Lecture 5 : The Z-Transform

Foundations of Digital Signal Processing

Outline

- Linear Difference Equations / LTI Systems Review
- The Z-Transform
- The Properties of the Z-Transform
- Poles, Zeros, and Region of Convergence

News

- **Homework #2**
  - Due *Today* by 11:59 PM
  - Submit via canvas
  - Solutions will be posted Wednesday next week
- **Coding Assignment #1**
  - Due *Today* by 11:59 PM
  - Submit via canvas
    - Submit answers as a PDF
    - Submit code as .m files

Lecture 4: Discrete -Time LTI Systems

Foundations of Digital Signal Processing

Outline

- **Linear Difference Equations / LTI Systems Review**
- The Z-Transform
- The Properties of the Z-Transform
- Poles, Zeros, and Region of Convergence

Complex Exponentials

- **Question: Consider the signals**
  - $x[n] = Ae^{j\frac{2\pi}{5}n}$
  - What is the magnitude of $x[n]$, i.e., $|x[n]|$?
  - What is the phase of $x[n]$, i.e., $\angle \mathrm{x}[n]$?

General LTI System

- **General form for an LTI system is:**

$$\sum_{m=-\infty}^{\infty} g[m]y[n-m] = \sum_{m=-\infty}^{\infty} r[m]x[n-m]$$

$$g[n] * y[n] = r[n] * x[n]$$

General LTI System

- **General form for an LTI system is:**

$$\sum_{m=-\infty}^{\infty} g[m]y[n-m] = \sum_{m=-\infty}^{\infty} r[m]x[n-m]$$

$$g[n] * y[n] = r[n] * x[n]$$

$$\ldots + g[n-1]y[n-1] + g[n]y[n] + g[n+1]y[n+1] + \cdots$$
$$= \cdots + r[n-1]x[n-1] + r[n]x[n] + r[n+1]x[n+1] + \cdots$$

General LTI System

- **General form for an LTI system is:**

$$\sum_{m=-\infty}^{\infty} g[m]y[n-m] = \sum_{m=-\infty}^{\infty} r[m]x[n-m]$$

$$g[n] * y[n] = r[n] * x[n]$$

$$\begin{aligned}&\ldots + g[-1]y[n+1] + g[n]y[n] + g[1]y[n-1] + \cdots \\ &= \cdots + r[-1]x[n+1] + r[n]x[n] + r[1]x[n-1] + \cdots\end{aligned}$$

$$\begin{aligned}&\ldots + g_{-1}y[n-1] + g_n y[n] + g_1 y[n-1] + \cdots \\ &= \cdots + r_{-1}x[n-1] + r_n x[n] + r_1 x[n-1] + \cdots\end{aligned}$$

General LTI System

- **General form for an LTI system is:**

$$\sum_{m=-\infty}^{\infty} g[m]y[n-m] = \sum_{m=-\infty}^{\infty} r[m]x[n-m]$$

$$g[n] * y[n] = r[n] * x[n]$$

- **Question: Consider the difference equations where**
  - $g[n] = \delta[n]$
  - What does the general form become?

General LTI System

- **General form for an LTI system is:**

$$\sum_{m=-\infty}^{\infty} g[m]y[n-m] = \sum_{m=-\infty}^{\infty} r[m]x[n-m]$$

$$g[n] * y[n] = r[n] * x[n]$$

- **Question: Consider the difference equations where**
  - $g[n] = \delta[n]$
  - What is the impulse response $h[n]$?

General LTI System

- **General form for an LTI system is:**

$$\sum_{m=-\infty}^{\infty} g[m]y[n-m] = \sum_{m=-\infty}^{\infty} r[m]x[n-m]$$

$$g[n] * y[n] = r[n] * x[n]$$

- **Question: Consider the difference equations where**
  - $g[n] = \delta[n],\ \ r[0] = 1,\ \ r[1] = 1/2,\ \ \ r[n] = 0$ otherwise
  - What is the impulse response?

