# Module 07

## 05

> What is $Pr[X \leq 2]$ example? Write down $Pr[X \leq k]$ above? Write down $Pr[X \leq k]$ for all relevant values of $k$. Next, suppose $Pr[heads] = p$ in the above example. Write down $Pr[X = i], i \in \{0, 1, 2, 3\}$, in terms of $p$.

$$\begin{aligned}
Pr[X \leq 0] &= Pr[X = 0] = 0.064 \\
Pr[X \leq 1] &= Pr[X \leq 0] + Pr[X = 1] = 0.352 \\
Pr[X \leq 2] &= Pr[X \leq 1] + Pr[X = 2] = 0.784 \\
Pr[X \leq 3] &= Pr[X \leq 2] + Pr[X = 3] = 1
\end{aligned}$$

Let $P[heads] = p$,

$$\begin{aligned}
Pr[X = 0] &= (1 - p)^3 \\
Pr[X = 1] &= 3p(1 - p)^2 \\
Pr[X = 2] &= 3p^2(1 - p) \\
Pr[X = 3] &= p^3
\end{aligned}$$

## 08

> If $p = 0.6$, what is the probability that the first heads appears on the 3rd flip? Verify your answer using Coin.java and CoinExample.java.

$$Pr = (1 - 0.6)^2 0.6 = 0.096$$

## 09

> Suppose I compare two parameter values for the Geometric distribution: $p = 0.6$ and $p = 0.8$. For which of the two values of $p$ is $Pr[X = 3]$ higher?

Let $p = 0.6$,

$$Pr[X = 3] = (1 - 0.6)^2 0.6 = 0.096$$

Let $p = 0.8$,

$$Pr[X = 3] = (1 - 0.8)^2 0.8 = 0.032$$

It seems that $P[X = 3]$ for $p = 0.8$ is higher.

## 10

> Compute (by hand) $Pr[X > k]$ when $X \sim Geometric(p)$.

$$Pr[X > k] = 1 - Pr[X \leq k] = 1 - \sum_{i=1}^{k} Pr[X = i] = 1 - \sum_{i=1}^{k} [(1-p)^{i-1} p]$$

## 11

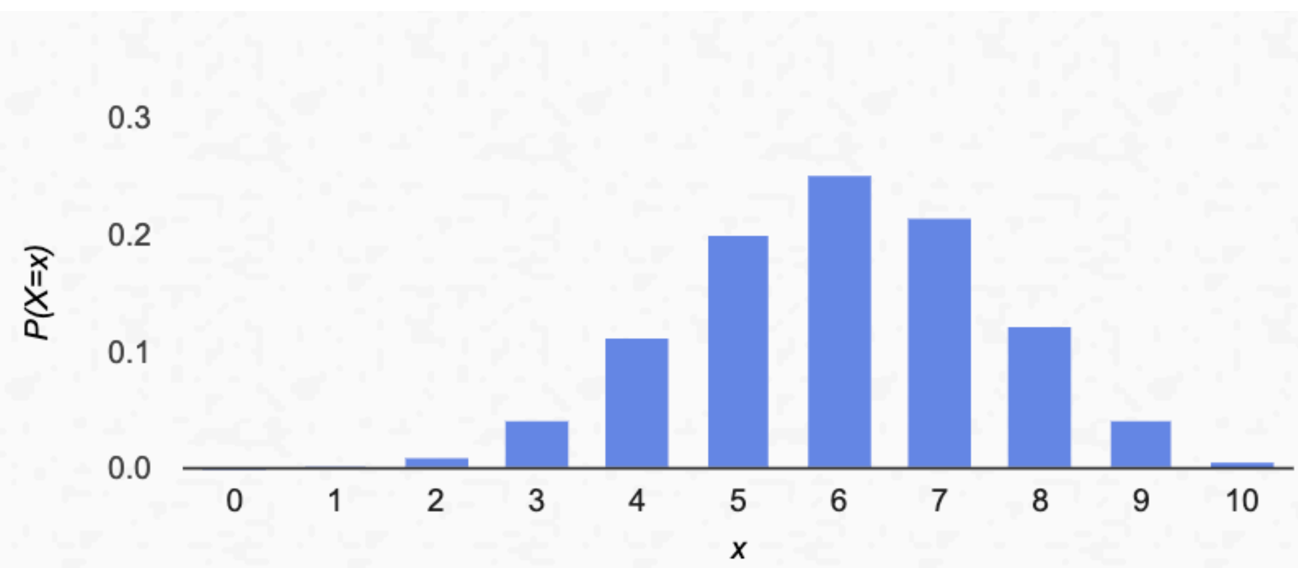
> Suppose we flip a coin $n$ times and count the number of heads using a coin for which $P[H] = p$
> ,
>
> - Write code to compute $Pr[X = k]$ using the formula
>
> $P[X = k] = \binom{n}{k} p^k (1-p)^{n-k}$
>
> Write your code in Binomial.java.
>
> - Plot a graph of $Pr[X = k]$ vs. $k$ for the case $n = 10, p = 0.6$ and for the case $n = 10, p = 0.2$.
> - Write a simulation to estimate $Pr[X = 3]$ when $n = 10, p = 0.6$. You can use Coin.java and CoinExample2.java for this purpose. Verify the estimate using the earlier formula.

