Calc2 Notes Draft

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Contents

1 Integration By Part

$$\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx \tag{1}$$

Simple Example 1

Let try to integrate:

$$\int (x+1)\cos(x)$$

Let:

$$f(x) = x + 1 \qquad g'(x) = cos(x)$$

So that:

$$f'(x) = 1 \qquad g(x) = \sin(x)$$

$$\begin{aligned}\int (x+1)\cos(x) &= (x+1) * \sin(x) - \int 1 * \sin(x)\ dx \\ &= (x+1) * \sin(x) - \int \sin(x)\ dx \\ &= (x+1) * \sin(x) + \cos(x)\end{aligned}$$

Simple Example 2

A weird case that you'll probably learn:

$$\int e^x \sin(x)\ dx$$

Let:

$$f(x) = e^x \qquad g'(x) = \sin(x)$$

So that:

$$f'(x) = e^x \qquad g(x) = -\cos(x)$$

Sub everything into the formula:

$$\int e^x \sin(x)\ dx = e^x * -cos(x) - \int e^x * -\cos(x)\ dx$$
$$\int e^x \sin(x)\ dx = -e^x * \cos(x) + \int e^x * cos(x)\ dx$$

Let:

$$f(x) = e^x \qquad g'(x) = \cos(x)$$

So that:

$$f'(x) = e^x \qquad g(x) = \sin(x)$$

$$\int e^x \sin(x)\ dx = -e^x * \cos(x) + \int e^x * cos(x)\ dx$$
$$\int e^x \sin(x)\ dx = -e^x * \cos(x) + \left[e^x * \sin(x) - \int e^x * \sin(x)\ dx\right]$$
$$\int e^x \sin(x)\ dx = -e^x * \cos(x) + e^x * \sin(x) - \int e^x * \sin(x)\ dx$$
$$2\int e^x \sin(x)\ dx = -e^x * \cos(x) + e^x * \sin(x)$$
$$\int e^x \sin(x)\ dx = -\frac{1}{2}e^x * \cos(x) + \frac{1}{2}e^x * \sin(x)$$

Theorem 1: LIATE

Here's a quick checklist when picking f(x)

1. L-Logarithmic
2. I-Inverse Trigonometric
3. A-Algebra
4. T-Trigonometric
5. E-Exponential

L is your most preferred f(x) and E is the worst choice.[1]

[1] From my experience, you almost never have to pick L or I. T and E are exchangeable, pick the one that makes solving the question easier

2 Trig

2.1 Trigonometric Formulas

$$sin^2x = \frac{1-\cos 2x}{2} \tag{2}$$

$$cos^2x = \frac{1+\cos 2x}{2} \tag{3}$$

Simple Example 3

Find

$$\int \sin^2(x)dx$$

using trigonometric formulas 2:

$$\begin{aligned}
\int \sin^2(x)dx &= \int \frac{1-\cos(2x)}{2}dx \\
&= \frac{1}{2}\int (1-\cos(2x))dx \\
&= \frac{1}{2}\left[\int 1dx - \int \cos(2x)dx\right] \\
&= \frac{1}{2}\left[x - \int \cos(2x)dx\right]
\end{aligned}$$

Let $u = 2x$, so that $\frac{du}{dx} = 2$

$$\begin{aligned}
\int \cos(2x)dx &= \frac{1}{2}\int \cos(u)du \\
&= \frac{1}{2}\sin(u) \\
&= \frac{1}{2}\sin(2x) + C
\end{aligned}$$

Back the the equation above:

$$\begin{aligned}
&= \frac{1}{2}\left[x - \int \cos(2x)dx\right] \\
&= \frac{1}{2}\left[x - \frac{1}{2}\sin(2x)\right] + C \\
&= \frac{1}{2}x - \frac{1}{4}\sin(2x) + C
\end{aligned}$$

2.2 Application In Integration

$$\int \sin^2 x\ dx = \frac{1}{2}x - \frac{1}{4}\sin 2x + C \tag{4}$$

$$\int \cos^2 x\ dx = \frac{1}{2}x + \frac{1}{4}\sin 2x + C \tag{5}$$

It is strongly recommended that you remember these two equations.

2.3 Inverse Trigonometry

$$\int \frac{1}{\sqrt{a^2-u^2}}\,du = \sin^{-1}\left(\frac{u}{a}\right) + C \tag{6}$$

$$\int \frac{1}{a^2+u^2}\,du = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C \tag{7}$$

$$\int \frac{1}{u\sqrt{u^2-a^2}}\,du = \frac{1}{a}\sec^{-1}\left(\frac{u}{a}\right) + C \tag{8}$$

2.4 Reduction Formula

$$\int \sin^n x\ dx = \frac{1}{n}\cos x \sin^{n-1} x + \frac{n-1}{n}\int \sin^{n-2} x\ dx \tag{9}$$

$$\int \cos^n x\ dx = \frac{1}{n}\sin x \cos^{n-1} x + \frac{n-1}{n}\int \cos^{n-2} x\ dx \tag{10}$$

Simple Example 4

Find

$$\int \sin^4(x)dx$$

using reduction formula
Solution:

$$\begin{aligned}
\int \sin^4(x) &= \frac{1}{4}\cos(x)\sin^{4-1}(x) + \frac{4-1}{4}\int \sin^{4-2}(x)dx \\
&= \frac{1}{4}\cos(x)\sin^3(x) + \frac{3}{4}\int \sin^2(x)dx \\
&= \frac{1}{4}\cos(x)\sin^3(x) + \frac{3}{4} * \left(\frac{1}{2}x - \frac{1}{4}\sin(2x)\right) + C \\
&= \frac{1}{4}\cos(x)\sin^3(x) + \frac{3}{8}x - \frac{3}{16}\sin(2x) + C
\end{aligned}$$

2.5 Double Angle Formula

$$2\sin\theta\cos\theta \ = \ \sin 2\theta \tag{11}$$

2.6 Products Of Sin and Cos

Let A and B represent real number

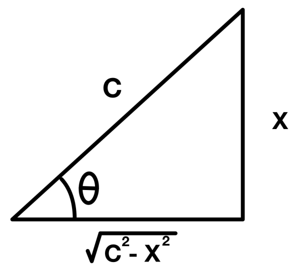
$$\sin(Ax)\cos(Bx) = \frac{1}{2}(\sin(Ax-Bx) + \sin(Ax+B)) \tag{12}$$

$$\sin(Ax)\sin(Bx) = \frac{1}{2}(\cos(Ax-Bx) - \cos(Ax+Bx)) \tag{13}$$

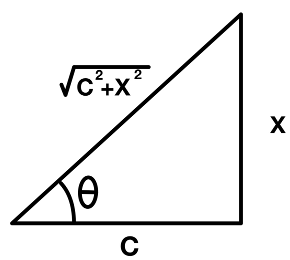
$$\cos(Ax)\cos(Bx) = \frac{1}{2}(\cos(Ax-Bx) + cos(Ax+Bx)) \tag{14}$$

2.7 Sub Into Trig

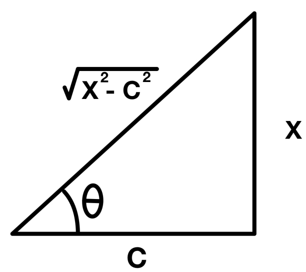
$$\sqrt{c^2 - x^2} \quad Let\ x = c\sin(\theta) \quad \sqrt{c^2 - x^2} = c\cos(\theta) \tag{15}$$



$$\sqrt{c^2 + x^2} \quad Let\ x = c\tan(\theta) \quad \sqrt{c^2 + x^2} = c\sec(\theta) \tag{16}$$



$$\sqrt{x^2 - c^2} \quad Let\ x = c\sec(\theta) \quad \sqrt{x^2 - c^2} = c\tan(\theta) \tag{17}$$



Simple Example 5

Find

$$\int \sqrt{25 - x^2}$$

Let c=5, so that

$$\int \sqrt{25 - x^2} = \int \sqrt{5^2 - x^2}$$

Let $x = 5\sin(\theta)$ and $\sqrt{5^2 - x^2} = 5\cos(\theta)$ so that $\frac{dx}{d\theta} = 5\cos(\theta)$ and $\theta = \sin^- 1(\frac{x}{5})$

$$\begin{aligned}\int \sqrt{5^2 - x^2}dx &= \int 5\cos(\theta) * 5\cos(\theta)d\theta \\ &= 25\int \cos^2(\theta) \\ &= 25 * \left(\frac{1}{2}\theta + \frac{1}{4}\sin(2\theta)\right) \\ &= 25 * \left(\frac{1}{2}\theta + \frac{1}{4} * 2 * \sin(\theta) * \cos(\theta)\right)\end{aligned}$$