General LTI System

- **General form for an LTI system is:**

$$\sum_{m=-\infty}^{\infty} g[m]y[n-m] = \sum_{m=-\infty}^{\infty} r[m]x[n-m]$$

$$g[n] * y[n] = r[n] * x[n]$$

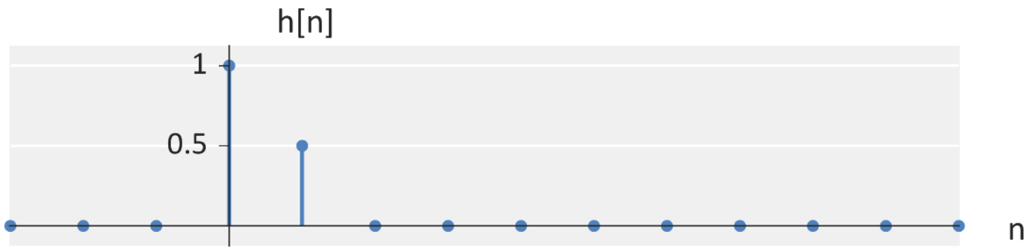
- **Question: Consider the difference equations where**
  - $r[n] = \delta[n], \;\; g[0] = 1, \;\; g[1] = 1/2, \;\; g[n] = 0$ otherwise
  - What is the impulse response?

General LTI System

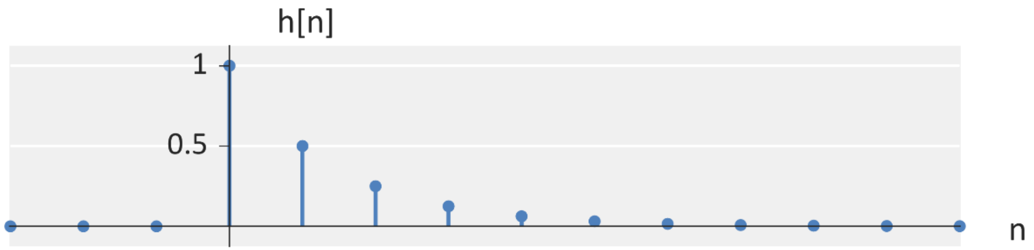
- **Observation:**
  - I previously said that all LTI systems can be represented by an impulse response $h[n]$… ***What is the impulse response here?***
  - $y[n] + (1/2)y[n-1] = \delta[n]$ (Assume $y[n] = 0$ for $n \leq 0$)

General LTI System

- **Finite Impulse Response System**
  - $h[n] = \delta[n] + (1/2)\delta[n-1]$



- **Infinite Impulse Response System**
  - $h[n] = 2^{-n}u[n]$



General LTI System

- **Finite Impulse Response System**

$$y[n] = \sum_{m=-\infty}^{\infty} h[m]x[n-m]$$

- **Infinite Impulse Response System**

$$\sum_{m=-\infty}^{\infty} g[m]y[n-m] = \sum_{m=-\infty}^{\infty} h[m]x[n-m]$$

Recursive components

General LTI System

- **We need more math**
  - How do I convert between

$$\sum_{m=-\infty}^{\infty} g[m]y[n-m] = \sum_{m=-\infty}^{\infty} r[m]x[n-m]$$

How do I convert between these?

$$h[n]$$

Lecture 4: Discrete -Time LTI Systems

Foundations of Digital Signal Processing

Outline

- Linear Difference Equations / LTI Systems Review
- **The Z-Transform**
- The Properties of the Z-Transform
- Poles, Zeros, and Region of Convergence

