Lecture 25: Filter Bank Reconstruction

Foundations of Digital Signal Processing

Outline

- Homework Questions / Discussions
- Review Short Time Fourier Transform
- Inefficient DFT Filter Banks
- Efficient DFT Filter Bank
- General Two-Channel Filter Bank Reconstruction

News

- **Schedule / Plan**
  - *Tomorrow:, Nov. 16 Homework #10*
  - *Monday, Nov. 19: Coding Assignment #6*
  - *Next Week: No Due Dates*
  - *Thursday, Nov. 29th: Homework #11*
  - *Tuesday, Dec. 4th: Exam #3*
  - *Wednesday, Dec. 5th: Coding Assignment #7 (short)*
  - *Wednesday, Dec. 12th: Final Exam*
  - *Friday, Dec. 14th: EEE5502 Reports Due*

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Filter Design

- **Impulse Invariant Method**

$$\frac{A}{s-p} \leftrightarrow \frac{A}{1-e^{pT}z^{-1}}$$

  - What is the time-domain representation?

Filter Design

- **Impulse Invariant Method**

$$\frac{A}{s-p} \leftrightarrow \frac{A}{1-e^{pT}z^{-1}}$$

  - What is the time-domain representation?

$$\frac{A}{s-p} \leftrightarrow Ae^{pt}u(t)$$

$$\frac{A}{1-(e^{pT})z^{-1}} \leftrightarrow Ae^{pTn}u[n]$$

Same impulse response!

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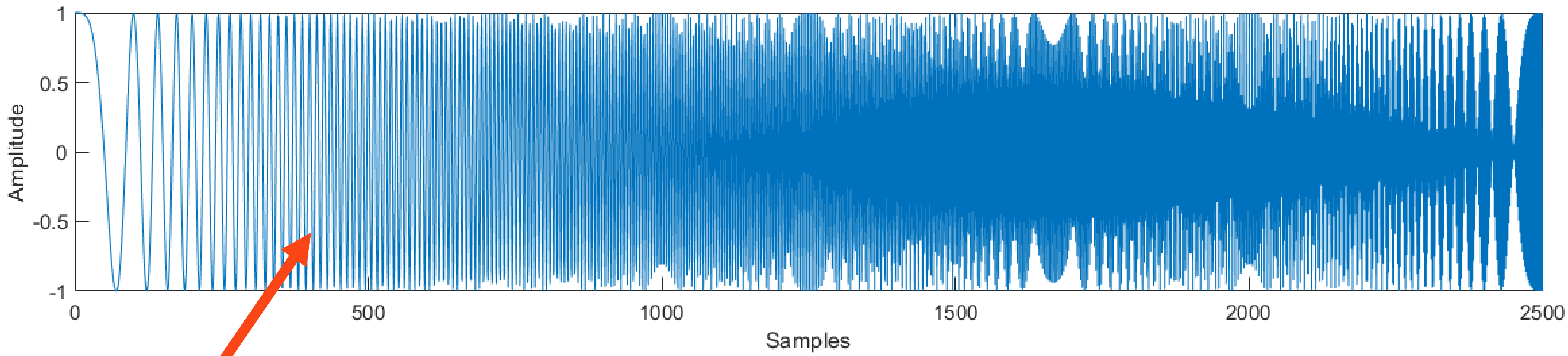
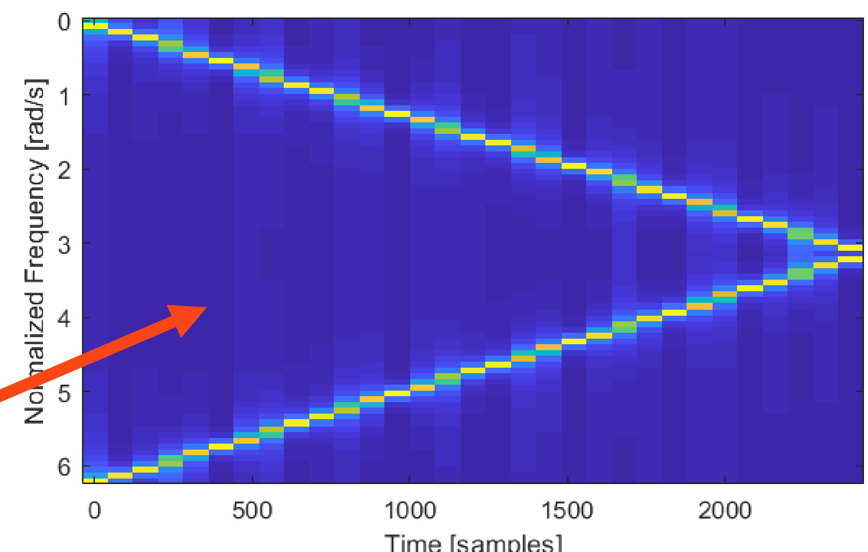
The Short-Time Fourier Transform

- **The Definition:**

$$X[Mm,k] = \mathrm{DFT}_{n\to k}(w[n]x[n-Mm])$$



Chirp



STFT

The Short-Time Fourier Transform

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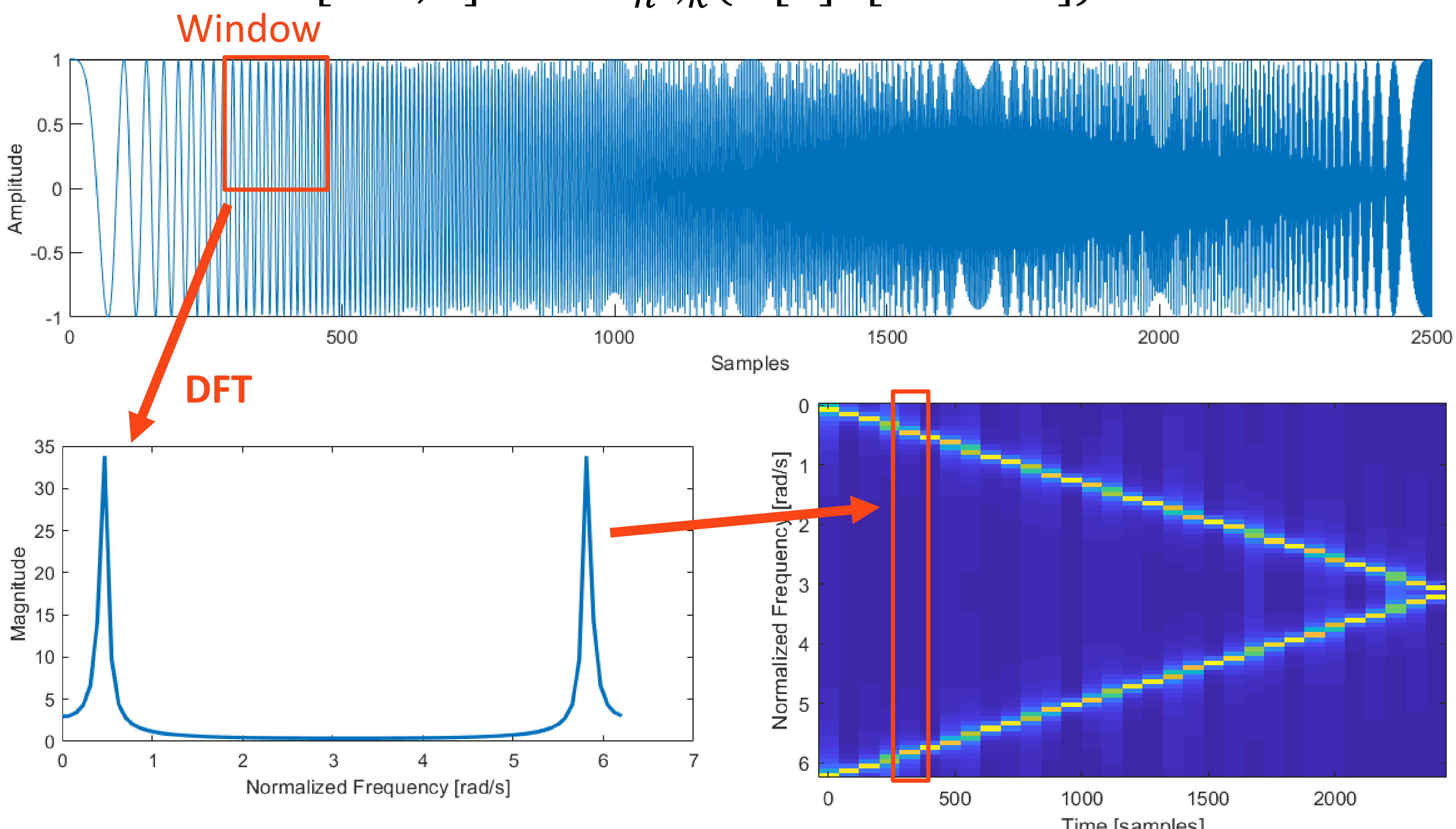

The Short-Time Fourier Transform

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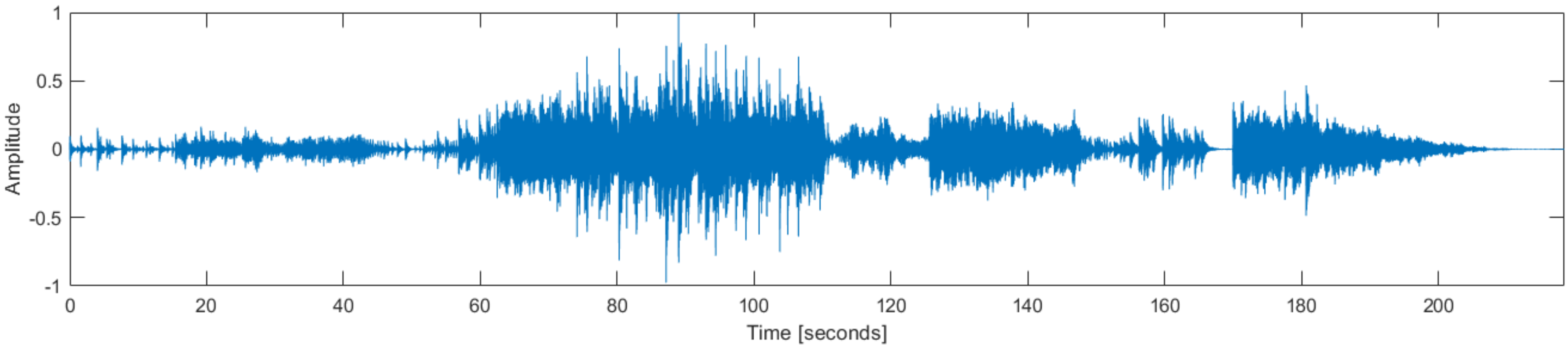

