

# PATTERN RECOGNITION AND MACHINE LEARNING

Slide Set 3: DetectionTheory

January 2017

#### Heikki Huttunen

heikki.huttunen@tut.fi

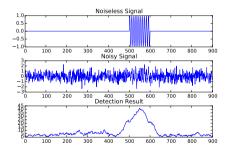
Department of Signal Processing Tampere University of Technology

## **Detection theory**

- In this section, we will briefly consider detection theory.
- Detection theory has many common topics with machine learning.
- The methods are based on estimation theory and attempt to answer questions such as
  - Is a signal of specific model present in our time series? E.g., detection of noisy sinusoid; beep or no beep?
  - Is the transmitted pulse present at radar signal at time t?
  - Does the mean level of a signal change at time t?
  - After calculating the mean change in pixel values of subsequent frames in video, is there something moving in the scene?
  - Is there a person in this video frame?

# **Detection theory**

- The area is closely related to hypothesis testing, which is widely used e.g., in medicine: Is the response in patients due to the new drug or due to random fluctuations?
- Consider the detection of a sinusoidal waveform



### **Detection theory**

In our case, the hypotheses could be

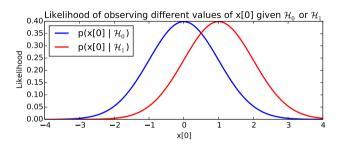
$$\mathcal{H}_1: x[n] = A\cos(2\pi f_0 n + \phi) + w[n]$$
  
$$\mathcal{H}_0: x[n] = w[n]$$

- This example corresponds to detection of noisy sinusoid.
- The hypothesis  $\mathcal{H}_1$  corresponds to the case that the sinusoid is present and is called *alternative hypothesis*.
- The hypothesis  $\mathcal{H}_0$  corresponds to the case that the measurements consists of noise only and is called *null hypothesis*.

- Neyman-Pearson approach is the classical way of solving detection problems in an optimal manner.
- It relies on so called Neyman-Pearson theorem.
- Before stating the theorem, consider a simplistic detection problem, where we observe one sample x[0] from one of two densities:  $\mathcal{N}(0,1)$  or  $\mathcal{N}(1,1)$ .
- The task is to choose the correct density in an optimal manner.
- Our hypotheses are now

$$\mathcal{H}_1 : \mu = 1$$
,  $\mathcal{H}_0 : \mu = 0$ ,

and the corresponding likelihoods are plotted below.



- An obvious approach for deciding the density would choose the one, which is higher for a particular x[0].
- More specifically, study the likelihoods and choose the more likely one.

The likelihoods are

$$\mathcal{H}_1: p(x[0] \mid \mu = 1) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(x[n] - 1)^2}{2}\right).$$

$$\mathcal{H}_0: p(x[0] \mid \mu = 0) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(x[n])^2}{2}\right).$$

- One should select  $\mathcal{H}_1$  if " $\mu = 1$ " is more likely than " $\mu = 0$ ".
- In other words,  $p(x[0] | \mu = 1) > p(x[0] | \mu = 0)$ .

• Let's state this in terms of x[0]:

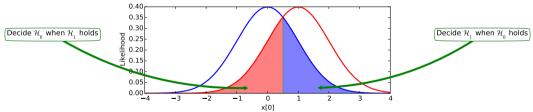
$$\begin{aligned} & p(x[0] \mid \mu = 1) > p(x[0] \mid \mu = 0) \\ & \Leftrightarrow \frac{p(x[0] \mid \mu = 1)}{p(x[0] \mid \mu = 0)} > 1 \\ & \Leftrightarrow \frac{\frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(x[n]-1)^2}{2}\right)}{\frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(x[n])^2}{2}\right)} > 1 \\ & \Leftrightarrow \exp\left(-\frac{(x[n]-1)^2 - x[n]^2}{2}\right) > 1 \end{aligned}$$

$$\Leftrightarrow (x[n]^2 - (x[n] - 1)^2) > 0$$
  
$$\Leftrightarrow 2x[n] - 1 > 0$$
  
$$\Leftrightarrow x[n] > \frac{1}{2}.$$

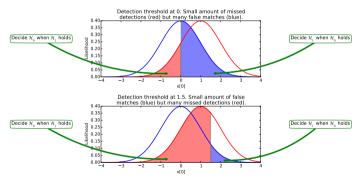
- In other words, choose  $\mathcal{H}_1$  if x[0] > 0.5 and  $\mathcal{H}_0$  if x[0] < 0.5.
- Studying the ratio of likelihoods on the second row of the derivation is the key.
- This ratio is called *likelihood ratio*, and comparison to a threshold  $\gamma$  (here  $\gamma=1$ ) is called *likelihood ratio test* (LRT).
- Of course the threshold  $\gamma$  may be chosen other than  $\gamma = 1$ .