### 1

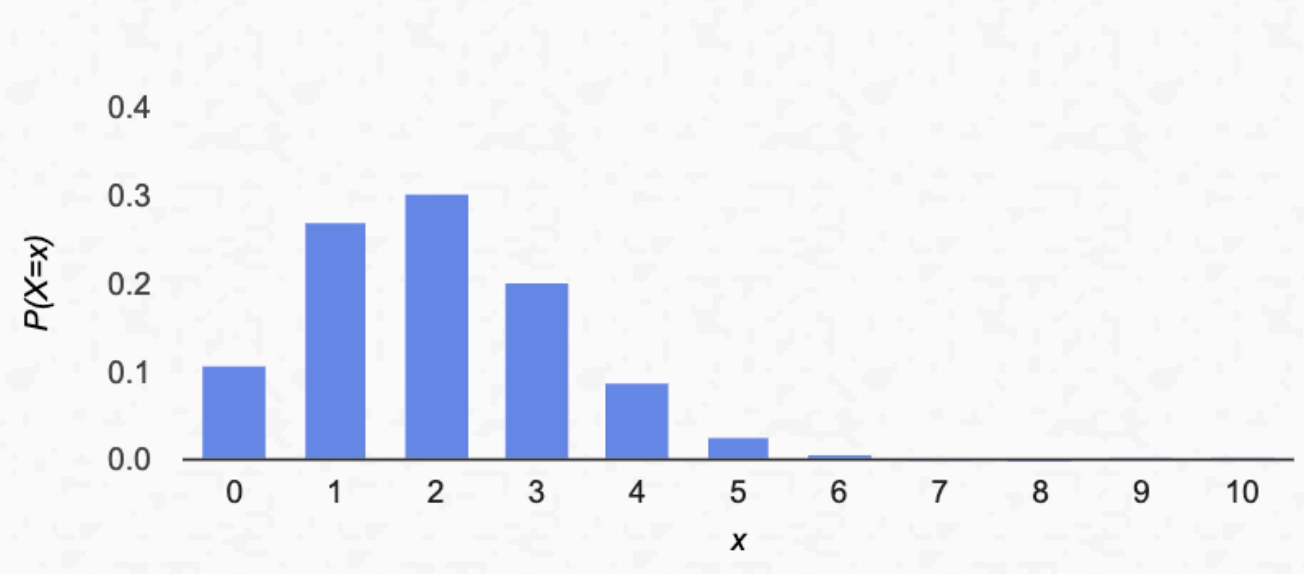
See file `Binomial.java`.

### 2

$n = 10$ and $p = 0.6$




$n = 10$ and $p = 0.2$




**3**

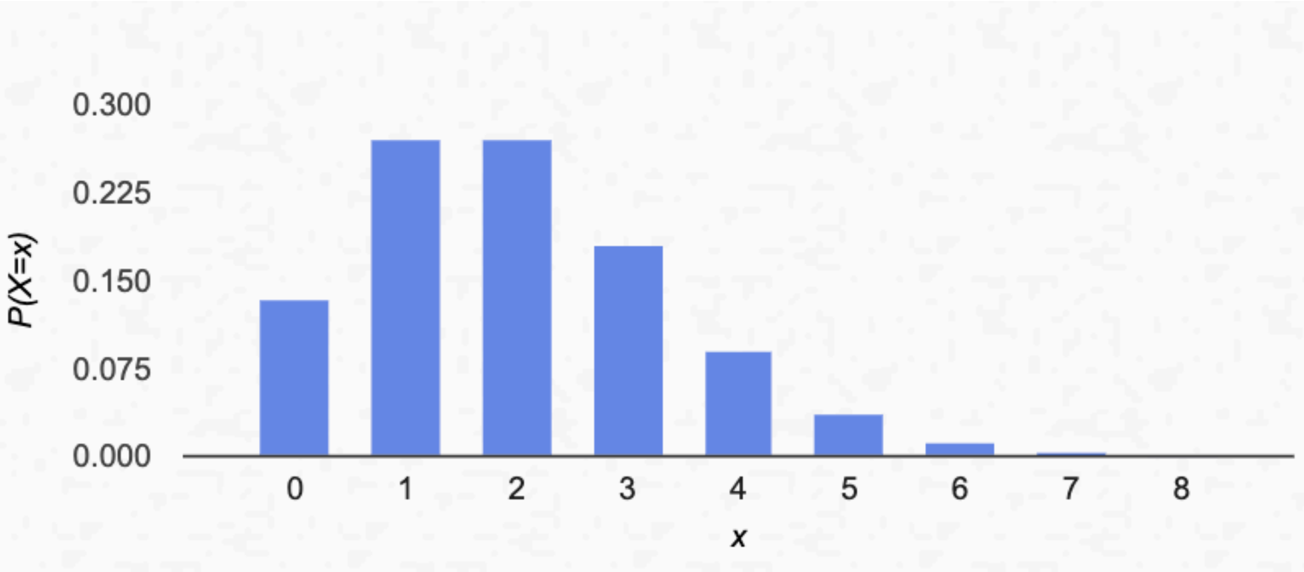
See file `CoinExample2.java`.

# 13

> Add code to [Poisson.java]() to compute $Pr[X = k]$ and plot a graph of $Pr[X = k]$ vs. $k$ when $\gamma = 2$. Use the Taylor series for $e^x$ to prove that $\sum_k Pr[X = k]$ adds up to 1.