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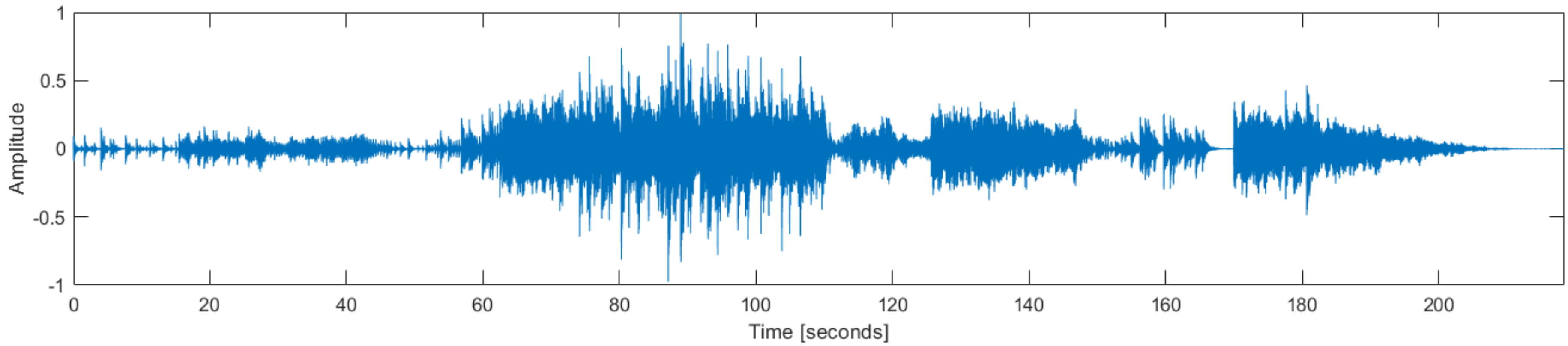
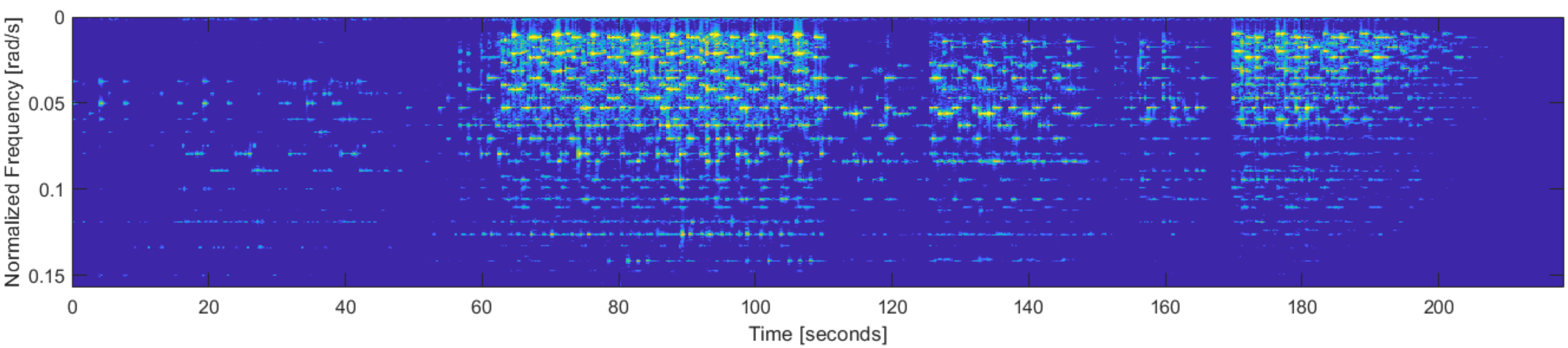
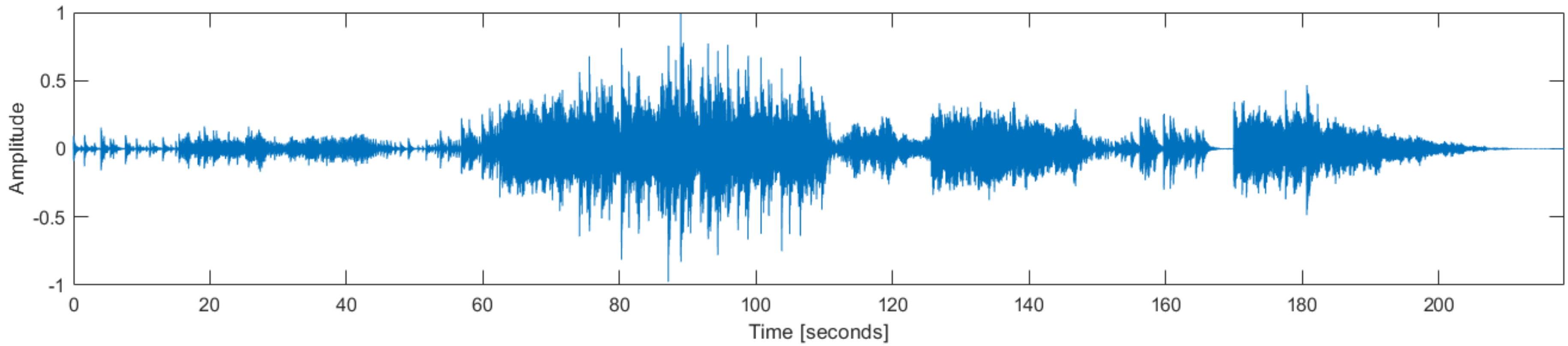
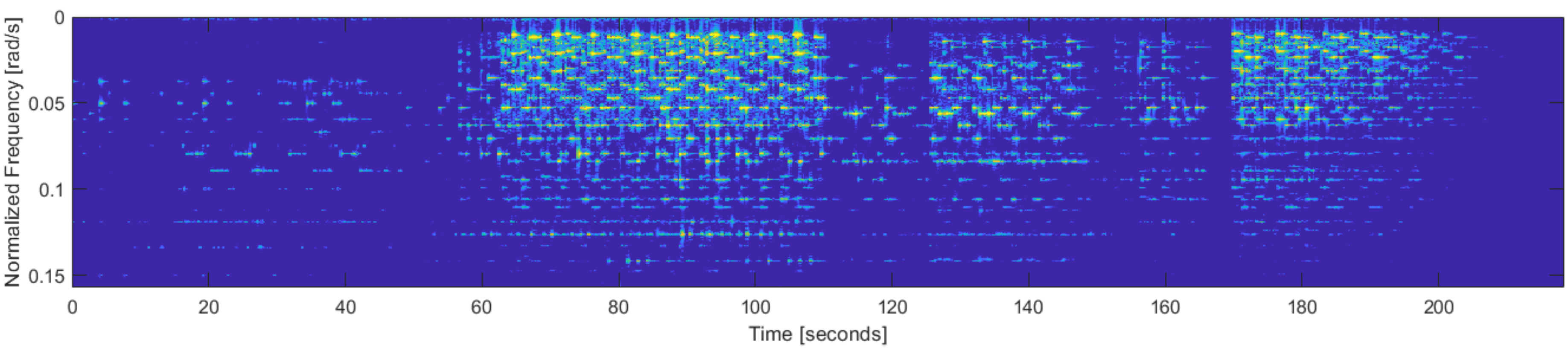




The Short-Time Fourier Transform

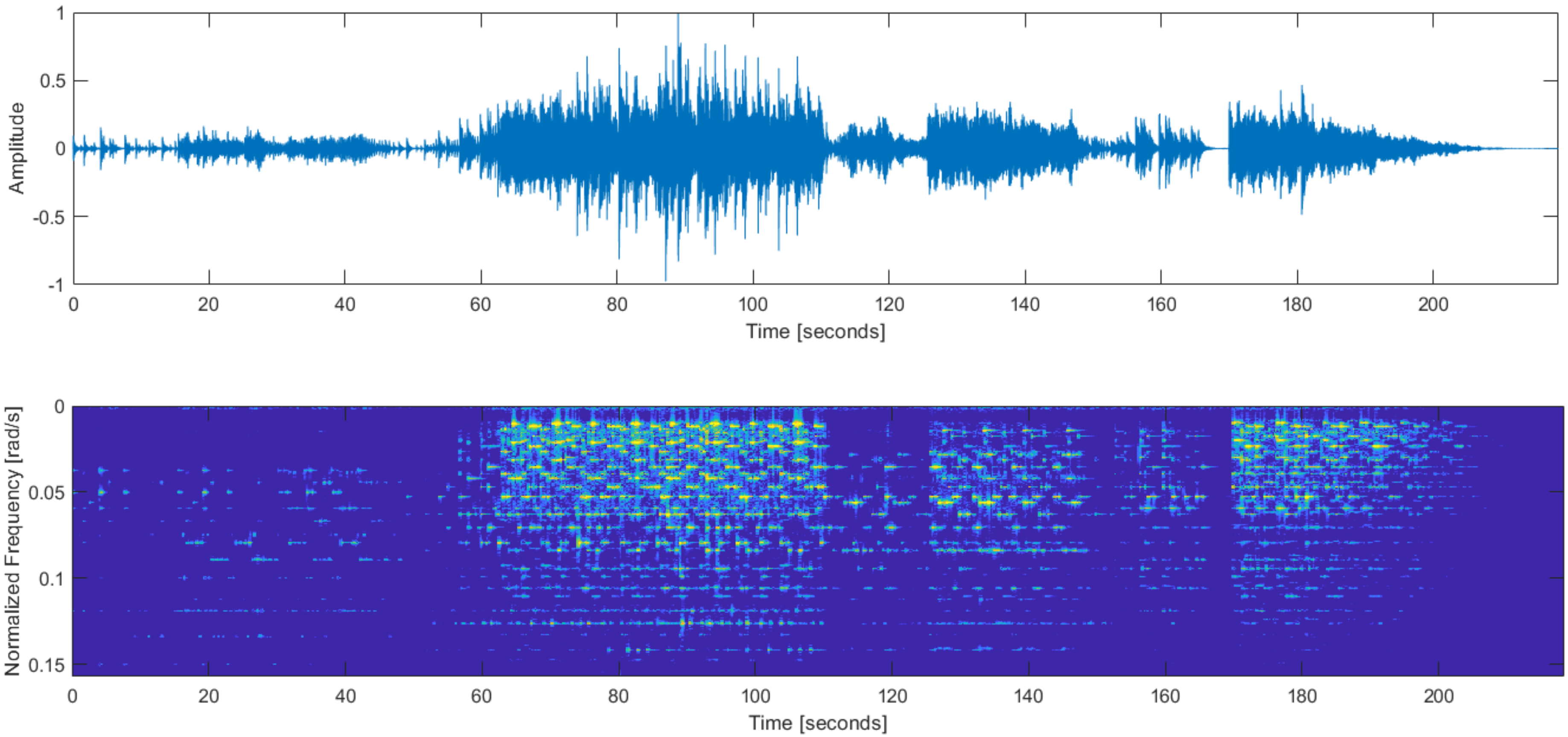
- **Question: Why do I care about the STFT?**





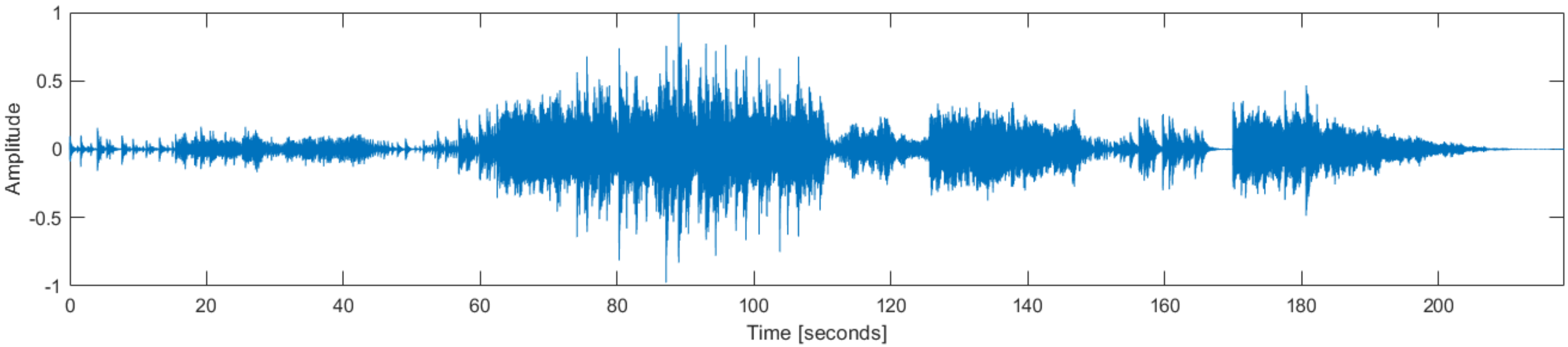
The Short-Time Fourier Transform

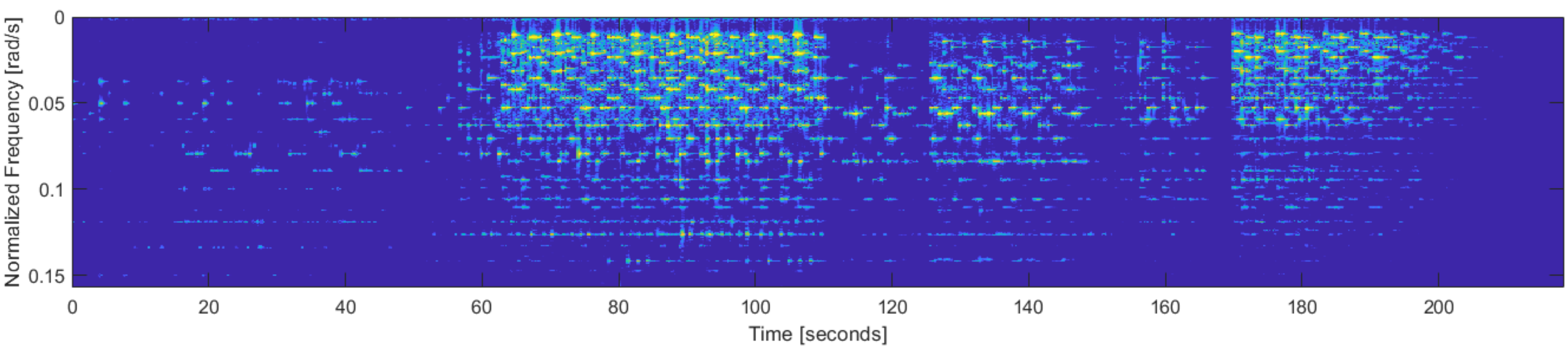
- **Question: How is this problematic for a real-time system?**