- It might be that the detection problem is not symmetric and some errors are more costly than others.
- For example, when detecting a disease, a missed detection is more costly than a false alarm.
- The tradeoff between misses and false alarms can be adjusted using the threshold of the LRT.

- The below figure illustrates the probabilities of the two kinds of errors.
- The blue area on the left corresponds to the probability of choosing  $\mathcal{H}_1$  while  $\mathcal{H}_0$  would hold (false match).
- The red area is the probability of choosing  $\mathcal{H}_0$  while  $\mathcal{H}_1$  would hold (missed detection).



• It can be seen that we can decrease either probability arbitrarily small by adjusting the detection threshold.



- Both probabilities can be calculated.
- Probability of false alarm for the threshold  $\gamma = 1.5$  is

$$P_{FA} = P(x[0] > \gamma \mid \mu = 0) = \int_{1.5}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(x[n])^2}{2}\right) dx[n] \approx 0.0668.$$

Probability of missed detection is

$$P_M = P(x[0] < \gamma \mid \mu = 1) = \int_{-\infty}^{1.5} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(x[n]-1)^2}{2}\right) dx[n] \approx 0.6915.$$

 An equivalent, but more useful term is the complement of P<sub>M</sub>: probability of detection:

$$P_D = 1 - P_M = \int_{1.5}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(x[n] - 1)^2}{2}\right) dx[n] \approx 0.3085.$$



#### Neyman-Pearson Theorem

- Since  $P_{FA}$  and  $P_D$  depend on each other, we would like to maximize  $P_D$  subject to given maximum allowed  $P_{FA}$ . Luckily the following theorem makes this easy.
- **Neyman-Pearson Theorem:** For a fixed  $P_{FA}$ , the likelihood ratio test maximizes  $P_D$  with the decision rule

$$L(\mathbf{x}) = \frac{\rho(\mathbf{x}; \mathcal{H}_1)}{\rho(\mathbf{x}; \mathcal{H}_0)} > \gamma,$$

with threshold  $\gamma$  is the value for which

$$\int_{\mathbf{x}:L(\mathbf{x})>\gamma} p(\mathbf{x};\mathcal{H}_0) d\mathbf{x} = P_{FA}.$$

#### Neyman-Pearson Theorem

- As an example, suppose we want to find the best detector for our introductory example, and we can tolerate 10% false alarms ( $P_{FA} = 0.1$ ).
- According to the theorem, the detection rule is:

Select 
$$\mathcal{H}_1$$
 if  $\frac{p(x \mid \mu = 1)}{p(x \mid \mu = 0)} > \gamma$ 

The only thing to find out now is the threshold  $\gamma$  such that

$$\int_{\gamma}^{\infty} p(x \mid \mu = 0) dx = 0.1.$$

#### Neyman-Pearson Theorem

 This can be done with Python function isf, which solves the inverse cumulative distribution function.

```
>>> import scipy.stats as stats
>>> # Compute threshold such that P_FA = 0.1
>>> T = stats.norm.isf(0.1, loc = 0, scale = 1)
>>> print T
1.28155156554
```

 The parameters loc and scale are the mean and standard deviation of the Gaussian density, respectively.

- The NP approach applies to all cases where likelihoods are available.
- An important special case is that of a known waveform s[n] embedded in WGN sequence w[n]:

$$\mathcal{H}_1: x[n] = s[n] + w[n]$$
  
$$\mathcal{H}_0: x[n] = w[n].$$

 An example of a case where the waveform is known could be detection of radar signals, where a pulse s[n] transmitted by us is reflected back after some propagation time.

For this case the likelihoods are

$$\rho(\mathbf{x} \mid \mathcal{H}_1) = \prod_{n=0}^{N-1} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x[n] - s[n])^2}{2\sigma^2}\right),$$

$$\rho(\mathbf{x} \mid \mathcal{H}_0) = \prod_{n=0}^{N-1} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x[n])^2}{2\sigma^2}\right).$$

The likelihood ratio test is easily obtained as

$$\frac{p(\mathbf{x} \mid \mathcal{H}_1)}{p(\mathbf{x} \mid \mathcal{H}_0)} = \exp\left[-\frac{1}{2\sigma^2} \left(\sum_{n=0}^{N-1} (x[n] - s[n])^2 - \sum_{n=0}^{N-1} (x[n])^2\right)\right] > \gamma.$$



This simplifies by taking the logarithm from both sides:

$$-\frac{1}{2\sigma^2}\left(\sum_{n=0}^{N-1}(x[n]-s[n])^2-\sum_{n=0}^{N-1}(x[n])^2\right)>\ln\gamma.$$

