Lecture 7: The Discrete-time Fourier Transform Properties

Foundations of Digital Signal Processing

Outline

- The Z-Transform Review
- The Discrete-time Fourier Transform (DTFT)
- The Properties of the Discrete-time Fourier Transform (DTFT)

News

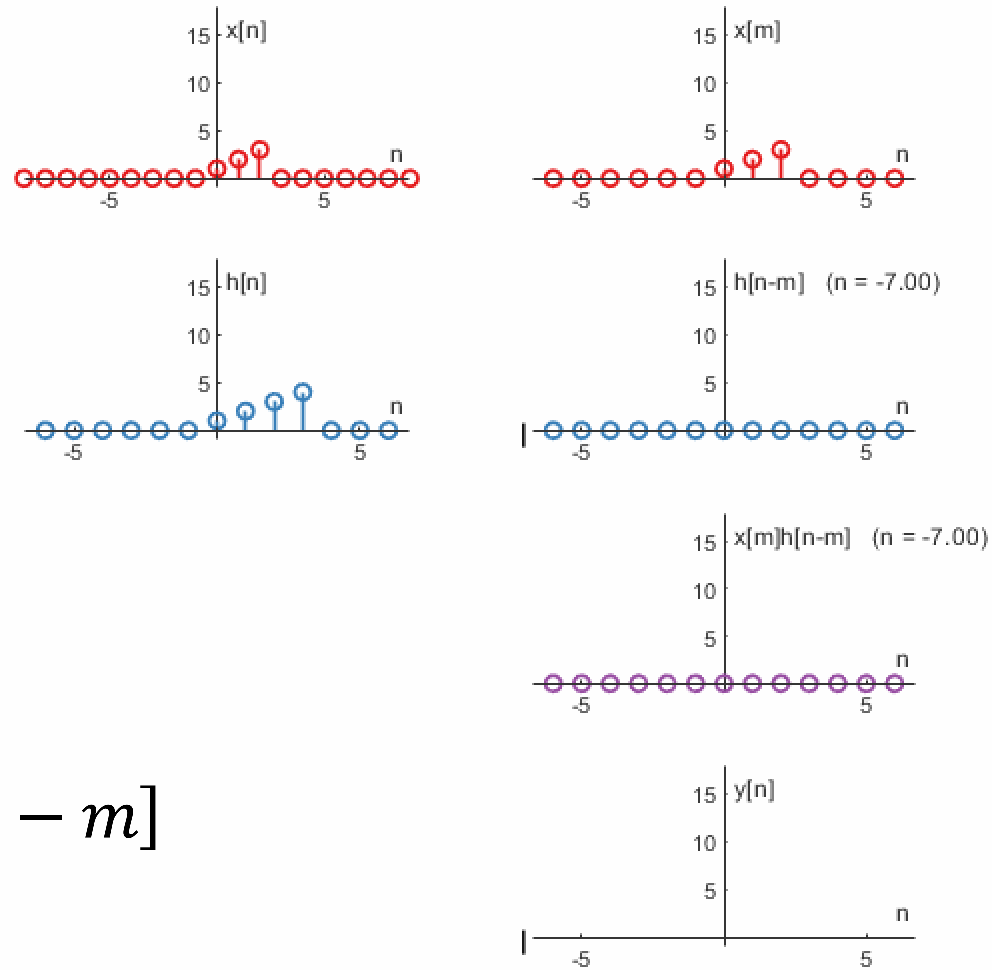
- **Homework #3**
  - Due *Today* by 11:59 PM
  - Submit via canvas
- **Coding Problem #2**
  - Due *Next Week* by 11:59 PM
  - Submit via canvas
- **Exam #1**
  - September 25th
  - 1.5 weeks away

News

- **Exam #1**
  - September 25th (1.5 weeks away)
  - Will cover all material up to today… such as
    - Signal properties
    - System properties
    - LTI Systems
    - Difference equations
    - Discrete-time convolution
    - The Z-transform and its properties
    - The Discrete-time Fourier Transform and its properties
    - Etc.

