Lecture 12: Sampling, Aliasing, and the Discrete Fourier Transform

Foundations of Digital Signal Processing

Outline

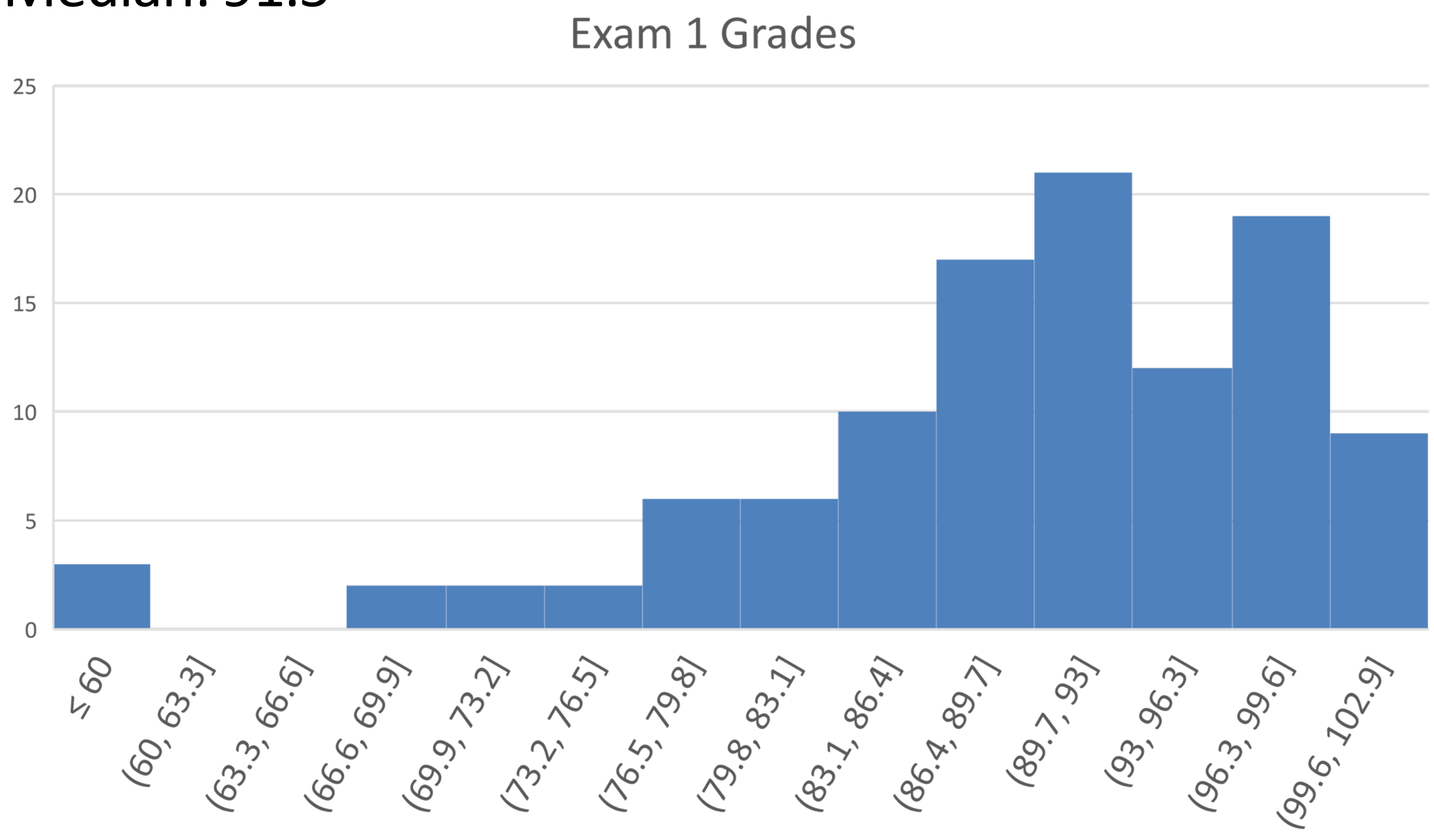
- **Review of Sampling**
- The Nyquist-Shannon Sampling Theorem
- Continuous-time Reconstruction / Interpolation
- Aliasing and anti-Aliasing
- Deriving Transforms from the Fourier Transform
  - Discrete-time Fourier Transform, Fourier Series, Discrete-time Fourier Series
- The Discrete Fourier Transform

News

- **Homework #5**
  - Due *this week*
  - Submit via canvas
- **Coding Problem #4**
  - Due *this week*
  - Submit via canvas

Exam 1 Grades

- **The class did exceedingly well**
  - Mean: 89.3
  - Median: 91.5



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Sampling

- Discrete-Time Fourier Transform

Sampling

- Sampling

Discrete-Time Fourier Transform

- **Example:** Consider the DTFT signal

$$X(\omega) = e^{-\frac{j\omega\pi}{8}} \sum_{k=-\infty}^{\infty} u\left(\omega + \frac{\pi}{2} - 2\pi k\right) - u\left(\omega + \frac{\pi}{4} - 2\pi k\right) + u\left(\omega - \frac{\pi}{2} - 2\pi k\right) - u\left(\omega - \frac{\pi}{4} - 2\pi k\right)$$

  - Sketch the magnitude of $X(\omega)$.

Discrete-Time Fourier Transform

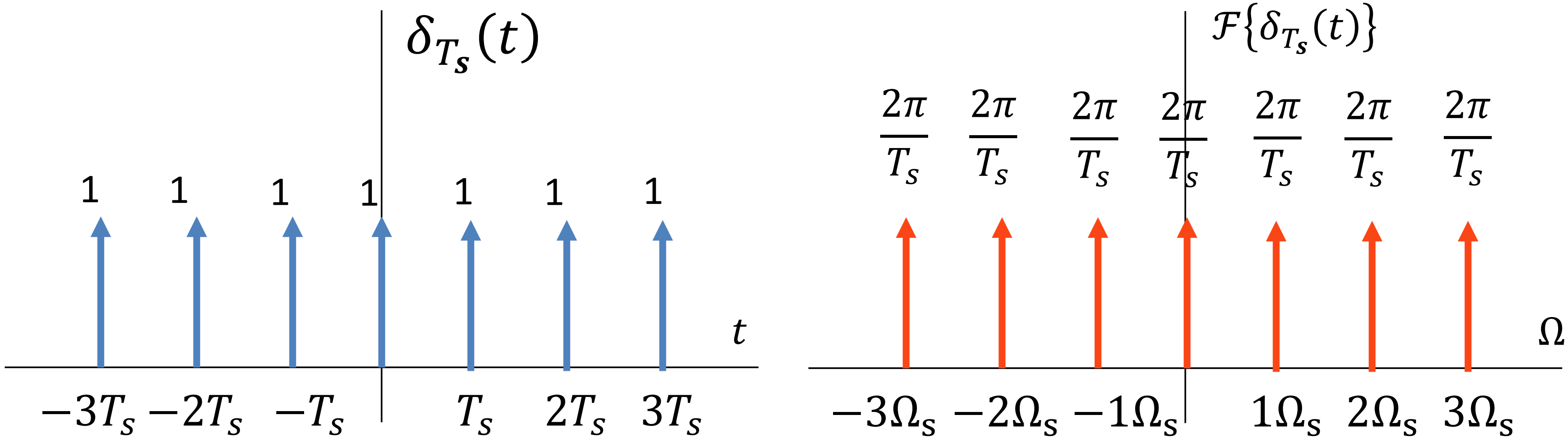
- **Example:** Consider the DTFT signal

$$X(\omega) = e^{-\frac{j\omega\pi}{8}} \sum_{k=-\infty}^{\infty} u\left(\omega + \frac{\pi}{2} - 2\pi k\right) - u\left(\omega + \frac{\pi}{4} - 2\pi k\right) + u\left(\omega - \frac{\pi}{2} - 2\pi k\right) - u\left(\omega - \frac{\pi}{4} - 2\pi k\right)$$

  - Sketch the phase of $X(\omega)$.

Sampling

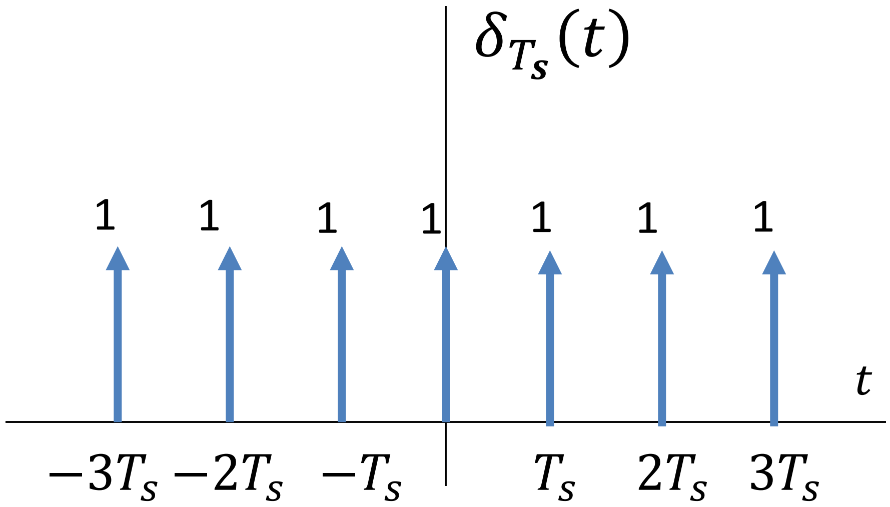
$$\delta_{T_s}(t) \longleftrightarrow \frac{2\pi}{T_s}\delta_{\Omega_s}(\Omega)$$

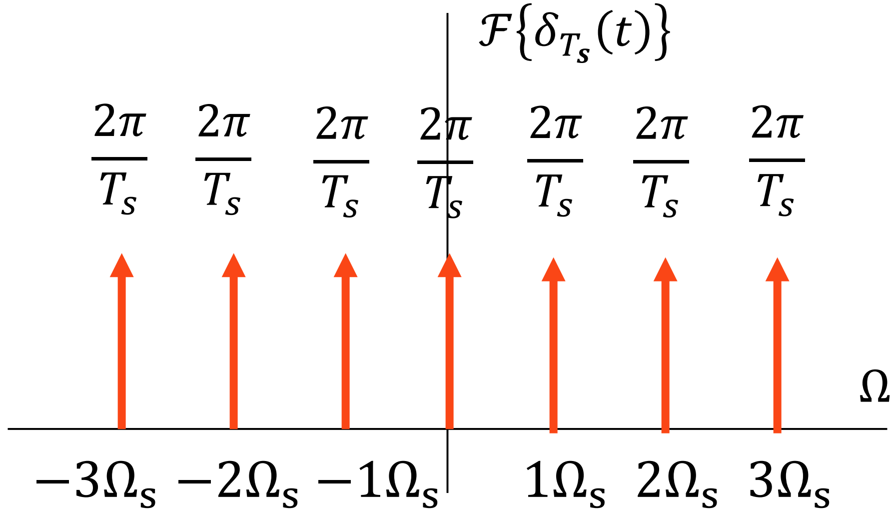


Sampling

$$\delta_{T_s}(t) \longleftrightarrow \frac{2\pi}{T_s}\delta_{\Omega_s}(\Omega)$$

$$x[n]\delta_{T_s}(t) \longleftrightarrow \frac{1}{2\pi}X(\Omega) * \left[\frac{2\pi}{T_s}\delta_{\Omega_s}(\Omega)\right]$$

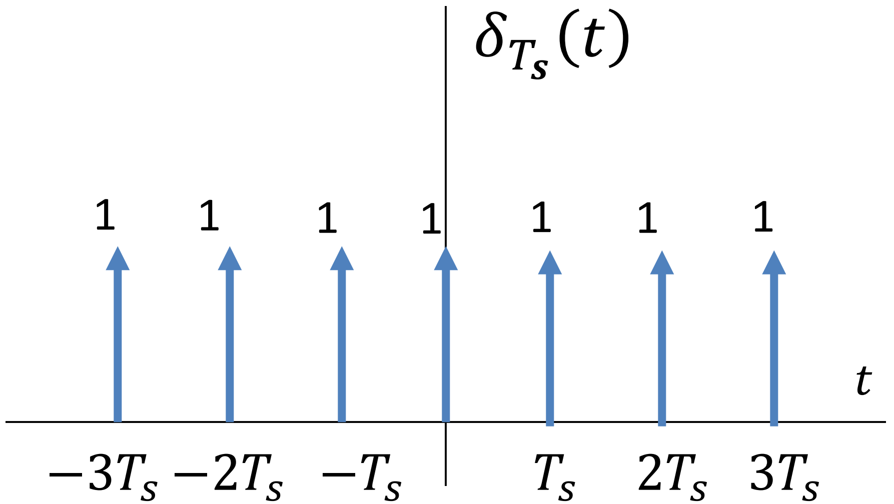


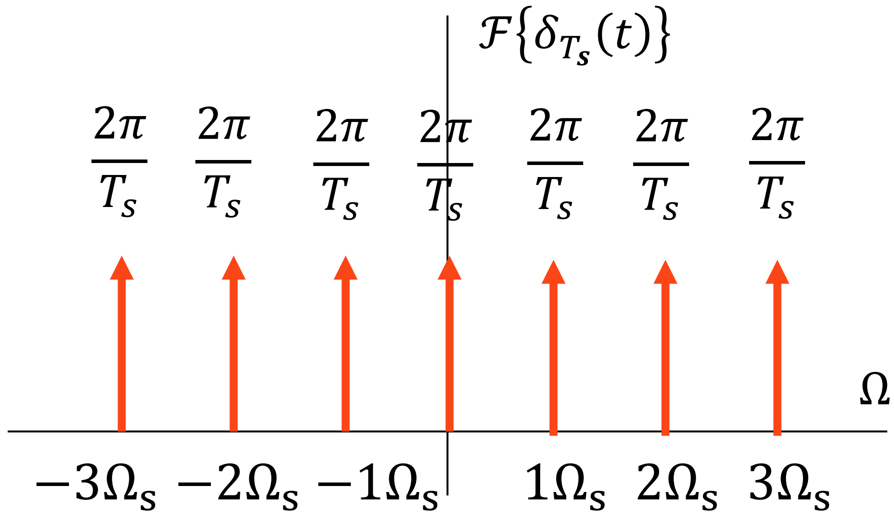


Sampling

$$\delta_{T_s}(t) \longleftrightarrow \frac{2\pi}{T_s}\delta_{\Omega_s}(\Omega)$$

$$x[n]\delta_{T_s}(t) \longleftrightarrow \frac{1}{T_s}\sum_{k=-\infty}^{\infty} X(\Omega - \Omega_s k)$$





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- Review of Sampling
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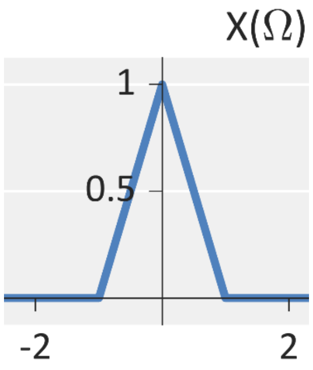
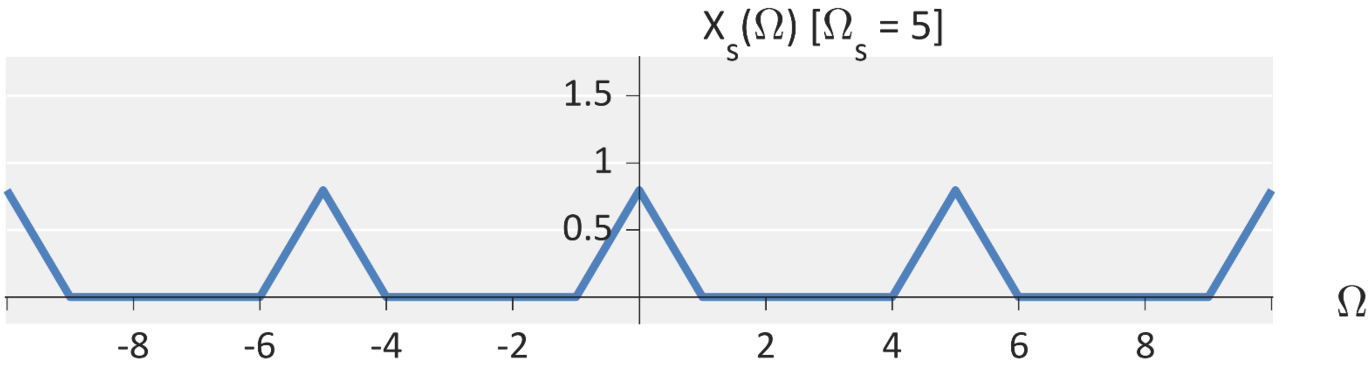
Sampling

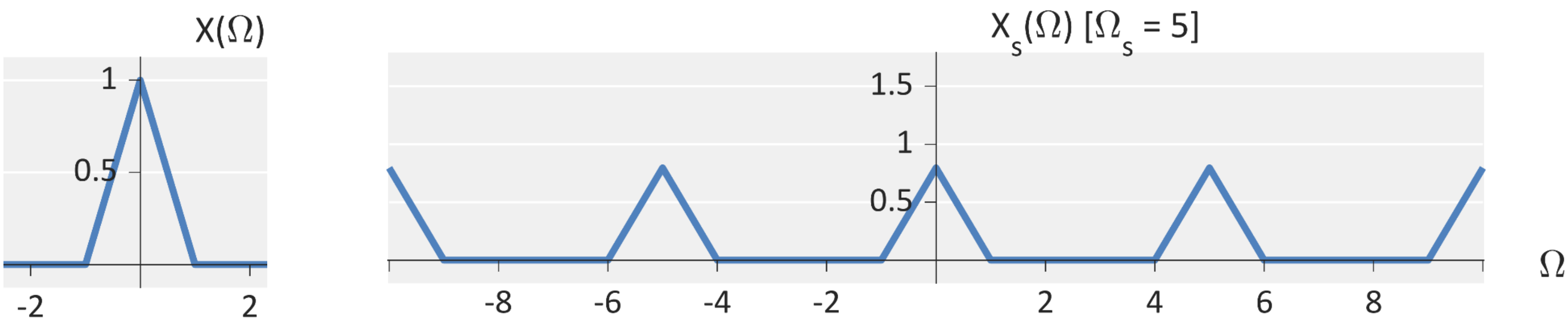
- **Question: Can I preserve all information when I sample?**
  - Yes!





Sampling

- **Question: How fast do I sample to preserve information?**



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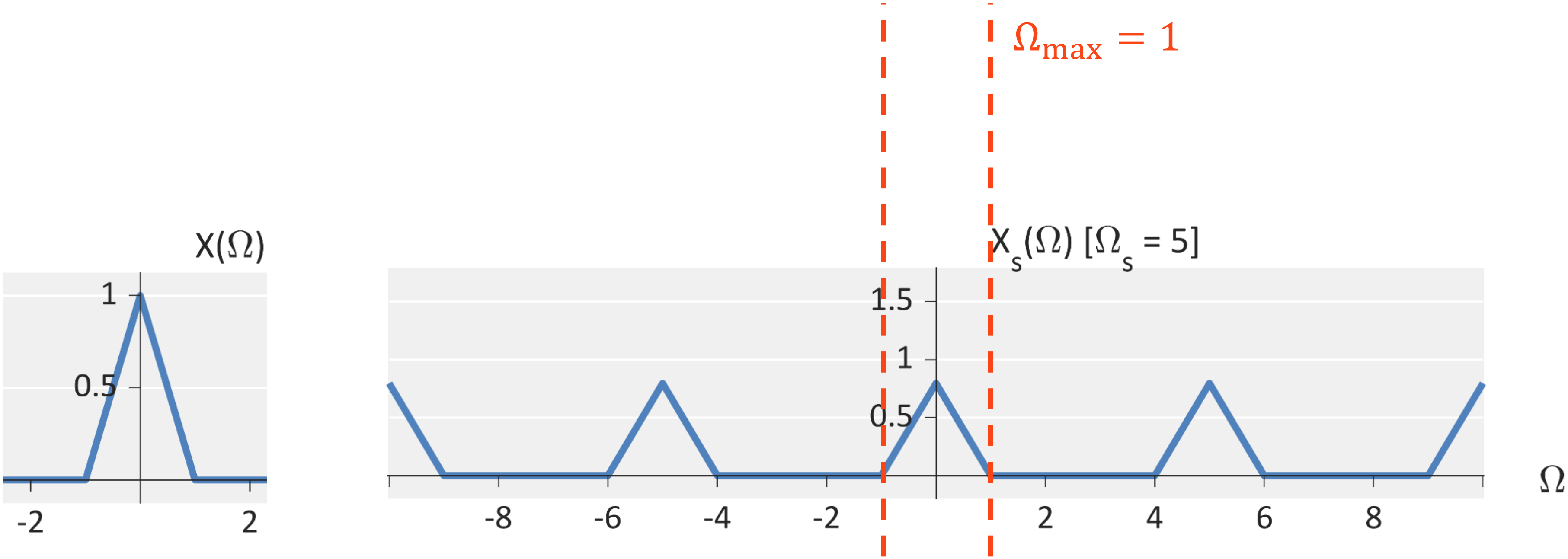

Sampling

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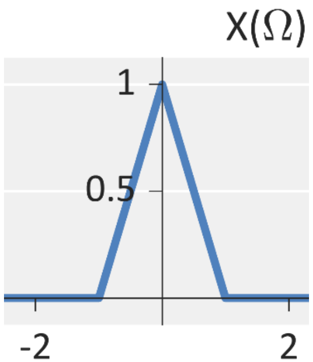

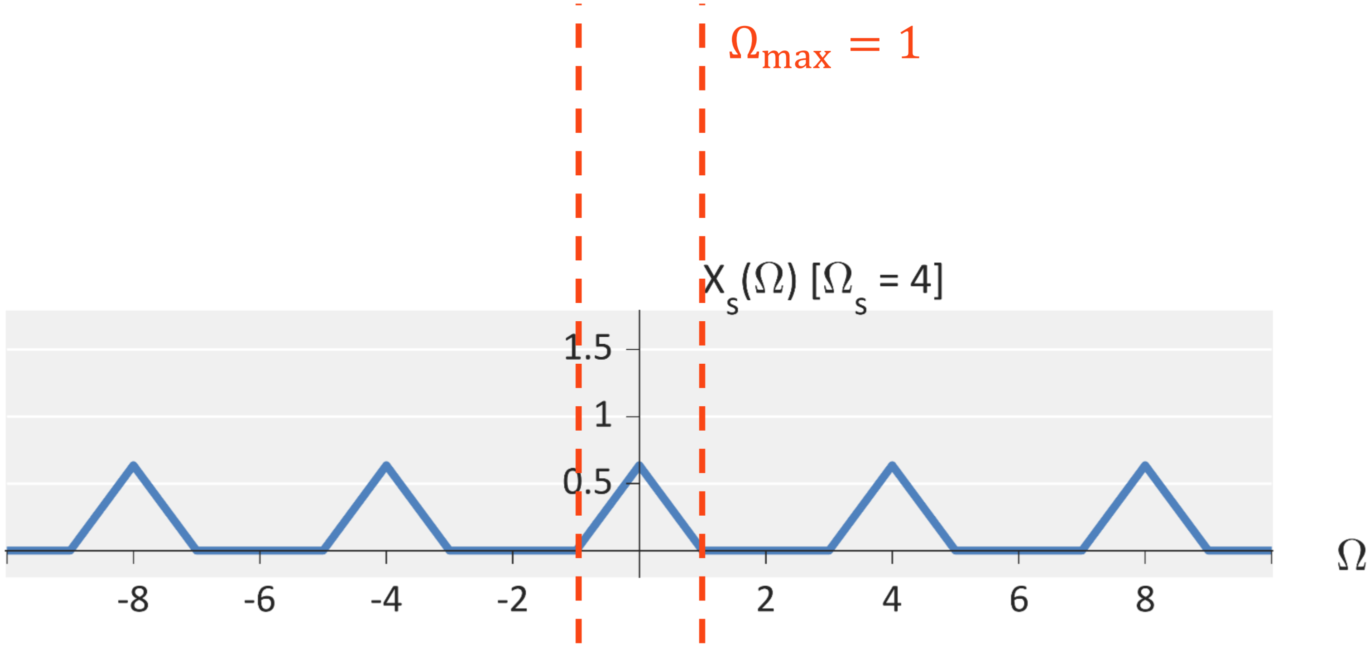

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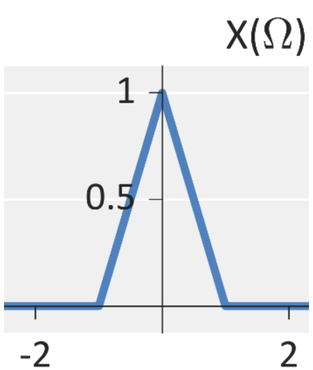

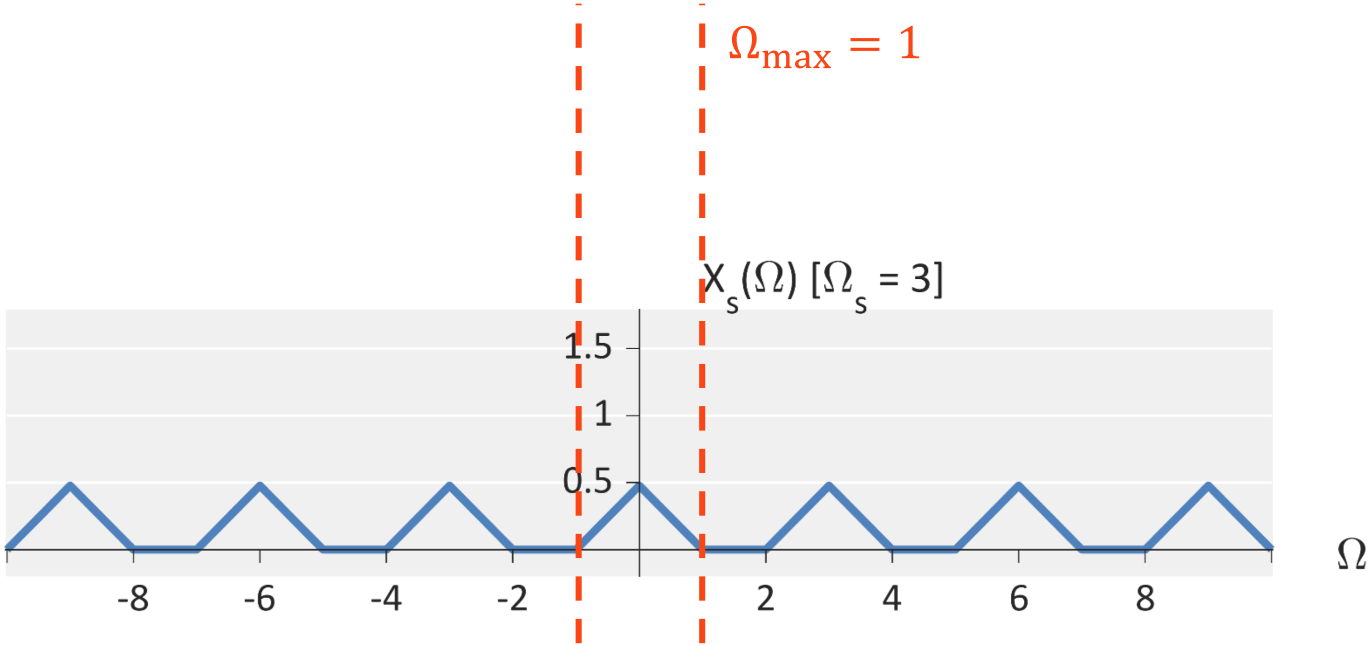

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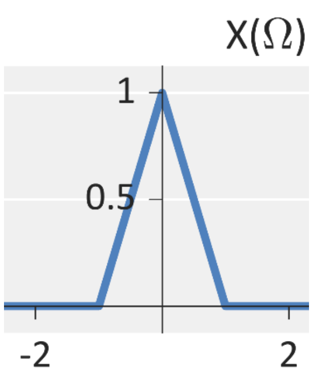
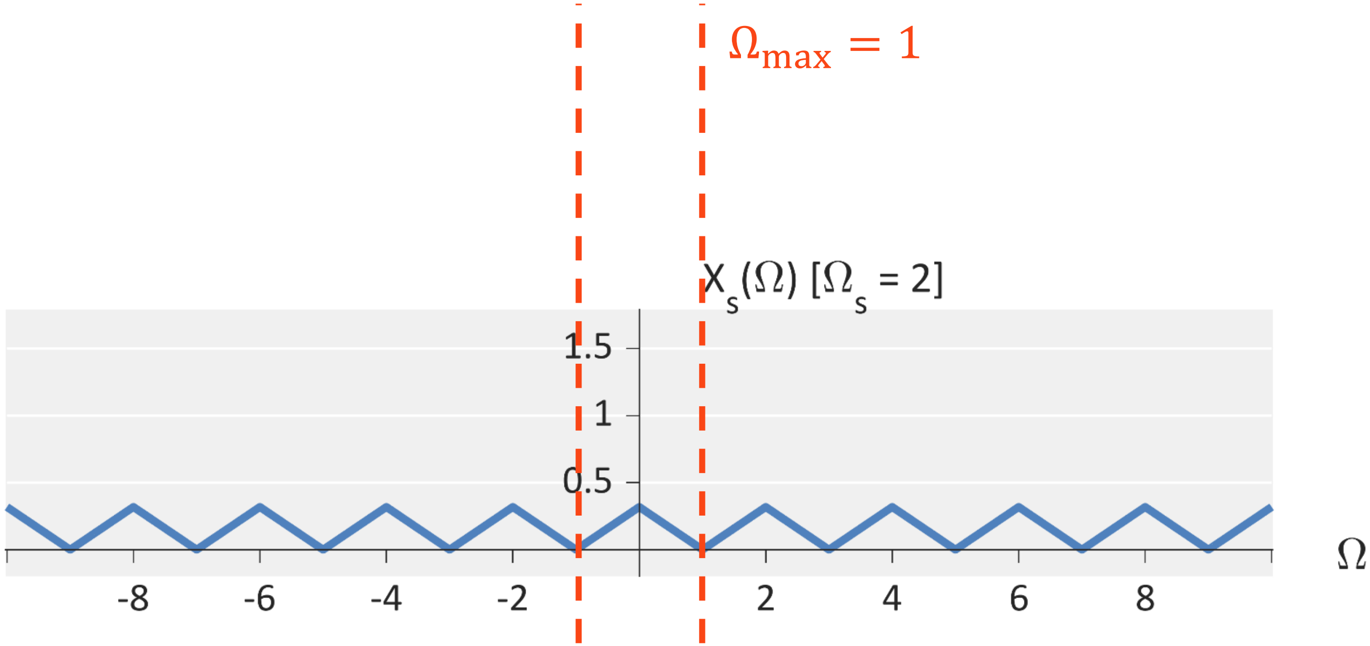
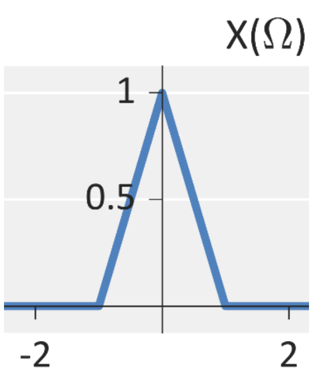


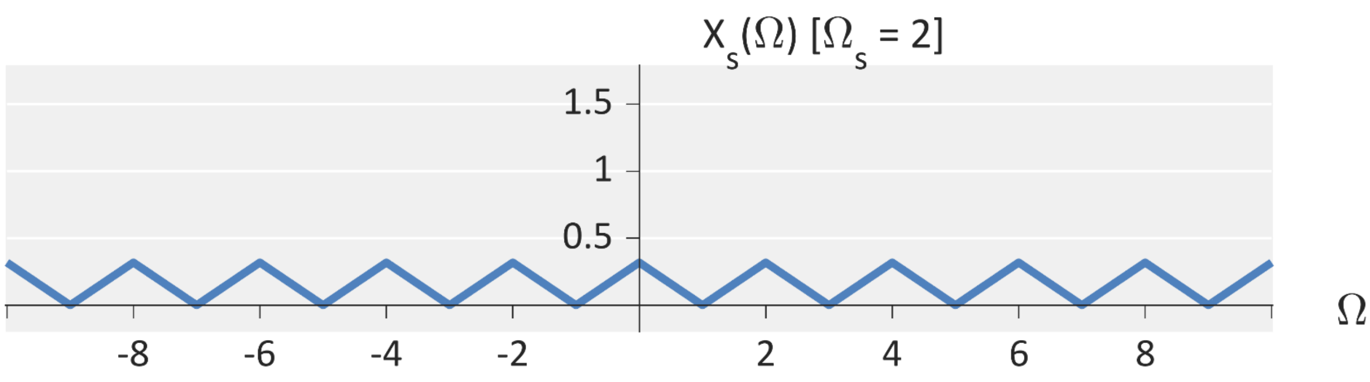


Sampling

- **Question: How fast do I sample to preserve information?**
  - We need to sample twice as fast as the maximum frequency

$$\Omega_s > 2\Omega_{max}$$





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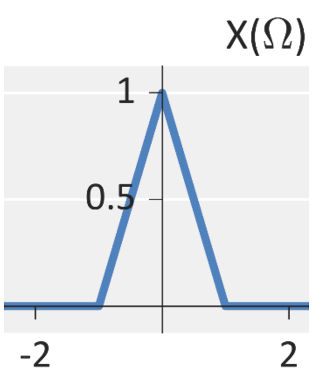
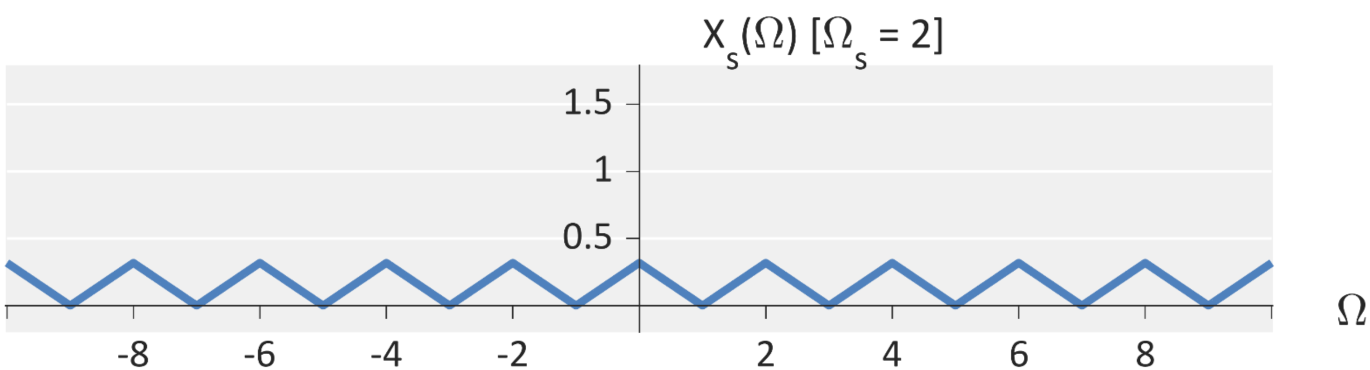
$$\Omega_s > 2\Omega_{max}$$

$$f_s > 2f_{max}$$

Nyquist-Shannon Sampling Theorem





Sampling

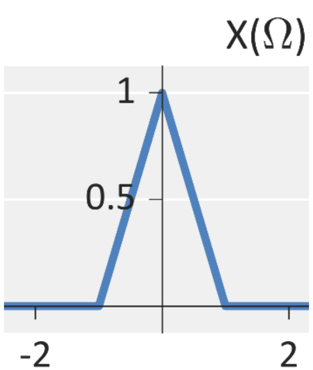
- **Question: How fast do I sample to preserve information?**
  - We need to sample twice as fast as the maximum frequency

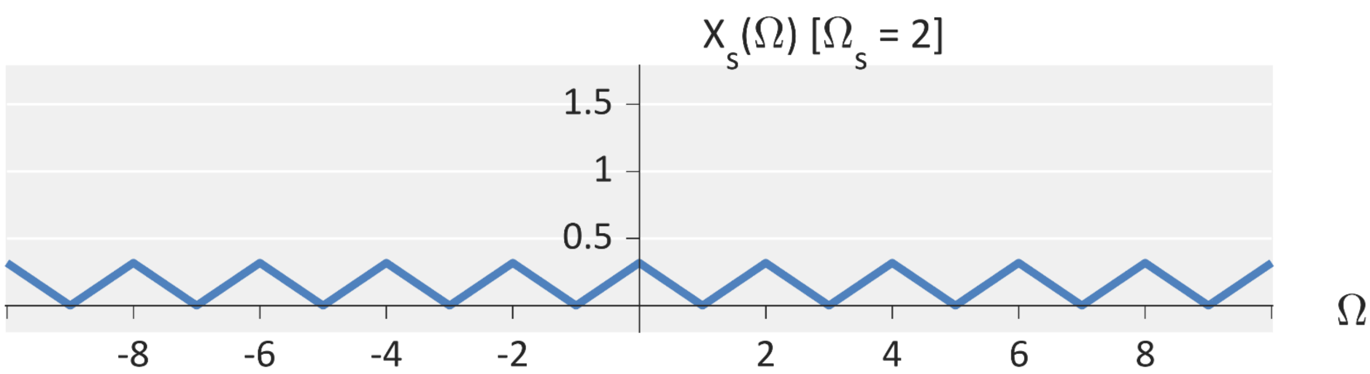
$\Omega_s > 2\Omega_{max}$

$f_s > 2f_{max}$

Nyquist-Shannon Sampling Theorem

$\Omega_N = 2\Omega_{max}$ <- *Nyquist Rate*





Example

- **Example: Consider the signal $x(t) = \cos(5\pi t)$.**
  - What is the Nyquist rate?
  - Sketch the Fourier transform $X_s(\Omega)$ of the sampled signal when $\Omega_s = 12\pi$

Example

- **Example: Consider the frequency signal**

$$X(\Omega) = u(\omega + 5\pi) - u(\omega - 5\pi)$$

  - What is the Nyquist rate?
  - Sketch the Fourier transform $X_s(\Omega)$ of the sampled signal when $\Omega_s = 20\pi$

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Sampling

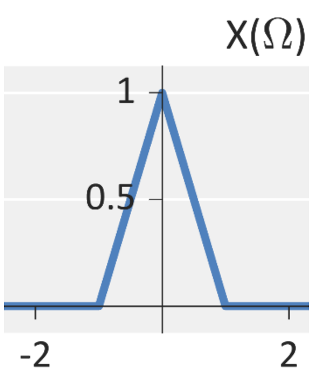
- **Question:** **How do I return to continuous–time?**

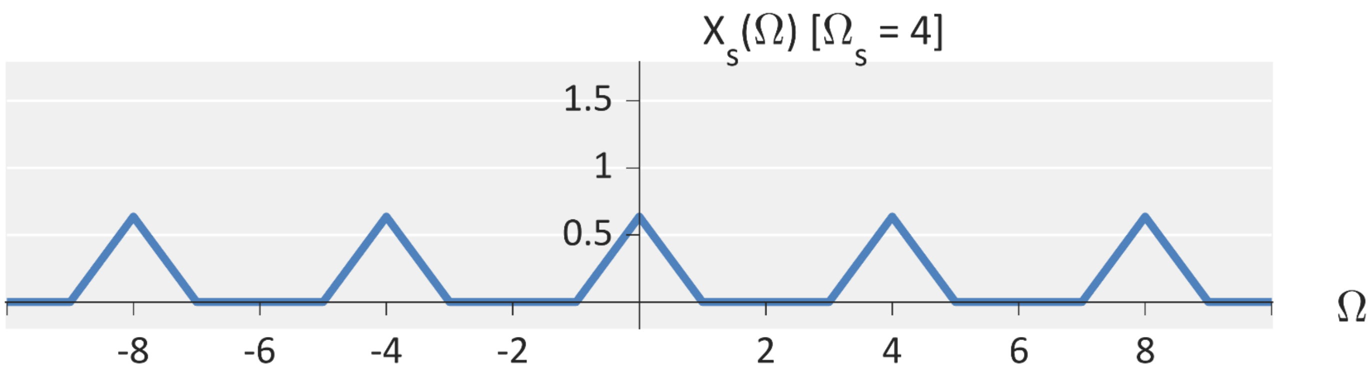
$\Omega_s > 2\Omega_{max}$

$f_s > 2f_{max}$

Nyquist-Shannon Sampling Theorem

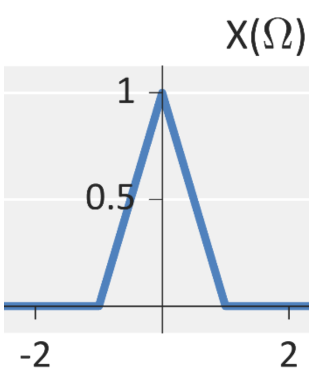
$\Omega_N = 2\Omega_{max}$ <- *Nyquist Rate*

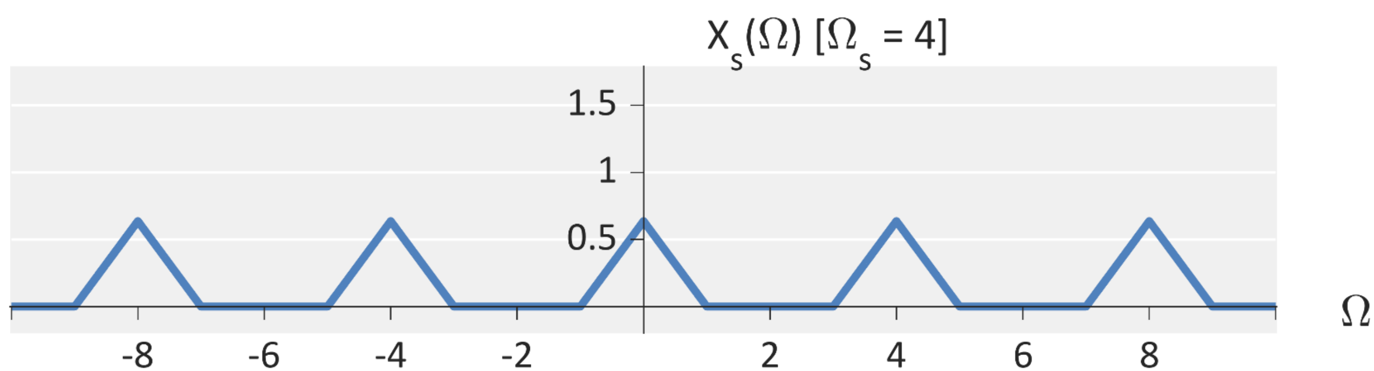




Sampling

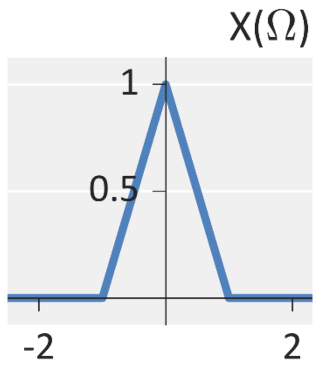
- **Question: How do I return to continuous–time?**
  - **Filter:** Low pass filter to keep $\Omega_s/2 \le \Omega \le \Omega_s/2$
  - **Amplify:** Amplify signal by T_s

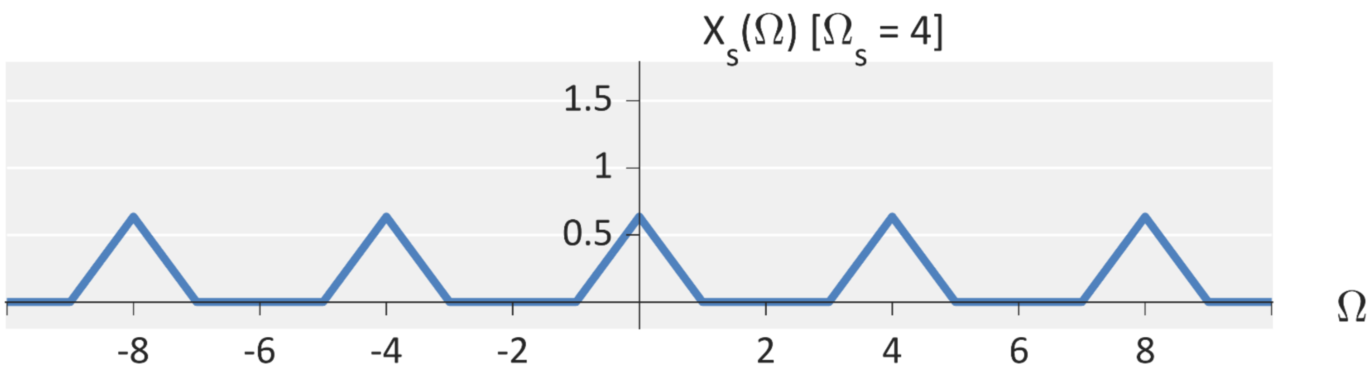




Sampling

- **Question: How do I return to continuous–time?**
  - Apply a low-pass reconstruction filter
    - ◊ Cut-off frequency: $\Omega_s/2$
    - ◊ Gain: T_s