The Short-Time Fourier Transform

- **Question: How do I solve this problem??**





The Short-Time Fourier Transform

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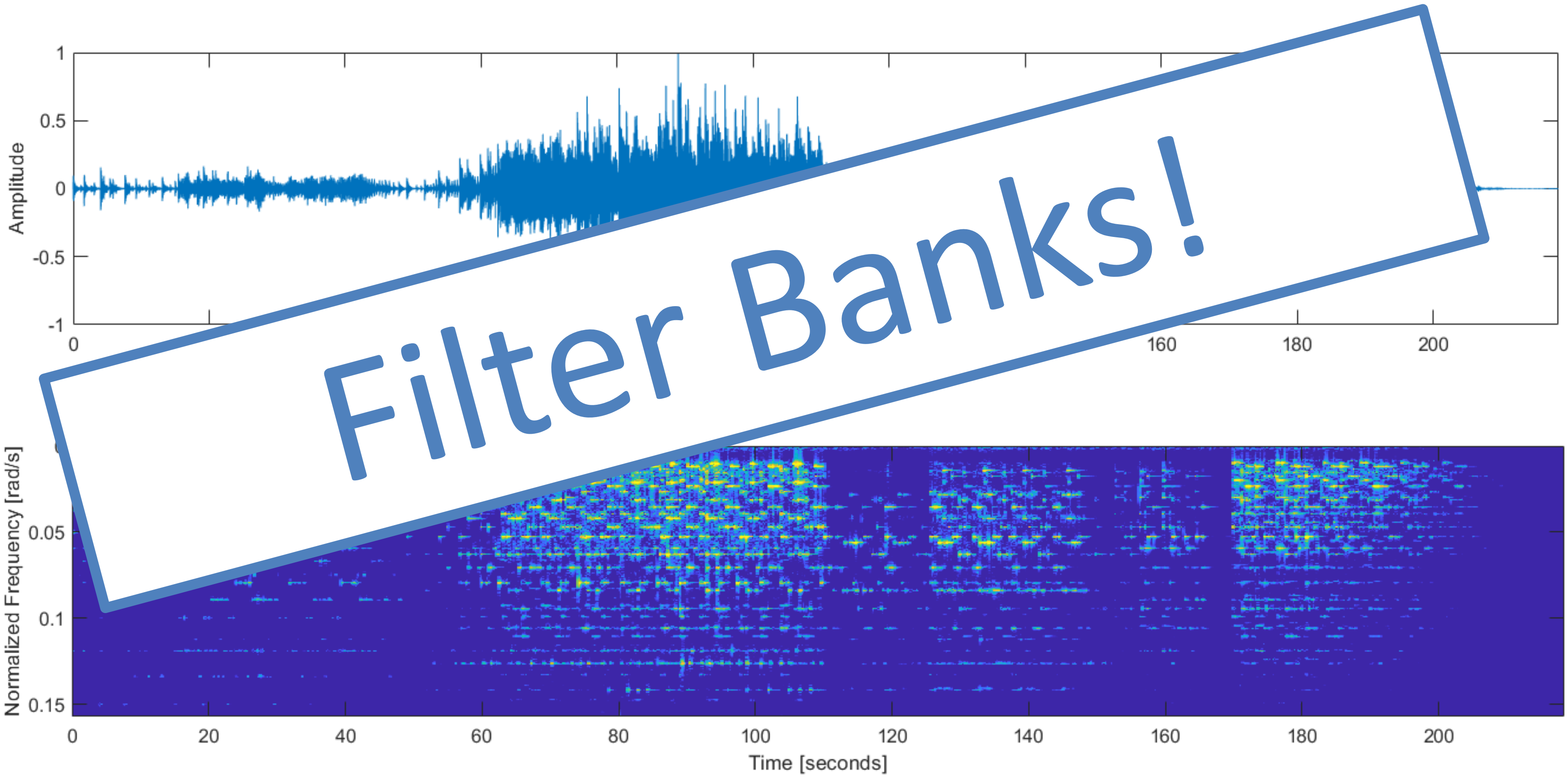

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- **Inefficient DFT Filter Banks**
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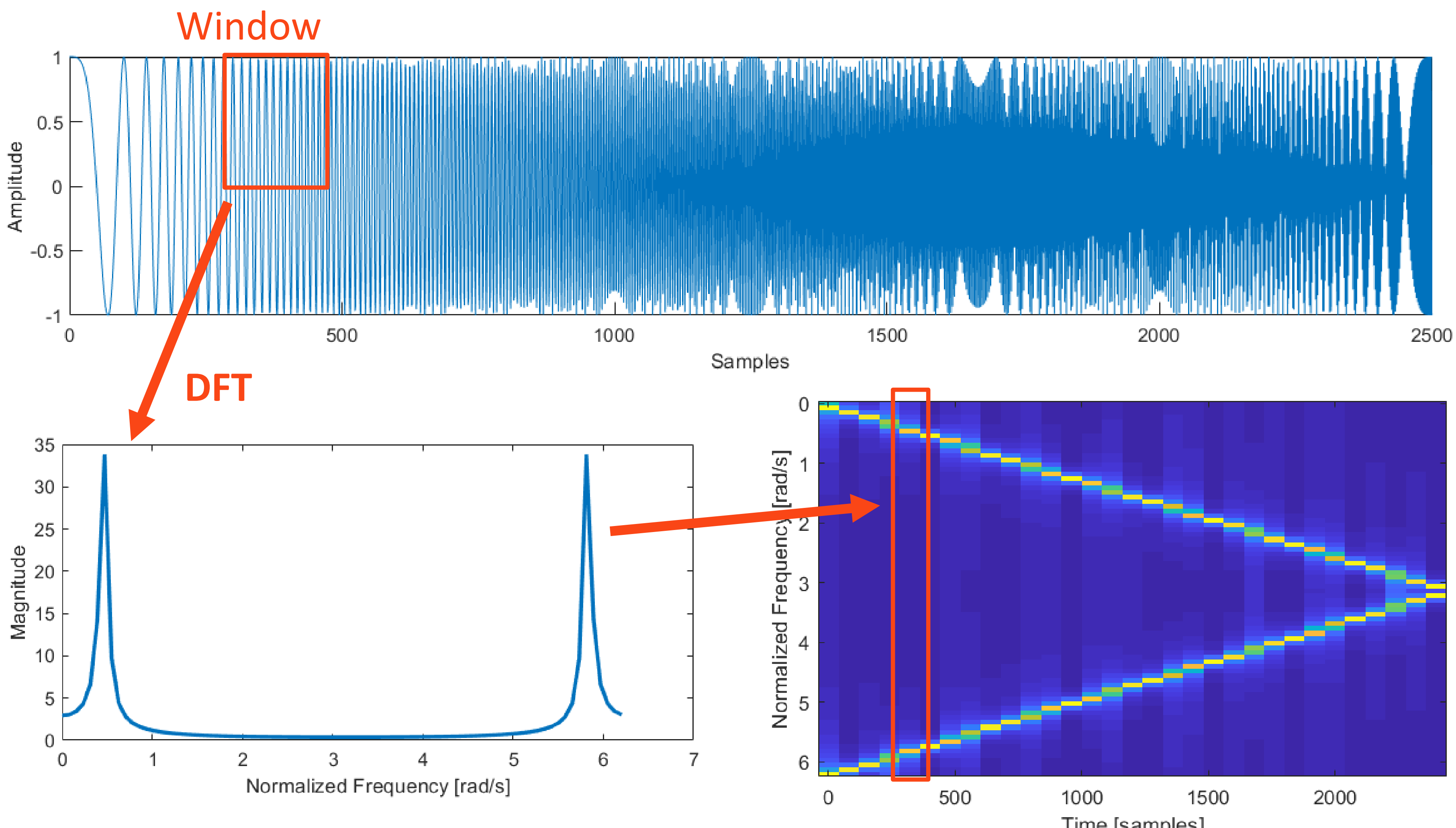
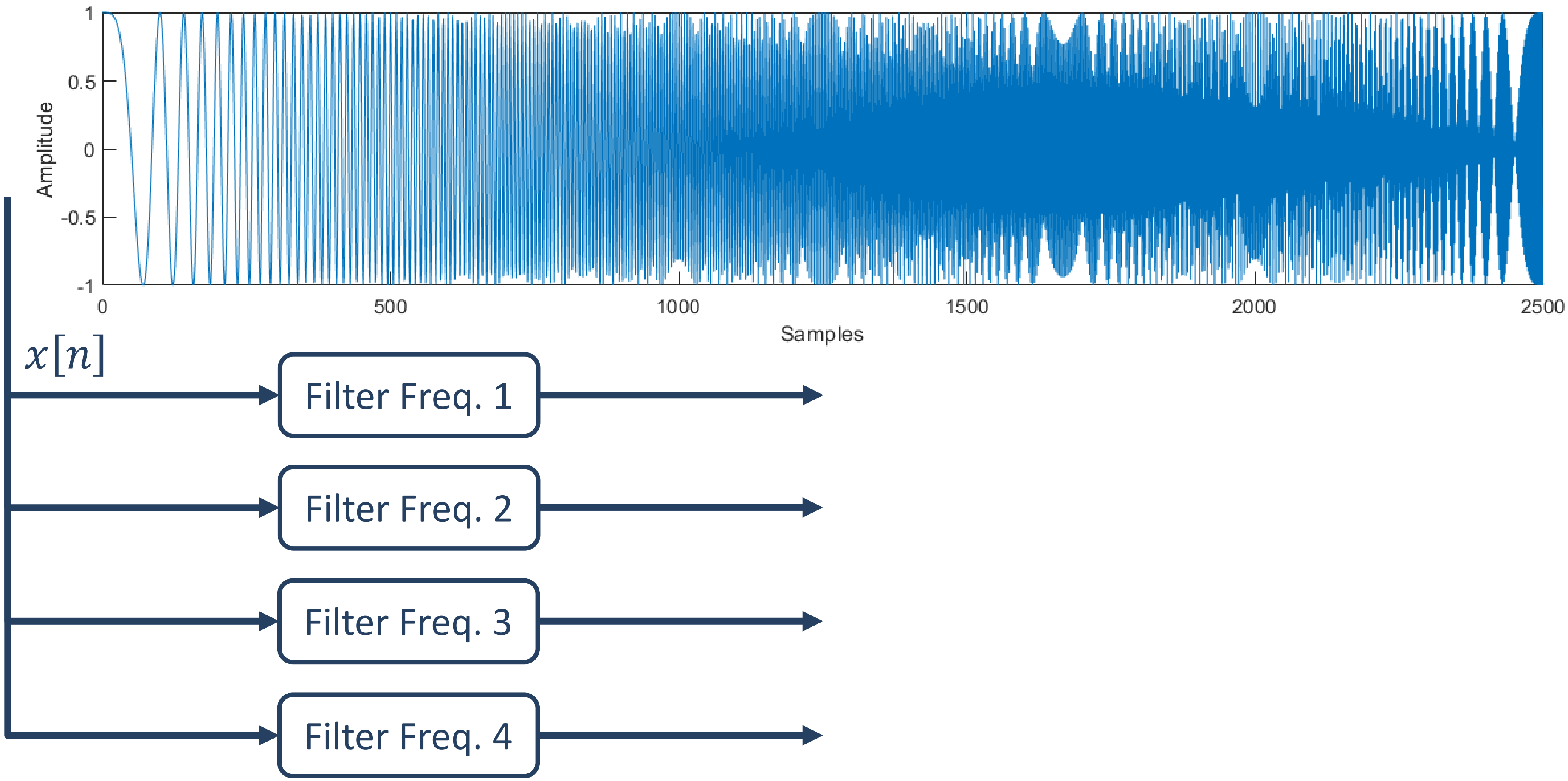
Filter Banks