Look at the triangle from theorem 15, we can see that $\sin(\theta) = \frac{x}{c}$ and $\cos(\theta) = \frac{\sqrt{5^2 - x^2}}{5}$

$$\begin{aligned}25\left(\frac{1}{2}\theta + \frac{1}{4} * 2 * \sin(x) * \cos(\theta)\right) &= 25\left(\frac{1}{2}\sin^{-1}(x) + \frac{1}{2} * \left(\frac{x}{5}\right) * \left(\frac{\sqrt{5^2 - x^2}}{5}\right)\right) \\ &= 25\left(\frac{1}{2}\sin^{-1}(\frac{x}{5}) + \frac{1}{50}x\sqrt{25 - x^2}\right)\end{aligned}$$

Challenging Example 1

Find

$$\int x\sqrt{16 - 2x^2}dx$$

Let $c = 4$ and $u = \sqrt{2}x$, so that $\frac{du}{dx} = \sqrt{2}$

$$\int x\sqrt{16 - 2x^2}dx = \frac{1}{2}\int u\sqrt{4^2 - u^2}du$$

Let $u = 4\sin(\theta)$ and $\sqrt{4^2 - u^2} = 4\cos(\theta)$
so that $\frac{du}{d\theta} = 4\cos(\theta)$

$$\begin{aligned}\frac{1}{2}\int u\sqrt{4^2 - u^2}du &= \frac{1}{2}\int 4\sin(\theta) * 4\cos(\theta) * 4\cos(\theta)d\theta \\ &= 32\int \sin(\theta)\cos^2(\theta)d\theta\end{aligned}$$

Let $v = \cos(\theta)$, so that $\frac{dv}{d\theta} = -\sin(\theta)$

$$\begin{aligned} 32\int \sin(\theta)\cos^2(\theta)d\theta &= -32\int v^2 dv \\ &= -32 * \frac{1}{3}v^3 + C \\ &= -\frac{32}{3}v^3 + C \\ &= -\frac{32}{3}\cos^3(\theta) + C \end{aligned}$$

Looking at the image of equation 15, we observe that

$$\cos(\theta) = \frac{\sqrt{c^2 - x^2}}{c}$$

$$\begin{aligned} -\frac{32}{3}\cos^3(\theta) &= -\frac{32}{3} * \left(\frac{\sqrt{4^2 - u^2}}{4}\right)^3 \\ &= -\frac{32}{3} * \left(\frac{\sqrt{16 - 2x^2}}{4}\right)^3 \\ &= -\frac{1}{6} * \left(\sqrt{16 - 2x^2}\right)^3 \end{aligned}$$

or use u-substitution lol.

Challenging Example 2

Find

$$\int \frac{x^2}{(4 - x^2)^{\frac{3}{2}}} dx$$

Let $x = 2\sin(\theta)$ and $\sqrt{4 - x^2} = 2\cos(\theta)$
so that $\theta = \sin^{-1}(x)$, $\dfrac{dx}{d\theta} = 2\cos(\theta)$

$$\begin{aligned}
\int \frac{x^2}{(4-x^2)^{\frac{3}{2}}}dx &= \int \frac{4\sin^2(\theta)}{8\cos^3(\theta)} * 2\cos(\theta)d\theta \\
&= \int \frac{8\sin^2(\theta)\cos(\theta)}{8\cos^3(\theta)}d\theta \\
&= \int \frac{\sin^2(\theta)}{\cos^2(\theta)}d\theta \\
&= \int \tan^2(\theta)d\theta \\
&= \int \left(\sec^2(\theta) - 1\right)d\theta \\
&= \tan(\theta) - \theta + C
\end{aligned}$$

Take a look back into the triangle in equation 15, we notice that $\tan(\theta) = \dfrac{x}{\sqrt{4-x^2}}$

$$\tan(\theta) - \theta + C = \tan(\theta) = \frac{x}{\sqrt{4-x^2}} - \sin^{-1}\left(\frac{x}{2}\right) + C$$

3 Partial Fraction

Checklist:

1. Check for long division
2. Factor
3. Write out all variables...
4. Cross multiply
5. Isolate for x, $x^2 \ldots$
6. Linear algebra
7. Sub everything back in

Simple Example 6

Find:

$$\int \frac{x+5}{x^2+3x+2}$$

1. Check for long division:

   We can see that no long division is required.

2. Factor:

$$x^2+3x+2=(x+1)(x+2)$$

3. Write out all variables

$$\frac{x+5}{(x+1)(x+2)}=\frac{A}{x+1}+\frac{B}{x+2}$$

4. Cross multiply

$$x+5=A(x+2)+B(x+1)$$

5. Isolate for x, $x^2 \ldots$

$$x+5=Ax+2A+Bx+B$$
$$x+5=x(A+B)+2A+B$$

6. Linear Algebra

$$=\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 2 & 1 & -5 \end{array}\right]$$
$$=\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & -1 & -7 \end{array}\right]$$
$$=\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 1 & 7 \end{array}\right]$$
$$=\left[\begin{array}{cc|c} 1 & 0 & -6 \\ 0 & 1 & 7 \end{array}\right]$$

   A=-6 B=7

7. Sub everything back in

$$\begin{aligned}
\int \frac{x+5}{(x+1)(x+2)} &= \int \frac{A}{x+1} + \int \frac{B}{x+2} \\
&= \int \frac{-6}{x+1} + \int \frac{7}{x+2} \\
&= -6 \int \frac{1}{x+1} + 7 \int \frac{1}{x+2} \\
&= -6 \ln(x+1) + 7 \ln(x+2) + C
\end{aligned}$$

4 Improper Integral

4.1 Normal Infinite Integral

Formula:

$$\int_c^{\infty} f(x) = \lim_{t \to -\infty} \int_c^t f(x) \tag{18}$$

$$\int_{-\infty}^{c} f(x) = \lim_{t \to -\infty} \int_t^c f(x) \tag{19}$$

c can be any constant.

Simple Example 7

Find:

$$\begin{aligned}
\int_1^{\infty} \frac{1}{x^2} &= \lim_{t \to \infty} \int_1^t \frac{1}{x^2}\, dx \\
&= \lim_{t \to \infty} \frac{1}{x^2}\Big|_1^t \\
&= \lim_{t \to \infty} \left(\frac{1}{t^2} + \frac{1}{1}\right) \\
&= \frac{1}{\infty} + \frac{1}{1} \\
&= 0 + 1 \\
&= 1
\end{aligned}$$

4.2 Double Infinite Integral

Formula:

$$\int_{-\infty}^{\infty} f(x) = \lim_{t \to \infty} \int_c^t f(x) + \lim_{t \to -\infty} \int_t^c f(x) \tag{20}$$

c can be any constant.

4.3 Discontinuity

Given f(x) is continuous at all points between a and b except at x=c

$$\int_a^b f(x)\, dx = \lim_{t \to c^+} \int_t^b f(x) + \lim_{t \to c^-} \int_a^t f(x)\, dx \tag{21}$$

4.4 Comparison Test

Theorem 2

If $f_1(x)$ is always bigger than $f_2(x)$, then if $f_2(x)$ diverges, $f_1(x)$ must diverge.
If $f_2(x)$ is always smaller than $f_1(x)$, then if $f_1(x)$ converge, $f_2(x)$ must converge.

Simple Example 8

In figure 1 blue line represents $\frac{1}{\sqrt{x}}$ and red line represents $\frac{1}{x}$. Since $\frac{1}{x}$ diverges[3] $\frac{1}{\sqrt{x}}$ has to diverge.

Figure 1

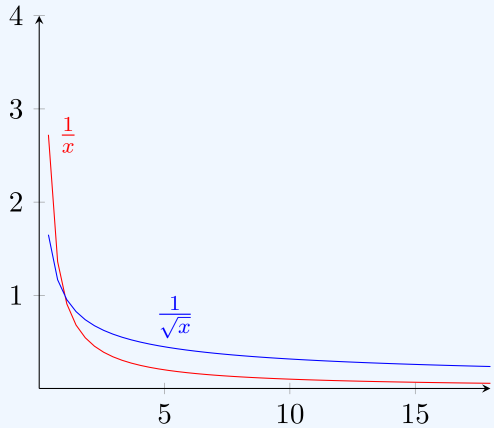
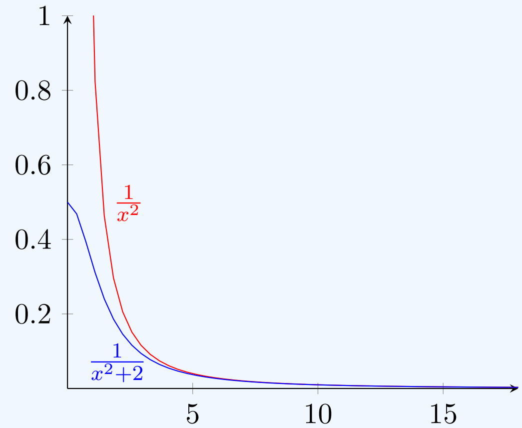


Figure 2



While in figure 2 blue line represents $\frac{1}{x^2+2}$ and red line represents $\frac{1}{x^2}$. Since $\frac{1}{x^2}$ converges[1] $\frac{1}{x^2+2}$ has to converge.