Convolution

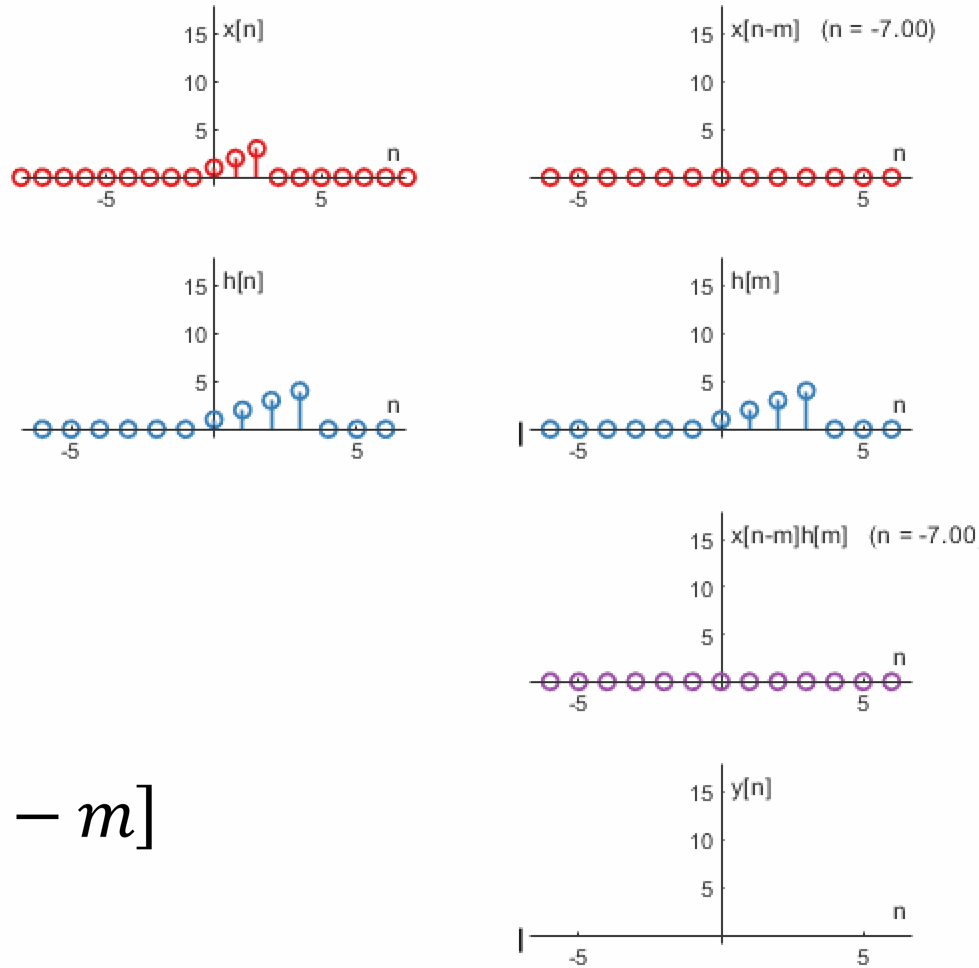
- **Convolution is defined by**



$$x[n] * h[n] = \sum_{m=-\infty}^{\infty} x[m]h[n-m]$$

Convolution

- **Convolution is defined by**



$$h[n] * x[n] = \sum_{m=-\infty}^{\infty} h[m]x[n-m]$$

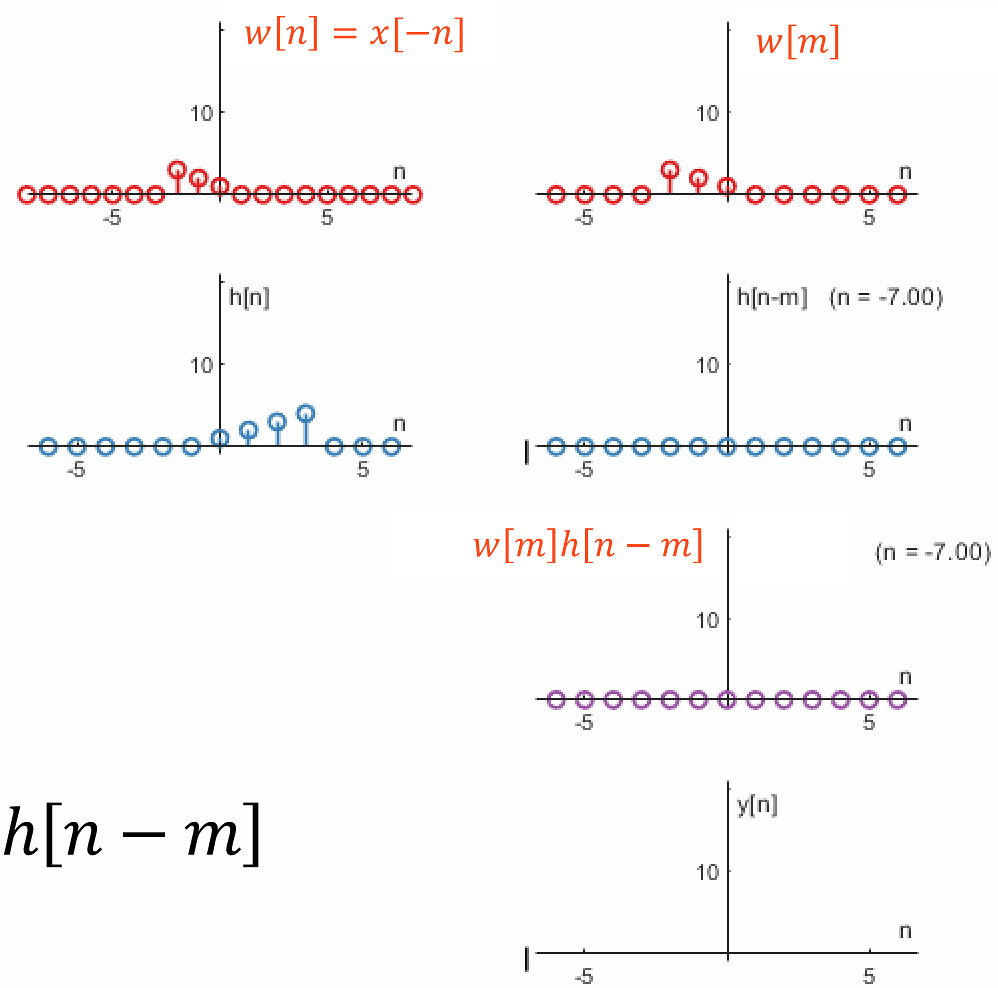
Correlation

- **Correlation is defined by**

$$x[-n] * h[n] = \sum_{m=-\infty}^{\infty} x[-m]h[n-m]$$

Correlation

- **Correlation is defined by**



$$x[-n] * h[n] = \sum_{m=-\infty}^{\infty} x[-m]h[n-m]$$

Correlation

- **Correlation is defined by**

$$h[n] * x[-n] = \sum_{m=-\infty}^{\infty} h[m]x[-(n-m)]$$