The Short Time Fourier Transform Process

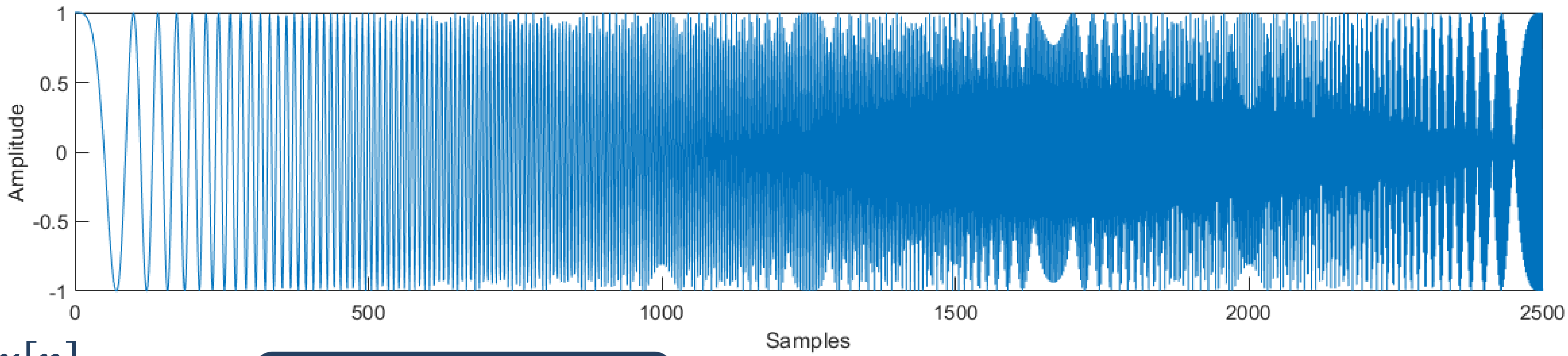


- **Inefficient Filter Bank Process**



Filter Banks

- **Inefficient Filter Bank Process**



$x[n]$

$$h_0[n] = e^{j\frac{2\pi}{4}n(0)}$$

$$h_1[n] = e^{j\frac{2\pi}{4}n(1)}$$

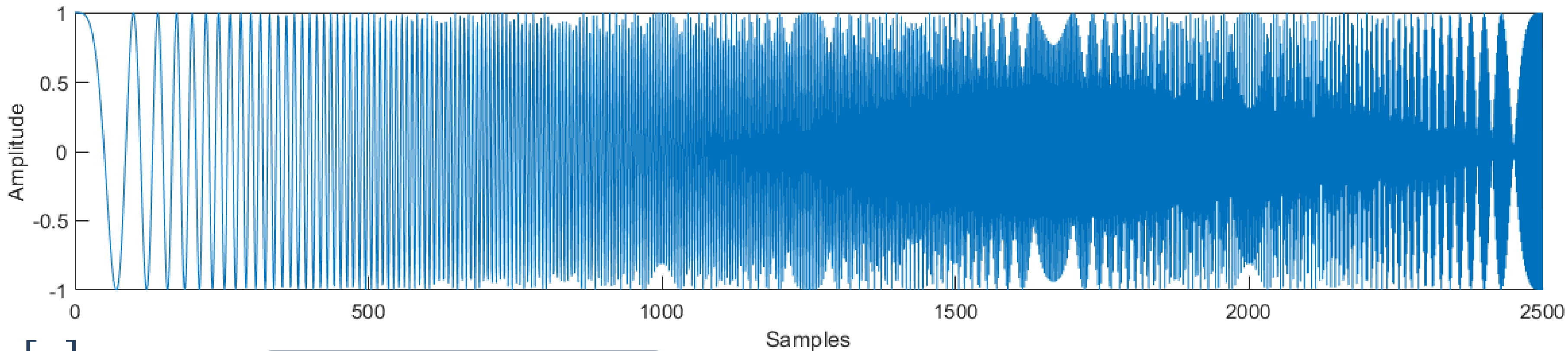
$$h_2[n] = e^{j\frac{2\pi}{4}n(2)}$$

$$h_3[n] = e^{j\frac{2\pi}{4}n(3)}$$

Assume all length-4 filters

Filter Banks

■ Inefficient Filter Bank Process



$x[n]$

$$h_0[n] = e^{j\frac{2\pi}{4}n(0)} \rightarrow e^{j\frac{2\pi}{N}n(0)} * x[n]$$

$$h_1[n] = e^{j\frac{2\pi}{4}n(1)} \rightarrow e^{j\frac{2\pi}{N}n(1)} * x[n]$$

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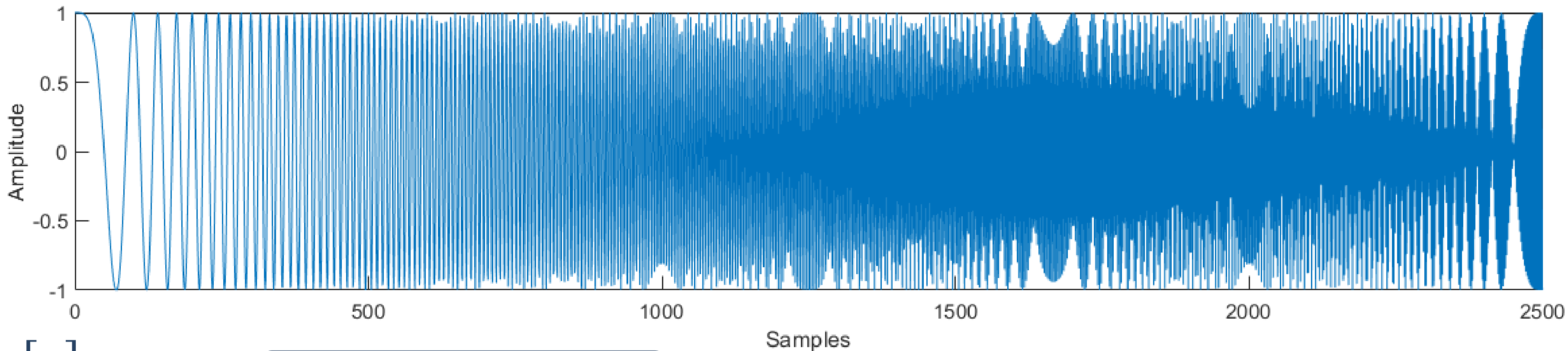
Assume all length-4 filters

Filter Banks

Inefficient Filter Bank Process



$x[n]$

$$h_0[n] = e^{j\frac{2\pi}{4}n(0)} \rightarrow \sum_{m=0}^{N-1} e^{j\frac{2\pi}{N}(n-m)(0)}\, x[m]$$

$$h_1[n] = e^{j\frac{2\pi}{4}n(1)} \rightarrow \sum_{m=0}^{N-1} e^{j\frac{2\pi}{N}(n-m)(1)}\, x[m]$$

$$h_2[n] = e^{j\frac{2\pi}{4}n(2)} \rightarrow \sum_{m=0}^{N-1} e^{j\frac{2\pi}{N}(n-m)(2)}\, x[m]$$

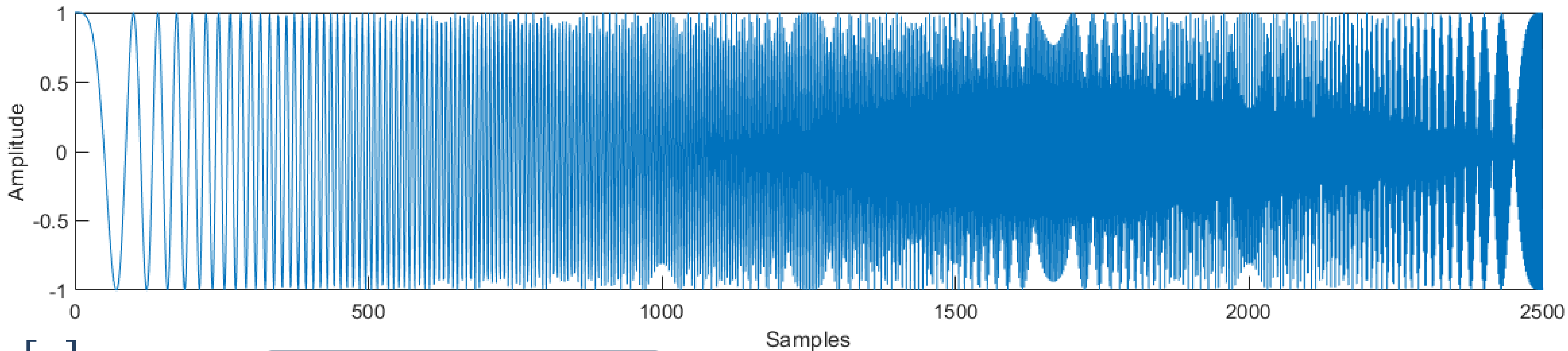
$$h_3[n] = e^{j\frac{2\pi}{4}n(3)} \rightarrow \sum_{m=0}^{N-1} e^{j\frac{2\pi}{N}(n-m)(3)}\, x[m]$$

Assume all length-4 filters

Hard solution: Use math

Filter Banks

Inefficient Filter Bank Process



$x[n]$

$$h_0[n] = e^{j\frac{2\pi}{4}n(0)} \rightarrow e^{j\frac{2\pi}{N}n(0)} \sum_{m=0}^{N-1} e^{-j\frac{2\pi}{N}m(0)}\, x[m]$$

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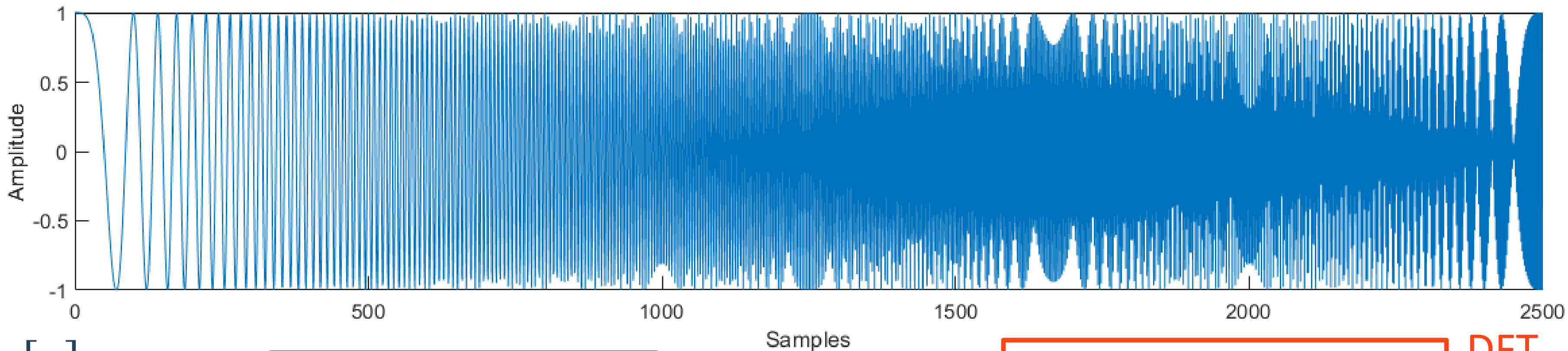
$$h_2[n] = e^{j\frac{2\pi}{4}n(2)} \rightarrow e^{j\frac{2\pi}{N}n(2)} \sum_{m=0}^{N-1} e^{-j\frac{2\pi}{N}m(2)}\, x[m]$$

$$h_3[n] = e^{j\frac{2\pi}{4}n(3)} \rightarrow e^{j\frac{2\pi}{N}n(3)} \sum_{m=0}^{N-1} e^{-j\frac{2\pi}{N}m(3)}\, x[m]$$

Assume all length-4 filters

Filter Banks

Inefficient Filter Bank Process



$x[n]$

$$h_0[n] = e^{j\frac{2\pi}{4}n(0)} \rightarrow e^{j\frac{2\pi}{N}n(0)} \boxed{\sum_{m=0}^{N-1} e^{-j\frac{2\pi}{N}m(0)}\, x[m]}$$

DFT for k = 0

$$h_1[n] = e^{j\frac{2\pi}{4}n(1)} \rightarrow e^{j\frac{2\pi}{N}n(1)} \sum_{m=0}^{N-1} e^{-j\frac{2\pi}{N}m(1)}\, x[m]$$