See file `Poisson.java`.

$\gamma = 2$




Based on Taylor series,

$$e^x = \sum_k \frac{x^k}{k!}$$

So,

$$\sum_k Pr[X=k] = \sum_k e^{-\gamma}\frac{\gamma^k}{k!} = e^{-\gamma}\sum_k \frac{\gamma^k}{k!} = e^{-\gamma}e^{\gamma} = 1$$

## 14

> Download BusStop.java and BusStopExample3.java, and modify the latter to estimate the probability that exactly three buses arrive during the interval $[0,2]$. Compare this with $Pr[X=3]$ when $X \sim Poisson(2)$.

See file `BusStopExample3.java`.

Result:

```
The probability that exactly three buses arrive during the interval [0,2] is
0.1813
```

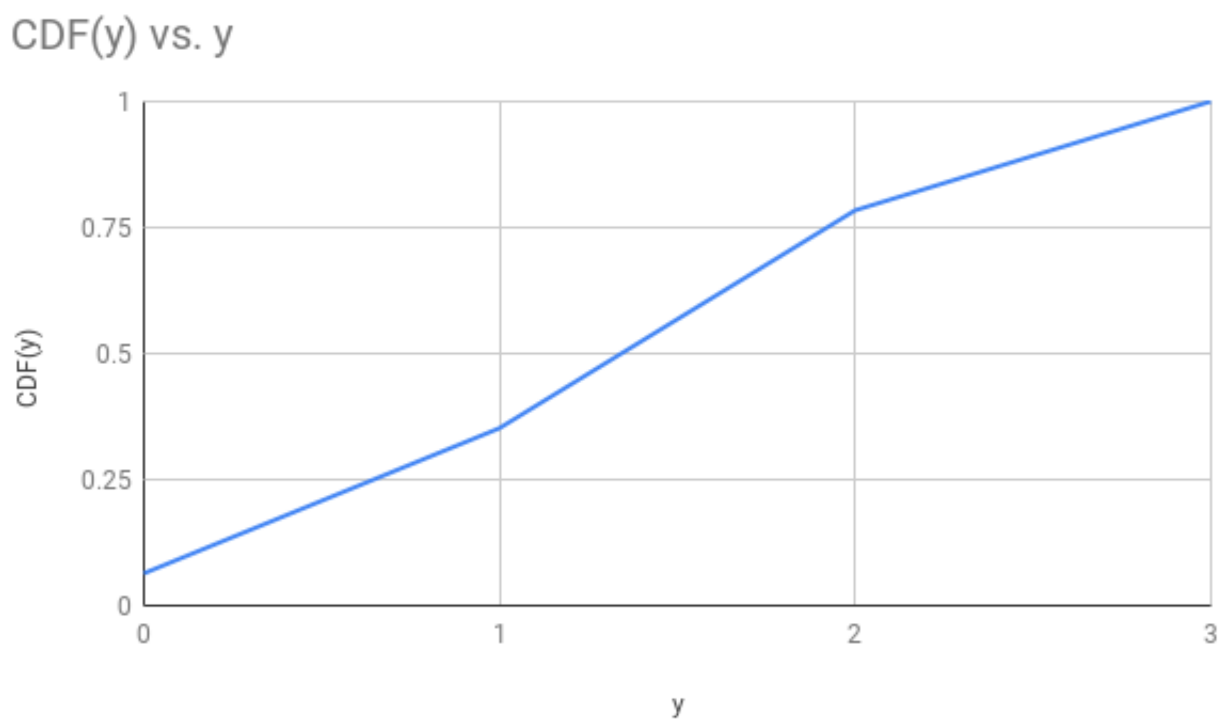
If $X \sim Poisson(2)$,

$$Pr[X=3] = e^{-\gamma}\frac{\gamma^k}{k!} = e^{-2}\frac{2^3}{3!} = 0.18045$$

## 19

> Consider the distribution for the 3-coin-flip example:
>
> $Pr[X=0] = 0.064$
>
> $Pr[X=1] = 0.288$
>
> $Pr[X=2] = 0.432$
>
> $Pr[X=3] = 0.216$
>
> Sketch the CDF on paper.

$$CDF(0) = Pr[X <= 0] = 0.064$$
$$CDF(1) = Pr[X <= 1] = 0.352$$
$$CDF(2) = Pr[X <= 2] = 0.784$$
$$CDF(3) = Pr[X <= 3] = 1$$




## 23

> What is an example of a continuous rv associated with the [QueueControl.java]() application?

the average service time for a customer

## 25

> The program [GaussianCDF.java]() estimates the CDF of a Gaussian rv. Execute the program to plot the CDF. Then, use this CDF to compute the following probabilities:
>
> - $Pr[0 < X \leq 2]$
> - $Pr[X > 0]$