This further simplifies into

$$\frac{1}{\sigma^2} \sum_{n=0}^{N-1} x[n] s[n] - \frac{1}{2\sigma^2} \sum_{n=0}^{N-1} (s[n])^2 > \ln \gamma.$$

• Since s[n] is a known waveform (= constant), we can simplify the procedure by moving it to the right hand side and combining it with the threshold:

$$\sum_{n=0}^{N-1} x[n]s[n] > \sigma^2 \ln \gamma + \frac{1}{2} \sum_{n=0}^{N-1} (s[n])^2.$$

We can equivalently call the right hand side as our threshold (say  $\gamma'$ ) to get the final decision rule

$$\sum_{n=0}^{N-1} x[n]s[n] > \gamma'.$$

### **Examples**

- This leads into some rather obvious results.
- The detector for a known DC level in WGN is

$$\sum_{n=0}^{N-1} x[n]A > \gamma \Rightarrow A \sum_{n=0}^{N-1} x[n] > \gamma$$

Equally well we can set a new threshold and call it  $\gamma' = \gamma/(AN)$ . This way the detection rule becomes:  $\bar{x} > \gamma'$ . Note that a negative A would invert the inequality.

### Examples

The detector for a sinusoid in WGN is

$$\sum_{n=0}^{N-1} x[n] A \cos(2\pi f_0 n + \phi) > \gamma \Rightarrow A \sum_{n=0}^{N-1} x[n] \cos(2\pi f_0 n + \phi) > \gamma.$$

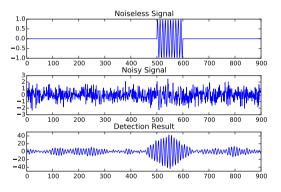
Again we can divide by A to get

$$\sum_{n=0}^{N-1} x[n]\cos(2\pi f_0 n + \phi) > \gamma'.$$

 In other words, we check the correlation with the sinusoid. Note that the amplitude A does not affect our statistic, only the threshold which is anyway selected according to the fixed P<sub>FA</sub> rate.

### Examples

• As an example, the below picture shows the detection process with  $\sigma = 0.5$ .



# Detection of random signals

- The problem with the previous approach was that the model was too restrictive; the results depend on how well the phases match.
- The model can be relaxed by considering random signals, whose exact form is unknown, but the correlation structure is known. Since the correlation captures the frequency (but not the phase), this is exactly what we want.
- In general, the detection of a random signal can be formulated as follows.
- Suppose  $\mathbf{s} \sim \mathcal{N}(0, \mathbf{C}_s)$  and  $\mathbf{w} \sim \mathcal{N}(0, \sigma^2 \mathbf{I})$ . Then the detection problem is a hypothesis test

$$\begin{aligned} \mathcal{H}_0: \boldsymbol{x} &\sim \mathcal{N}(0, \sigma^2 \boldsymbol{I}) \\ \mathcal{H}_1: \boldsymbol{x} &\sim \mathcal{N}(0, \boldsymbol{C}_{\scriptscriptstyle S} + \sigma^2 \boldsymbol{I}) \end{aligned}$$

# Detection of random signals

• It can be shown, that the decision rule becomes

Decide 
$$\mathcal{H}_1$$
, if  $\mathbf{x}^T \hat{\mathbf{s}} > \gamma$ ,

where

$$\hat{\mathbf{s}} = \mathbf{C}_{s}(\mathbf{C}_{s} + \sigma^{2}\mathbf{I})^{-1}\mathbf{x}.$$

# Example of Random Signal Detection

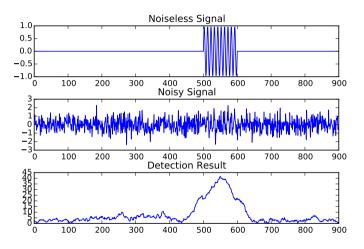
 Without going into the details, let's jump directly to the derived decision rule for the sinusoid:

$$\left|\sum_{n=0}^{N-1} x[n] \exp(-2\pi i f_0 n)\right| > \gamma.$$

- As an example, the below picture shows the detection process with  $\sigma = 0.5$ .
- Note the simplicity of Python implementation:

```
import numpy as np
h = np.exp(-2 * np.pi * 1j * f0 * n)
y = np.abs(np.convolve(h, xn, 'same'))
```

# Example of Random Signal Detection



- A usual way of illustrating the detector performance is the Receiver Operating Characteristics curve (ROC curve).
- This describes the relationship between  $P_{FA}$  and  $P_D$  for all possible values of the threshold  $\gamma$ .
- The functional relationship between  $P_{FA}$  and  $P_D$  depends on the problem and the selected detector.