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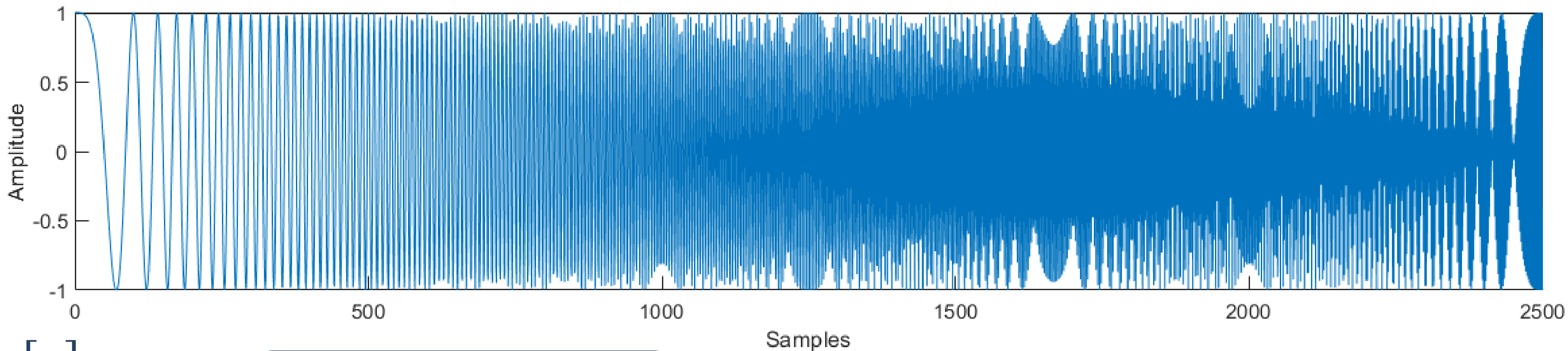
Assume all length-4 filters

Filter Banks

Inefficient Filter Bank Process



$x[n]$

$$h_0[n] = e^{j\frac{2\pi}{4}n(0)} \rightarrow e^{j\frac{2\pi}{N}n(0)} X[n,0]$$

$$h_1[n] = e^{j\frac{2\pi}{4}n(1)} \rightarrow e^{j\frac{2\pi}{N}n(1)} \boxed{\sum_{m=0}^{N-1} e^{-j\frac{2\pi}{N}m(1)} x[m]}$$

DFT for k = 1

$$h_2[n] = e^{j\frac{2\pi}{4}n(2)} \rightarrow e^{j\frac{2\pi}{N}n(2)} \sum_{m=0}^{N-1} e^{-j\frac{2\pi}{N}m(2)} x[m]$$

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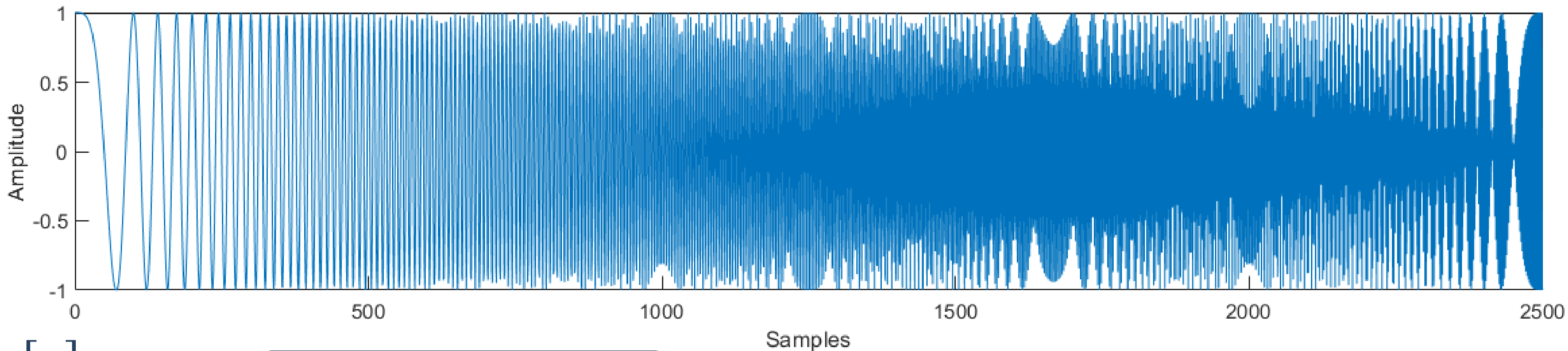
Assume all length-4 filters

Inefficient Filter Bank Process



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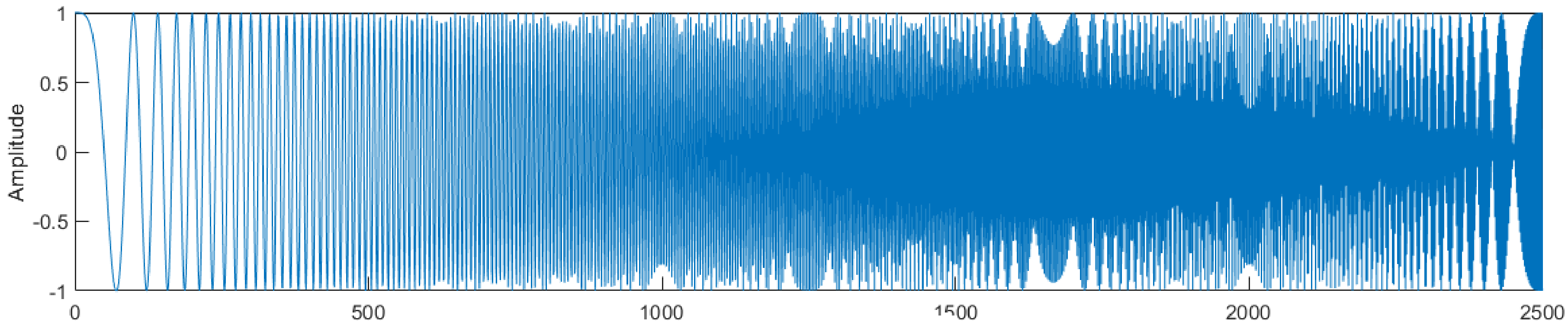
Assume all length-4 filters

Filter Banks

Inefficient Filter Bank Process



$x[n]$

$h_0[n] = e^{j\frac{2\pi}{4}n(0)}$ → $e^{-j\frac{2\pi}{4}n(0)}$ → $X[n,0]$

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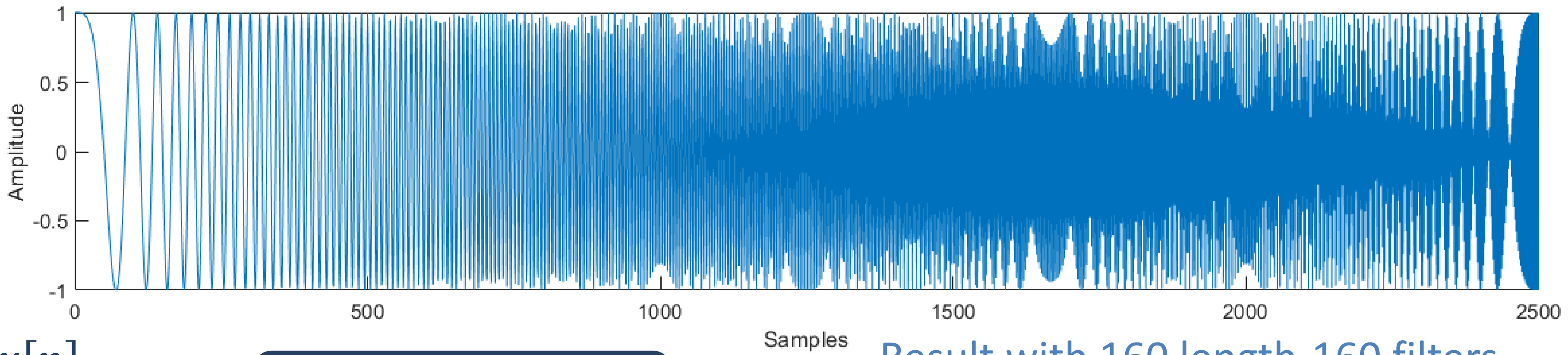
$h_3[n] = e^{j\frac{2\pi}{4}n(3)}$ → $e^{-j\frac{2\pi}{4}n(3)}$ → $X[n,3]$

We have the DFT coefficients!

Assume all length-4 filters

Filter Banks

Inefficient Filter Bank Process



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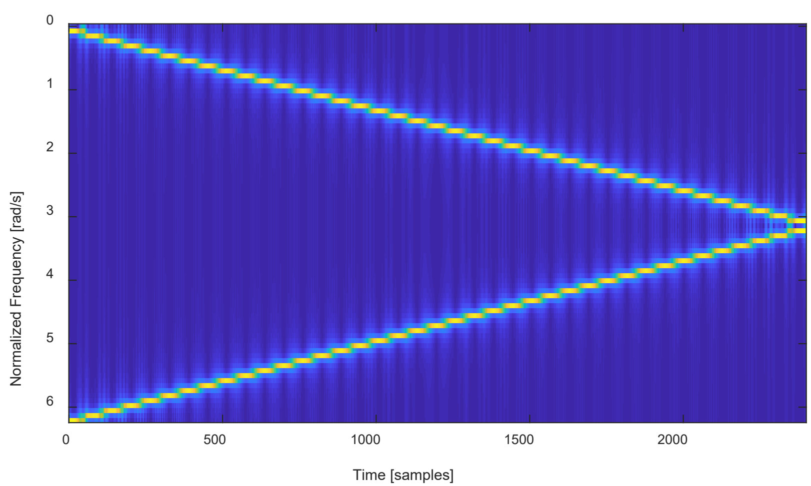
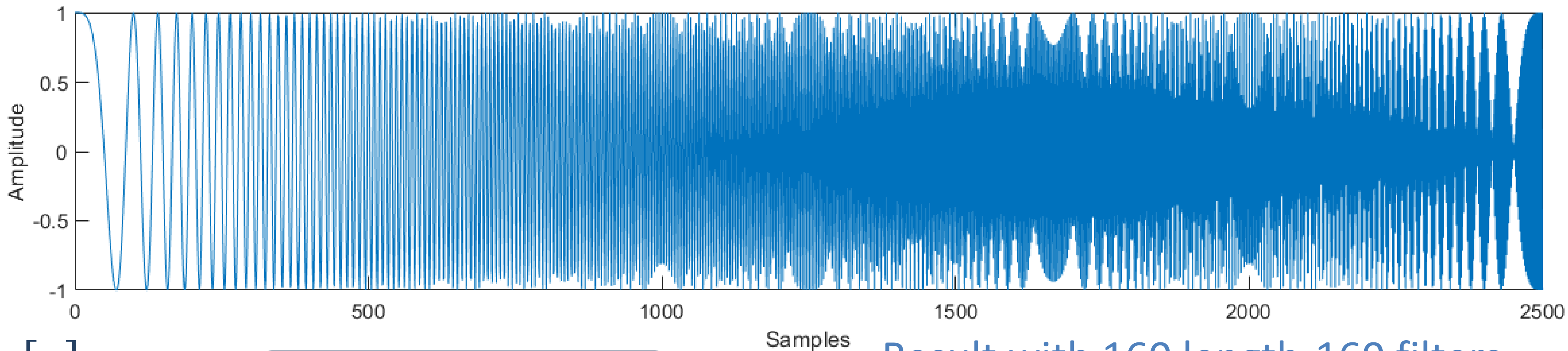
Assume all length-4 filters

Result with 160 length-160 filters



Filter Banks

- **Question: Why is this <u>not</u> a preferred approach?**



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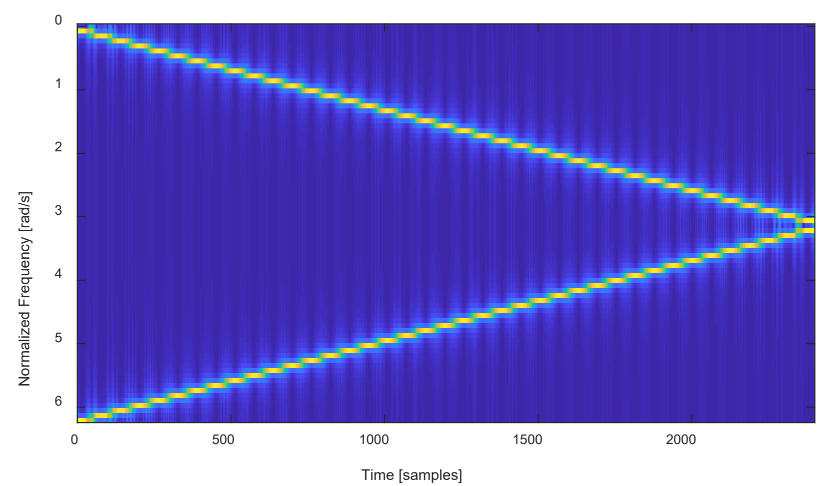
Assume all length-4 filters

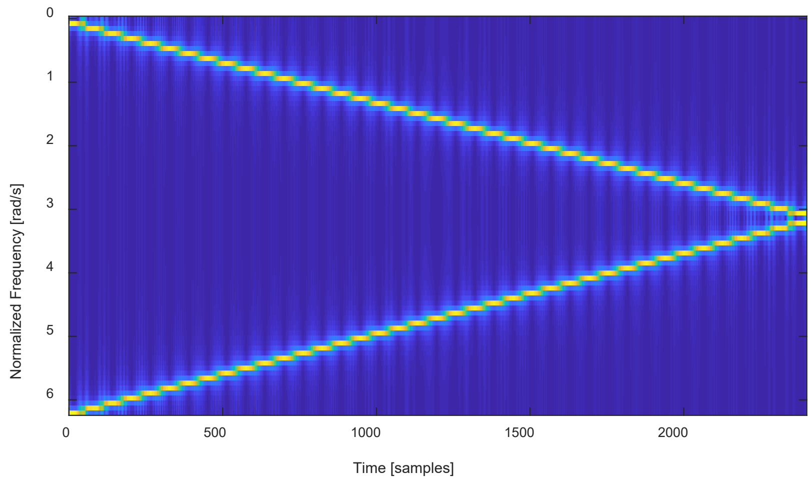
Result with 160 length-160 filters



Filter Banks

- **Question: Why is this not a preferred approach?**
  - It is really expensive!