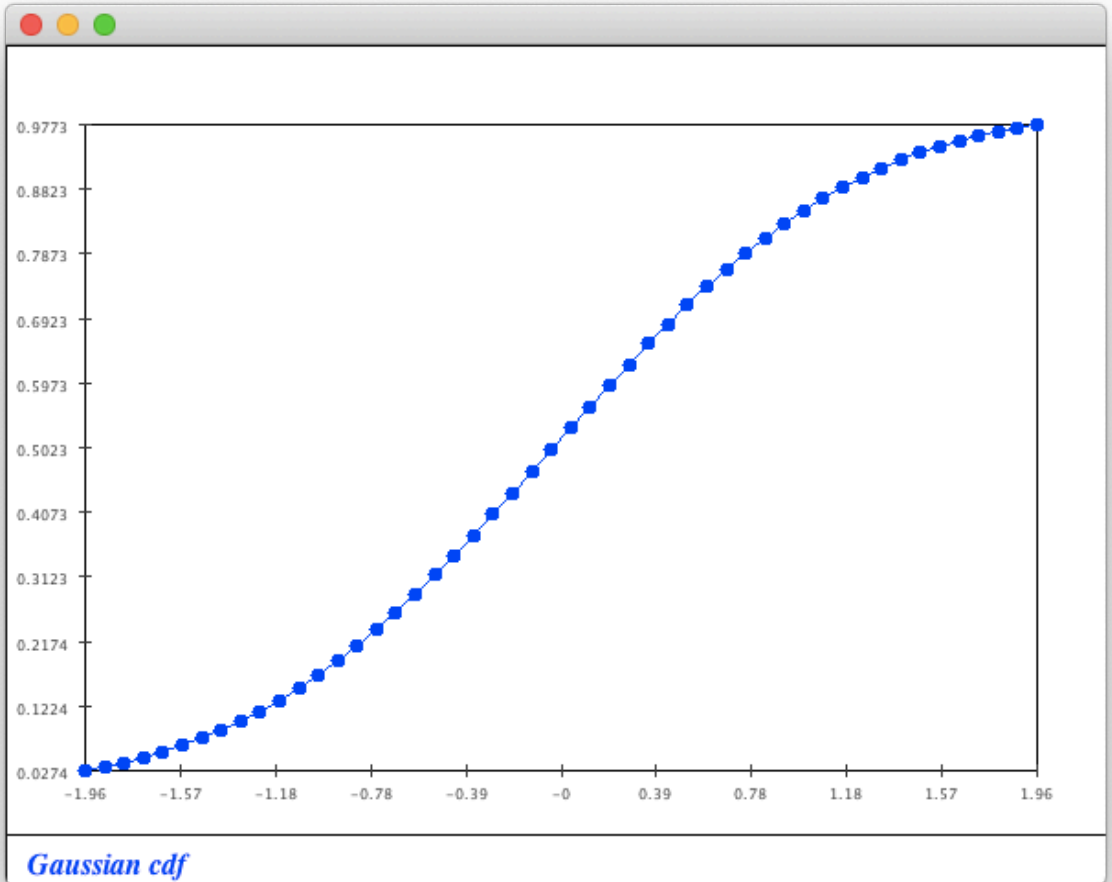
Modify file `GaussianCDF.java` as:

```
public class GaussianCDF {
    public static void main(String[] argv) {
        Function F = makeGaussianCDF();
        F.show();
        System.out.println("Pr[0 < x <= 2] = " + (F.get(2) - F.get(0)));
        System.out.println("Pr[x > 0] = " + (F.get(2) - F.get(0)));
    }
    static Function makeGaussianCDF() {
        double a = -2, b = 2;
        int M = 50;                    // Number of intervals.
```

```
        double delta = (b - a) / M;      // Interval size.
        double[] intervalCounts = new double[M];
        double numTrials = 1000000;
        for (int t = 0; t < numTrials; t++) {
            // Random sample:
            double y = RandTool.gaussian();
            // Truncate:
            if (y < a) {
                y = a;
            }
            if (y > b) {
                y = b;
            }
            // Find the right interval:
            int k = (int) Math.floor((y - a) / delta);
            // Increment the count for every interval above and including k.
            if (k < 0) {
                System.out.println("k=" + k + " y=" + y + " (y-a)=" + (y - a));
            }
            for (int i = k; i < M; i++) {
                intervalCounts[i]++;
            }
        }
        // Now compute probabilities for each interval.
        double[] cdf = new double[M];
        for (int k = 0; k < M; k++) {
            cdf[k] = intervalCounts[k] / numTrials;
        }
        // Build the CDF. Use mid-point of each interval.
        Function F = new Function("Gaussian cdf");
        for (int k = 0; k < M; k++) {
            double midPoint = a + k * delta + delta / 2;
            F.add(midPoint, cdf[k]);
        }
        return F;
    }
}
```