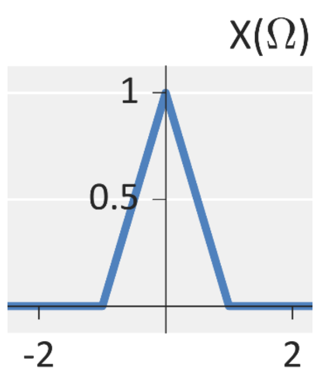


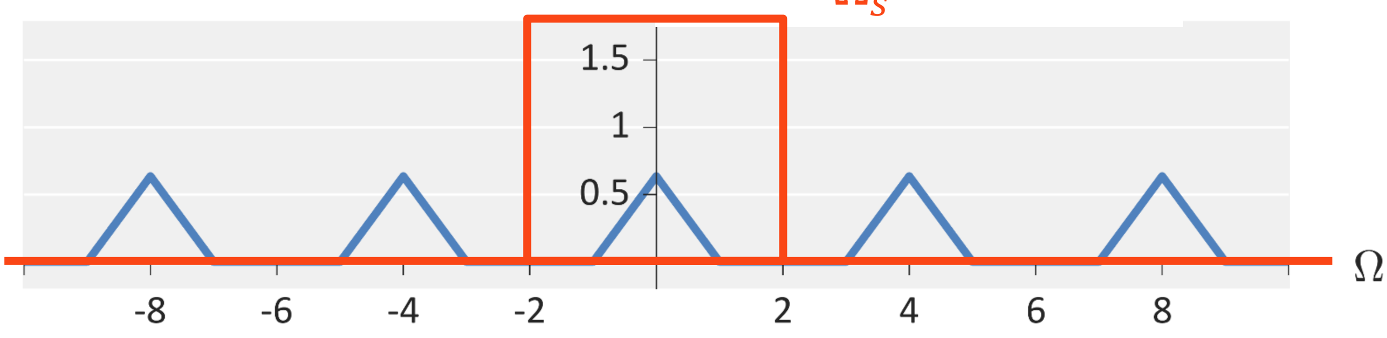


Sampling

- **Question: How do I return to continuous–time?**
  - Apply a low-pass reconstruction filter
    - Cut-off frequency: $\Omega_s/2$
    - Gain: T_s

$$T_s = \frac{2\pi}{\Omega_s} = 1.5708$$





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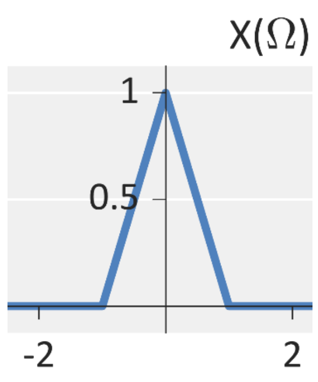
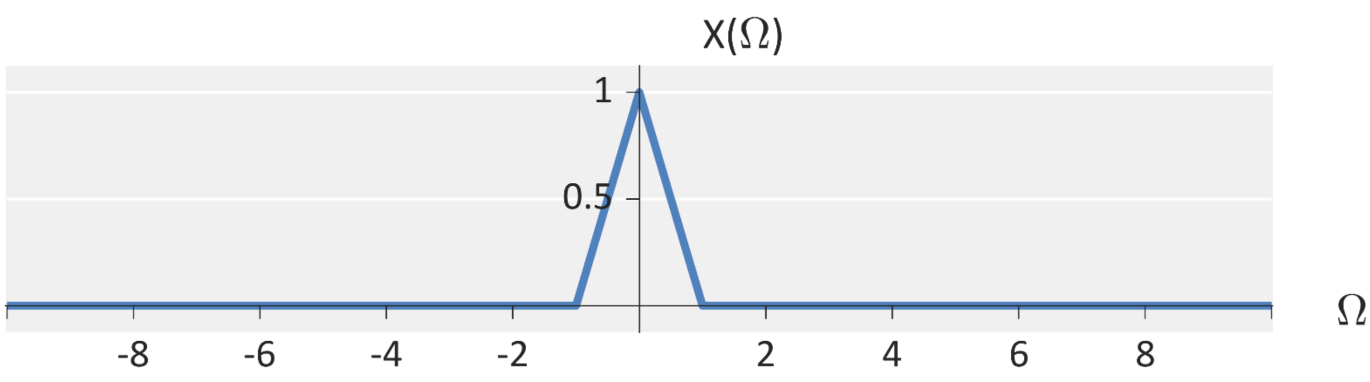


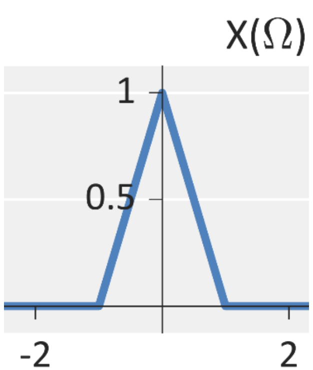


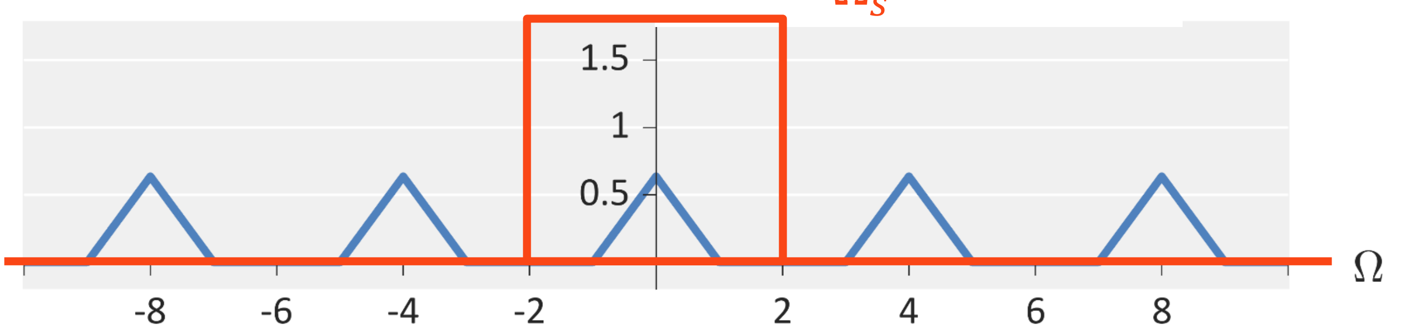
Sampling

- **Question: What is happening when I multiply in frequency?**

$$X(\Omega)\left[T_s[u(\Omega+\Omega_s/2)-u(\Omega-\Omega_s/2)]\right]$$

$$T_s = \frac{2\pi}{\Omega_s} = 1.5708$$

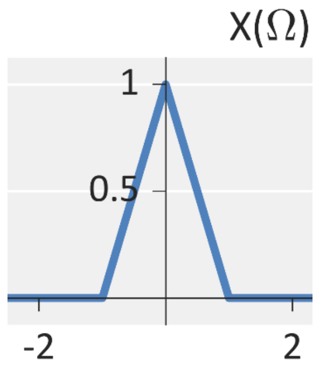




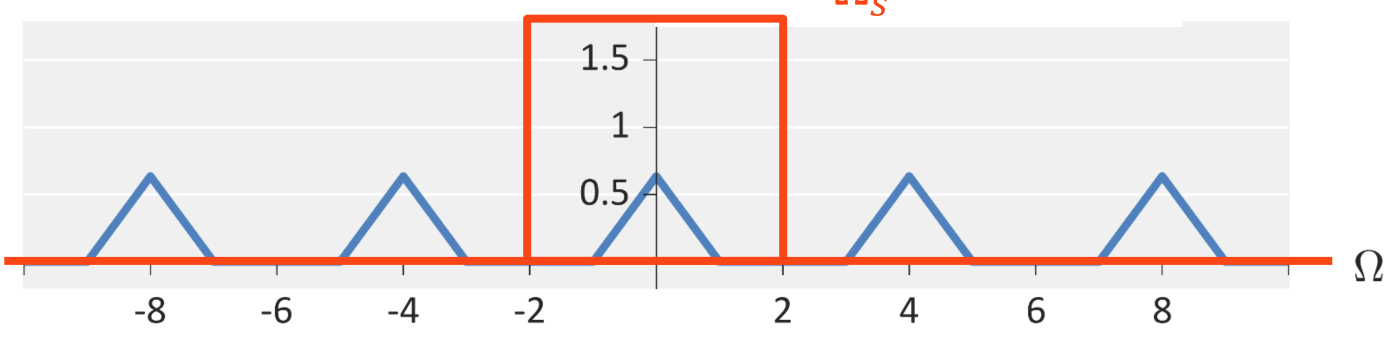
Sampling

- **Question:** **What is happening when I multiply in frequency?**

$$X(\Omega)\left[T_s X(\Omega)\text{rect}\left(\frac{\Omega}{\Omega_s}\right)\right]$$



$$T_s = \frac{2\pi}{\Omega_s} = 1.5708$$



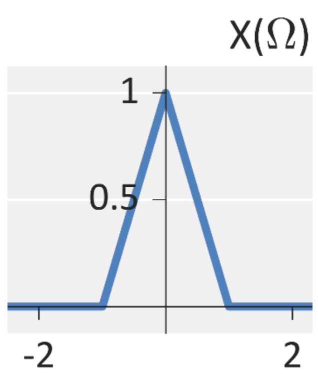
Sampling

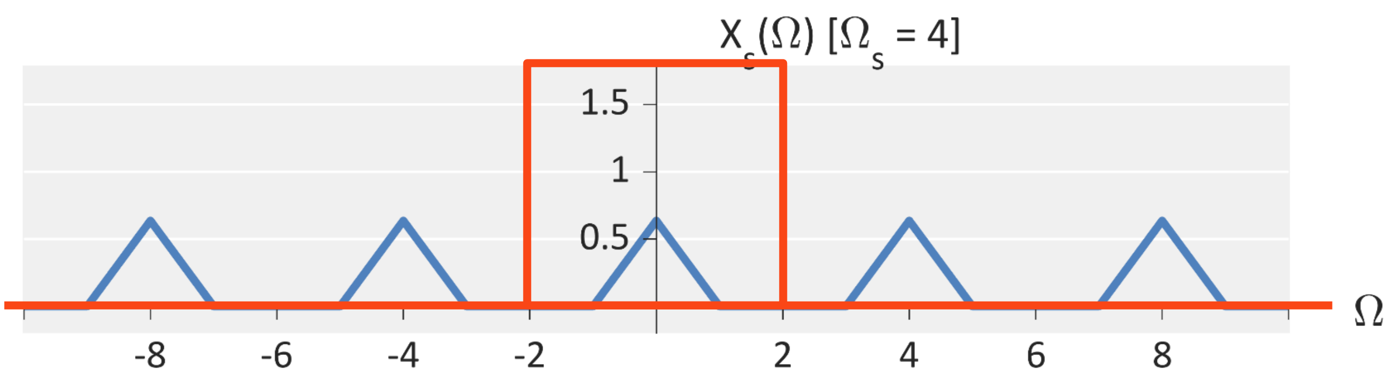
- **Question: What is happening when I multiply in frequency?**

$$T_s X(\Omega)\mathrm{rect}\left(\frac{\Omega}{2(\Omega_s/2)}\right)$$

$$\Updownarrow$$

$$x[n] * \frac{T_s(\Omega_s/2)}{\pi}\mathrm{sinc}\big((\Omega_s/2)t\big)$$





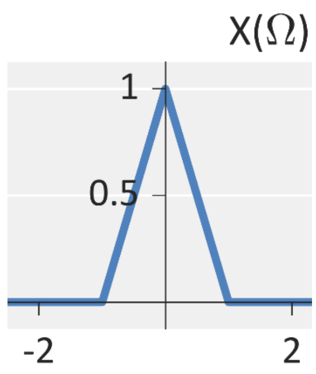
Sampling

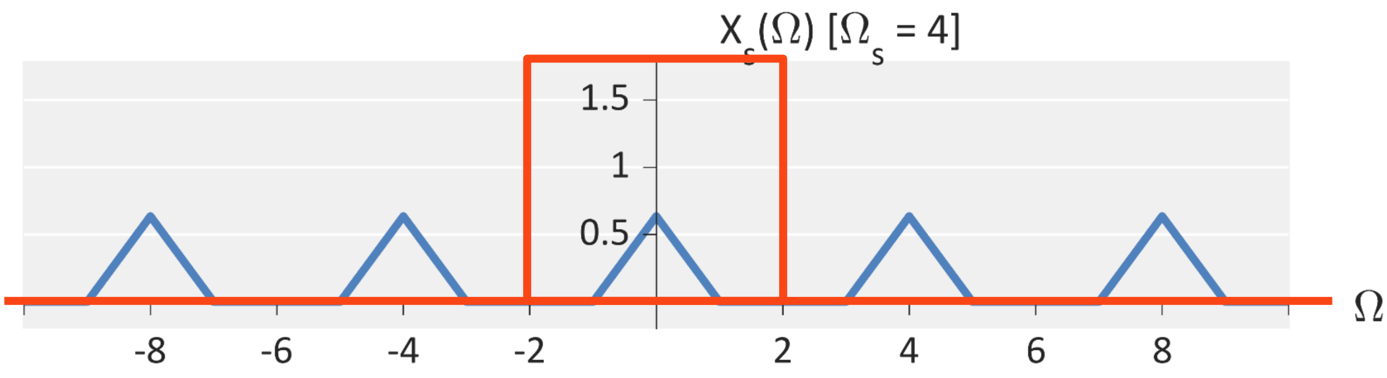
- **Question: What is happening when I multiply in frequency?**

$$T_s X(\Omega)\mathrm{rect}\left(\frac{\Omega}{2(\Omega_s/2)}\right)$$

$$\updownarrow$$

$$x[n] * \frac{T_s(\Omega_s/2)}{\pi}\mathrm{sinc}\big((\Omega_s/2)t\big) = x[n] * \mathrm{sinc}\big((\Omega_s/2)t\big)$$





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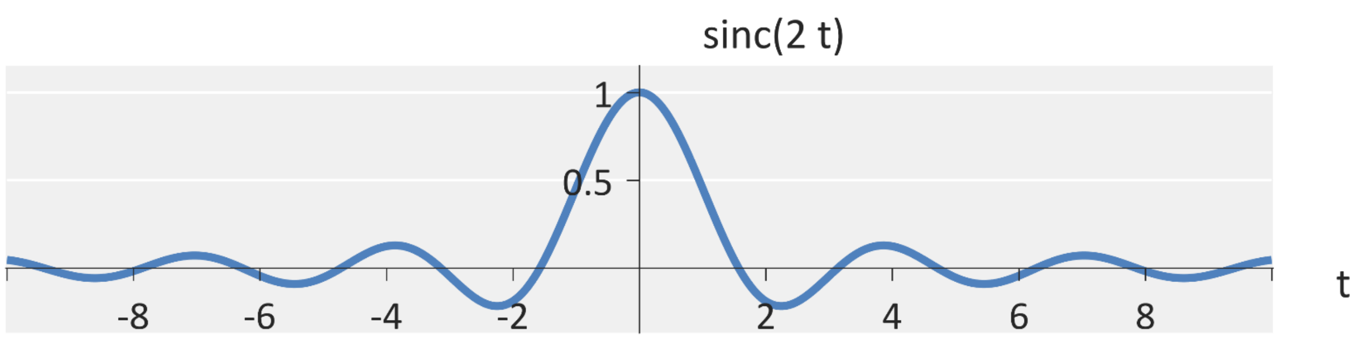

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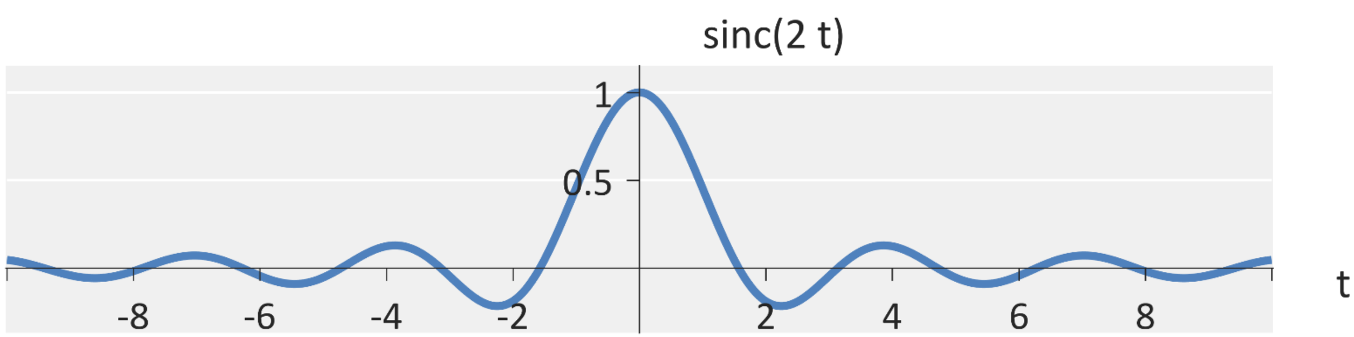
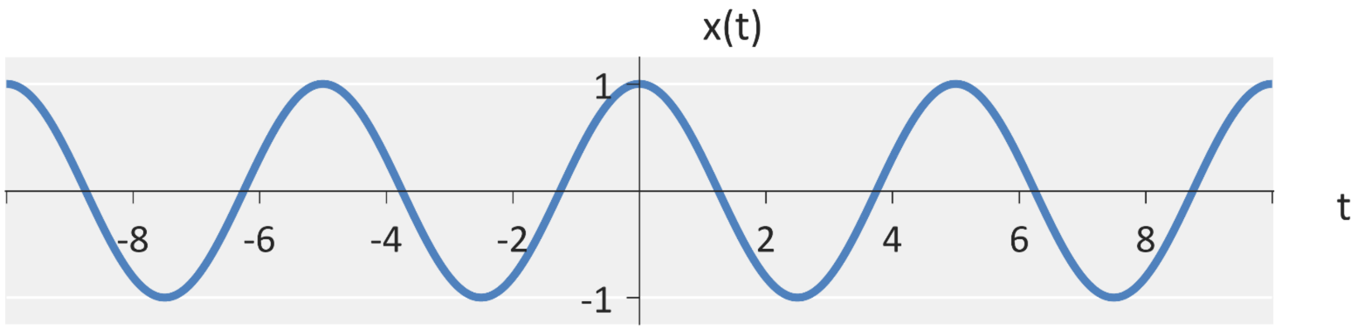
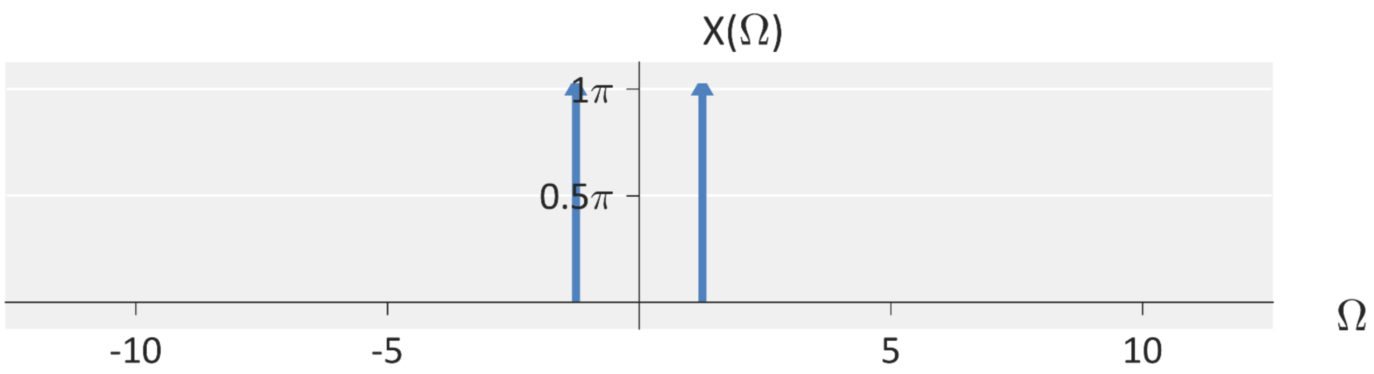


Reconstruction

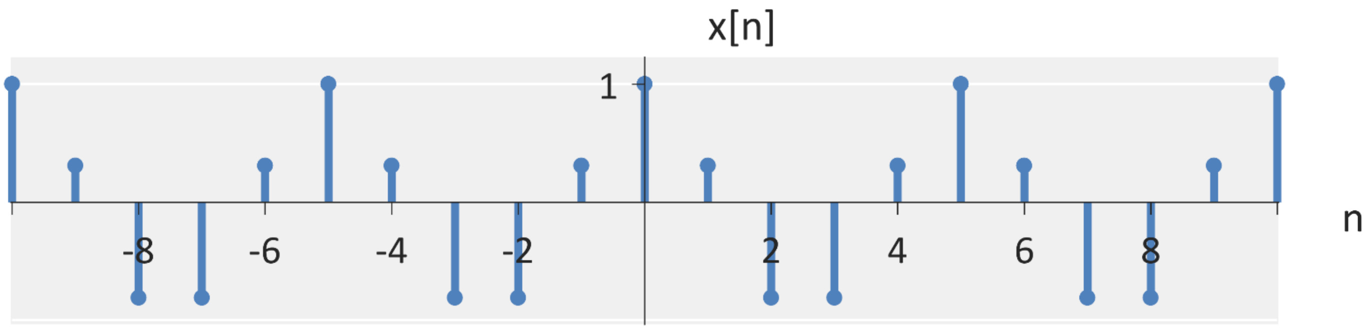
- **Consider a cosine**

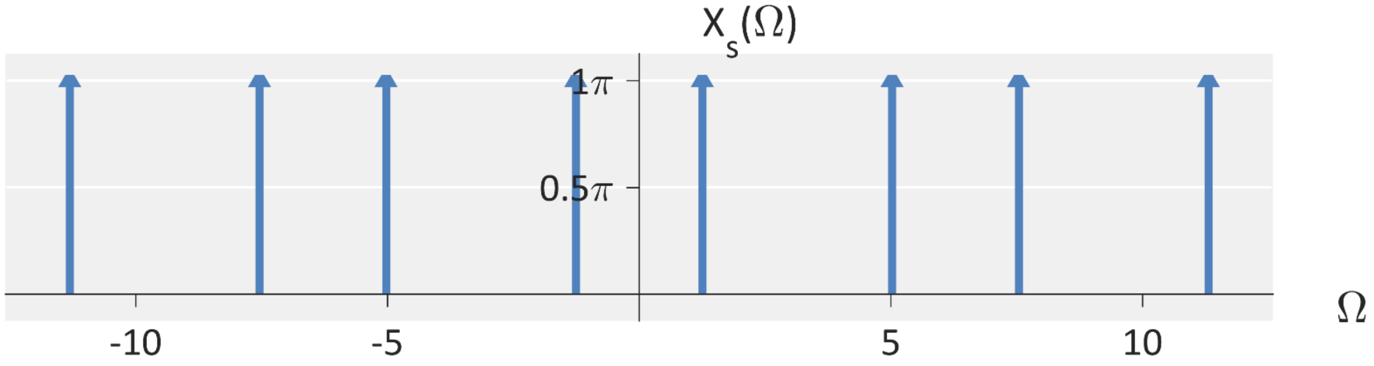




Reconstruction

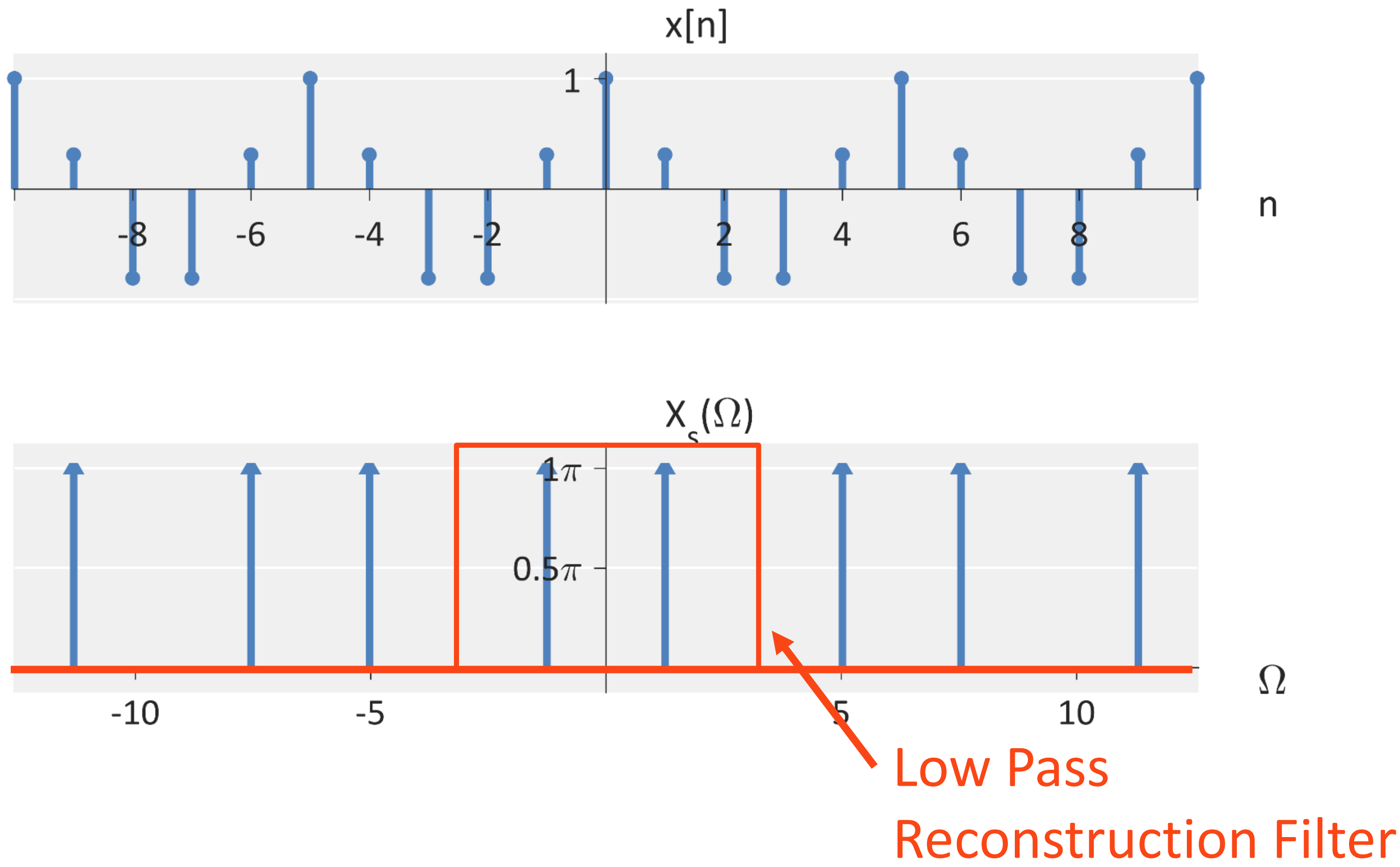
- **Consider a cosine – Sampled at $T_s = 1$**





Reconstruction

- **Consider a cosine –** **Low pass filter to keep $-\Omega_s/2$ to $\Omega_s/2$**



Reconstruction

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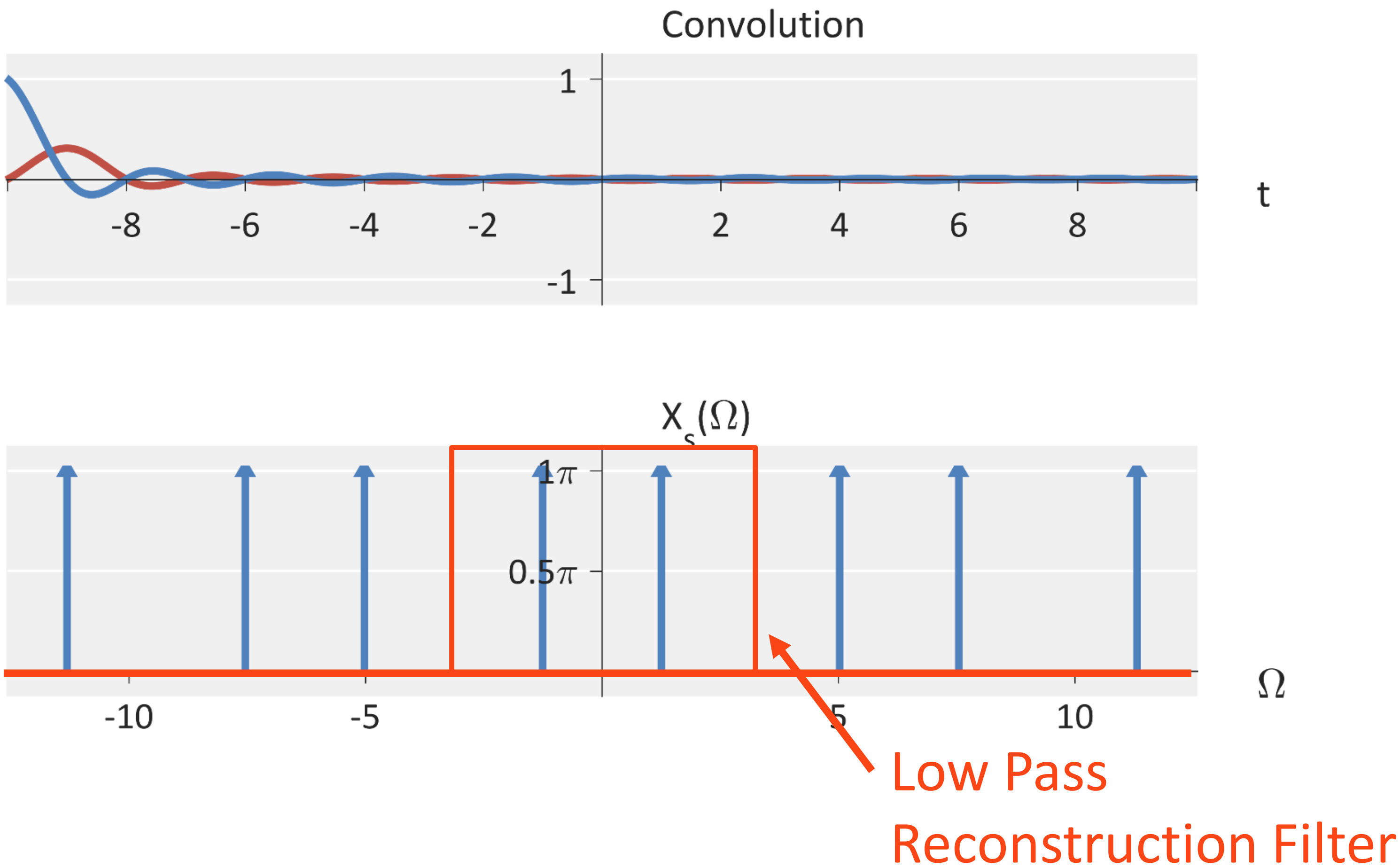

Reconstruction

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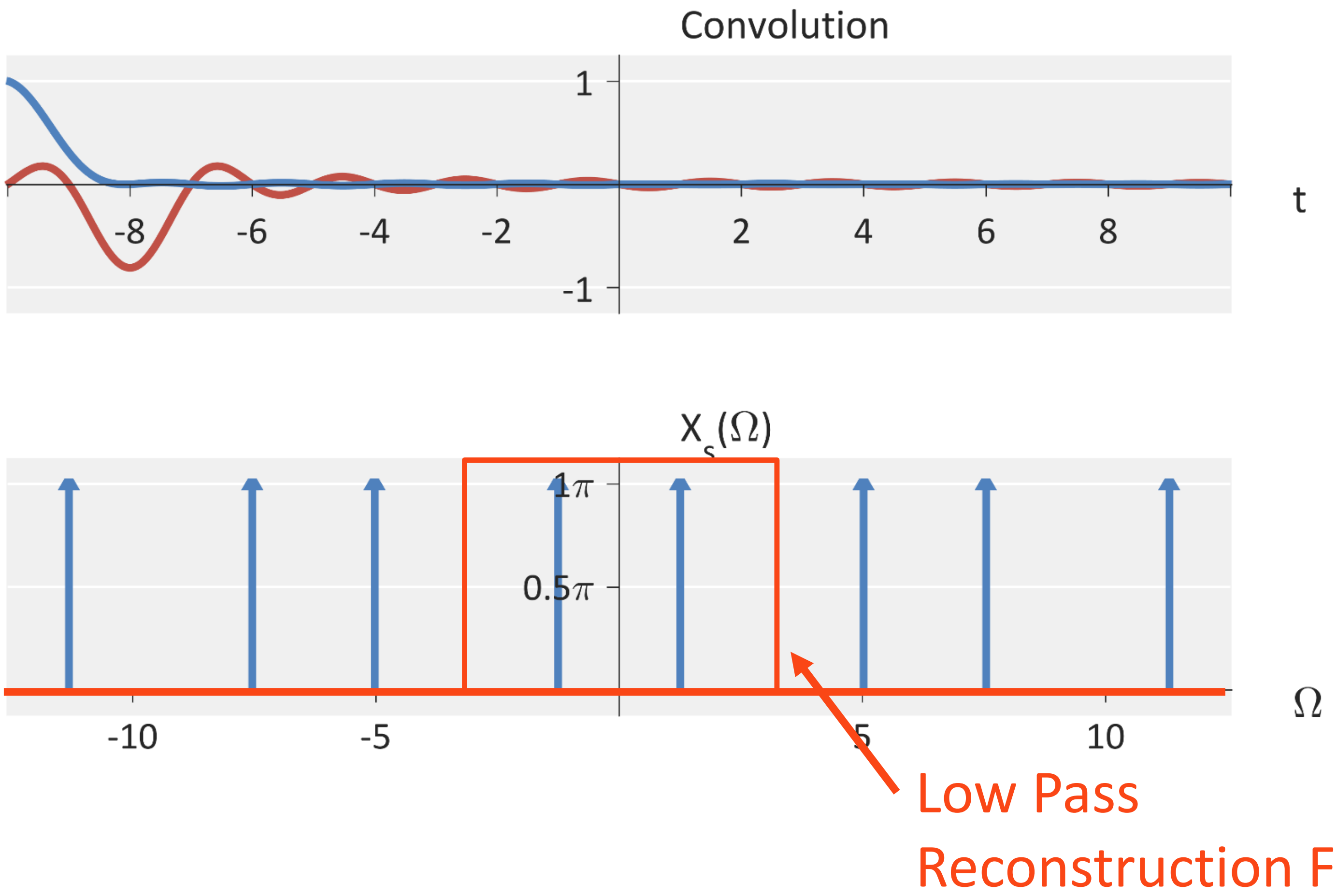

Reconstruction

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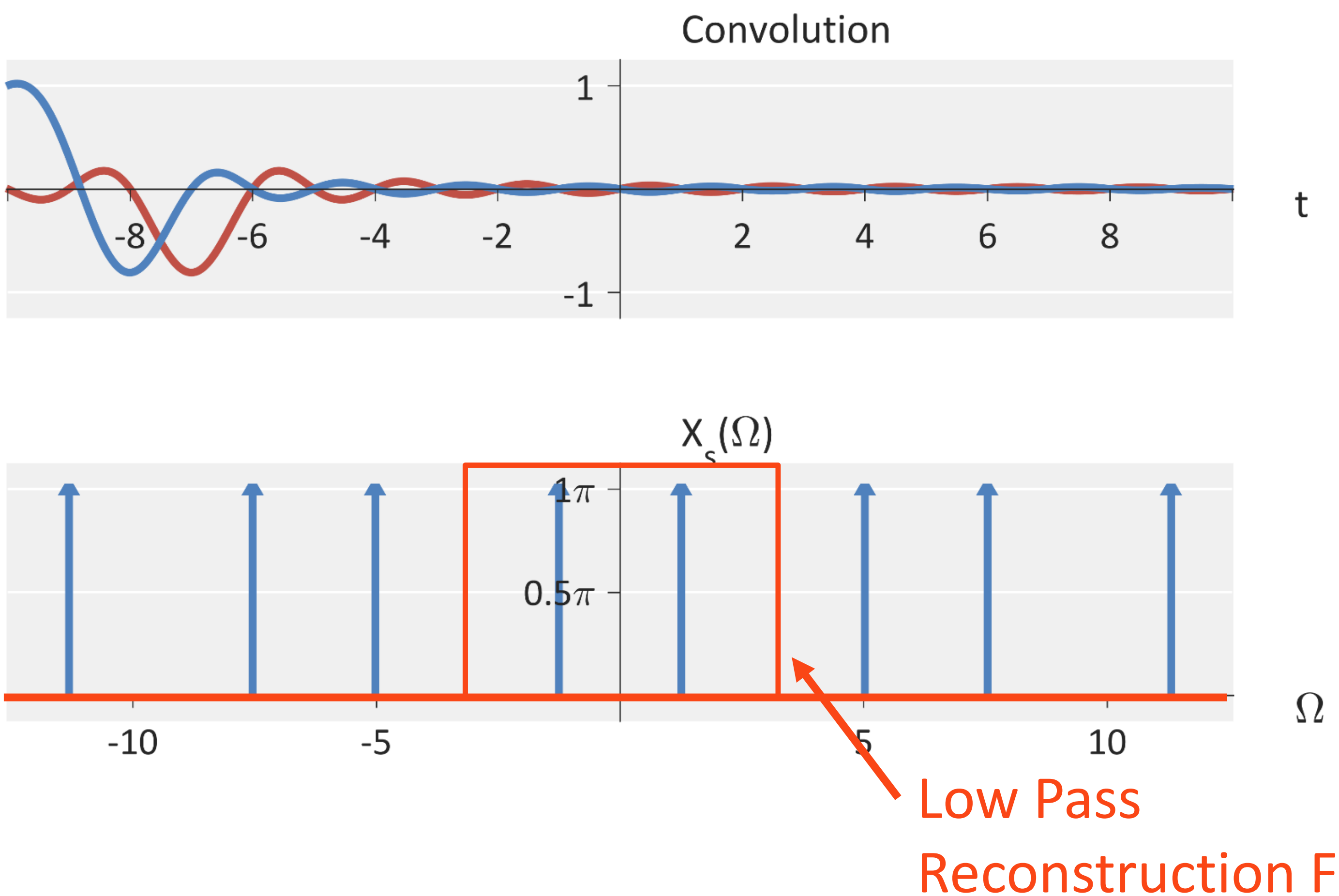

Reconstruction

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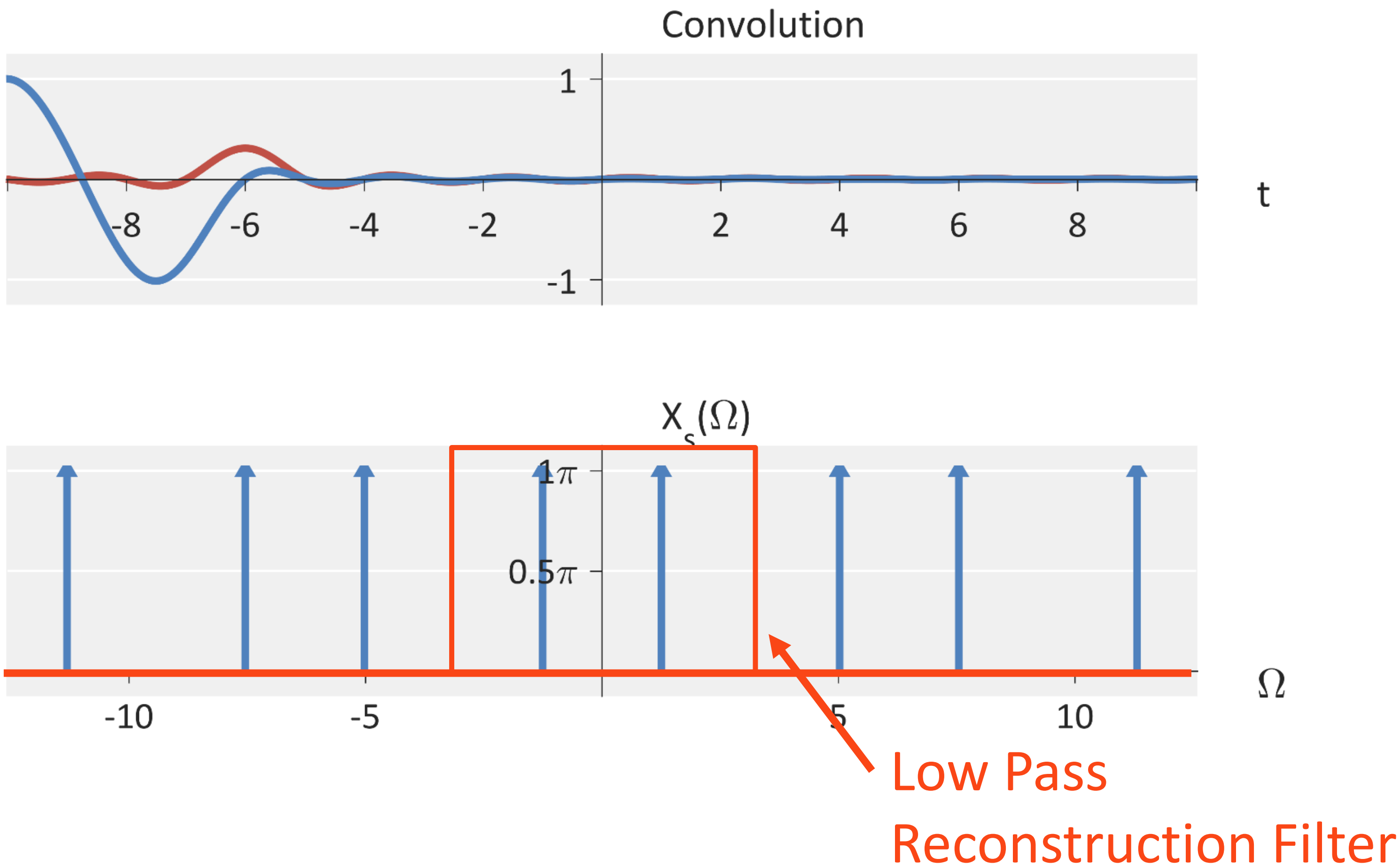

Reconstruction

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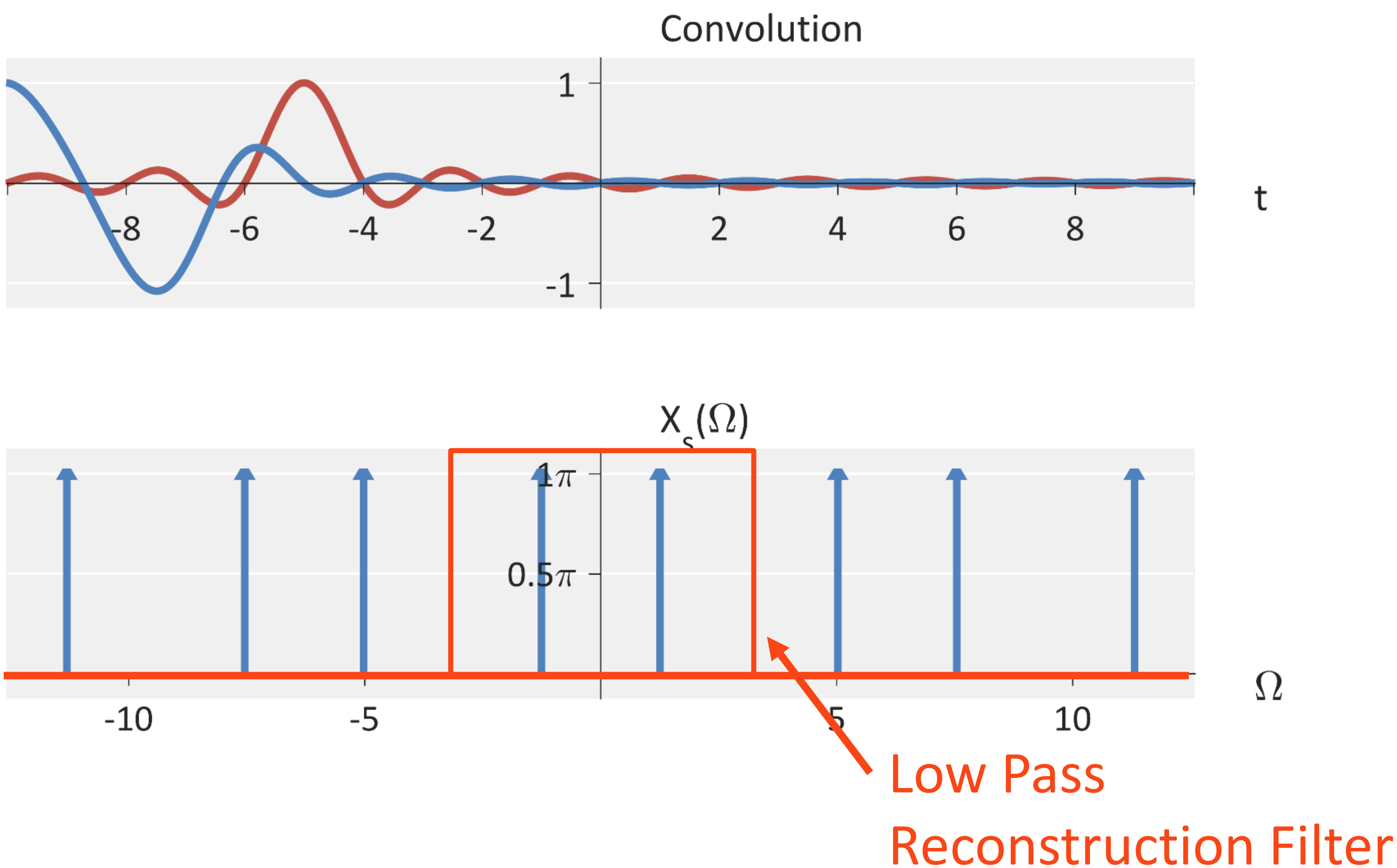
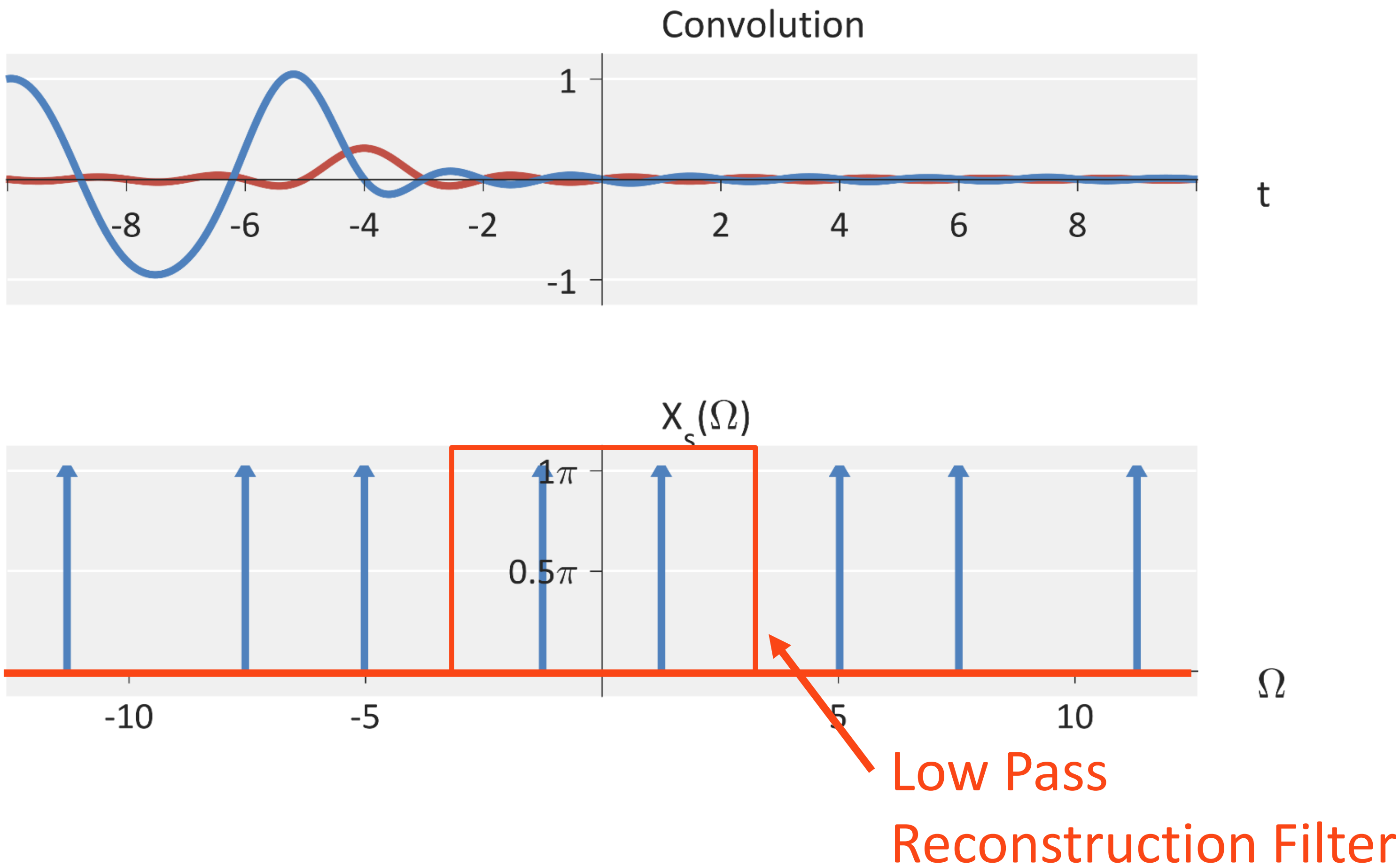