Result with 160 length-160 filters



- **STFT Approach**
  - W^2 multiplications for every W samples
  - $W^2 = 160^2 = 25{,}600$
- **Filter Bank Approach**
  - W^3 multiplications for every W samples
  - $W^3 = 160^3 = 4{,}086{,}000$

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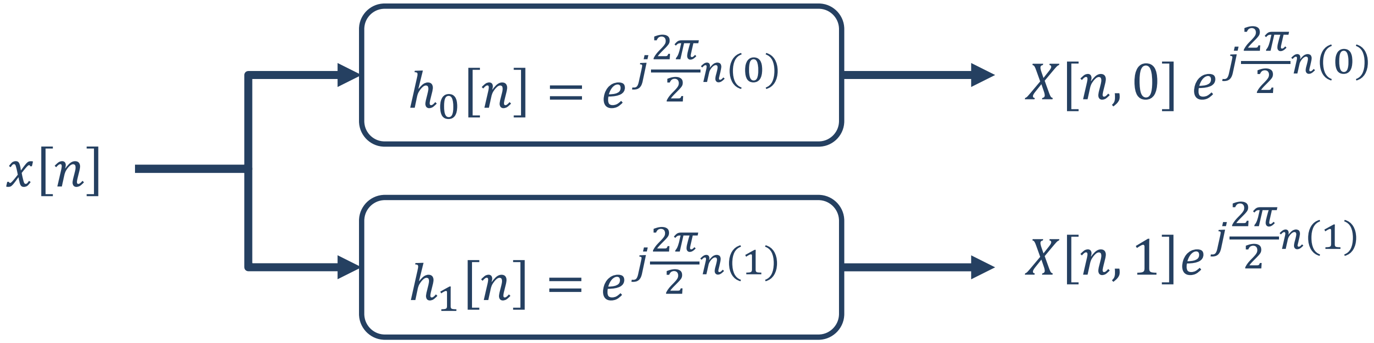
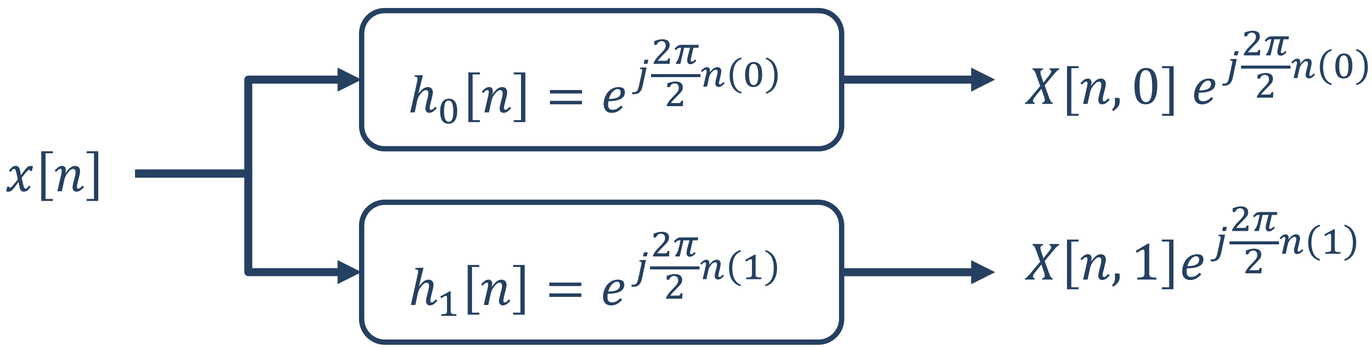
Filter Banks

- **Consider the following filter bank**



Filter Banks

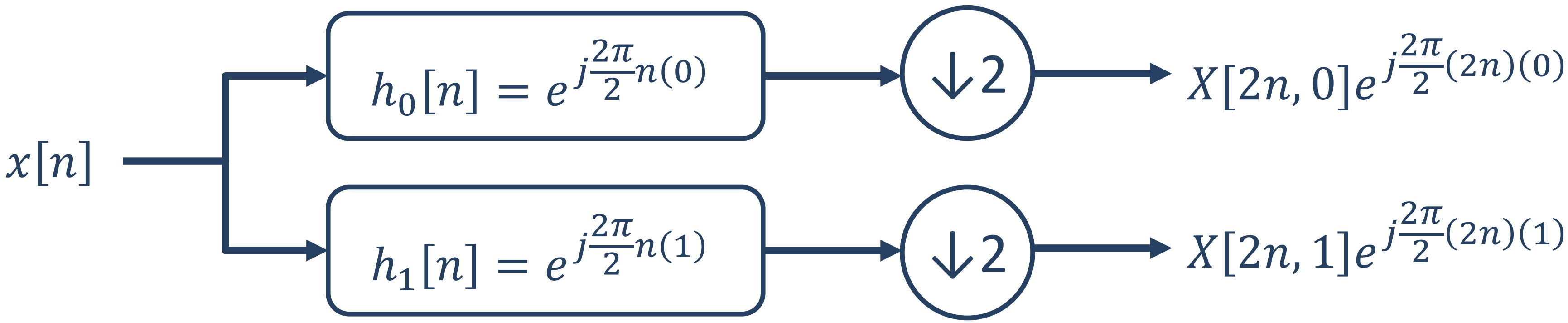
- **Consider the following filter bank**
  - **Question:** How do I make this like the STFT????



  - **Now recall:** The short-time Fourier Transform gave us
    - $X[Mn, 0]$ <- M = shift amount (often window length)

Filter Banks

- **Consider the following filter bank**
  - **Question:** How do I make this like the STFT????



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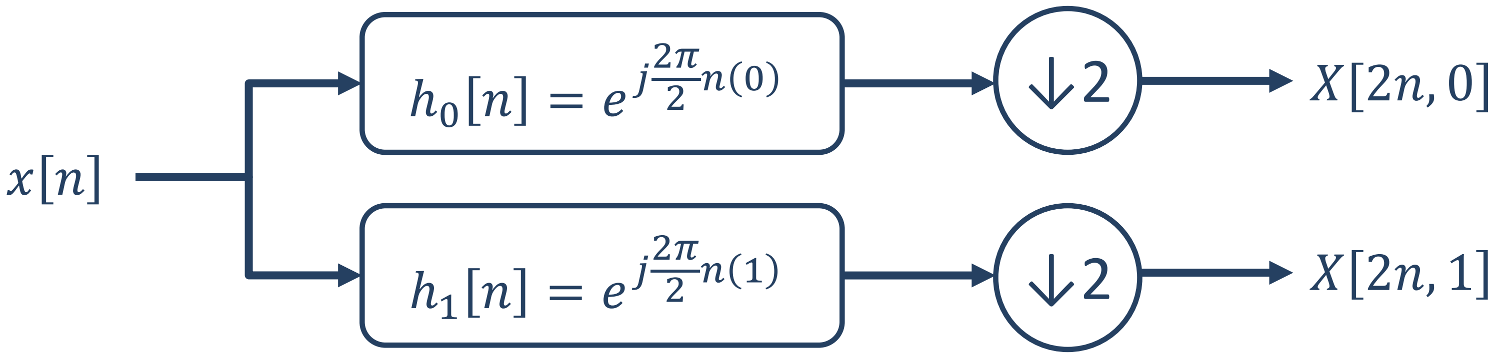

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Filter Banks

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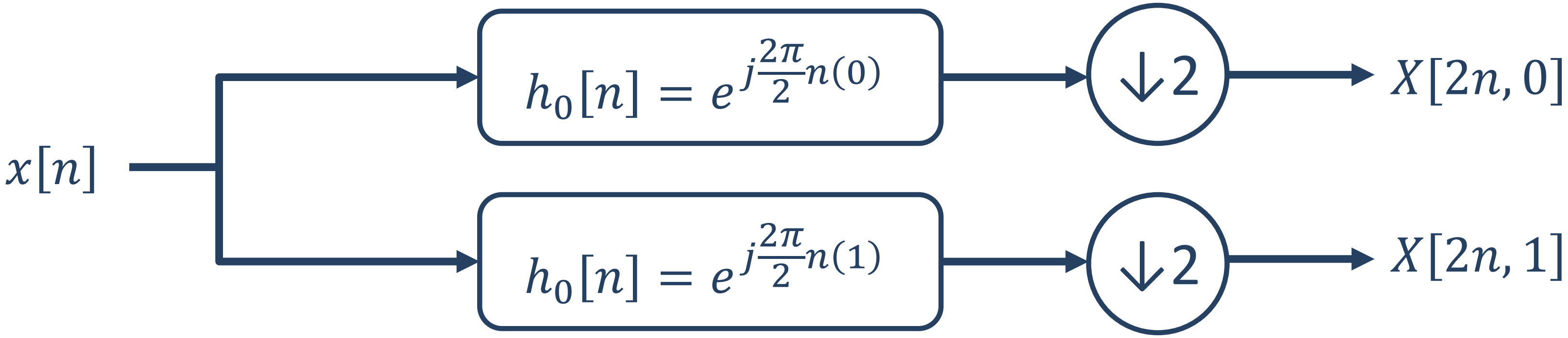
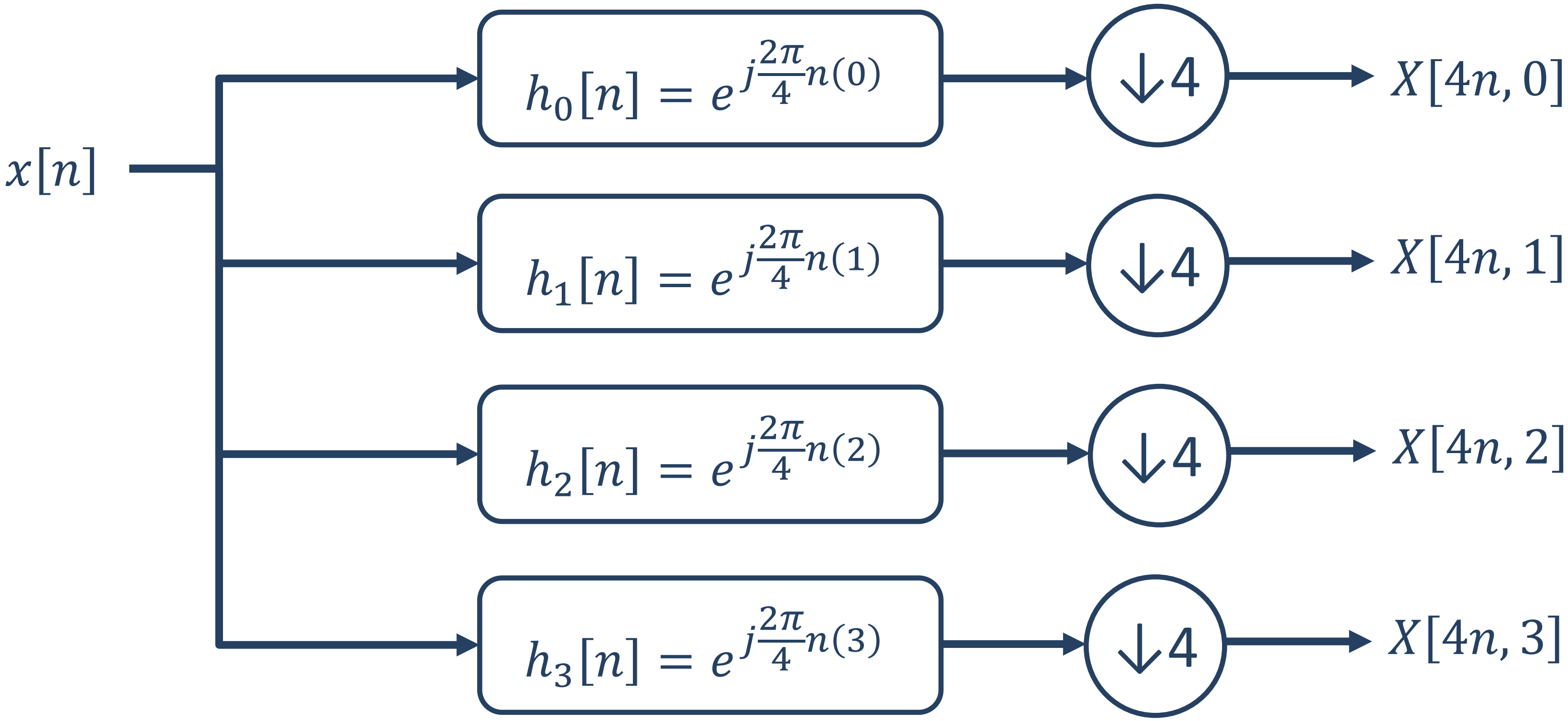