Result:

```
Pr[0 < x <= 2] = 0.462294
Pr[x > 0] = 0.462294
```

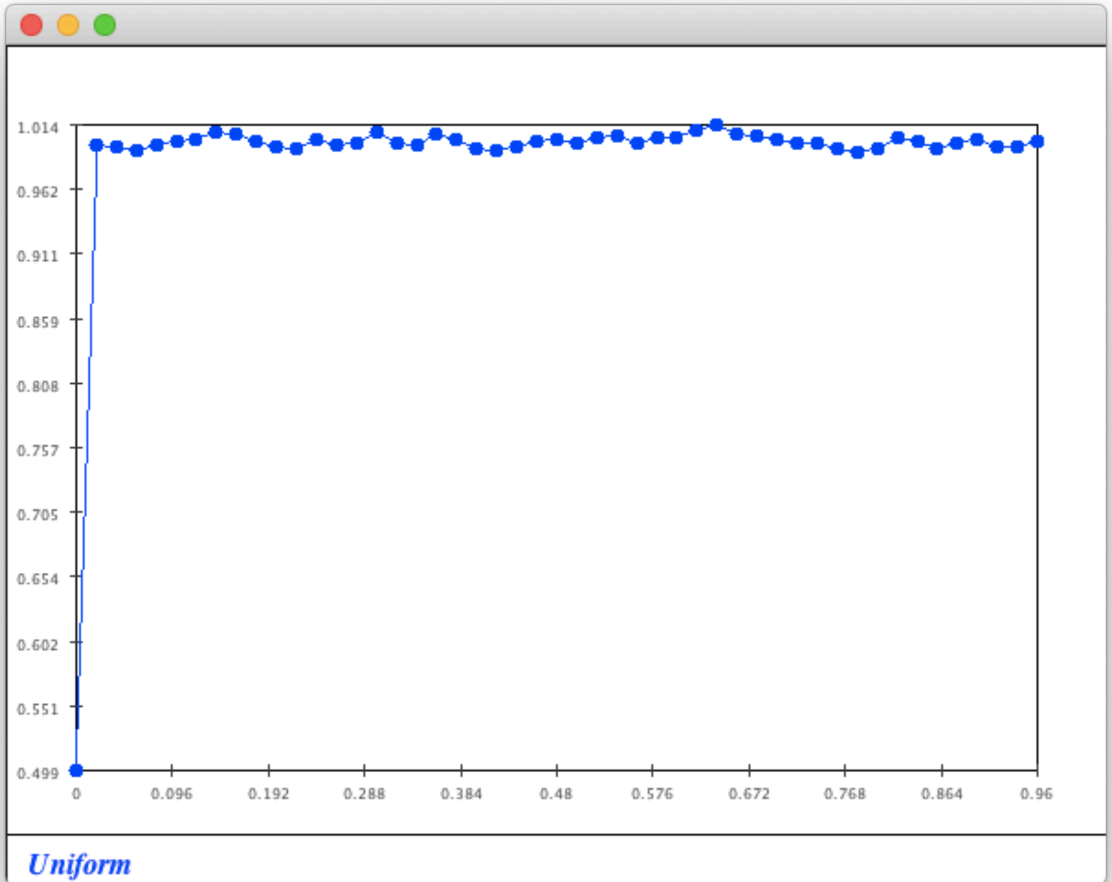



## 26

> Modify [UniformCDF.java]() and [GaussianCDF.java]() to compute the derivative of each. What is the shape of $F'(y)$ in each case?

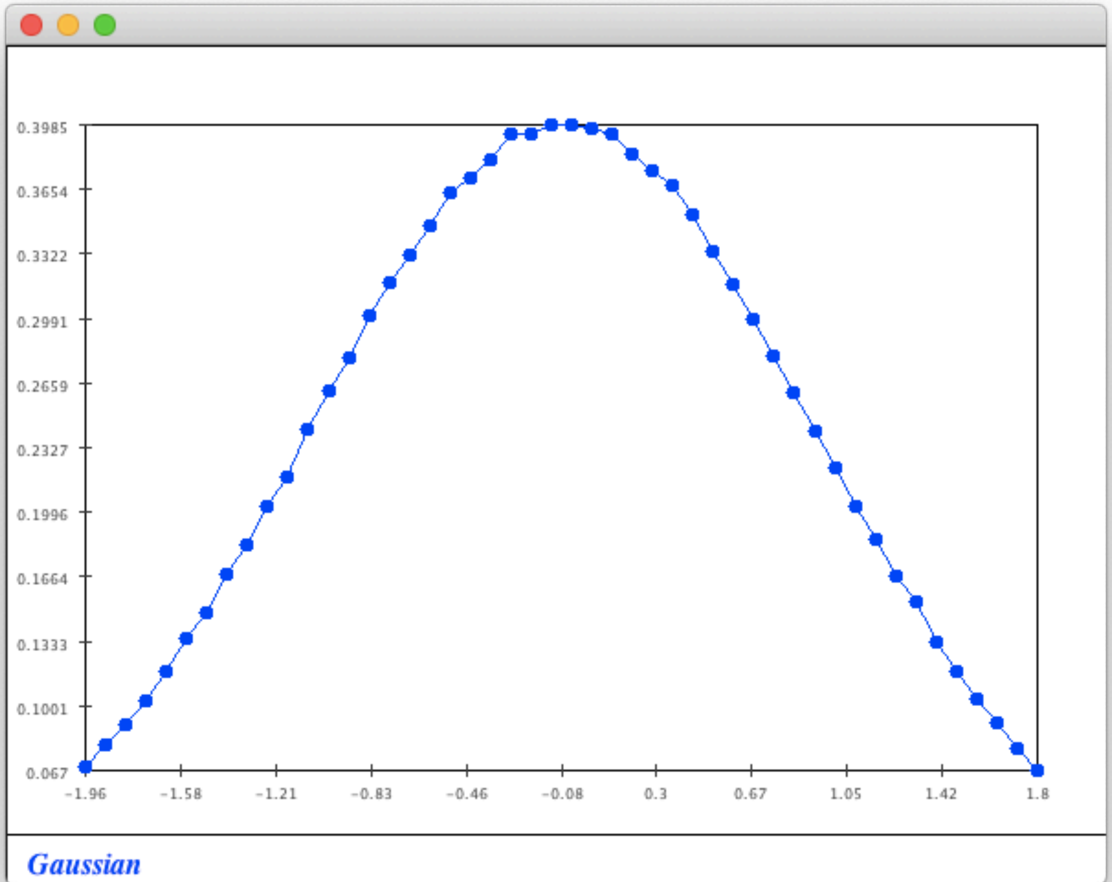
See file `UniformCDF.java`.