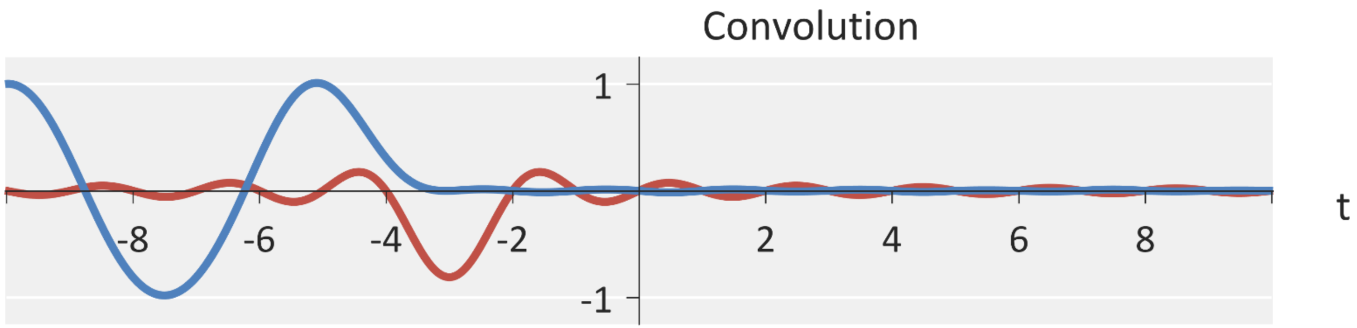
Reconstruction

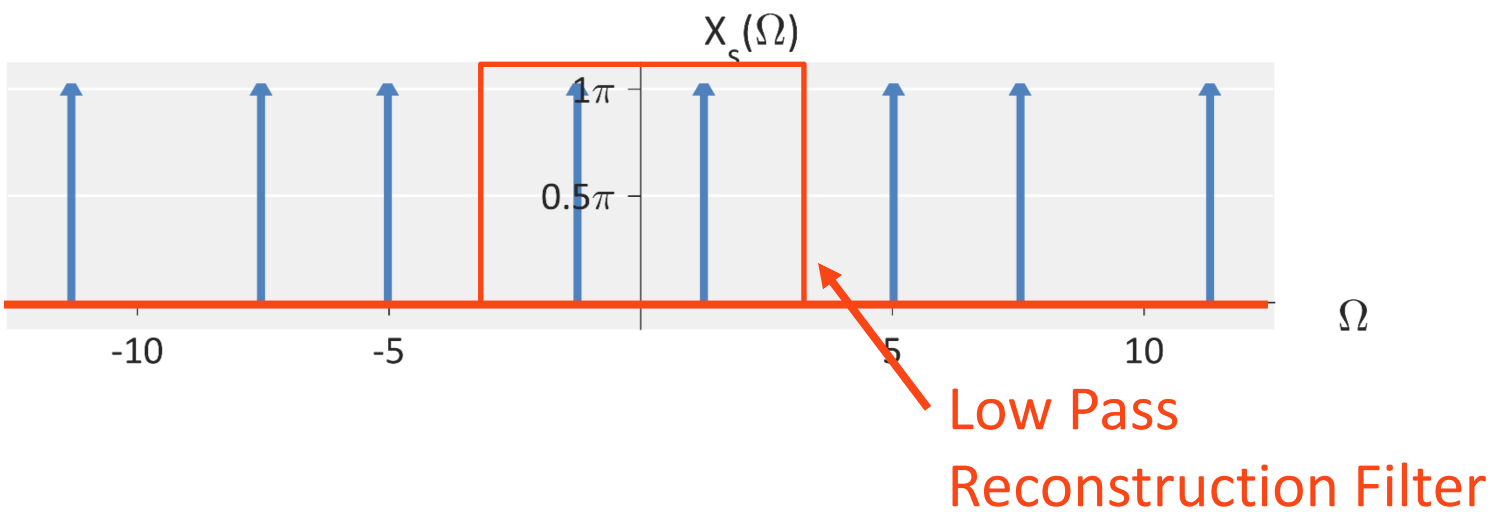
- **Consider a cosine – Low pass filter to keep $-\Omega_s/2$ to $\Omega_s/2$**



Reconstruction

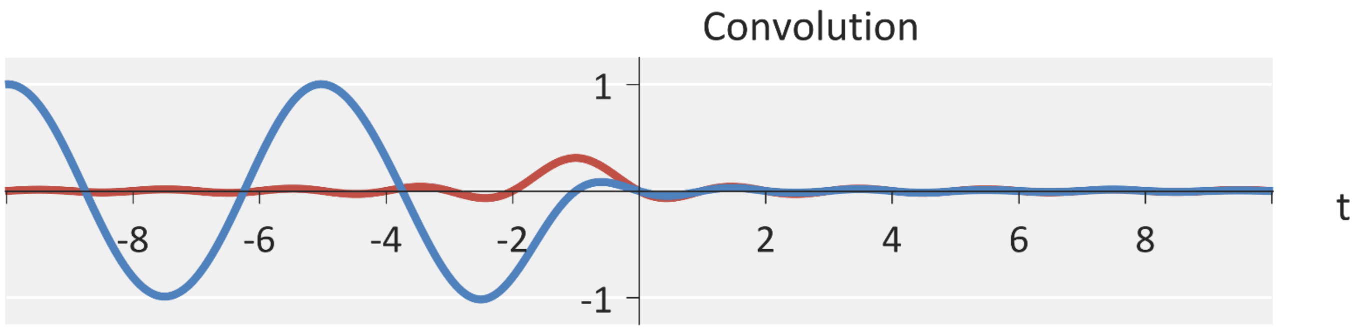
- **Consider a cosine –** **Low pass filter to keep $-\Omega_s/2$ to $\Omega_s/2$**

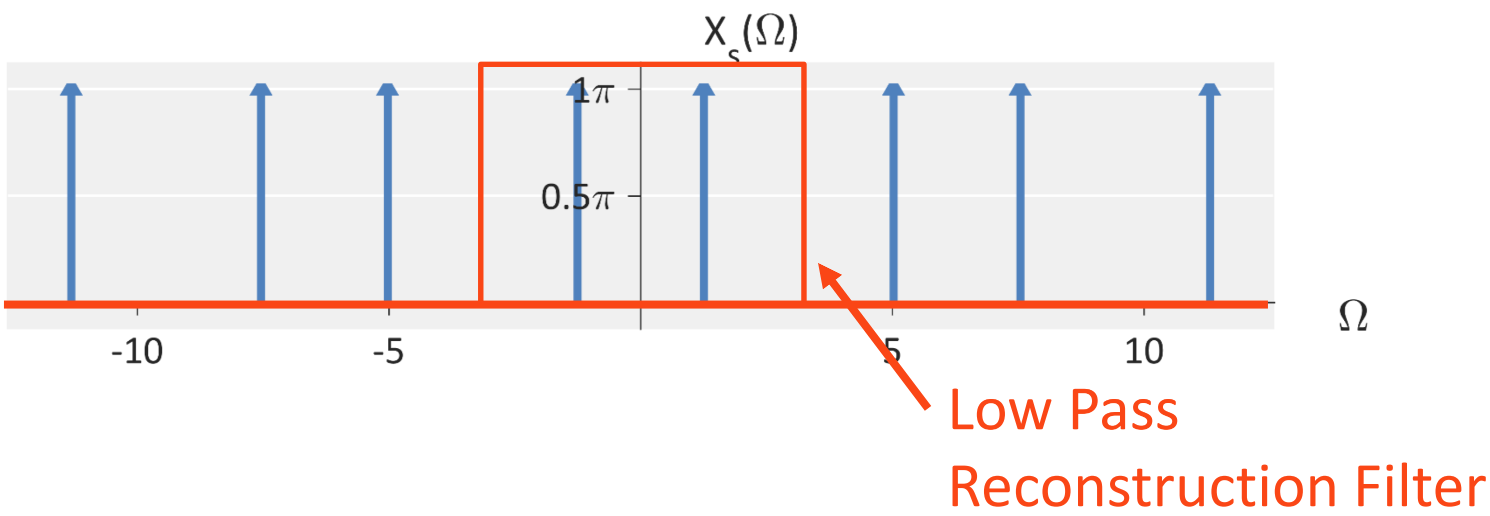




Reconstruction

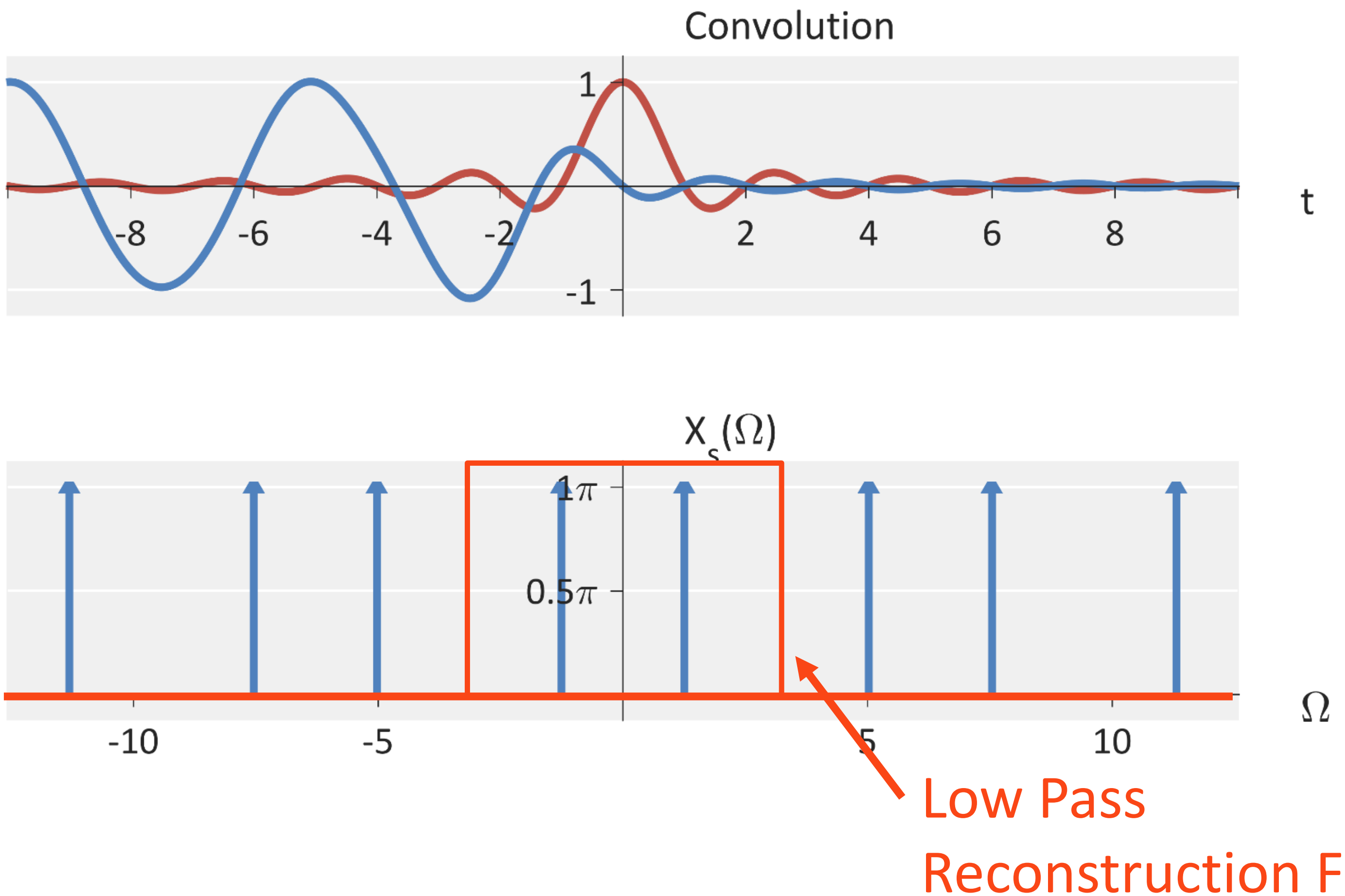
- **Consider a cosine – Low pass filter to keep $-\Omega_s/2$ to $\Omega_s/2$**





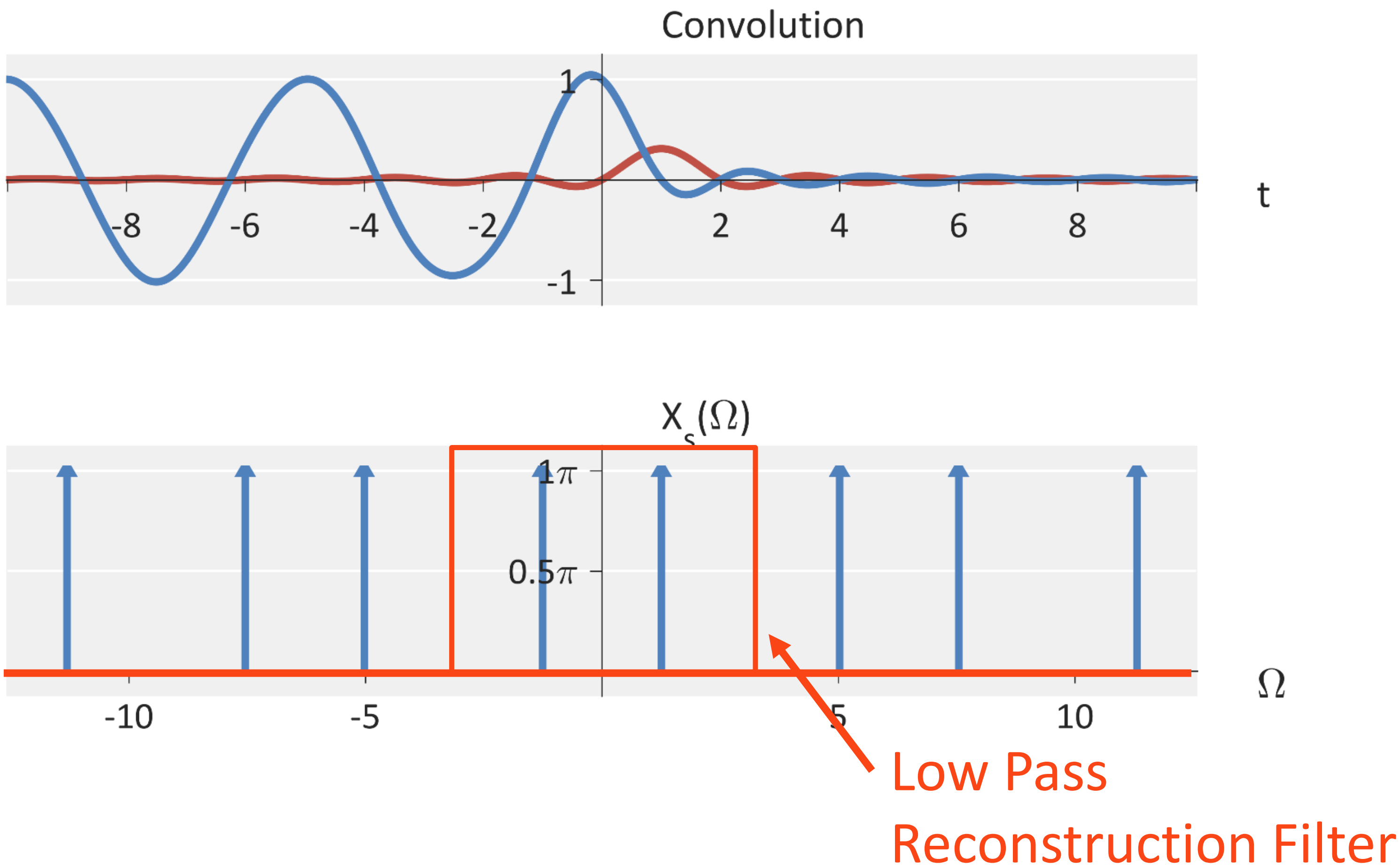
Reconstruction

- **Consider a cosine – Low pass filter to keep $-\Omega_s/2$ to $\Omega_s/2$**



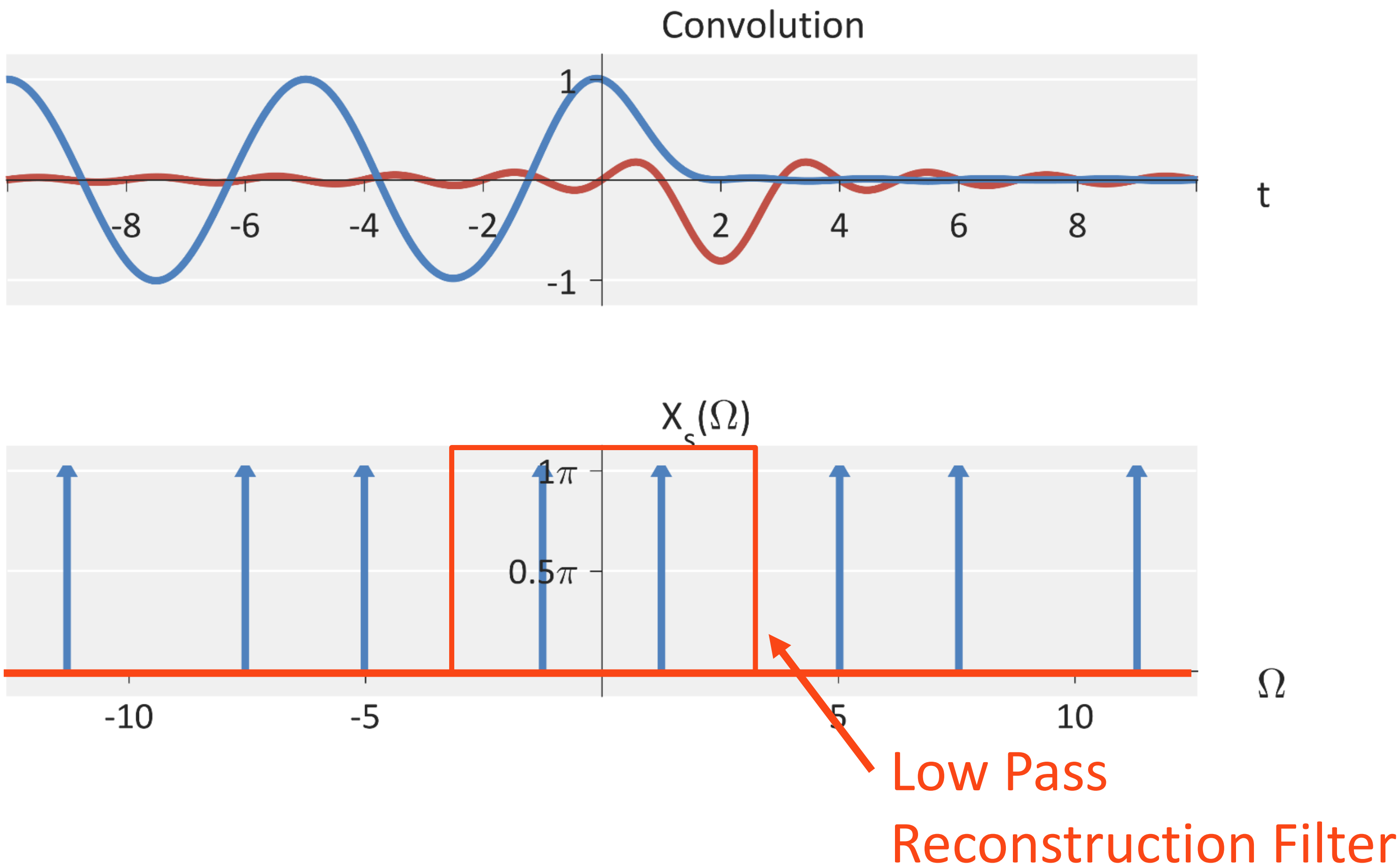
Reconstruction

- **Consider a cosine – Low pass filter to keep $-\Omega_s/2$ to $\Omega_s/2$**



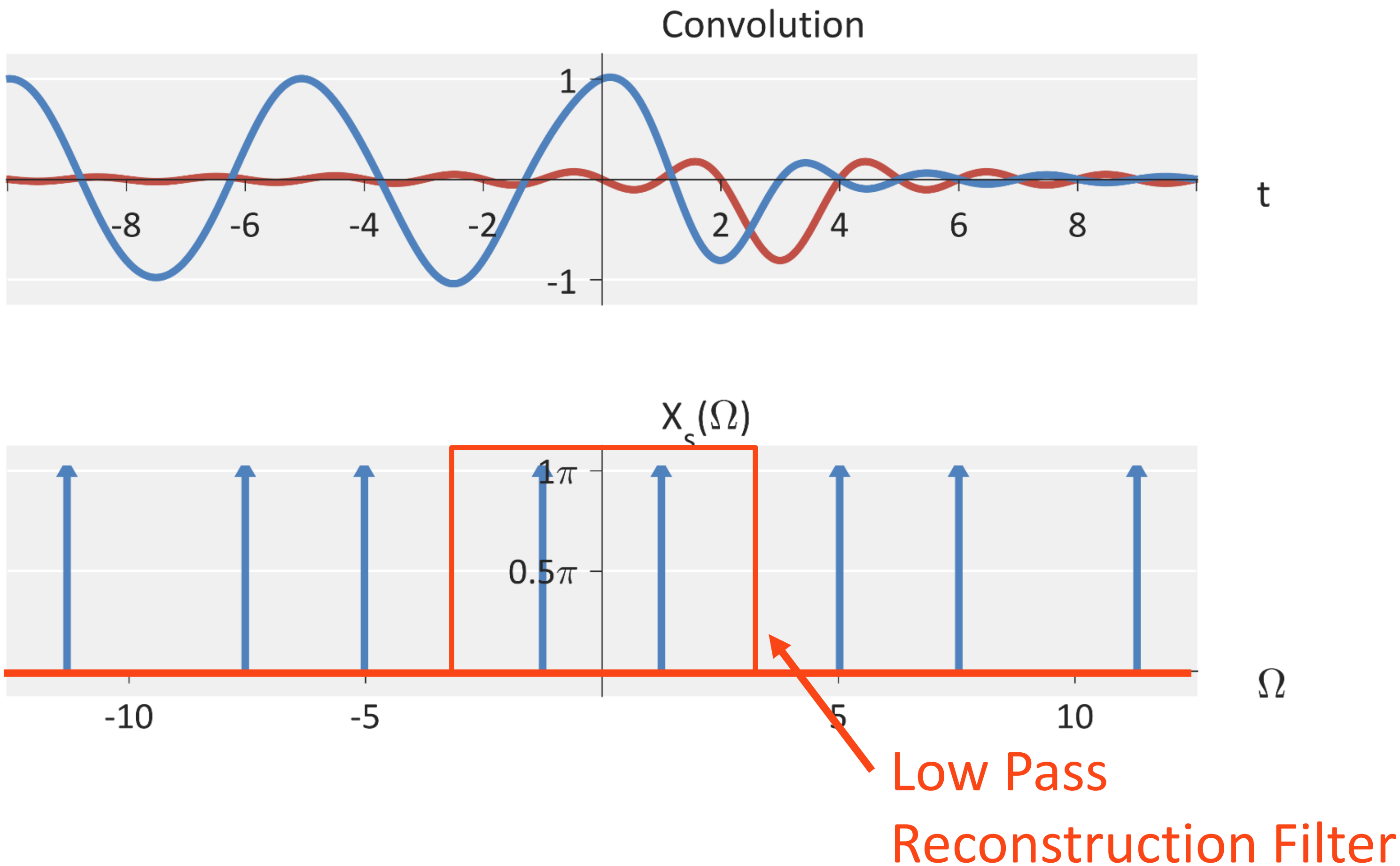
Reconstruction

- **Consider a cosine – Low pass filter to keep $-\Omega_s/2$ to $\Omega_s/2$**



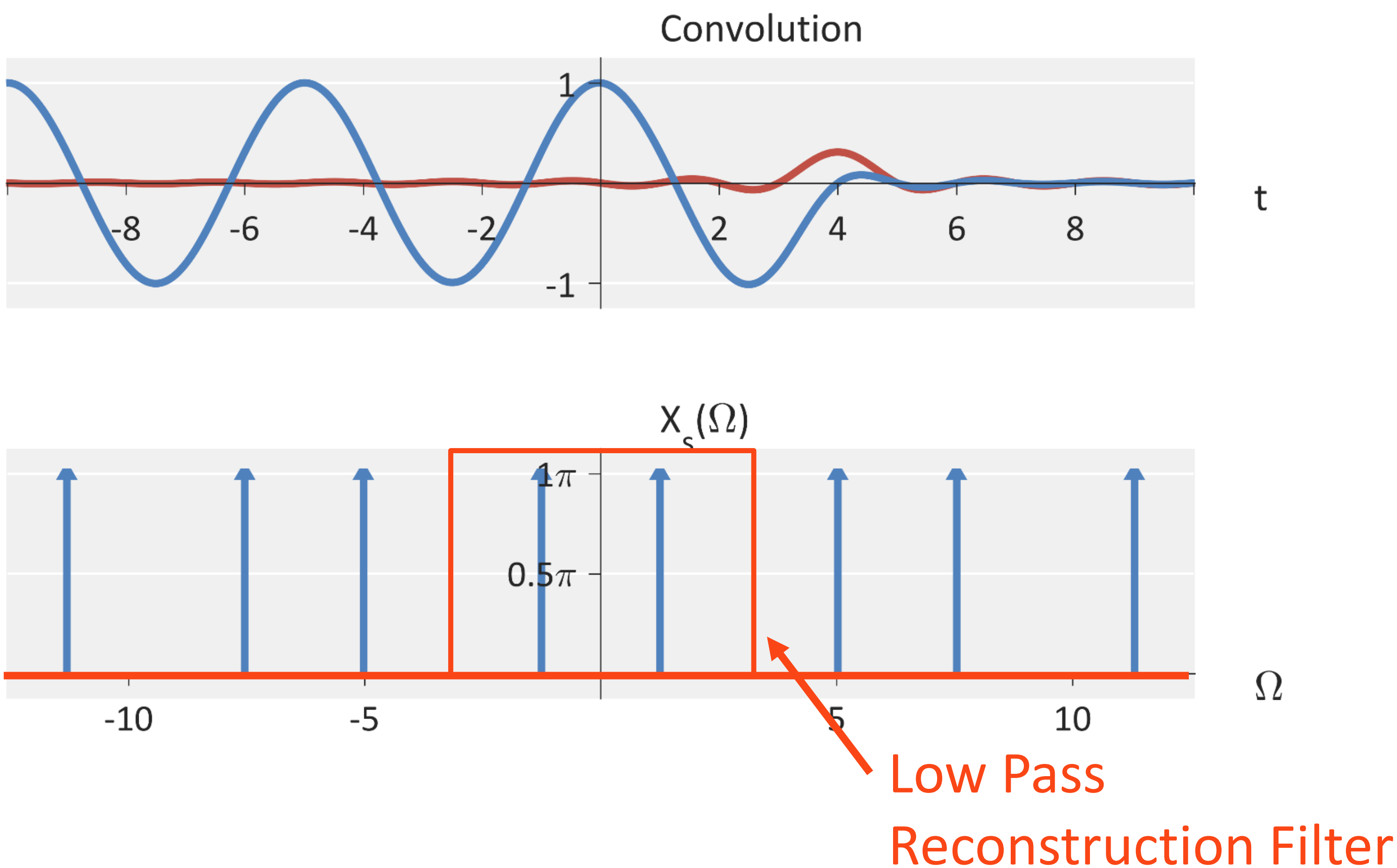
Reconstruction

- **Consider a cosine – Low pass filter to keep $-\Omega_s/2$ to $\Omega_s/2$**



Reconstruction

- **Consider a cosine – Low pass filter to keep $-\Omega_s/2$ to $\Omega_s/2$**



Reconstruction

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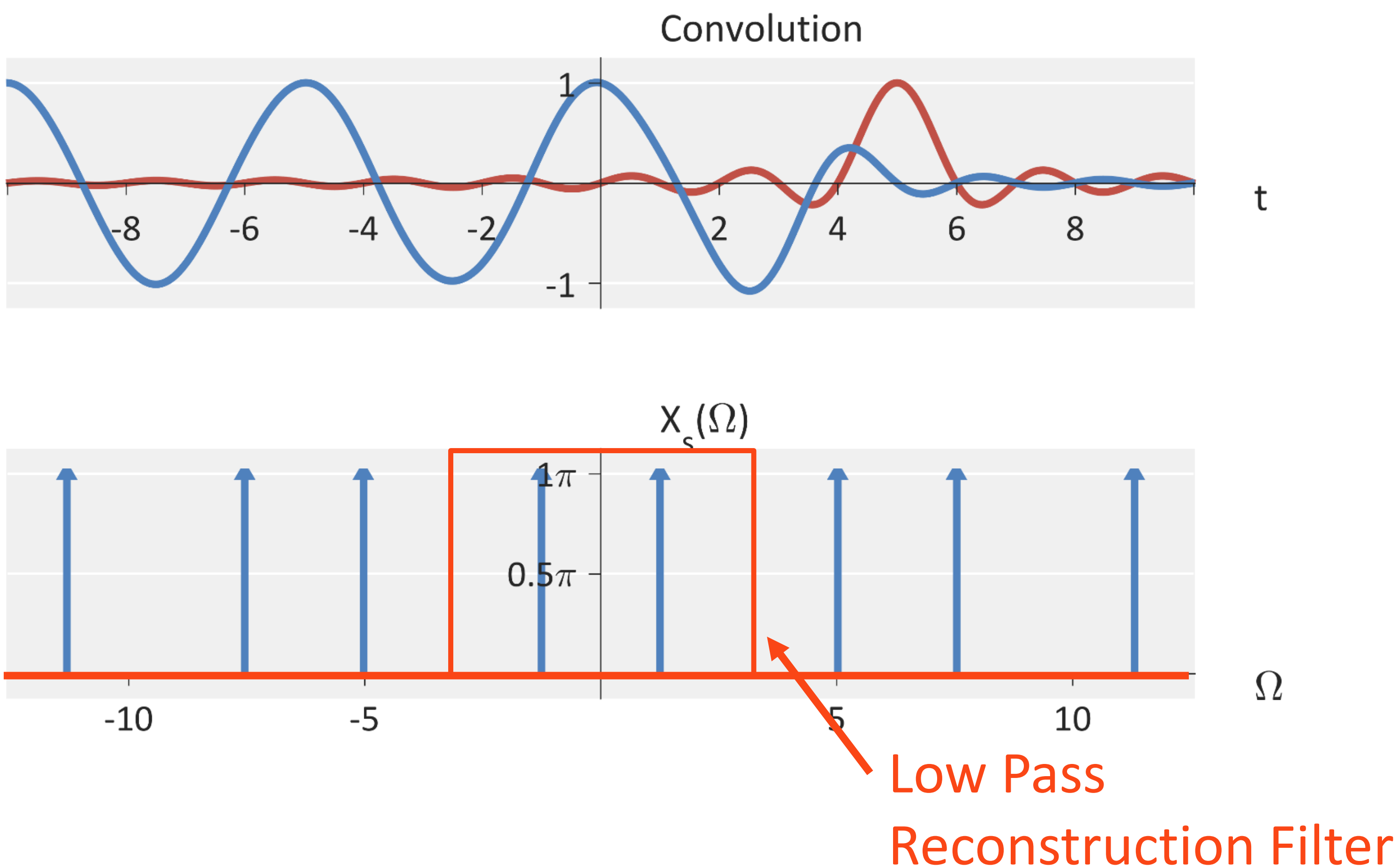
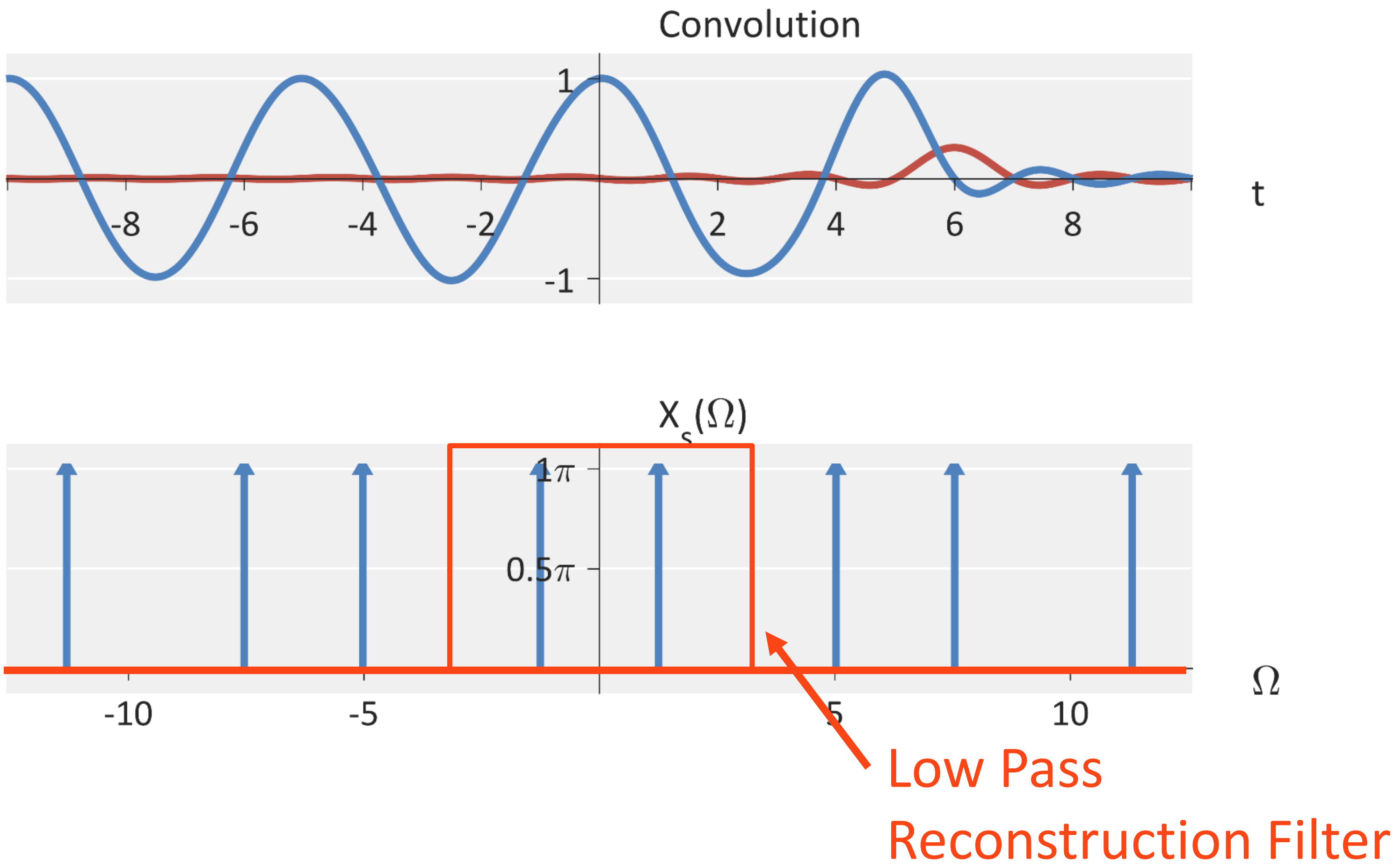


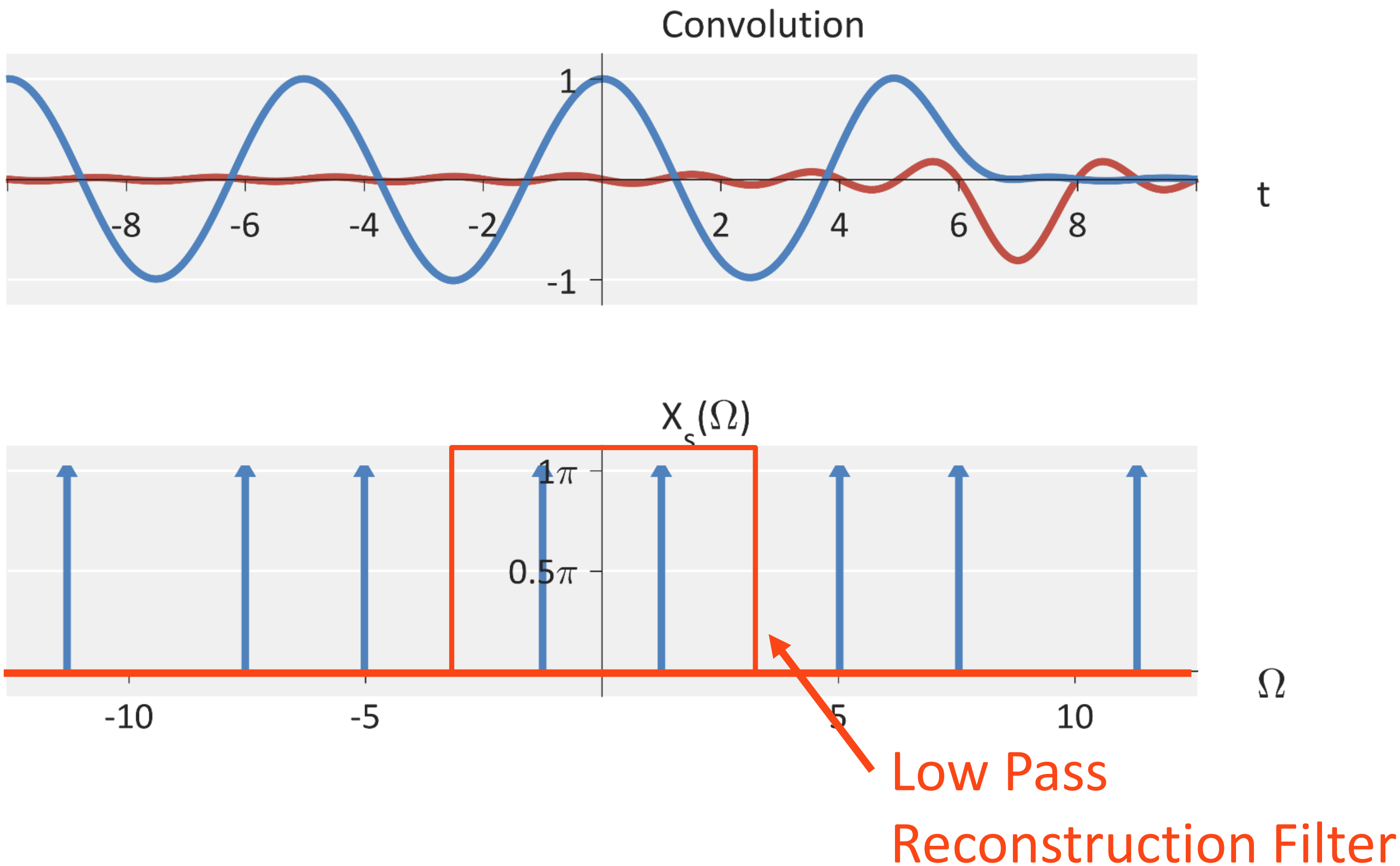
Reconstruction

- **Consider a cosine – Low pass filter to keep $-\Omega_s/2$ to $\Omega_s/2$**



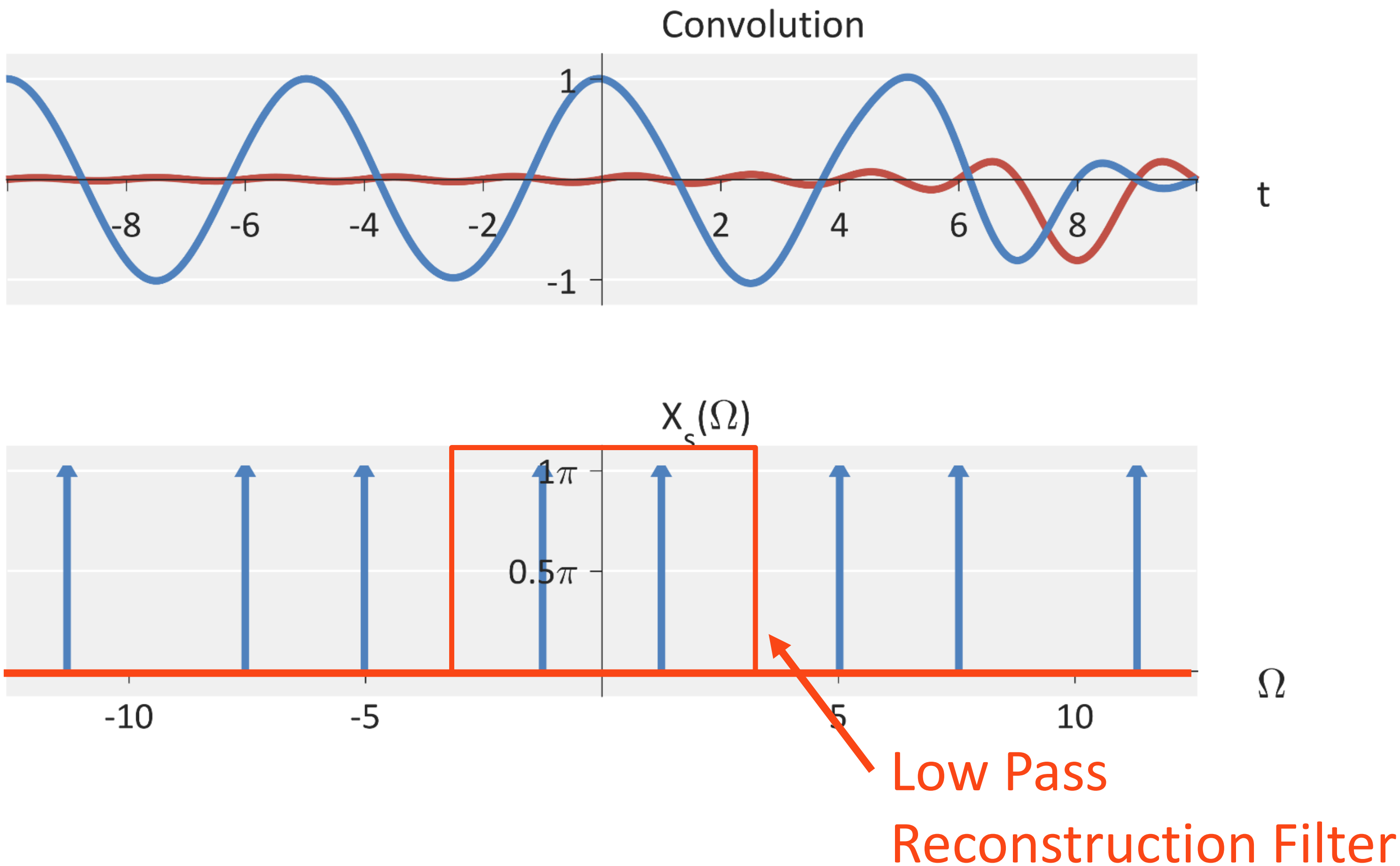
Reconstruction

- **Consider a cosine –** **Low pass filter to keep $-\Omega_s/2$ to $\Omega_s/2$**