[1]Because of p-series

5 Application of Integration

1. Method of disk(x axis):
$$\int_a^b \pi(f(x))^2 \; dx \tag{22}$$
2. Method of Washer(x axis):
$$\pi \int_a^b (f(x))^2 - (g(x))^2 \; dx \tag{23}$$
3. Method of Shell(y axis):
$$2\pi \int_a^b xf(x) \; dx \tag{24}$$
4. Arc Length:
$$L = \int_a^b \sqrt{1 + (f'(x))^2} \; dx \tag{25}$$
5. Surface Area:
$$A = 2\pi \int_a^b f(x)\sqrt{1 + (f'(x))^2} \; dx \tag{26}$$
6. Work:
$$W = \int_a^b f(x) \; dx \tag{27}$$
   In equation 27 f(x) represent force applied, x is distance travelled.
7. Mass:
$$M = \int_a^b p(x) \; dx \tag{28}$$
   In equation 28 p(x) represent density, x is distance travelled.
8. Moments:
$$M_x = p \int_a^b \frac{1}{2} \left((f(x))^2 - (g(x))^2\right) \tag{29}$$
$$M_y = p \int_a^b x\left(f(x) - g(x)\right) \; dx \tag{30}$$
   p represent density.
9. Center of Mass
$$\bar{x} = \frac{1}{A} \int_a^b x(f(x) - g(x)) \; dx \tag{31}$$
$$\bar{y} = \frac{1}{A} \int_a^b \frac{1}{2} \left((f(x))^2 - (g(x))^2\right) \; dx \tag{32}$$
   A is the area between two functions.

6 First Order ODE

6.1 Separable ODE

Checklist:

1. Seperate it into $f(y)g(x)$
$$\frac{dy}{dx} = f(y) * g(x) \tag{33}$$
2. Move $f(y)$ to other side:
$$\frac{1}{f(y)}\ dy = g(x)\ dx \tag{34}$$
3. Integrate both sides:
$$\int \frac{1}{f(y)}\ dy = \int g(x)\ dx \tag{35}$$
4. Isolate for y if possible

Simple Example 9

Find:

$$\frac{dy}{dx} = \frac{x}{e^y}, \qquad y(2) = 3$$

Solution:

$$\begin{aligned} e^y\ dy &= x\ dx \\ \int e^y\ dy &= \int x\ dx \\ e^y + C_1 &= \frac{1}{2}x^2 + C_2 \\ e^y &= \frac{1}{2}x^2 + C \\ y &= C \ln\left(\frac{1}{2}x^2\right) \end{aligned}$$

Using initial condition:

$$\begin{aligned} 3 &= C \ln\left(\frac{1}{2} * 2^2\right) \\ 3 &= C \ln(2) \\ C &= \frac{3}{\ln(2)} \end{aligned}$$

6.2 Linear ODE

1. Put the given equation into standard form:
$$\frac{dy}{dx} + f(x)y = g(x) \tag{36}$$

2. Find the integrating factor
$$I(x) = e^{\int f(x)\ dx} \tag{37}$$
3. Multiply both sides by integrating factor (Math magic)
$$I(x) * \left(\frac{dy}{dx} + f(x) * y\right) = I(x) * g(x)$$
$$\frac{dy}{dx}\left(I(x) * y\right) = I(x) * g(x)$$
4. Integrate both sides
$$I(x) * y = \int I(x) * g(x) \tag{38}$$
5. Find C

7 Second Order Differential equation

7.1 Step 1: Find solution to homogeneous equation

You are given:

$$ay'' + by' + c = f(x)$$

Find the complementary equation:

$$a\lambda^2 + b\lambda + c = 0$$
$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Three types of solutions:

1. Two different real solutions (λ_1, λ_2)
$$y = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x} \tag{39}$$
2. One real solution (λ)
$$y = C_1 e^{\lambda x} + C_2 x e^{\lambda x} \tag{40}$$
3. Two different imagery solutions $\left(\alpha = -\frac{b}{2a}, \beta = \frac{\sqrt{4ac-b^2}}{2a}\right)$
$$y = e^{\alpha x}\left(C_1 \cos(\beta x) + C_2 \sin(\beta x)\right) \tag{41}$$

7.2 Step 2: Find solution to non-homogeneous equation

Skip this step if f(x) equals 0.

1. Make an educated guess[4]:

f(x)	Educated Guess
c_1x+c_2	Ax+B
c_1x^2+c_2x+c_3	$Ax^2 + Bx + C$
$c_1 e^{\lambda x}$	$Ae^{\lambda x}$
$c_1 e^{\alpha x}\cos\beta x + c_2 e^{\alpha x}\sin\beta x$	$Ae^{\alpha x}\cos\beta x + Be^{\alpha x}\sin\beta x$
$\left(c_1x^2 + c_2x + c_3\right)e^{\lambda x}$	$\left(Ax^2 + Bx + C\right)e^{\lambda x}$
$(c_1x + c_2)\cos(\lambda x)$	$(Ax + B)\cos(\lambda x) + (Cx + D)\sin(\lambda x)$
$(c_1x + c_2)\sin(\lambda x)$	$(Ax + B)\cos(\lambda x) + (Cx + D)\sin(\lambda x)$
$\left(c_1x^2 + x_2x + c_3\right)e^{\lambda x}\cos(\beta x)$	$(Ax^2 + Bx + C)e^{\lambda x}\cos(\beta x) + (Dx^2 + Ex + F)e^{\lambda x}\sin(\beta x)$
$\left(c_1x^2 + x_2x + c_3\right)e^{\lambda x}\sin(\beta x)$	$(Ax^2 + Bx + C)e^{\lambda x}\cos(\beta x) + (Dx^2 + Ex + F)e^{\lambda x}\sin(\beta x)$

Simplified version:

f(x)	Educated Guess
Ploynomial	Same order polynomial
$e^{\lambda x}$	$Ae^{\lambda x}$
$\sin(\lambda x)$ *and/or* $\cos(\lambda x)$	$A\sin(\lambda x) + B\cos(\lambda x)$

[4]Must be same λx in f(x) and guess

2. Let g(x) represent your educated guess, find g'(x) and g''(x):

3. Sub your guess back into the ODE For example:

$$y'' + 2y' + 3y = 10e^{2x}$$

In this case, g(x)=Ae^{2x}, g'(x)=$2Ae^{2x}$, g''(x)=$4Ae^{2x}$:

$$\begin{aligned} g''(x) + 2g'(x) + 3g(x) &= 10e^{2x} \\ 4Ae^{2x} + 2 * 2Ae^{2x} + 3 * Ae^{2x} &= 10e^{2x} \end{aligned}$$

4. Find undetermined coefficient
Simple example:

$$\begin{aligned} 4A + 4A + 3 * A &= 10 \\ A &= \frac{10}{11} \end{aligned}$$

5. Find the particular solution: Sub the coefficients into your guess

$$g(x) = \frac{10}{11}e^{2x}$$

6. The general solution is combination of homogeneous solution and a particular solution:

$$y_{general} = y_{homogeneous} + y_{particular} \tag{42}$$

Challenging Example 3

Find:

$$2y'' + 13y' + 6y = -13te^{4t}$$

1. Find the solution for homo-generous equation

   Complementary equation

$$\begin{aligned} 2\lambda^2 + 13\lambda + 6 &= 0 \\ (2\lambda + 1)(\lambda + 6) &= 0 \\ \lambda_1 = -\frac{1}{2} \quad \lambda_2 &= -6 \end{aligned}$$

   Solution for homo-generous equation:

$$y = C_1 e^{-\frac{1}{2}t} + C_2 e^{-6t}$$

2. Make an educated guess

   Let $g(x) =$

$$(At + B)e^{4t}$$

   so that $g'(x) =$

$$\begin{gathered} (4At + 4B)e^{4t} + Ae^{4t} \\ 4Ate^{4t} + 4Be^{4t} + Ae^{4t} \end{gathered}$$

   and $g''(x) =$

$$\begin{gathered} (16At + 16B)e^{4t} + 4Ae^{4t} + 4Ae^{4t} \\ 16Ate^{4t} + 16Be^{4t} + 8Ae^{4t} \end{gathered}$$

   Sub everything back into the equation:

$$\begin{aligned} 16Ate^{4t} + 16Be^{4t} + 8Ae^{4t} + 6(4Ate^{4t} + 4Be^{4t} + Ae^{4t}) + 5((At + B)e^{4t}) &= -13te^{4t} \\ 16Ate^{4t} + 16Be^{4t} + 8Ae^{4t} + 24Ate^{4t} + 24Be^{4t} + 6Ae^{4t} + 5Ate^{4t} + 5Be^{4t} &= -13te^{4t} \\ 45Ate^{4t} + 45Be^{4t} + 14Ae^{4t} &= -13te^{4t} \\ 45At + 45B + 14A &= -13t \end{aligned}$$