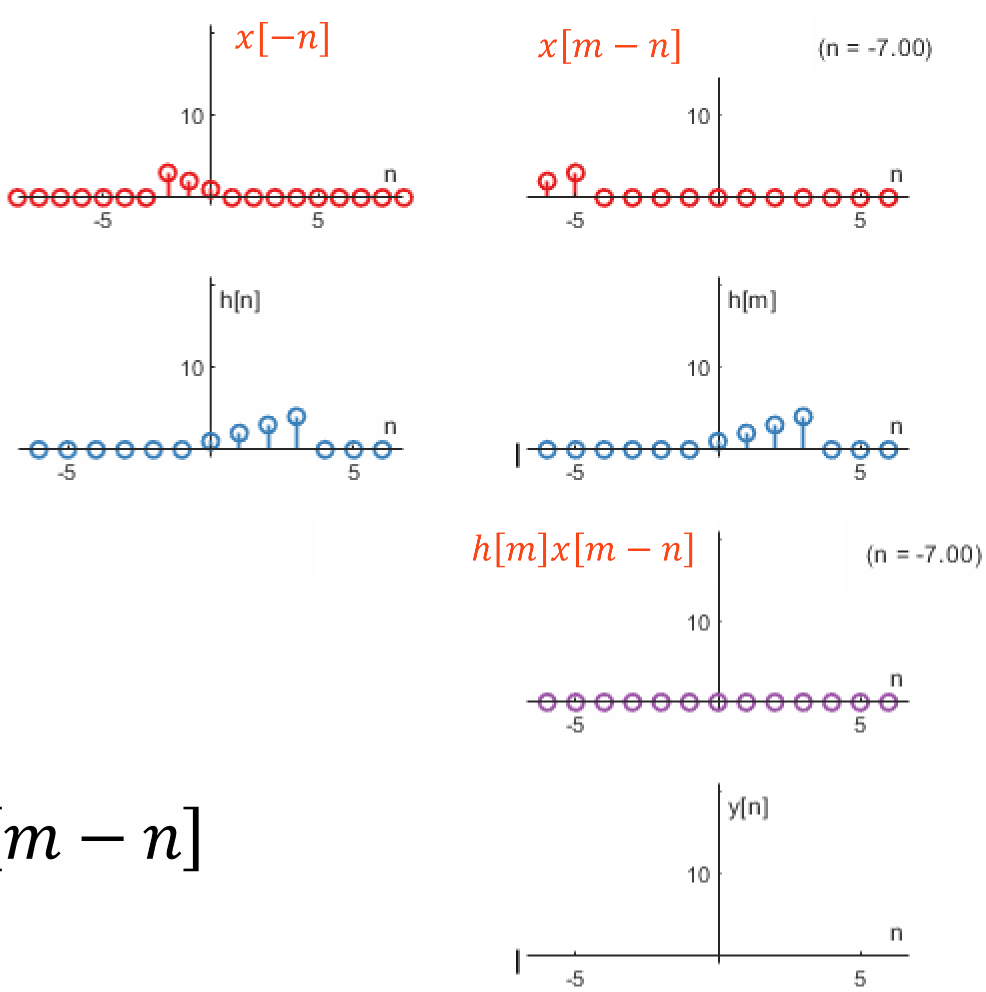
Correlation

- **Correlation is defined by**

$$h[n] * x[-n] = \sum_{m=-\infty}^{\infty} h[m]x[m-n]$$

Correlation

- **Correlation is defined by**



$$h[n] * x[-n] = \sum_{m=-\infty}^{\infty} h[m]x[m-n]$$

Correlation

- **Correlation is defined by**

$$x[-n] * h[n] = \sum_{m=-\infty}^{\infty} x[-m]h[n-m]$$

Correlation

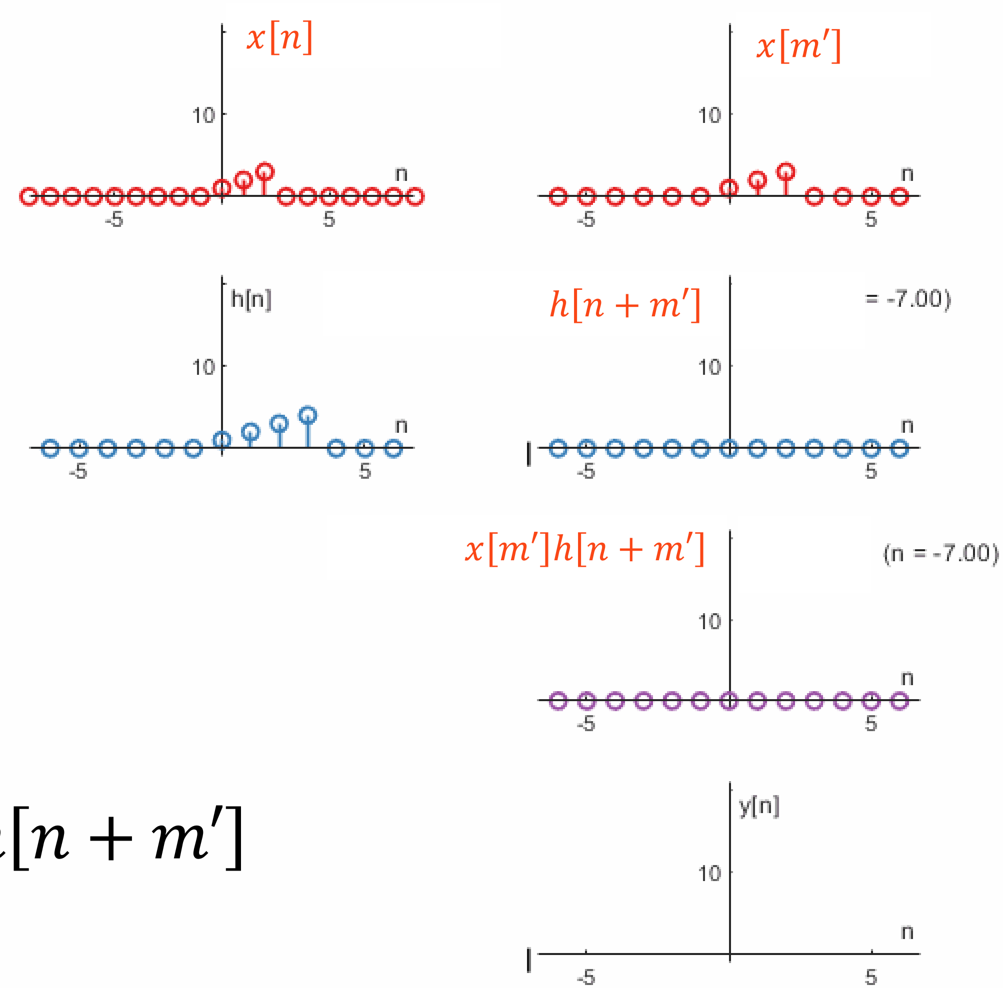
- **Correlation is defined by**

Choose $m' = -m$

$$x[-n] * h[n] = \sum_{m=-\infty}^{\infty} x[m']h[n+m']$$

Correlation

- **Correlation is defined by**

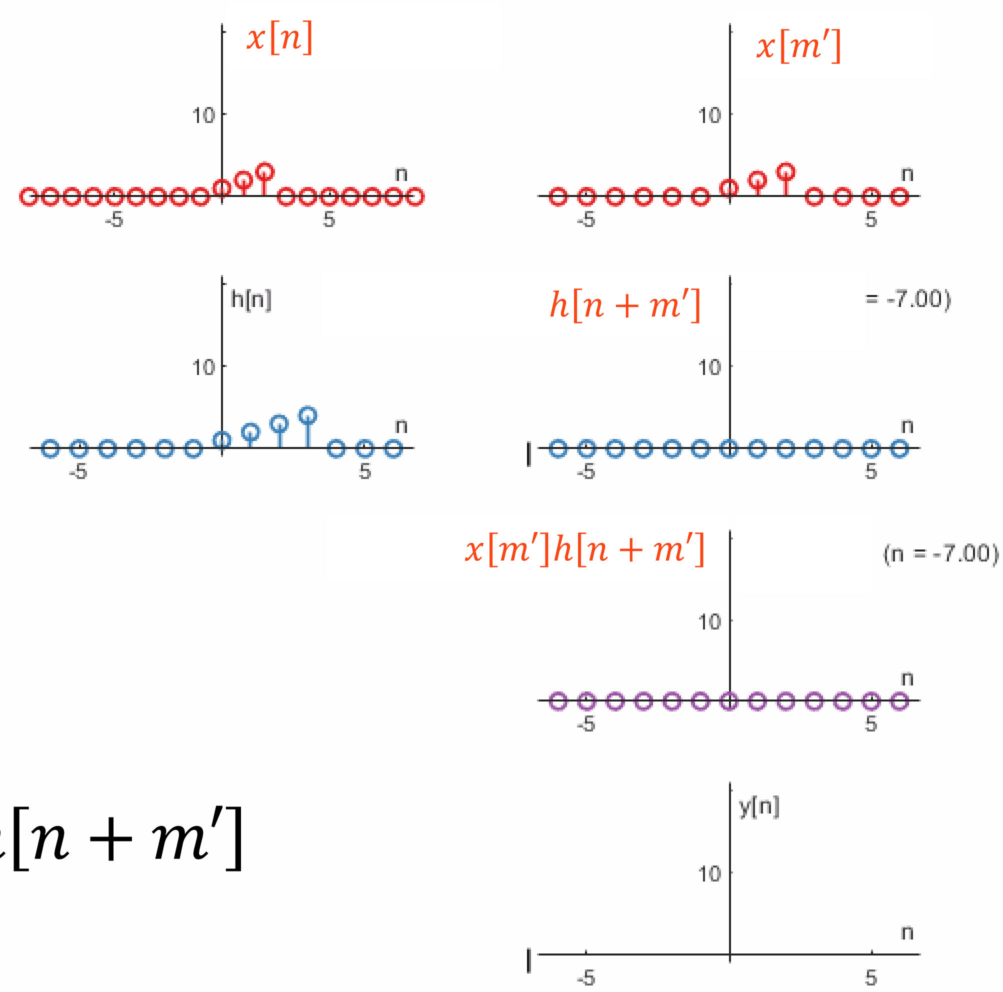


Choose $m' = -m$

$$x[-n] * h[n] = \sum_{m=-\infty}^{\infty} x[m']h[n+m']$$

Correlation

- **Correlation is defined by**



Choose $m' = -m$

$$x[-n] * h[n] = \sum_{m=-\infty}^{\infty} x[m']h[n+m']$$

Correlation

- **Note that for correlation:**

$$x[-n] * h[n] \neq x[n] * h[-n]$$

$$\sum_{m=-\infty}^{\infty} x[m]h[n+m] \neq \sum_{m=-\infty}^{\infty} x[n+m]h[m]$$

Lecture 7: The Discrete-time Fourier Transform Properties

Foundations of Digital Signal Processing

Outline

- **The Z-Transform Review**
- The Discrete-time Fourier Transform (DTFT)
- The Properties of the Discrete-time Fourier Transform (DTFT)

