Chapter 1

Mathematical Background and Linear Systems

1.1 Fourier Transformation

In electrical engineering, we are most concerned with a signal as a function of time, $f(t)$. The signal in question could be a voltage or a current. The forward temporal *Fourier transform* of $f(t)$ is given as

$$\mathcal{F}\{f(t)\} = F(\omega) = \int_{-\infty}^{\infty} f(t)\exp(-j\omega t)\,dt, \qquad (1.1\text{-}1a)$$

where the transform variables are time, t [second], and temporal radian frequency, ω [radian/second]. In Eq. (1.1a), $j = \sqrt{-1}$. The inverse Fourier transform is

$$\mathcal{F}^{-1}\{F(\omega)\} = f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\exp(j\omega t)\,d\omega. \qquad (1.1\text{-}1b)$$

In optics, we are most interested in dealing with a two-dimensional (2-D) signal. Examples include images or the transverse profile of an electromagnetic or optical field at some plane of spatial variables x and y. Hence, the two-dimensional spatial *Fourier transform* of a signal $f(x, y)$ is given as [Banerjee and Poon (1991), Poon and Banerjee (2001)]

$$\mathcal{F}_{xy}\{f(x,y)\} = F(k_x,k_y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\exp(jk_x x + jk_y y)\,dxdy, \qquad (1.1\text{-}2a)$$

and the inverse Fourier transform is

$$\begin{aligned}
&\mathcal{F}_{xy}{}^{-1}\{F(k_x,k_y)\} \\
&= f(x,y) \\
&= \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(k_x,k_y)\exp(-jk_x x - jk_y y)\,dk_x dk_y,
\end{aligned} \qquad (1.1\text{-}2b)$$

where the transform variables are spatial variables, x, y [meter], and spatial radian frequencies, k_x, k_y [radian/meter]. $f(x,y)$ and $F(k_x,k_y)$ are a Fourier transform pair and the statement is symbolically represented by

$$f(x,y) \Leftrightarrow F(k_x,k_y).$$

Note that the definitions for the forward and inverse transforms [see Eqs. (1.1-2a) and (1.1-2b)] are consistent with the engineering convention for a traveling wave, as explained in *Principles of Applied Optics* [Banerjee and Poon (1991)]. Common properties and examples of 2-D Fourier transform appear in the Table below.

Table 1.1 Properties and examples of some two-dimensional Fourier Transforms.

Function in (x, y)	**Fourier transform in (k_x, k_y)**
1. $f(x,y)$	$F(k_x,k_y)$
2. $f(x-x_0, y-y_0)$	$F(k_x,k_y)\exp(jk_x x_0 + jk_y y_0)$
3. $f(ax, by)$; a, b complex constants	$\frac{1}{\lvert ab\rvert}F(\frac{k_x}{a}, \frac{k_y}{b})$
4. $f^*(x,y)$	$F^*(-k_x, -k_y)$
5. $\partial f(x,y)/\partial x$	$-jk_x F(k_x,k_y)$
6. $\partial^2 f(x,y)/\partial x \partial y$	$-k_x k_y F(k_x,k_y)$
7. ***delta function***	
$\delta(x,y) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{\pm jk_x x \pm jk_y y} dk_x dk_y$	1
8. 1	$4\pi^2\delta(k_x,k_y)$
9. ***rectangle function***	***sinc function***
$\text{rect}(x,y) = \text{rect}(x)\text{rect}(y)$,	$\text{sinc}(\frac{k_x}{2\pi}, \frac{k_y}{2\pi}) = \text{sinc}(\frac{k_x}{2\pi})\text{sinc}(\frac{k_y}{2\pi})$,
where $\text{rect}(x) = \begin{pmatrix} 1, \lvert x\rvert<1/2 \\ 0, \text{otherwise} \end{pmatrix}$	where $\text{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$
10. ***Gaussian function***	***Gaussian function***
$\exp[-\alpha(x^2+y^2)]$	$\frac{\pi}{\alpha}\exp[-\frac{k_x^2+k_y^2}{4\alpha}]$

Example 1.1 Fourier Transform of $\text{rect}(x, y)$ plus MATLAB

The one-dimensional (1-D) rectangular function or simply *rect function*, $\text{rect}(x/a)$, is given by

$$\text{rect}(x/a) = \begin{pmatrix} 1, \lvert x\rvert < a/2 \\ 0, \text{ otherwise} \end{pmatrix}, \tag{1.1-3a}$$

where a is the width of the function. The function is shown in Fig. 1.1a). The two-dimensional version of the rectangular function is given by

$$\text{rect}(x/a, y/b) = \text{rect}(x/b)\text{rect}(y/b). \tag{1.1-3b}$$

Figure 1.1b) and 1.1c) show the three-dimensional plot and the gray scale plot of the function. In the gray scale plot, we have assumed that an amplitude of 1 translates to "white" and an amplitude of zero to "black" Therefore, from the definition of Eq. (1.1-3b), the white area is $a \times b$.

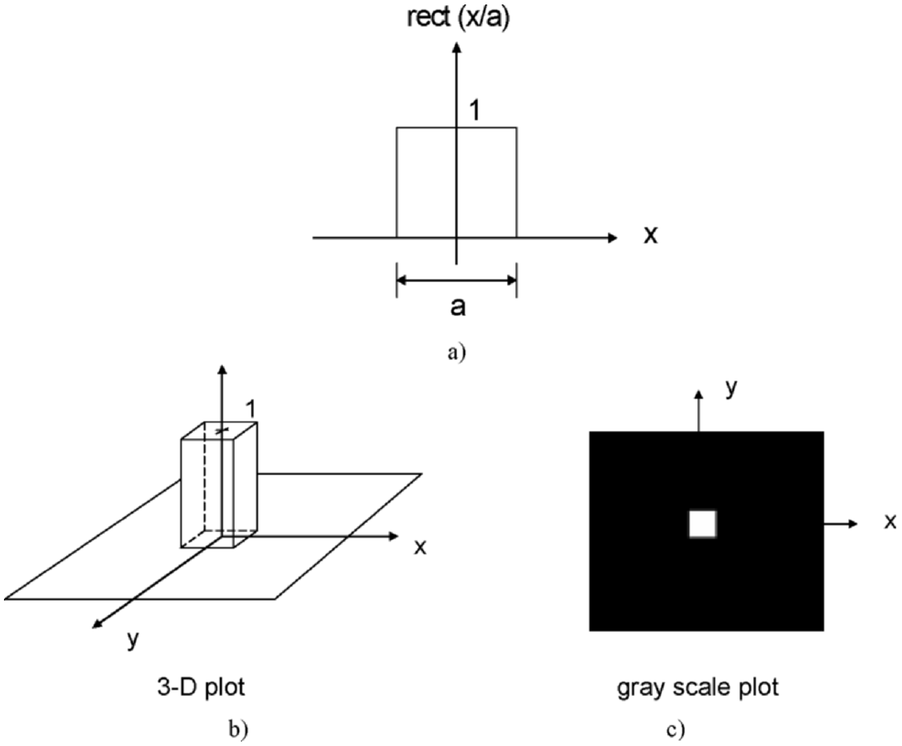

Fig. 1.1 Rect function.

To find the Fourier transform of the 2-D rectangular function, we simply evaluate the integral given by Eq. (1.1-2a) by recognizing that $f(x, y) = \text{rect}(x/a, y/b)$. Therefore, we write

$$\mathcal{F}_{xy}\{f(x, y)\} = \mathcal{F}_{xy}\{\text{rect}(x/a, y/b)\}$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \text{rect}(x/a, y/b)\exp(jk_x x + jk_y y)dxdy. \qquad (1.1\text{-}4)$$

Since $\text{rect}(x/a, y/b)$ is a *separable function* [see Eq. 1.1-3b)], we re-write Eq. (1.1-4) as follows:

$$\mathcal{F}_{xy}\{\text{rect}(x/a, y/b)\}$$

$$= \int_{-\infty}^{\infty} \text{rect}(x/a)\exp(jk_x x)dx \times \int_{-\infty}^{\infty} \text{rect}(y/b)\exp(jk_y y)dy$$

$$= \int_{-a/2}^{a/2} 1\exp(jk_x x)dx \times \int_{-b/2}^{b/2} 1\exp(jk_y y)dy. \qquad (1.1\text{-}5)$$

By writing the last step, Eq. (1.1-5), we have used the definition of the rectangular function given by Eq. (1.1-3a). We can now evaluate Eq. (1.1-5) by using

$$\int \exp(cx)dx = \frac{1}{c}\exp(cx). \tag{1.1-6}$$

Therefore,

$$\int_{-a/2}^{a/2} 1 \exp(jk_x x)dx = a\mathrm{sinc}(\frac{ak_x}{2\pi}), \tag{1.1-7}$$

where $\mathrm{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$ is defined as the *sinc function*. Table 1.2 shows the m-file for plotting the sinc function and its output is shown in Fig. 1.2. Note that the sinc function has zeros at $x = \pm 1, \pm 2, \pm 3,$ ...

Table 1.2 Plot_sinc.m: m-file for plotting the sinc function.

```
%Plot_sinc.m   Plotting of sinc(x) function
x= -5.5:0.01:5.5;
sinc=sin(pi*x)./(pi*x);
plot(x,sinc)
axis([-5.5 5.5 -0.3 1.1])
grid on
xlabel('x')
ylabel('sinc (x)')
```

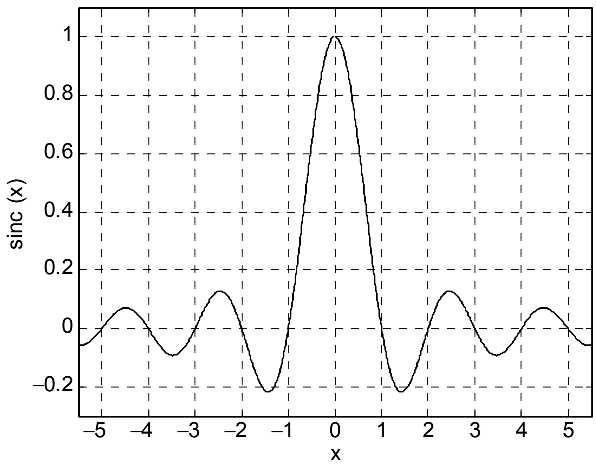

Fig. 1.2 Sinc function.

To complete the original problem of determining the Fourier transform of a rect function, we take advantage of the result of Eq. (1.1-7); Eq. (1.1-5) becomes

$$\mathcal{F}_{xy}\{\text{rect}(x/a, y/b)\} = ab\text{sinc}(\frac{ak_x}{2\pi})\text{sinc}(\frac{bk_y}{2\pi})$$

$$= ab\text{sinc}(\frac{ak_x}{2\pi}, \frac{bk_y}{2\pi}). \quad (1.1\text{-}8a)$$

Hence, we may write

$$\text{rect}(x/a, y/b) \Leftrightarrow ab\text{sinc}(\frac{ak_x}{2\pi}, \frac{bk_y}{2\pi}). \quad (1.1\text{-}8b)$$

Note that when the width of the rect function along x is a, the first zero along k_x is $k_{x,0} = 2\pi/a$. Figure 1.3 shows the transform pair of Eq. (1.1-8b). The top figures are 2-D gray-scale plots, and the bottom figures are line traces along the horizontal axis through the center of the top figures. These figures are generated using the m-file shown in Table 1.3 where M $=$ 11. For this value of M, $a = 0.0429$ units of length and the first zero $k_{x,0} = 146.23$ radian/(unit of length). Note that the area of display in the x-y plane has been scaled to 1 unit of length by 1 unit of length.

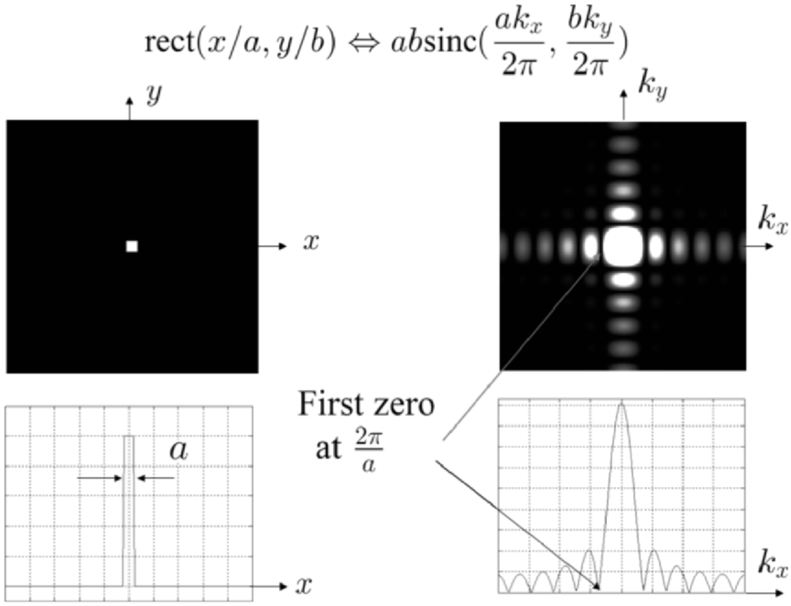

Fig. 1.3 Rect function and its Fourier transform.

Table 1.3 fft2Drect.m: m-file for 2-D Fourier transform of rect$(x/a, y/b)$.

```
%fft2Drect.m %Simulation of Fourier transformation of a 2-D rect function
%
clear
```

```
L=1; %display area is L by L, L has unit of length
N=256; % number of sampling points
dx=L/(N-1); % dx : step size

% Create square image, M by M square, rect(x/a,y/a), M=odd number
M=input ('M (size of rect(x/a,y/a), enter odd numbers from 3-33)=');
a=M/256;
kx0=2*pi/a;
sprintf('a = %0.5g[unit of length]',a)
sprintf('kx0 (first zero)= %0.5g[radian/unit of length]',kx0)
R=zeros(256); %assign a matrix (256x256) of zeros
r=ones(M); % assign a matrix (MxM) of ones
n=(M-1)/2;
R(128-n:128+n,128-n:128+n)=r;
%End of creating square input image M by M

%Axis Scaling
for k=1:256
  X(k)=1/255*(k-1)-L/2;
  Y(k)=1/255*(k-1)-L/2;
   %Kx=(2*pi*k)/((N-1)*dx)
  %in our case, N=256, dx=1/255
  Kx(k)=(2*pi*(k-1))/((N-1)*dx)-((2*pi*(256-1))/((N-1)*dx))/2;
  Ky(k)=(2*pi*(k-1))/((N-1)*dx)-((2*pi*(256-1))/((N-1)*dx))/2;
end

%Image of the rect function
figure(1)
image(X+dx/2,Y+dx/2,255*R);
title('rect function: gray-scale plot')
xlabel('x')
ylabel('y')
colormap(gray(256));
axis square

%Computing Fourier transform
FR=(1/256)^2*fft2(R);
FR=fftshift(FR);

% plot of cross-section of rect function
figure(2)
plot(X+dx/2,R(:,127))
title('rect function: cross-section plot')
xlabel('x')
ylabel('rect(x/a)')
grid
axis([-0.5 0.5 -0.1 1.2])

%Centering the axis and plot of cross-section of transform along kx
figure(3)
plot(Kx-pi/(dx*(N-1)),10*abs(FR(:,127)))
title('Square-absolute value of Fourier transform of rect function: cross-section plot')
xlabel('kx')
ylabel('|a*b*sinc(a*kx/2pi)|')
```

```
axis([-800 800 0 max(max(abs(FR)))*10.1])
grid

%Mesh the Fourier transformation
figure(4);
mesh(Kx,Ky,(abs(FR)).^2)
title('Square-absolute value of Fourier transform of rect function: 3-D plot,scale
arbitrary')
xlabel('kx')
ylabel('ky')
axis square

%Image of the Fourier transformation of rectangular function
figure(5);
gain=10000;
image(Kx,Ky,gain*(abs(FR)).^2/max(max(abs(FR))).^2)
title('Square-absolute value of Fourier transform of the rect function: gray-scale plot')
xlabel('kx')
ylabel('ky')
axis square
colormap(gray(256))
```

Example 1.2 MATLAB Example:
Fourier Transform of Bitmap Images

When the 2-D function or image is given with a bitmap file, we can use the m-file given in Table 1.4 to find its Fourier transform. Figure 1.4a) is the bitmap image used when the image file of the size is 256 by 256. It is easily generated with Microsoft® Paint. Figure 1.4b) is the corresponding absolute value of the transformed image.

Table 1.4 fft2Dbitmap_image.m: m-file for 2-D Fourier transform of bitmap image.

```
%fft2Dbitmap_image.m
%Simulation of Fourier transformation of bitmap images
clear

I=imread('triangle.bmp','bmp'); %Input bitmap image
I=I(:,:,1);
figure(1)  %displaying input
colormap(gray(255));
image(I)
axis off
FI=fft2(I);
FI=fftshift(FI);
max1=max(FI);
max2=max(max1);
scale=1.0/max2;
FI=FI.*scale;
figure(2)  %Gray scale image of the absolute value of transform
```

```
colormap(gray(255));
image(10*(abs(256*FI)));
axis off
-----------------------------------------------------
```

a) Bitmap image of a triangle. b) Displaying the absolute value of the transform of a).

Fig. 1.4 Bitmap image and its transform generated using the m-file in Table 1.4.

Example 1.3 Delta Function and its Transform

The *delta function*, $\delta(x)$, is one of the most important functions in the study of systems. We can define the delta function as follows:

$$\delta(x) = \lim_{a \to 0} \{\frac{1}{a}\mathrm{rect}(\frac{x}{a})\}. \qquad (1.1\text{-}9)$$

The situation is shown graphically in Fig. 1.5.

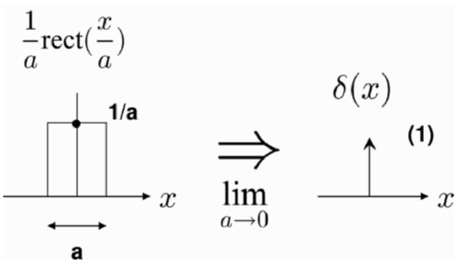

Fig. 1.5 Illustration of the definition of the delta function graphically.

The delta function has three important properties, which are listed as follows:

Property #1: Unit Area

$$\int_{-\infty}^{\infty} \delta(x - x_0)dx = 1. \qquad (1.1\text{-}10a)$$

The delta function has a unit area (or strength), which is denoted by a "(1)" beside the arrow, as shown in Fig. 1.5. This unit area property is clearly demonstrated by the definition illustrated on the left hand side of Fig. 1.5. The area is always a unity regardless of the value of a.

Property #2: Product Property

$$f(x)\delta(x-x_0) = f(x_0)\delta(x-x_0). \tag{1.1-10b}$$

The result of this property can be confirmed graphically by the illustration shown in Fig. 1.6 where an arbitrary function, $f(x)$, is shown to be overlapped with the offset delta function, $\delta(x-x_0)$, located at $x=x_0$. The product of the two functions is clearly equal to $f(x_0)$ multiplied by $\delta(x-x_0)$. Therefore, the result has become an offset delta function with its strength given by $f(x_0)$.

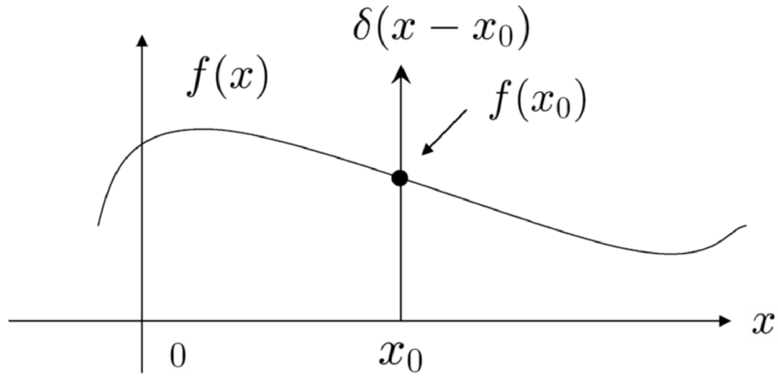

Fig. 1.6 Illustrating the result of the Product Property.

Property #3: Sampling Property

$$\int_{-\infty}^{\infty} f(x)\delta(x-x_0)dx = f(x_0). \tag{1.1-10c}$$

To obtain the result above, we simply use Properties #1 and #2. From Eq. (1.1-10c) and by using Property #2, we have

$$\int_{-\infty}^{\infty} f(x)\delta(x-x_0)dx = \int_{-\infty}^{\infty} f(x_0)\delta(x-x_0)dx$$

$$= f(x_0)\int_{-\infty}^{\infty} \delta(x-x_0)dx = f(x_0),$$

where we have used Property #1 to obtain the last step of the result. Equation (1.1-10c) is known as the *sampling property* because the delta function selects, or samples, a particular value of the function, $f(x)$, at the location of the delta function (i.e., x_0) in the integration process.

While a 1-D delta function is called an impulse function in electrical engineering, the 2-D version of a delta function, $\delta(x,y)=\delta(x)\delta(y)$, represents an idealized point source of light in optics. According to Eq. (1.1-2a), the 2-D Fourier transform of $\delta(x,y)$ is given by

$$\begin{aligned}\mathcal{F}_{xy}\{\delta(x,y)\} &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\delta(x,y)\exp(jk_x x+jk_y y)\,dxdy \\ &= \int_{-\infty}^{\infty}\delta(x)\exp(jk_x x)dx\int_{-\infty}^{\infty}\delta(y)\mathrm{exp}(jk_y y)dy \\ &= 1,\end{aligned}$$

where we have used the sampling property of the delta function to evaluate the above integrals. Figure 1.7 shows the 2-D delta function as well as its corresponding Fourier transform.

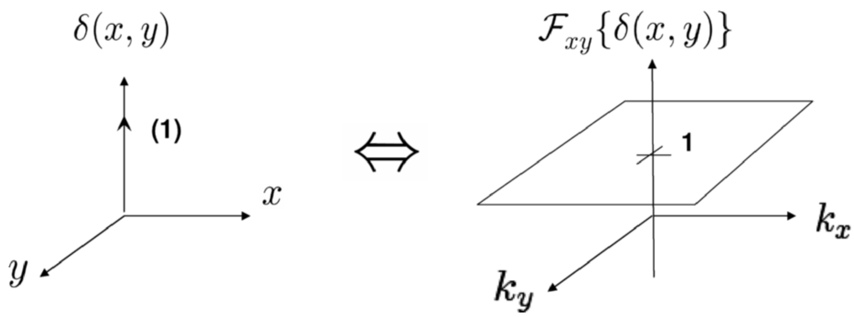

Fig. 1.7 Two-dimensional delta function and its Fourier transform.

1.2 Linear and Invariant Systems

1.2.1 Linearity and Invariance

A system is defined as the mapping of an input or set of inputs into an output or set of outputs. A system is linear if *superposition* applies. For a single-input – single-output system, if an input $f_1(t)$ gives an output of $g_1(t)$, and if another input $f_2(t)$ gives an output of $g_2(t)$, then superposition means if the input is given by $af_1(t)+bf_2(t)$, the system's output is $ag_1(t)+bg_2(t)$, where a and b are some constants. The situation of a *linear system* is further illustrated in Fig. 1.8.

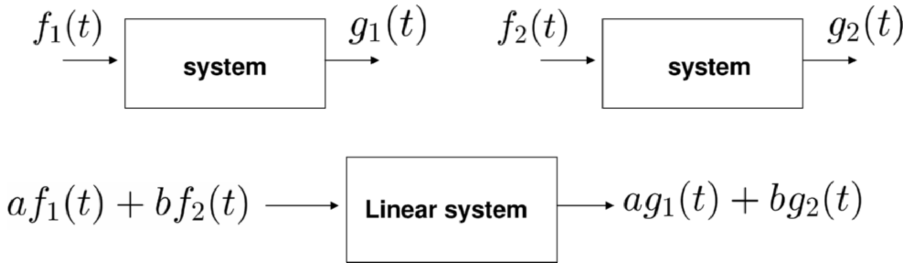

Figure 1.8 Linear system.

Systems with parameters that do not change with time are *time-invariant systems*. Consequently, a time delay in the input results in a corresponding time delay in the output. This property of the system is shown graphically in Fig. 1.9, where t_0 is the time delay.

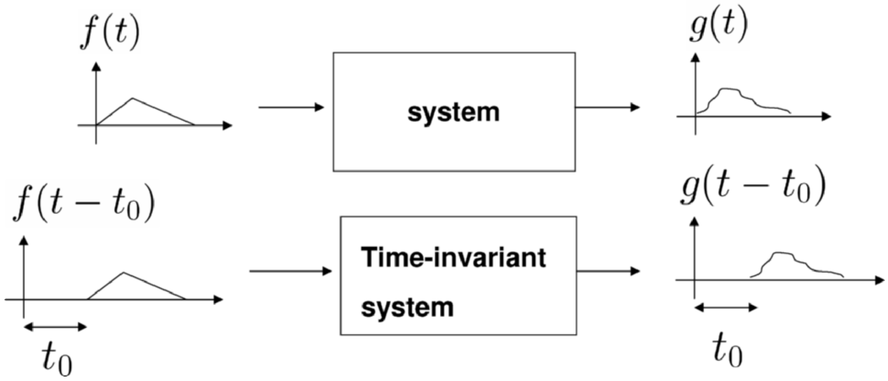

Fig. 1.9 Time-invariant system.

As it turns out, if a system is linear and time-invariant (LTI) with all initial conditions being zero, there is a definite relationship between the input and output. The relationship is given by the so-called *convolution integral*,

$$g(t) = \int_{-\infty}^{\infty} f(t')h(t-t')dt' = f(t)*h(t)\,, \tag{1.2-1}$$

where $h(t)$ is called the *impulse response* of the LTI system, and $*$ is a symbol denoting the convolution of $f(t)$ and $h(t)$. The expression $f*g$ reads as f convolves with g. To see why $h(t)$ is called the impulse response, if we let the input be a delta function, $\delta(t)$, then the output, according to Eq. (1.2-1), is

$$g(t) = \delta(t)*h(t) = \int_{-\infty}^{\infty} \delta(t')h(t-t')dt' = h(t),$$

where we use the sampling property of the delta function to obtain the last step of the result. Once we know $h(t)$ of the LTI system, which can be determined experimentally by simply applying an impulse to the input of the system, we can find the response to any arbitrary input, say, $f(t)$, to the system through the calculation of Eq. (1.2-1).

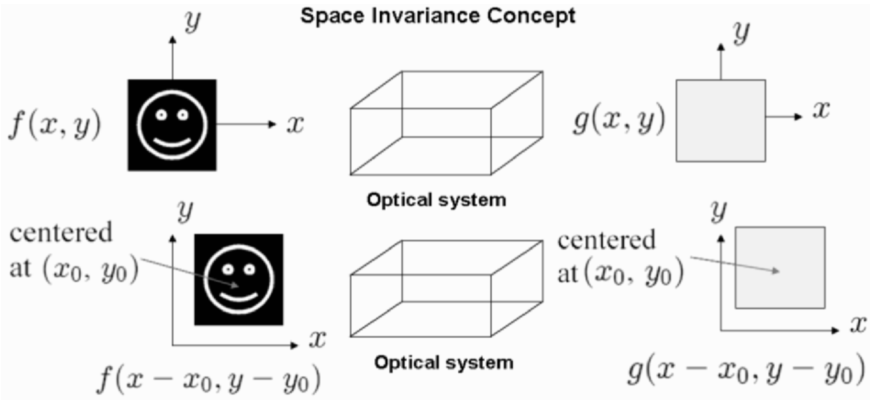

Fig. 1.10 Concept of space-invariance.

In optics, when we are dealing with signals of spatial coordinates, we can extend the concept of LTI systems to the so-called *linear space-invariant (LSI) system*. Hence we can extend the 1-D convolution integral to two dimensions as follows:

$$\begin{aligned} g(x,y) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x',y')h(x-x',y-y')dx'dy' \\ &= f(x,y)*h(x,y), \end{aligned} \qquad (1.2\text{-}2)$$

where $f(x,y)$ is the 2-D input to the LSI system. $h(x,y)$ and $g(x,y)$ are the corresponding impulse response and output of the system, respectively. While the concept of time-invariance is clearly delineated by Fig. 1.9 for electrical signals, the concept of space-invariance for optical signals is not immediately clear. In Fig. 1.10, we can clarify this concept. We see that as the input image, $f(x,y)$, is shifted or translated to a new origin, (x_0, y_0), its output, $g(x,y)$, is shifted accordingly on the x-y plane. Hence, we see that the delay of an input signal in electrical systems corresponds to the translation of an output image over the output plane.

Figure 1.11 shows the block diagrams of a LSI optical system both in spatial and frequency domain. To analyze the LSI system in frequency domain, we simply take the Fourier transform of Eq. (1.2-2) to obtain

$$\mathcal{F}_{xy}\{g(x,y)\} = \mathcal{F}_{xy}\{f(x,y)*h(x,y)\}, \qquad (1.2\text{-}3a)$$

which is shown to be

$$G(k_x,k_y) = F(k_x,k_y)H(k_x,k_y), \tag{1.2-3b}$$

where $G(k_x,k_y)$ and $H(k_x,k_y)$ are the Fourier transform of $g(x,y)$ and $h(x,y)$, respectively. While $h(x,y)$ is called the *spatial impulse response* or *point spread function (PSF)* of the LSI system, its Fourier transform, $H(k_x,k_y)$, is called the *spatial frequency response* or the system's *frequency transfer function*. The proof of Eq. (1.2-3b) is demonstrated in Example 1.4.

Block diagram of LSI optical system in spatial domain

$$f(x,y) \longrightarrow \boxed{h(x,y)} \longrightarrow g(x,y)$$

$$g(x,y) = f(x,y)*h(x,y)$$

$h(x,y)$: spatial impulse response or point spread function

Block diagram of LSI optical system in frequency domain

$$F(k_x,k_y) \longrightarrow \boxed{H(k_x,k_y)} \longrightarrow G(k_x,k_y)$$

$$\mathcal{F}_{xy}\{g(x,y)\} = \mathcal{F}_{xy}\{f(x,y)*h(x,y)\}$$
or $G(k_x,k_y) = F(k_x,k_y)H(k_x,k_y)$

$H(k_x,k_y)$: spatial frequency response or frequency transfer function

Fig. 1.11 Block diagrams of LSI system.

Example 1.4 Fourier Transform of the Convolution of Two Functions

From Eq. (1.2-3a), we have

$$\begin{aligned}\mathcal{F}_{xy}\{g(x,y)\} &= \mathcal{F}_{xy}\{f(x,y)*h(x,y)\} \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[f(x,y)*h(x,y)]\exp(jk_x x + jk_y y)\,dxdy\end{aligned}$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Big[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x',y')h(x-x',y-y')dx'dy'\Big]$$

$$\times \exp(jk_x x + jk_y y)\, dxdy,$$

where we have utilized the definition of convolution. After grouping the x and y variables together, the above equation can be written as

$$\mathcal{F}_{xy}\{f(x,y)*h(x,y)\}$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x',y')$$

$$\times \Big[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(x-x',y-y')\exp(jk_x x + jk_y y)dxdy\Big]dx'dy'.$$

The inner integral is the Fourier transform of $h(x-x',y-y')$. Using Table 1.1 (item #2), the transform is given by $H(k_x,k_y)\exp(jk_x x' + jk_y y')$. Hence

$$\mathcal{F}_{xy}\{f(x,y)*h(x,y)\}$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x',y')\Big[H(k_x,k_y)\exp(jk_x x' + jk_y y')\Big]dx'dy'$$

$$= H(k_x,k_y)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x',y')\exp(jk_x x' + jk_y y')dx'dy'$$

$$= F(k_x,k_y)H(k_x,k_y).$$

1.2.2 Convolution and Correlation Concept

In the last section, we have demonstrated that in the LSI system, the convolution integral is involved. In this section we will first explain the concept of convolution, and then, we will discuss another important operation called *correlation.* Finally we will make distinction between the two processes.

In Fig. 1.12, we illustrate the convolution of two images, $f(x,y)$ and $h(x,y)$. According to the definition in Eq. (1.2-2), the convolution of the two images involves the calculation of the area under the product of two functions, $f(x',y')$ and $h(x-x',y-y')$, for different shifts, (x,y).

$$g(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x', y')h(x - x', y - y')dx'dy'$$

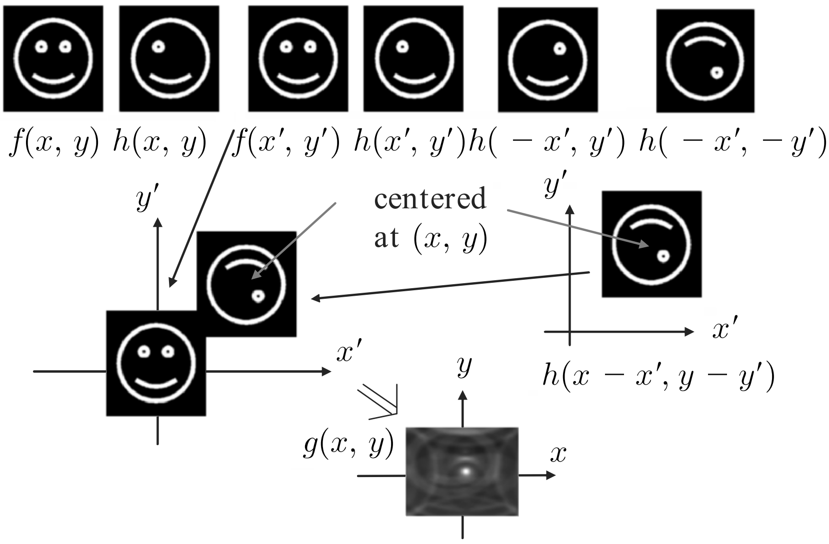

Fig. 1.12 Concept of 2-D convolution.

$$C_{fh}(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f^*(x',y')h(x+x',y+y')dx'dy'$$

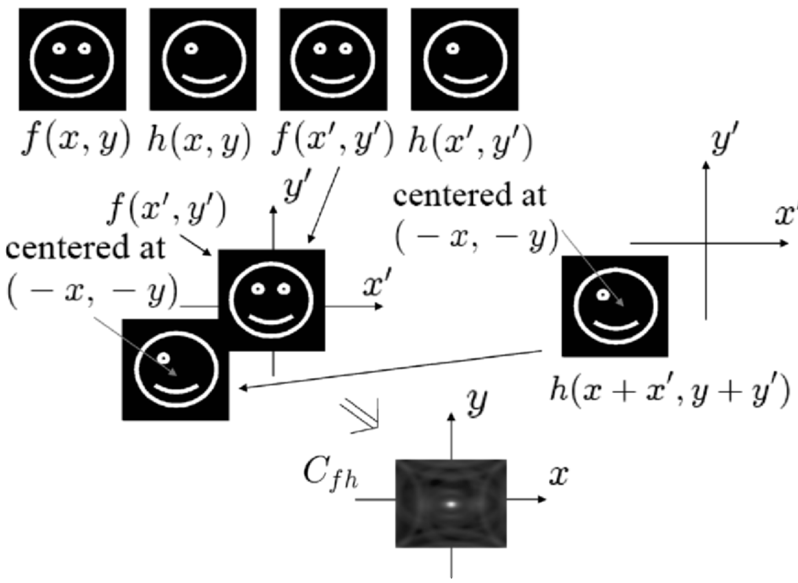

Fig. 1.13 Concept of 2-D correlation (assuming f is real).

The first row of figures in Fig. 1.12 shows the construction of $f(x', y')$ and $h(-x', -y')$ from the original images $f(x, y)$ and $h(x, y)$. We then construct $h(x - x', y - y')$ as shown in Fig. 1.12 by translating $h(-x', -y')$ to a center at (x, y) to form $h(x - x', y - y')$. Once we have $f(x', y')$ and $h(x - x', y - y')$, we superimpose them on the x'-y' plane as illustrated in Fig. 1.12. Finally, we need to calculate the area of the product of $f(x', y')$ and $h(x - x', y - y')$ for different shifts (x, y) to obtain a 2-D gray-scale plot of $g(x, y)$.

Another important integral is called the *correlation integral*. The correlation, $C_{fh}(x, y)$, of two functions $f(x, y)$ and $h(x, y)$, is defined as

$$C_{fh}(x, y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f^*(x', y')h(x + x', y + y')dx'dy'$$

$$= f(x, y) \otimes h(x, y). \qquad (1.2\text{-}4)$$

This integral is useful when comparing the similarity of two functions, and it has been knowingly used for applications in pattern recognition. For simplicity, if we assume in Fig. 1.13 that $f(x, y)$ is real, we can illustrate the correlation of the two images, $f(x, y)$ and $h(x, y)$. Similar to the convolution of the two images, the correlation involves the calculation of the area under the product of two functions, $f(x', y')$ and $h(x + x', y + y')$, for different shifts, (x, y). The first row of images in Fig. 1.13 shows the construction of $f(x', y')$ and $h(x', y')$ from the original images, $f(x, y)$ and $h(x, y)$. Unlike convolution, to calculate he area of the product of $f(x', y')$ and $h(x + x', y + y')$ for different shifts ,(x, y), there is no need to flip the image, $h(x', y')$, upon the x'- axis and the y'-axis to obtain the 2-D plot of $C_{fh}(x, y)$.

Example 1.5 Relationship between Convolution and Correlation

In this example, we will show that correlation can be expressed in terms of convolution through the following relationship:

$$f(x, y) \otimes h(x, y) = f^*(-x, -y) * h(x, y). \qquad (1.2\text{-}5)$$

According to the definition of convolution [see Eq. (1.2-2)], we write

$$f^*(-x, -y) * h(x, y)$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f^*(-x', -y')h(x - x', y - y')dx'dy'$$

$$= \int_{\infty}^{-\infty}\int_{\infty}^{-\infty} f^*(x''-x,y''-y)h(x'',y'')(-dx'')(-dy''),$$

where we have made the substitutions $x-x'=x''$ and $y-y'=y''$ to obtain the last step of the equation. By re-arranging the last step and substituting the equivalents for $x''-x=\widetilde{x}$ and $y''-y=\widetilde{y}$, we obtain

$$f^*(-x,-y)*h(x,y)$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f^*(\widetilde{x},\widetilde{y})h(\widetilde{x}+x,\widetilde{y}+y)d\widetilde{x}d\widetilde{y},$$

$$= f(x,y)\otimes h(x,y)$$

by the definition of correlation. Therefore, we have proven Eq. (1.2-5).

With reference to Eq. (1.2-4), when $f\neq h$, the result is known as *cross-correlation*, C_{fh}. When $f=h$, the result is known as *auto-correlation*, C_{ff}, of the function f. As it turns out, we can show that

$$|C_{ff}(0,0)| \geq |C_{ff}(x,y)|, \tag{1.2-6}$$

i.e., autocorrelation always has a central maximum. The use of this fact has been employed by *pattern recognition.* Pioneering schemes of optical pattern recognition, implementing Eq. (1.2-5), are due to Vander Lugt [1964], and Weaver and Goodman [1966]. The book, *Optical Pattern Recognition*, provides a comprehensive review of optical pattern recognition, covering theoretical aspects and details of some practical implementations [Yu and Jutamulia (1998)]. For some of the most novel approaches to optical pattern recognition, the reader is encouraged to refer to the article by Poon and Qi [2003].

Example 1.6 MATLAB Example: Pattern Recognition

For pattern recognition applications, one implements correlation given by Eq. (1.2-4). In this example, we implement the equation in the frequency domain. To do this, we realize that

$$\mathcal{F}_{xy}\{f(x,y)\otimes h(x,y)\} = F^*(k_x,k_y)H(k_x,k_y), \tag{1.2-7}$$

which can be shown using the procedure similar to Example 1.4. For the given images f and h, we first find their corresponding 2-D Fourier transforms, and then the correlation is evident when we take the inverse transform of Eq. (1.2-7):

$$f(x,y) \otimes h(x,y) = \mathcal{F}_{xy}{}^{-1}\{F^*(k_x,k_y)H(k_x,k_y)\}. \qquad (1.2\text{-}8)$$

Figure 1.14 shows the result of auto-correlation for two identical images, while Fig. 1.15 shows the cross-correlation result for two different images. These figures are generated using the m-file shown in Table 1.5. Two 256 by 256 smiley.bmp files have been used for the auto-correlation calculation. Note that in auto-correlation, shown in Fig. 1.14, a bright spot in the center of the correlation output represents the "match" of the two patterns, as suggested by Eq. (1.2-6), whereas in Fig. 1.15, there is no discernible bright spot in the center.

Fig. 1.14 Auto-correlation.

Fig. 1.15 Cross-correlation.

Table 1.5 correlation.m: m-file for performing 2-D correlation.

```
%correlation.m
clear

I1=imread('smiley.bmp','bmp'); %Input image 1 (reference image)
I1=I1(:,:,1);
figure(1)  %displaying input image 1
colormap(gray(255));
image(I1)
axis off

FI1=fft2(I1);
max1=max(FI1);
max2=max(max1);
scale=1.0/max2;
FI1=FI1.*scale;

I2=imread('smiley.bmp','bmp'); %Input image 2 (image to be recognized)
```

```
I2=I2(:,:,1);
figure(2)  %displaying input image 2
colormap(gray(255));
image(I2)
axis off

FI2=fft2(I2);
max1=max(FI2);
max2=max(max1);
scale=1.0/max2;
FI2=FI2.*scale;

FPR=FI1.*conj(FI2);%calculating correlation
PR=ifft2(FPR);
PR=fftshift(PR);

max1=max(PR);
max2=max(max1);
scale=1.0/max2;
PR=PR.*scale;

figure(3)%display of correlation in spatial domain
colormap(gray(255));
image(abs(256*PR));
axis off
```
- -

References

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1.3 Poon, T.-C. and Y. Qi (2003). "Novel real-time joint-transform correlation by use of acousto-optic heterodyning," *Applied Optics*, 42, 4663-4669.

1.4 VanderLugt, A. (1964). "Signal detection by complex spatial filter," *IEEE Trans. Inf. Theory* IT-10, 139–146.

1.5 Weaver, C.S. and J. W. Goodman (1969). "A technique for optical convolving two functions," *Applied Optics*, 5, 1248-1249.

1.6 Yu, F.T.S. and S. Jutamulia, ed. (1998). *Optical Pattern Recognition.* Cambridge University Press, Cambridge, UK.