   Using some simple linear algebra:

$$\begin{aligned} &= \left[\begin{array}{cc|c} 45 & 0 & -13 \\ 14 & 45 & 0 \end{array}\right] \\ &= \left[\begin{array}{cc|c} 1 & 0 & -\frac{13}{45} \\ 14 & 45 & 0 \end{array}\right] \\ &= \left[\begin{array}{cc|c} 1 & 0 & -\frac{13}{45} \\ 0 & 45 & \frac{182}{45} \end{array}\right] \\ &= \left[\begin{array}{cc|c} 1 & 0 & -\frac{13}{45} \\ 0 & 1 & \frac{182}{2025} \end{array}\right] \end{aligned}$$

$g(x) =$

$$-\frac{13}{45}te^{4t} + \frac{182}{2025}e^{4t}$$

3. The general solution to the question is:

$$y = C_1e^{-\frac{1}{2}t} + C_2e^{-6t} - \frac{13}{45}te^{4t} + \frac{182}{2025}e^{4t}$$

8 Applications of Second Order Differential Equations

8.1 Simple Oscillation

$$my'' + ky = 0 \tag{43}$$

8.2 Damped Oscillation

$$my'' + by' + ky = 0 \tag{44}$$

8.3 Forced Oscillation

$$my'' + by' + ky = F(t) \tag{45}$$

8.4 Circuit

$$Lq'' + Rq' + \frac{1}{C}q = V(t) \tag{46}$$

9 Series

9.1 Divergence Test

If the series is:

$$\sum_{n=0}^{\infty} a_n$$

if

$$\lim_{n\to\infty} a_n \neq 0$$

then the series diverges.

9.2 Geometric Series

Geometry series takes the form of:

$$\sum_{n=0}^{\infty} ar^n \tag{47}$$

The series only converges to

$$\frac{a}{1-r} \tag{48}$$

if and only if $|r| < 1$

9.3 p-series

p-series takes the form of

$$\frac{1}{n^p} \tag{49}$$

p-series only converges when $p > 1$

9.4 Comparison Test

Theorem 3

If there are two series

$$\sum_{n\to\infty}^{\infty} a_n \tag{50}$$

$$\sum_{n\to\infty}^{\infty} b_n \tag{51}$$

and

$$a_n \text{ is always bigger than or equal to } b_n \; as \; n \to \infty$$

or

$$b_n \text{ is always smaller than or equal to } a_n \; as \; n \to \infty$$

if $\sum_{n\to\infty}^{\infty} a_n$ converges then $\sum_{n\to\infty}^{\infty} b_n$ must also converge.

if $\sum_{n\to\infty}^{\infty} b_n$ diverges then $\sum_{n\to\infty}^{\infty} a_n$ must also diverge. Its really similar to the comparison test of infinite integral.

Theorem 4

The bigger the denominator gets, the smaller the value become

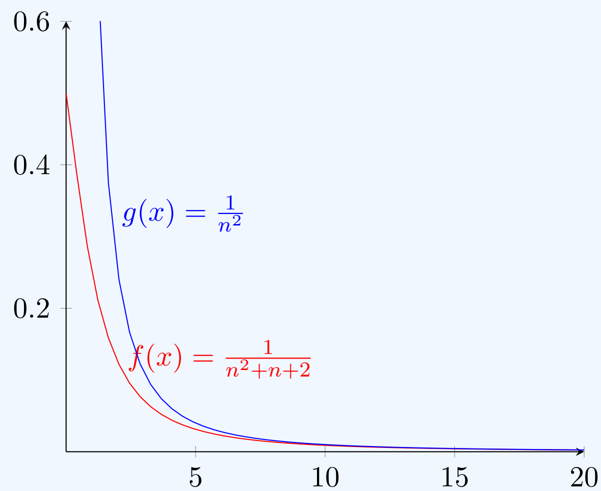
Simple Example 10

Find if

$$\frac{1}{x^2+x+2}$$

will converge.

1. Think of a similar sequence that might help It takes a lot of practice.
   In this case we will use $\frac{1}{x^2}$ for converge
2. Make sure it is actually smaller/bigger than the series of interest.
   We need $\frac{1}{x^2}$ to be smaller than $\frac{1}{x^2+x+2}$



   In this case, $\frac{1}{n^2}$ is always bigger than $\frac{1}{n^2+n+2}$
3. Since $\frac{1}{x^2}$ converges$_{p-series}$, $\frac{1}{x^2+x+2}$ must also converge.

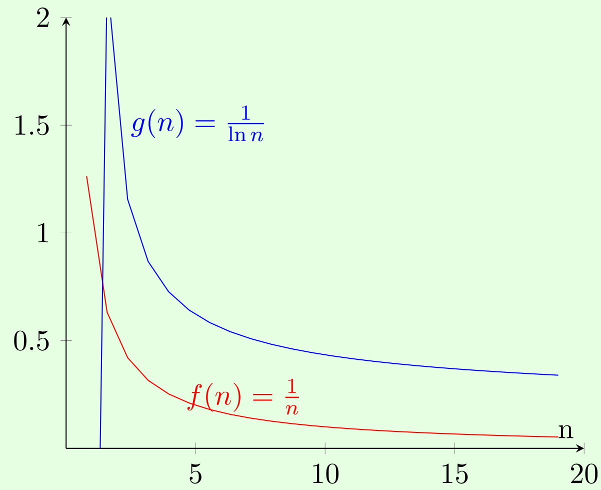
Challenging Example 4

Find if

$$\sum_{n=0}^{\infty} \frac{1}{\ln n}$$

converges.

1. We can use $\frac{1}{n}$ to compare
2. $\frac{1}{n}$ is always going to be smaller than $\frac{1}{\ln n}$



$\frac{1}{\ln n}$ is always bigger than $\frac{1}{n}$ in the longer run.

3. $\frac{1}{n}$ diverges$_p - series$, so $\frac{1}{\ln n}$ has to diverge.

Challenging Example 5

Find if

$$\sum_{n=0}^{\infty} \frac{1}{0.789^n + 0.789^{n-1} + 0.789^{n-2} + \dots}$$

converges

1. Compare it to the equation:
$$\frac{1}{0.789^n}$$

2. Using theorem 2 stated above, we can conclude $\frac{1}{0.789^n+0.789^{n-1}+0.789^{n-2}+\dots}$ is always smaller than $\frac{1}{0.789^n}$ in the long run.

3. $\sum_{n=0}^{\infty} \frac{1}{0.789^n}$ can be written in the form of a geometric series:

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{1}{0.789^n} &= \sum_{n=0}^{\infty} 1 * \frac{1}{0.789^n}{}_{Hint:\sum_{n=0}^{\infty} a*r^n} \\ &= \sum_{n=0}^{\infty} 1 * \left(\frac{1}{0.789}\right)^n\end{aligned}$$

In this case
$$a = 1, \quad r = \frac{1}{0.789}$$

$|r|$ is smaller than 1, subbing variables into $\dfrac{a}{1-r}$

$$\sum_{n=0}^{\infty} \frac{1}{0.789^n} = \frac{1}{1-0.789} \approx 4.73$$

4. Since $\sum_{n=0}^{\infty} \frac{1}{0.789^n}$ converges, $\sum_{n=0}^{\infty} \frac{1}{0.789^n+0.789^{n-1}+0.789^{n-2}+...}$ must also converge

9.5 Integral Test

Given $f(n) = a_n$ for all integer values of n and f(x) is a continuous and decreasing function.

Theorem 5

if $\displaystyle\int_c^{\infty} f(x)$ converges then $\displaystyle\sum_{n=0}^{\infty} a_n$ will converge.

if $\displaystyle\int_c^{\infty} f(x)$ diverges then $\displaystyle\sum_{n=0}^{\infty} a_n$ will diverge.

c can be any finite integer

Simple Example 11

Find if

$$\frac{1}{n^2}$$

converges?

We can take a look at the integral:

$$\begin{aligned}
\int_1^{\infty} \frac{1}{n^2} dn &= \lim_{t\to\infty} \int_0^t \frac{1}{n^2} dn \\
&= \lim_{t\to\infty} -\frac{1}{n}\bigg|_1^t \\
&= \lim_{t\to\infty} -\left(\frac{1}{t} - \frac{1}{1}\right) \\
&= -(0-1) \\
&= 1
\end{aligned}$$

Since the integral $\displaystyle\int_1^{\infty} \frac{1}{n^2}$ converges, the series will converge.

The challenge to this kind of questions is to solve the integral, not apply integral test.