The Z-Transform

- **The [Bi-lateral] Z-Transform**

The Z-Transform

- **Z-Transform Properties**

The Z-Transform

- **Z-Transform Region of Convergence**

The Z-Transform

- **The Discrete-Time Fourier Transform**

The Z-Transform

- **The Discrete-Time Fourier Transform**

The Z-Transform

- **Example: Evaluate this convolution via the Z-transform.**

$$x[n] = \left(\frac{1}{2}\right)^n u[n] * \left(\frac{1}{4}\right)^n u[n]$$

The Z-Transform

- **Example: Evaluate this convolution via the Z-transform.**

$$x[n] = \left(\frac{1}{2}\right)^n u[n] * \left(\frac{1}{4}\right)^n u[n]$$

$$X(z) = \frac{1}{\left(1 - \left(\frac{1}{2}\right) z^{-1}\right)\left(1 - \left(\frac{1}{4}\right) z^{-1}\right)}$$

The Z-Transform

- **Example: Evaluate this convolution via the Z-transform.**

$$x[n] = \left(\frac{1}{2}\right)^n u[n] * \left(\frac{1}{4}\right)^n u[n]$$

$$X(z) = \frac{1}{\left(1-\left(\frac{1}{2}\right)z^{-1}\right)\left(1-\left(\frac{1}{4}\right)z^{-1}\right)} = \frac{A}{\left(1-\left(\frac{1}{2}\right)z^{-1}\right)} + \frac{B}{\left(1-\left(\frac{1}{4}\right)z^{-1}\right)}$$

The Z-Transform

- **Example: Evaluate this convolution via the Z-transform.**

$$x[n] = \left(\frac{1}{2}\right)^n u[n] * \left(\frac{1}{4}\right)^n u[n]$$

$$X(z) = \frac{1}{\left(1 - \left(\frac{1}{2}\right) z^{-1}\right)\left(1 - \left(\frac{1}{4}\right) z^{-1}\right)} = \frac{A}{\left(1 - \left(\frac{1}{2}\right) z^{-1}\right)} + \frac{B}{\left(1 - \left(\frac{1}{4}\right) z^{-1}\right)}$$

$$1 = A\left(1 - \left(\frac{1}{4}\right) z^{-1}\right) + B\left(1 - \left(\frac{1}{2}\right) z^{-1}\right)$$

$$z^{-1} = 4, \qquad 1 = B(1-2) \rightarrow B = -1$$

$$z^{-1} = 2, \qquad 1 = A(1-1/2) \rightarrow A = 2$$

The Z-Transform

- **Example: Evaluate this convolution via the Z-transform.**

$$x[n] = \left(\frac{1}{2}\right)^n u[n] * \left(\frac{1}{4}\right)^n u[n]$$

$$X(z) = \frac{2}{\left(1 - \left(\frac{1}{2}\right) z^{-1}\right)} - \frac{1}{\left(1 - \left(\frac{1}{4}\right) z^{-1}\right)}$$

$$x[n] = 2\left(\frac{1}{2}\right)^n u[n] - \left(\frac{1}{4}\right)^n u[n]$$

The Z-Transform

- **Example: Evaluate this convolution via the Z-transform.**

$$x[n] = \left(\frac{1}{2}\right)^n u[n] * \left(\frac{1}{4}\right)^n u[n]$$

$$X(z) = \frac{2}{\left(1 - \left(\frac{1}{2}\right) z^{-1}\right)} - \frac{1}{\left(1 - \left(\frac{1}{4}\right) z^{-1}\right)}$$

$$x[n] = 2\left(\frac{1}{2}\right)^n u[n] - \left(\frac{1}{4}\right)^n u[n]$$

Confirm this?!

The Z-Transform

- **Example: Evaluate this convolution via the Z-transform.**

$$x[n] = \left(\frac{1}{2}\right)^n u[n] * \left(\frac{1}{4}\right)^n u[n]$$

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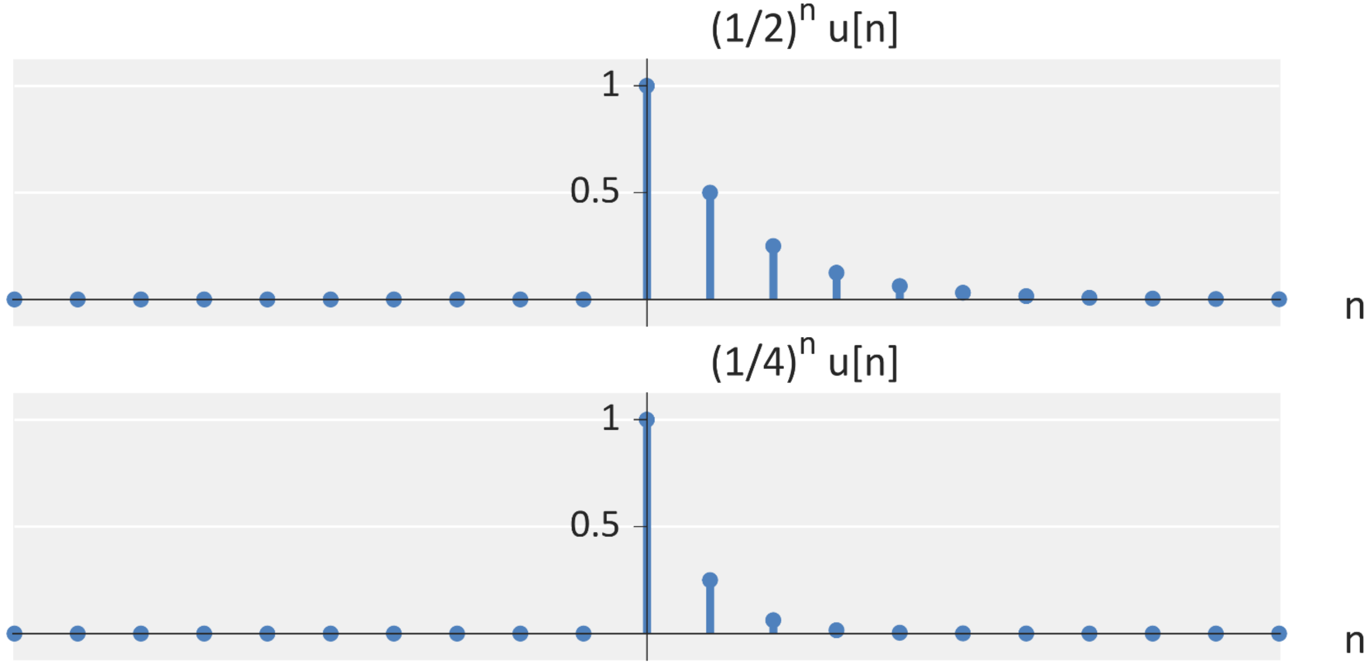