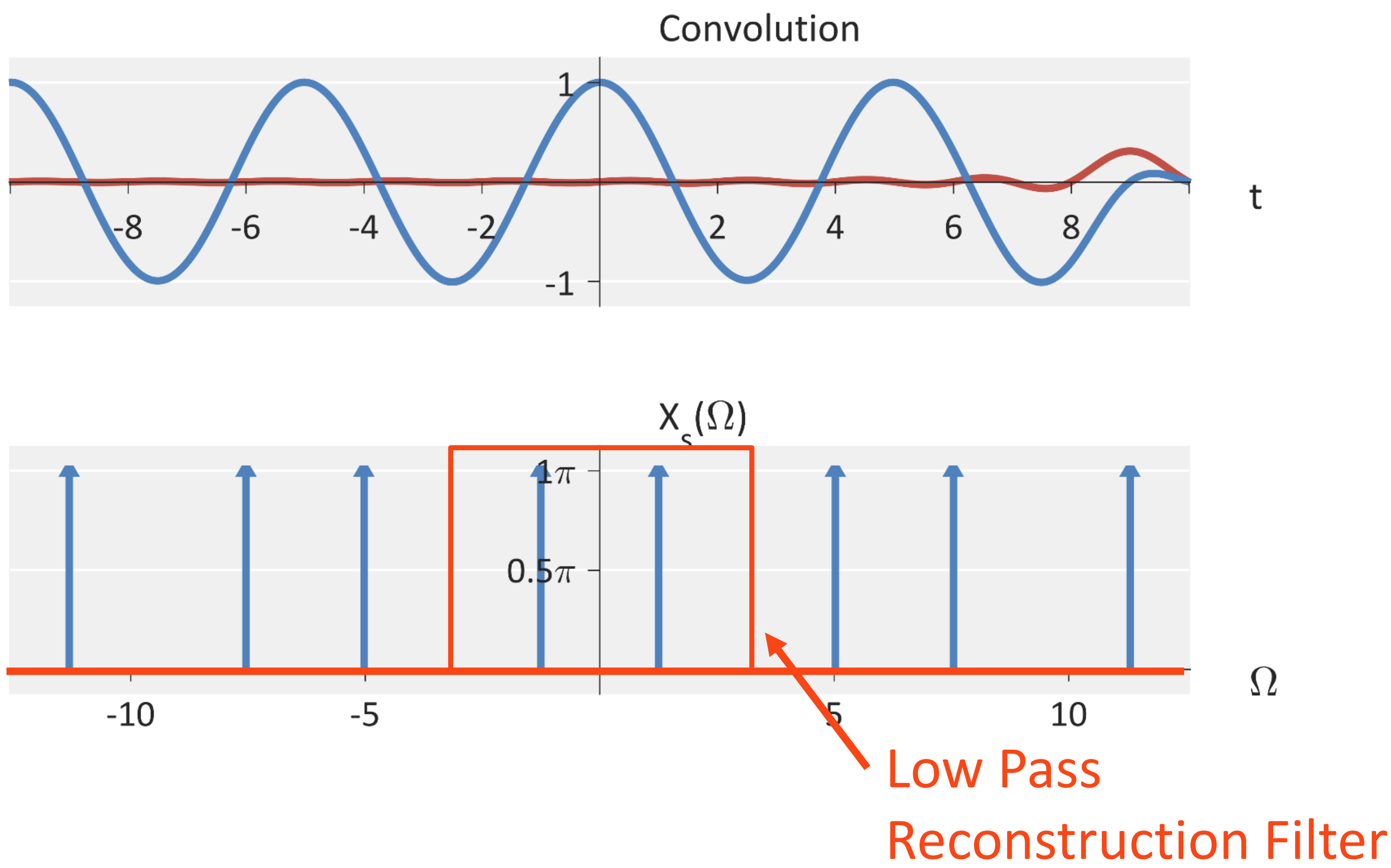
Reconstruction

- **Consider a cosine – Low pass filter to keep $-\Omega_s/2$ to $\Omega_s/2$**



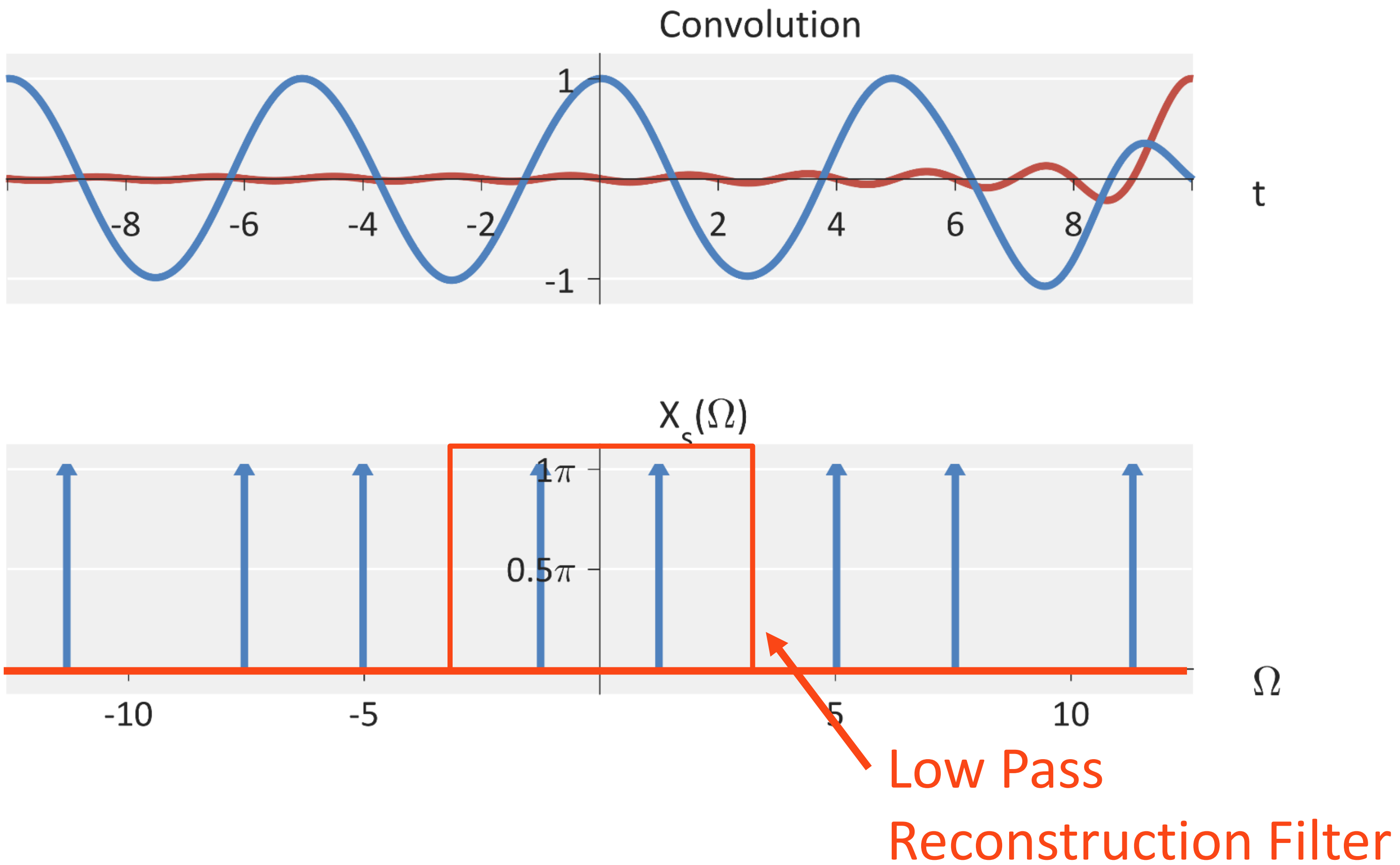
Reconstruction

- **Consider a cosine – Low pass filter to keep $-\Omega_s/2$ to $\Omega_s/2$**



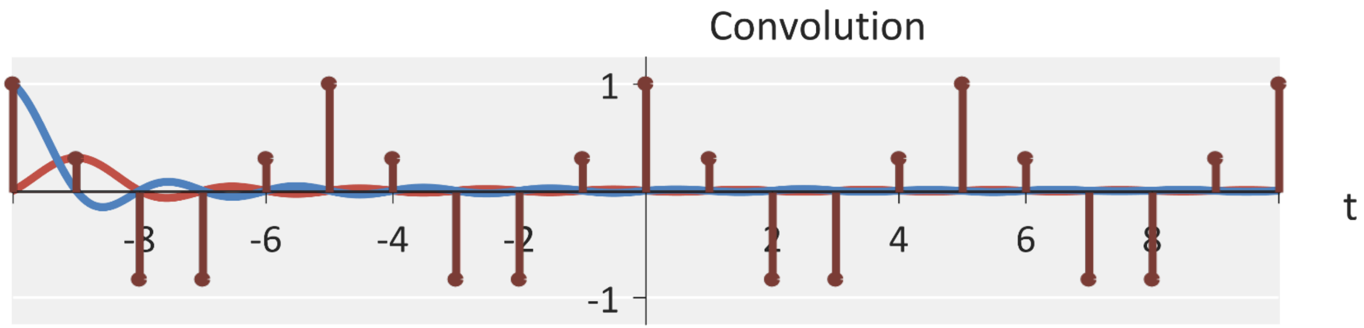
Reconstruction

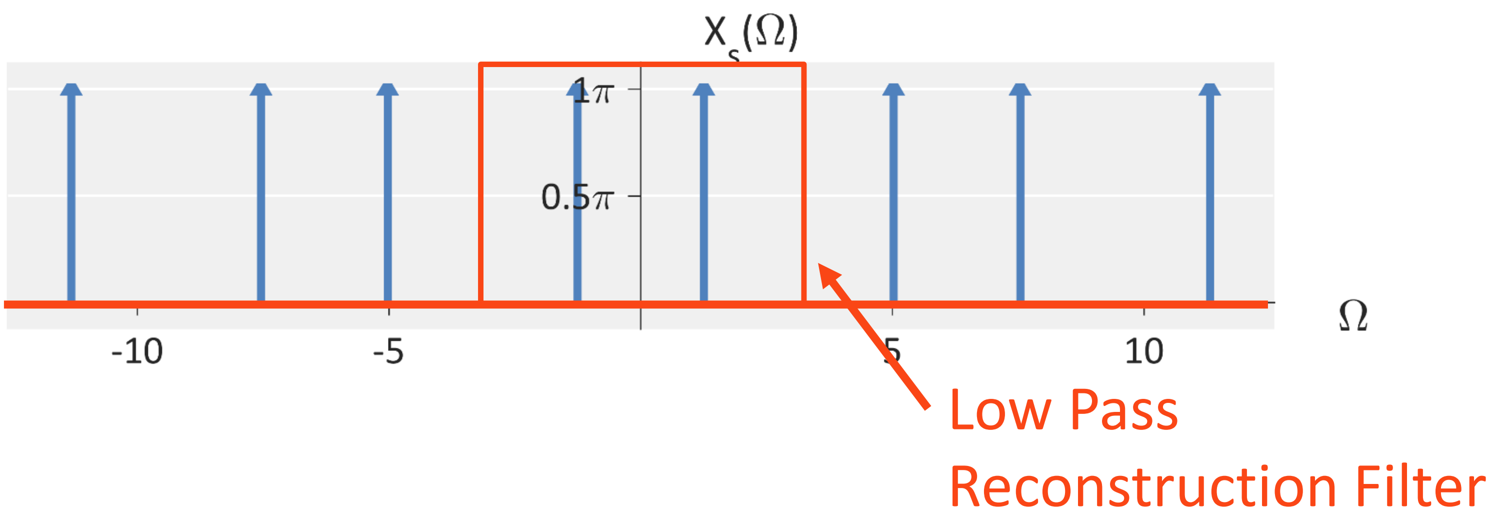
- **Consider a cosine – Low pass filter to keep $-\Omega_s/2$ to $\Omega_s/2$**



Reconstruction

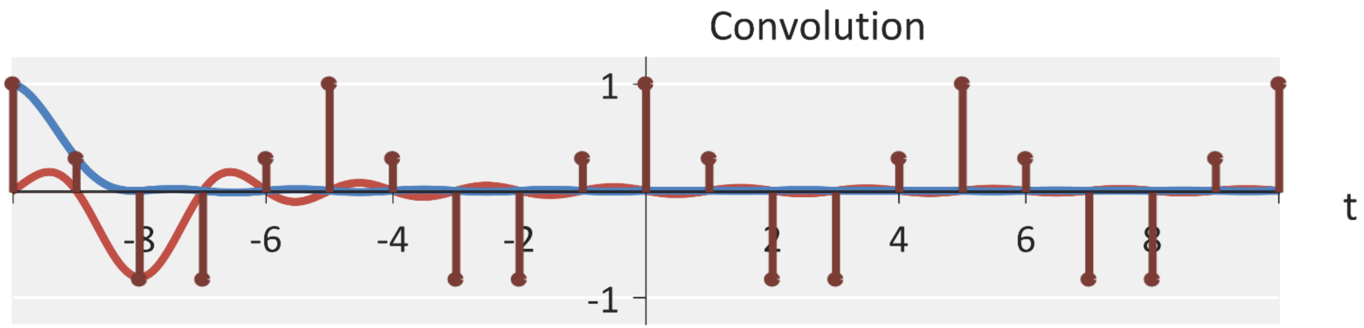
- **Consider a cosine – Low pass filter to keep $-\Omega_s/2$ to $\Omega_s/2$**

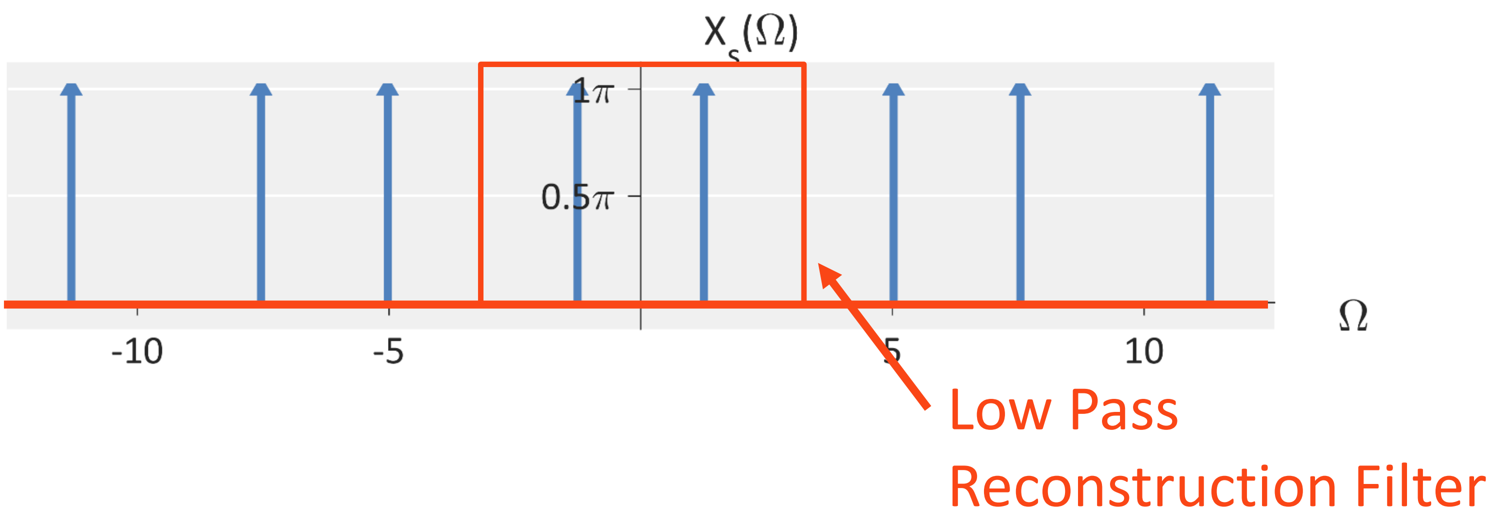




Reconstruction

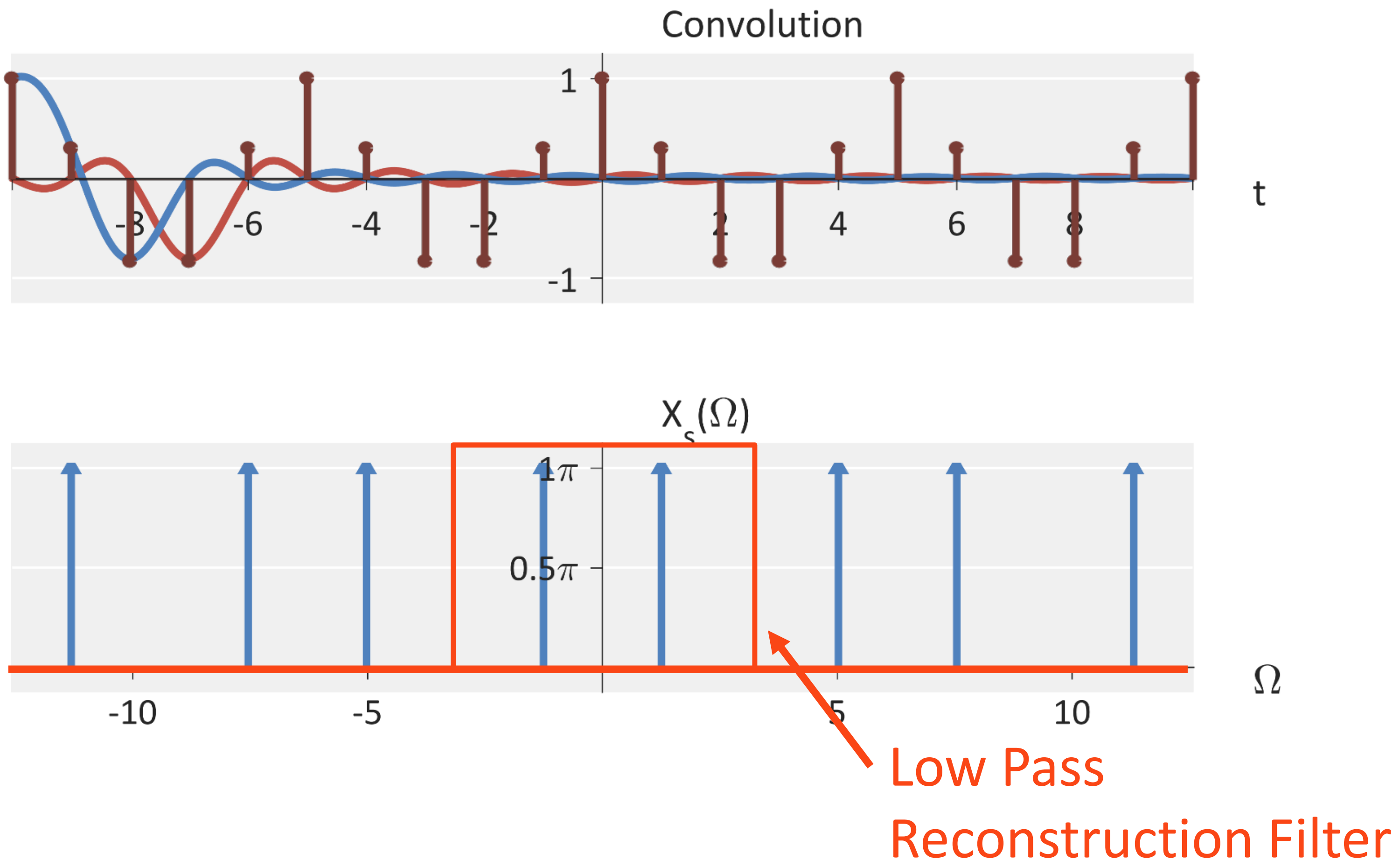
- **Consider a cosine – Low pass filter to keep $-\Omega_s/2$ to $\Omega_s/2$**





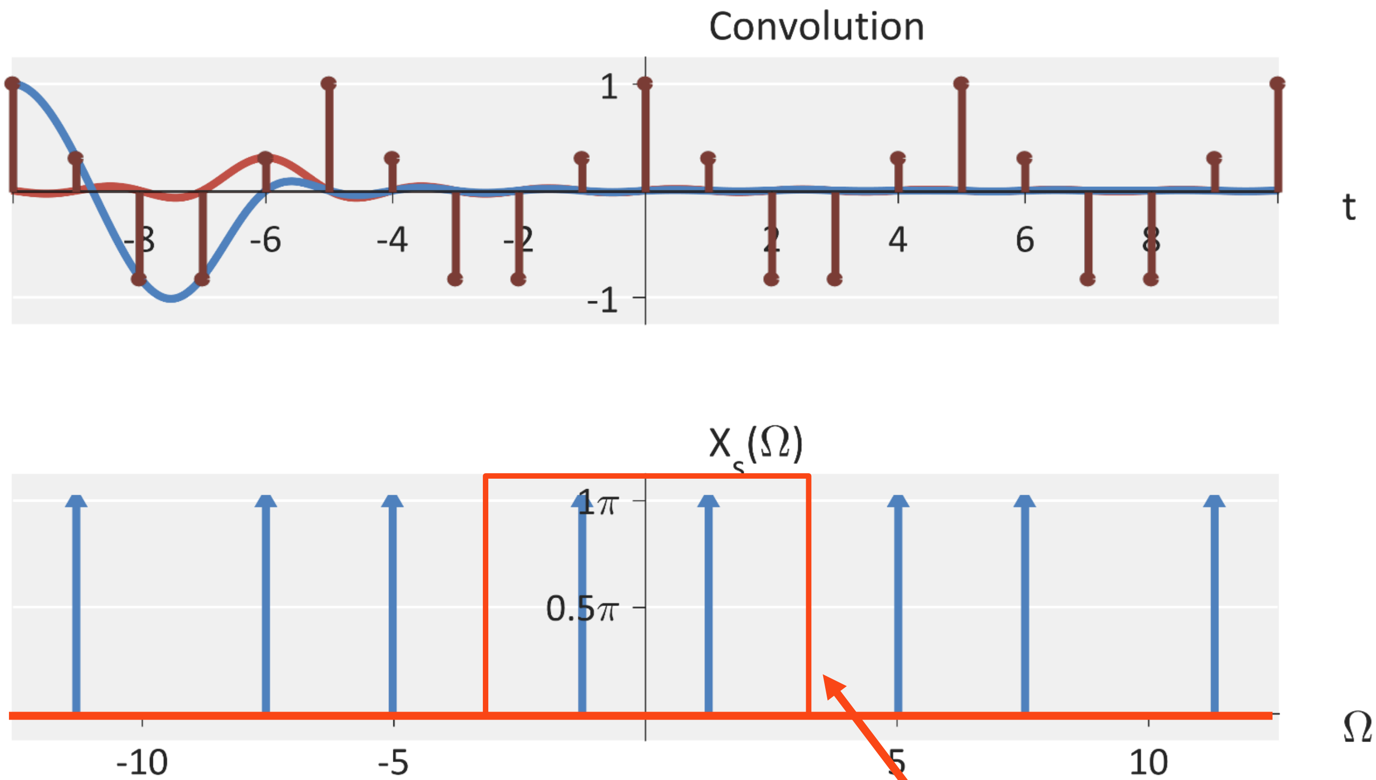
Reconstruction

- **Consider a cosine –** Low pass filter to keep $-\Omega_s/2$ to $\Omega_s/2$



Reconstruction

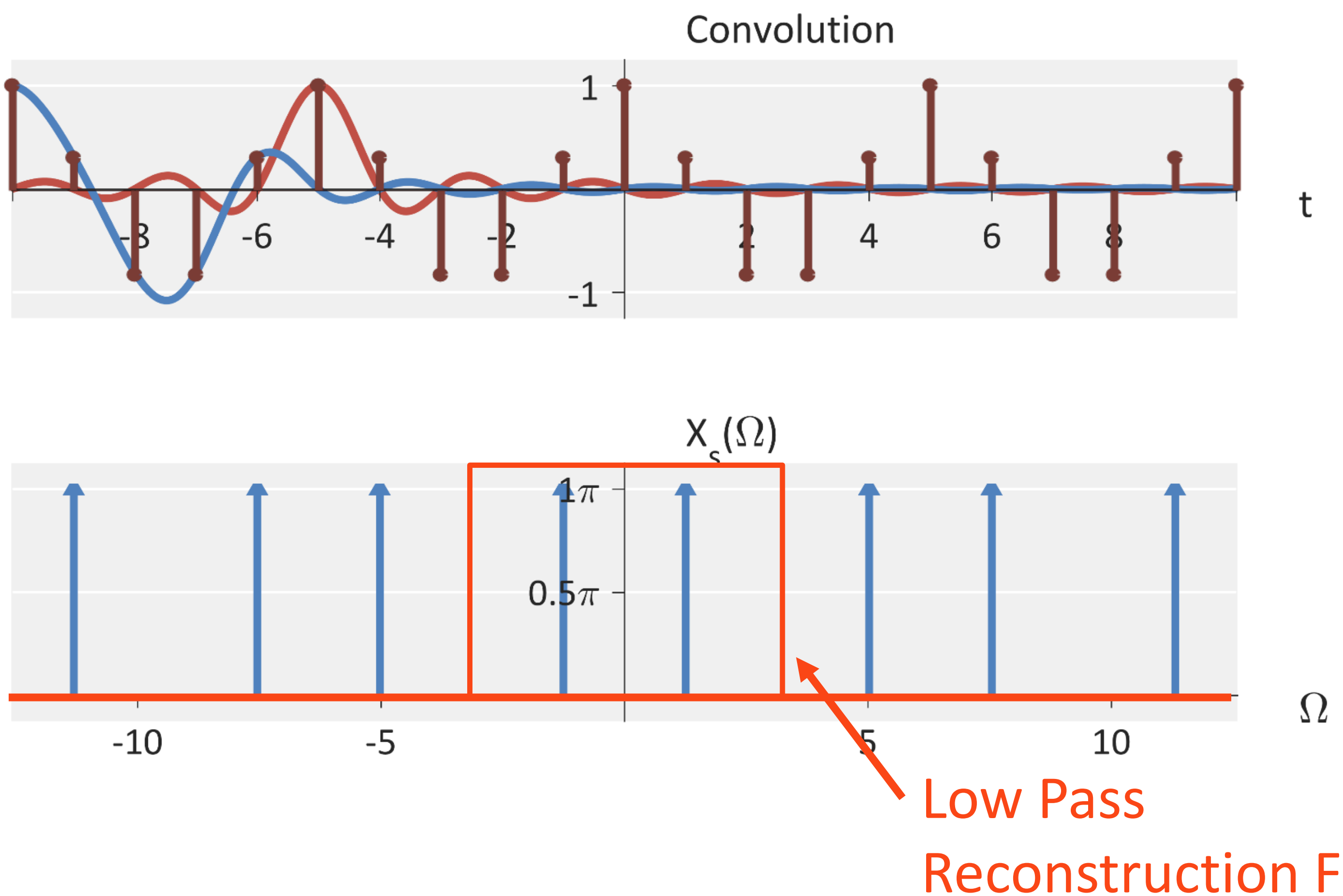
- **Consider a cosine –** **Low pass filter to keep $-\Omega_s/2$ to $\Omega_s/2$**



Low Pass
Reconstruction Filter

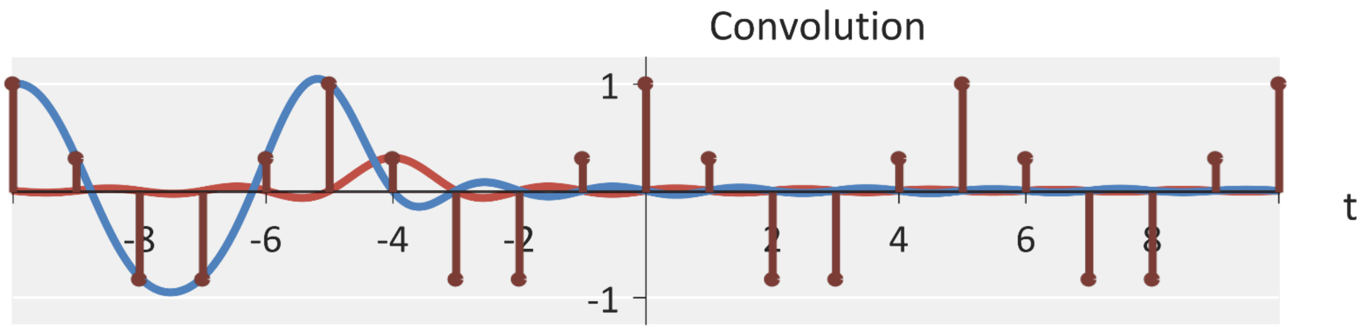
Reconstruction

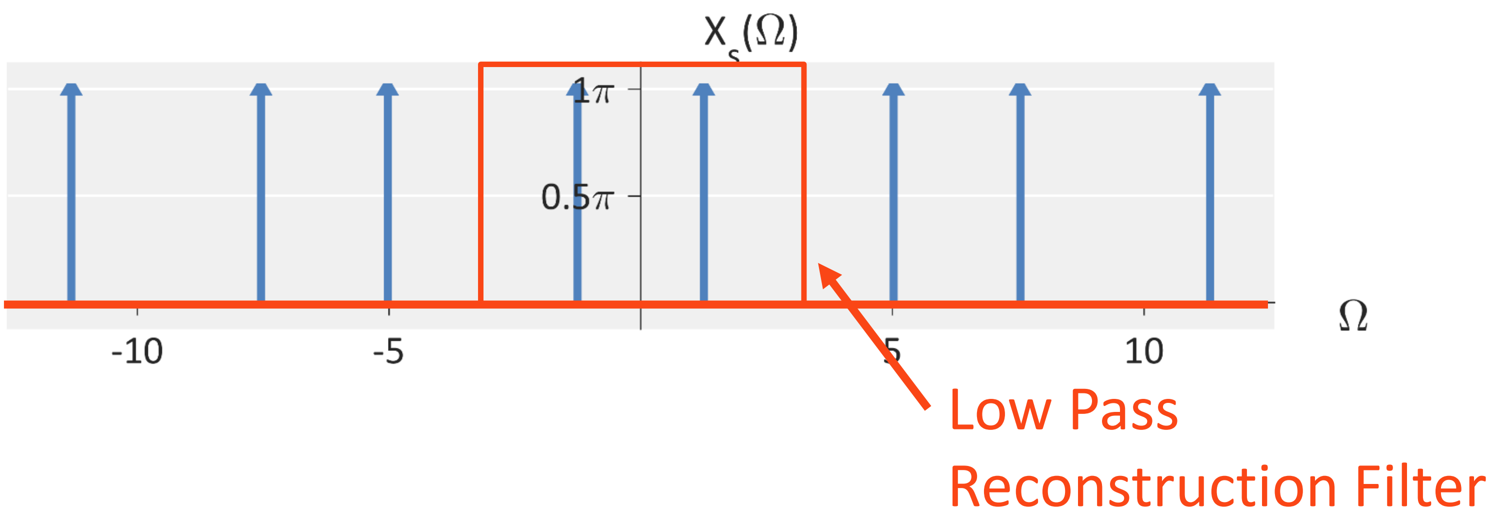
- **Consider a cosine – Low pass filter to keep $-\Omega_s/2$ to $\Omega_s/2$**



Reconstruction

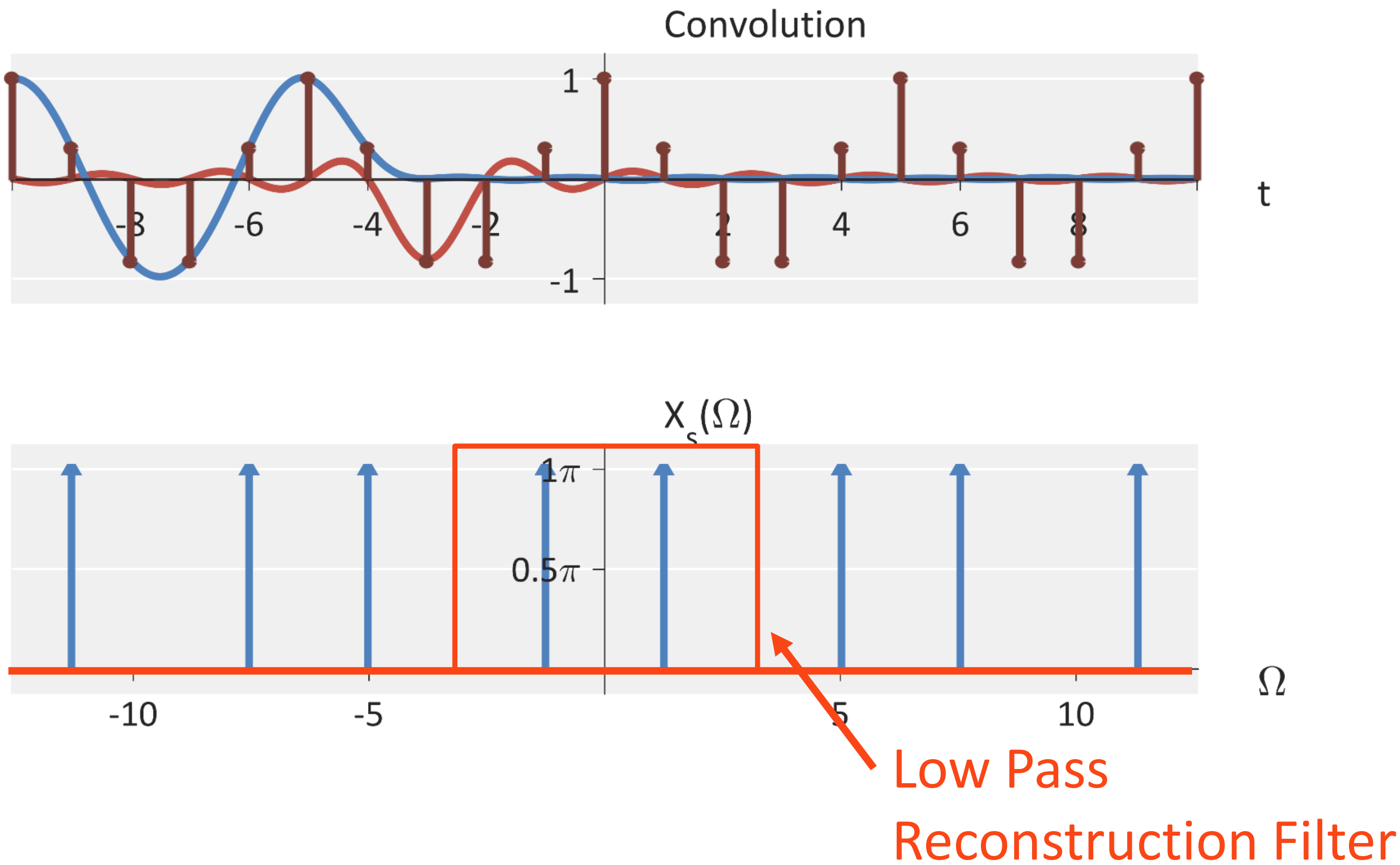
- **Consider a cosine –** **Low pass filter to keep $-\Omega_s/2$ to $\Omega_s/2$**





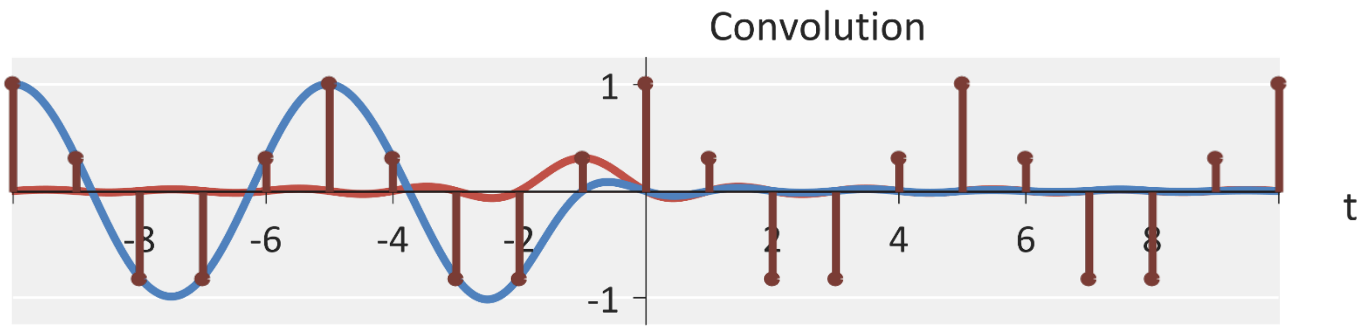
Reconstruction

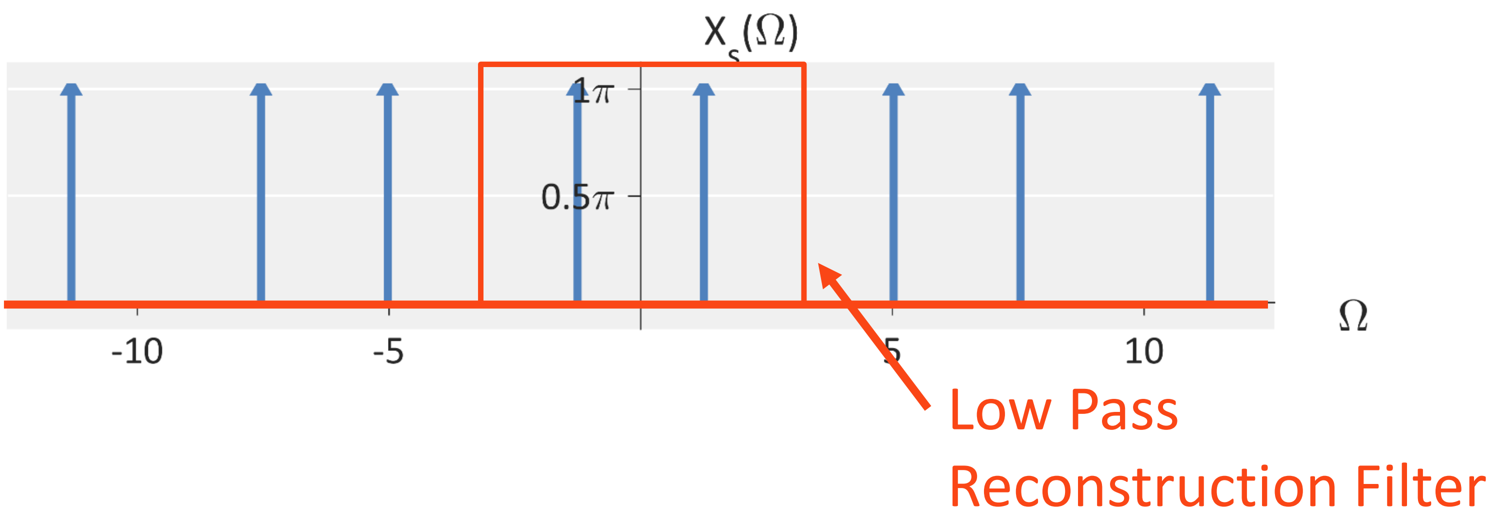
- **Consider a cosine – Low pass filter to keep $-\Omega_s/2$ to $\Omega_s/2$**



Reconstruction

- **Consider a cosine – Low pass filter to keep $-\Omega_s/2$ to $\Omega_s/2$**





Reconstruction

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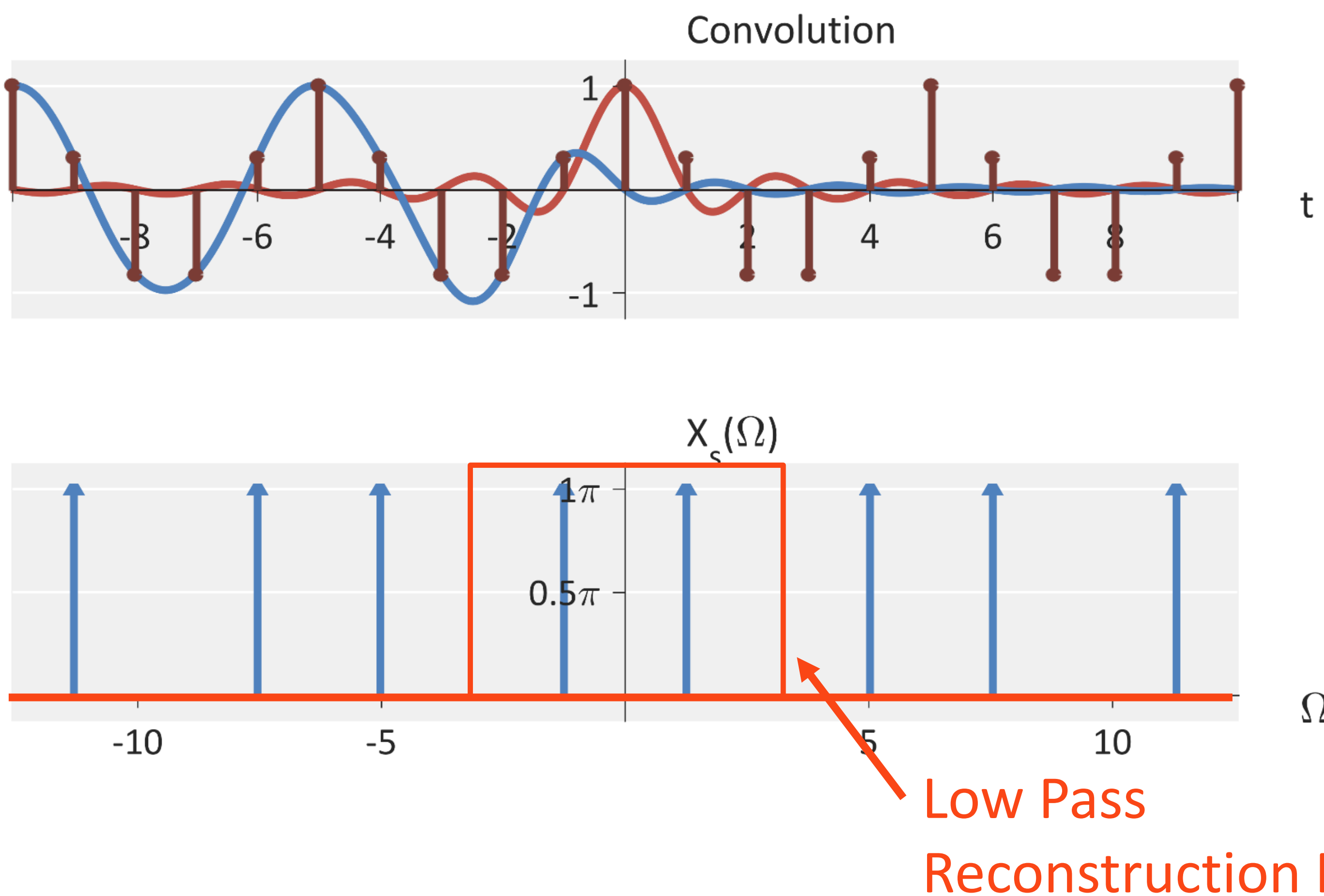
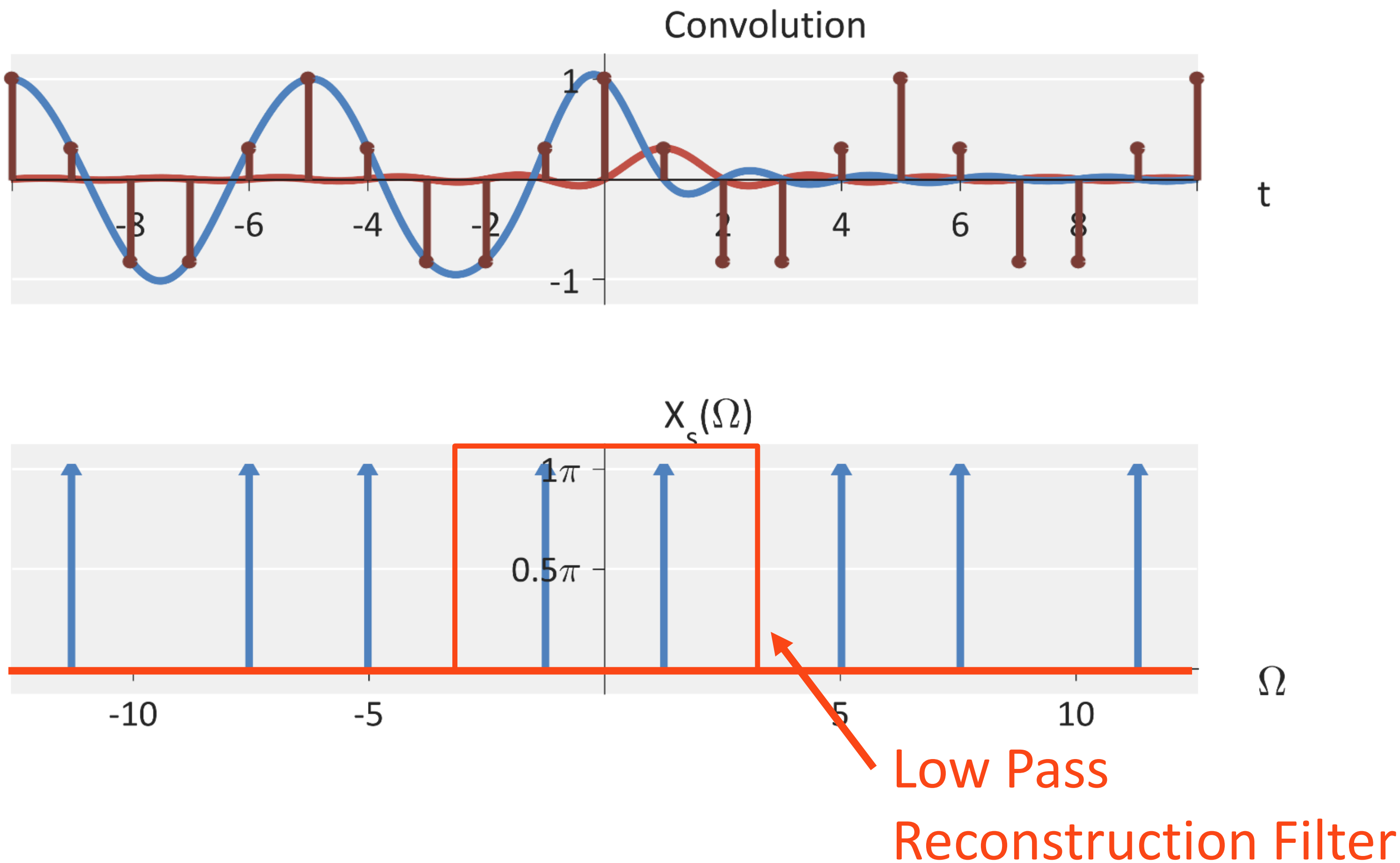