9.6 Alternating Series

Theorem 6

If a series takes one of these forms:

$$\sum_{n=0}^{\infty}(-1)^n a_n \tag{52}$$

$$\sum_{n=0}^{\infty}(-1)^{n+1} a_n \tag{53}$$

and a_n is a constantly decreasing towards 0.
The series is going to converge.

In simpler words, the requirements for an alternating series are:

1. Constantly alternating between positive and negative
2. $\lim\limits_{n\to\infty} = 0$
3. $a_n \geq a_{n+1}$

Simple Example 12

Will

$$\sum_{n=0}^{\infty}\frac{(-1)^n n}{n^3}$$

converge?
Solution: We will check if it satisfies the conditions of an alternating series.

1. The function is going to switch between positive and negative
2. $\lim\limits_{n\to\infty}\frac{n}{n^3} = 0$
3. $\frac{1}{(n+1)^2} \leq \frac{1}{n^2}$

The series qualifies for all three conditions, therefore it is an alternating series and must converge.

9.7 Ratio Test

Theorem 7: Absolute Convergence

When $\sum\limits_{n=0}^{\infty}|a_n|$ converges, $\sum\limits_{n=0}^{\infty}a_n$ must also converge.

Theorem 8: Conditional Convergence

When $\sum_{n=0}^{\infty} a_n$ converges, $\sum_{n=0}^{\infty} |a_n|$ might not converge.

Theorem 9

Given a function in the form:

$$\sum_{n=0}^{\infty} a_n$$

Find

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| \tag{54}$$

There are three possible outcomes

1. The limit is smaller than $1_{including\ 0}$, the series is absolute convergent.
2. The limit is equal to 1, the test is inconclusive.
3. the limit is greater than $1_{including\ \infty}$, the series diverges

Ratio test is really helpful when you have exponents or factorial

9.8 Root Test

This test is really similar to ratio test.

Theorem 10

Given a function in the form:

$$\sum_{n=0}^{\infty} a_n$$

Find

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} \tag{55}$$

There are three possible outcomes

1. The limit is smaller than $1_{including\ 0}$, the series is absolute convergent.
2. The limit is equal to 1, the test is inconclusive.
3. the limit is greater than $1_{including\ \infty}$, the series diverges

9.9 Checklist of Tests

1. Comparison Test (p-series/geometric series)
2. Alternating Series Test ($(-1)^n$)
3. Ratio Test (exponents or factorial)

4. Integral Test (decreasing, positive and continuous)
5. Divergent Test (Last resort)

9.10 Power Series

Theorem 11

A power series centered at x_0 is a series takes the form:

$$\sum_{n=0}^{\infty} c_n(x-x_0)^n \tag{56}$$

Unlike before, power series has a variable. x can be any value and it will influence whether the power series converges.

Simple Example 13

We can take a look at

$$\sum_{n=0}^{\infty} \frac{(x)^n}{2^n}$$

If we let x=5

$$\sum_{n=0}^{\infty} \frac{5^n}{2^n} = \sum_{n=0}^{\infty} \left(\frac{5}{2}\right)^n$$

It becomes a geometric series. Since $|r|$ is bigger than 1, the series diverges.

If we let x=1

$$\sum_{n=0}^{\infty} \frac{1^n}{2^n} = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n$$

In this case $|r|$ is smaller than 1, the series converges.
Note that not all power series are this simple!

Theorem 12

There are three possible cases of x that will convergence the series:

1. Only converge at x_0
2. Converges within an interval
3. Converges for all x

It should be noted the series will always converge at x_0.

When you are finding interval of convergence for some series, always try ratio test.

Simple Example 14

Find the interval of convergence for

$$\sum_{n=0}^{\infty} \frac{x^n}{n * 3^n}$$

Solution:

$$\begin{aligned}
\lim_{n\to\infty} \left| \frac{\frac{x^{n+1}}{(n+1)3^{n+1}}}{\frac{x^n}{n*3^n}} \right| &= \lim_{n\to\infty} \left| \frac{x * n}{(n+1)3} \right| \\
&= \lim_{n\to\infty} \left| \frac{x}{3} \frac{n}{n+1} \right| \\
&= \left| \frac{x}{3} \right|
\end{aligned}$$

$\left| \frac{x}{3} \right|$ has to be smaller than 1 for the series will converge.

$$\begin{aligned}
\left| \frac{x}{3} \right| &< 1 \\
-1 < \frac{x}{3} &< 1 \\
-3 < x &< 3
\end{aligned}$$

Now that we know the radius of convergence, we need to find if the boundaries converge.

When x=-3:

$$\begin{aligned}
&= \sum_{n=0}^{\infty} \frac{(-3)^n}{n * 3^n} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n 3^n}{n * 3^n} \\
&= \sum_{n=0}^{\infty} (-1)^n \frac{1}{n}
\end{aligned}$$

Alternating series, the series converges when x=-3

When x=3:

$$\begin{aligned}
&= \sum_{n=0}^{\infty} \frac{(3)^n}{n * 3^n} \\
&= \sum_{n=0}^{\infty} \frac{1}{n}
\end{aligned}$$

p-series, the series diverges at x=3

The interval of convergence is $[-3 < x < 3)$

Geometric series can be considered as a power series centered at 0 and all constant $c_n = a$.

$$\sum_{n=0}^{\infty} ar^n = \sum_{n=0}^{\infty} c_n x^n$$

Simple Example 15

Express

$$f(x) = \frac{3}{1+x^3}$$

as a series.
Solution:

$$f(x) = \frac{3}{1+x^3} = \frac{3}{1-(-x^3)}$$
$$= \sum_{n=0}^{\infty} 3(-x^3)^n$$

9.11 Term by Term Differentiation and Integration

Theorem 13

$$f(x) = \sum_{n=0}^{\infty} c_n (x - x_0)^n \tag{57}$$

$$f'(x) = \sum_{n=1}^{\infty} c_n * n * (x - x_0)^{n-1} \tag{58}$$

$$\int f(x)dx = \sum_{n=1}^{\infty} c_n \frac{(x-x_0)^{n+1}}{n+1} + C \tag{59}$$

P.S. $f(0) = C$ and remember n=1 for differentiation of series.

9.12 Taylor Series

9.12.1 Function to Taylor Series

Theorem 14

Taylor series of f(x) is:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^n(x_0)}{n!}(x - x_0)^n \tag{60}$$

Simple Example 16

$$f(x) = \sin(x)$$

Find f(x) represented using power series centered at $x_0 = \frac{\pi}{2}$

$$
\begin{aligned}
f(x) &= \sin(x) & f(\tfrac{\pi}{2}) &= 1\\
f'(x) &= \cos(x) & f'(\tfrac{\pi}{2}) &= 0\\
f''(x) &= -\sin(x) & f''(\tfrac{\pi}{2}) &= -1\\
f^3(x) &= -\cos(x) & f^3(\tfrac{\pi}{2}) &= 0\\
f^4(x) &= \sin(x) & f^4(\tfrac{\pi}{2}) &= 1
\end{aligned}
$$

$$
\begin{aligned}
f(x) = \sin(\frac{\pi}{2})& \\
&+ \frac{\cos(\frac{\pi}{2})}{1!}\left(x-\frac{\pi}{2}\right)\\
&+ \frac{-\sin(\frac{\pi}{2})}{2!}\left(x-\frac{\pi}{2}\right)^2\\
&+ \frac{-\cos(\frac{\pi}{2})}{3!}\left(x-\frac{\pi}{2}\right)^3\\
&+ \frac{\sin(x)}{4!}\left(x-\frac{\pi}{2}\right)^4\dots\\
f(x) = 1 + 0 - \frac{1}{2}\left(x-\frac{\pi}{2}\right)^2 &+ \frac{1}{24}\left(x-\frac{\pi}{2}\right)^4\dots
\end{aligned}
$$

By looking at the pattern we observe that

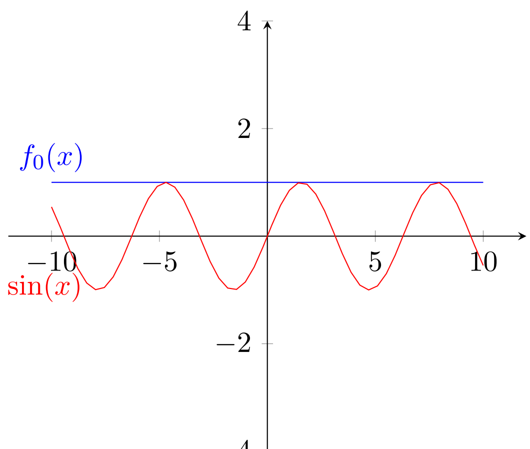
$$f(x) = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\left(x-\frac{\pi}{2}\right)^{2n}$$

