taylor Series Error Bound

To prove that a series is equal to a function:

$$\lim_{n\to\infty} \frac{M}{(n+1)!}(x - x_0)^{n+1} = 0 \tag{64}$$

and

$$-\lim_{n\to\infty} \frac{M}{(n+1)!}(x - x_0)^{n+1} = 0 \tag{65}$$

Proof. Using theorem Taylor Series Error Bound:

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x - x_0|^{n+1}$$

We can conclude:

$$-\frac{M}{(n+1)!}|x-x_0|^{n+1} \leq R_n(x) \leq \frac{M}{(n+1)!}|x-x_0|^{n+1}$$

If both sides equal to 0 as n approaches infinity, it forces $R_n(x)$ to be equal to zero according to squeezing theorem. If there is no error, the series and the function are equivalent. □

Simple Example 18

Prove the Taylor series of $\sin(x)$ is

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\left(x - \frac{\pi}{2}\right)^{2n}$$

Solution:
$f'(x) = \cos(x)$ We know that the maximum for a cosine function is going to be 1, therefore, M=1.

$$\begin{aligned} \lim_{n\to\infty} \frac{1}{(n+1)!}|x-x_0|^{n+1} &= \lim_{n\to\infty} \frac{x^n}{n!} \\ &= 0 \end{aligned}$$

Similar calculation can be perform on $-\frac{M}{(n+1)!}|x-x_0|^{n+1}$.
Since both conditions are satisfied, this Taylor series is equivalent to the function.

10 Parametric Equation

10.1 Eliminate the parameter

Checklist:

1. Express x or y in terms of t
2. Sub t value

Simple Example 19

Eliminate the parameter

$$y = 2t + 1, \; x = 1 + t$$

Solution:

$$\begin{aligned} x &= 1 + t \\ x - 1 &= t \end{aligned}$$

Sub $t = x - 1$ into $y = 2t + 1$

$$\begin{aligned} y &= 2t + 1 \\ y &= 2(x-1) + 1 \\ y &= 2x - 1 \end{aligned}$$

10.2 Derivative of a parametric equation

$$\frac{dy}{dx} = \frac{\frac{d}{dt}y(t)}{\frac{d}{dt}x(t)} \tag{66}$$

Simple Example 20

Find the derivative of:

$$x = \sin(t), \; y = \cos(t)$$

Solution:

$$\begin{aligned} \frac{dy}{dx} &= \frac{\frac{d}{dt}\sin(t)}{\frac{d}{dt}\cos(t)} \\ &= \frac{\cos(t)}{-\sin(t)} \\ &= \cot(t) \end{aligned}$$

10.3 Second derivative of a parametric equation

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dx}\left(\frac{dy}{dx}\right)}{\frac{d}{dt}x(t)} \tag{67}$$

Simple Example 21

Find second derivative of:

$$x = \sin(t),\ \ y = \cos(t)$$

Solution:
Using what we calculated in the previous simple example:

$$\frac{dy}{dx} = \cot(t)$$

Hint: $\frac{d}{dt}\cot(t) = -\csc^2(t)$

10.4 Area of a parametric equation

$$A = \int_a^b y(t)x'(t)dt \tag{68}$$

$x(a)$ must be less than $x(b)$ to use this equation

10.5 Length of parametric equation curve

$$\int_a^b \sqrt{(x'(t))^2 + (y'(t))^2} \tag{69}$$

Simple Example 22

Let

$$x(t) = 2\cos(t),\ \ y(t) = 2\sin(t)$$

Find curve length between t=4 and t=10

Solution:
$\frac{d}{dt}x(t) = -2\sin(t)$ $\frac{d}{dt}y(t) = 2\cos(t)dt$

$$\begin{aligned}
\int^1 0_4\sqrt{(-2\sin(t))^2 + (2\cos(t))^2}dt &= \int^1 0_4\sqrt{4\sin^2(t) + 4\cos^2(t)}dt \\
&= \int^1 0_4\sqrt{4}dt \\
&= \int^1 0_4 2dt \\
&= 2\ |_4^{10} \\
&= 20 - 8 \\
&= 12
\end{aligned}$$

11 Polar Form

11.1 Derivative of Polar Form

$$\frac{dy}{dx} = \frac{\frac{dr}{d\theta}\sin(\theta) + r\cos(\theta)}{\frac{dr}{d\theta}\cos(\theta) - r\sin(\theta)} \tag{70}$$

11.2 Switching to coordinates

$$r\sin(\theta) = y \tag{71}$$

$$r\cos(\theta) = x \tag{72}$$

11.3 Switching from coordinates

$$r = \sqrt{x^2 + y^2} \tag{73}$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) \tag{74}$$