The Z-Transform

- **The Bilateral Z-Transform**

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

The Z-Transform

- **The Bilateral Z-Transform**

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

- **The Inverse Z-Transform**

$$x[n] = \frac{1}{2\pi j} \oint_C X(z) z^{n-1}\, dz$$

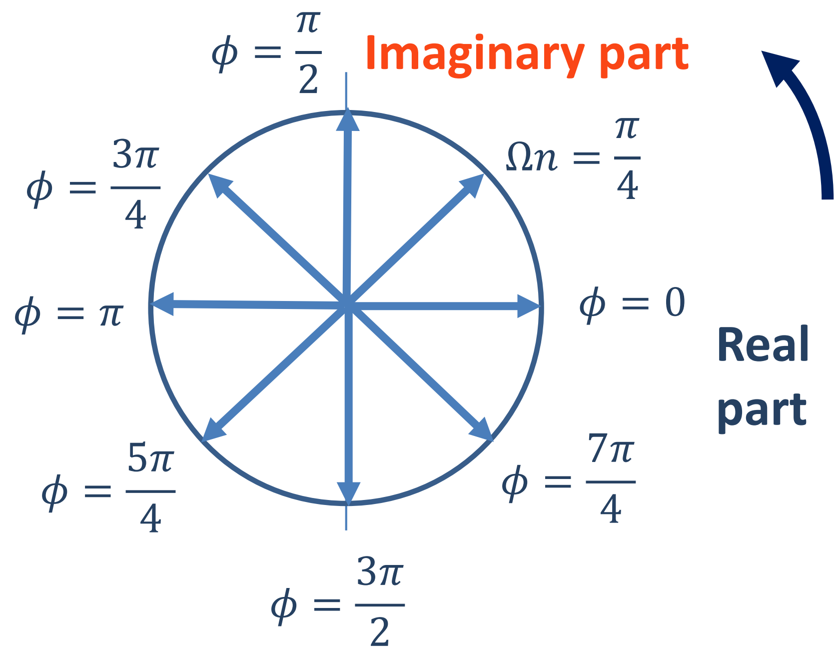
The Z-Transform

- **The Bilateral Z-Transform**

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

A complex number

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]\left(Re^{j\phi}\right)^{-n}$$



The Z-Transform

The Bilateral Z-Transform

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

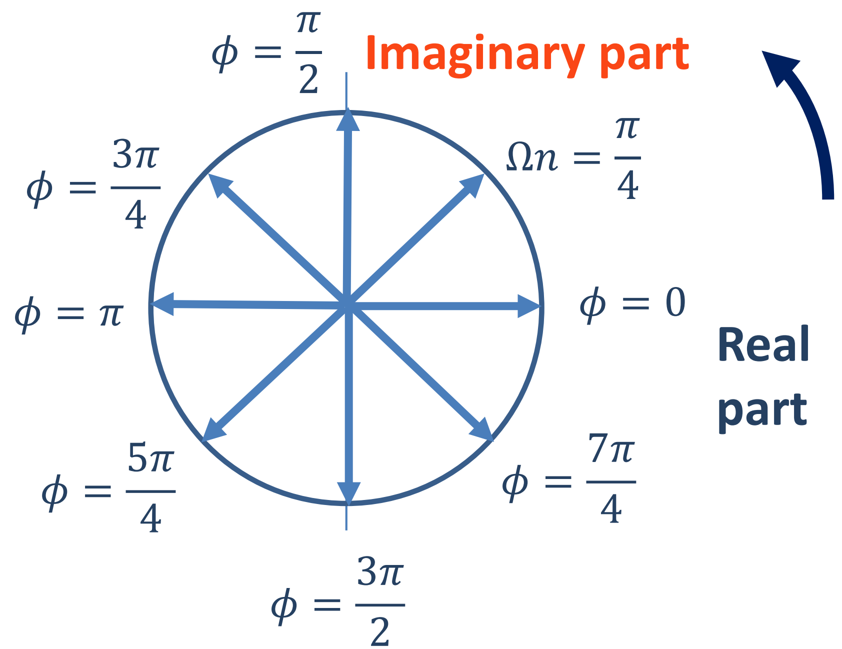
A complex number

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]\left(Re^{j\phi}\right)^{-n}$$

Decay

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]R^{-n}e^{-j\phi n}$$

Sinusoids



The Z-Transform

- **Question:** **Why is the decay part necessary?**

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]\underbrace{R^{-n}}_{\text{Decay}}\underbrace{e^{-j\phi n}}_{\text{Sinusoids}}$$

The Z-Transform

- **Example Problem: Compute the Z-transform of**

$$x[n] = \delta[n]$$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

The Z-Transform

- **Example Problem: Compute the Z-transform of**

$$x[n] = \delta[n - 78]$$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

The Z-Transform

- **Example Problem: Compute the Z-transform of**
  $x[n] = 10\delta[n] + 12\delta[n-1] - 5\delta[n-2] + 8\delta[n-3]$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$