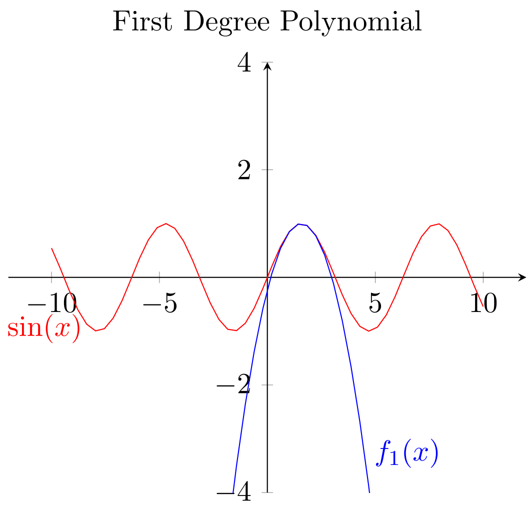
9.12.2 Visualization of Taylor Series

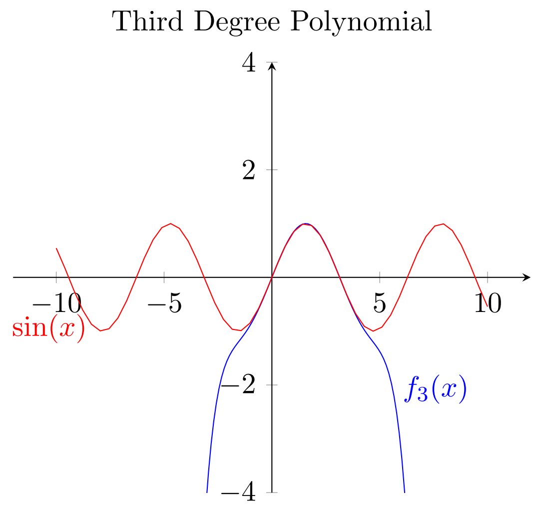
You don't need to know this for the course, but its pretty fun.

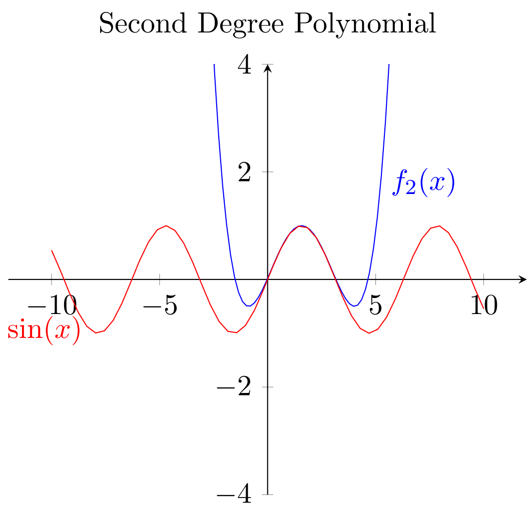
$$
\begin{aligned}
f_0(x) &= 1\\
f_1(x) &= 1 - \frac{1}{2}\left(x-\frac{\pi}{x}\right)^2\\
f_2(x) &= 1 - \frac{1}{2}\left(x-\frac{\pi}{x}\right)^2 + \frac{1}{24}\left(x-\frac{\pi}{2}\right)^4\\
f_3(x) &= 1 - \frac{1}{2}\left(x-\frac{\pi}{x}\right)^2 + \frac{1}{24}\left(x-\frac{\pi}{2}\right)^4 - \frac{1}{720}\left(x-\frac{\pi}{2}\right)^6\\
f_4(x) &= 1 - \frac{1}{2}\left(x-\frac{\pi}{x}\right)^2 + \frac{1}{24}\left(x-\frac{\pi}{2}\right)^4 - \frac{1}{720}\left(x-\frac{\pi}{2}\right)^6 + \frac{1}{40320}\left(x-\frac{\pi}{2}\right)^8
\end{aligned}
$$

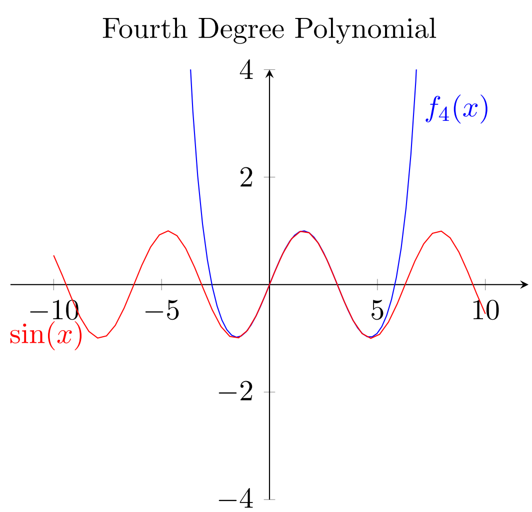
Zero Degree Polynomial











As the number of Taylor polynomial terms increases, the series approaches the original function.

Theorem 15

Maclaurin Series are Taylor Series centered at x=0

$$\sum_{n=0}^{\infty} \frac{f^n(0)}{n!} x^n \tag{61}$$

9.13 Estimation using Series

9.13.1 Alternating Series

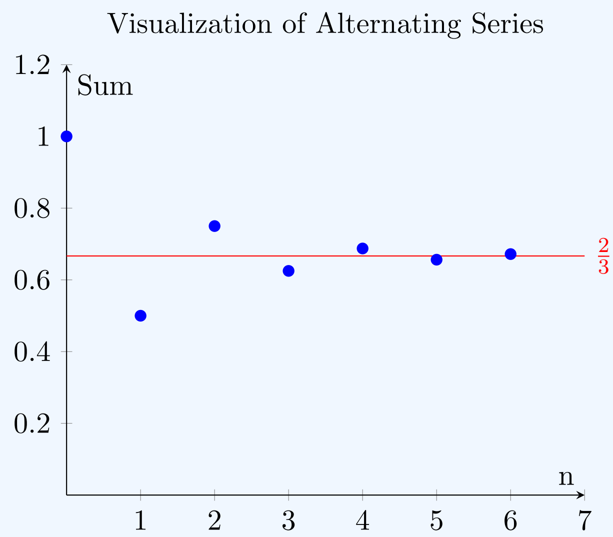
Let's do another visualization:

Simple Example 17

Let S_N denote the sum of N terms of the alternating series. We will try to approximate:

$$\sum_{n=0}^{\infty} (-1)^n \frac{1}{2^n}$$

$$\begin{aligned}
S_0 &= 1 \\
S_1 &= 1 - \frac{1}{2} \\
S_2 &= 1 - \frac{1}{2} + \frac{1}{4} \\
S_3 &= 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} \\
S_4 &= 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} \\
S_5 &= 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \frac{1}{32} \\
S_6 &= 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \frac{1}{32} + \frac{1}{64}
\end{aligned}$$



As you can verify, the series sums to $\frac{2}{3}$.

Theorem 16

Let S represent the sum of an alternating series.

$$\sum_{n=0}^{\infty}(-1)^n a_n$$

Let R_N represent the remainder of the series so that $R_N = S - S_N$.

$$|R_N| \leq a_{n+1} \tag{62}$$

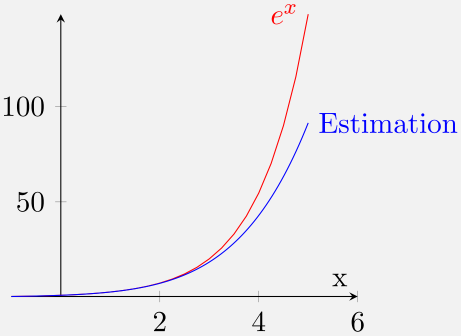
In simpler words, the distance from S_N to series value is always smaller than distance to S_{N+1}.

9.13.2 Taylor Series

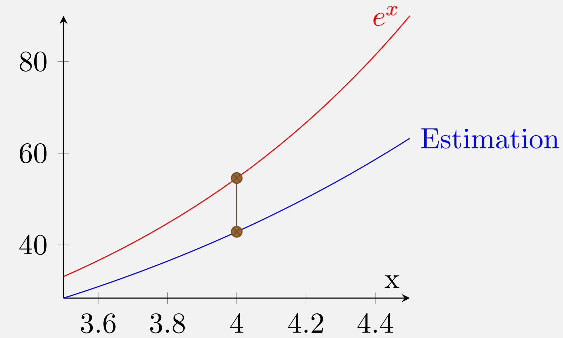
Theorem 17

Let S denote the value of f(x) at x(Top dot).

Fifth order Taylor Series estimation of e^x



Zoom in at the error at x



Let S_N represent the sum of the series evaluated at x for N terms.(Bottom dot)
Let R_N represent the error such that $R_n = S - S_N$(The line connecting two dots).