11.4 Polar Equation That You Need To Remember

11.4.1 Lines

$$r\cos(\theta) = a \tag{75}$$

$$r\sin(\theta) = b \tag{76}$$

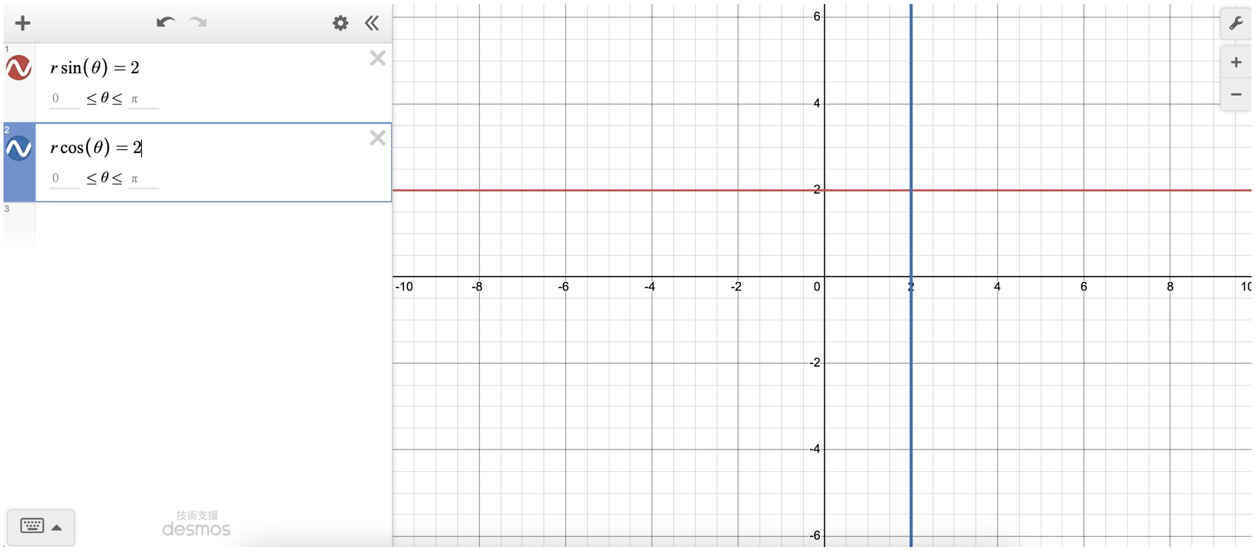


Figure 1: Polar Lines [2]

[2]Created using Desmos

11.4.2 Cardioid

$$a - b\cos(\theta) \tag{77}$$

There are 3 cases with cardioid: $a > b$, $a = b$ and $a < b$

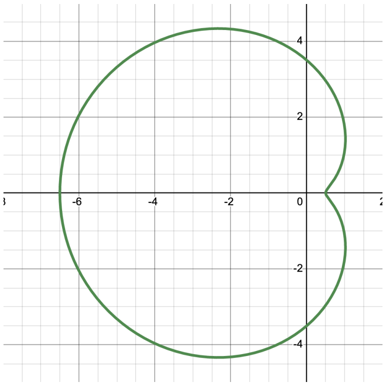


Figure 2: $a > b$ [2]

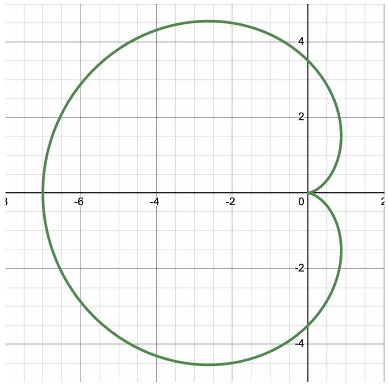


Figure 3: $a = b$ [2]

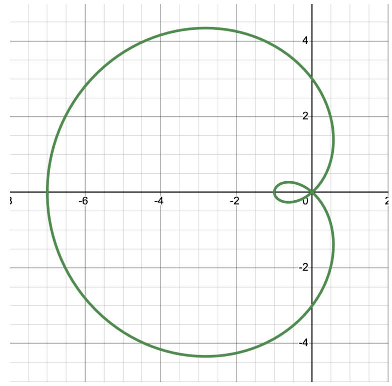


Figure 4: $a < b$ [2]

That's footnote, not exponent.