  - **Result:** It is now exactly the same as the STFT with a window of length 2 and shift of 2 between windows
  - **But,** we do not need to buffer $x[n]$

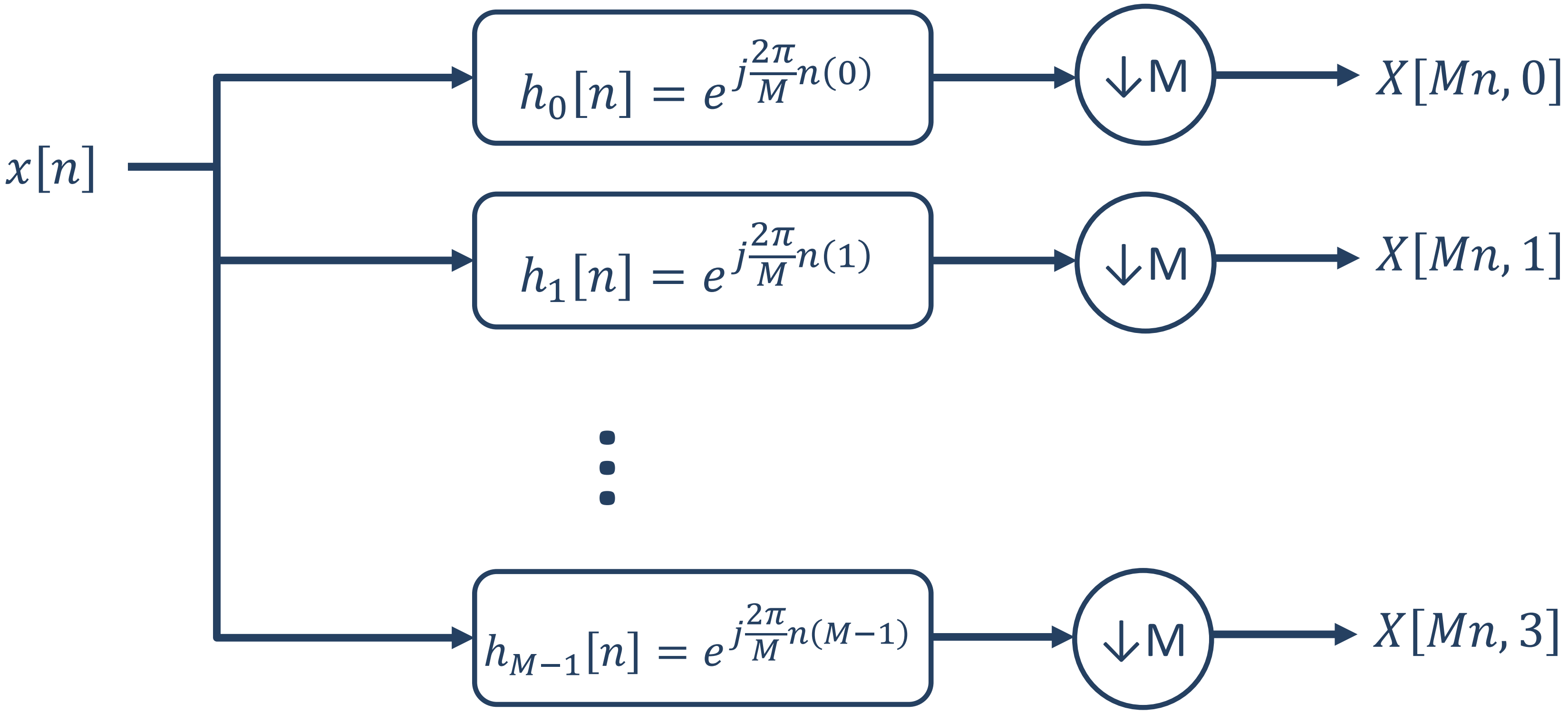
Filter Banks

- **Consider the following filter bank**



Filter Banks

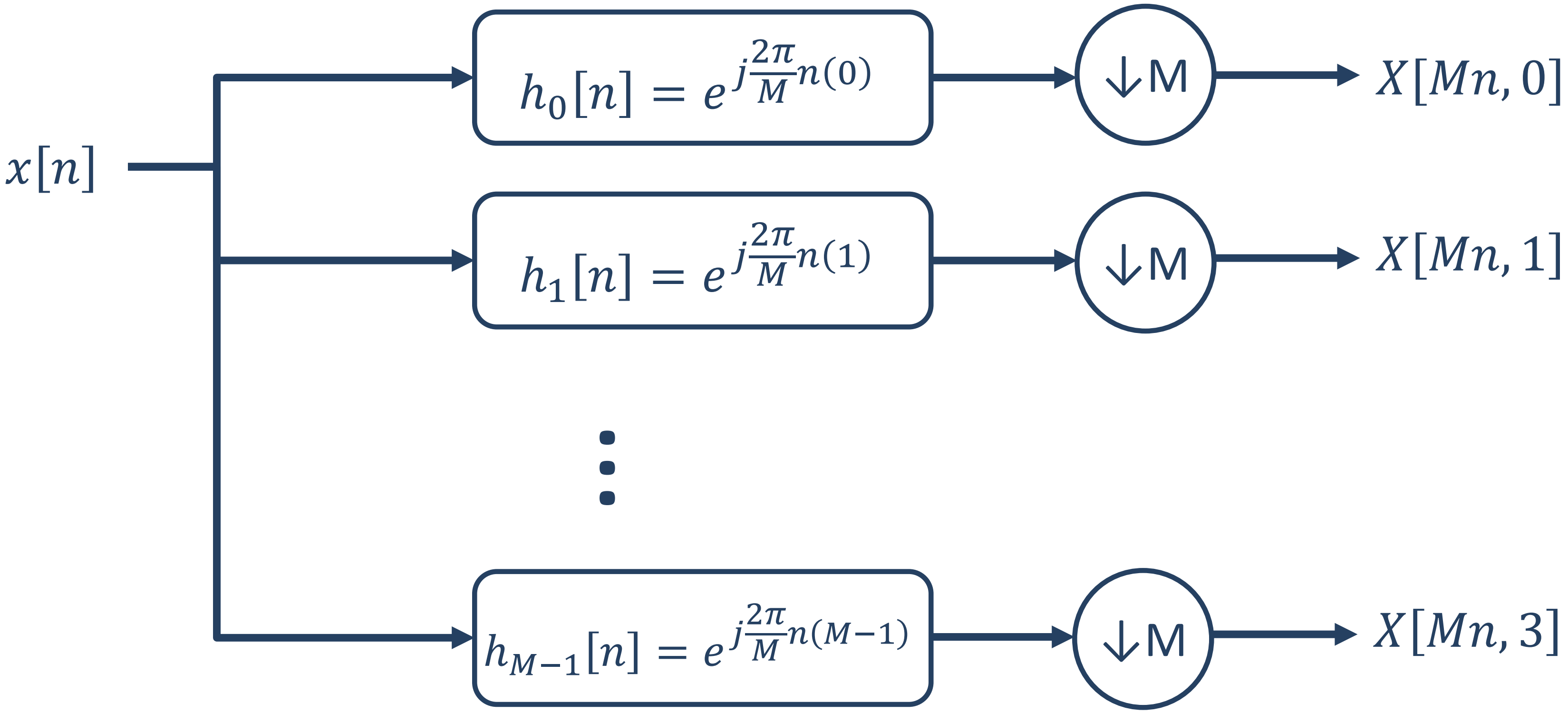
- **Hence, this is an M-point DFT**



- **So, I can implement the STFT as a filter bank…**
  - Can I do more?

Filter Banks

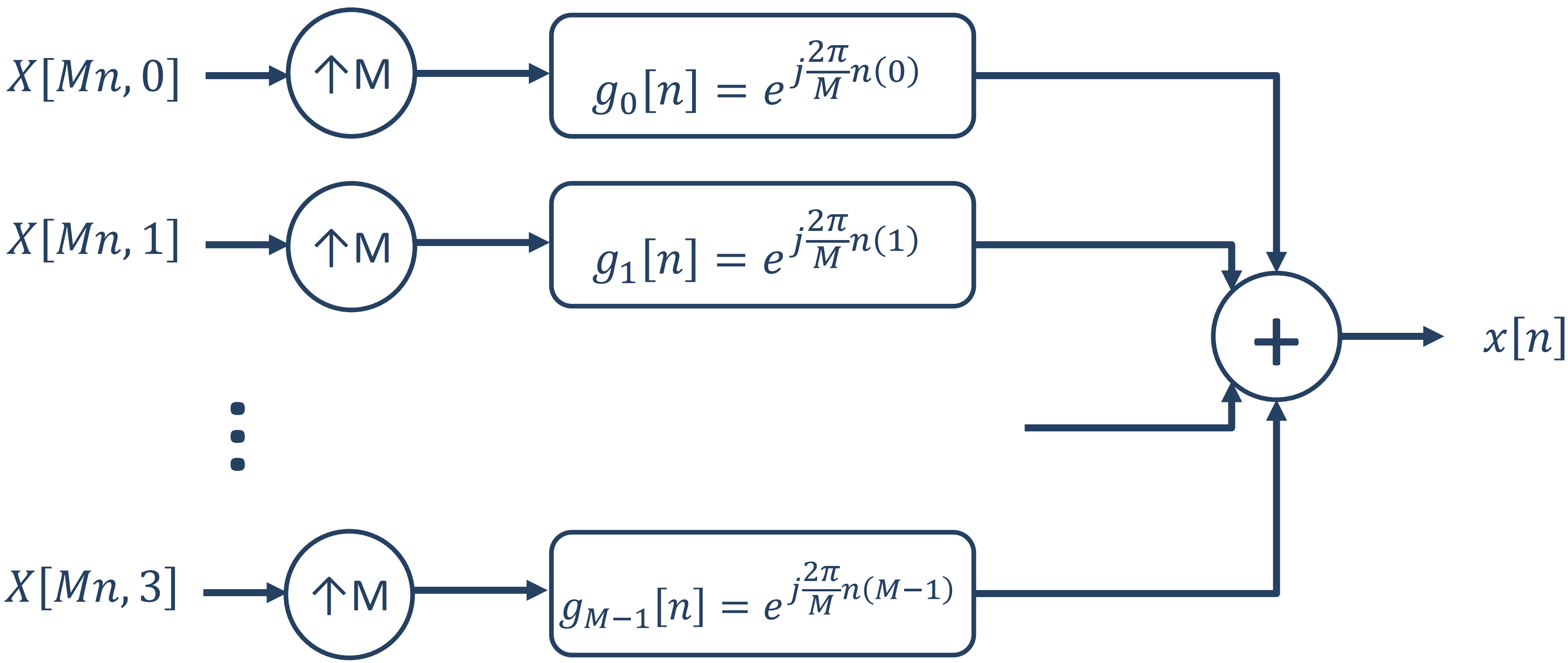
- **Hence, this is an M-point DFT**



- **So, I can implement the STFT as a filter bank…**
  - **Question:** How do I get back into the time domain?