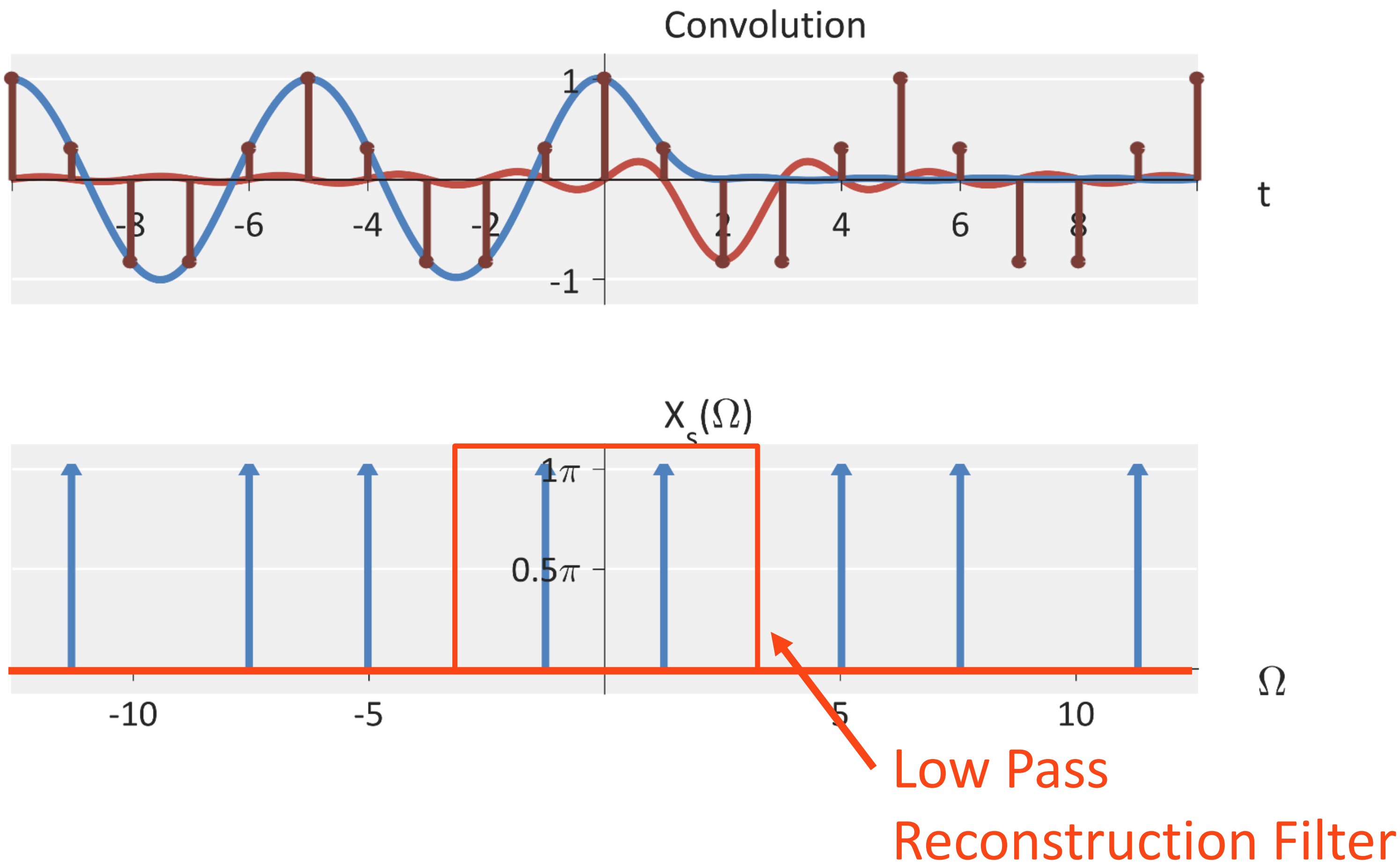
Reconstruction

- **Consider a cosine –** **Low pass filter to keep $-\Omega_s/2$ to $\Omega_s/2$**



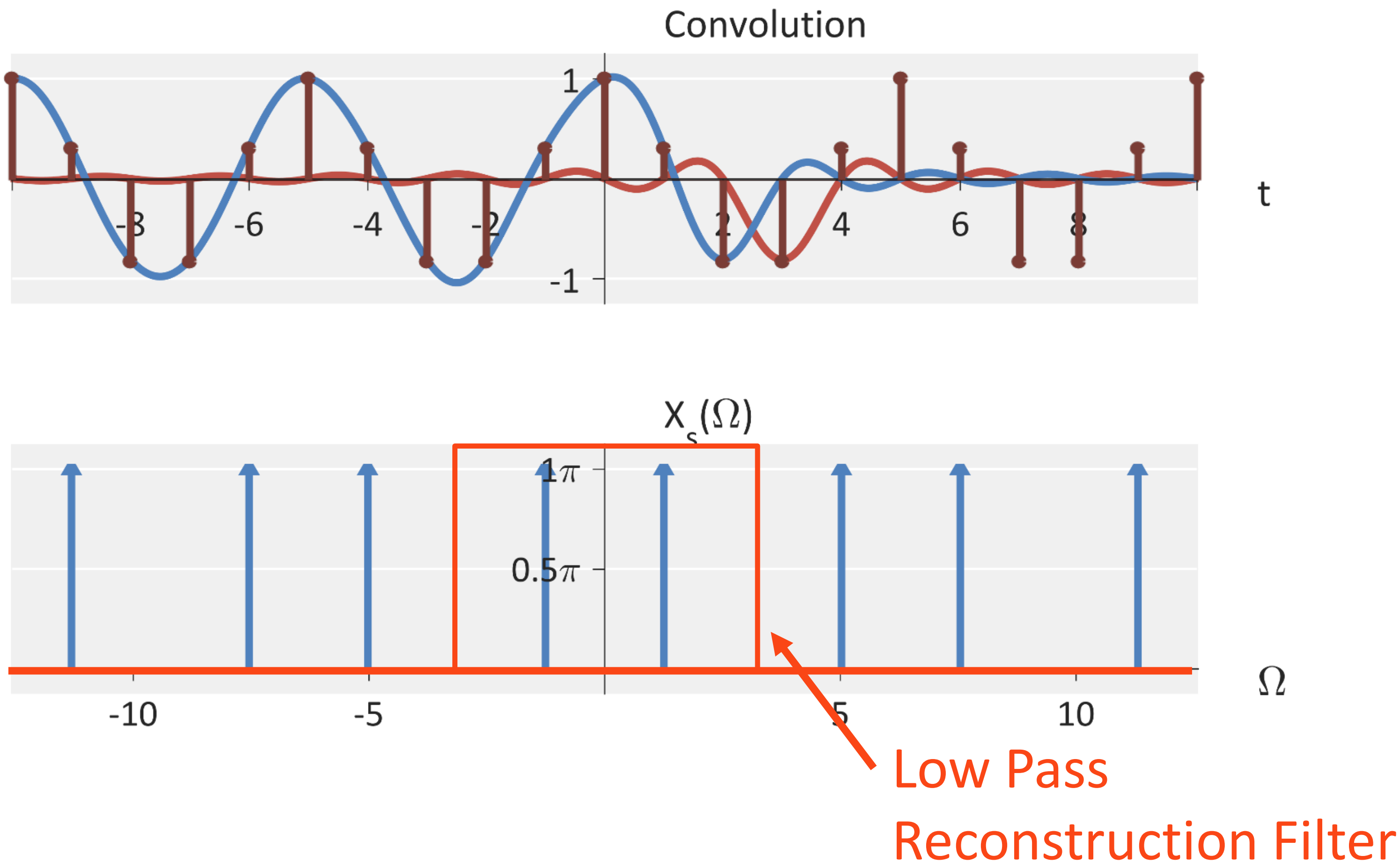
Reconstruction

- **Consider a cosine – Low pass filter to keep $-\Omega_s/2$ to $\Omega_s/2$**



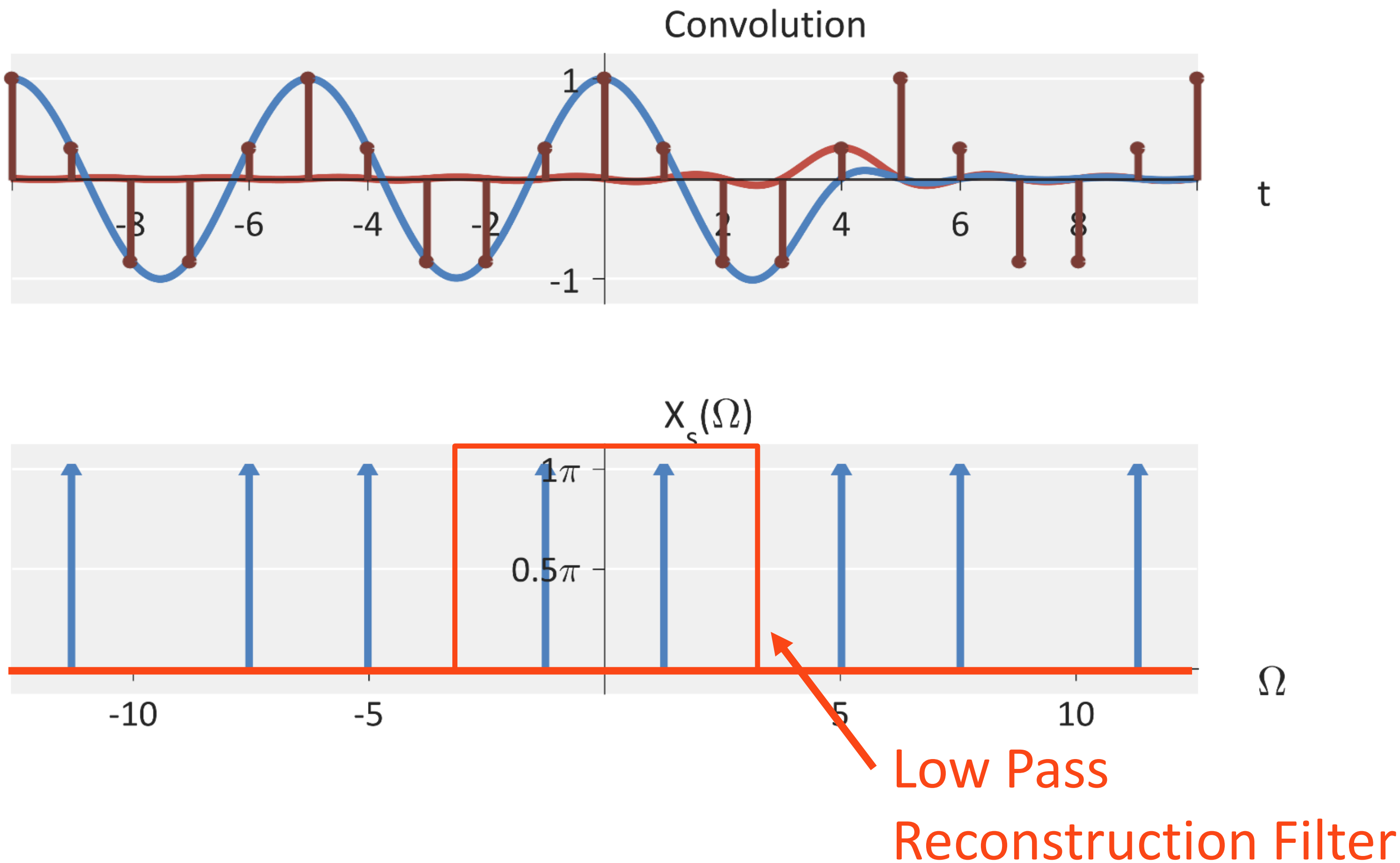
Reconstruction

- **Consider a cosine –** **Low pass filter to keep $-\Omega_s/2$ to $\Omega_s/2$**



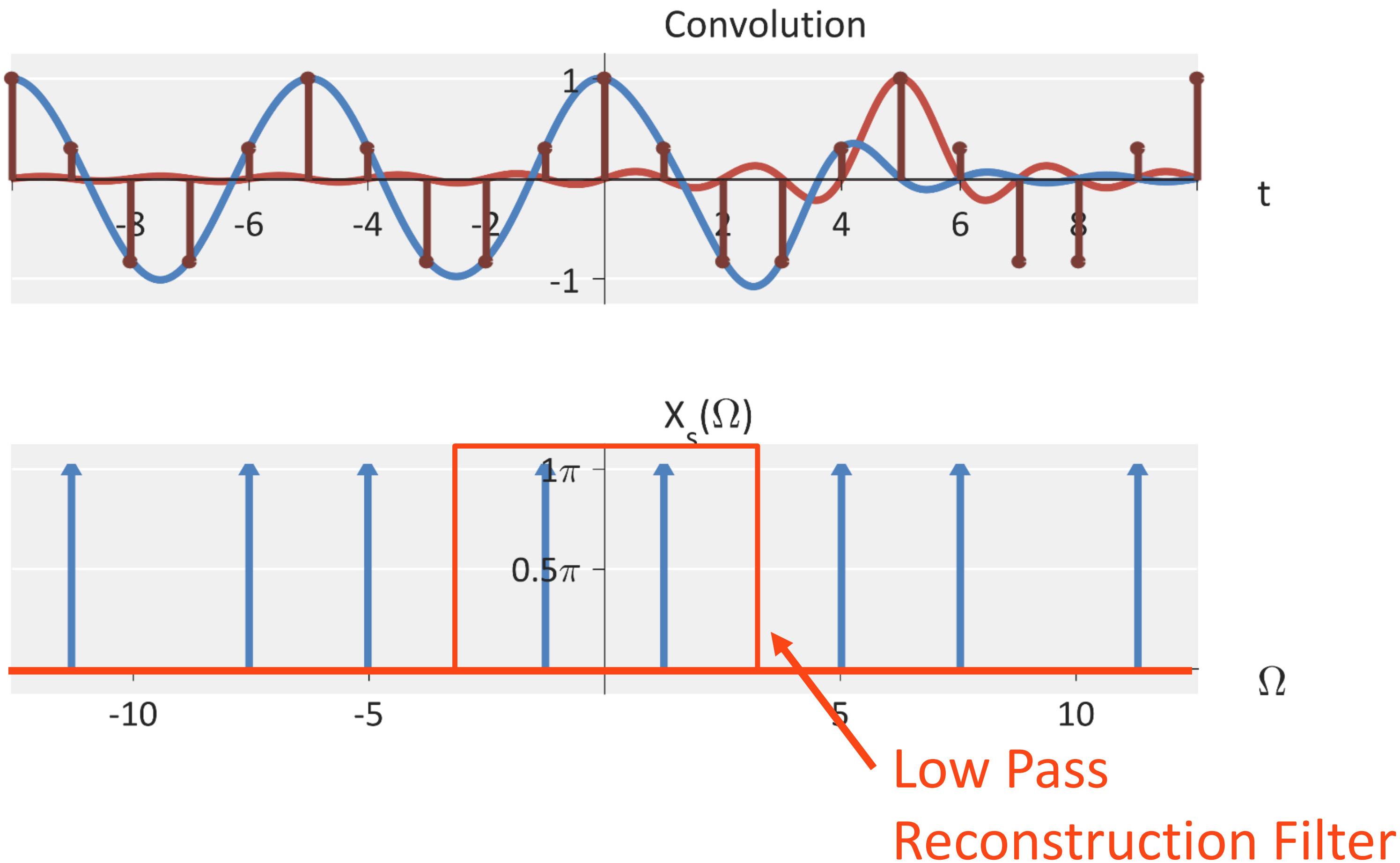
Reconstruction

- **Consider a cosine – Low pass filter to keep $-\Omega_s/2$ to $\Omega_s/2$**



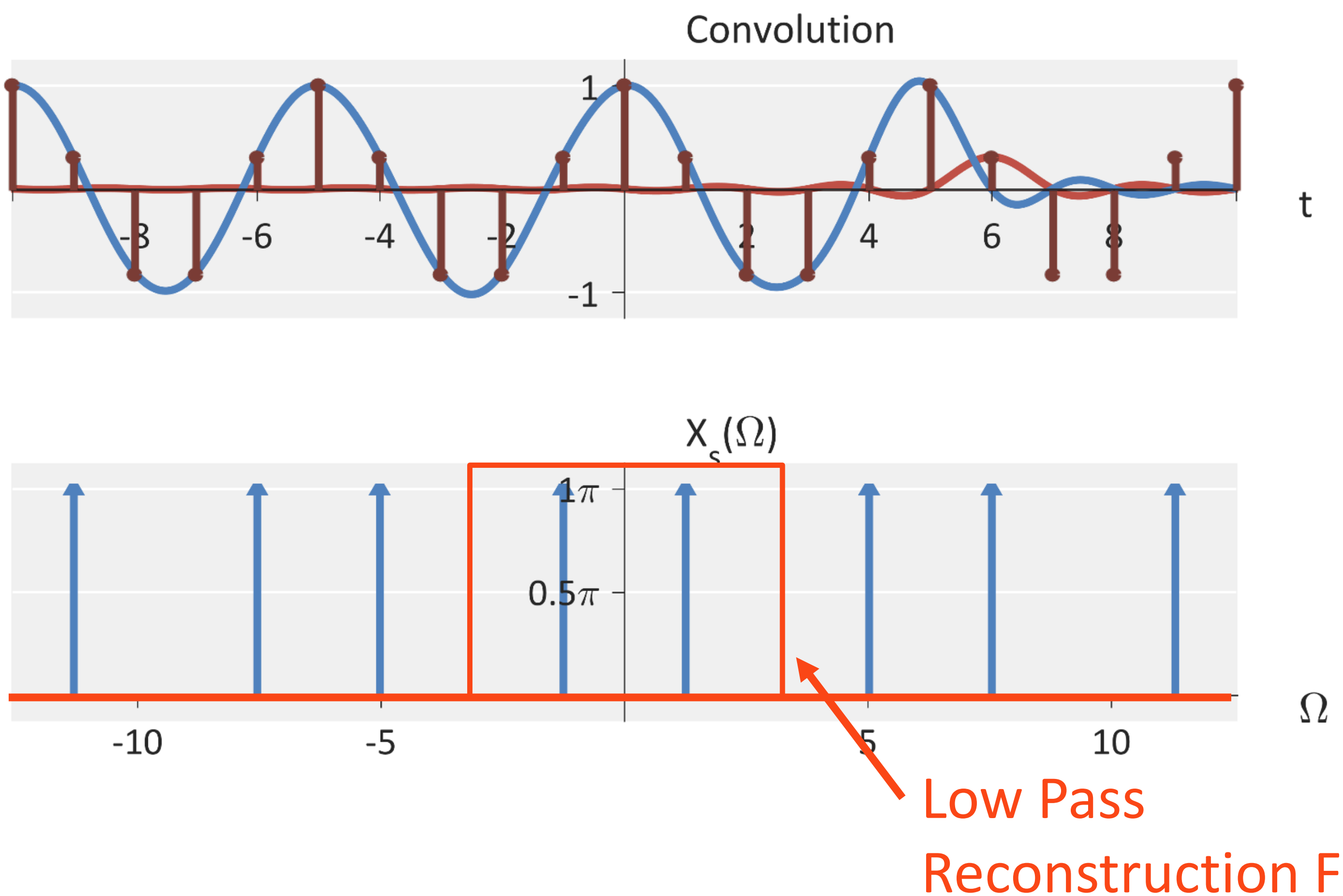
Reconstruction

- **Consider a cosine –** **Low pass filter to keep $-\Omega_s/2$ to $\Omega_s/2$**



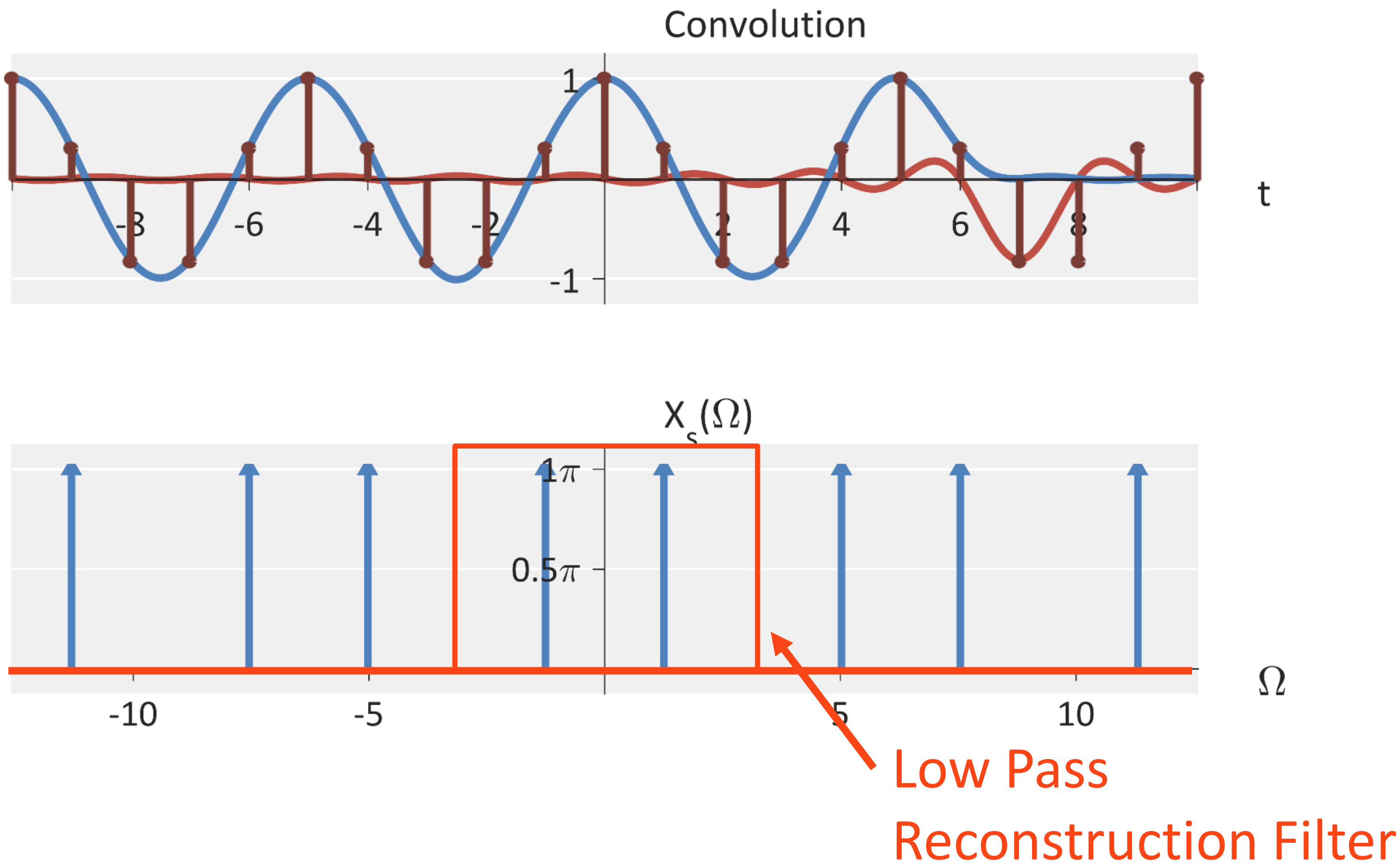
Reconstruction

- **Consider a cosine – Low pass filter to keep $-\Omega_s/2$ to $\Omega_s/2$**



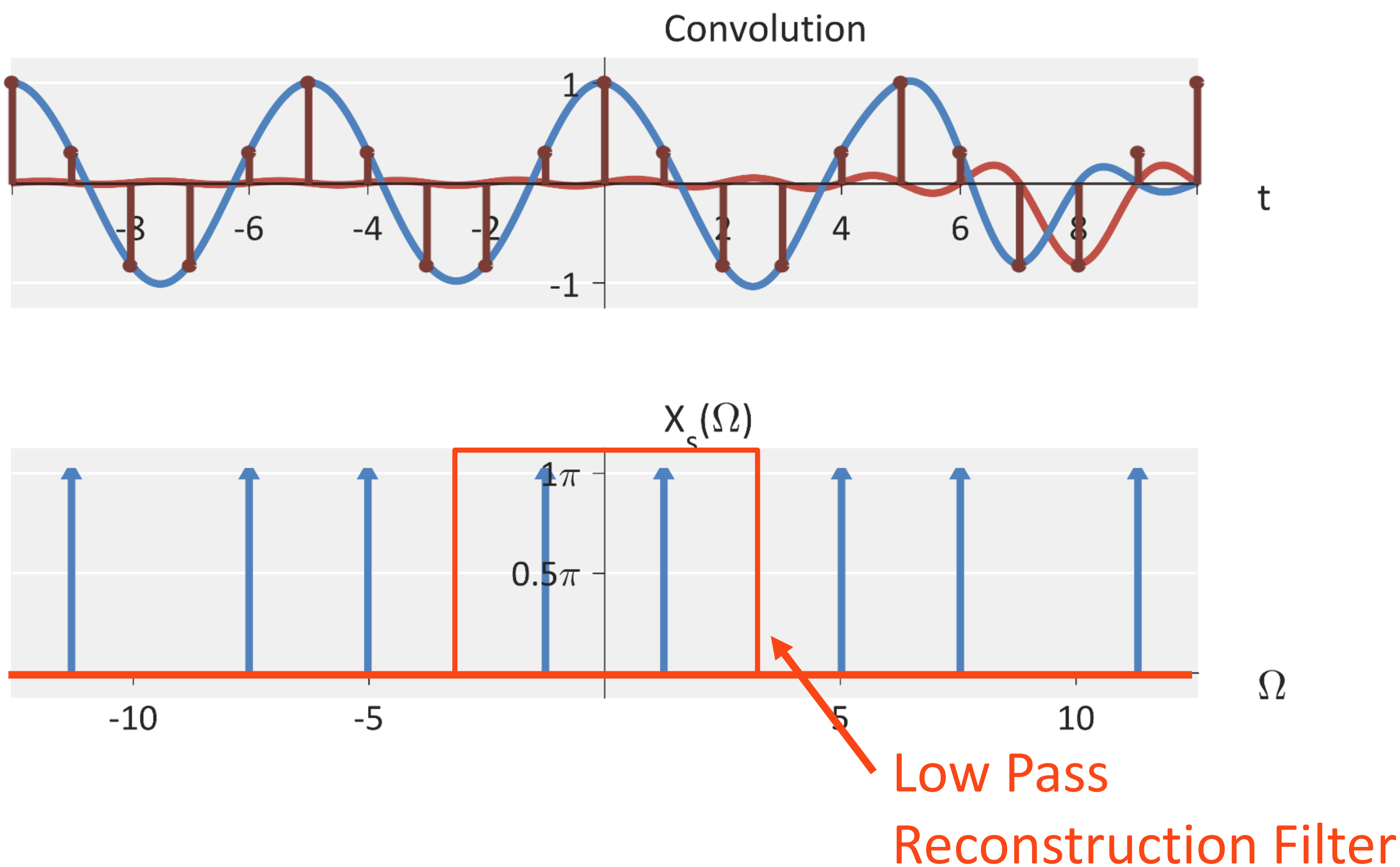
Reconstruction

- **Consider a cosine – Low pass filter to keep $-\Omega_s/2$ to $\Omega_s/2$**



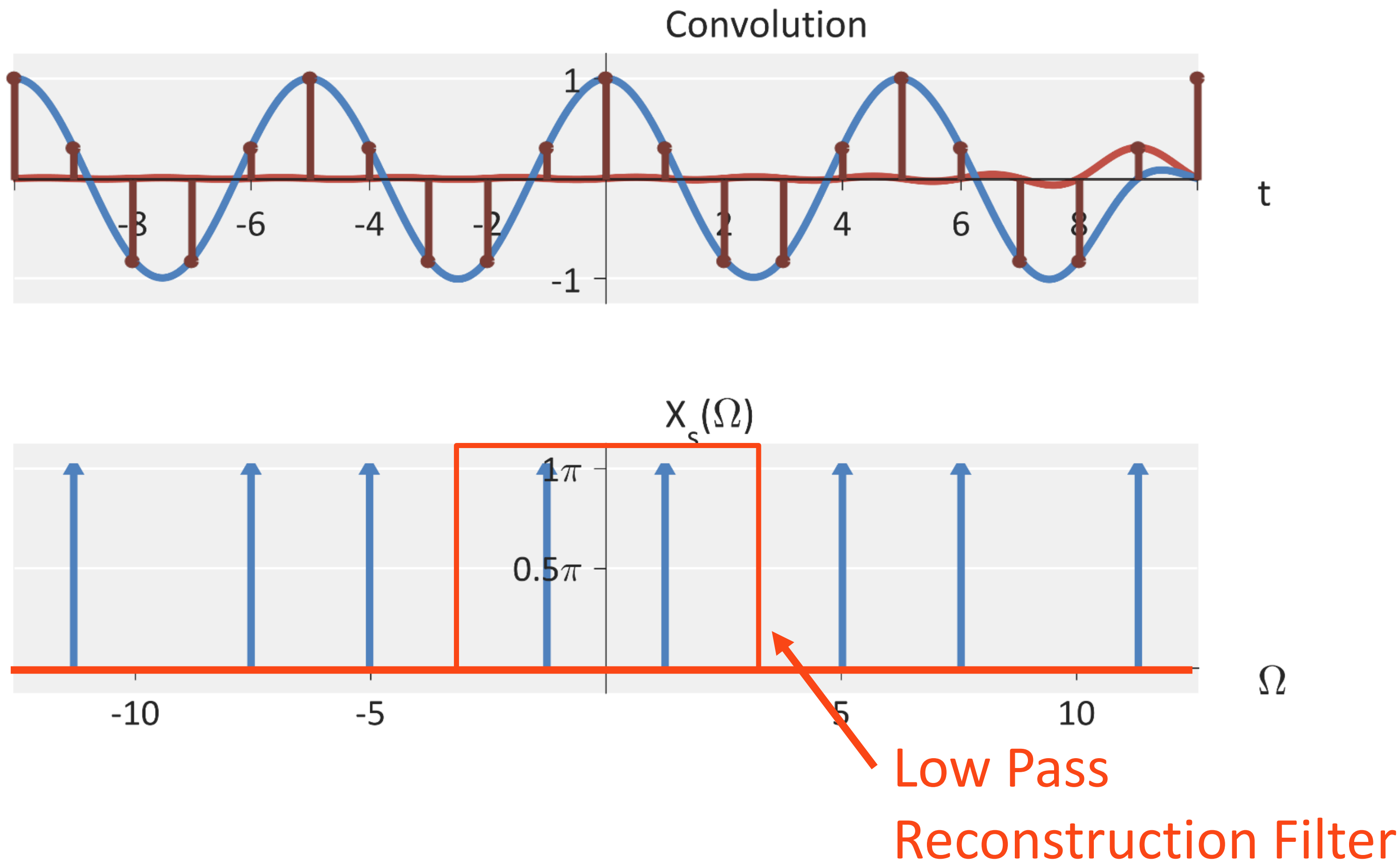
Reconstruction

- **Consider a cosine – Low pass filter to keep $-\Omega_s/2$ to $\Omega_s/2$**



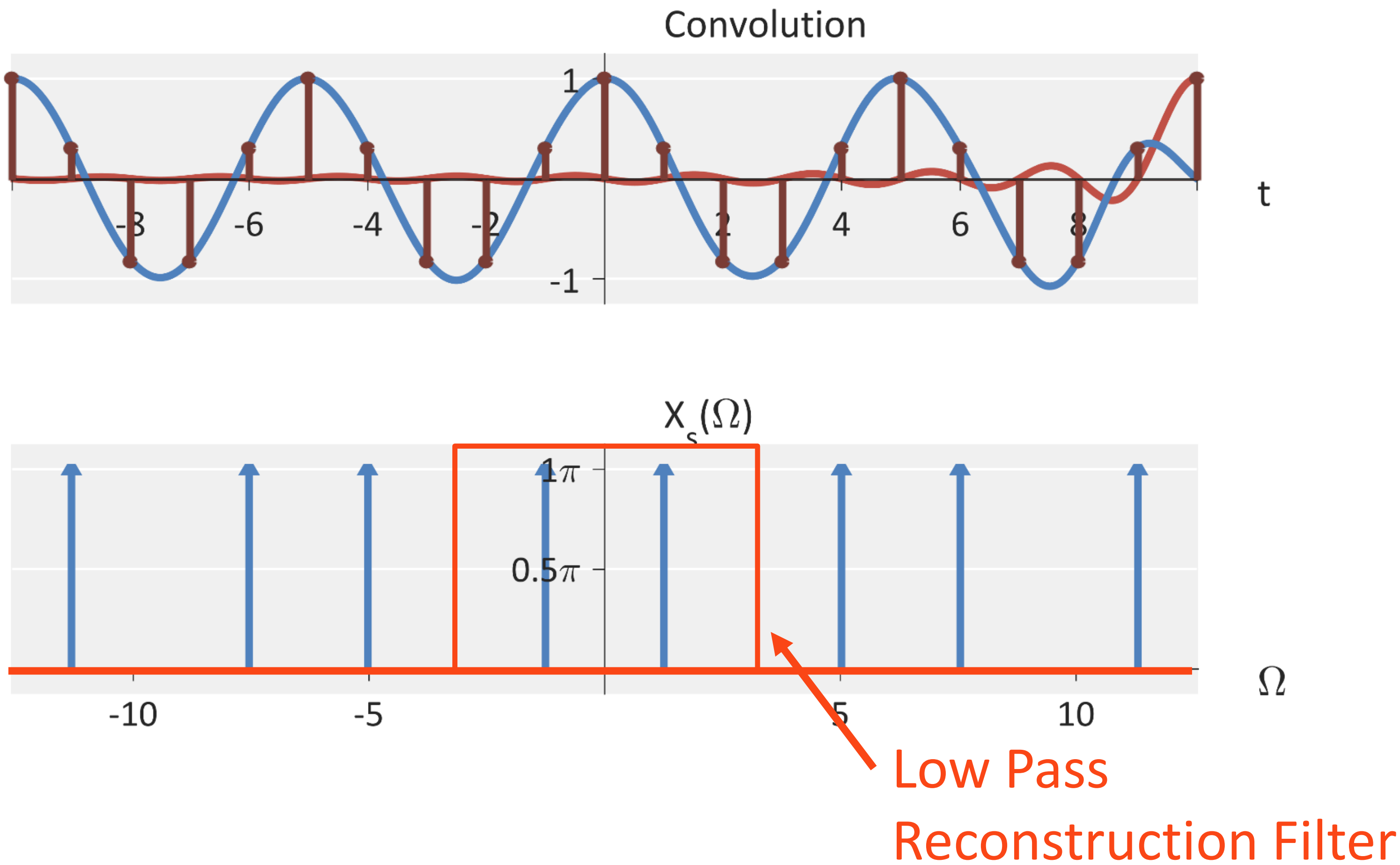
Reconstruction

- **Consider a cosine –** **Low pass filter to keep $-\Omega_s/2$ to $\Omega_s/2$**



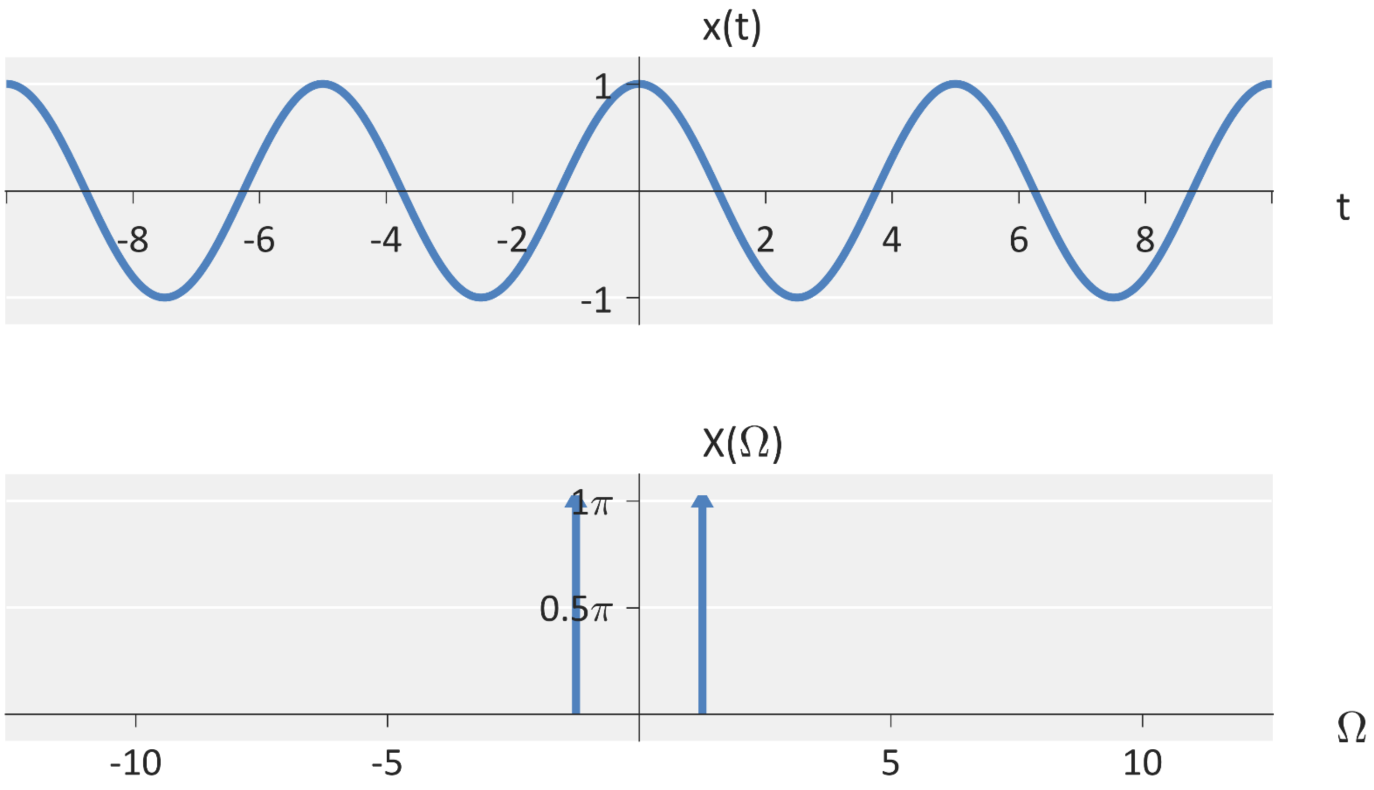
Reconstruction

- **Consider a cosine –** **Low pass filter to keep $-\Omega_s/2$ to $\Omega_s/2$**



Reconstruction

- **Consider a cosine – After the reconstruction filter**



Lecture 12: Sampling, Aliasing, and the Discrete Fourier Transform

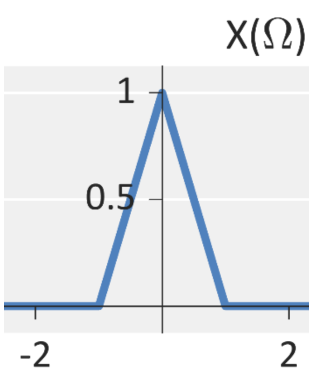
Foundations of Digital Signal Processing

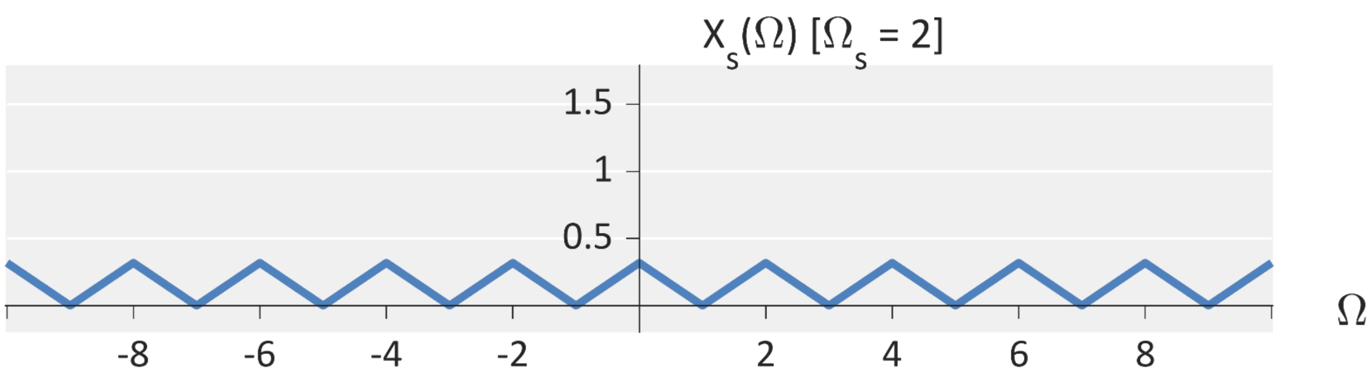
Outline

- Review of Sampling
- The Nyquist-Shannon Sampling Theorem
- Continuous-time Reconstruction / Interpolation
- **Aliasing and anti-Aliasing**
- Deriving Transforms from the Fourier Transform
    - Discrete-time Fourier Transform, Fourier Series, Discrete-time Fourier Series
- The Discrete Fourier Transform

Sampling

- **Aliasing occurs when we do not satisfy the sampling theorem**
- **Question: What can happen when there is aliasing?**

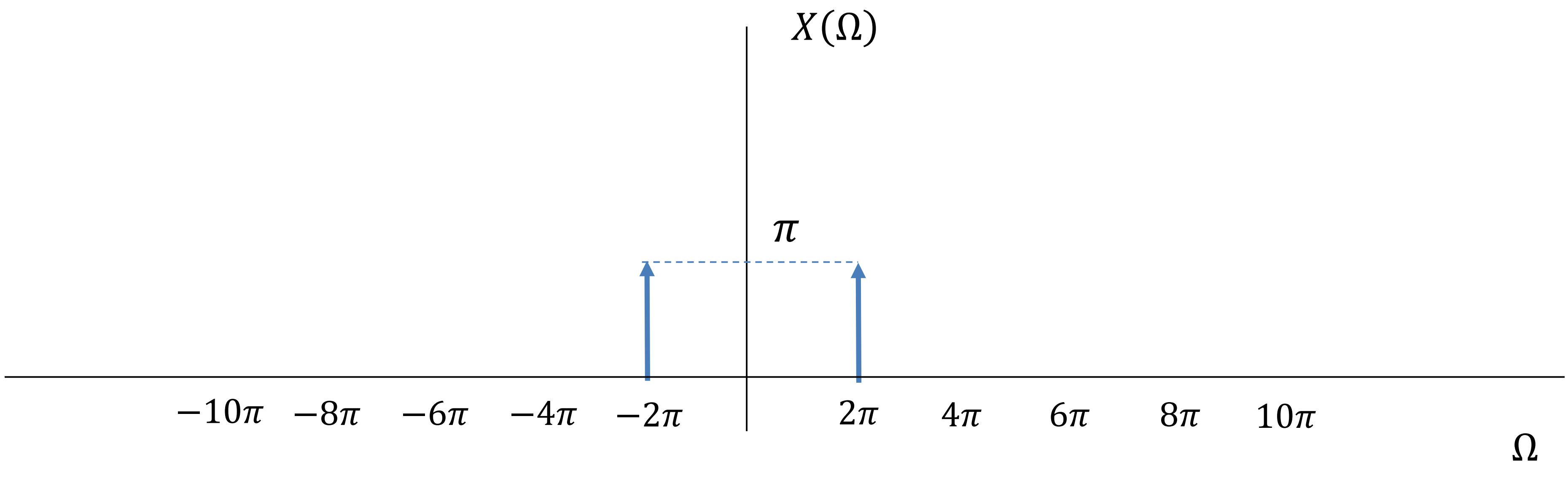




Example 1

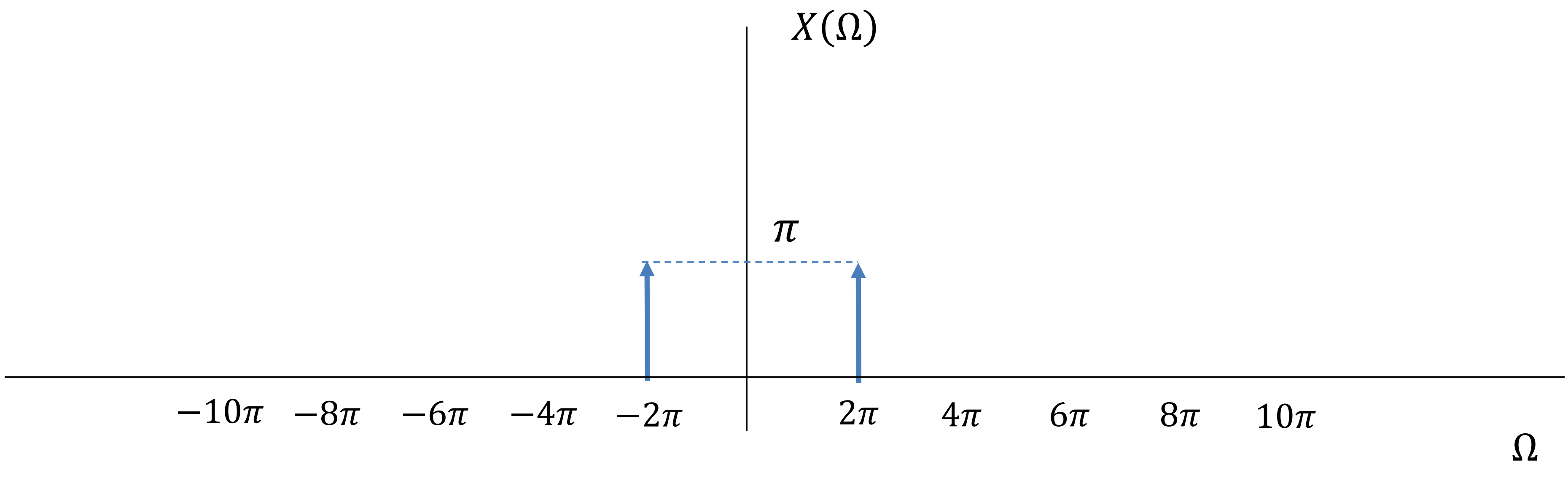
Aliasing

- **Example: Consider** $x(t) = \cos(2\pi t)$
  - **Sample this at a rate** of $\Omega_s = 8\pi$
    - ◊ **What is the Nyquist rate?**
    - ◊ **What is the cutoff frequency for the low-pass filter?**



Aliasing

- **Example: Consider** $x(t) = \cos(2\pi t)$
  - **Sample this at a rate** of $\Omega_s = 8\pi$
    - **What is the Nyquist rate?** $\Omega_s > 4\pi$
    - **What is the cutoff frequency for the low-pass filter?** $\Omega_c = 4\pi$