If it is a plus sign, simply flip everything

$$a + b\cos(\theta) \tag{78}$$

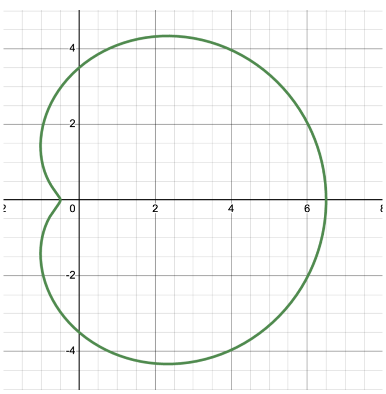


Figure 5: $a > b$ [2]

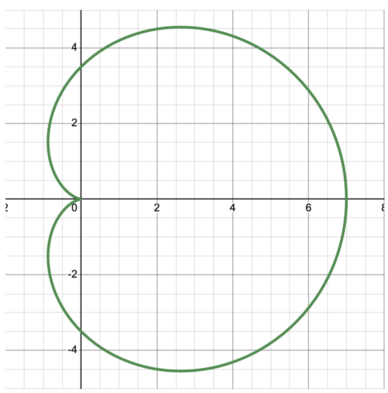


Figure 6: $a = b$ [2]

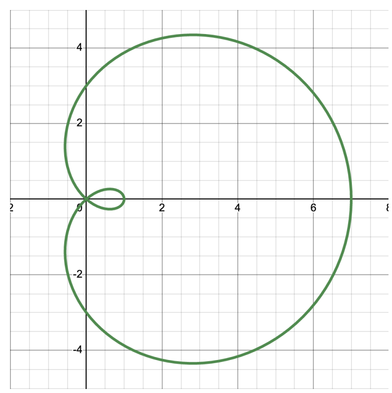


Figure 7: $a < b$ [2]

[2] Created using Desmos

$$a - b\sin(\theta) \tag{79}$$

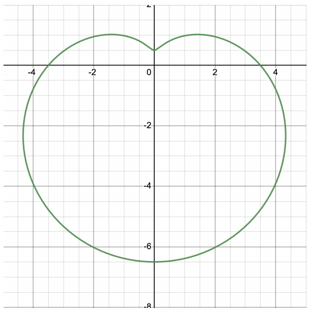


Figure 8: $a > b$ [2]

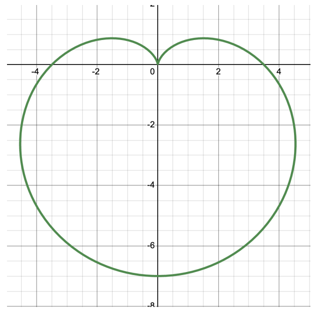


Figure 9: $a = b$ [2]

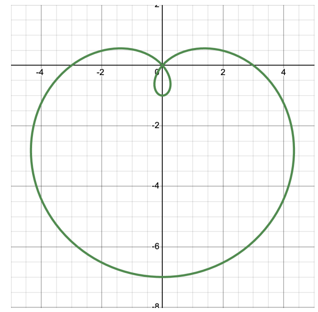


Figure 10: $a < b$ [2]

$$a + b\sin(\theta) \tag{80}$$

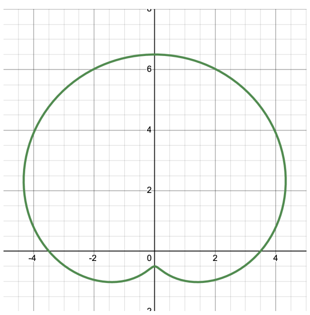


Figure 11: $a > b$ [2]

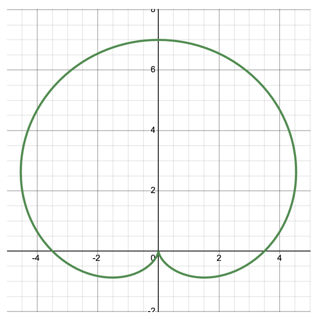


Figure 12: $a = b$ [2]

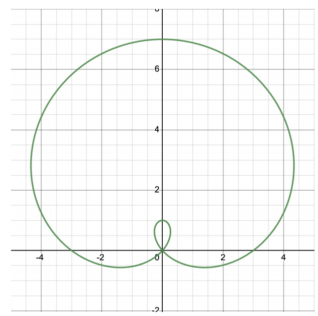


Figure 13: $a < b$ [2]

11.5 Area of Polar Form

11.5.1 Area of one polar equation

$$A = \frac{1}{2}\int_a^b r^2 d\theta \tag{81}$$

11.5.2 Area between two polar equations

$$A = \frac{1}{2}\int_a^b r_1^2 - r_2^2 d\theta \tag{82}$$

11.6 Curve Length of Polar Form

$$L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta \tag{83}$$

[2]Created using Desmos

12 Vector

I trust you took linear algebra before.