· For example, in the DC level example,

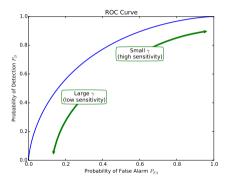
$$P_D(\gamma) = \int_{\gamma}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(x-1)^2}{2}\right) dx$$
$$P_{FA}(\gamma) = \int_{\gamma}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx$$

It is easy to see the relationship:

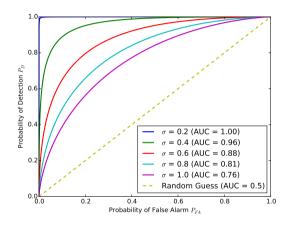
$$P_D(\gamma) = \int_{\gamma-1}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx = P_{FA}(\gamma-1).$$



ullet Plotting the ROC curve for all  $\gamma$  results in the following curve.



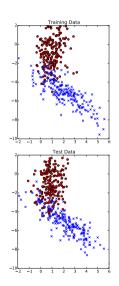
- The higher the ROC curve, the better the performance.
- A random guess has diagonal ROC curve.
- This gives rise to a widely used measure for detector performance: the *Area Under (ROC) Curve*, or AUC criterion.
- The benefit of AUC is that it is threshold independent, and tests the accuracy for all thresholds.
- In the DC level case, the performance increases if the noise variance  $\sigma^2$  decreases (since the problem becomes easier).
- Below are the ROC plots for various values of  $\sigma^2$ .



#### **Empirical AUC**

- Initially, AUC and ROC stem from radar and radio detection problems.
- More recently, AUC has become one of the standard measures of classification performance, as well.
- Usually a closed form expression for  $P_D$  and  $P_{FA}$  can not be derived.
- Thus, ROC and AUC are most often computed empirically; *i.e.*, by evaluating the prediction results on a holdout test set.

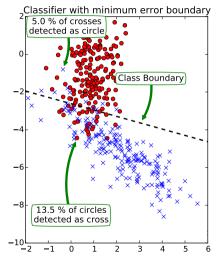
- For example, consider the 2-dimensional dataset on the right.
- The data is split to training and test sets, which are similar but not exactly the same.
- Let's train 4 classifiers on the upper data and compute the ROC for each on the bottom data.



- A linear classifier trained with the training data produces the shown class boundary.
- The class boundary has the orientation and location that minimizes the overall classification error for the training data.
- The boundary is defined by:

$$V = C_1X + C_0$$

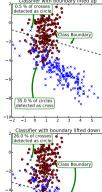
with parameters  $c_1$  and  $c_0$  learned from data.

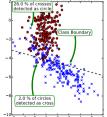


- We can adjust the sensitivity of classification by moving the decision boundary up or down.
- In other words, slide the parameter  $c_0$  in

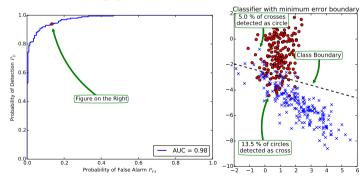
$$y = c_1 x + c_0$$

• This can be seen as a *tuning parameter* for plotting the ROC curve.



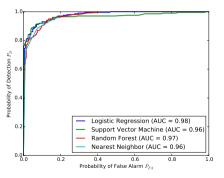


- When the boundary slides from bottom to top, we plot the *empirical ROC* curve.
- Plotting starts from upper right corner.
- Every time the boundary passes a blue cross, the curve moves left.
- Every time the boundary passes a red circle, the curve moves down.





- Real usage is for comparing classifiers.
- Below is a plot of ROC curves for 4 widely used classifiers.
- Each classifier produces a class membership score over which the tuning parameter slides.



#### ROC and AUC code in Python

```
classifiers = [(LogisticRegression(), "Logistic Regression").
               (SVC(probability = True), "Support Vector Machine"),
               (RandomForestClassifier(n_estimators = 100), "Random Forest"),
               (KNeighborsClassifier(), "Nearest Neighbor")]
for clf, name in classifiers:
    clf.fit(X, v)
    ROC = [1]
    for gamma in np.linspace(0, 1, 1000):
        err1 = np.count_nonzero(clf.predict_proba(X_test[v_test == 0, :])[:,1] <= gamma)
        err2 = np.count_nonzero(clf.predict_proba(X_test[v_test == 1, :])[:.1] > gamma)
        err1 = float(err1) / np.count_nonzero(v_test == 0)
        err2 = float(err2) / np.count_nonzero(v_test == 1)
        ROC.append([errl. err2])
    ROC = np.array(ROC)
    ROC = ROC[::-1.:]
    auc = roc_auc_score(v_test, clf.predict_proba(X_test)[:.1])
    plt.plot(1-ROC[:. 0]. ROC[:. 1]. linewidth = 2. label="%s (AUC = %.2f)" % (name, auc))
```