Aliasing

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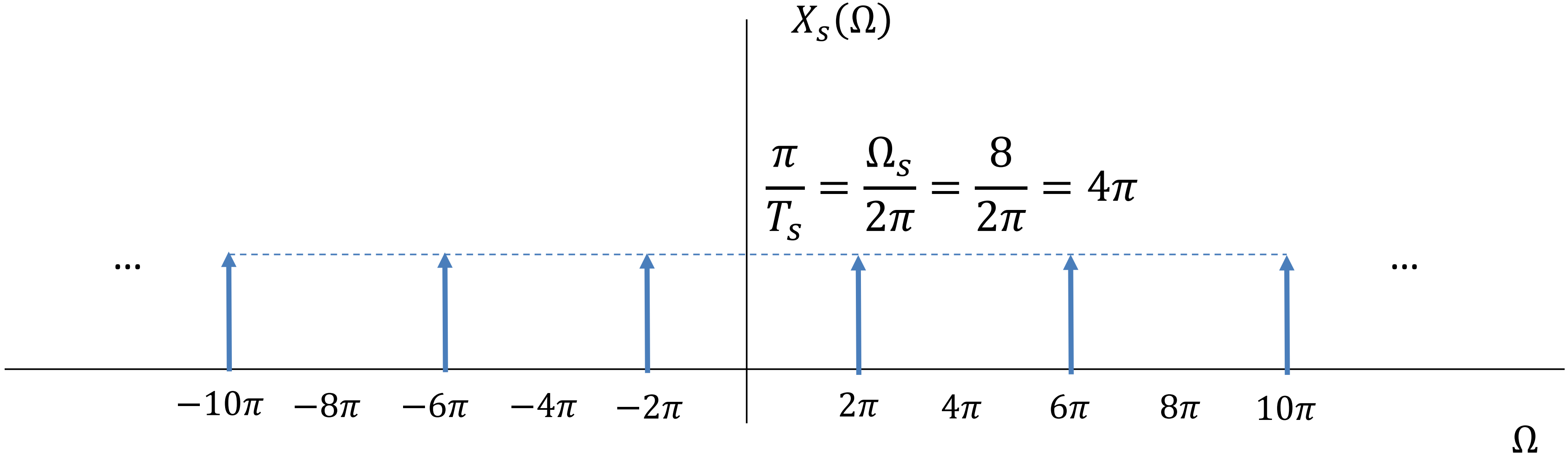

Aliasing

- **Example: Consider** $x(t) = \cos(2\pi t)$
  - **Sample this at a rate** of $\Omega_s = 8\pi$
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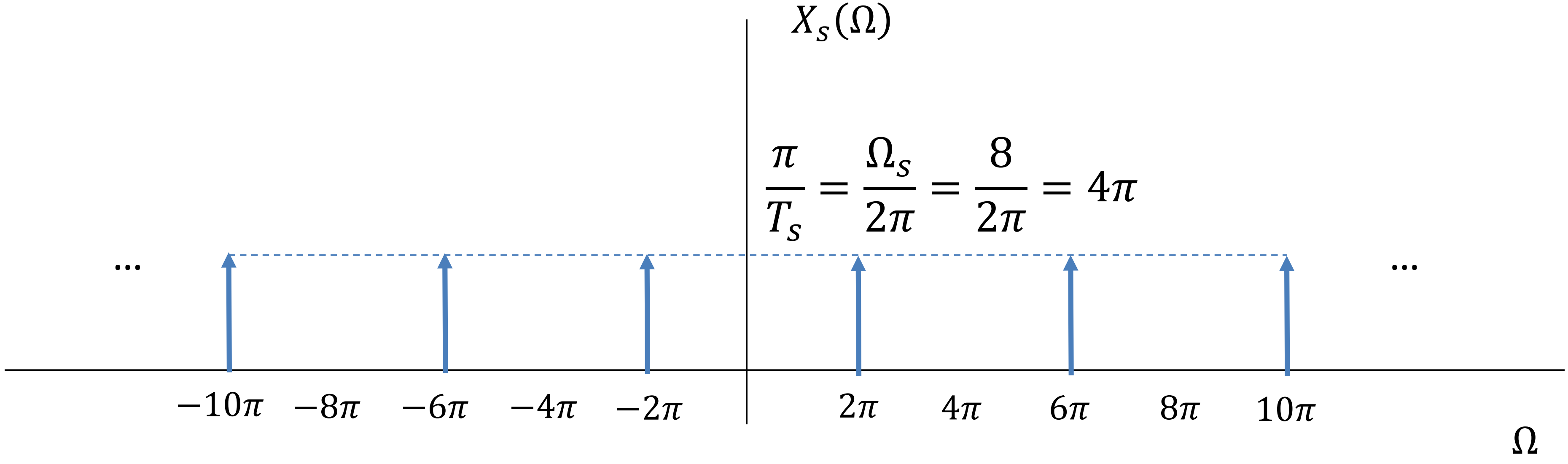
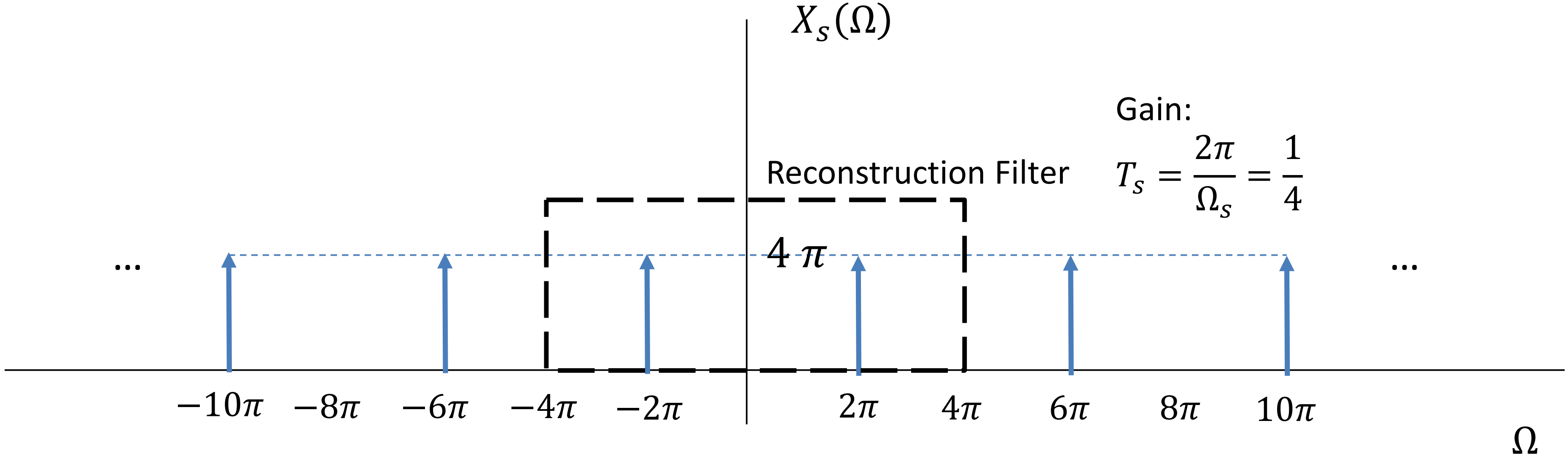
    - ◊ **What is the cutoff frequency for the low-pass filter?** $\Omega_c = 4\pi$



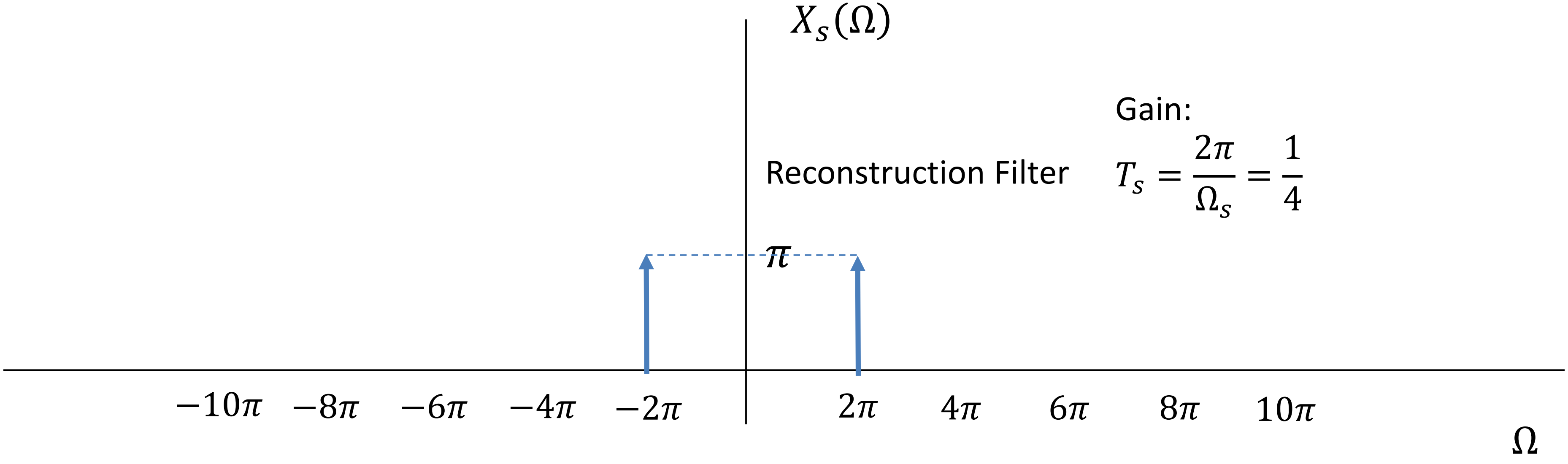
Aliasing

- **Example: Consider** $x(t) = \cos(2\pi t)$
  - **Sample this at a rate** of $\Omega_s = 8\pi$
    - **What is the Nyquist rate?** $\Omega_s > 4\pi$
    - **What is the cutoff frequency for the low-pass filter?** $\Omega_c = 4\pi$



Aliasing

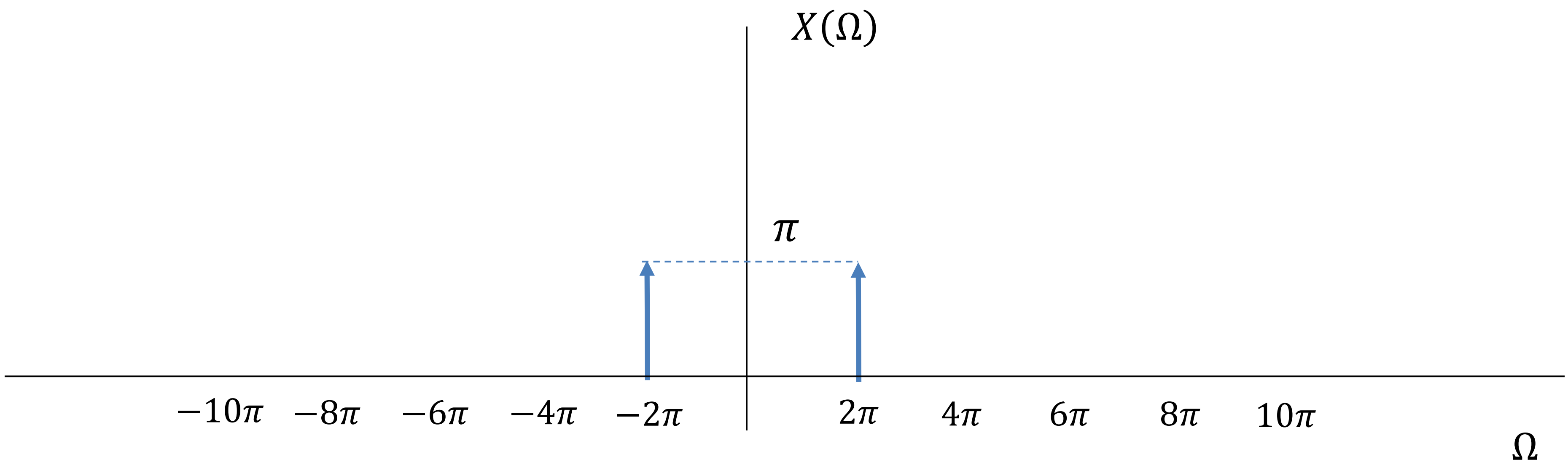
- **Example: Consider** $x(t) = \cos(2\pi t)$
  - **Sample this at a rate** of $\Omega_s = 8\pi$
    - **What is the Nyquist rate?** $\Omega_s > 4\pi$
    - **What is the cutoff frequency for the low-pass filter?** $\Omega_c = 4\pi$
    - **Reconstructed Signal:** $x(t) = \cos(2\pi t)$



Example 2

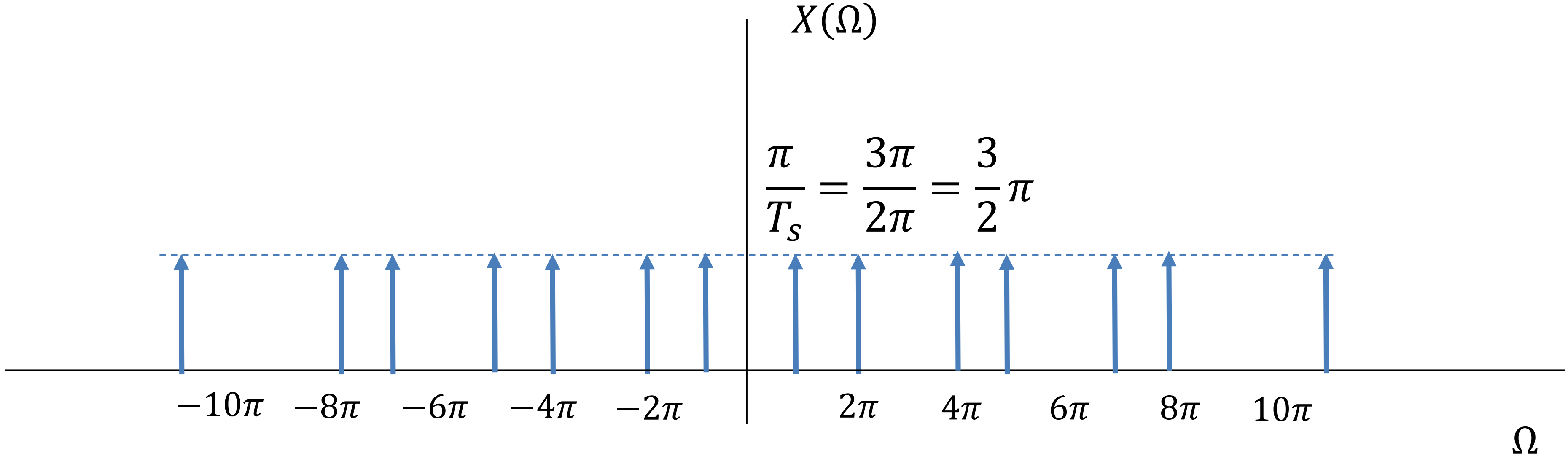
Aliasing

- **Example: Consider** $x(t) = \cos(2\pi t)$
  - **Sample this at a rate** of $\Omega_s = 3\pi$
    - **What is the Nyquist rate?** $\Omega_s > 4\pi$
    - **What is the cutoff frequency for the low-pass filter?** $\Omega_c = 1.5\pi$



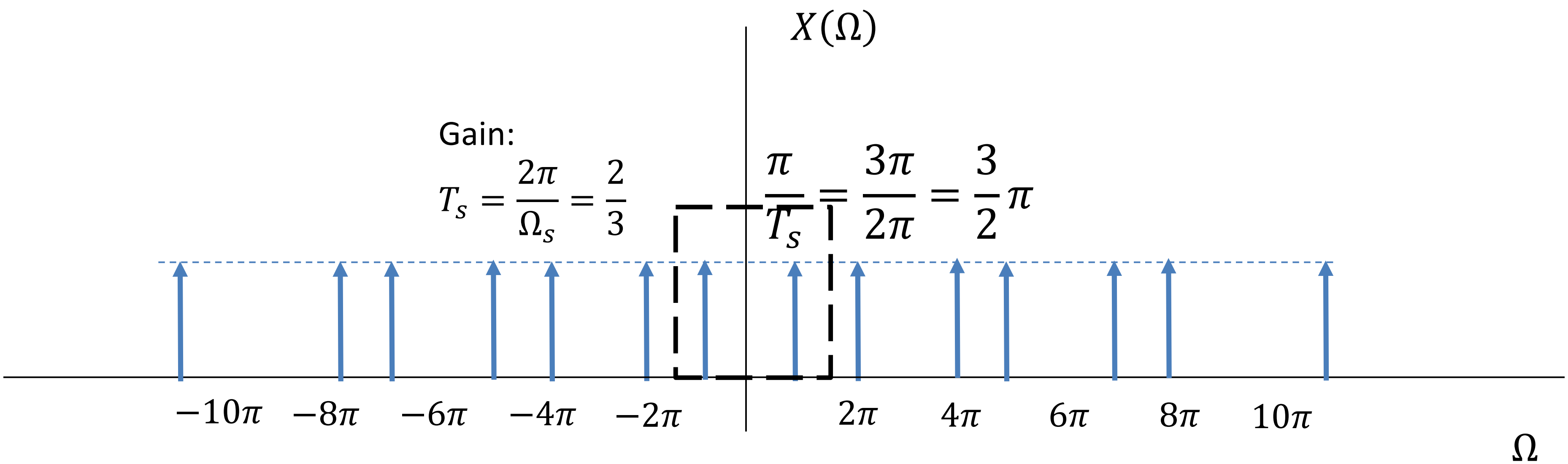
Aliasing

- **Example: Consider** $x(t) = \cos(2\pi t)$
  - **Sample this at a rate** of $\Omega_s = 3\pi$
    - **What is the Nyquist rate?** $\Omega_s > 4\pi$
    - **What is the cutoff frequency for the low-pass filter?** $\Omega_c = 1.5\pi$



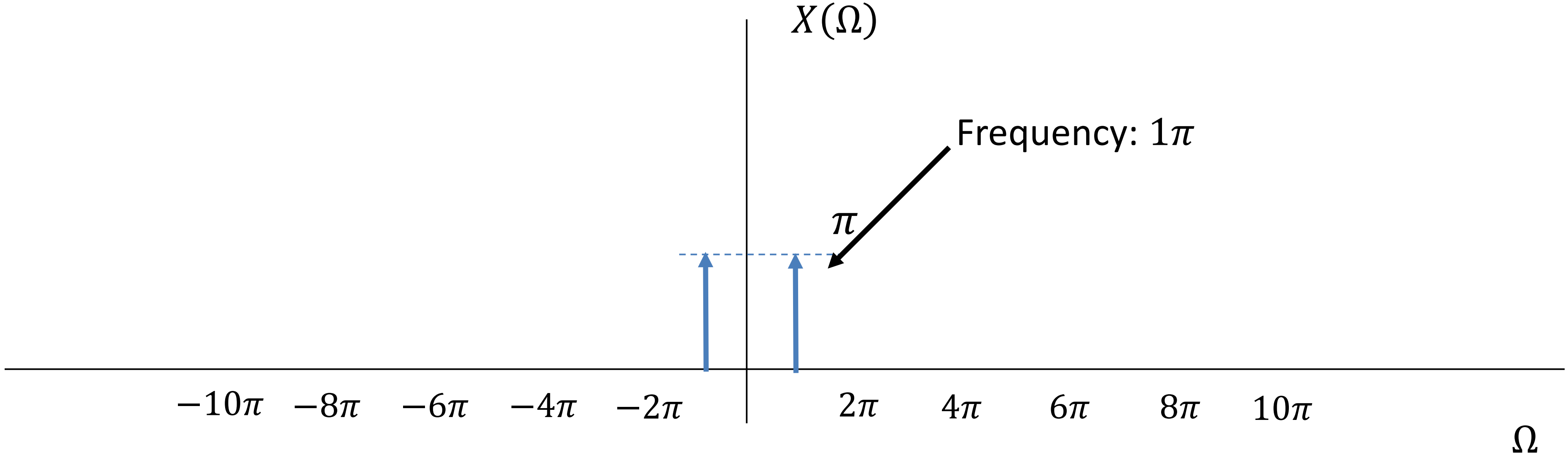
Aliasing

- **Example: Consider** $x(t) = \cos(2\pi t)$
  - **Sample this at a rate** of $\Omega_s = 3\pi$
    - **What is the Nyquist rate?** $\Omega_s > 4\pi$
    - **What is the cutoff frequency for the low-pass filter?** $\Omega_c = 1.5\pi$



Aliasing

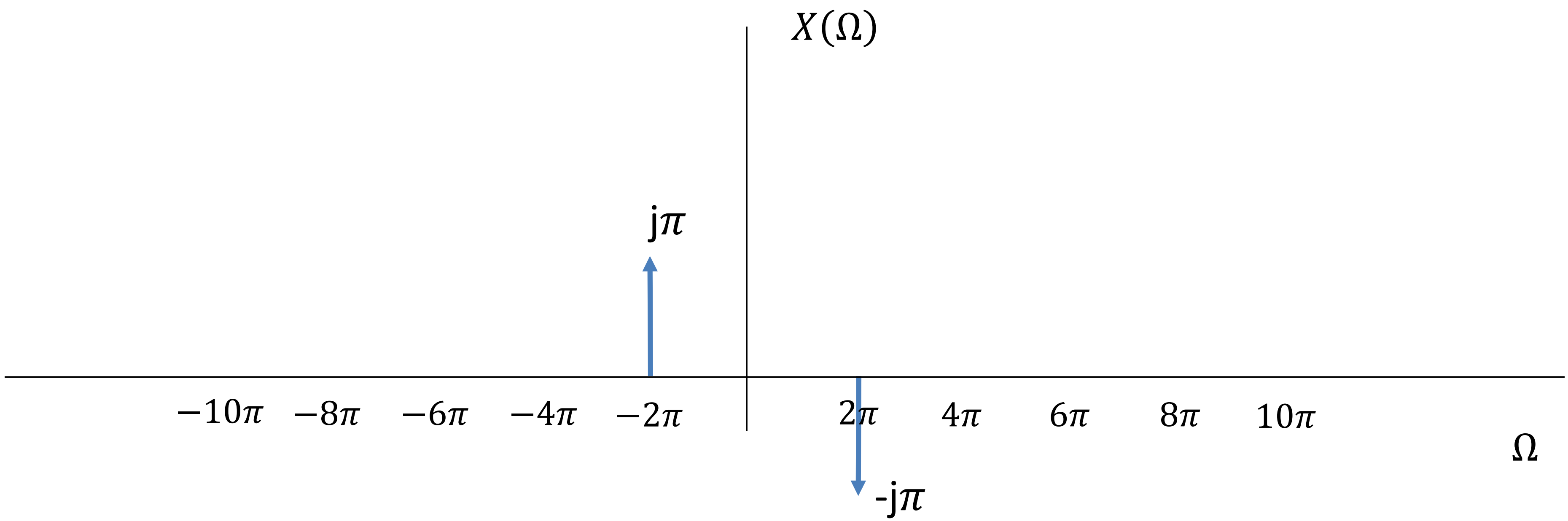
- **Example: Consider** $x(t) = \cos(2\pi t)$
  - **Sample this at a rate** of $\Omega_s = 3\pi$
    - **What is the Nyquist rate?** $\Omega_s > 4\pi$
    - **What is the cutoff frequency for the low-pass filter?** $\Omega_c = 1.5\pi$
    - **Reconstructed Signal:** $x(t) = \cos(\pi t)$



Example 3

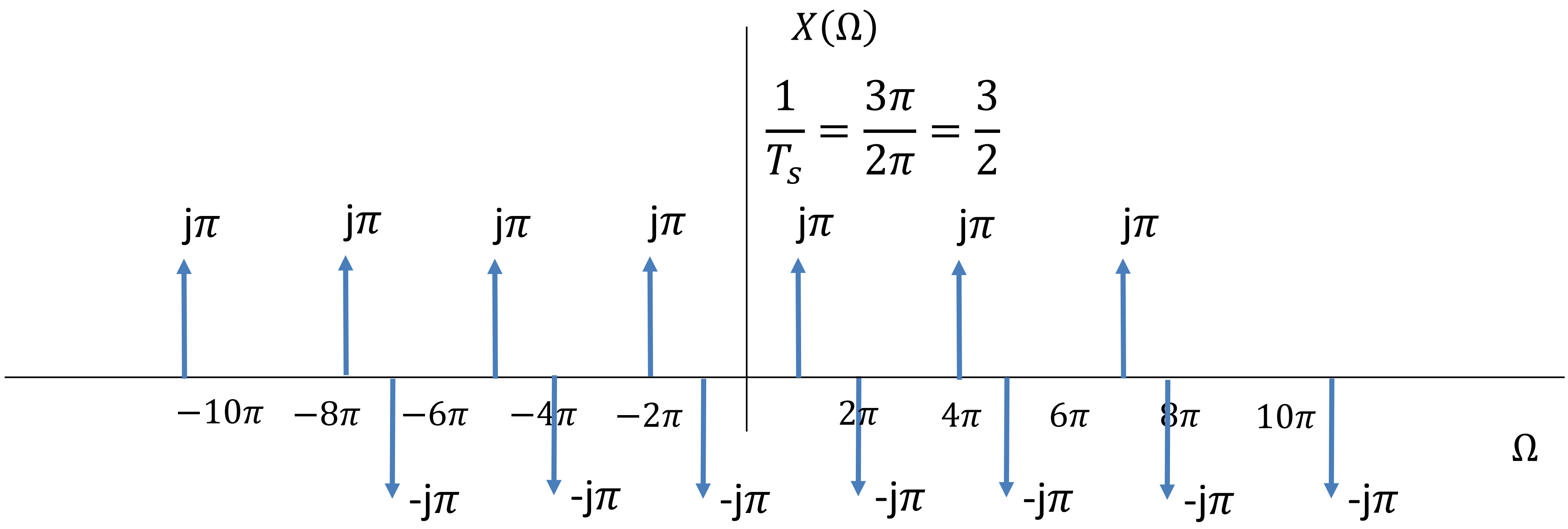
Aliasing

- **Consider** $x(t) = \sin(2\pi t)$
  - **Sample this at a rate** of $\Omega_s = 3\pi$
    - ◊ **What is the Nyquist rate?** $\Omega_s > 4\pi$
    - ◊ **What is the cutoff frequency for the low-pass filter?** $\Omega_c = 1.5\pi$



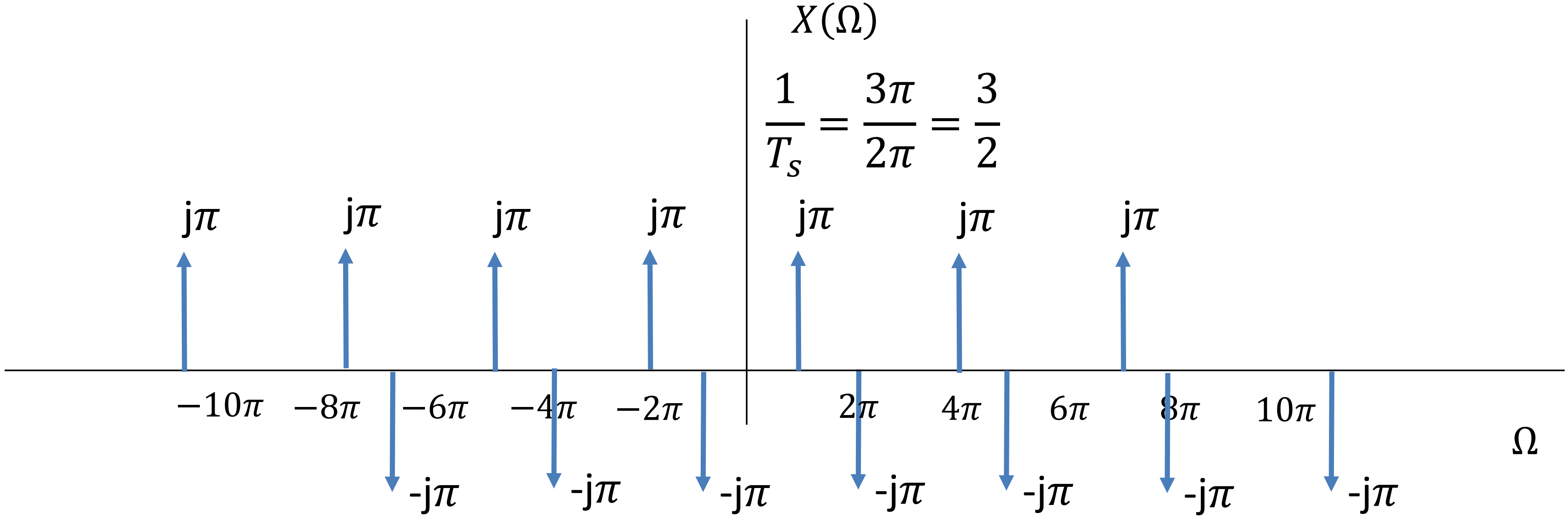
Aliasing

- **Consider** $x(t) = \sin(2\pi t)$
  - **Sample this at a rate** of $\Omega_s = 3\pi$
    - **What is the Nyquist rate?** $\Omega_s > 4\pi$
    - **What is the cutoff frequency for the low-pass filter?** $\Omega_c = 1.5\pi$



Aliasing

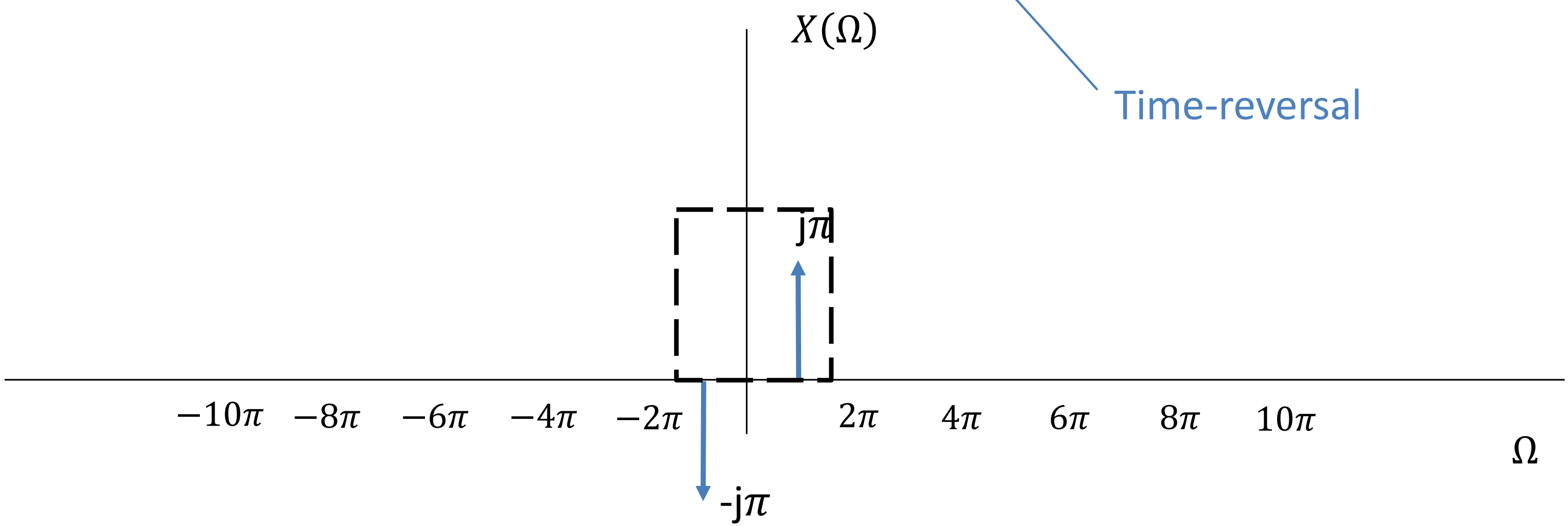
- **Consider** $x(t) = \sin(2\pi t)$
  - **Sample this at a rate** of $\Omega_s = 3\pi$
    - **What is the Nyquist rate?** $\Omega_s > 4\pi$
    - **What is the cutoff frequency for the low-pass filter?** $\Omega_c = 1.5\pi$



Aliasing

- **Consider** $x(t) = \sin(2\pi t)$
  - **Sample this at a rate** of $\Omega_s = 3\pi$
    - **What is the Nyquist rate?** $\Omega_s > 4\pi$
    - **What is the cutoff frequency for the low-pass filter?** $\Omega_c = 1.5\pi$
    - **Reconstructed Signal:** $x(t) = \sin(-\pi t) = -\sin(\pi t)$

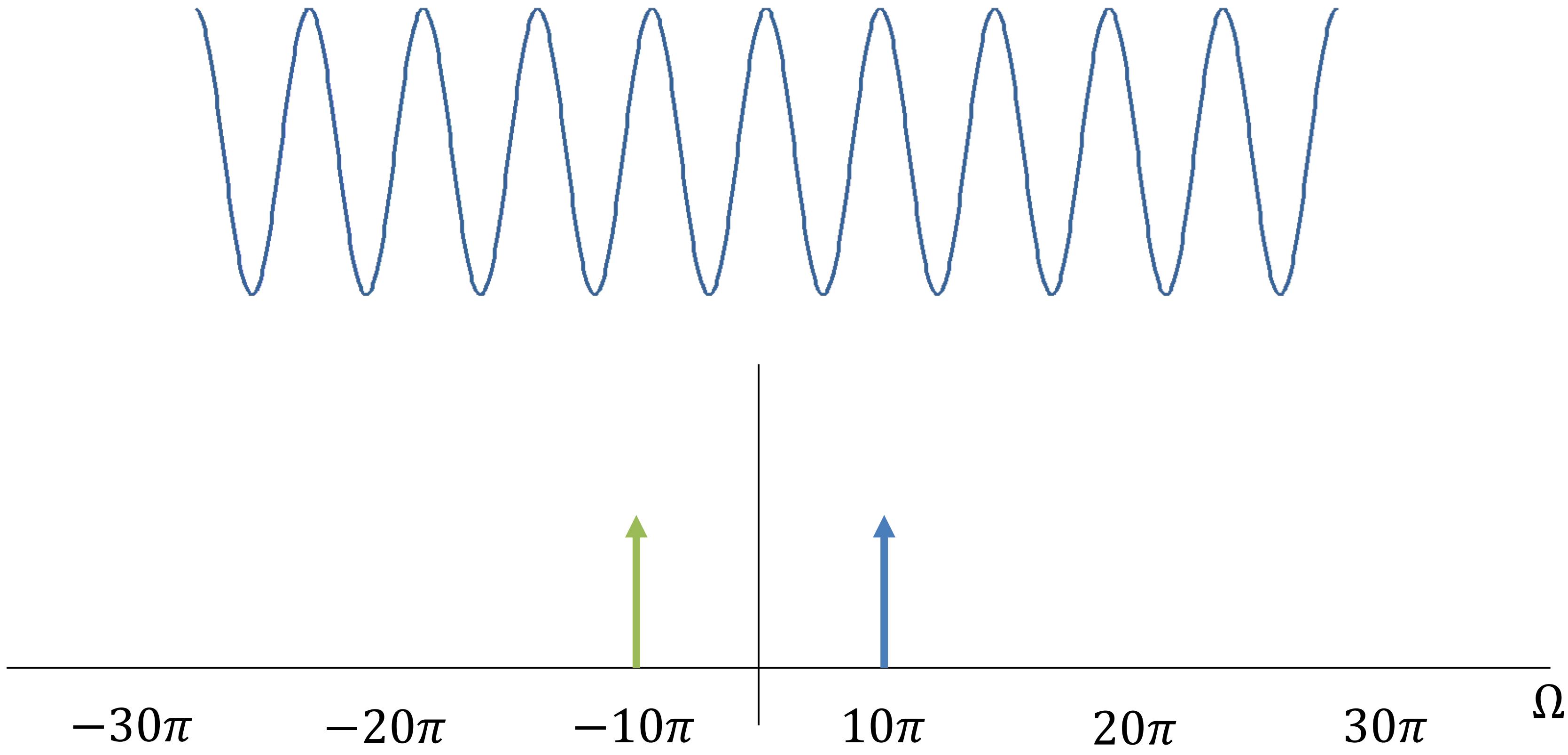
Time-reversal



Example 4

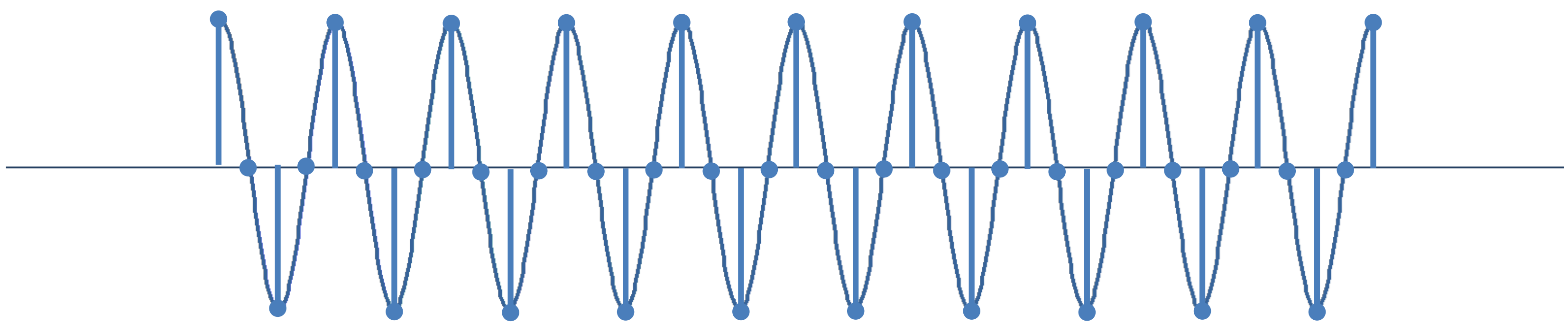
Aliasing with a sinusoid

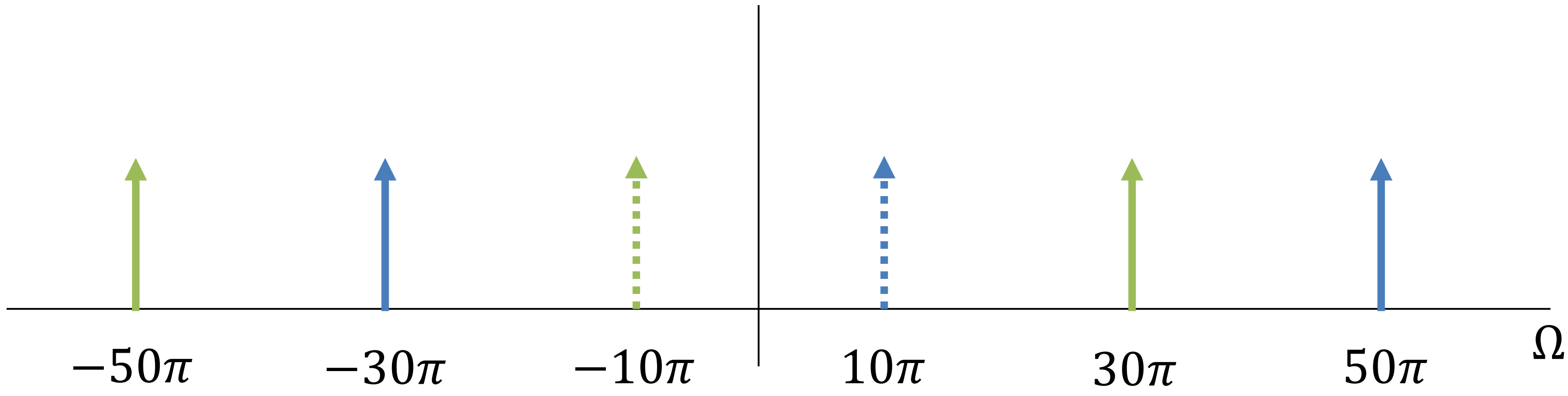
- $x(t) = \cos(10\pi t) \quad (T = 1/5)$



Aliasing with a sinusoid

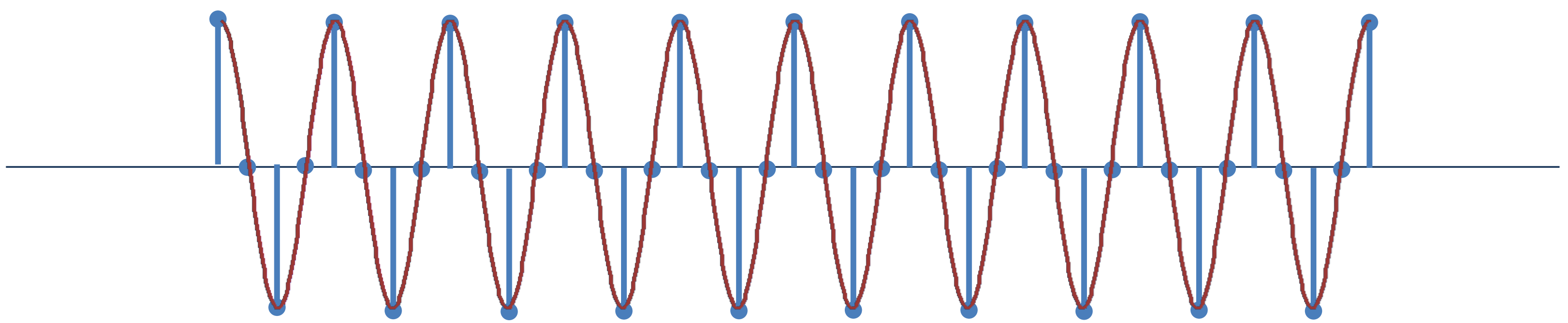
- $x(t) = \cos(10\pi t) \quad (T = 1/5)$
  - Sample with a sampling rate of $\Omega_s = 40\pi \ (T_s = 1/20)$

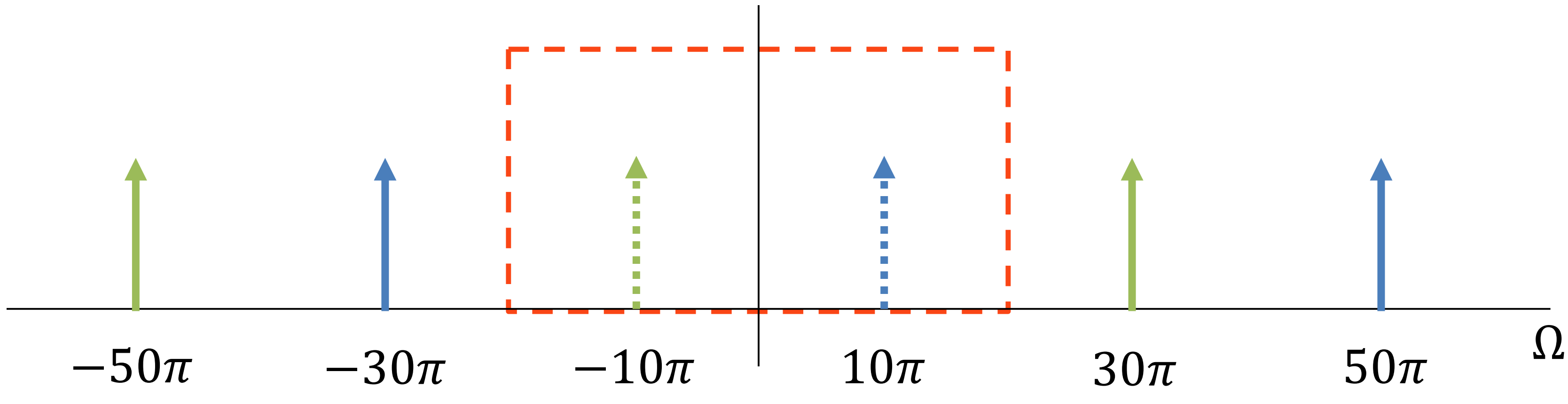




Aliasing with a sinusoid

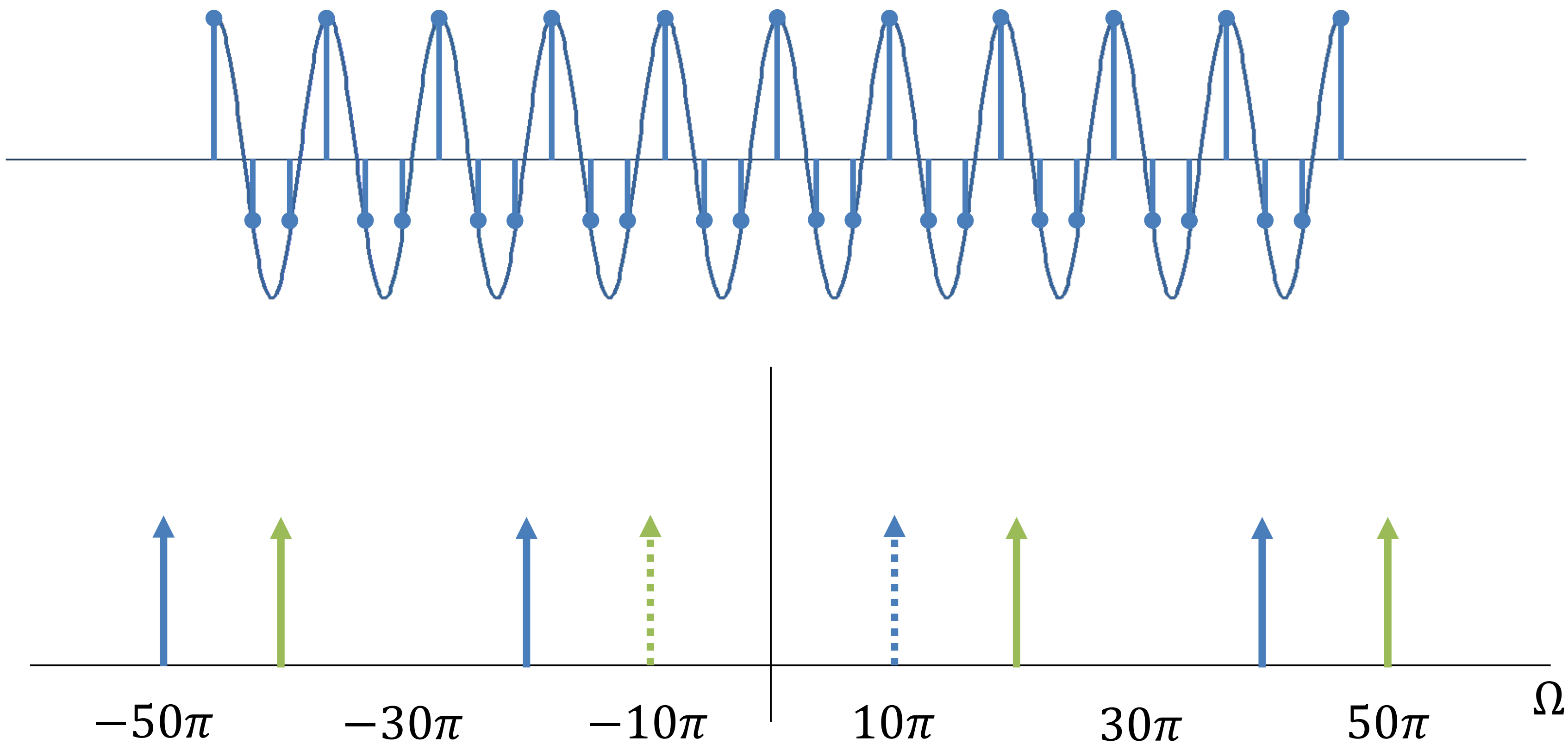
- $x(t) = \cos(10\pi t) \quad (T = 1/5)$
  - Sample with a sampling rate of $\Omega_s = 40\pi \;\; (T_s = 1/20)$