Result:




See file `GaussianCDF.java`.

Result:

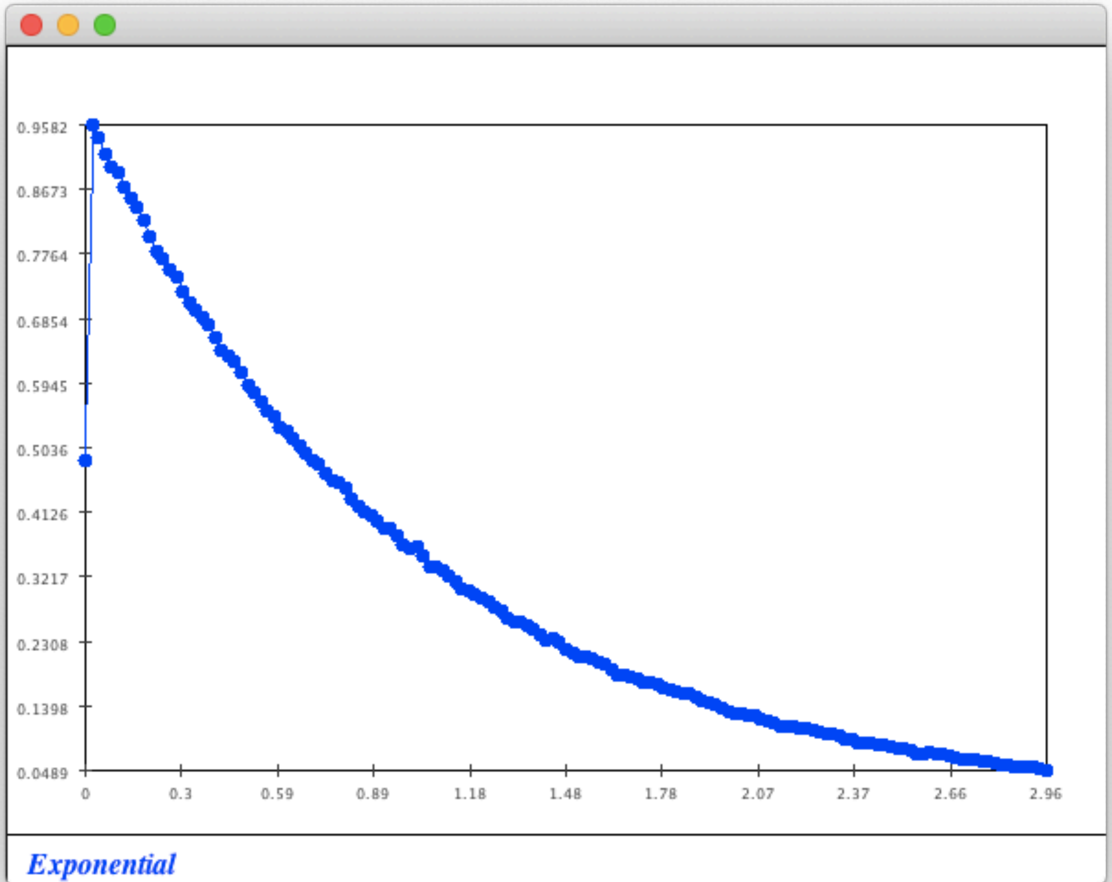



## 27

> If $X$ denotes the first interarrival time in the bus-stop problem, estimate the CDF of $X$ as follows:
>
> - Assume that values fall in the range $[0, 3]$ (i.e., disregard values outside this range).
> - Use ExponentialCDF.java as a template, and add modified code from UniformCDF.java.
> - Next, compute the *derivative* of this function and display it.

See file `ExponentialCDF.java`.

Result:




## 28

> Complete the calculation above. What would you get if $Pr[H] = 0.5$?

$$\begin{aligned}
E[X] &= \sum_{k \in \{0,1,2,3\}} k \cdot Pr[X = k] \\
&= 0 \cdot Pr[X = 0] + 1 \cdot Pr[X = 1] + 2 \cdot Pr[X = 2] + 3 \cdot Pr[X = 3] \\
&= 0 \times 0.064 + 1 \times 0.288 + 2 \times 0.432 + 3 \times 0.216 \\
&= 1.8 \\
&= 3 \times 0.6 \\
&= np
\end{aligned}$$

If $Pr[H] = 0.5$ which means $X \sim Binomial(3, 0.5)$,

$$P[X = k] = \binom{3}{k} 0.5^k (1 - 0.5)^{3-k} = \binom{3}{k} 0.5^3$$

So,

$$
\begin{aligned}
Pr[X = 0] &= 0.125 \\
Pr[X = 1] &= 0.375 \\
Pr[X = 2] &= 0.375 \\
Pr[X = 3] &= 0.125
\end{aligned}
$$

Expected value:

$$
\begin{aligned}
E[X] = &\sum_{k \in \{0,1,2,3\}} k \cdot Pr[X = k] \\
&= 0 \cdot Pr[X = 0] + 1 \cdot Pr[X = 1] + 2 \cdot Pr[X = 2] + 3 \cdot Pr[X = 3] \\
&= 0 * 0.125 + 1 * 0.375 + 2 * 0.375 + 3 * 0.125 \\
&= 1.5 \\
&= 3 \times 0.5 \\
&= np
\end{aligned}
$$

## 29

> How does this relate to the 3-coin-flip example?