Theorem 19

Given a vector equation:

$$r(t) = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k} \tag{84}$$

Derivative of the equation is:

$$r'(t) = f'(t)\vec{i} + g'(t)\vec{j} + h'(t)\vec{k} \tag{85}$$

Integral of the equation is:

$$\int r(t) = \int f(t)\vec{i} + \int g(t)\vec{j} + \int h(t)\vec{k} + C \tag{86}$$

C is any vector <a, b, c> where a, b and c are all constants.

12.1 Magnitude of Vector

$$||r'(t)|| = \sqrt{(f'(t))^2 + (g'(t))^2 + (h'(t))^2} \tag{87}$$

12.2 Unit vector

We are interested in unit tangent vector:

$$T(t) = \frac{r'(t)}{||r'(t)||} \tag{88}$$

12.3 Principal/Unit Normal Vector

$$N(t) = \frac{T'(t)}{||T'(t)||} \tag{89}$$

12.4 Binormal Vector

$$B(t) = T(t) \times N(t) \tag{90}$$

12.5 Arc Length

$$L = \int_a^b \sqrt{(f'(t))^2 + (g'(t))^2 + (h'(t))^2}dt = \int_a^b ||r'(t)||dt \tag{91}$$

12.6 Arc Length Function

Arc length function is basically expressing the function with distance traveled as the domain.

Checklist:

1. Find the distance traveled with respect to t
2. Isolate for t
3. Sub the equation back into r(t)

Simple Example 23

Given

$$r(t) =< 2\sin(t), 2\cos(t) >$$

find arc length function.
Solution:
Step 1:

$$\begin{aligned} S &= \int_0^t ||r'(\theta)||d\theta \\ &= \int_0^t \sqrt{4\sin^2(\theta) + 4\cos^2(\theta)}d\theta \\ &= \int_0^t \sqrt{4}d\theta \\ &= \int_0^t 2d\theta \\ &= 2t \end{aligned}$$

Step 2:

$$\begin{aligned} S =& 2t \\ \frac{S}{2} &= t \end{aligned}$$

Step 3:

$$r(t) = \left\langle 2\sin\left(\frac{S}{2}\right), 2\cos\left(\frac{S}{2}\right)\right\rangle$$

If starting at another point:
Checklist:

1. Find t_0 at that point
2. $\int_{t_0}^t ||r'(t)||dt$

Simple Example 24

Given the equation:

$$r(t) =< t^2 + 3, 2\cos\pi t >$$

find t_0 when $r(t) =< 12, -2 >$
Solution: t_0=3

12.7 Curvature

$$\kappa = \frac{||T'(t)||}{||r'(t)||} = \frac{||r'(t) \times r''(t)||}{||r'(t)||^3}$$

12.8 Acceleration

Theorem 20

Acceleration contains two components:

a_T is tangential acceleration a_n is normal/perpendicular acceleration

$$\vec{a}(t) = a_T\vec{T} + a_n\vec{N} \tag{92}$$

Where $\vec{T}$ is unit tangent vector and $\vec{N}$ is unit normal vector.
Magnitude of tangential component is:

$$a_T = \frac{r'(t) \cdot r''(t)}{||r'(t)||} \tag{93}$$

Magnitude of normal component is:

$$a_n = \frac{||r'(t) \times r''(t)||}{||r'(t)||} \tag{94}$$

Theorem 21

If a cylinder can be expressed with equation $x^2 + y^2 = a$, think of it as a cylinder with a radius of $\frac{a}{2}$ centered at origin.

Challenging Example 6

Find the intersection of cylinder:

$$x^2 + y^2 = a$$

and plane

$$x + y + z = b$$

Solution:
We can create a variable t such that x and y value can be express in the form of $\frac{a}{2}\cos(t)$ and $\frac{a}{2}\sin(t)$ respectively.Taking these values back into plane equation:

$$x + y + z = b$$
$$\frac{a}{2}\cos(t) + \frac{a}{2}\sin(t) + z = b$$
$$z = b + \frac{a}{2}\cos(t) + \frac{a}{2}\sin(t)$$

Therefore the intersection can be described by vector equation:

$$r(t) = \left\langle \frac{a}{2}\cos(t), \frac{a}{2}\sin(t), b + \frac{a}{2}\cos(t) + \frac{a}{2}\sin(t) \right\rangle$$