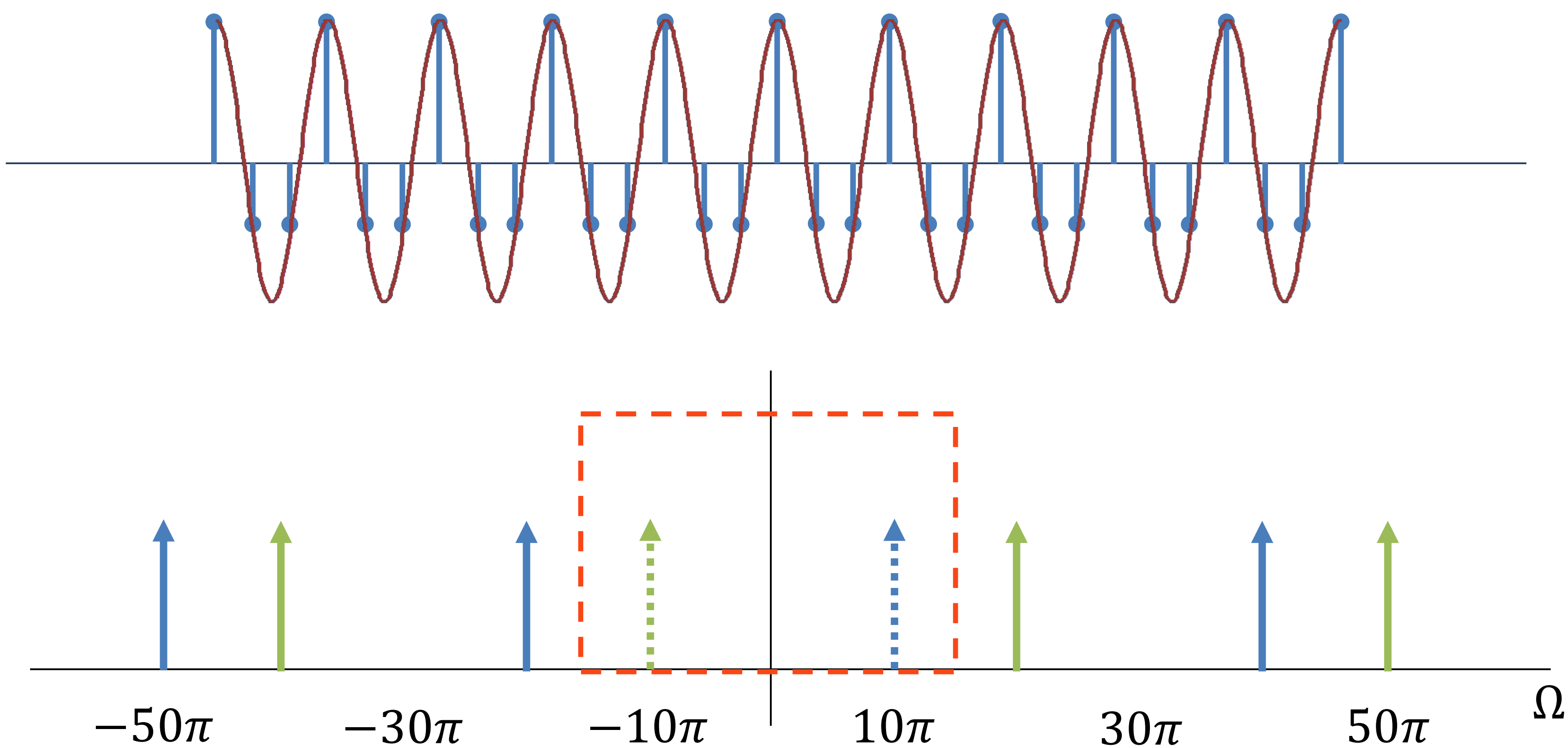
Aliasing with a sinusoid

- $x(t) = \cos(10\pi t) \quad (T = 1/5)$
  - Sample with a sampling rate of $\Omega_s = 30\pi \; (T_s = 1/15)$



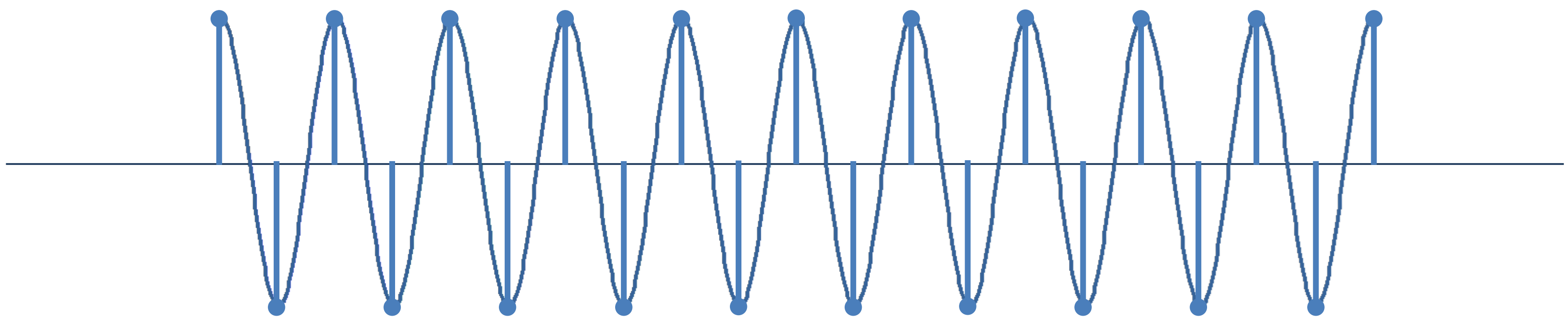
Aliasing with a sinusoid

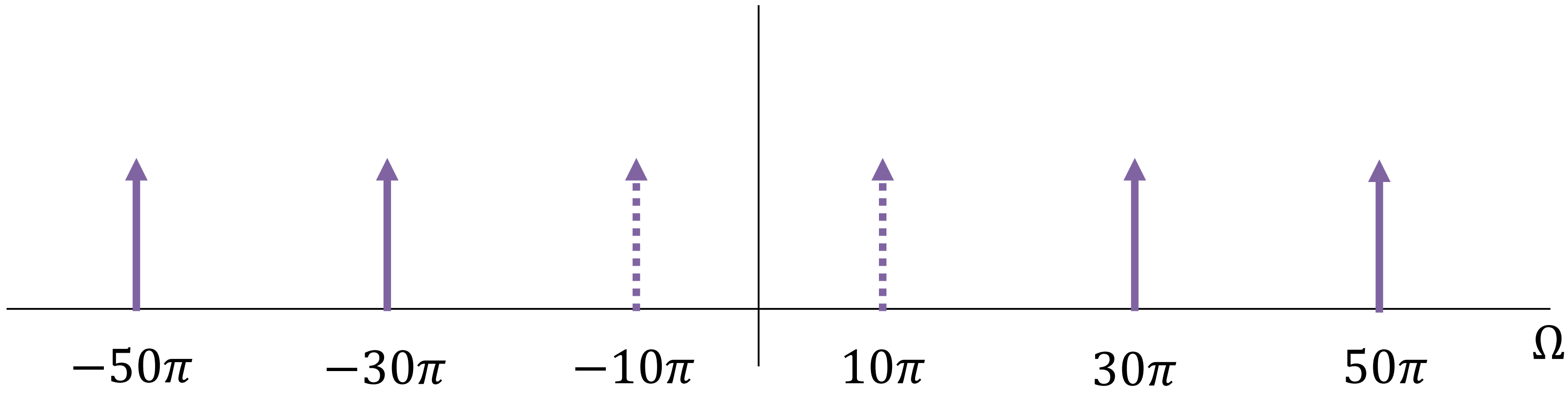
- $x(t) = \cos(10\pi t) \quad (T = 1/5)$
  - Sample with a sampling rate of $\Omega_s = 30\pi \;\; (T_s = 1/15)$



Aliasing with a sinusoid

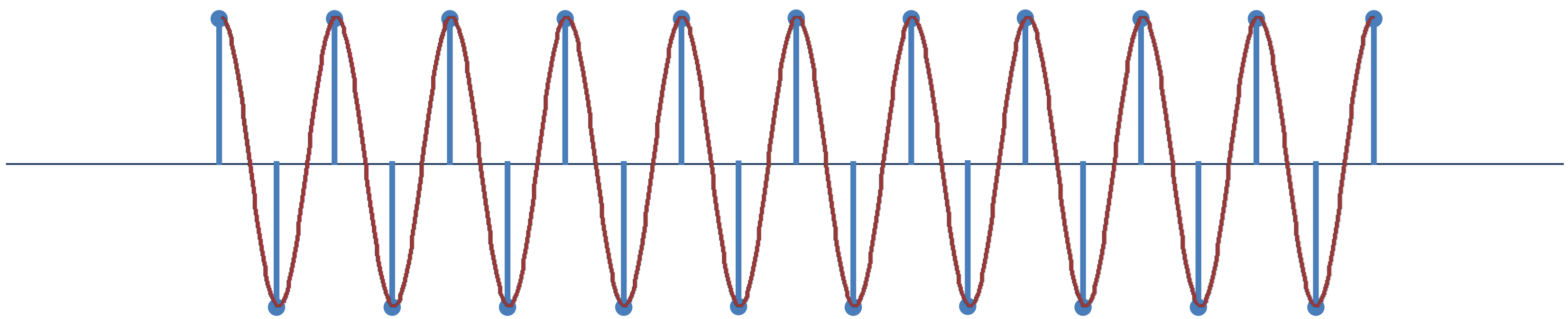
- $x(t) = \cos(10\pi t) \quad (T = 1/5)$
  - Sample with a sampling rate of $\Omega_s = 20\pi \ (T_s = 1/10)$

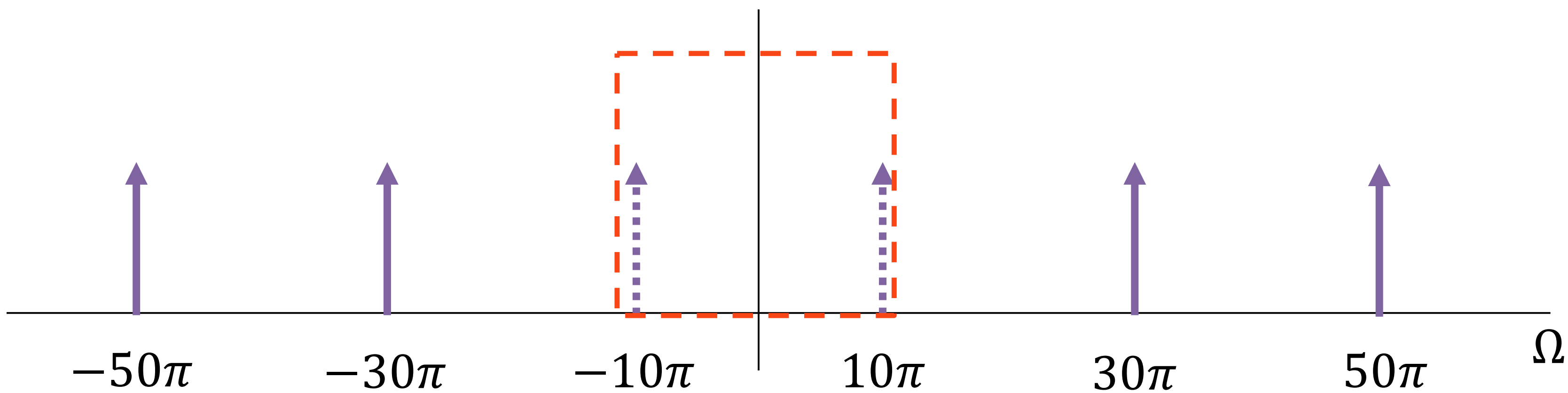




Aliasing with a sinusoid

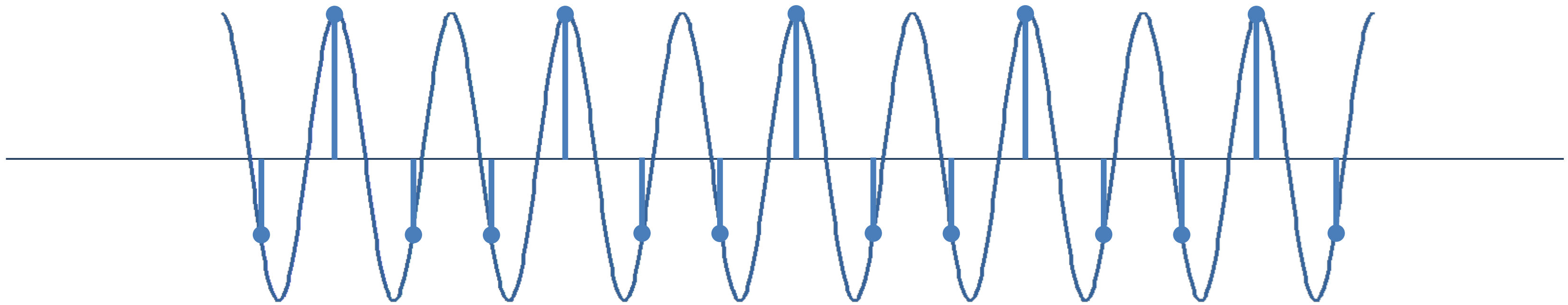
- $x(t) = \cos(10\pi t) \quad (T = 1/5)$
  - Sample with a sampling rate of $\Omega_s = 20\pi \;\; (T_s = 1/10)$

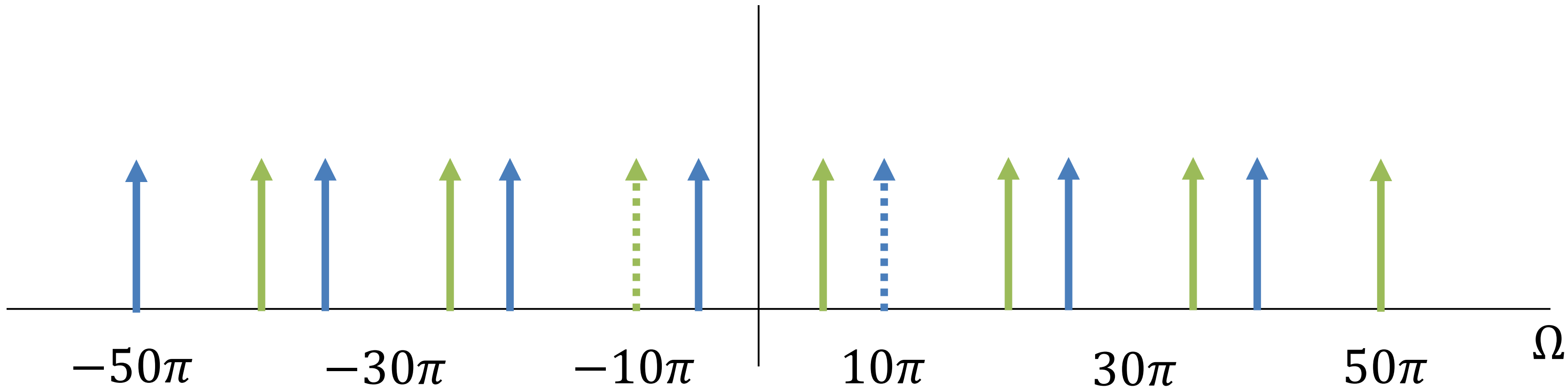




Aliasing with a sinusoid

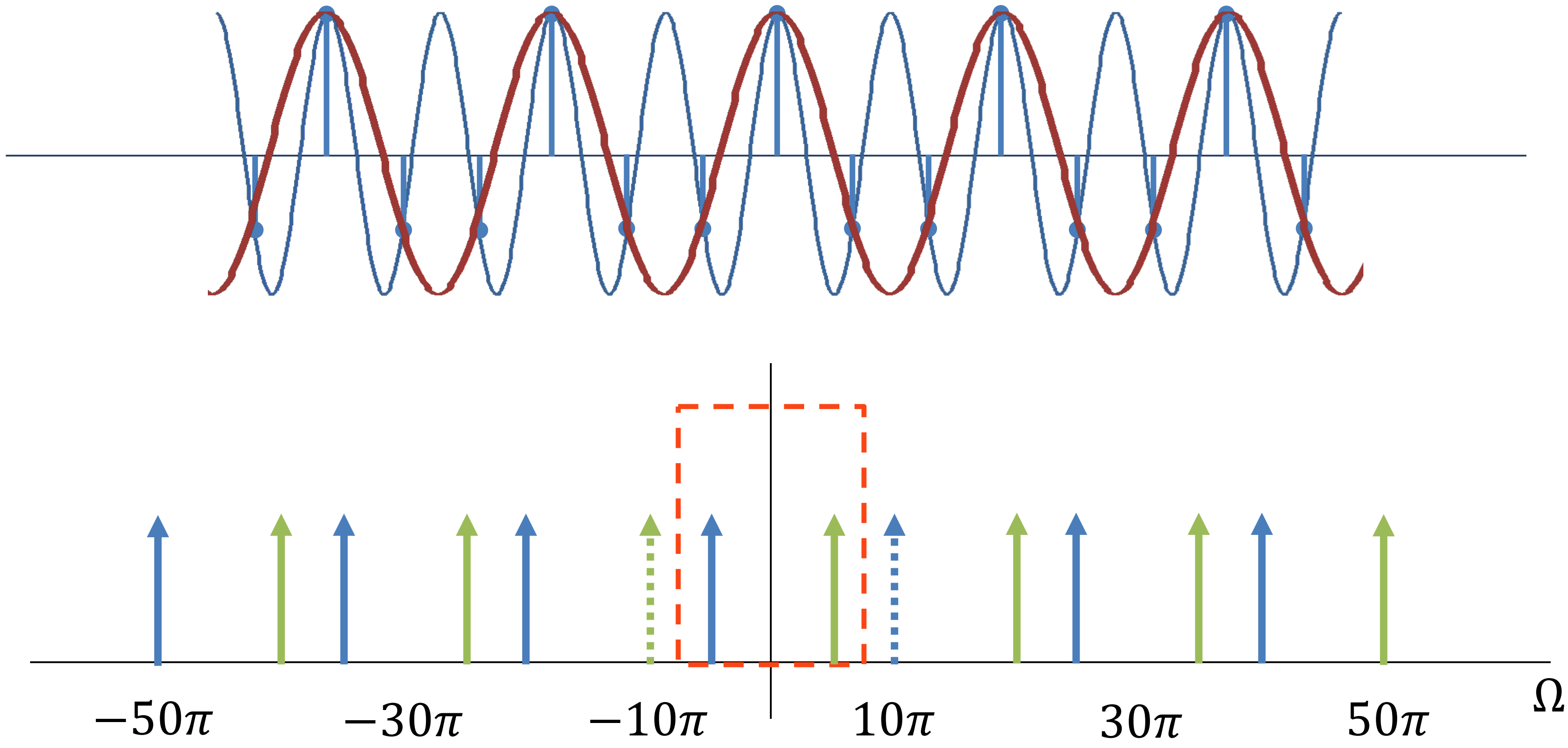
- $x(t) = \cos(10\pi t) \quad (T = 1/5)$
  - Sample with a sampling rate of $\Omega_s = 15\pi \ (T_s = 2/15)$





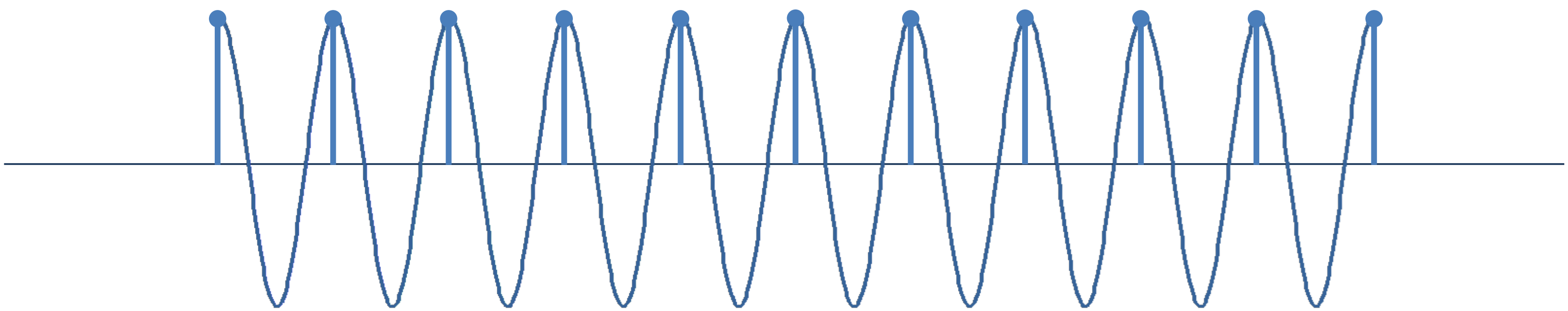
Aliasing with a sinusoid

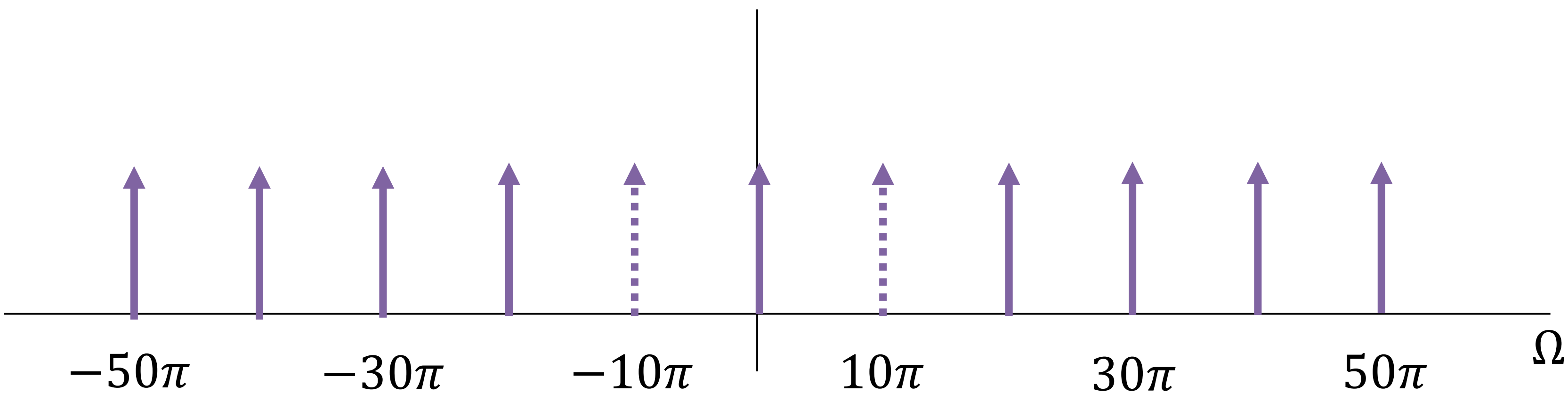
- $x(t) = \cos(10\pi t) \quad (T = 1/5)$
  - Sample with a sampling rate of $\Omega_s = 15\pi \ (T_s = 2/15)$



Aliasing with a sinusoid

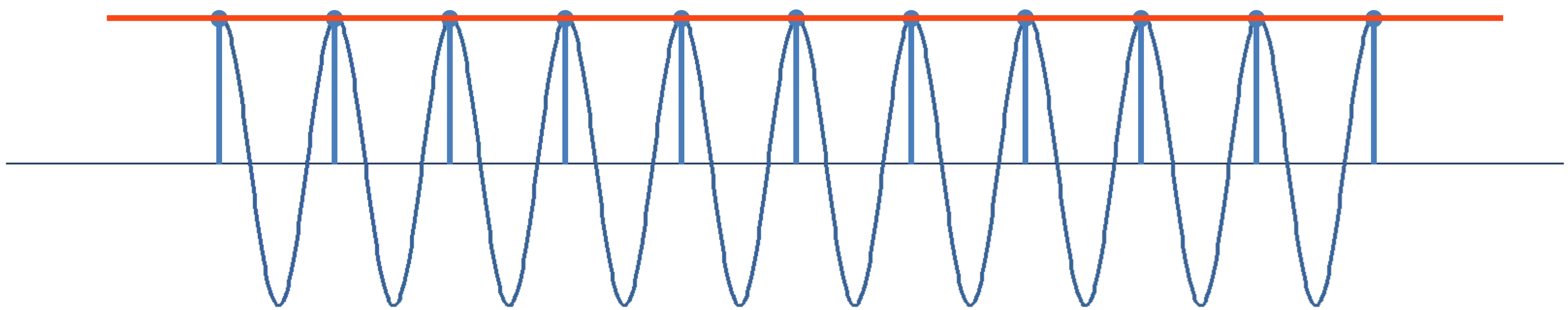
- $x(t) = \cos(10\pi t) \quad (T = 1/5)$
  - Sample with a sampling rate of $\Omega_s = 10\pi \ (T_s = 1/5)$

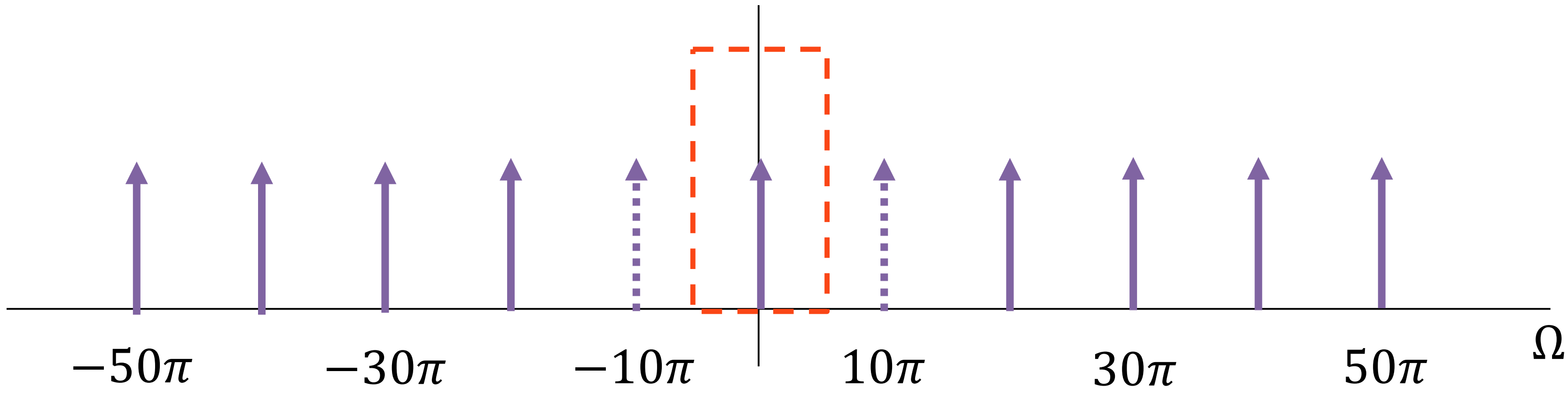




Aliasing with a sinusoid

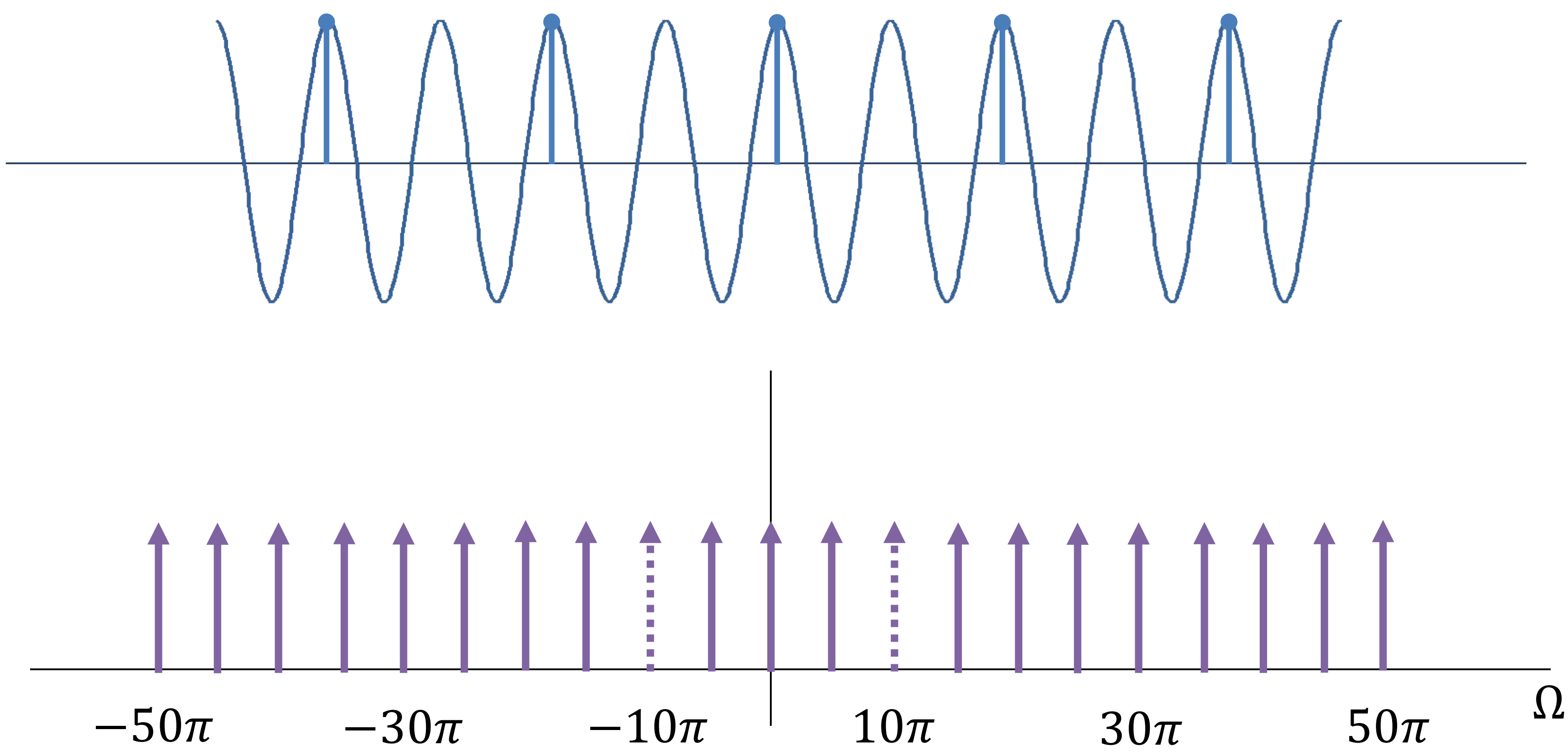
- $x(t) = \cos(10\pi t) \quad (T = 1/5)$
  - Sample with a sampling rate of $\Omega_s = 10\pi \ (T_s = 1/5)$





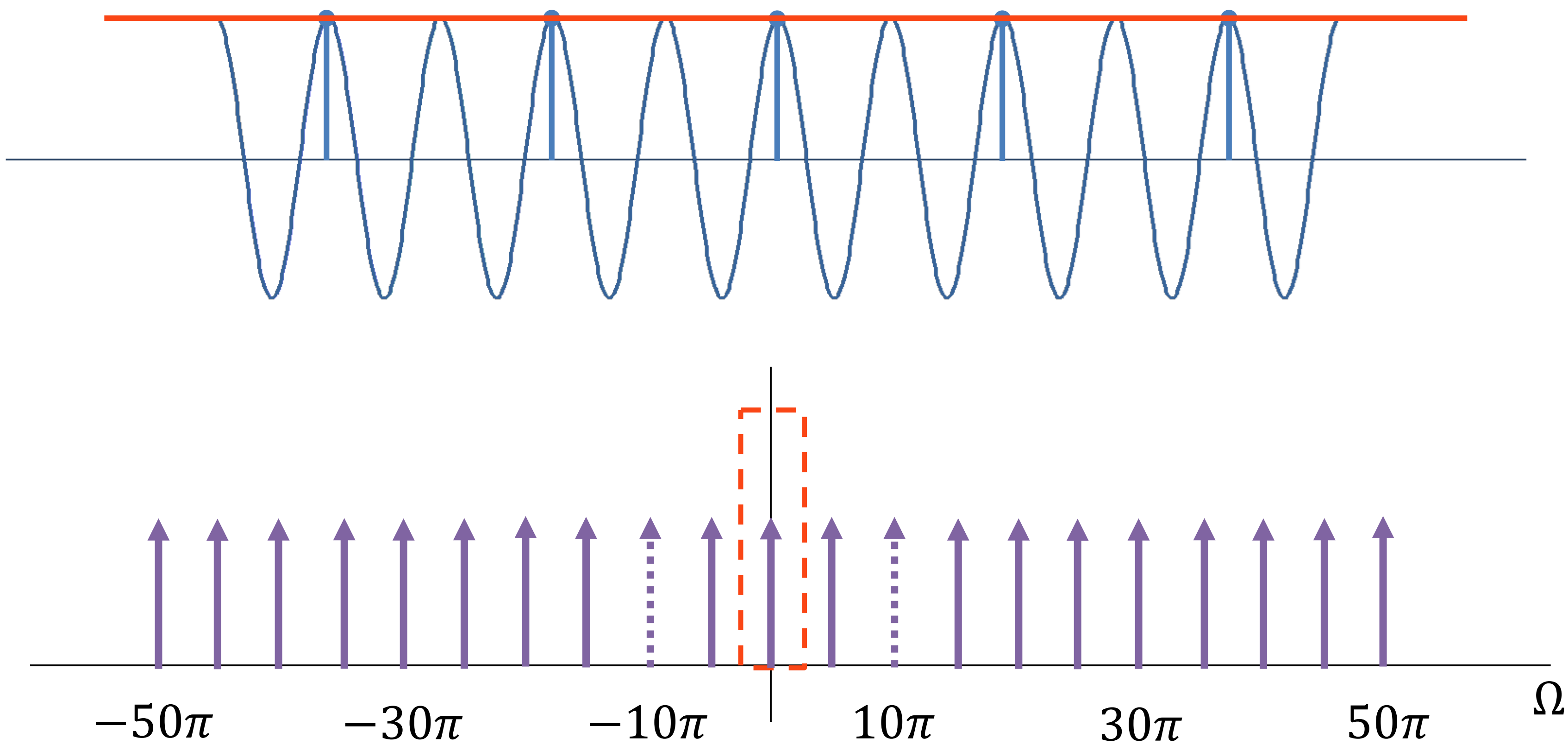
Aliasing with a sinusoid

- $x(t) = \cos(10\pi t) \quad (T = 1/5)$
  - Sample with a sampling rate of $\Omega_s = 5\pi \; (T_s = 2/5)$



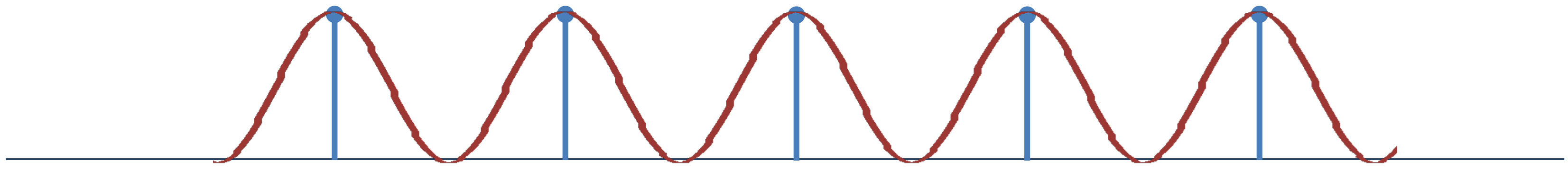
Aliasing with a sinusoid

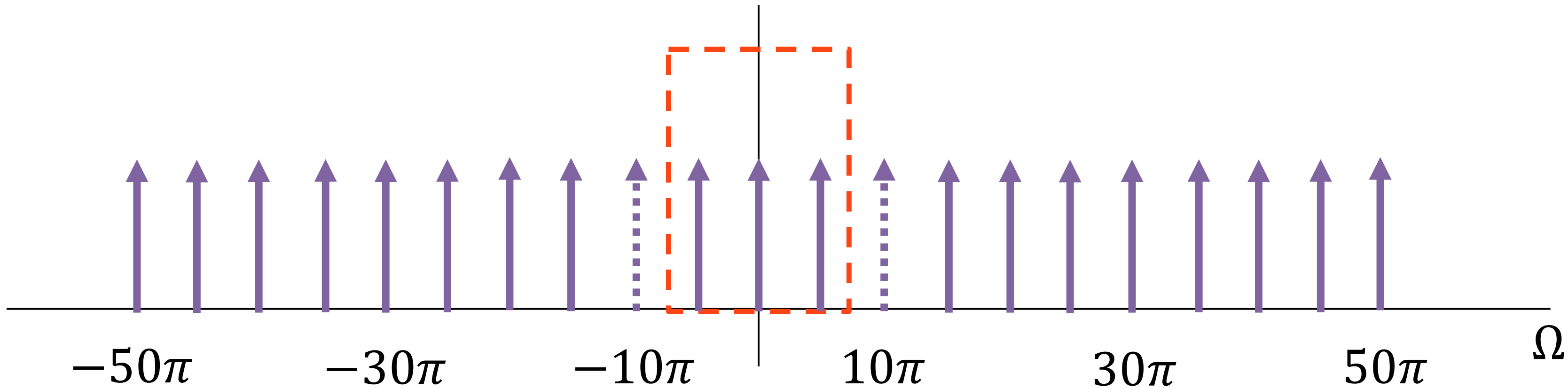
- $x(t) = \cos(10\pi t) \quad (T = 1/5)$
  - Sample with a sampling rate of $\Omega_s = 5\pi \;\; (T_s = 2/5)$



Aliasing with a sinusoid

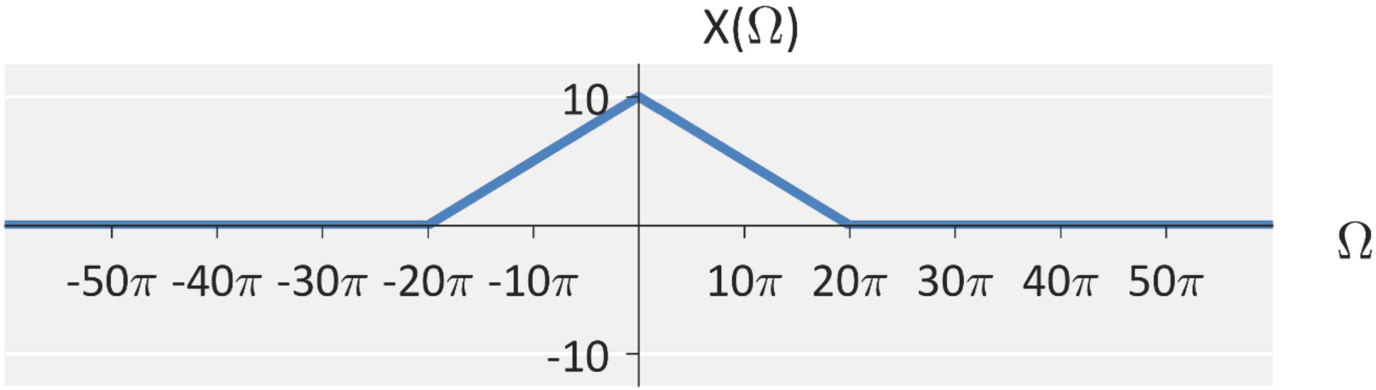
- $x(t) = \cos(10\pi t) \quad (T = 1/5)$
  - Sample with a sampling rate of $\Omega_s = 5\pi \;\; (T_s = 2/5)$





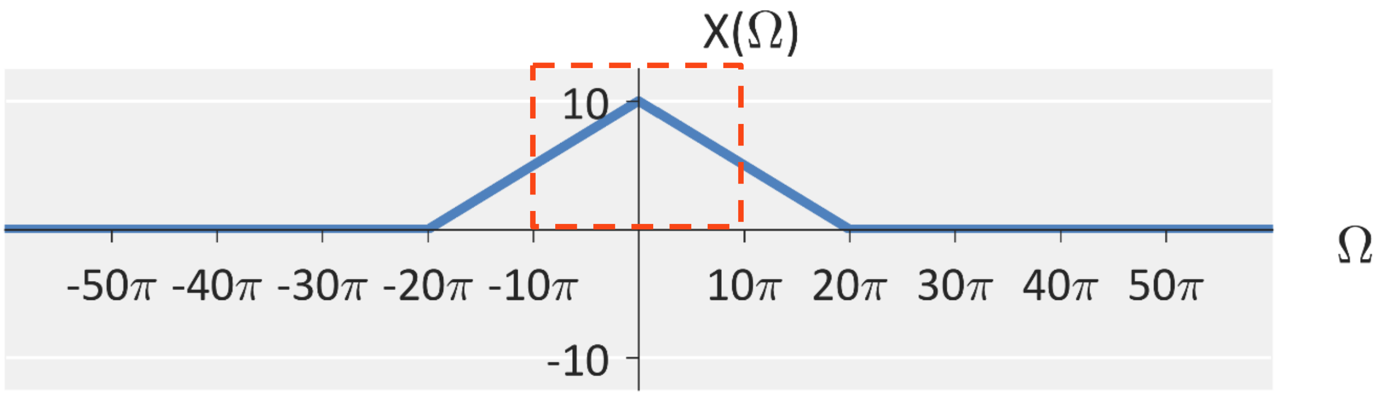
Anti-Aliasing Filter

- **Question: How do we reduce the effects of aliasing?**



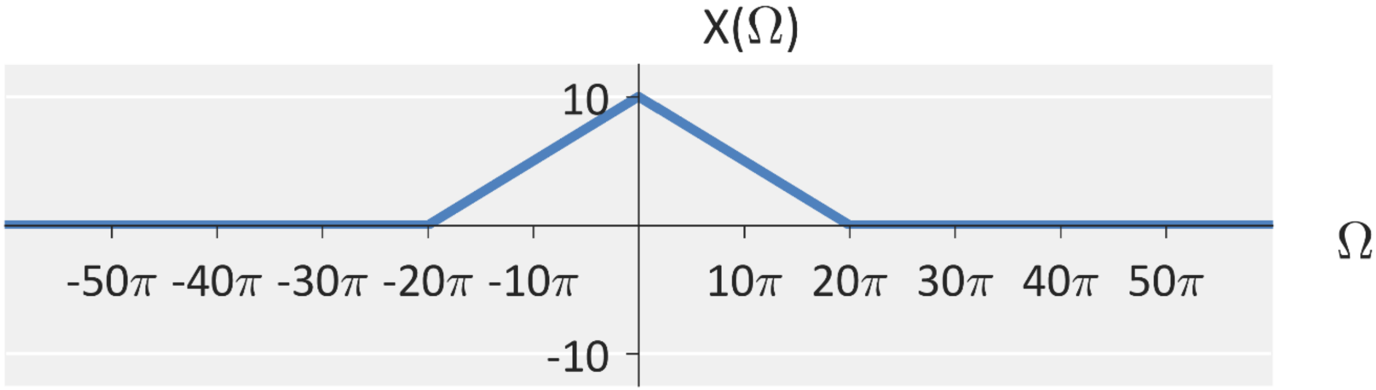
Anti-Aliasing Filter

- **Question: How do we reduce the effects of aliasing?**
  - Apply a low-pass anti-aliasing filter
    - Cut-off frequency: $\Omega_s/2$
    - Gain: 1



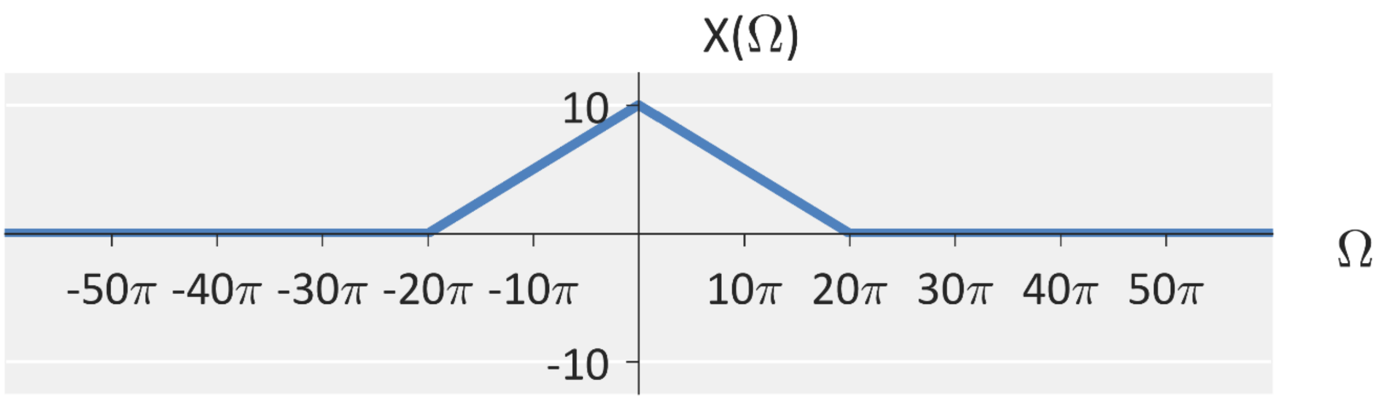
Anti-Aliasing Filter

- **Example: Consider the following signal.**
  - What is the Nyquist rate?