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x - x_0|^{n+1} \tag{63}$$

Where M is the maximum value of f'(x) between x and x_0

Proof. Bruh, just trust me. □

Theorem 18: Taylor Series Error Bound

To prove that a series is equal to a function:

$$\lim_{n\to\infty} \frac{M}{(n+1)!}(x - x_0)^{n+1} = 0 \tag{64}$$

and

$$-\lim_{n\to\infty} \frac{M}{(n+1)!}(x - x_0)^{n+1} = 0 \tag{65}$$

Proof. Using theorem Taylor Series Error Bound:

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x - x_0|^{n+1}$$

We can conclude:

$$-\frac{M}{(n+1)!}|x - x_0|^{n+1} \leq R_n(x) \leq \frac{M}{(n+1)!}|x - x_0|^{n+1}$$

If both sides equal to 0 as n approaches infinity, it forces $R_n(x)$ to be equal to zero according to squeezing theorem. If there is no error, the series and the function are equivalent. □

Simple Example 18

Prove the Taylor series of $\sin(x)$ is

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \left(x - \frac{\pi}{2}\right)^{2n}$$

Solution:
$f'(x) = \cos(x)$ We know that the maximum for a cosine function is going to be 1, therefore, M=1.

$$\begin{aligned} \lim_{n\to\infty} \frac{1}{(n+1)!}|x - x_0|^{n+1} &= \lim_{n\to\infty} \frac{x^n}{n!} \\ &= 0 \end{aligned}$$

Similar calculation can be perform on $-\frac{M}{(n+1)!}|x - x_0|^{n+1}$.
Since both conditions are satisfied, this Taylor series is equivalent to the function.

10 Parametric Equation

10.1 Eliminate the parameter

Checklist:

1. Express x or y in terms of t
2. Sub t value

Simple Example 19

Eliminate the parameter

$$y = 2t + 1,\ x = 1 + t$$

Solution:

$$\begin{aligned} x &= 1 + t \\ x - 1 &= t \end{aligned}$$

Sub $t = x - 1$ into $y = 2t + 1$

$$\begin{aligned} y &= 2t + 1 \\ y &= 2(x-1) + 1 \\ y &= 2x - 1 \end{aligned}$$

10.2 Derivative of a parametric equation

$$\frac{dy}{dx} = \frac{\frac{d}{dt}y(t)}{\frac{d}{dt}x(t)} \tag{66}$$

Simple Example 20

Find the derivative of:

$$x = \sin(t),\ y = \cos(t)$$

Solution:

$$\begin{aligned} \frac{dy}{dx} &= \frac{\frac{d}{dt}\sin(t)}{\frac{d}{dt}\cos(t)} \\ &= \frac{\cos(t)}{-\sin(t)} \\ &= \cot(t) \end{aligned}$$

10.3 Second derivative of a parametric equation

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dx}\left(\frac{dy}{dx}\right)}{\frac{d}{dt}x(t)} \tag{67}$$

Simple Example 21

Find second derivative of:

$$x = \sin(t),\ \ y = \cos(t)$$

Solution:
Using what we calculated in the previous simple example:

$$\frac{dy}{dx} = \cot(t)$$

Hint: $\frac{d}{dt}\cot(t) = -\csc^2(t)$

10.4 Area of a parametric equation

$$A = \int_a^b y(t)x'(t)dt \tag{68}$$

$x(a)$ must be less than $x(b)$ to use this equation

10.5 Length of parametric equation curve

$$\int_a^b \sqrt{(x'(t))^2 + (y'(t))^2} \tag{69}$$

Simple Example 22

Let

$$x(t) = 2\cos(t),\ \ y(t) = 2\sin(t)$$

Find curve length between t=4 and t=10

Solution:
$\frac{d}{dt}x(t) = -2\sin(t)$ $\frac{d}{dt}y(t) = 2\cos(t)dt$

$$\begin{aligned}\int^1 0_4\sqrt{(-2\sin(t))^2 + (2\cos(t))^2}dt &= \int^1 0_4\sqrt{4\sin^2(t) + 4\cos^2(t)}dt \\ &= \int^1 0_4\sqrt{4}dt \\ &= \int^1 0_4 2dt \\ &= 2\ |_4^{10} \\ &= 20 - 8 \\ &= 12\end{aligned}$$

11 Polar Form

11.1 Derivative of Polar Form

$$\frac{dy}{dx} = \frac{\frac{dr}{d\theta}\sin(\theta) + r\cos(\theta)}{\frac{dr}{d\theta}\cos(\theta) - r\sin(\theta)} \tag{70}$$

11.2 Switching to coordinates

$$r\sin(\theta) = y \tag{71}$$

$$r\cos(\theta) = x \tag{72}$$

11.3 Switching from coordinates

$$r = \sqrt{x^2 + y^2} \tag{73}$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) \tag{74}$$

11.4 Polar Equation That You Need To Remember

11.4.1 Lines

$$r\cos(\theta) = a \tag{75}$$

$$r\sin(\theta) = b \tag{76}$$

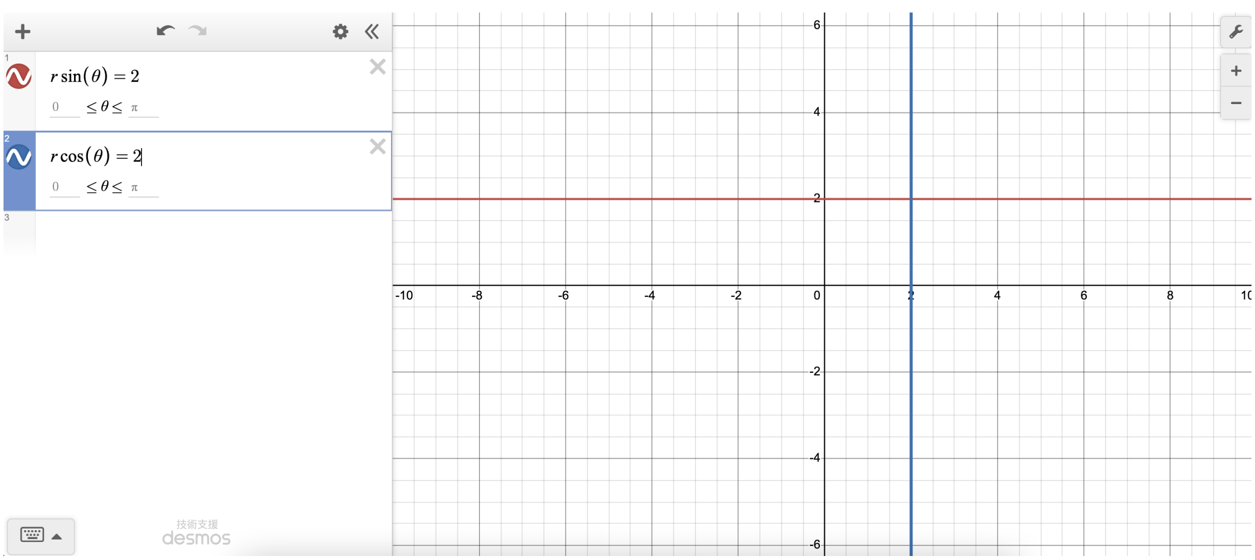


Figure 1: Polar Lines [2]

[2]Created using Desmos

11.4.2 Cardioid

$$a - b\cos(\theta) \tag{77}$$

There are 3 cases with cardioid: $a > b$, $a = b$ and $a < b$

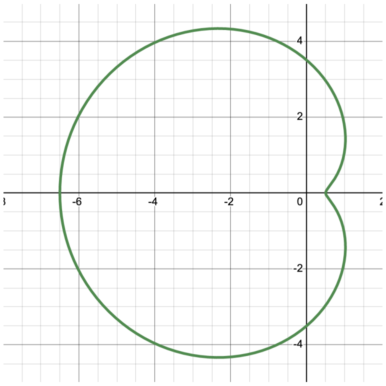


Figure 2: $a > b$ [2]

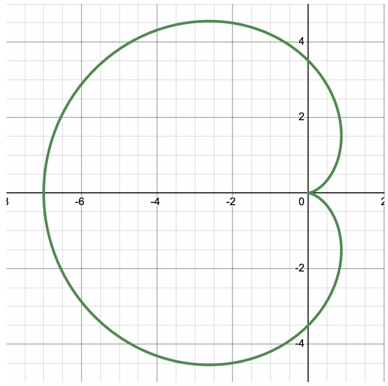


Figure 3: $a = b$ [2]

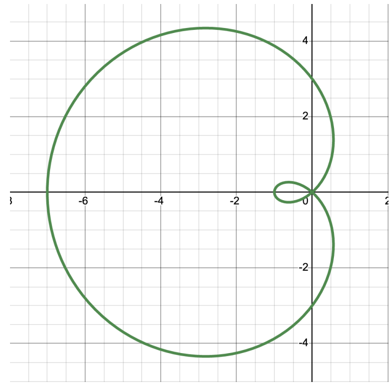


Figure 4: $a < b$ [2]

That's footnote, not exponent.