The Z-Transform

- **Example:** **Evaluate this convolution via the Z-transform.**

$$x[n] = \left(\frac{1}{2}\right)^n u[n] * \left(\frac{1}{4}\right)^n u[n]$$

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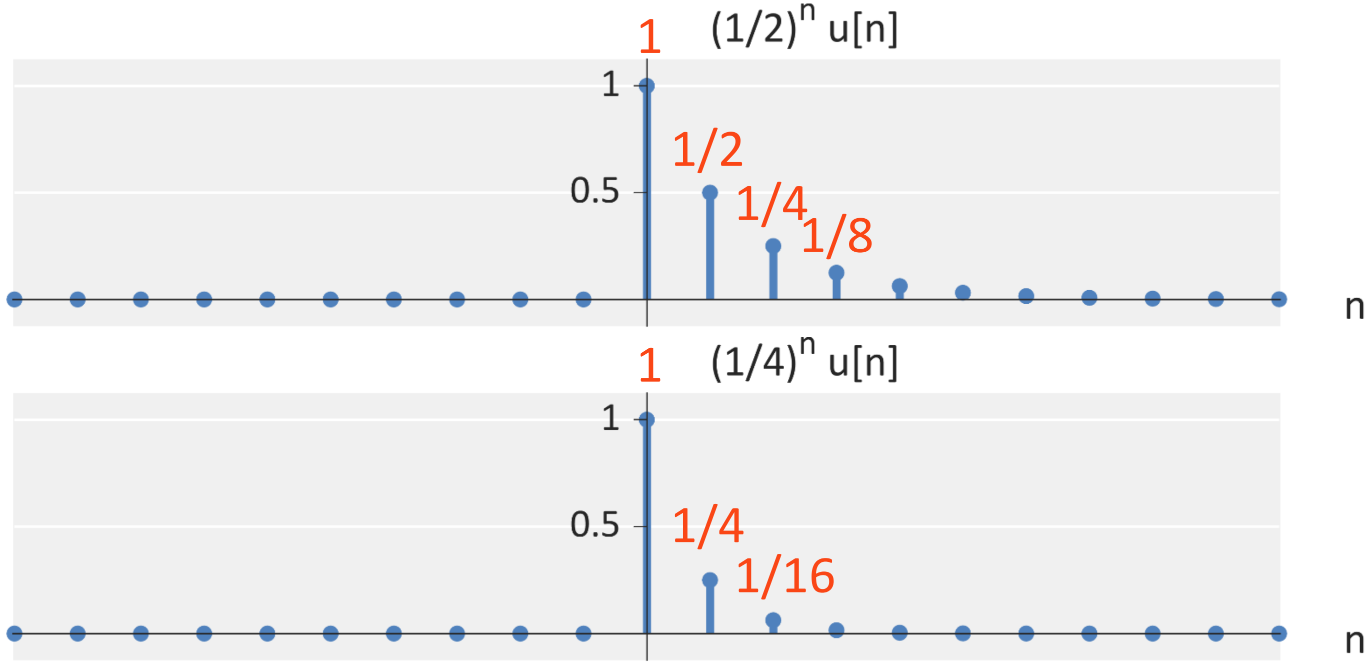

The Z-Transform

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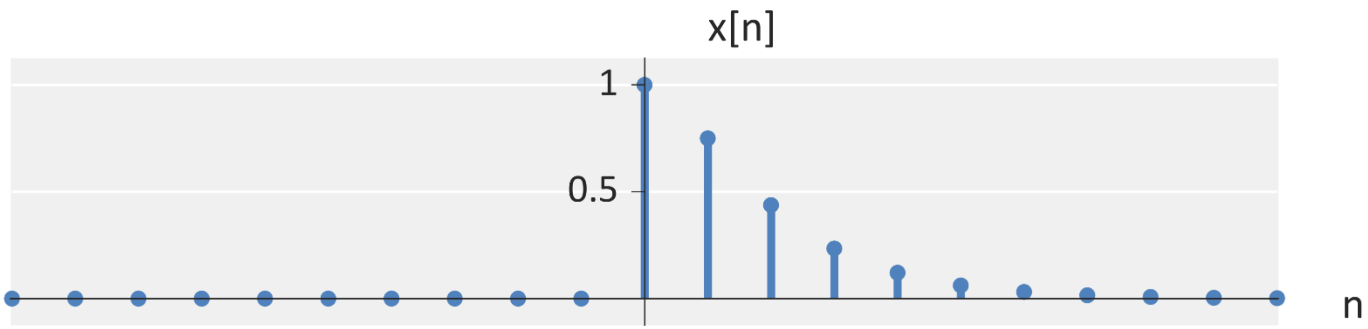

The Z-Transform

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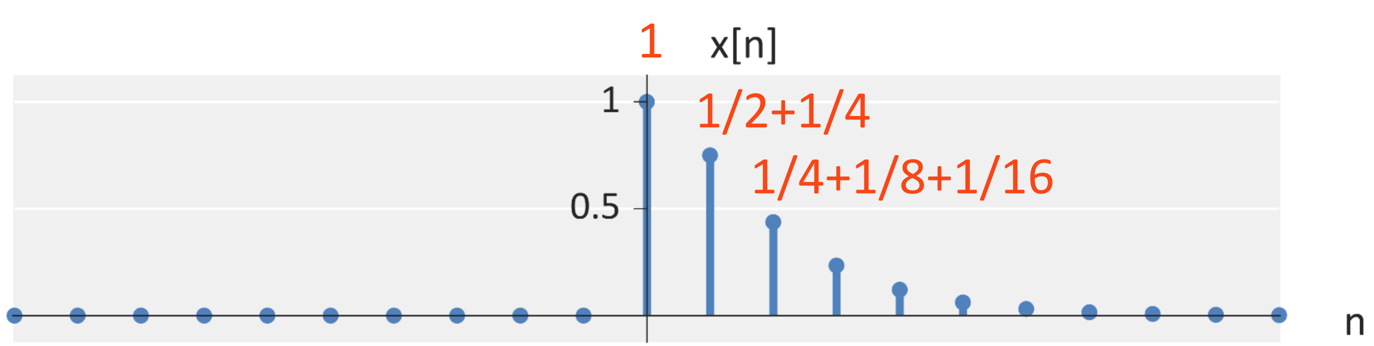

The Z-Transform

- **Example: Evaluate this convolution via the Z-transform.**

$$x[n] = \left(\frac{1}{2}\right)^n u[n] * \left(\frac{1}{4}\right)^n u[n]$$

Solution:

$$X(z) = \frac{z^2}{(z - 1/2)(z - 1/4)}$$

The Z-Transform

- **Example: Evaluate this convolution via the Z-transform.**

$$x[n] = \left(\frac{1}{2}\right)^{n-1} u[n-1] * \left(\frac{1}{4}\right)^{n} u[n]$$

The Z-Transform

- **Example: Evaluate this convolution via the Z-transform.**

$$x[n] = \left(\frac{1}{2}\right)^{n-1} u[n-1] * \left(\frac{1}{4}\right)^{n} u[n]$$

$$X(z) = \frac{z^{-1}}{\left(1 - \left(\frac{1}{2}\right) z^{-1}\right)\left(1 - \left(\frac{1}{4}\right) z^{-1}\right)}$$

  - **What are the multiple ways to solve this??**

The Z-Transform

- **Example: Evaluate this convolution via the Z-transform.**

$$x[n] = \left(\frac{1}{2}\right)^{n-1} u[n-1] * \left(\frac{1}{4}\right)^{n} u[n]$$

$$X(z) = z^{-1}\left[\frac{1}{\left(1 - \left(\frac{1}{2}\right)z^{-1}\right)\left(1 - \left(\frac{1}{4}\right)z^{-1}\right)}\right]$$

Option 1: Use the shifting property.

The Z-Transform

- **Example: Evaluate this convolution via the Z-transform.**

$$x[n] = \left(\frac{1}{2}\right)^{n-1} u[n-1] * \left(\frac{1}{4}\right)^{n} u[n]$$

$$X(z) = z^{-1}\left[\frac{1}{\left(1-\left(\frac{1}{2}\right)z^{-1}\right)\left(1-\left(\frac{1}{4}\right)z^{-1}\right)}\right]$$

Option 1: Use the shifting property.

$$x[n] = 2\left(\frac{1}{2}\right)^{n-1} u[n-1] - \left(\frac{1}{4}\right)^{n-1} u[n-1]$$

$$x[n] = 4\left(\frac{1}{2}\right)^{n} u[n-1] - 4\left(\frac{1}{4}\right)^{n} u[n-1]$$

The Z-Transform

- **Example: Evaluate this convolution via the Z-transform.**

$$x[n] = \left(\frac{1}{2}\right)^{n-1} u[n-1] * \left(\frac{1}{4}\right)^{n} u[n]$$

$$X(z) = \frac{z^{-1}}{\left(1-\left(\frac{1}{2}\right)z^{-1}\right)\left(1-\left(\frac{1}{4}\right)z^{-1}\right)} = \frac{A}{\left(1-\left(\frac{1}{2}\right)z^{-1}\right)} + \frac{B}{\left(1-\left(\frac{1}{4}\right)z^{-1}\right)}$$

$$z^{-1} = A\left(1-\left(\frac{1}{4}\right)z^{-1}\right) + B\left(1-\left(\frac{1}{2}\right)z^{-1}\right)$$

Option 2: Use partial fractions

$$z^{-1} = 4, \qquad 4 = B(1-2) \rightarrow B = -4$$

$$z^{-1} = 2, \qquad 2 = A(1-1/2) \rightarrow A = 4$$

The Z-Transform

- **Example: Evaluate this convolution via the Z-transform.**

$$x[n] = \left(\frac{1}{2}\right)^{n-1} u[n-1] * \left(\frac{1}{4}\right)^{n} u[n]$$

$$X(z) = \frac{4}{\left(1 - \left(\frac{1}{2}\right) z^{-1}\right)} - \frac{4}{\left(1 - \left(\frac{1}{4}\right) z^{-1}\right)}$$

$$x[n] = 4\left(\frac{1}{2}\right)^{n} u[n] - 4\left(\frac{1}{4}\right)^{n} u[n]$$

Option 2: Use partial fractions

- **Example:** **Evaluate this convolution via the Z-transform.**

$$x[n] = \left(\frac{1}{2}\right)^{n-1} u[n-1] * \left(\frac{1}{4}\right)^{n} u[n]$$

$$X(z) = \frac{4}{\left(1 - \left(\frac{1}{2}\right) z^{-1}\right)} - \frac{4}{\left(1 - \left(\frac{1}{4}\right) z^{-1}\right)}$$

Option 2: Use partial fractions

$$x[n] = 4\left(\frac{1}{2}\right)^{n} u[n] - 4\left(\frac{1}{4}\right)^{n} u[n]$$

$$= 4\left(\frac{1}{2}\right)^{n} u[n-1] - 4\left(\frac{1}{4}\right)^{n} u[n-1]$$ **Since n=0 yields 0**

The Z-Transform

- **Example: Determine the poles and zeros for**

$$x[n] = \left(\frac{1}{2}\right)^n u[n] * \left(\frac{1}{4}\right)^n u[n]$$

The Z-Transform

- **Example:** **Determine the region of convergence for**

$$x[n] = \left(\frac{1}{2}\right)^n u[n] * \left(\frac{1}{4}\right)^n u[n]$$

The Z-Transform

- **Example: Let $h[n]$ define the impulse response of a system. Find the output for an input $x[n] = 2\delta[n-2] + 4\delta[n-3]$.**

$$h[n] = \left(\frac{1}{2}\right)^n u[n] * \left(\frac{1}{4}\right)^n u[n]$$

The Z-Transform

- **Example:** **Let $h[n]$ define the impulse response of a system. Find the output for an input $x[n] = 2\delta[n-2] + 4\delta[n-3]$.**

$$h[n] = \left(\frac{1}{2}\right)^n u[n] * \left(\frac{1}{4}\right)^n u[n]$$

Solution

$$x[n] = 2\left[2\left(\frac{1}{2}\right)^{n-2} u[n-2] - \left(\frac{1}{4}\right)^{n-2} u[n-2]\right]$$
$$+4\left[2\left(\frac{1}{2}\right)^{n-3} u[n-3] - \left(\frac{1}{4}\right)^{n-3} u[n-3]\right]$$

Lecture 7: The Discrete -time Fourier Transform Properties

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- The Z-Transform Review
- **The Discrete-time Fourier Transform (DTFT)**
- The Properties of the Discrete-time Fourier Transform (DTFT)

The Discrete-Time Fourier Transform

- **The Discrete-Time Fourier Transform (DTFT)**

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

- **The Inverse Discrete-Time Fourier Transform**

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(\omega)e^{j\omega n}\, d\omega$$

The Discrete-Time Fourier Transform

- **The Discrete-Time Fourier Transform (DTFT)**

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

- **Question: How do I interpret this DTFT?**

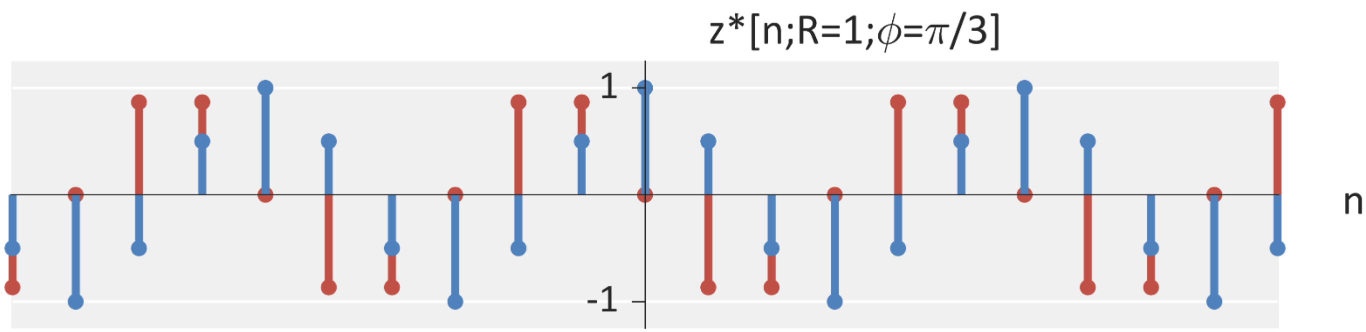
The Discrete-Time Fourier Transform

- **The Discrete-Time Fourier Transform (DTFT)**

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

Inner product of signal and sinusoids!

- **Question: How do I interpret this DTFT?**



The Discrete-Time Fourier Transform

- **Question:** Why am I interested in the Discrete-Time Fourier Transform?

The Discrete-Time Fourier Transform

- **The Discrete-Time Fourier Transform (DTFT)**

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

- **Example:** Compute the DTFT of $x[n] = a^n u[n]$

The Discrete-Time Fourier Transform

- **The Discrete-Time Fourier Transform (DTFT)**

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

- **Example:** Compute the DTFT of $x[n] = a^n u[n]$

$$X(\omega) = \sum_{n=-\infty}^{\infty} a^n u[n]\, e^{-j\omega n} = \sum_{n=0}^{\infty} \left(ae^{-j\omega}\right)^n = \frac{1}{1 - ae^{-j\omega}}$$

If $|a| < 1$

The Discrete-Time Fourier Transform

- **The Discrete-Time Fourier Transform (DTFT) Table**
  - http://smartdata.ece.ufl.edu/eee5502/eee5502_DiscreteTransforms.pdf