See exercise #28.

## 31

> What does $\frac{n}{k_n}$ become in the limit? Unfold the sum for the 3-coin-flip example to see why this is true.

$\frac{n}{k_n}$ in the limit will become as the probability.

## 32

> Download [Coin.java]() and [CoinExample3.java]() and and let $X =$ the number of heads in 3 coin flips.
>
> - Compute the average value of $X$ using $\frac{1}{n} S_n$
> - Estimate $Pr[X = k]$ using $\frac{n_k}{n}$.
> - Compute $\sum_k \frac{n_k}{n}$ using the estimates of $\frac{n_k}{n}$.
>
> Compare with the $E[X]$ calculation you made earlier.

See file `CoinExample3.java`.

Result:

```
average value of X: 1.800082
Pr[X=0] = 0.064165
Pr[X=1] = 0.287286
Pr[X=2] = 0.432851
Pr[X=3] = 0.215698
\sum_{k}k\frac{n_{k}}{n} = 1.8000819999999997
```

# 33

> Use Coin.java and CoinExample4.java and let $X =$ the number of flips needed to get the first heads when $Pr[Heads] = 0.1$. Compute the average value of $X$ using $\frac{1}{n}S_n$ as you did in the previous exercise. Compare with the $E[X]$ calculation from earlier.

See file `CoinExample4.java`.

Result:

```
range: 0 - 1000
average value of X: 10.008696
Pr[X=1] = 0.100442
Pr[X=2] = 0.089667
Pr[X=3] = 0.081536
\sum_{k}k\frac{n_{k}}{n} = 10.008696000000002
```

It is clear that $X \sim Geometric(0.1)$ and $E(X) = \frac{1}{0.1} = 10$.

# 34

> Try this computation with the uniform, Gaussian and exponential distributions using UniformCDF2.java, GaussianCDF2.java, and ExponentialCDF2.java. Explore what happens when more intervals are used in the expectation computation than in the CDF estimation.

See file `UniformCDF2.java`.

Result:

```
Uniform ex: 0.4900223200000005
```

See file `GaussianCDF2.java`.

Result:

```
Gaussian ex: -0.027875039999999858
```

See file `Exponential.java`.

Result:

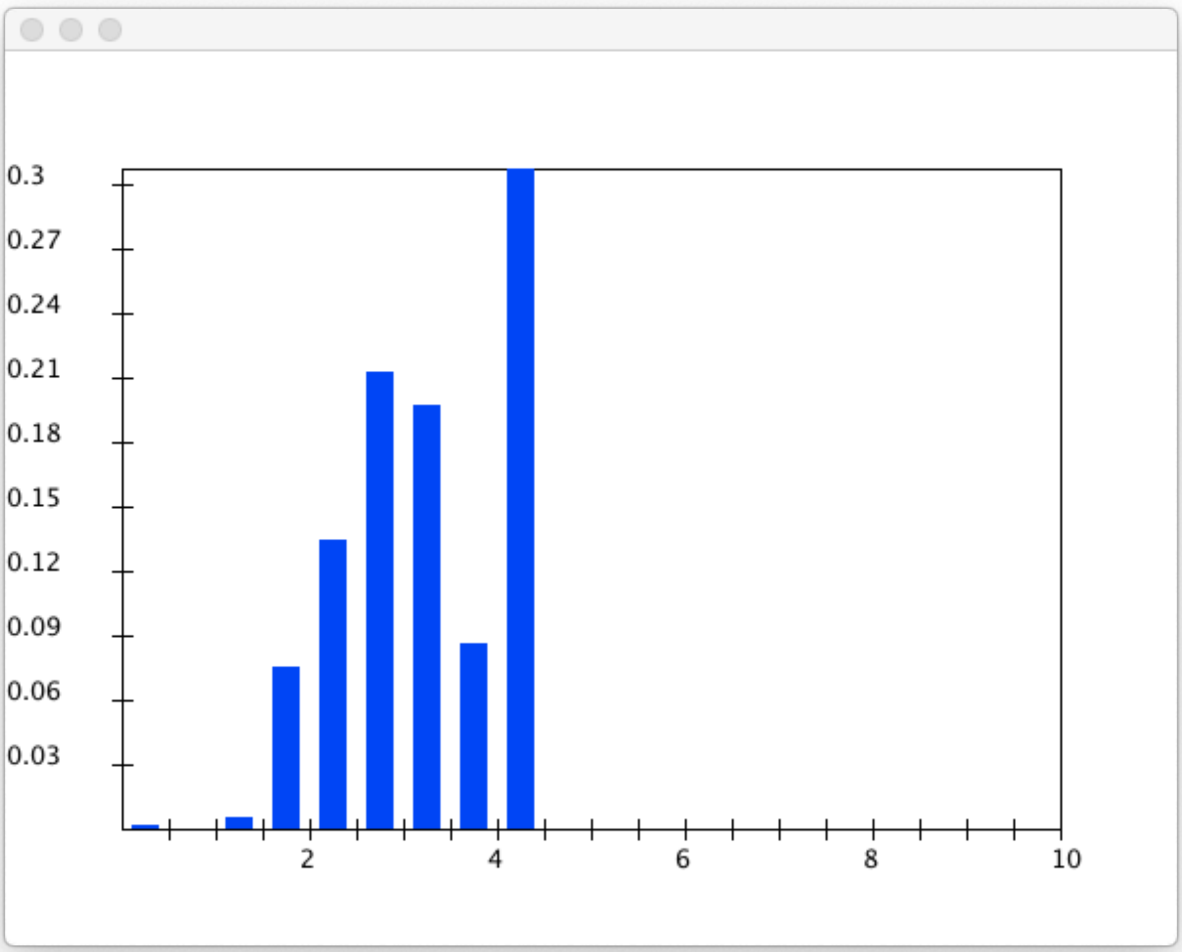
```
Exponential ex: 0.4621379250000003
```