If it is a plus sign, simply flip everything

$$a + b\cos(\theta) \tag{78}$$

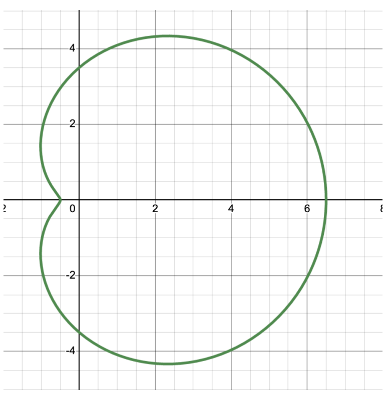


Figure 5: $a > b$ [2]

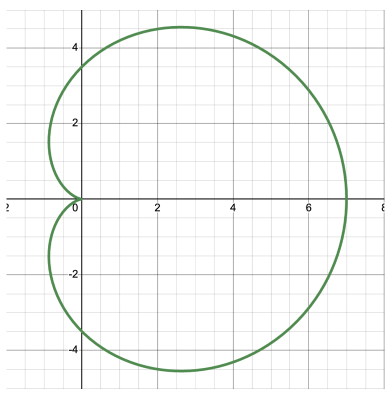


Figure 6: $a = b$ [2]

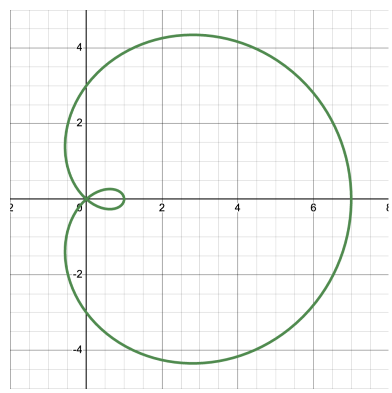


Figure 7: $a < b$ [2]

[2]Created using Desmos

$$a - b\sin(\theta) \tag{79}$$

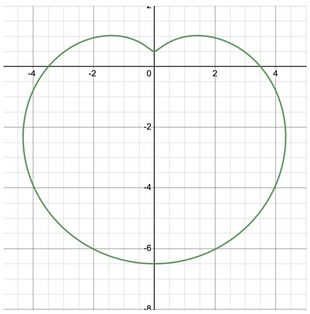


Figure 8: $a > b$ [2]

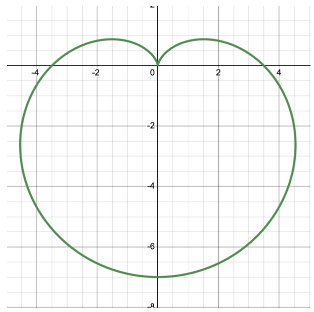


Figure 9: $a = b$ [2]

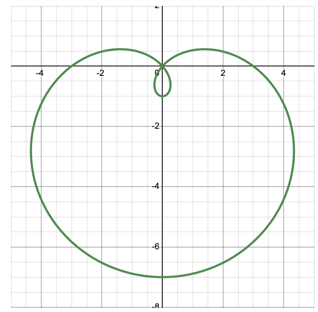


Figure 10: $a < b$ [2]

$$a + b\sin(\theta) \tag{80}$$

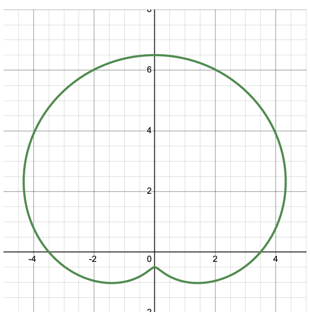


Figure 11: $a > b$ [2]

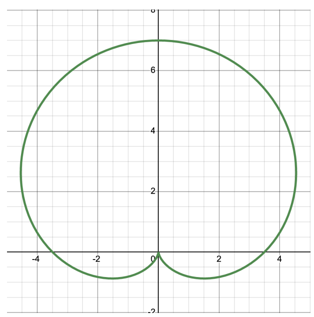


Figure 12: $a = b$ [2]

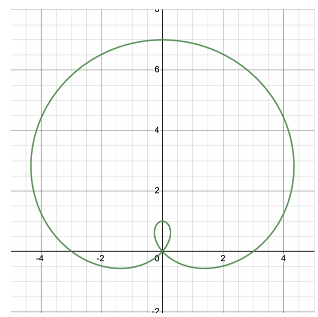


Figure 13: $a < b$ [2]

11.5 Area of Polar Form

11.5.1 Area of one polar equation

$$A = \frac{1}{2}\int_a^b r^2 d\theta \tag{81}$$

11.5.2 Area between two polar equations

$$A = \frac{1}{2}\int_a^b r_1^2 - r_2^2 d\theta \tag{82}$$

11.6 Curve Length of Polar Form

$$L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta \tag{83}$$

[2]Created using Desmos

12 Vector

I trust you took linear algebra before.

Theorem 19

Given a vector equation:

$$r(t) = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k} \tag{84}$$

Derivative of the equation is:

$$r'(t) = f'(t)\vec{i} + g'(t)\vec{j} + h'(t)\vec{k} \tag{85}$$

Integral of the equation is:

$$\int r(t) = \int f(t)\vec{i} + \int g(t)\vec{j} + \int h(t)\vec{k} + C \tag{86}$$

C is any vector <a, b, c> where a, b and c are all constants.

12.1 Magnitude of Vector

$$||r'(t)|| = \sqrt{(f'(t))^2 + (g'(t))^2 + (h'(t))^2} \tag{87}$$

12.2 Unit vector

We are interested in unit tangent vector:

$$T(t) = \frac{r'(t)}{||r'(t)||} \tag{88}$$

12.3 Principal/Unit Normal Vector

$$N(t) = \frac{T'(t)}{||T'(t)||} \tag{89}$$

12.4 Binormal Vector

$$B(t) = T(t) \times N(t) \tag{90}$$

12.5 Arc Length

$$L = \int_a^b \sqrt{(f'(t))^2 + (g'(t))^2 + (h'(t))^2} dt = \int_a^b ||r'(t)|| dt \tag{91}$$

12.6 Arc Length Function

Arc length function is basically expressing the function with distance traveled as the domain.

Checklist:

1. Find the distance traveled with respect to t
2. Isolate for t
3. Sub the equation back into r(t)

Simple Example 23

Given

$$r(t) =< 2\sin(t), 2\cos(t) >$$

find arc length function.
Solution:
Step 1:

$$\begin{aligned} S &= \int_0^t ||r'(\theta)||d\theta \\ &= \int_0^t \sqrt{4\sin^2(\theta) + 4\cos^2(\theta)}d\theta \\ &= \int_0^t \sqrt{4}d\theta \\ &= \int_0^t 2d\theta \\ &= 2t \end{aligned}$$

Step 2:

$$\begin{aligned} S &= 2t \\ \frac{S}{2} &= t \end{aligned}$$

Step 3:

$$r(t) = \left\langle 2\sin\left(\frac{S}{2}\right), 2\cos\left(\frac{S}{2}\right)\right\rangle$$

If starting at another point:
Checklist:

1. Find t_0 at that point
2. $\int_{t_0}^t ||r'(t)||dt$

Simple Example 24

Given the equation:

$$r(t) =< t^2 + 3, 2\cos\pi t >$$

find t_0 when $r(t) =< 12, -2 >$
Solution: t_0=3

12.7 Curvature

$$\kappa = \frac{||T'(t)||}{||r'(t)||} = \frac{||r'(t) \times r''(t)||}{||r'(t)||^3}$$

12.8 Acceleration

Theorem 20

Acceleration contains two components:

a_T is tangential acceleration $\qquad$ a_n is normal/perpendicular acceleration

$$\vec{a}(t) = a_T\vec{T} + a_n\vec{N} \tag{92}$$

Where $\vec{T}$ is unit tangent vector and $\vec{N}$ is unit normal vector.
Magnitude of tangential component is:

$$a_T = \frac{r'(t) \cdot r''(t)}{||r'(t)||} \tag{93}$$

Magnitude of normal component is:

$$a_n = \frac{||r'(t) \times r''(t)||}{||r'(t)||} \tag{94}$$

Theorem 21

If a cylinder can be expressed with equation $x^2 + y^2 = a$, think of it as a cylinder with a radius of $\frac{a}{2}$ centered at origin.

Challenging Example 6

Find the intersection of cylinder:

$$x^2 + y^2 = a$$

and plane

$$x + y + z = b$$

Solution:
We can create a variable t such that x and y value can be express in the form of $\frac{a}{2}\cos(t)$ and $\frac{a}{2}\sin(t)$ respectively.Taking these values back into plane equation:

$$x + y + z = b$$
$$\frac{a}{2}\cos(t) + \frac{a}{2}\sin(t) + z = b$$
$$z = b + \frac{a}{2}\cos(t) + \frac{a}{2}\sin(t)$$

Therefore the intersection can be described by vector equation:

$$r(t) = \left\langle \frac{a}{2}\cos(t), \frac{a}{2}\sin(t), b + \frac{a}{2}\cos(t) + \frac{a}{2}\sin(t) \right\rangle$$