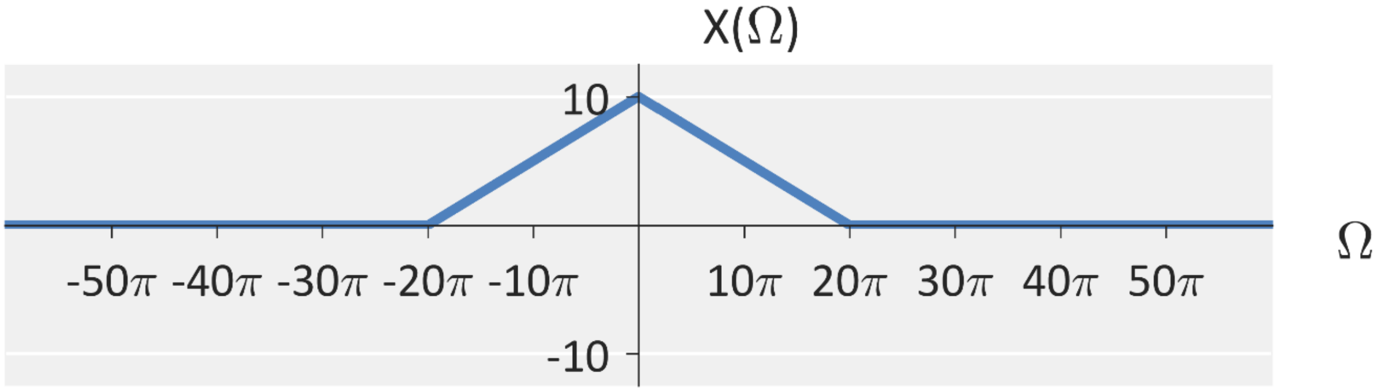
Anti-Aliasing Filter

- **Example: Consider the following signal.**
  - What is the Nyquist rate? 40π



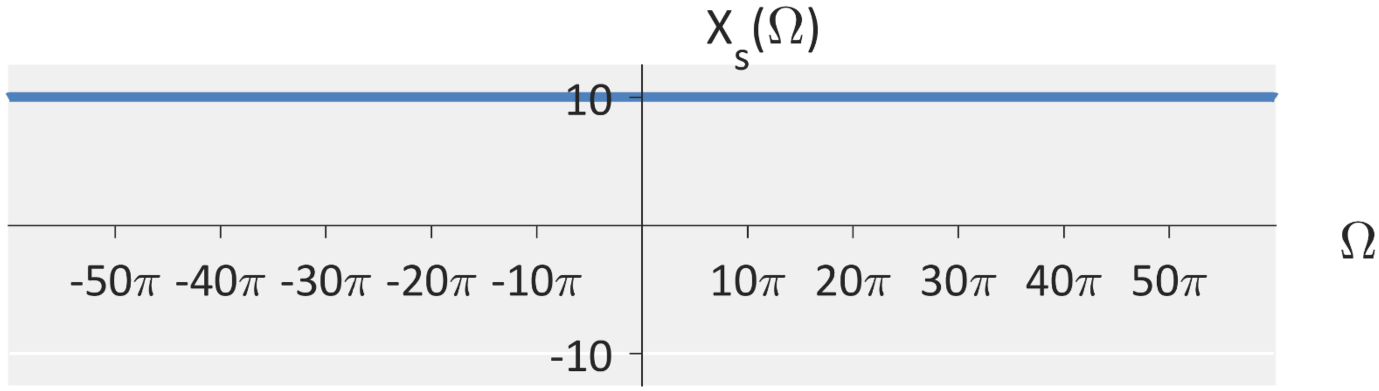
Anti-Aliasing Filter

- **Example: Consider the following signal.**
  - What is the Nyquist rate? 40π
  - Sketch the Fourier transform after sampling at $\Omega_s = 20\pi$.
  - Use no anti-aliasing filter



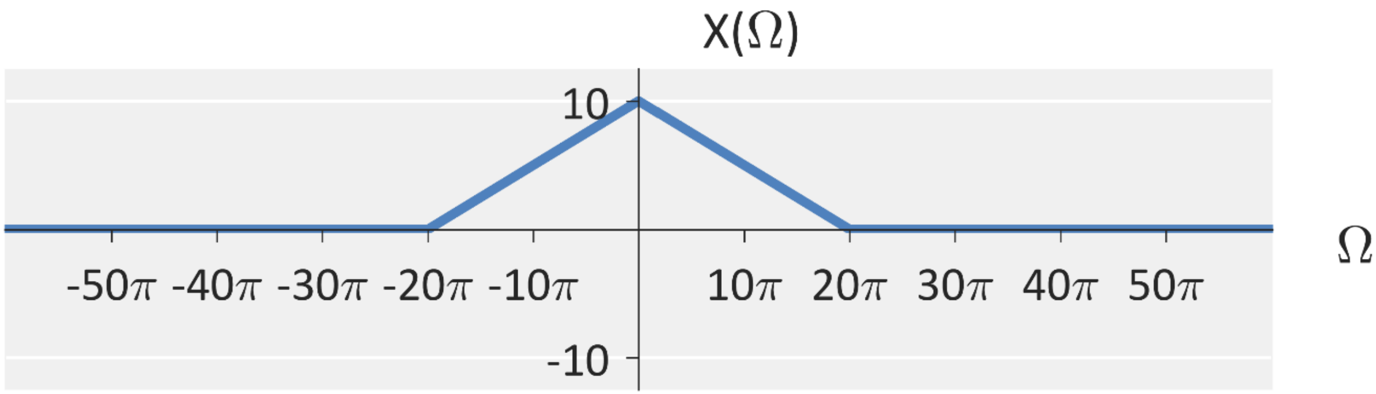
Anti-Aliasing Filter

- **Example: Consider the following signal.**
  - What is the Nyquist rate? 40π
  - Sketch the Fourier transform after sampling at $\Omega_s = 20\pi$.
  - Use no anti-aliasing filter



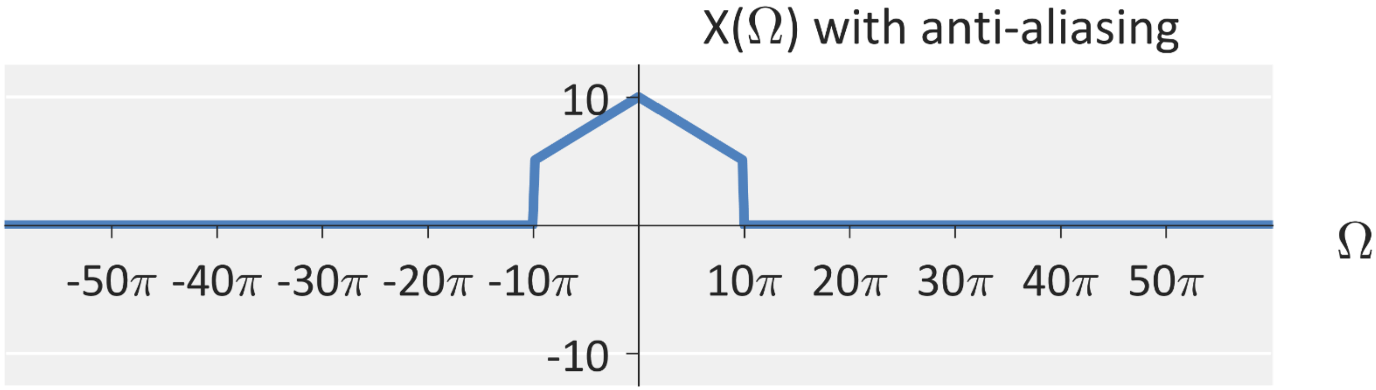
Anti-Aliasing Filter

- **Example: Consider the following signal.**
  - What is the Nyquist rate? 40π
  - Sketch the Fourier transform after sampling at $\Omega_s = 20\pi$.
  - Use an anti-aliasing filter



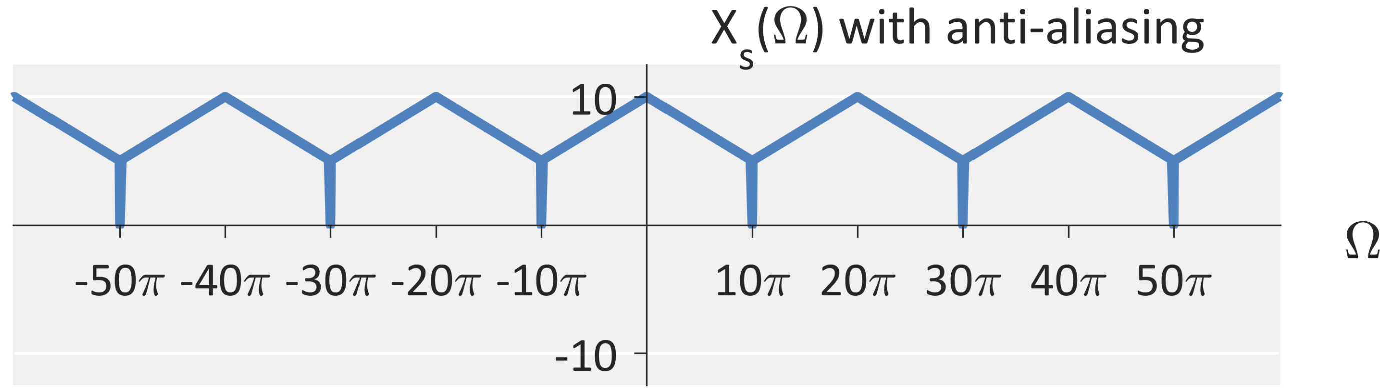
Anti-Aliasing Filter

- **Example: Consider the following signal.**
  - What is the Nyquist rate? 40π
  - Sketch the Fourier transform after sampling at $\Omega_s = 20\pi$.
  - Use an anti-aliasing filter



Anti-Aliasing Filter

- **Example: Consider the following signal.**
  - What is the Nyquist rate? 40π
  - Sketch the Fourier transform after sampling at $\Omega_s = 20\pi$.
  - Use an anti-aliasing filter



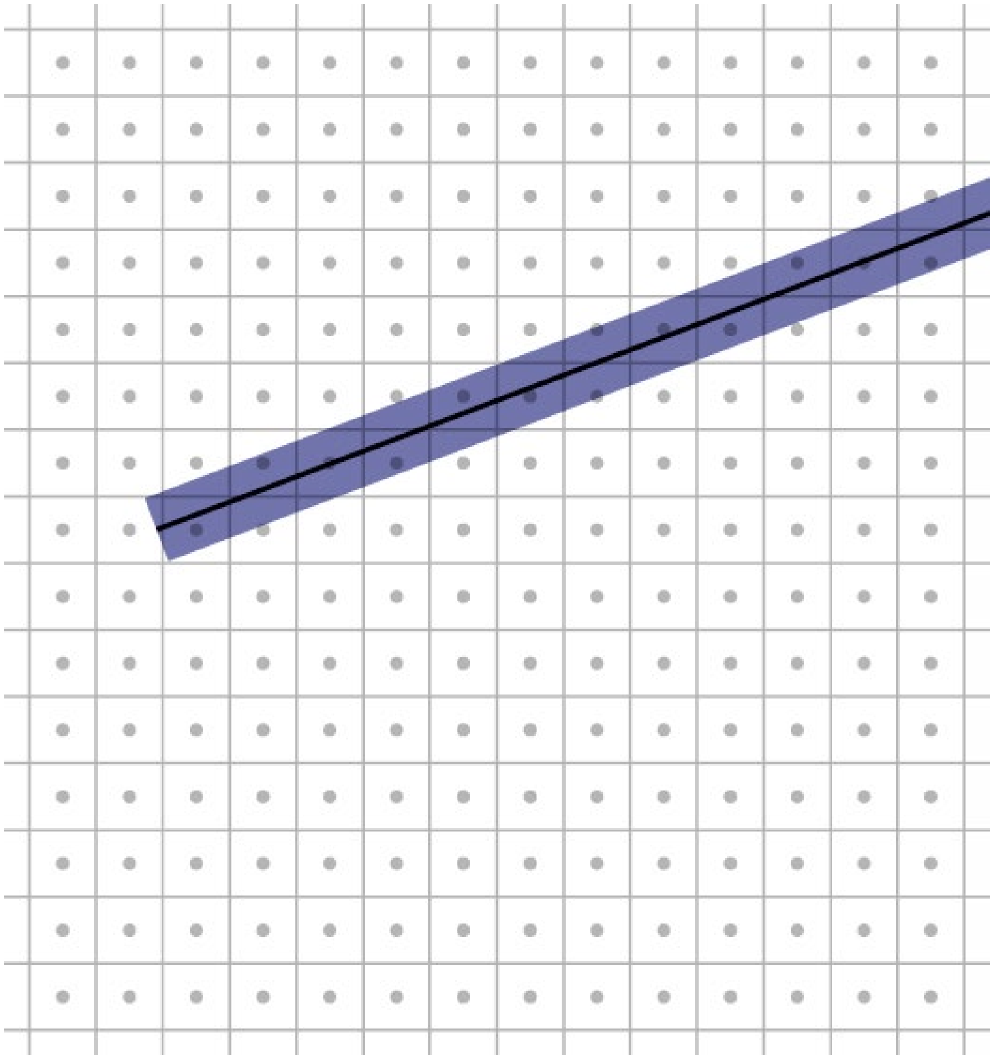
Sampling and Aliasing in Real Life

Aliasing

- **Where have we seen such effects before in real life?**

Sampling

- **Sampling a line**
  - We sample a continuous image at each point

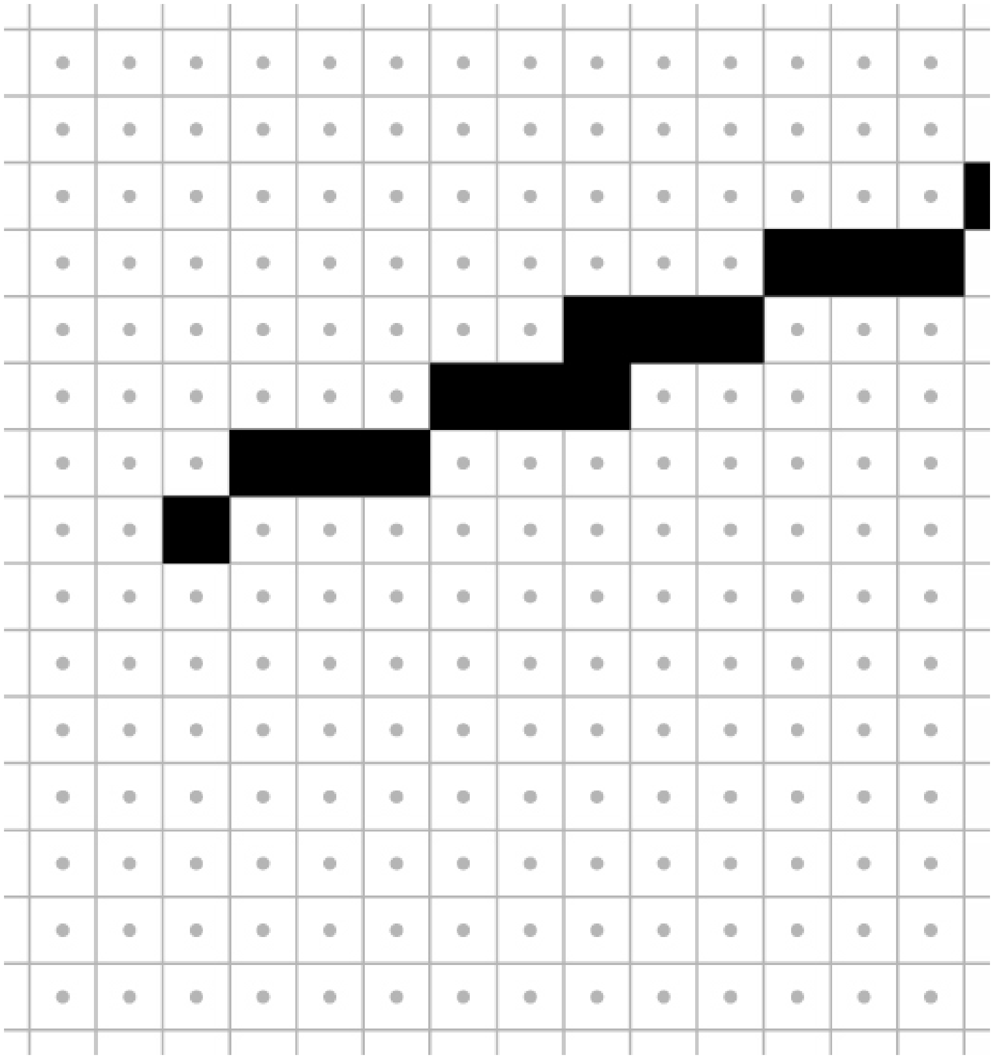


Illustrations from Cornell CS465 Spring 2006 Slides

Sampling

- **Sampling a line**
  - We sample a continuous image at each point

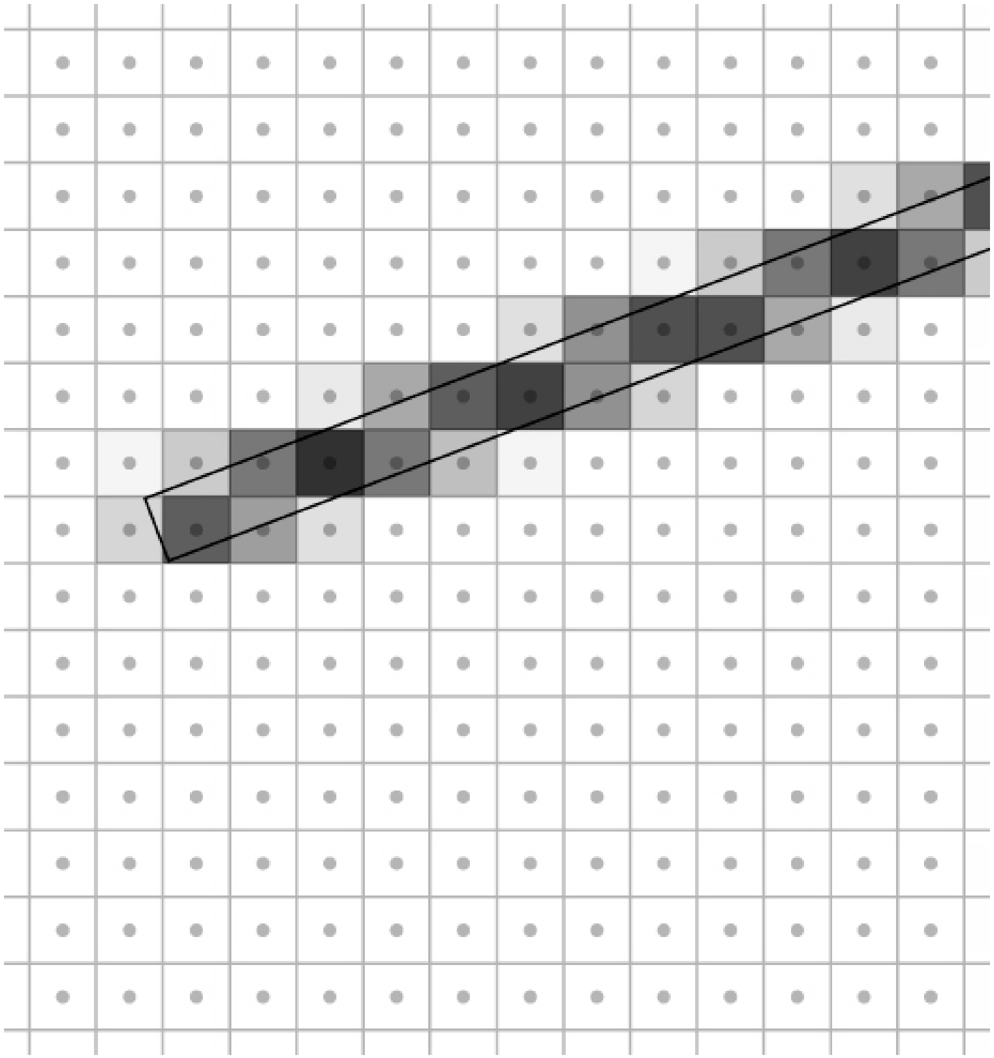
- **Aliasing**
  - Creates undesired high frequency information



Illustrations from Cornell CS465 Spring 2006 Slides

Anti-Aliasing

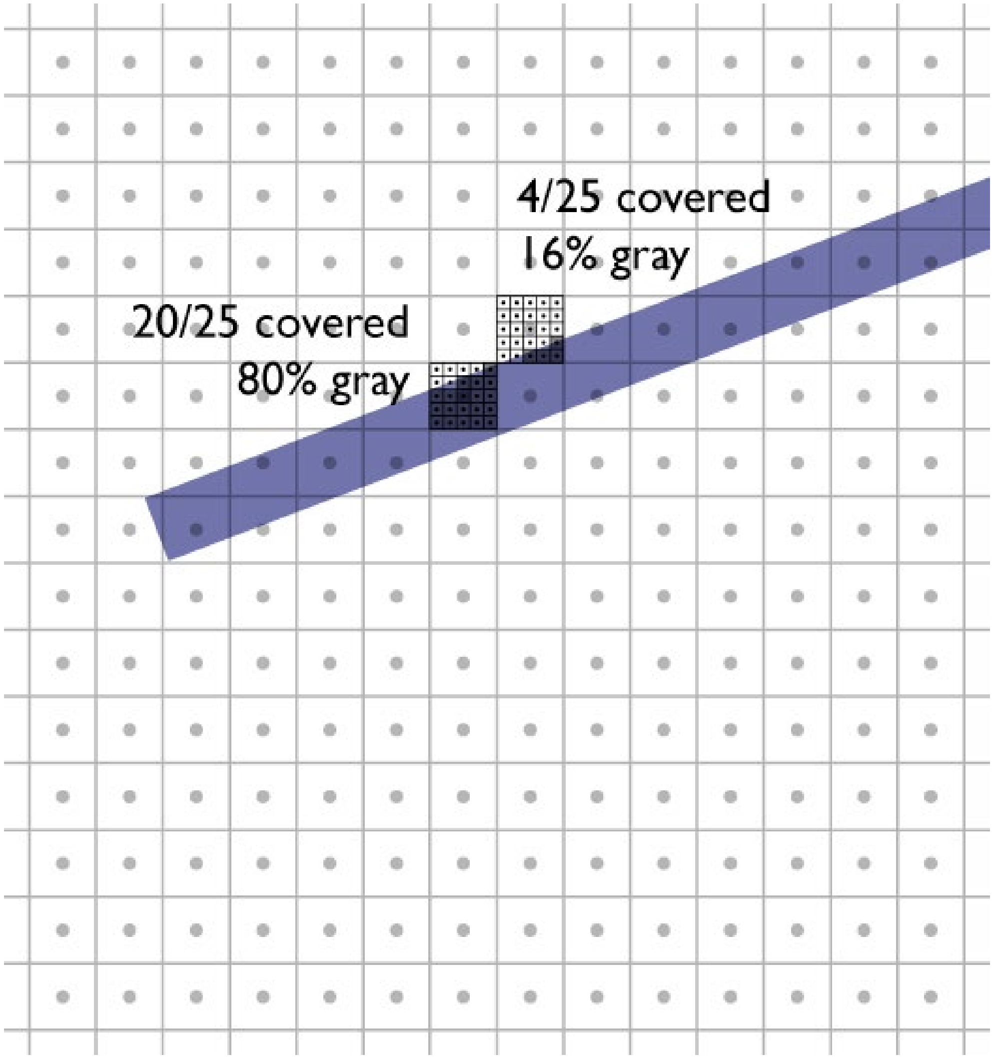
- **Sampling a line**
  - We sample a continuous image at each point
- **Aliasing**
  - Creates undesired high frequency information
- **Anti-aliasing**
  - Create a smooth image



Illustrations from Cornell CS465 Spring 2006 Slides

Anti-Aliasing

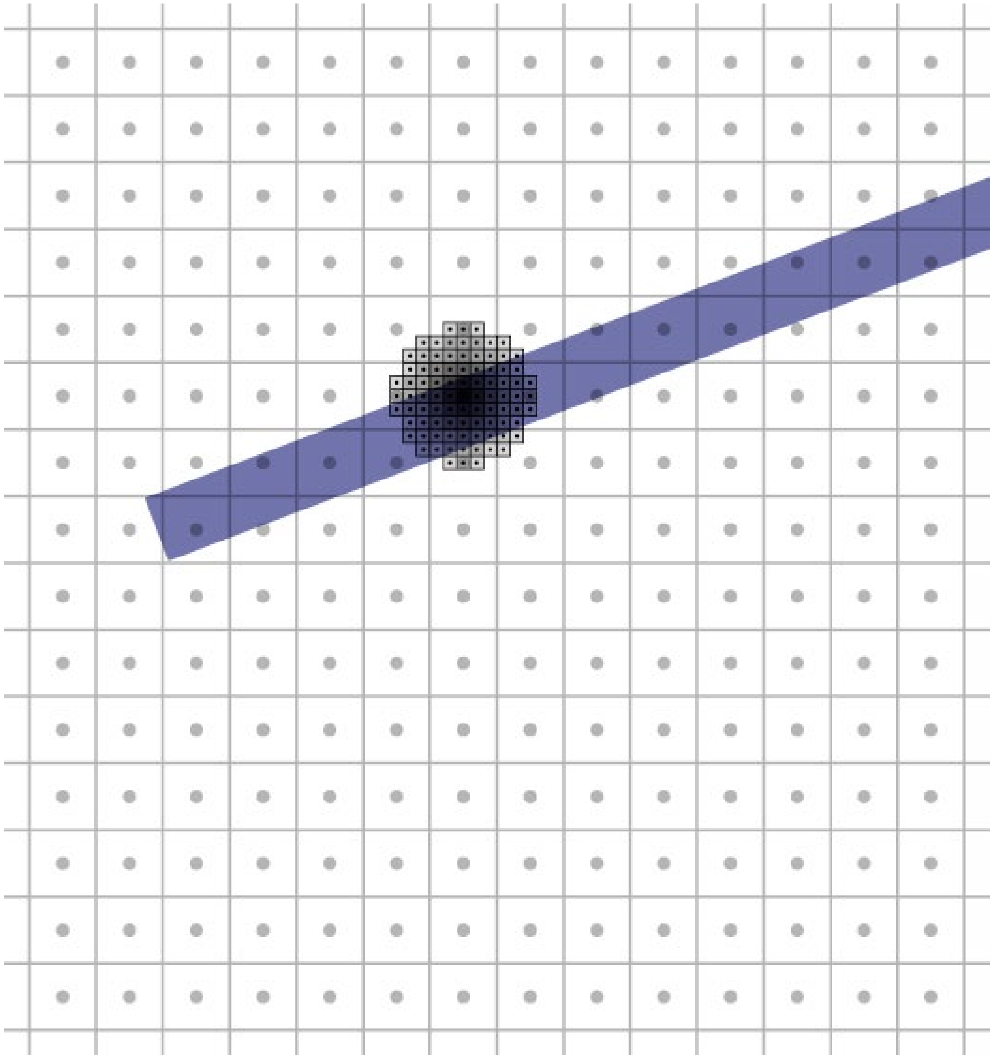
- **Box Filter**
  - Convolve the continuous (high resolution) image with a box



Illustrations from Cornell CS465 Spring 2006 Slides

Anti-Aliasing

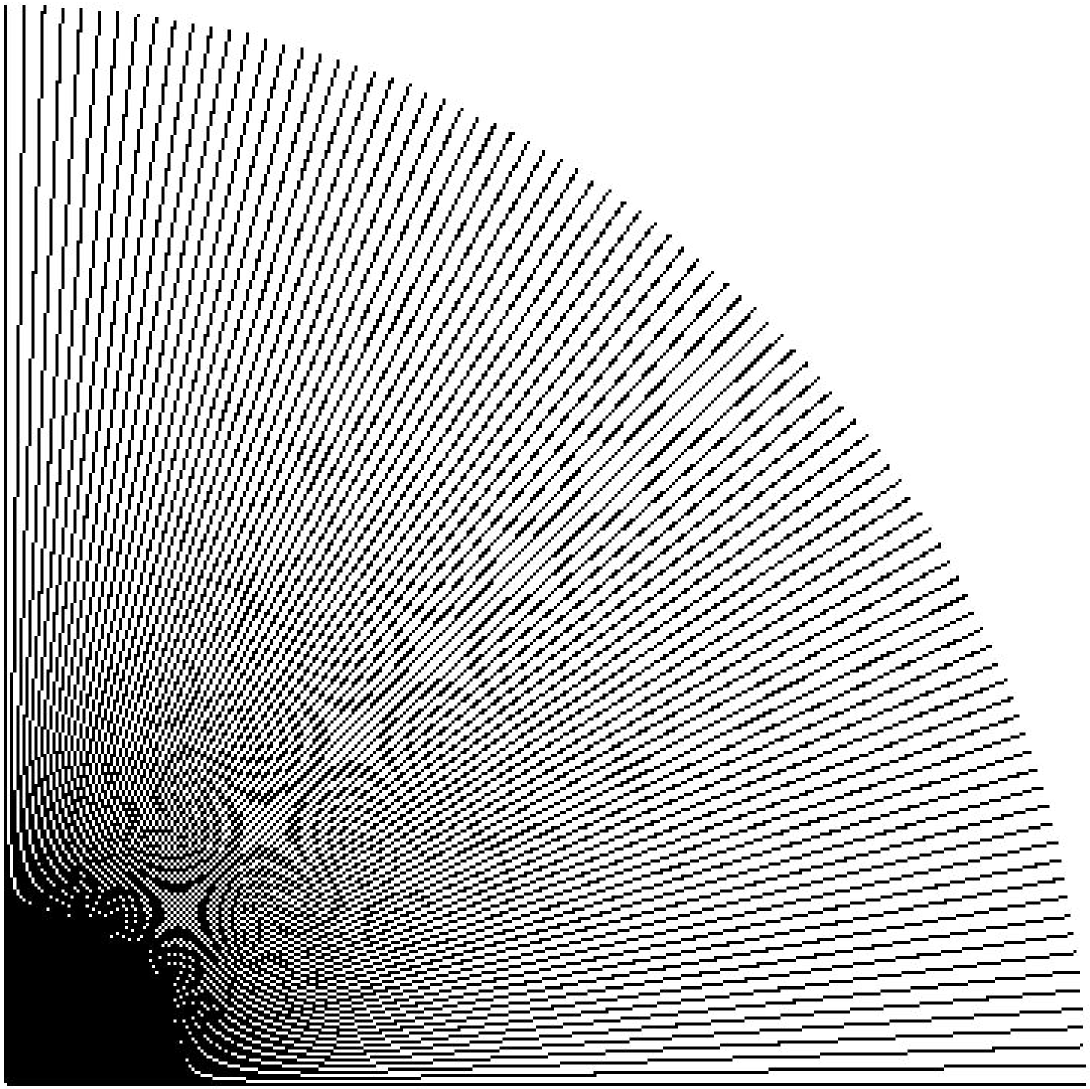
- **Gaussian Filter**
  - Convolve the continuous (high resolution) image with a Gaussian



Illustrations from Cornell CS465 Spring 2006 Slides

Anti-Aliasing

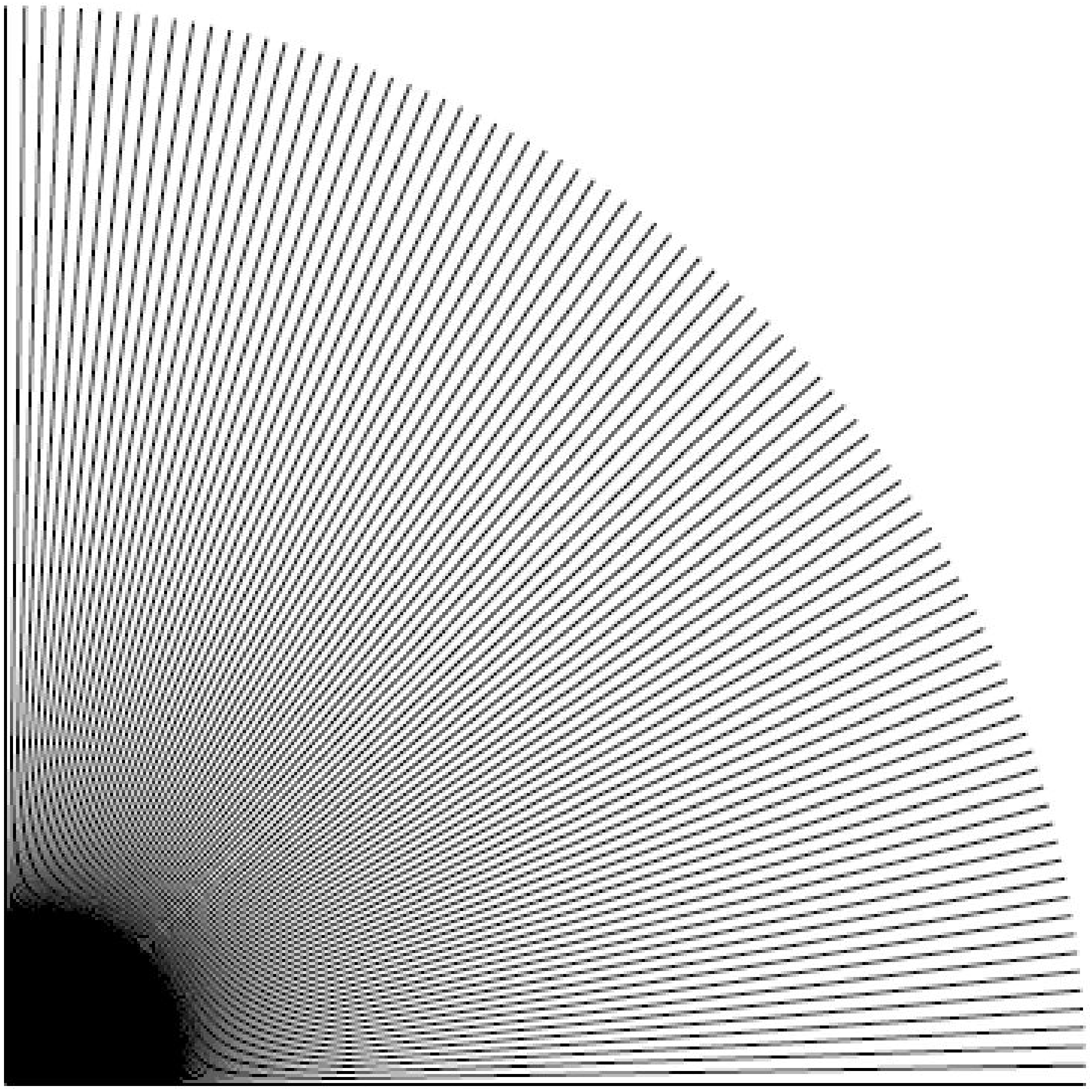
- **Sampling of lines**



Illustrations from Cornell CS465 Spring 2006 Slides

Anti-Aliasing

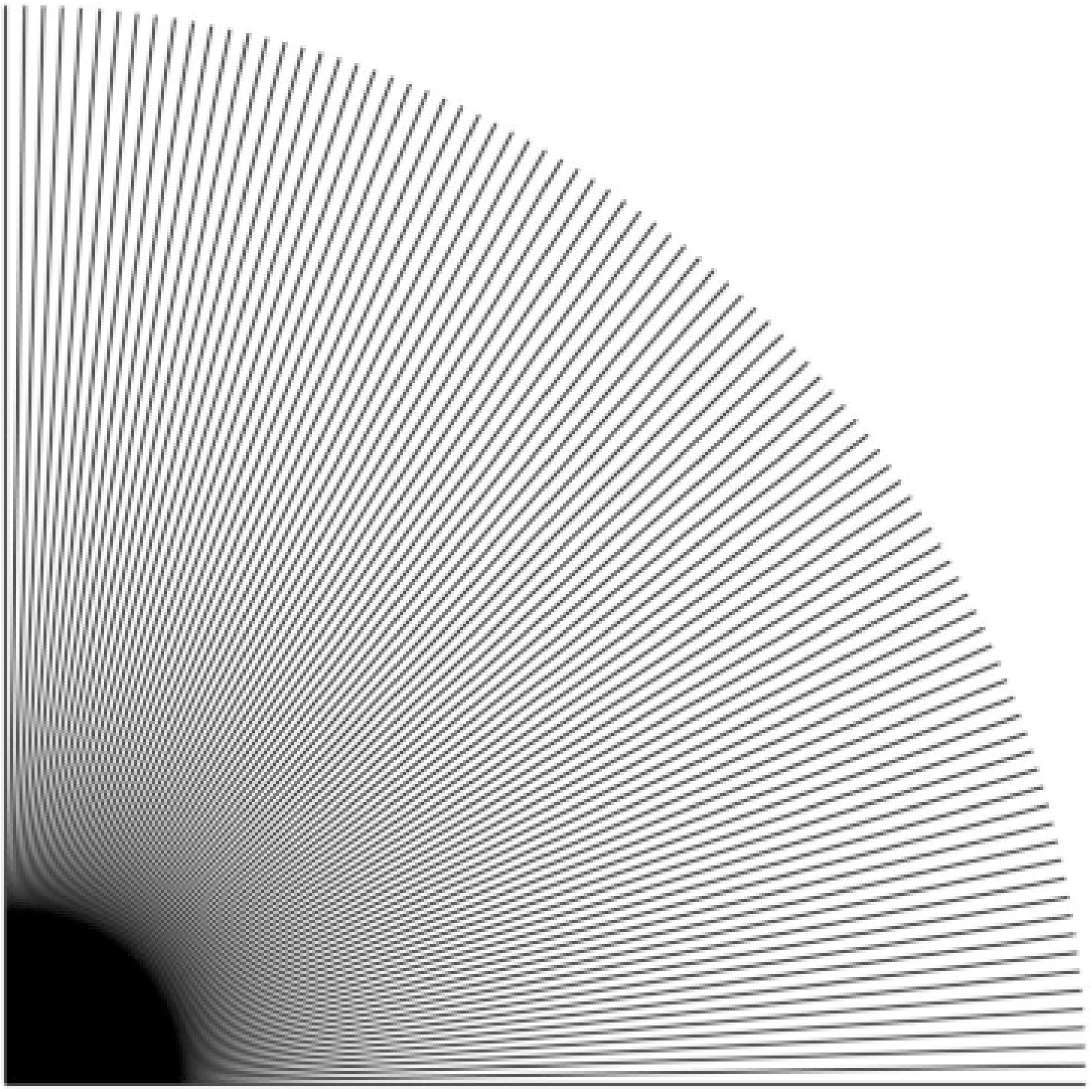
- **Sampling of lines**
  - Anti-aliasing with a box filter



Illustrations from Cornell CS465 Spring 2006 Slides

Anti-Aliasing

- **Sampling of lines**
  - Anti-aliasing with a Gaussian filter



Illustrations from Cornell CS465 Spring 2006 Slides

Aliasing Example

- **A Wheel**

From: https://www.youtube.com/watch?v=bI8lrqBBAXQ

Aliasing Example

- **Making water float**

From: https://www.youtube.com/watch?v=mODqQvlrgIQ

Lecture 12: Sampling, Aliasing, and the Discrete Fourier Transform

Foundations of Digital Signal Processing

Outline

- Review of Sampling
- The Nyquist-Shannon Sampling Theorem
- Continuous-time Reconstruction / Interpolation
- Aliasing and anti-Aliasing
- **Deriving Transforms from the Fourier Transform**
  - Discrete-time Fourier Transform, Fourier Series, Discrete-time Fourier Series
- The Discrete Fourier Transform

Deriving Transforms

- **Consider the Fourier Transform....**
  - What happens if we sample $x(t)$?

$$X(\Omega) = \int_{-\infty}^{\infty} x(t)e^{-j\Omega \mathrm{t}}\,dt$$

Deriving Transforms

- **Consider the Fourier Transform....**
  - What happens if we sample $x(t)$?

$$X(\Omega) = \int_{-\infty}^{\infty} x(t)\delta_{T_s}(t)e^{-j\Omega t}\,dt$$

Deriving Transforms

- **Consider the Fourier Transform....**
  - What happens if we sample $x(t)$?

$$X(\Omega) = \int_{-\infty}^{\infty} x(t)\delta_{T_s}(t)e^{-j\Omega \mathrm{t}}\,dt$$

$$= \int_{-\infty}^{\infty} x(t)\left[\sum_{n=-\infty}^{\infty} \delta(t - nT_s)\right] e^{-j\Omega \mathrm{t}}\,dt$$

$$= \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} x(t)\delta(t - nT_s)e^{-j\Omega \mathrm{t}}\,dt$$

Deriving Transforms

- **Consider the Fourier Transform….**
  - What happens if we sample $x(t)$?

$$X(\Omega) = \int_{-\infty}^{\infty} x(t)\delta_{T_s}(t)e^{-j\Omega \mathrm{t}}\,dt$$

$$= \int_{-\infty}^{\infty} x(t)\left[\sum_{n=-\infty}^{\infty} \delta(t - nT_s)\right]e^{-j\Omega \mathrm{t}}\,dt$$

$$= \sum_{n=-\infty}^{\infty} x(nT_s)e^{-j\Omega nT_s}$$

Deriving Transforms

- **Consider the Fourier Transform....**
  - What happens if we sample $x(t)$?

$$X(\Omega) = \int_{-\infty}^{\infty} x(t)\delta_{T_s}(t)e^{-j\Omega \mathrm{t}}\,dt$$

$$= \int_{-\infty}^{\infty} x(t)\left[\sum_{n=-\infty}^{\infty} \delta(t - nT_s)\right] e^{-j\Omega \mathrm{t}}\,dt$$

$$= \sum_{n=-\infty}^{\infty} x(nT_s)e^{-j\Omega nT_s}$$

Choose $T_s = 1, \Omega = \omega$

Deriving Transforms

- **Consider the Fourier Transform....**
  - What happens if we sample $x(t)$?

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

  - The Fourier Transform becomes the DTFT

Deriving Transforms

- **Consider the Inverse Fourier Transform….**
  - What happens if we sample $X(\Omega)$?

$$x(t) = \int_{-\infty}^{\infty} X(\Omega)e^{+j\Omega t}\,d\Omega$$

Deriving Transforms

- **Consider the Inverse Fourier Transform....**
  - What happens if we sample $X(\Omega)$?

$$x(\mathrm{t}) = \int_{-\infty}^{\infty} X(\Omega) e^{+j\Omega\mathrm{t}}\, d\Omega$$

$$= \int_{-\infty}^{\infty} X(\Omega) \left[\sum_{k=-\infty}^{\infty} \delta(\Omega - k\Omega_s) \right] e^{+j\Omega\mathrm{t}}\, d\Omega$$

$$= \sum_{k=-\infty}^{\infty} X(k\Omega_s) e^{+jk\Omega_s\mathrm{t}}$$

Choose $X(k\Omega_s) = c_k$

Deriving Transforms

- **Consider the Inverse Fourier Transform….**
  - What happens if we sample $\mathrm{X}(\Omega)$?

$$x(\mathrm{t}) = \sum_{k=-\infty}^{\infty} c_k e^{+jk\Omega_s \mathrm{t}}$$

  - The Fourier Transform becomes the Fourier Series

Deriving Transforms

- **Consider the Inverse Discrete-Time Fourier Transform….**
  - What happens if we sample $\mathrm{X}(\omega)$?

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(\omega) e^{+j\omega \mathrm{n}} \, d\omega$$

Deriving Transforms

- **Consider the Inverse Discrete-Time Fourier Transform....**
  - What happens if we sample $\mathrm{X}(\omega)$?

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(\omega) e^{+j\omega\mathrm{n}}\, d\omega$$

$$= \frac{1}{2\pi} \int_{2\pi} X(\omega) \left[2\pi \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_s) \right] e^{+j\omega\mathrm{n}}\, d\omega$$

Deriving Transforms

- **Consider the Inverse Discrete-Time Fourier Transform….**
  - What happens if we sample $\mathrm{X}(\omega)$?

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(\omega) e^{+j\omega \mathrm{n}} \, d\omega$$

$$= \frac{1}{2\pi} \int_{0}^{2\pi} X(\omega) \left[2\pi \sum_{k=-\infty}^{\infty} \underbrace{\delta(\omega - k\omega_s)} \right] e^{+j\omega \mathrm{n}} \, d\omega$$

$$k\omega_s \geq 0 \qquad k\omega_s < 2\pi$$

$$k \geq 0 \qquad k < \frac{2\pi}{\omega_s}$$

Deriving Transforms

- **Consider the Inverse Discrete-Time Fourier Transform….**
  - What happens if we sample $X(\omega)$?

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(\omega) e^{+j\omega \mathrm{n}}\, d\omega$$

$$= \frac{1}{2\pi} \int_{0}^{2\pi} X(\omega) \left[2\pi \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_s) \right] e^{+j\omega \mathrm{n}}\, d\omega$$

$$= \sum_{k=0}^{\frac{2\pi}{\omega_s}-1} X(k\omega) e^{+jk\omega_s \mathrm{n}}$$

Let $2\pi/\omega_s = N$

$\omega_s = 2\pi/N$

Deriving Transforms

- **Consider the Inverse Discrete-Time Fourier Transform....**
  - What happens if we sample $X(\omega)$?

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(\omega) e^{+j\omega \mathrm{n}}\, d\omega$$

$$= \frac{1}{2\pi} \int_{0}^{2\pi} X(\omega) \left[2\pi \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_s) \right] e^{+j\omega \mathrm{n}}\, d\omega$$

$$= \sum_{k=0}^{N-1} X(k\omega) e^{+j\frac{2\pi k}{N}\mathrm{n}}$$

Let $2\pi/\omega_s = K$

$\omega_s = 2\pi/K$

Deriving Transforms

- **Consider the Inverse Discrete-Time Fourier Transform….**
  - What happens if we sample $\mathrm{X}(\omega)$?

$$x[n] = \sum_{k=0}^{N-1} X[k] e^{+j\frac{2\pi}{N}\mathrm{kn}}$$

  - The DTFT becomes the Discrete-Time Fourier Series