## 40

> Estimate the density of the time spent in the system by a random customer in the QueueControl example. To do this, you need to build a density histogram of values of the variable `timeInSystem` in [QueueControl.java](QueueControl.java).

See file `exercise40.java`.

Result:




## 44

> Suppose $X \sim Exponential(\gamma)$ with CDF $F(x)$. Write down an expression for $F^{-1}(y)$, the inverse of $F$.

$$F(x) = \begin{cases} 0 & x \le 0 \\ 1 - e^{-\gamma x} & x > 0 \end{cases}$$

Only consider when $x > 0$,

$$y = F(x) = 1 - e^{-\gamma x}$$

So,

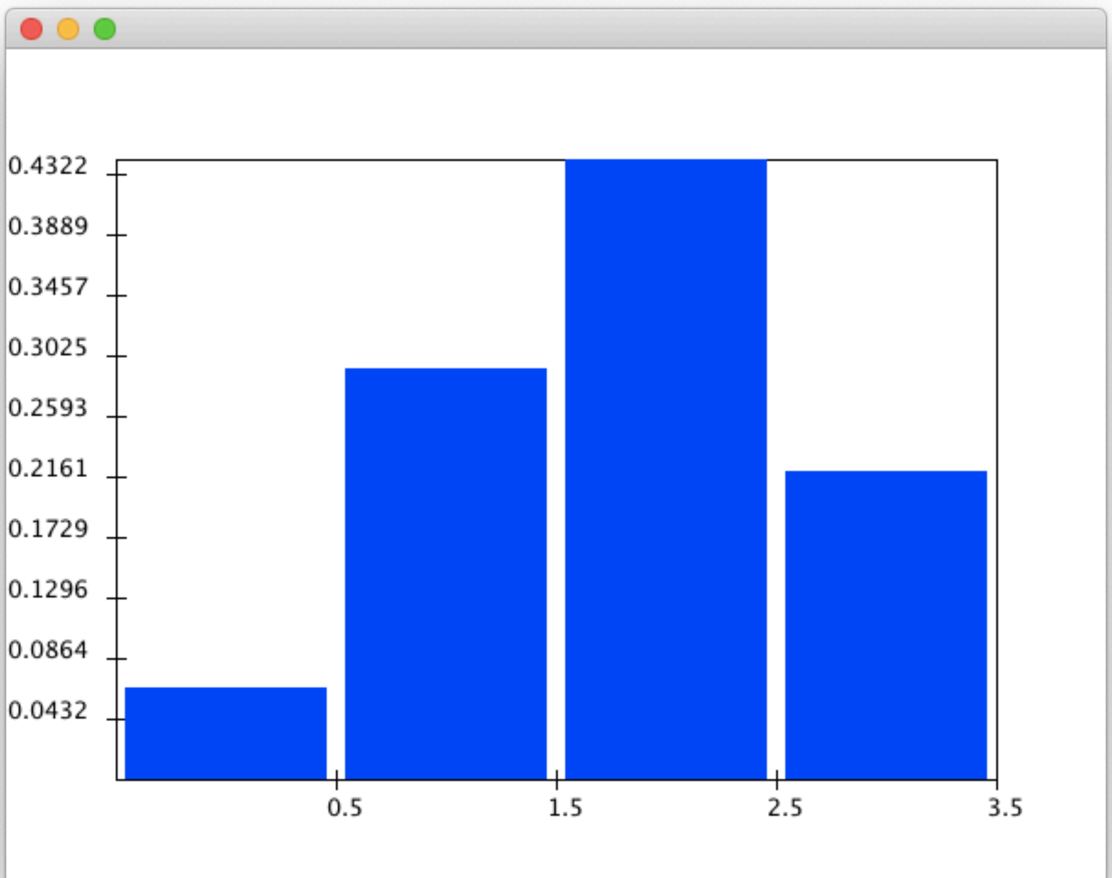
$$x = F^{-1}(y) = -\frac{\log(1-y)}{\gamma}$$

## 46

> Add code to DiscreteGenExample.java to implement the above generator, and to test it by building a histogram.

See file `DiscreteGenExample.java`.

Result:



## 48

> Add code to [ExponentialGenerator.java]() to implement the above idea. Use the inverse-CDF you computed earlier. The test code is written to produce a histogram. Use your modified version of [PropHistogram.java]() to make a density histogram. Compare the result with the actual density (using $\gamma = 4$). How do you know your code worked?

See file `ExponentialGenerator.java`.

Result: