Lecture 23: Introduction to Filter Banks

Foundations of Digital Signal Processing

Outline

- Review of Filter Design
- Short Time Fourier Transform
- Inefficient Filter Banks
- DFT Filter Bank

News

- **Homework #9**
  - Due *Today*
  - Submit via canvas
- **Coding Assignment #6**
  - Due *on Tuesday, Nov. 20th*
  - Submit via canvas

Lecture 23: Introduction to Filter Banks

Foundations of Digital Signal Processing

Outline

- **Review of Filter Design**
- Short Time Fourier Transform
- Inefficient Filter Banks
- DFT Filter Bank

Filter Design

- **Problem**
  - Consider the ideal filter (assume ω is periodic with period 2π)

$$|H(\omega)| = \begin{cases} 1 & \text{for} \quad |\omega| \le \dfrac{\pi}{3} \\ 0 & \text{for} \quad \text{otherwise} \end{cases} \qquad w[n] = \frac{1}{2}\left(1 - \cos\left(\frac{2\pi n}{N-1}\right)\right)$$

  - Compute a discrete-time FIR filter with a 12-point Hann window defined by $w[n]$.

Filter Design

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  - Consider the ideal filter (assume ω is periodic with period 2π)

$$|H(\omega)| = \begin{cases} 1 & \text{for} \quad |\omega| \le \frac{\pi}{3} \\ 0 & \text{for} \quad \text{otherwise} \end{cases} \qquad w[n] = \frac{1}{2}\left(1 - \cos\left(\frac{2\pi n}{N-1}\right)\right)$$

  - Compute a discrete-time FIR filter with a 12-point Hann window defined by $w[n]$. Need to add shift for causality.

$$|H(\omega)| = \sum_{k=-\infty}^{\infty} u\left(\omega + \frac{\pi}{3} - 2\pi k\right) - u\left(\omega - \frac{\pi}{3} - 2\pi k\right)$$

$$h[n] = \frac{\sin\left(\left(\frac{\pi}{3}\right)\left(n - \frac{N-1}{2}\right)\right)}{\pi\left(n - \frac{N-1}{2}\right)}$$

Filter Design

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  - Compute a discrete-time FIR filter with a 12-point Hann window defined by $w[n]$.

$$h[n]w[n] = \begin{cases} \dfrac{\sin\left(\left(\frac{\pi}{3}\right)(n-5.5)\right)}{2\pi(n-5.5)}\left(1 - \cos\left(\dfrac{2\pi n}{N-1}\right)\right) & \text{for} \quad 0 \le n \le 11 \\ 0 & \text{for} \quad \text{otherwise} \end{cases}$$

Filter Design

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  - Consider the ideal filter (assume ω is periodic with period 2π)

$$H(\omega) = \begin{cases} 1 & \text{for} \quad |\omega| \le \frac{\pi}{3} \\ 0 & \text{for} \quad \text{otherwise} \end{cases}$$

  - Determine a discrete-time FIR filter with a 12-point frequency sampling method

Filter Design

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  - Determine a discrete-time FIR filter with a 12-point frequency sampling method

$$H[k] = [1 \quad 1 \quad 1 \quad 0 \quad 0 \quad 0 \quad 0]$$

$$0, \frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{5\pi}{6}, \pi$$

Filter Design

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  - Consider the ideal filter (assume ω is periodic with period 2π)

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  - Determine a discrete-time FIR filter with a 12-point frequency sampling method

$$\text{Even N: } h[n] = H[0] + 2\sum_{k=1}^{N/2} H[k]\cos\left(\frac{2\pi}{N}\left(n - \frac{N-1}{2}\right)k\right)$$

$$\text{Odd N: } h[n] = H[0] + 2\sum_{k=1}^{(N-1)/2} H[k]\cos\left(\frac{2\pi}{N}\left(n - \frac{N-1}{2}\right)k\right)$$

Filter Design

- **Problem**
  - Consider the ideal filter (assume ω is periodic with period 2π)

$$H(\omega) = \begin{cases} 1 & \text{for} \quad |\omega| \le \dfrac{\pi}{3} \\ 0 & \text{for} \quad \text{otherwise} \end{cases}$$

  - Determine a discrete-time FIR filter with a 12-point frequency sampling method

$$h[n] = H[0] + 2\sum_{k=1}^{N/2} H[k] \cos\left(\frac{2\pi}{12}(n-5.5)k\right)$$

Filter Design

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  - Consider the ideal filter (assume ω is periodic with period 2π)

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  - Determine a discrete-time FIR filter with a 12-point frequency sampling method

$$h[n] = H[0] + 2\sum_{k=1}^{N/2} H[k]\cos\left(\frac{2\pi}{12}(n-5.5)k\right)$$

$$\boxed{h[n] = 1 + 2\cos\left(\frac{2\pi}{12}(n-5.5)\right) + 2\cos\left(\frac{2\pi}{6}(n-5.5)\right)}$$

Filter Design

- **Problem**
  - The squared magnitude of a Butterworth filter is defined by

$$|H(s)|^2 = \frac{1}{1 - s^{2N}}$$

  - Determine the squared magnitude for a discrete-time IIR filter of an N=2 order Butterworth filter by using the discrete approximation of the derivative.

Filter Design

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$$s \to \frac{1}{T}(1 - z^{-1})$$

Filter Design

- **Problem**
  - The squared magnitude of a Butterworth filter is defined by

$$|H(s)|^2 = \frac{1}{1 - s^{2N}}$$

  - Determine the squared magnitude for a discrete-time IIR filter of an N=2 order Butterworth filter by using the discrete approximation of the derivative and a sampling rate of T=1.

$$s \rightarrow \frac{1}{T}(1 - z^{-1})$$

$$|H(z)|^2 = \left|\frac{1}{1 - \frac{1}{T^4}(1 - z^{-1})^4}\right| = \left|\frac{1}{1 - (1 - z^{-1})^4}\right|$$

Filter Design

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  - The squared magnitude of a Butterworth filter is defined by

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  - Determine the squared magnitude for a discrete-time IIR filter of an N=2 order Butterworth filter by using the discrete approximation of the derivative and a sampling rate of T=1.

$$s \rightarrow \frac{1}{T}(1 - z^{-1})$$

$$|H(z)|^2 = \left|\frac{1}{1 - [z^{-4} - 4z^{-3} + 6z^{-2} - 4z^{-1} + 1]}\right|$$

Filter Design

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  - The squared magnitude of a Butterworth filter is defined by

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  - Determine the squared magnitude for a discrete-time IIR filter of an N=2 order Butterworth filter by using the discrete approximation of the derivative and a sampling rate of T=1.

$$s \rightarrow \frac{1}{T}(1 - z^{-1})$$

$$|H(z)|^2 = \left|\frac{1}{z^{-1}(2 - z^{-1})(1 - j - z^{-1})(1 + j - z^{-1})}\right|$$

Filter Design

- **Problem**
  - The squared magnitude of a Butterworth filter is defined by

$$|H(s)|^2 = \frac{1}{1 - s^{2N}}$$

  - Determine the squared magnitude for a discrete-time IIR filter of an N=2 order Butterworth filter by using the discrete approximation of the derivative and a sampling rate of T=1.

Poles: $\infty, \frac{1}{2}, \frac{1}{2} - \frac{j}{2}, \frac{1}{2} + \frac{j}{2}$ **Zeros:** $0,0,0,0$

$$|H(z)|^2 = \left| \frac{1}{z^{-1}(2 - z^{-1})(1 - j - z^{-1})(1 + j - z^{-1})} \right|$$

Filter Design

- **Problem**
  - The squared magnitude of a Butterworth filter is defined by

$$|H(s)|^2 = \frac{1}{1 - s^{2N}}$$

  - Determine the squared magnitude for a discrete-time IIR filter of an N=2 order Butterworth filter with impulse invariance and T=1.

Filter Design

- **Problem**
  - The squared magnitude of a Butterworth filter is defined by

$$|H(s)|^2 = \frac{1}{1 - s^{2N}}$$

  - Determine the squared magnitude for a discrete-time IIR filter of an N=2 order Butterworth filter with impulse invariance and T=1.

Definition:

$$H(z) = \sum_{k=1}^{K} \frac{c_k}{1 - e^{p_k T} z^{-1}}$$

Filter Design

- **Problem**
  - The squared magnitude of a Butterworth filter is defined by

$$|H(s)|^2 = \frac{1}{1 - s^{2N}}$$

  - Determine the squared magnitude for a discrete-time IIR filter of an N=2 order Butterworth filter with impulse invariance and T=1.

$$|H(s)|^2 = \frac{1}{1 - s^{2N}} = \frac{1}{1 - s^4}$$

Poles:

Filter Design

- **Problem**
  - The squared magnitude of a Butterworth filter is defined by

$$|H(s)|^2 = \frac{1}{1 - s^{2N}}$$

  - Determine the squared magnitude for a discrete-time IIR filter of an N=2 order Butterworth filter with impulse invariance and T=1.

$$|H(s)|^2 = \frac{1}{1 - s^{2N}} = \frac{1}{1 - s^4}$$

Poles: $s = e^{-j\frac{2\pi}{4}(0)}, e^{-j\frac{2\pi}{4}(1)}, e^{-j\frac{2\pi}{4}(2)}, e^{-j\frac{2\pi}{4}(3)}$

Filter Design

- **Problem**
  - The squared magnitude of a Butterworth filter is defined by

$$|H(s)|^2 = \frac{1}{1 - s^{2N}}$$

  - Determine the squared magnitude for a discrete-time IIR filter of an N=2 order Butterworth filter with impulse invariance and T=1.

$$|H(s)|^2 = \frac{1}{1 - s^{2N}} = \frac{1}{1 - s^4}$$

Poles: $s = 1, -j, -1, j$

Filter Design

- **Problem**
  - The squared magnitude of a Butterworth filter is defined by

$$|H(s)|^2 = \frac{1}{1 - s^{2N}}$$

  - Determine the squared magnitude for a discrete-time IIR filter of an N=2 order Butterworth filter with impulse invariance and T=1.

$$|H(s)|^2 = \frac{1}{(s-1)(s+1)(s^2+1)} = \frac{c_1}{(s-1)} + \frac{c_2}{(s+1)} + \frac{b}{(s^2+1)}$$

$$1 = c_1(s+1)(s^2+1) + c_2(s-1)(s^2+1) + b(s-1)(s+1)$$

s = 1

$$1 = 4c_1 \quad , \quad c_1 = 1/4$$

Filter Design

- **Problem**
  - The squared magnitude of a Butterworth filter is defined by

$$|H(s)|^2 = \frac{1}{1 - s^{2N}}$$

  - Determine the squared magnitude for a discrete-time IIR filter of an N=2 order Butterworth filter with impulse invariance and T=1.

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$$1 = c_1(s+1)(s^2+1) + c_2(s-1)(s^2+1) + b(s-1)(s+1)$$

s = -1

$$1 = -4c_2 \qquad , \qquad c_2 = -1/4$$

Filter Design

- **Problem**
  - The squared magnitude of a Butterworth filter is defined by

$$|H(s)|^2 = \frac{1}{1 - s^{2N}}$$

  - Determine the squared magnitude for a discrete-time IIR filter of an N=2 order Butterworth filter with impulse invariance and T=1.

$$|H(s)|^2 = \frac{c_1}{(s-1)} + \frac{c_2}{(s+1)} + \frac{c_3}{(s+j)} + \frac{c_4}{(s-j)}$$

$$1 = c_1(s+1)(s^2+1) + c_2(s-1)(s^2+1)$$
$$+c_3(s-1)(s+1)(s-j) + c_4(s-1)(s+1)(s+j)$$

s = -j

$$1 = (-j-1)(-j+1)(-2j)c_3 = 4jc_3 \qquad , \qquad c_3 = -j/4$$

Filter Design

- **Problem**
  - The squared magnitude of a Butterworth filter is defined by

$$|H(s)|^2 = \frac{1}{1 - s^{2N}}$$

  - Determine the squared magnitude for a discrete-time IIR filter of an N=2 order Butterworth filter with impulse invariance and T=1.

$$|H(s)|^2 = \frac{c_1}{(s-1)} + \frac{c_2}{(s+1)} + \frac{c_3}{(s+j)} + \frac{c_4}{(s-j)}$$

$$1 = c_1(s+1)(s^2+1) + c_2(s-1)(s^2+1)$$
$$+c_3(s-1)(s+1)(s-j) + c_4(s-1)(s+1)(s+j)$$

s = j

$$1 = (j-1)(j+1)(2j)c_4 = -4jc_4 \qquad , \qquad c_4 = j/4$$

Filter Design

- **Problem**
  - The squared magnitude of a Butterworth filter is defined by

$$|H(s)|^2 = \frac{1}{1 - s^{2N}}$$

  - Determine the squared magnitude for a discrete-time IIR filter of an N=2 order Butterworth filter with impulse invariance and T=1.

$$|H(s)|^2 = \frac{1/4}{(s-1)} - \frac{1/4}{(s+1)} - \frac{j/4}{(s+j)} + \frac{j/4}{(s-j)}$$

Poles: $s = 1, -j, -1, j$

Filter Design

- **Problem**
  - The squared magnitude of a Butterworth filter is defined by

$$|H(s)|^2 = \frac{1}{1 - s^{2N}}$$

  - Determine the squared magnitude for a discrete-time IIR filter of an N=2 order Butterworth filter with impulse invariance and T=1.

Poles: $s = 1, -j, -1, j$

$$H(z) = \sum_{k=1}^{K} \frac{c_k}{1 - e^{p_k T} z^{-1}} = \frac{1/4}{1 - e^{1} z^{-1}} - \frac{1/4}{1 - e^{-1} z^{-1}} - \frac{j/4}{1 - e^{-j} z^{-1}} + \frac{j/4}{1 - e^{+j} z^{-1}}$$

Poles: $z = e, e^{-1}, e^{-j}, e^{j}$

Filter Design

- **Problem**
  - The squared magnitude of a Butterworth filter is defined by

$$|H(s)|^2 = \frac{1}{1 - s^{2N}}$$

  - Determine the squared magnitude for a discrete-time IIR filter of an N=2 order Butterworth filter with the bilinear transform and T=1.

Filter Design

- **Problem**
  - The squared magnitude of a Butterworth filter is defined by

$$|H(s)|^2 = \frac{1}{1 - s^{2N}}$$

  - Determine the squared magnitude for a discrete-time IIR filter of an N=2 order Butterworth filter w/ the bilinear transform and T=1.

$$s \rightarrow \frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}$$

Filter Design

- **Problem**
  - The squared magnitude of a Butterworth filter is defined by

$$|H(s)|^2 = \frac{1}{1 - s^{2N}}$$

  - Determine the squared magnitude for a discrete-time IIR filter of an N=2 order Butterworth filter w/ the bilinear transform and T=1.

$$s \rightarrow \frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}$$

$$|H(z)|^2 = \frac{1}{1 - s^4} = \frac{1}{1 - \left(2\frac{1 - z^{-1}}{1 + z^{-1}}\right)^4} = \frac{(1 + z^{-1})^4}{(1 + z^{-1})^4 - 16(1 - z^{-1})^4}$$

Filter Design

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  - Determine the squared magnitude for a discrete-time IIR filter of an N=2 order Butterworth filter w/ the bilinear transform and T=1.

$$s \rightarrow \frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}$$

$$|H(z)|^2 = \frac{(1+z^{-1})^4}{(3z^{-1}-1)(3-z^{-1})(5z^{-2}-6z^{-1}+5)}$$

Poles: $z = \frac{1}{3}, 3, \frac{3}{5} - \frac{4}{5}j, \frac{3}{5} + \frac{4}{5}j$

Lecture 23: Introduction to Filter Banks

Foundations of Digital Signal Processing

Outline

- Review of Filter Design
- **Short Time Fourier Transform**
- Inefficient Filter Banks
- DFT Filter Bank

The Short-Time Fourier Transform

- **The Definition:**

$$X[Mm, k] = \mathrm{DFT}_{n\to k}(w[n]x[n - Mm])$$

Frequency

Time

Shifted Data

Window

The Short-Time Fourier Transform

- **The Definition:**

$$X[Mm, k] = \mathrm{DFT}_{n \to k}(w[n]x[n - Mm])$$

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$$X[m, k] = \sum_{n=0}^{N-1} w[n]x[n - Mm]\, e^{-j\frac{2\pi}{N}nk}$$

The Short-Time Fourier Transform

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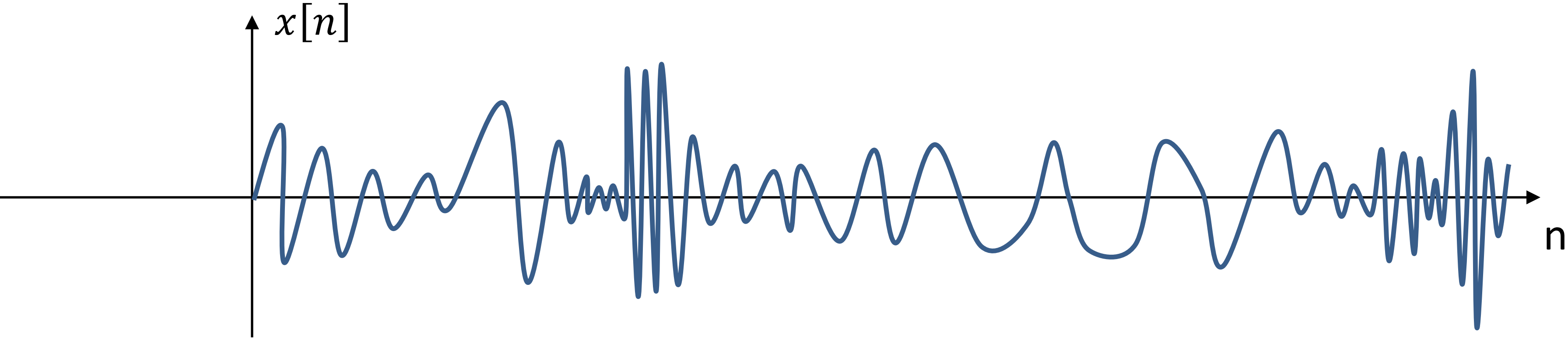

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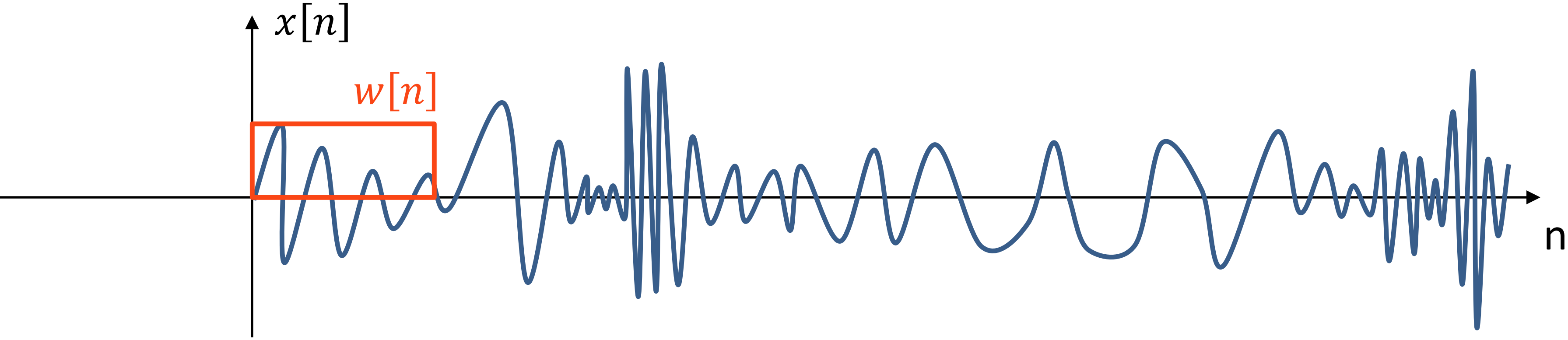

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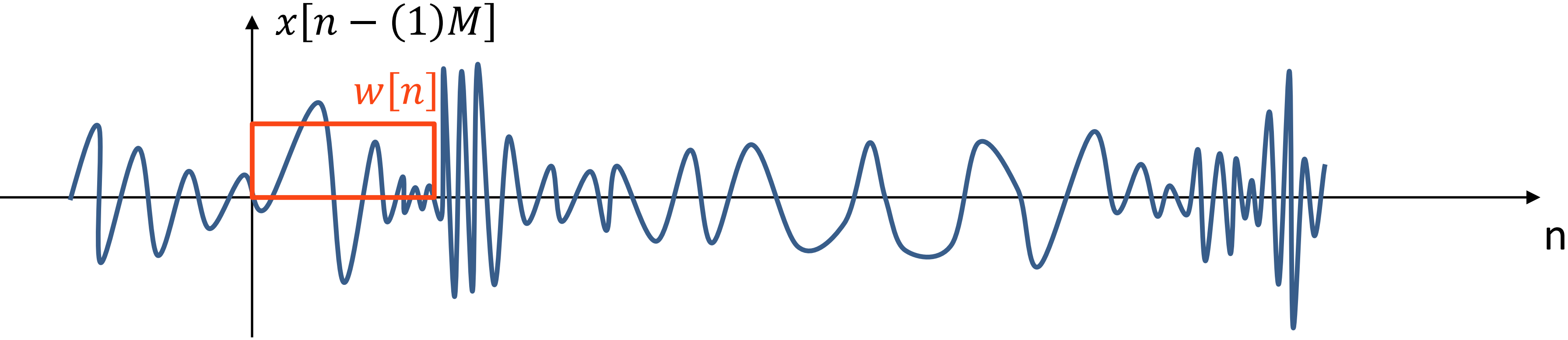

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Time Frequency Window Shifted Data

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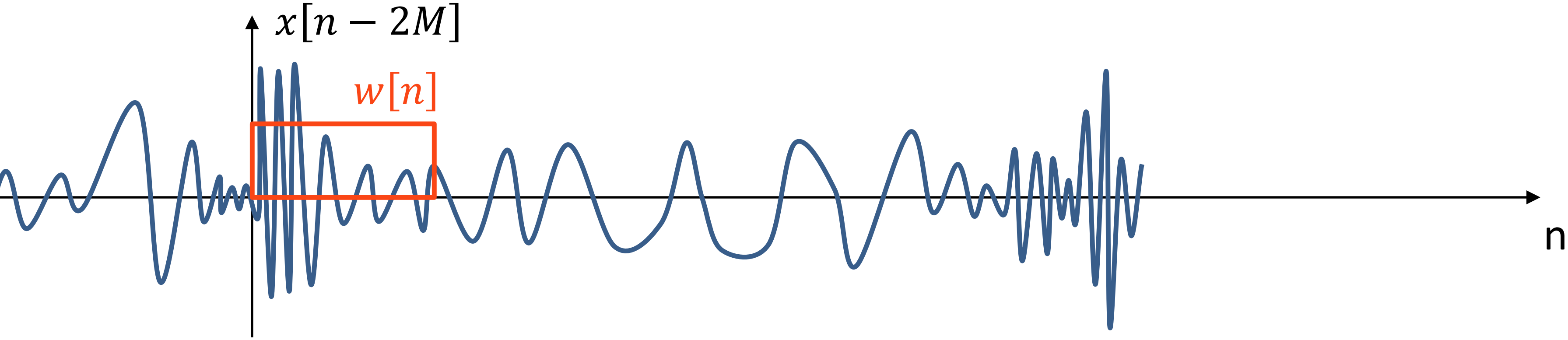

The Short-Time Fourier Transform

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Time Frequency Window Shifted Data

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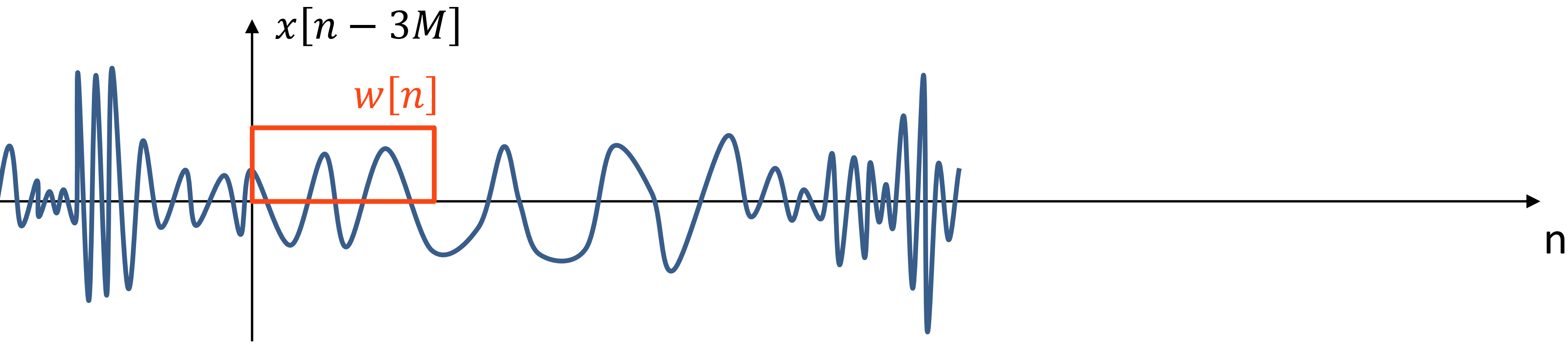

The Short-Time Fourier Transform

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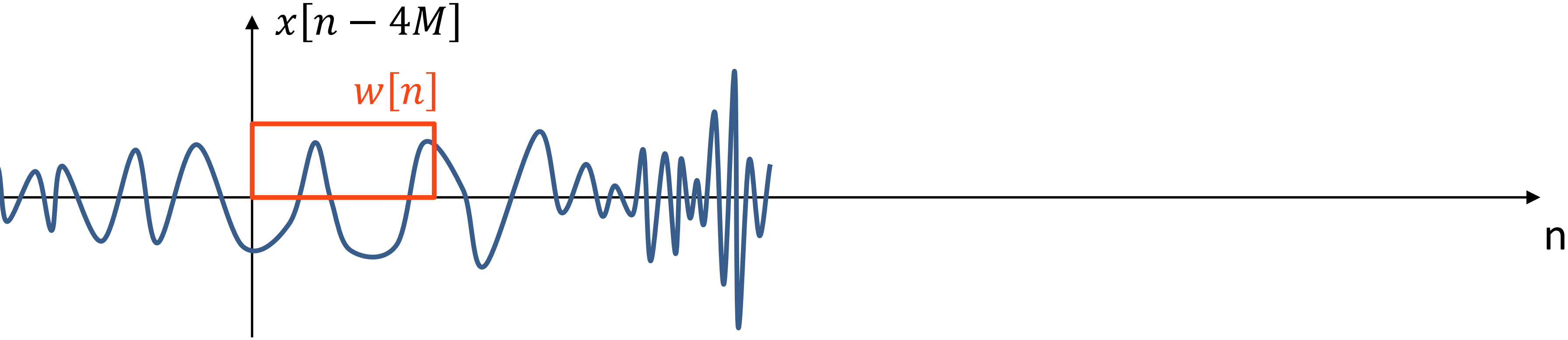

The Short-Time Fourier Transform

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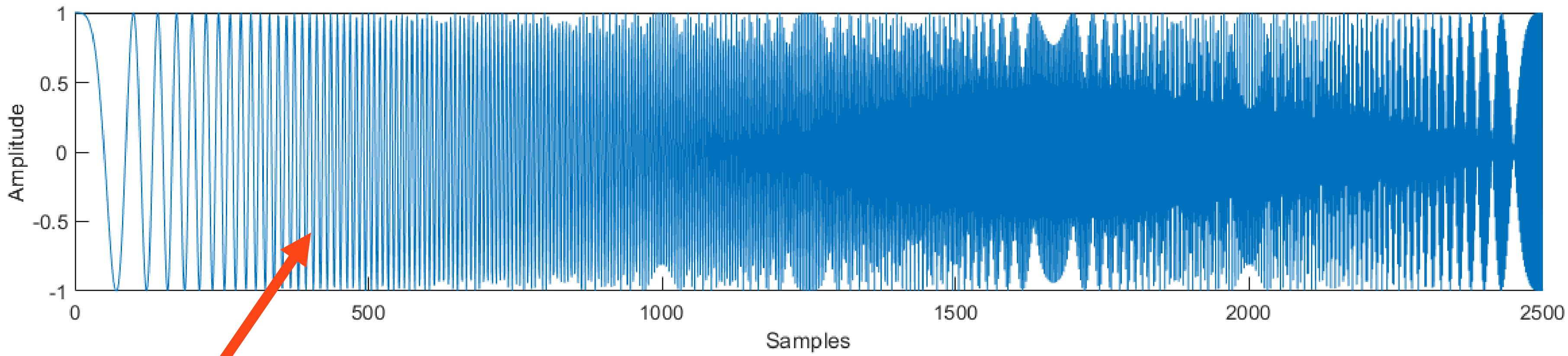
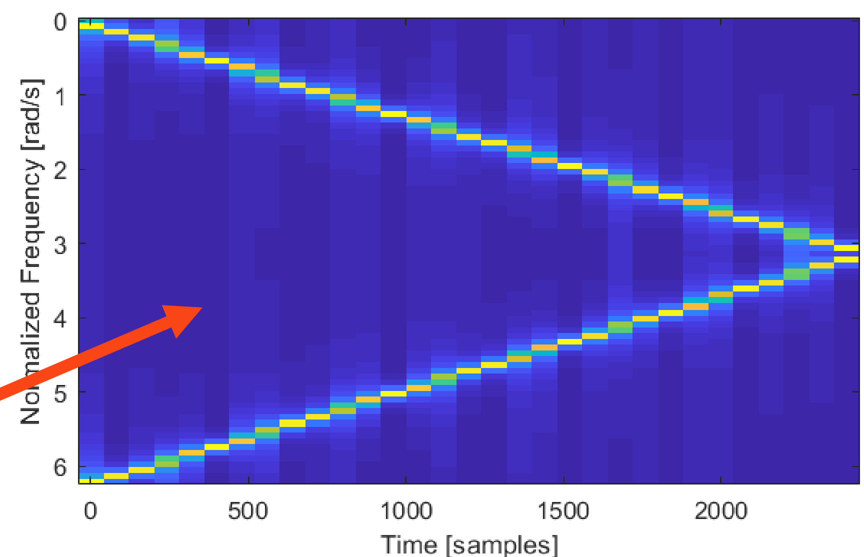


Chirp

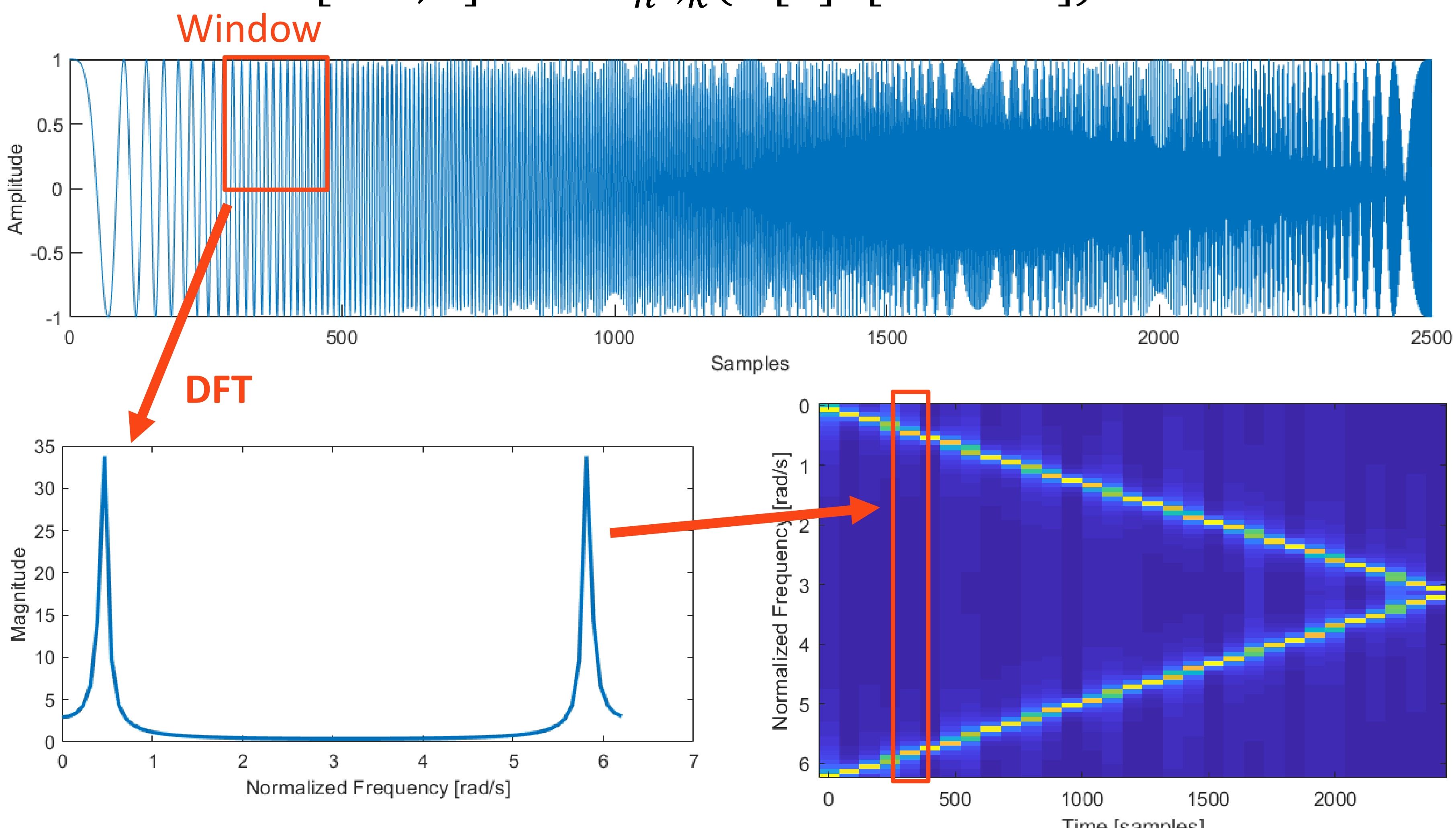


STFT

The Short-Time Fourier Transform

- **The Definition:**

$$X[Mm, k] = \mathrm{DFT}_{n \to k}(w[n]x[n - Mm])$$



The Short-Time Fourier Transform

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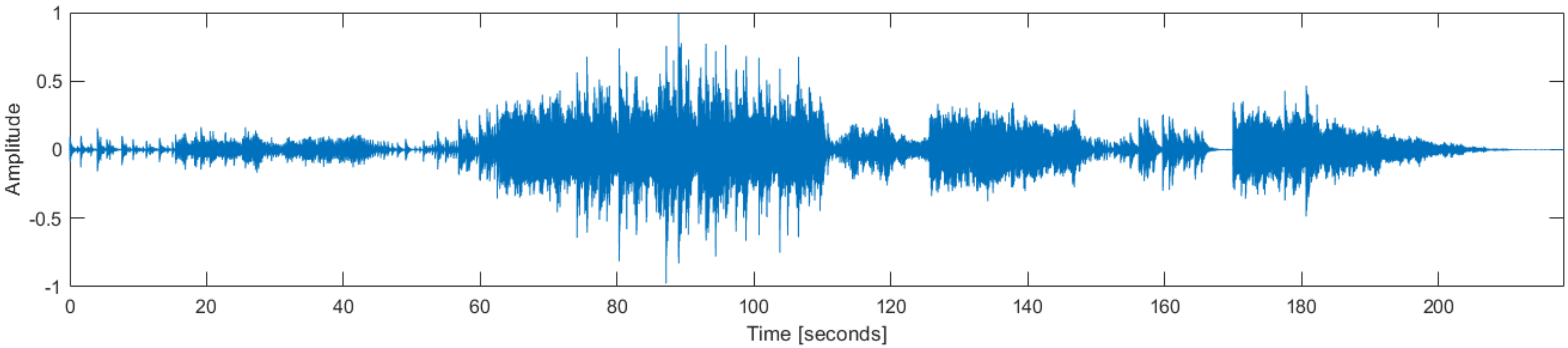

The Short-Time Fourier Transform

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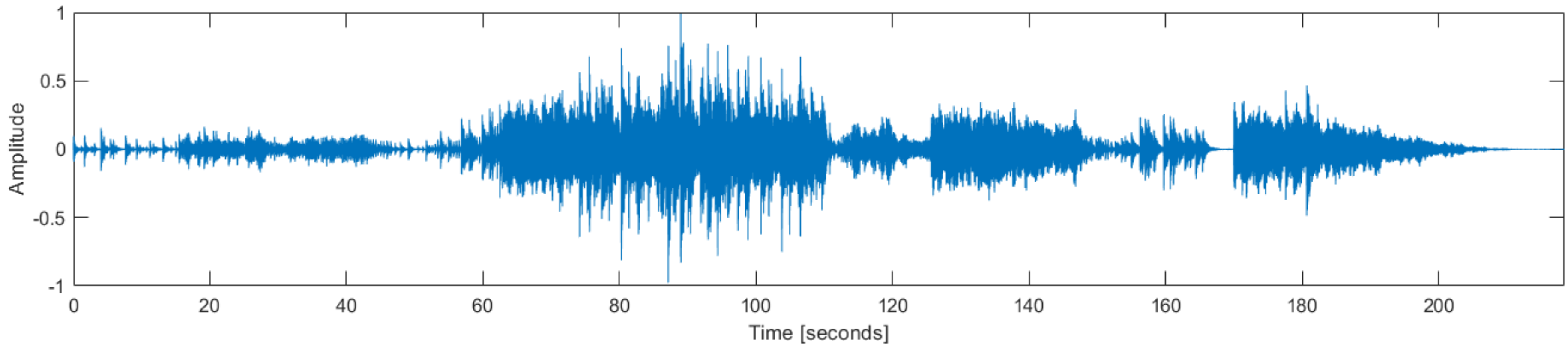
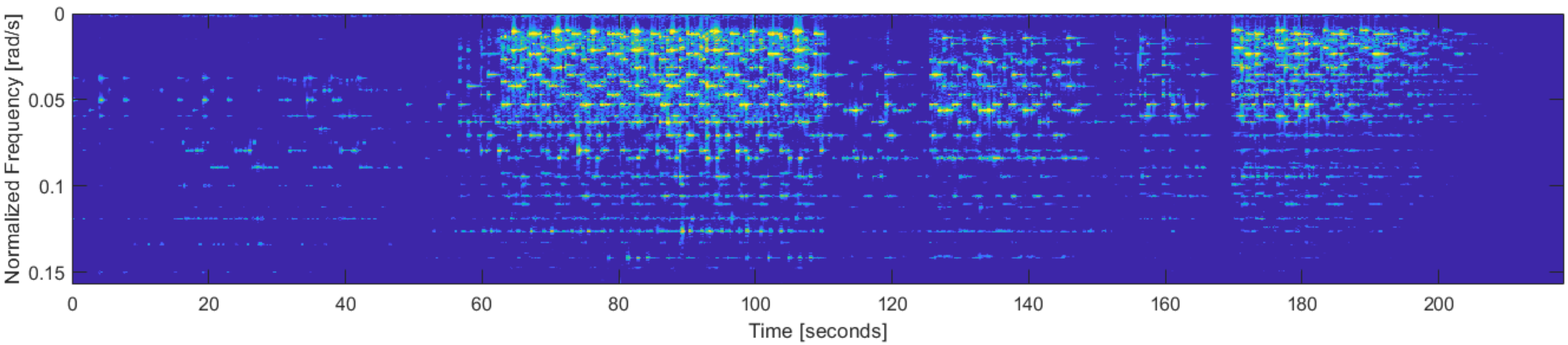
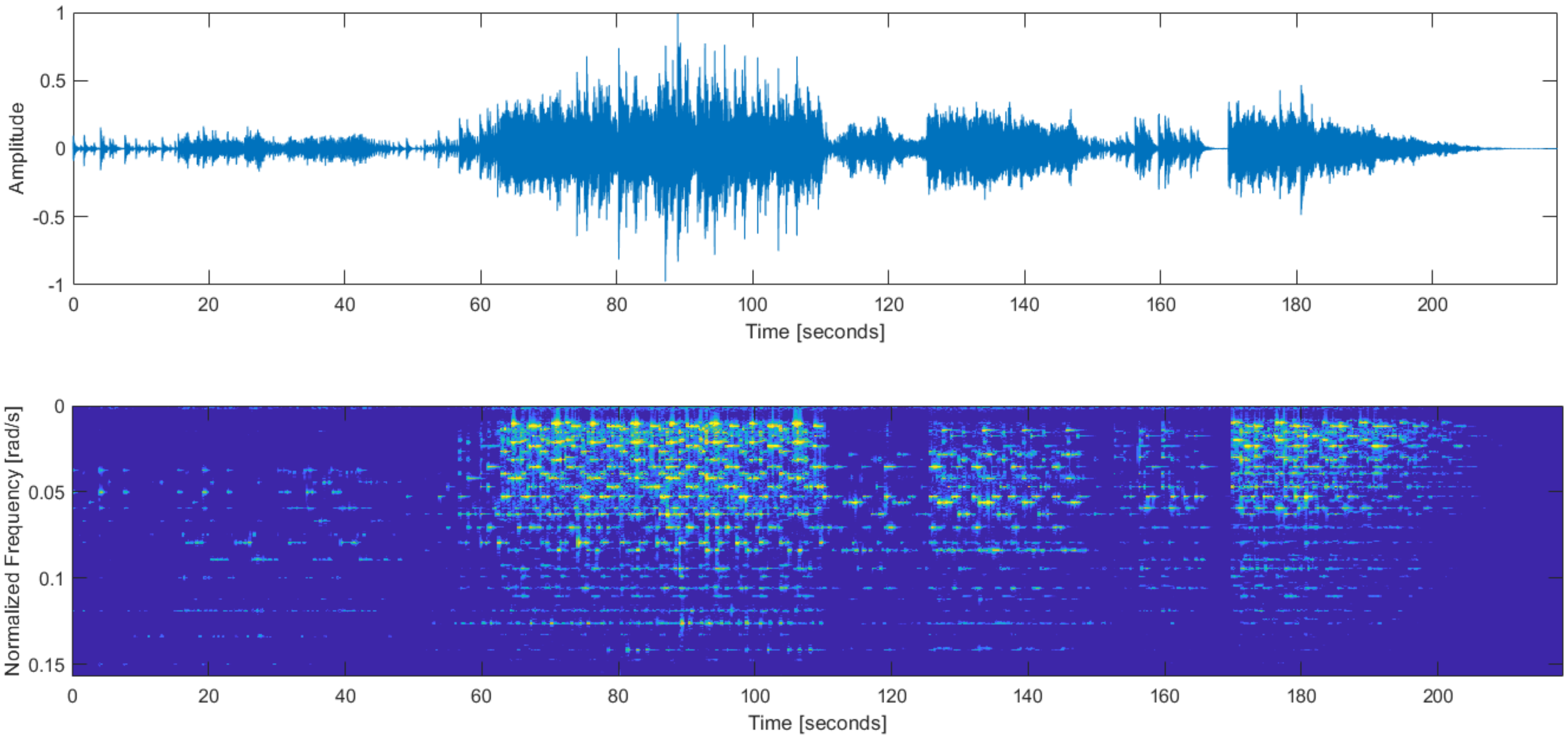




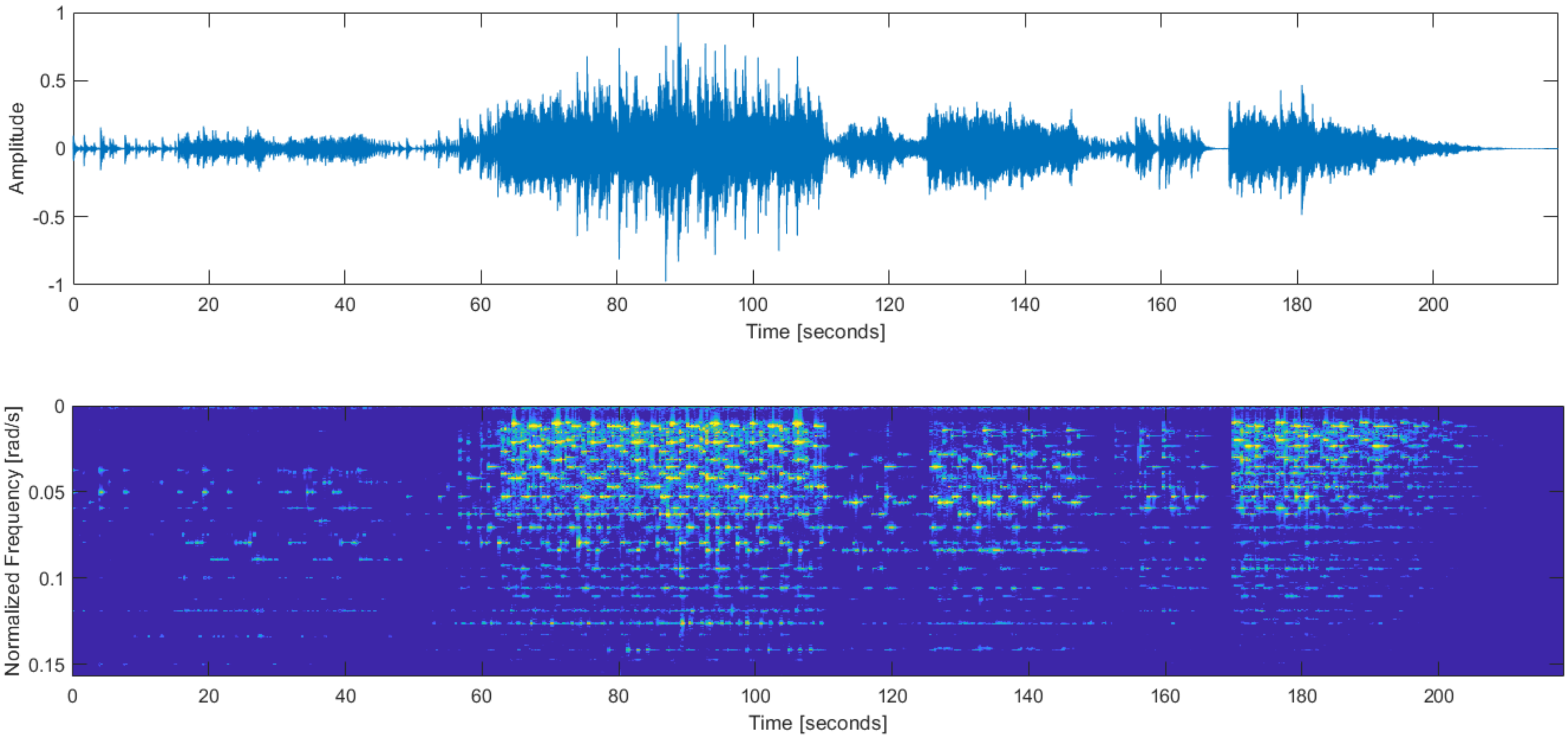
The Short-Time Fourier Transform

- **Question: Why do I care about the STFT?**



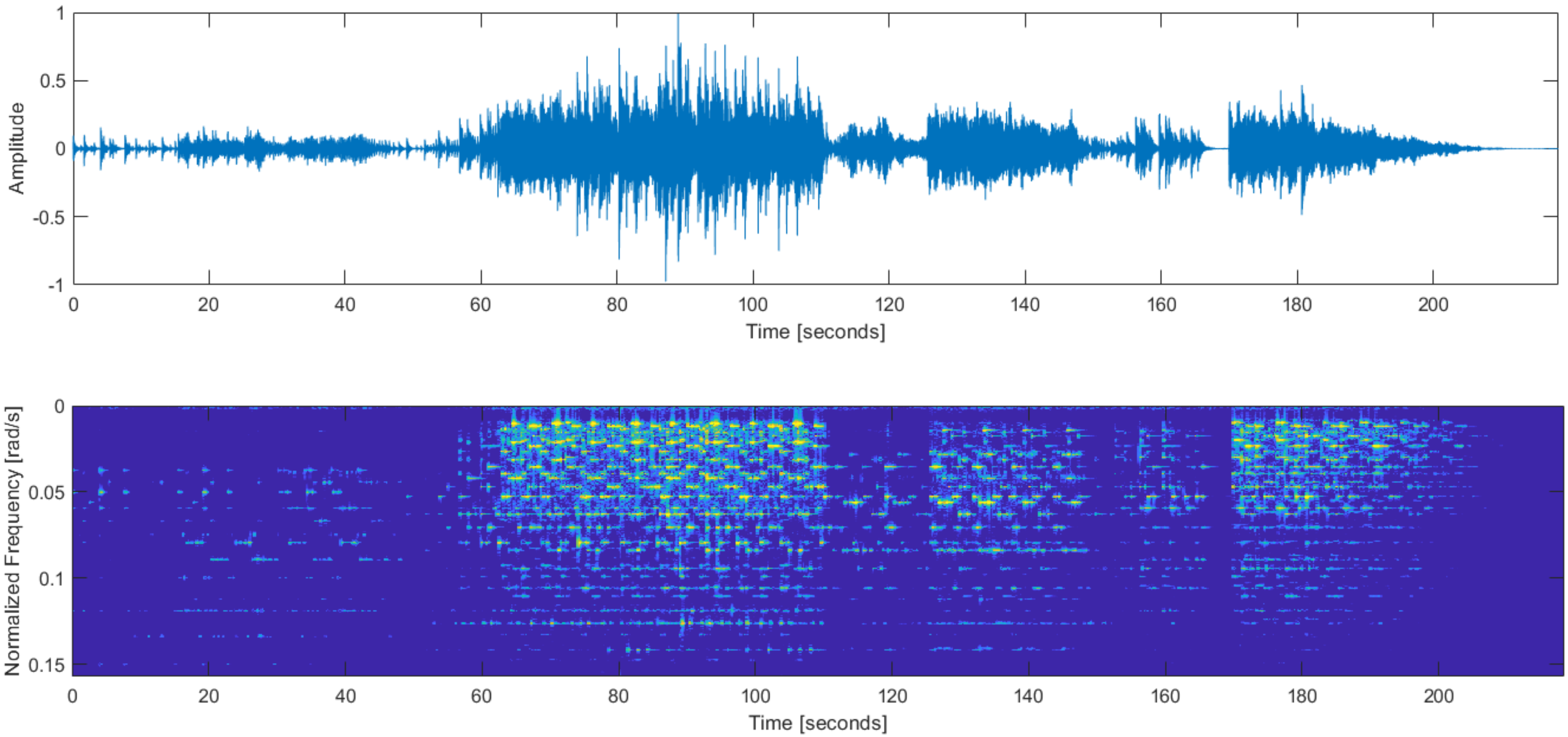
The Short-Time Fourier Transform

- **Question: How is this problematic for a real-time system?**



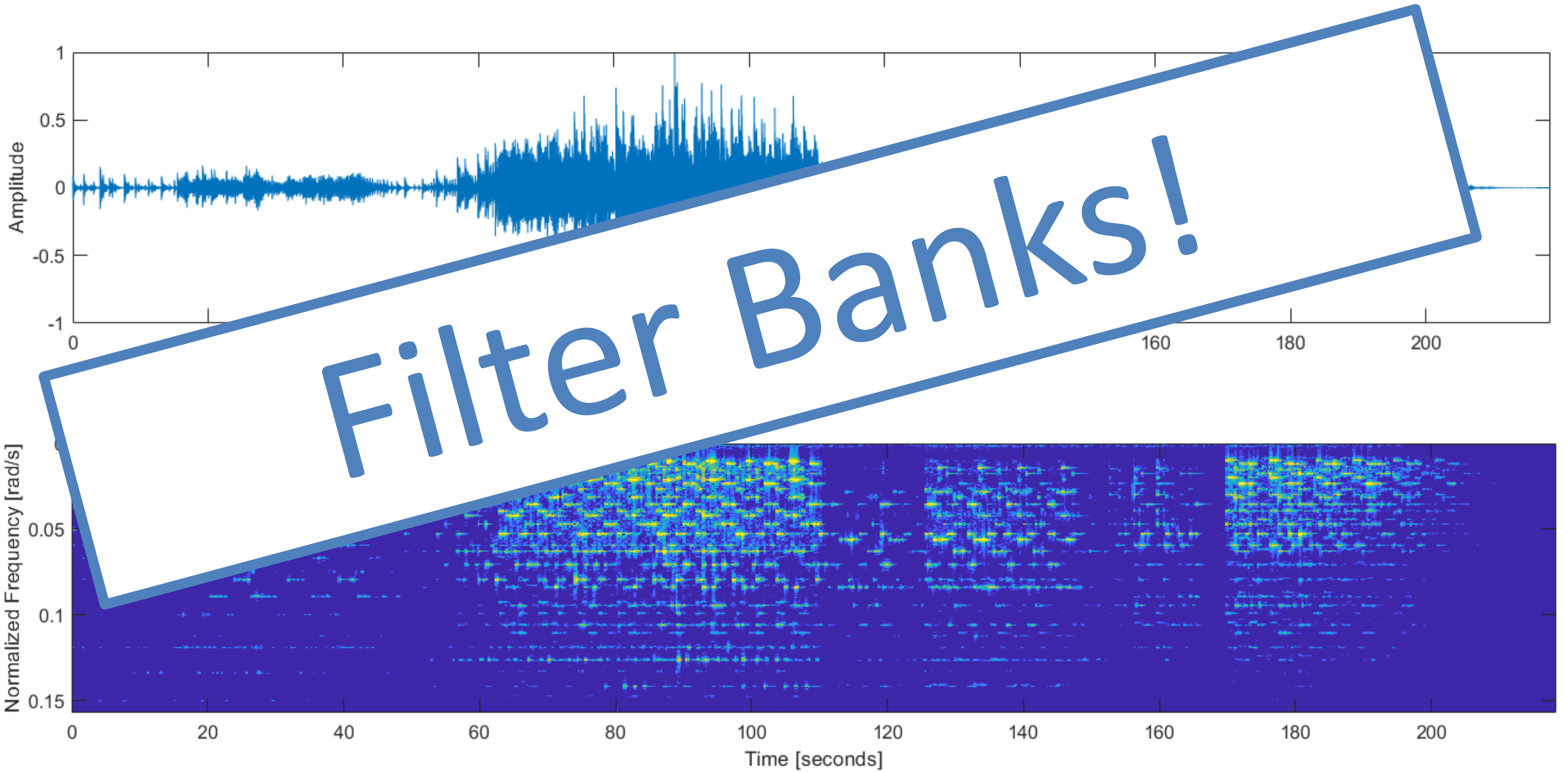
The Short-Time Fourier Transform

- **Question: How do I solve this problem??**



The Short-Time Fourier Transform

- **Question:** **How do I solve this problem??**



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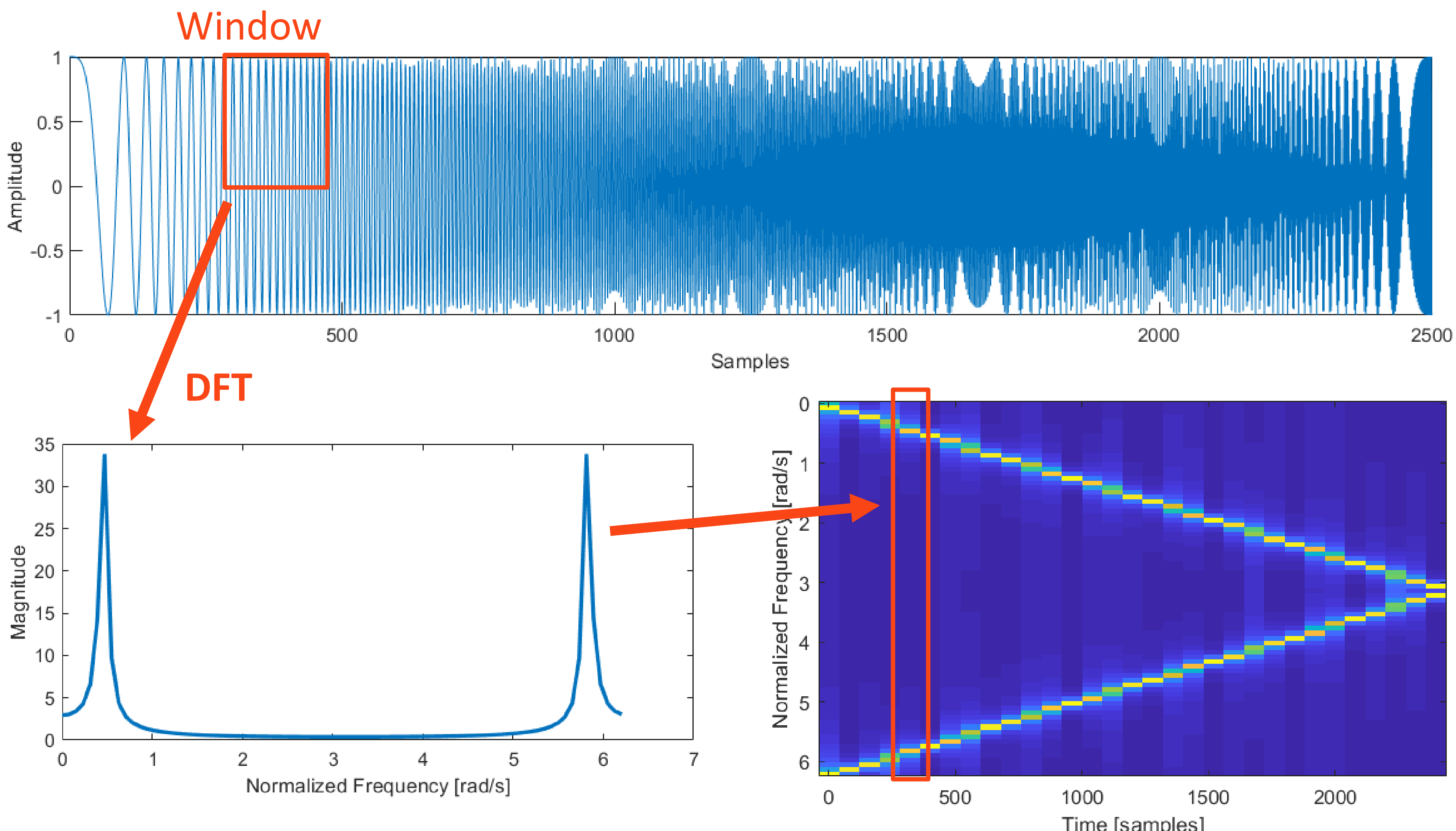
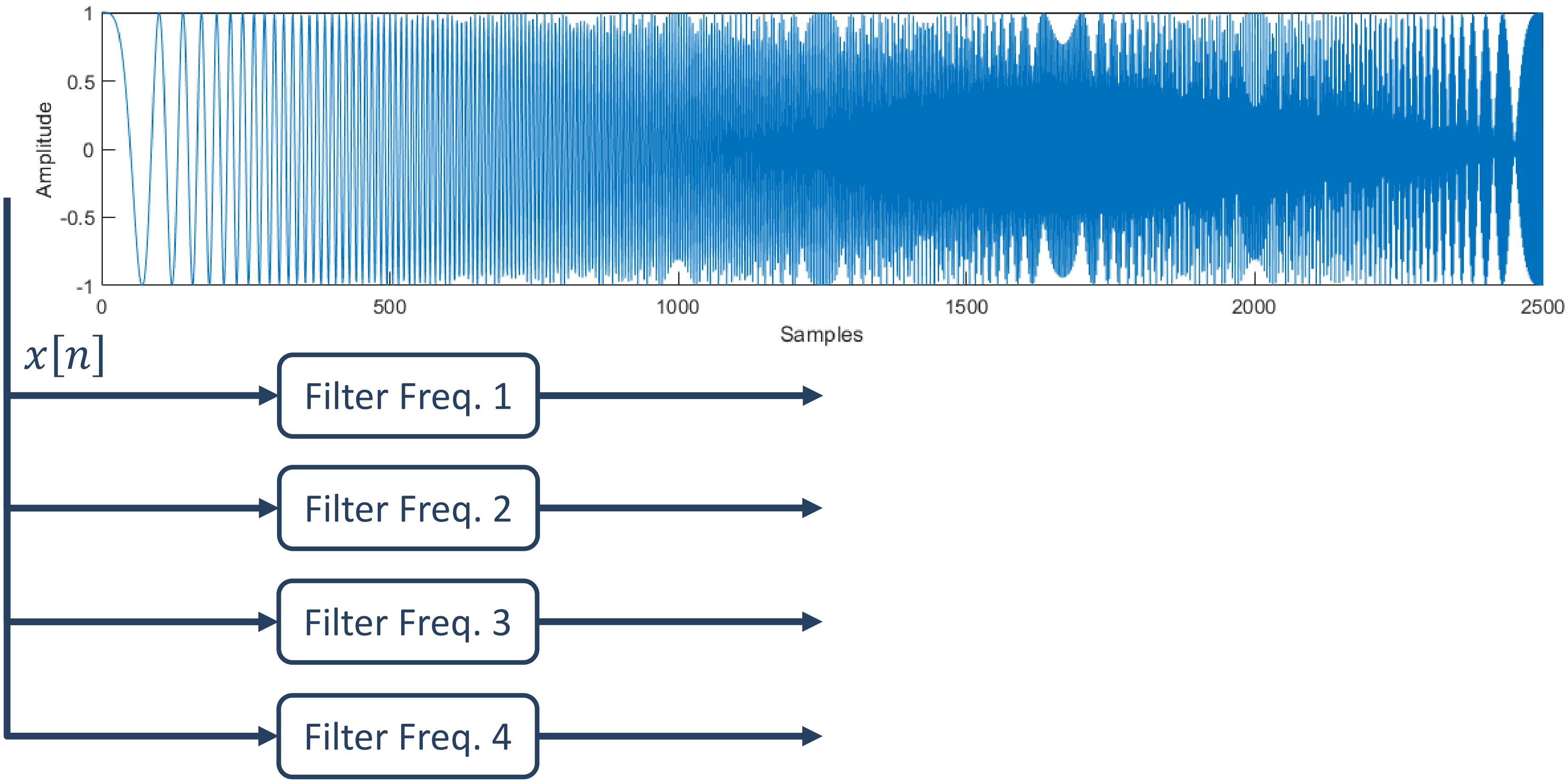
The Short Time Fourier Transform Process



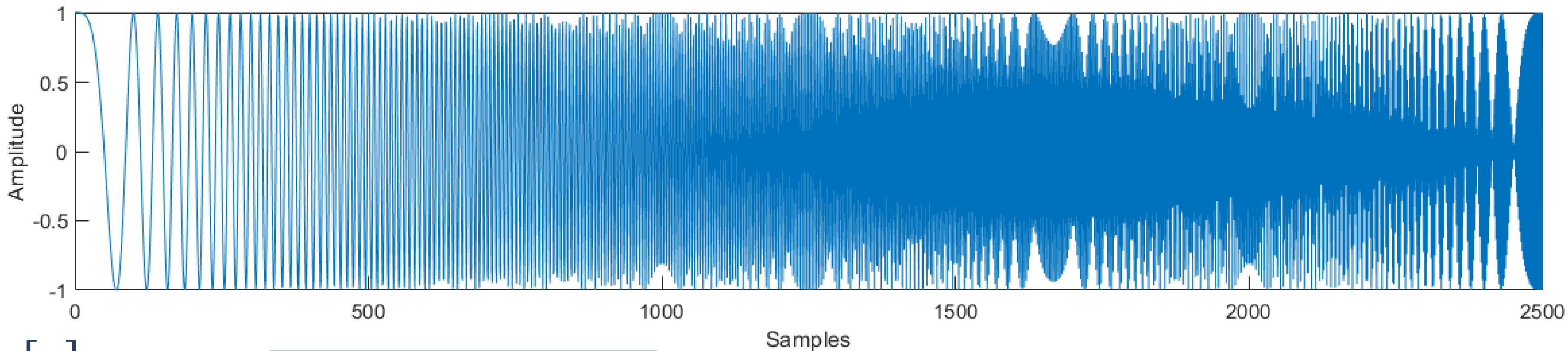
Filter Banks

Inefficient Filter Bank Process



Filter Banks

Inefficient Filter Bank Process



$x[n]$

$$h_0[n] = e^{j\frac{2\pi}{4}n(0)}$$

$$h_1[n] = e^{j\frac{2\pi}{4}n(1)}$$

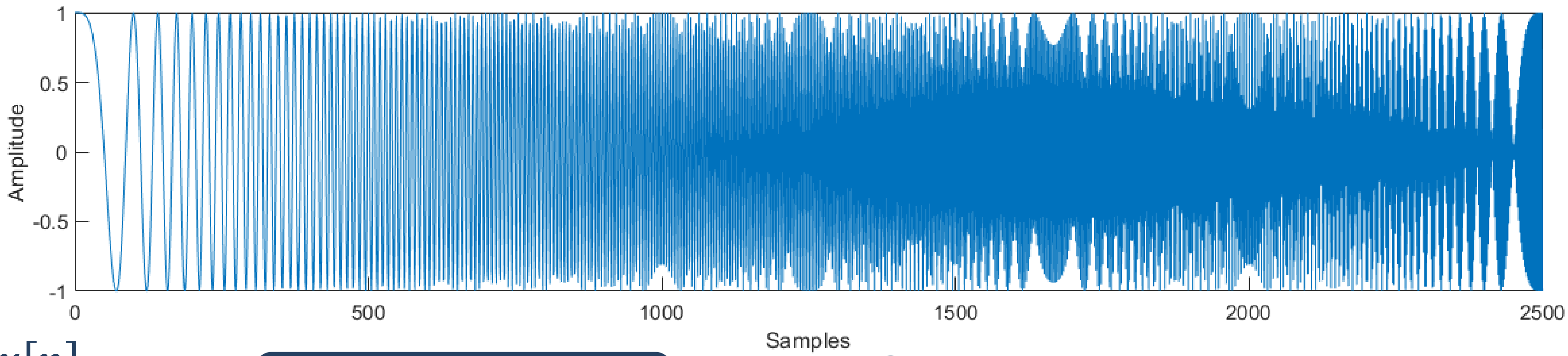
$$h_2[n] = e^{j\frac{2\pi}{4}n(2)}$$

$$h_3[n] = e^{j\frac{2\pi}{4}n(3)}$$

Assume all length-4 filters

Filter Banks

Inefficient Filter Bank Process



$x[n]$

$$h_0[n] = e^{j\frac{2\pi}{4}n(0)} \rightarrow e^{j\frac{2\pi}{N}n(0)} * x[n]$$

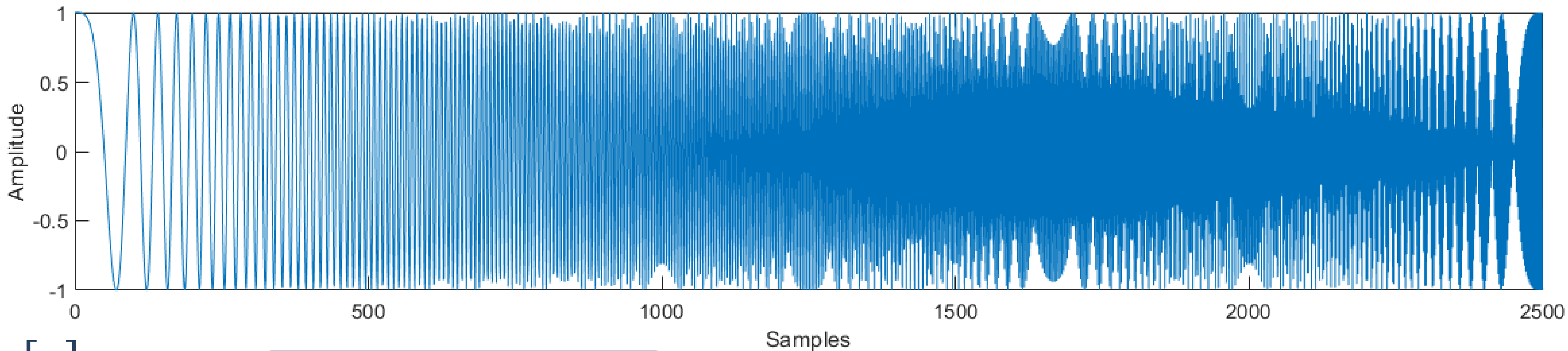
$$h_1[n] = e^{j\frac{2\pi}{4}n(1)} \rightarrow e^{j\frac{2\pi}{N}n(1)} * x[n]$$

$$h_2[n] = e^{j\frac{2\pi}{4}n(2)} \rightarrow e^{j\frac{2\pi}{N}n(2)} * x[n]$$

$$h_3[n] = e^{j\frac{2\pi}{4}n(3)} \rightarrow e^{j\frac{2\pi}{N}n(3)} * x[n]$$

Assume all length-4 filters

Ineffiecient Filter Bank Process



$x[n]$

$$h_0[n] = e^{j\frac{2\pi}{4}n(0)} \rightarrow X[n,0]e^{j\frac{2\pi}{N}n(0)}$$

$$h_1[n] = e^{j\frac{2\pi}{4}n(1)} \rightarrow X[n,1]e^{j\frac{2\pi}{N}n(1)}$$

$$h_2[n] = e^{j\frac{2\pi}{4}n(2)} \rightarrow X[n,2]e^{j\frac{2\pi}{N}n(2)}$$

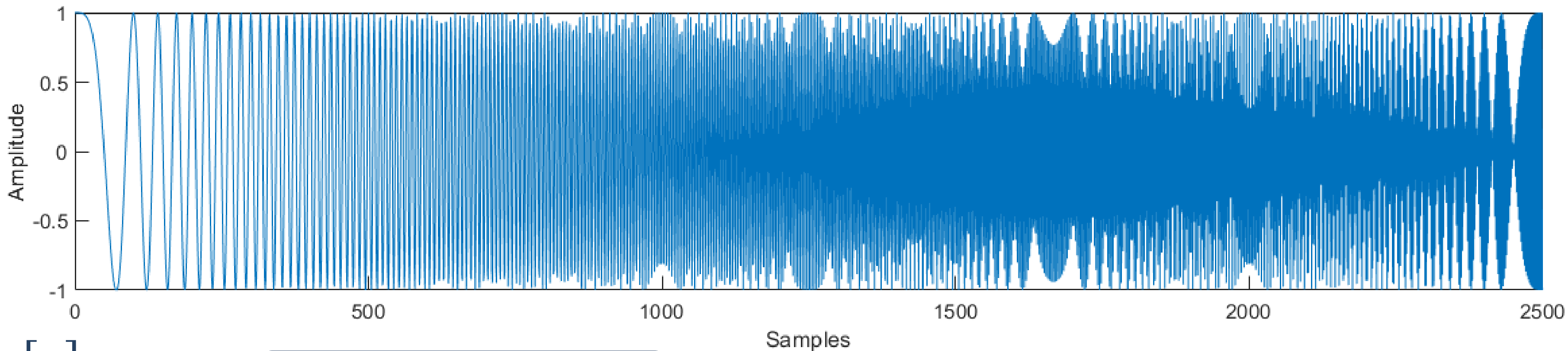
$$h_3[n] = e^{j\frac{2\pi}{4}n(3)} \rightarrow X[n,3]e^{j\frac{2\pi}{N}n(3)}$$

Assume all length-4 filters

Easy solution: Use the DTFT

Filter Banks

- **Inefficient Filter Bank Process**



$x[n]$

$$h_0[n] = e^{j\frac{2\pi}{4}n(0)} \rightarrow \sum_{m=0}^{N-1} e^{j\frac{2\pi}{N}(n-m)(0)}\, x[m]$$

$$h_1[n] = e^{j\frac{2\pi}{4}n(1)} \rightarrow \sum_{m=0}^{N-1} e^{j\frac{2\pi}{N}(n-m)(1)}\, x[m]$$

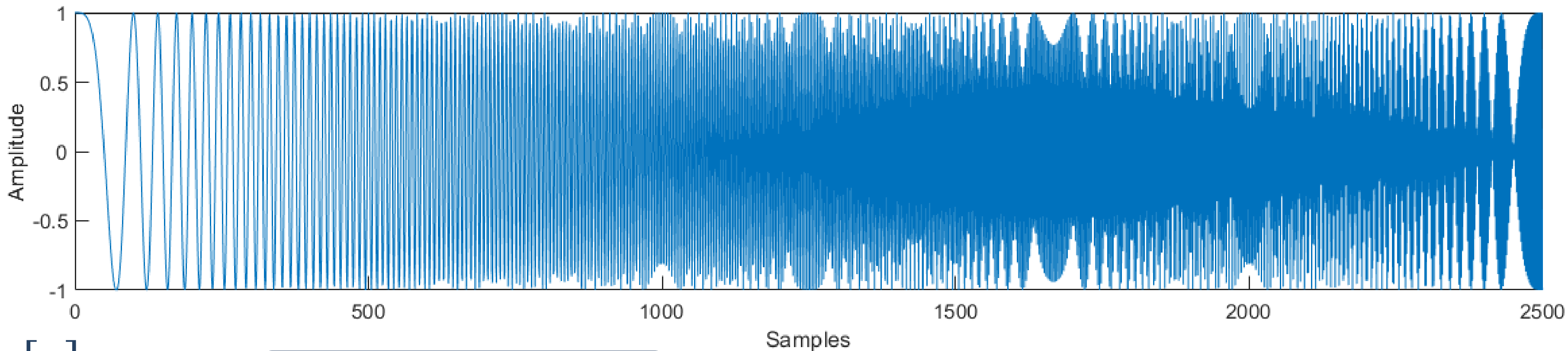
$$h_2[n] = e^{j\frac{2\pi}{4}n(2)} \rightarrow \sum_{m=0}^{N-1} e^{j\frac{2\pi}{N}(n-m)(2)}\, x[m]$$

$$h_3[n] = e^{j\frac{2\pi}{4}n(3)} \rightarrow \sum_{m=0}^{N-1} e^{j\frac{2\pi}{N}(n-m)(3)}\, x[m]$$

Assume all length-4 filters

Hard solution: Use math

Inefficient Filter Bank Process



$x[n]$

$$h_0[n] = e^{j\frac{2\pi}{4}n(0)} \rightarrow e^{j\frac{2\pi}{N}n(0)} \sum_{m=0}^{N-1} e^{-j\frac{2\pi}{N}m(0)}\, x[m]$$

$$h_1[n] = e^{j\frac{2\pi}{4}n(1)} \rightarrow e^{j\frac{2\pi}{N}n(1)} \sum_{m=0}^{N-1} e^{-j\frac{2\pi}{N}m(1)}\, x[m]$$

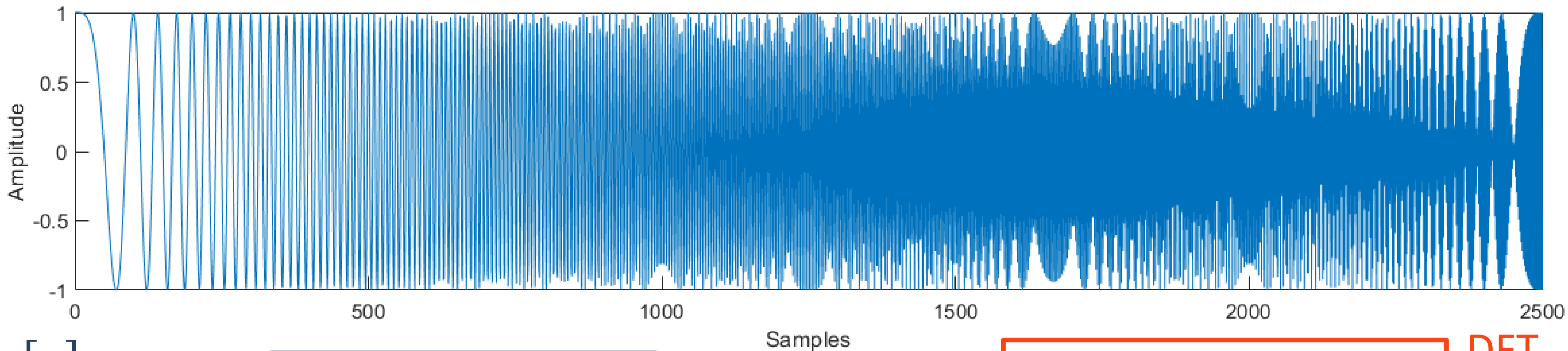
$$h_2[n] = e^{j\frac{2\pi}{4}n(2)} \rightarrow e^{j\frac{2\pi}{N}n(2)} \sum_{m=0}^{N-1} e^{-j\frac{2\pi}{N}m(2)}\, x[m]$$

$$h_3[n] = e^{j\frac{2\pi}{4}n(3)} \rightarrow e^{j\frac{2\pi}{N}n(3)} \sum_{m=0}^{N-1} e^{-j\frac{2\pi}{N}m(3)}\, x[m]$$

Assume all length-4 filters

Filter Banks

Inefficient Filter Bank Process



$x[n]$

$$h_0[n] = e^{j\frac{2\pi}{4}n(0)} \rightarrow e^{j\frac{2\pi}{N}n(0)} \boxed{\sum_{m=0}^{N-1} e^{-j\frac{2\pi}{N}m(0)}\, x[m]}$$

DFT for k = 0

$$h_1[n] = e^{j\frac{2\pi}{4}n(1)} \rightarrow e^{j\frac{2\pi}{N}n(1)} \sum_{m=0}^{N-1} e^{-j\frac{2\pi}{N}m(1)}\, x[m]$$

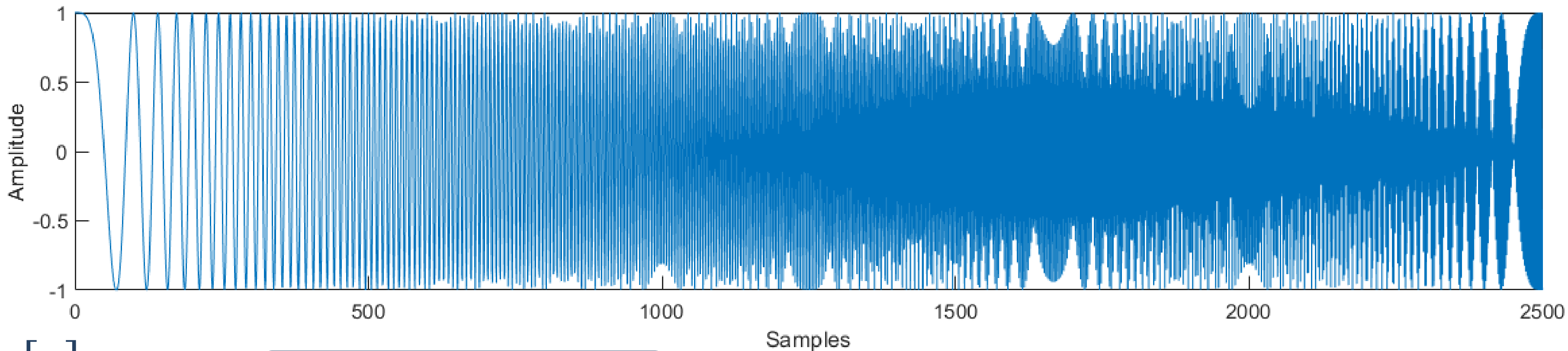
$$h_2[n] = e^{j\frac{2\pi}{4}n(2)} \rightarrow e^{j\frac{2\pi}{N}n(2)} \sum_{m=0}^{N-1} e^{-j\frac{2\pi}{N}m(2)}\, x[m]$$

$$h_3[n] = e^{j\frac{2\pi}{4}n(3)} \rightarrow e^{j\frac{2\pi}{N}n(3)} \sum_{m=0}^{N-1} e^{-j\frac{2\pi}{N}m(3)}\, x[m]$$

Assume all length-4 filters

Filter Banks

Inefficient Filter Bank Process



$x[n]$

$$h_0[n] = e^{j\frac{2\pi}{4}n(0)} \rightarrow e^{j\frac{2\pi}{N}n(0)} X[n,0]$$

$$h_1[n] = e^{j\frac{2\pi}{4}n(1)} \rightarrow e^{j\frac{2\pi}{N}n(1)} \boxed{\sum_{m=0}^{N-1} e^{-j\frac{2\pi}{N}m(1)}\, x[m]}$$

DFT for k = 1

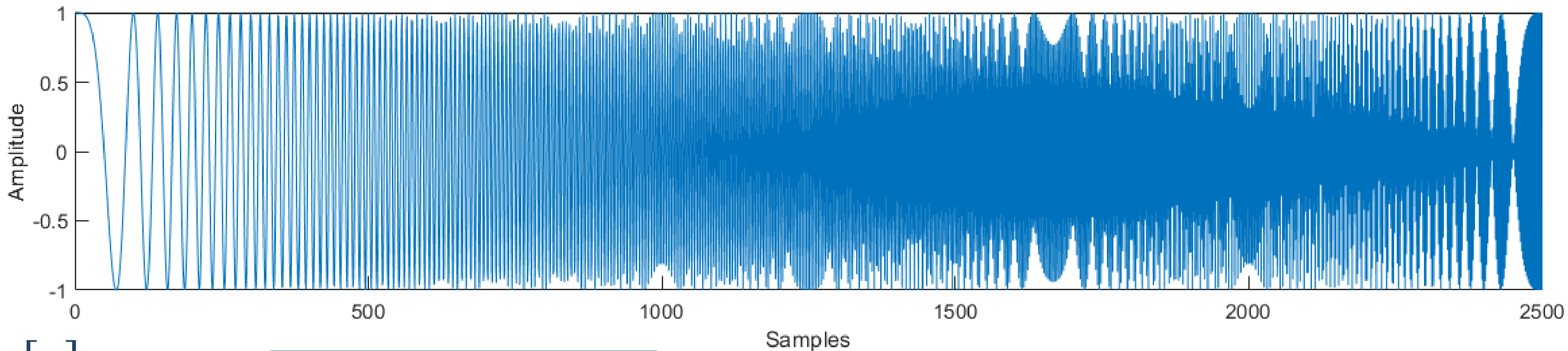
$$h_2[n] = e^{j\frac{2\pi}{4}n(2)} \rightarrow e^{j\frac{2\pi}{N}n(2)} \sum_{m=0}^{N-1} e^{-j\frac{2\pi}{N}m(2)}\, x[m]$$

$$h_3[n] = e^{j\frac{2\pi}{4}n(3)} \rightarrow e^{j\frac{2\pi}{N}n(3)} \sum_{m=0}^{N-1} e^{-j\frac{2\pi}{N}m(3)}\, x[m]$$

Assume all length-4 filters

Filter Banks

Ineffficient Filter Bank Process



$x[n]$

$$h_0[n] = e^{j\frac{2\pi}{4}n(0)} \rightarrow e^{j\frac{2\pi}{N}n(0)}X[n,0]$$

$$h_1[n] = e^{j\frac{2\pi}{4}n(1)} \rightarrow e^{j\frac{2\pi}{N}n(1)}X[n,1]$$

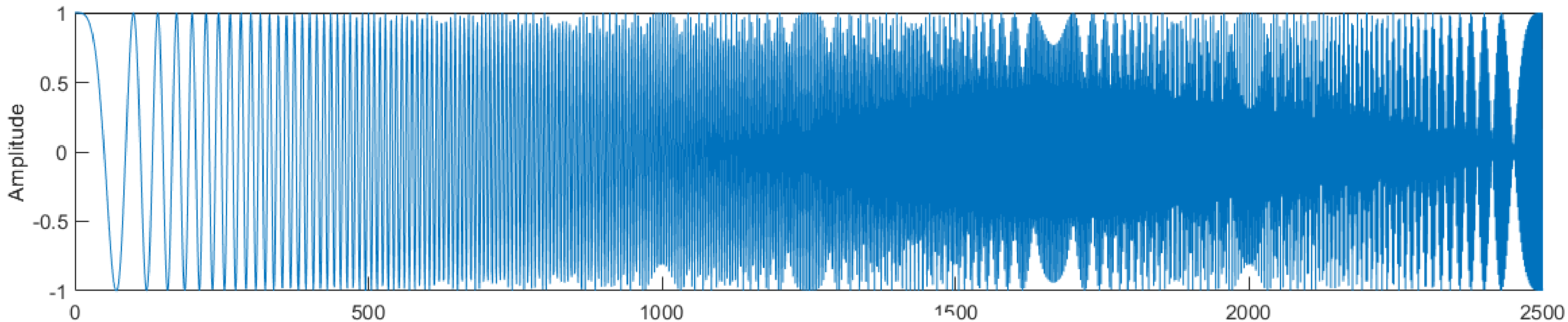
$$h_2[n] = e^{j\frac{2\pi}{4}n(2)} \rightarrow e^{j\frac{2\pi}{N}n(2)}X[n,2]$$

$$h_3[n] = e^{j\frac{2\pi}{4}n(3)} \rightarrow e^{j\frac{2\pi}{N}n(3)}X[n,3]$$

Assume all length-4 filters

Filter Banks

Inefficient Filter Bank Process



$x[n]$

$h_0[n] = e^{j\frac{2\pi}{4}n(0)}$ $\xrightarrow{e^{-j\frac{2\pi}{4}n(0)}}$ $X[n,0]$

$h_1[n] = e^{j\frac{2\pi}{4}n(1)}$ $\xrightarrow{e^{-j\frac{2\pi}{4}n(1)}}$ $X[n,1]$

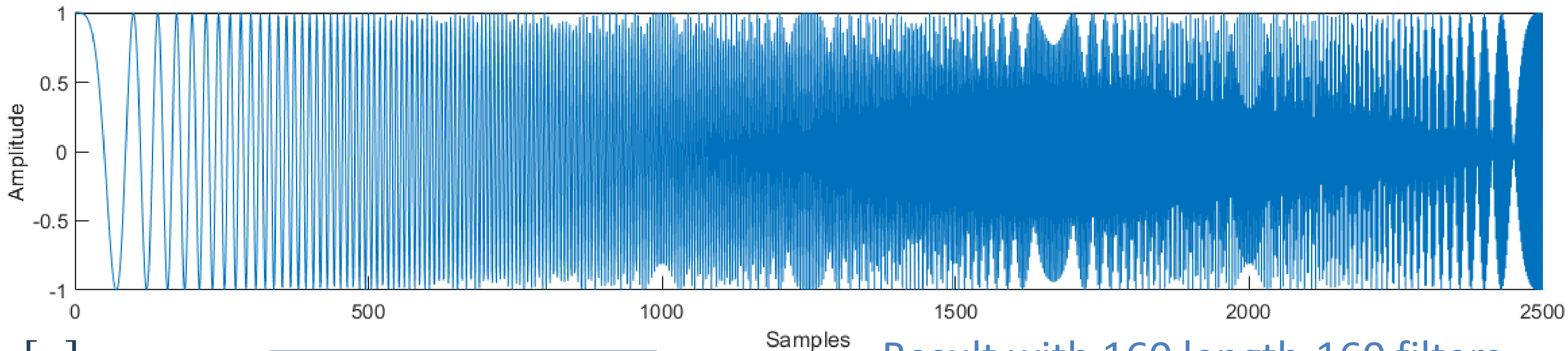
$h_2[n] = e^{j\frac{2\pi}{4}n(2)}$ $\xrightarrow{e^{-j\frac{2\pi}{4}n(2)}}$ $X[n,2]$

$h_3[n] = e^{j\frac{2\pi}{4}n(3)}$ $\xrightarrow{e^{-j\frac{2\pi}{4}n(3)}}$ $X[n,3]$

We have the DFT coefficients!

Assume all length-4 filters

Inefficient Filter Bank Process



$x[n]$

$$h_0[n] = e^{j\frac{2\pi}{4}n(0)}$$

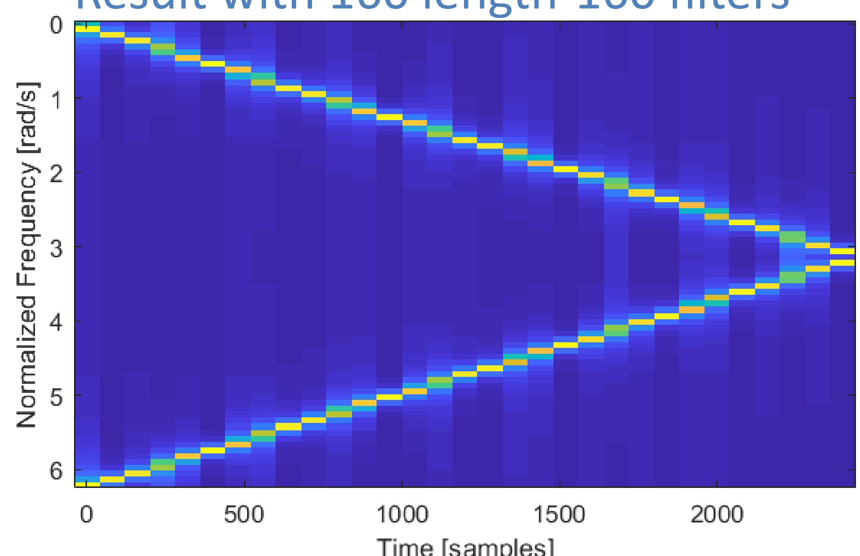
$$h_1[n] = e^{j\frac{2\pi}{4}n(1)}$$

$$h_2[n] = e^{j\frac{2\pi}{4}n(2)}$$

$$h_3[n] = e^{j\frac{2\pi}{4}n(3)}$$

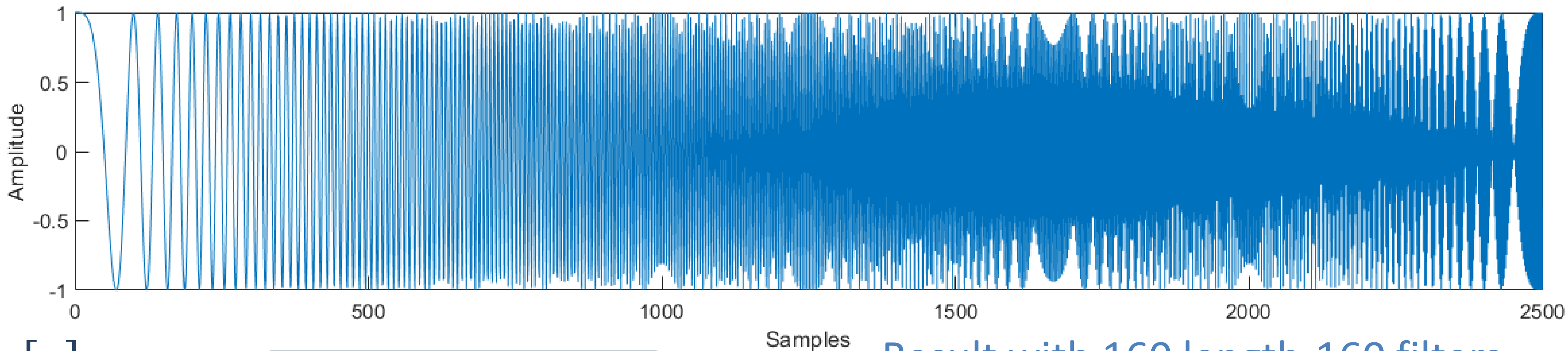
Assume all length-4 filters

Result with 160 length-160 filters



Filter Banks

- **Question: Why is this <u>not</u> a preferred approach?**



$x[n]$

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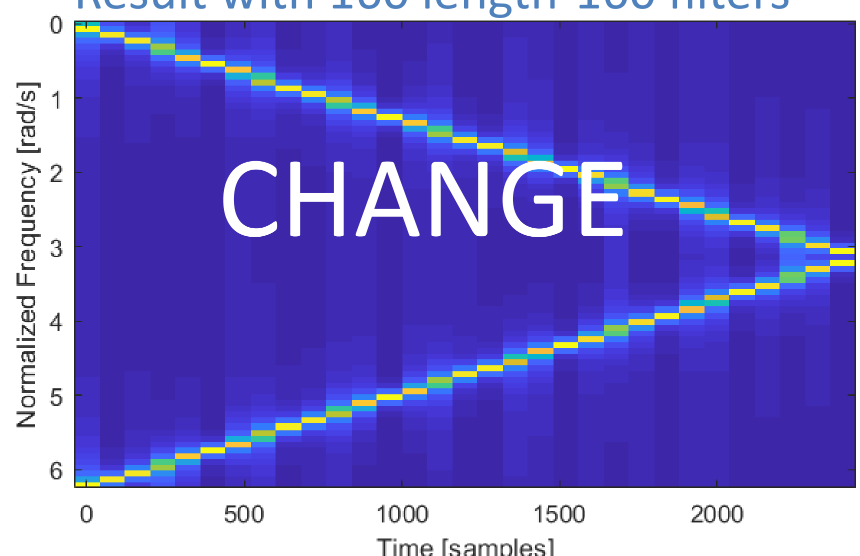
$$h_3[n] = e^{j\frac{2\pi}{4}n(3)}$$

Assume all length-4 filters

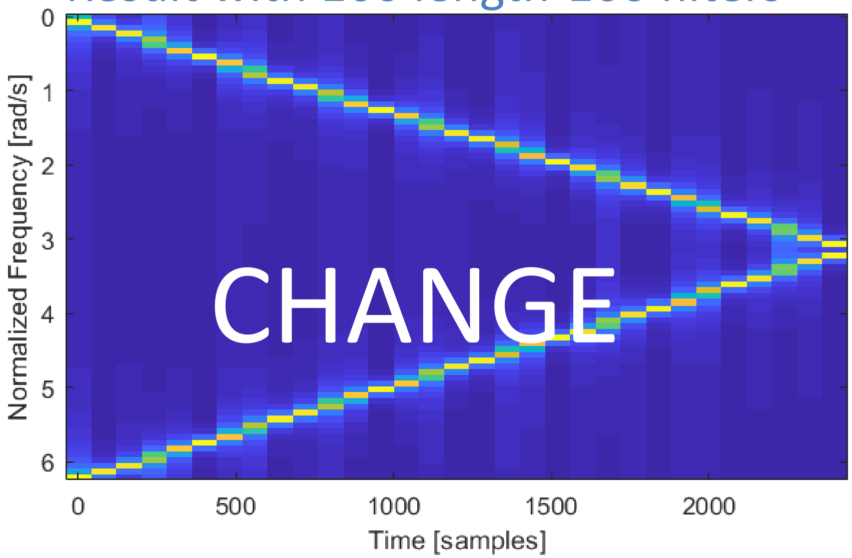
Result with 160 length-160 filters



Filter Banks

- **Question: Why is this not a preferred approach?**
  - It is really expensive!

Result with 160 length-160 filters



- **STFT Approach**
  - W^2 multiplications for every W samples
  - $W^2 = 160^2 = 25{,}600$
- **Filter Bank Approach**
  - W^3 multiplications for every W samples
  - $W^3 = 160^3 = 4{,}086{,}000$

Lecture 23: Introduction to Filter Banks

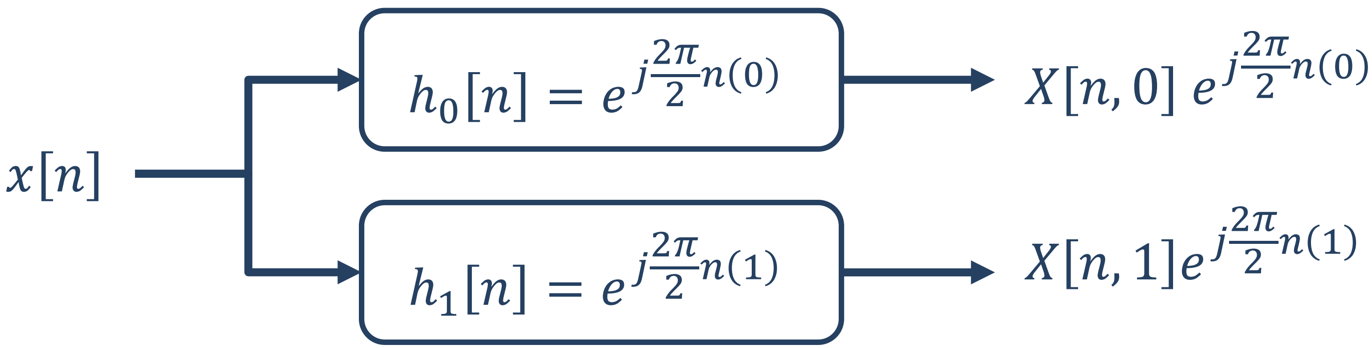
Foundations of Digital Signal Processing

Outline

- Review of Filter Design
- Short Time Fourier Transform
- Inefficient Filter Banks
- **DFT Filter Bank**

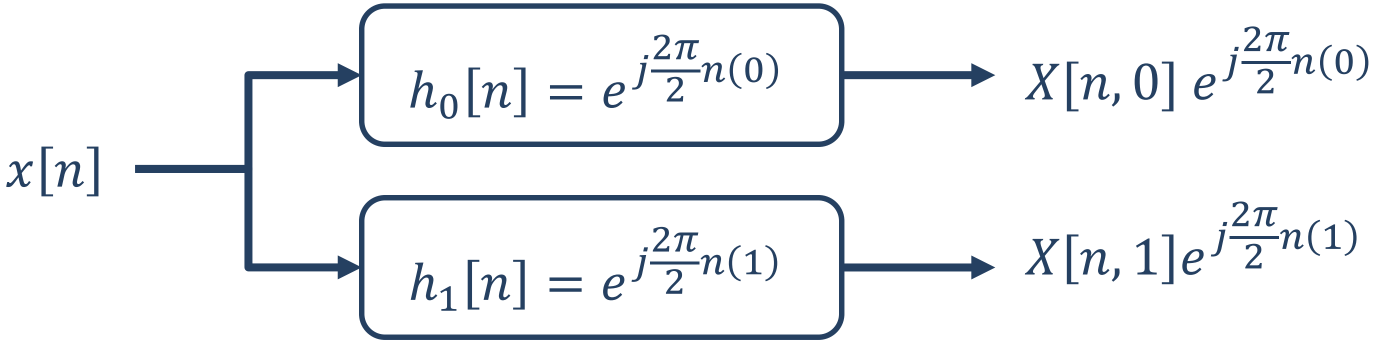
Filter Banks

- **Consider the following filter bank**



Filter Banks

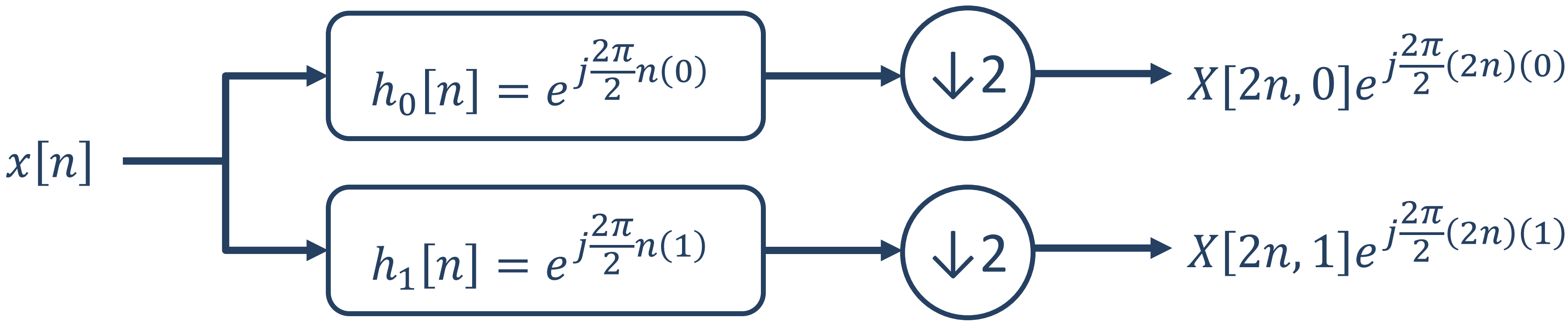
- **Consider the following filter bank**
  - **Question:** How do I make this like the STFT????



  - **Now recall:** The short-time Fourier Transform gave us
    - $X[Mn, 0]$ <- M = shift amount (often window length)

Filter Banks

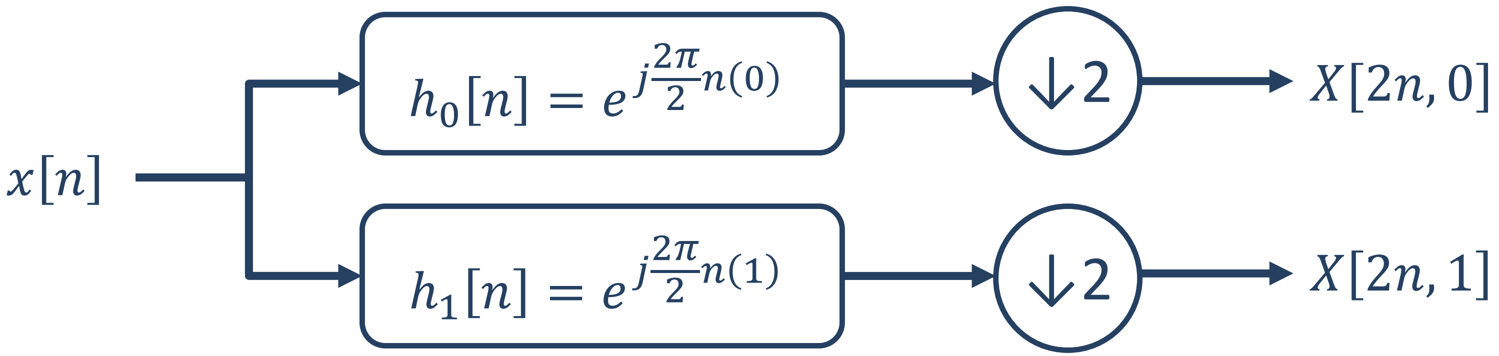
- **Consider the following filter bank**
  - **Question:** How do I make this like the STFT????



  - **Now recall:** The short-time Fourier Transform gave us
    - ◊ $X[Mn, 0]$ <- M = shift amount (often window length)

Filter Banks

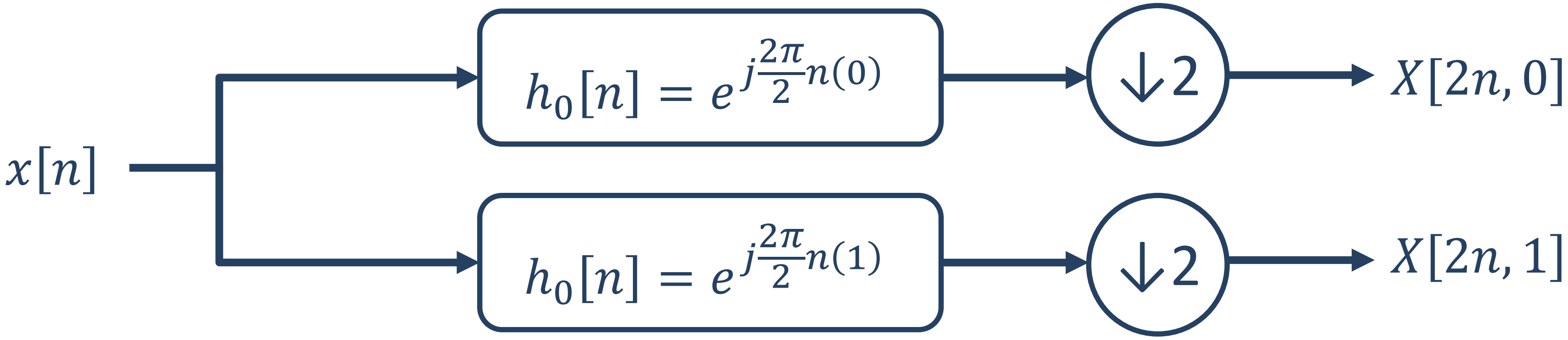
- **Consider the following filter bank**
  - **Question:** How do I make this like the STFT????



  - **Now recall:** The short-time Fourier Transform gave us
    - $X[Mn, 0]$ <- M = shift amount (often window length)

Filter Banks

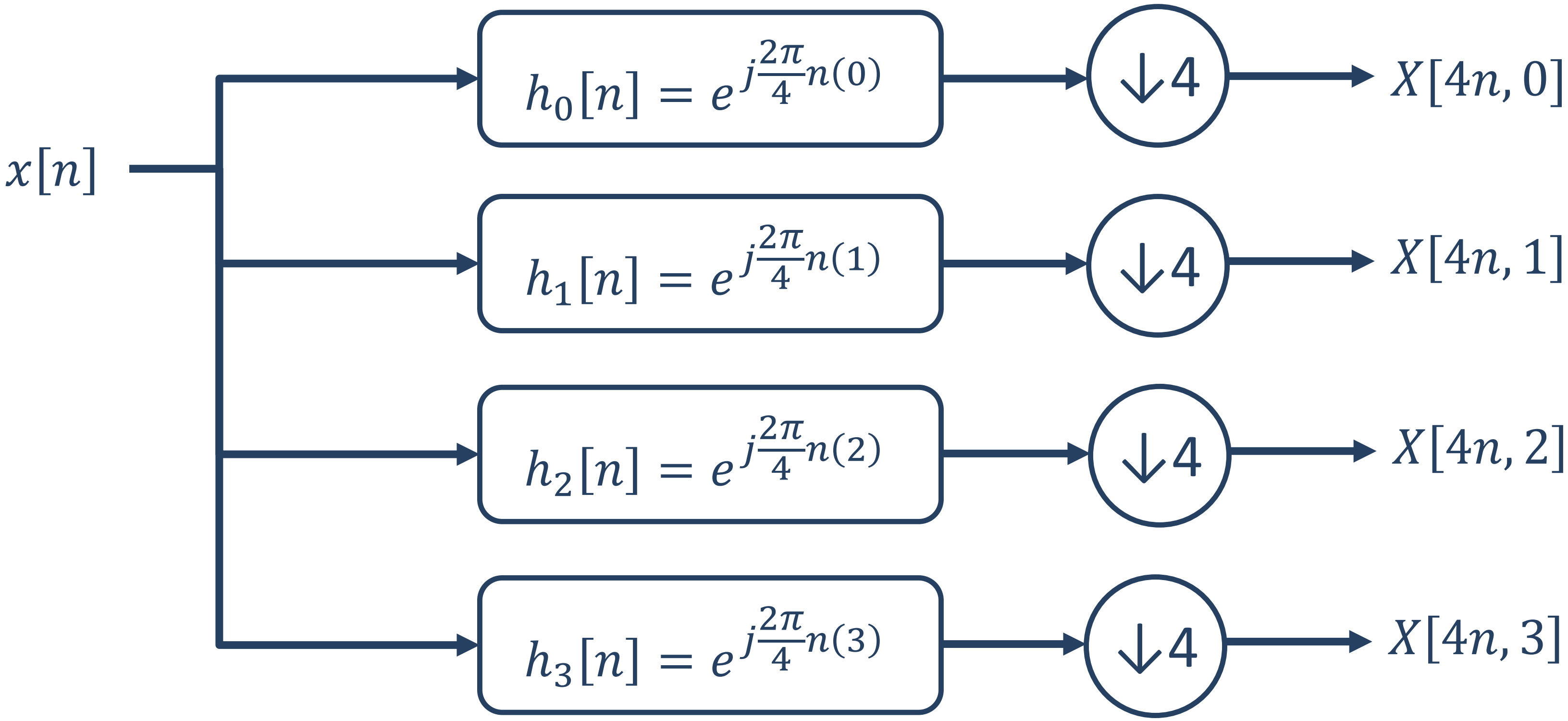
- **Consider the following filter bank**



  - **Result:** It is now exactly the same as the STFT with a window of length 2 and shift of 2 between windows
  - **But,** we do not need to buffer $x[n]$

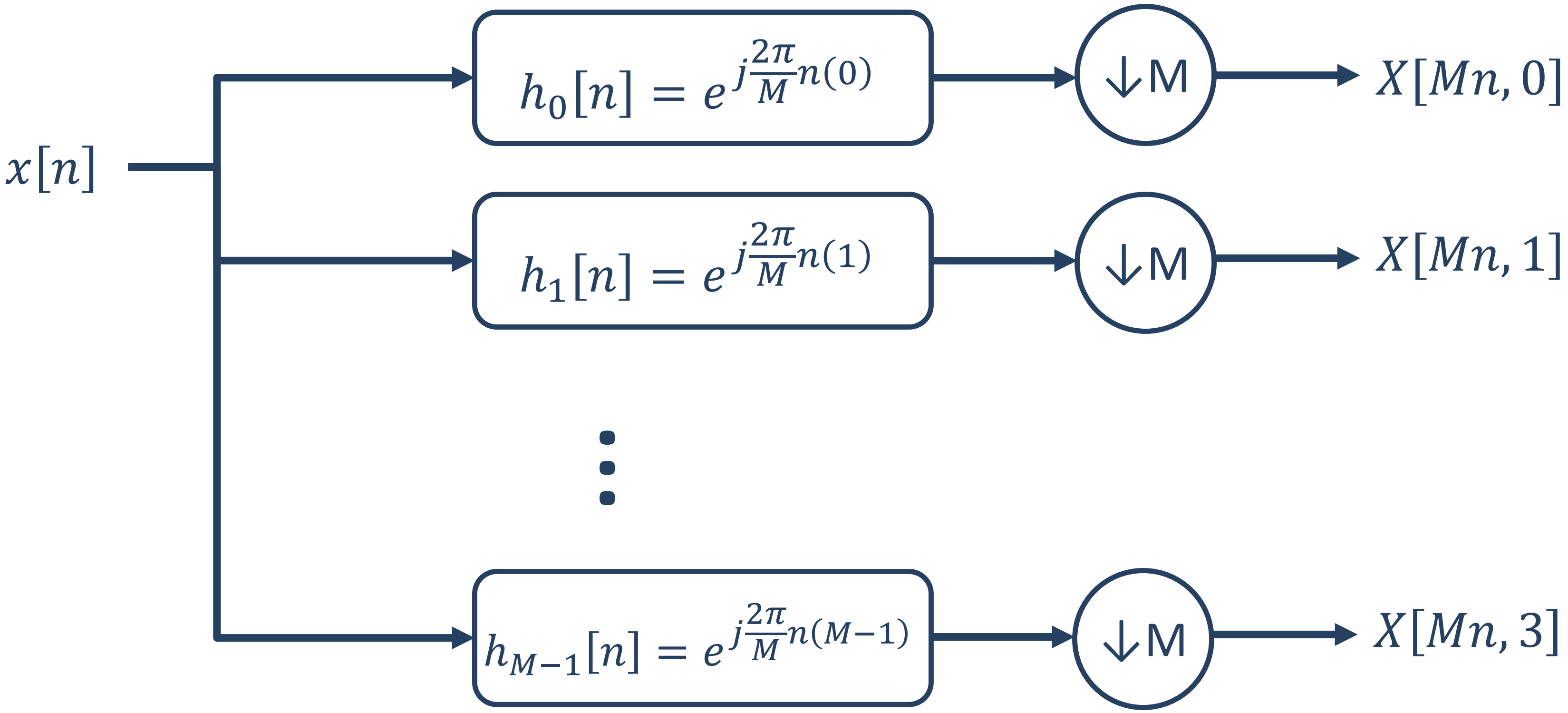
Filter Banks

- **Consider the following filter bank**



Filter Banks

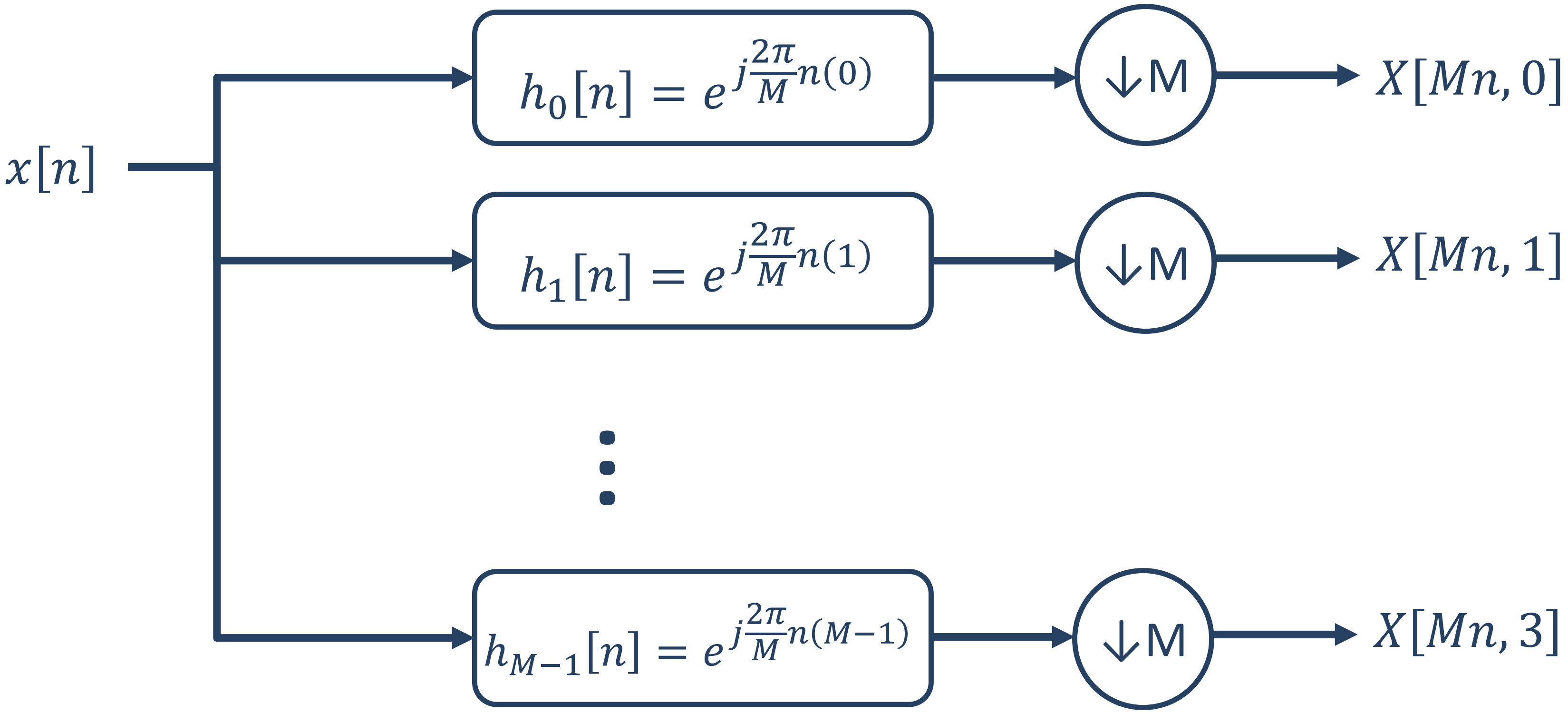
- **Hence, this is an M-point DFT**



- **So, I can implement the STFT as a filter bank…**
  - Can I do more?

Filter Banks

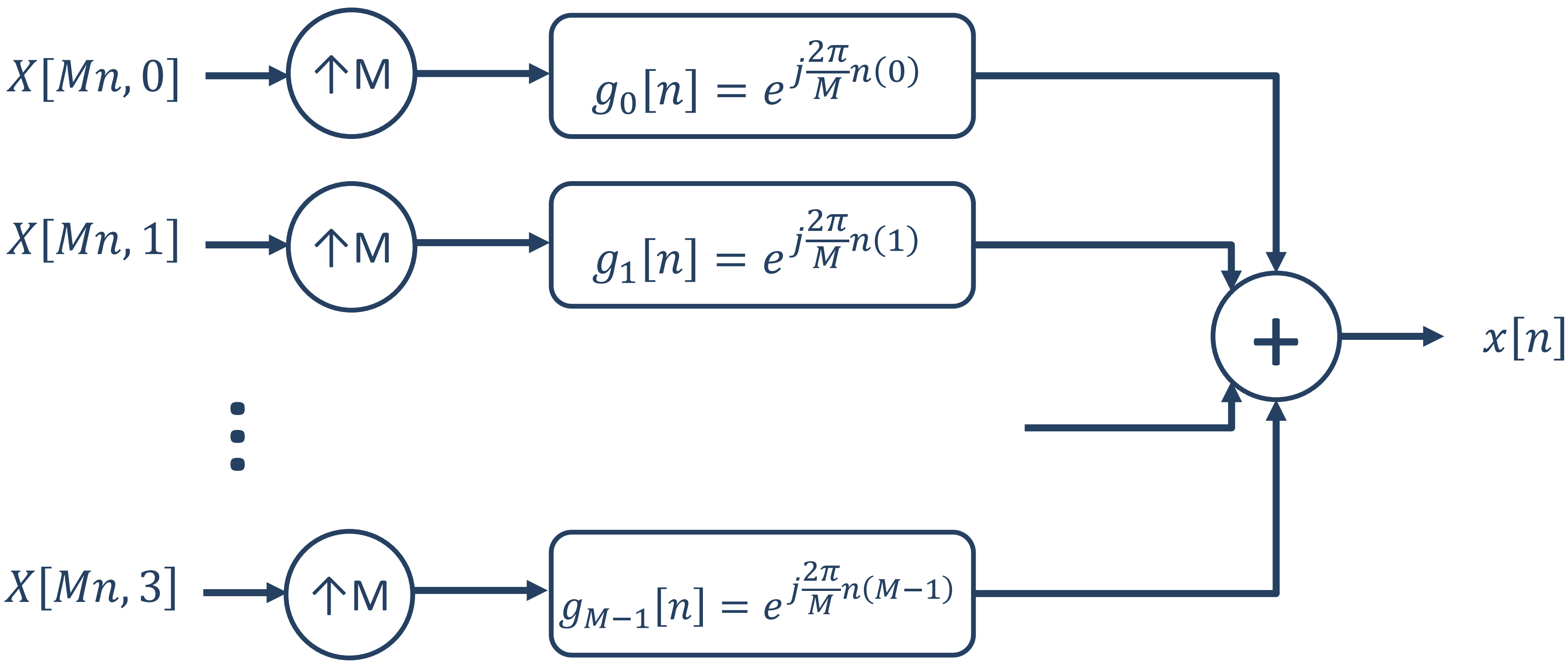
- **Hence, this is an M-point DFT**



- **So, I can implement the STFT as a filter bank…**
  - **Question:** How do I get back into the time domain?

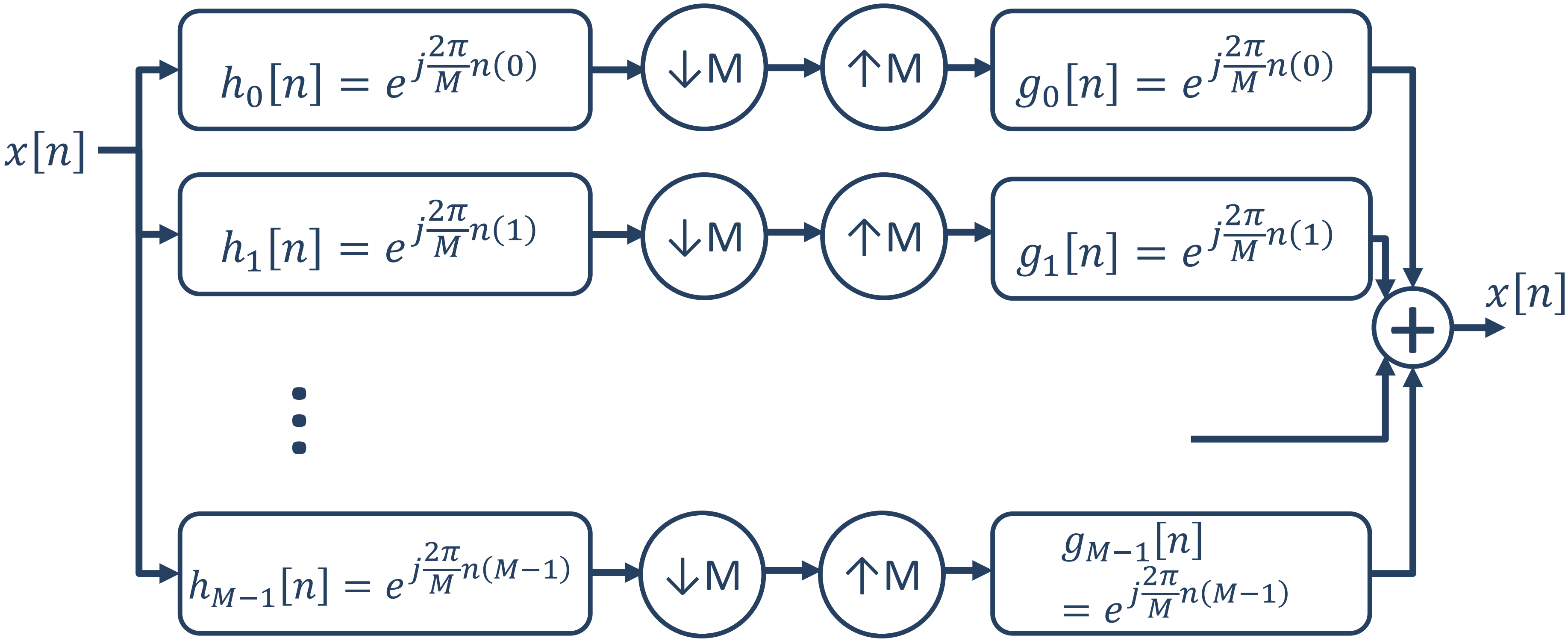
Filter Banks

- **Hence, this is an M-point IDFT**



Filter Banks

- **Hence, this is an M-point DFT and IDFT**



Filter Banks

- **Hence, this is an M=2 point DFT and IDFT**