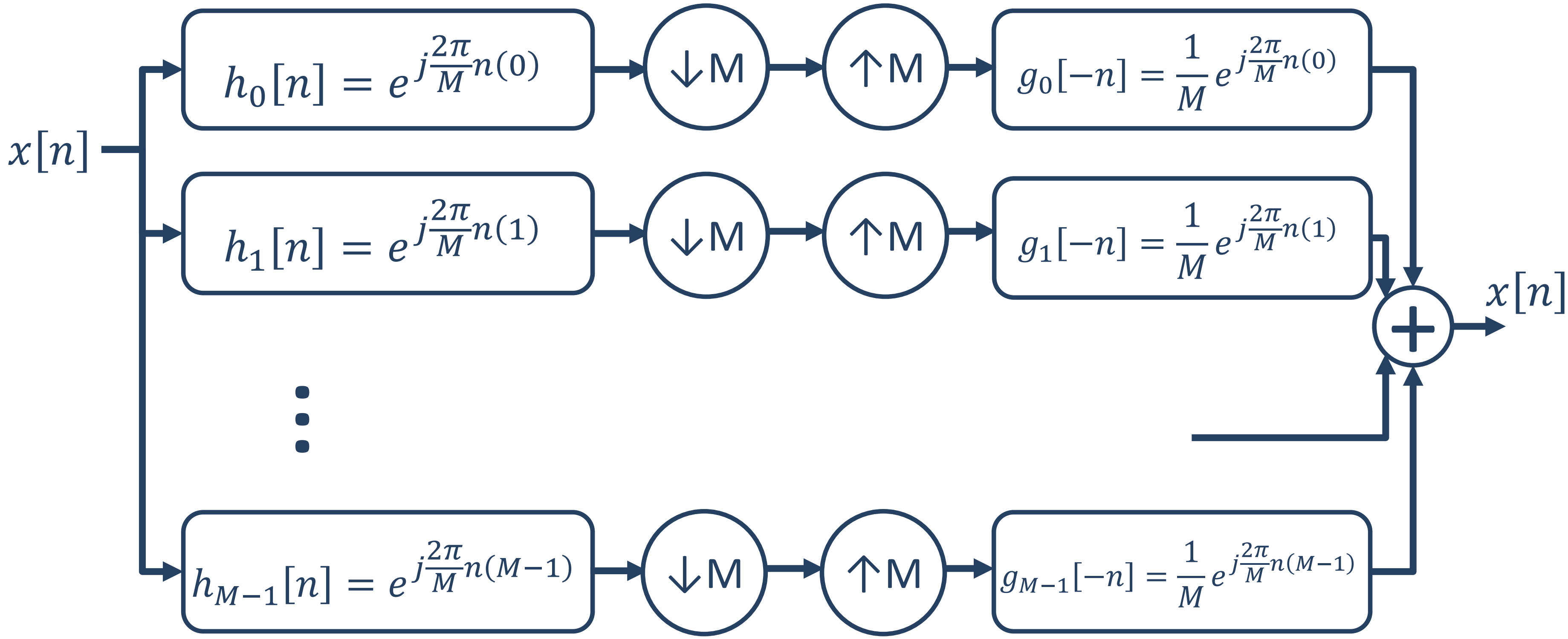
Filter Banks

- **Hence, this is an M-point IDFT**

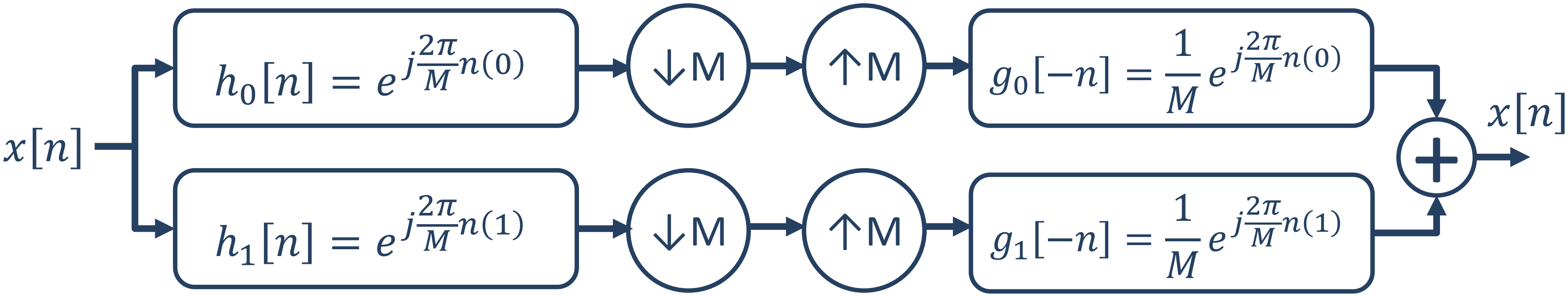


Filter Banks

- **Hence, this is an M-point DFT and IDFT**

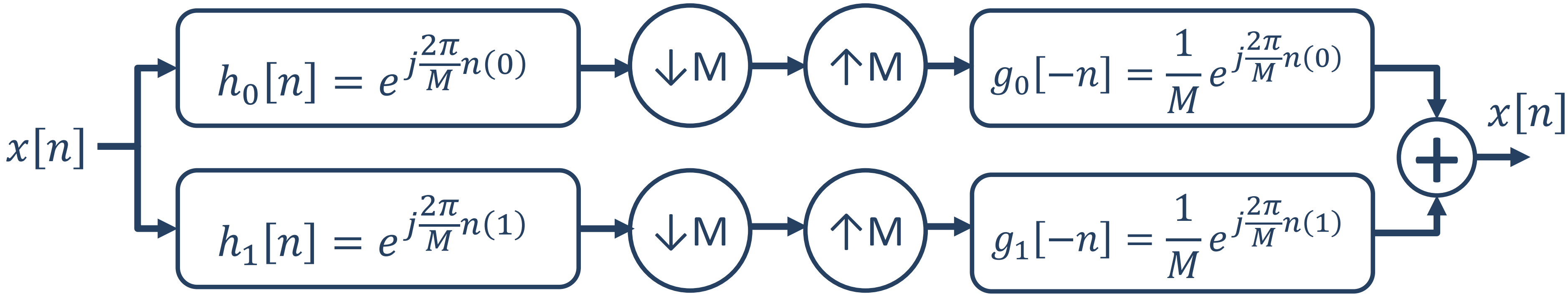


- **Example: The M=2 point DFT and IDFT**



Filter Banks

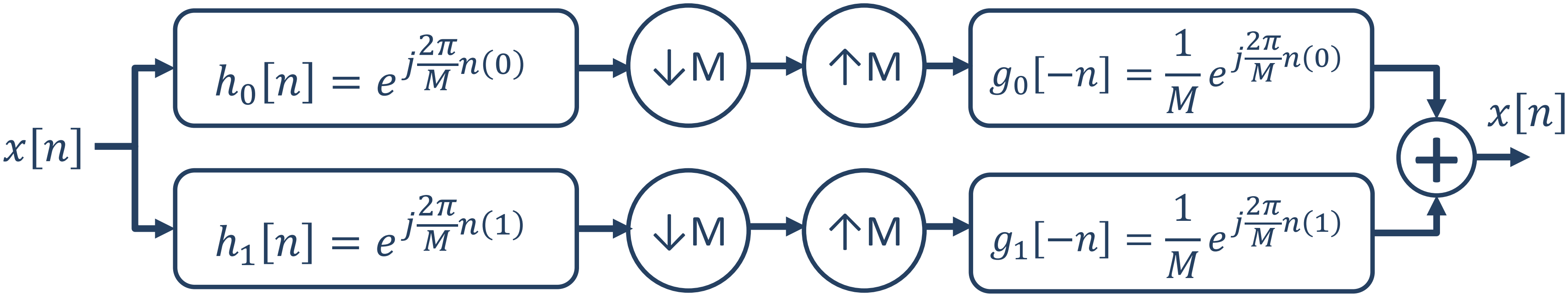
- **Example: The M=2 point DFT and IDFT**



  - What are the filter coefficients for $h_0[n]$ and $h_1[n]$?

Filter Banks

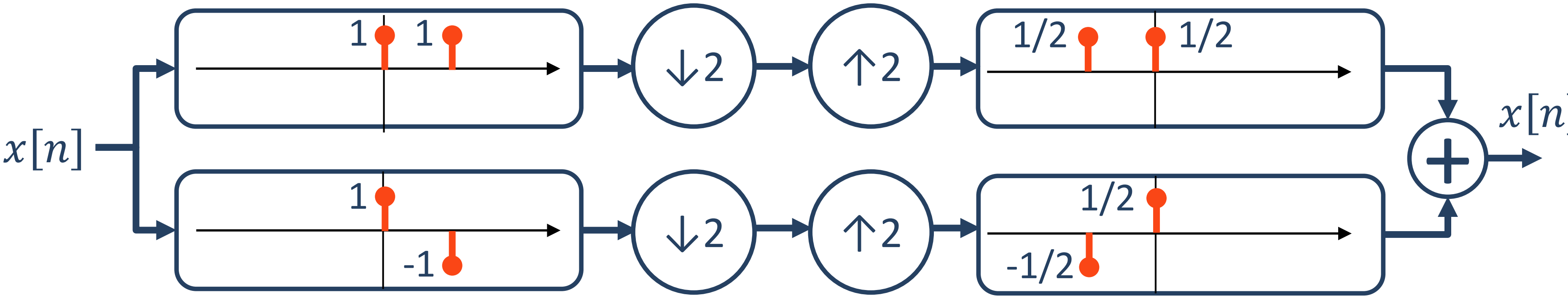
- **Example: The M=2 point DFT and IDFT**



- $h_0[n] = e^{j\frac{2\pi}{2}n(0)} = 1$ for $0 \le n \le 1$
- $h_1[n] = e^{j\frac{2\pi}{2}n(1)} = e^{\pi n}$ for $0 \le n \le 1$
- $g_0[n] = \frac{1}{M}e^{-j\frac{2\pi}{2}n(0)} = \frac{1}{M}$ for $-1 \le n \le 0$
- $g_1[n] = \frac{1}{M}e^{-j\frac{2\pi}{2}n(1)} = \frac{1}{M}e^{\pi n}$ for $-1 \le n \le 0$

Filter Banks

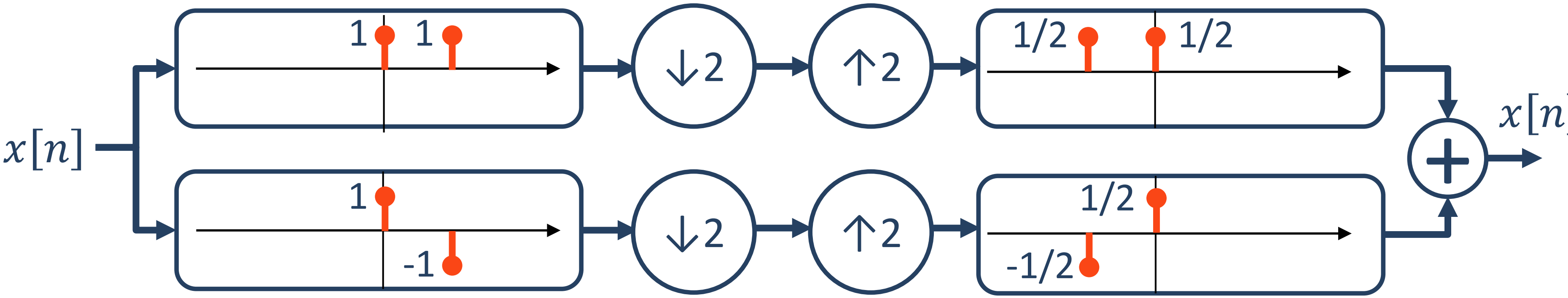
- **Example: The M=2 point DFT and IDFT**



- $h_0[n] = \delta[n] + \delta[n-1]$
- $h_1[n] = \delta[n] - \delta[n-1]$
- $g_0[n] = \frac{1}{2}[\delta[n+1] + \delta[n]]$
- $g_1[n] = \frac{1}{2}[-\delta[n+1] + \delta[n]]$

Filter Banks

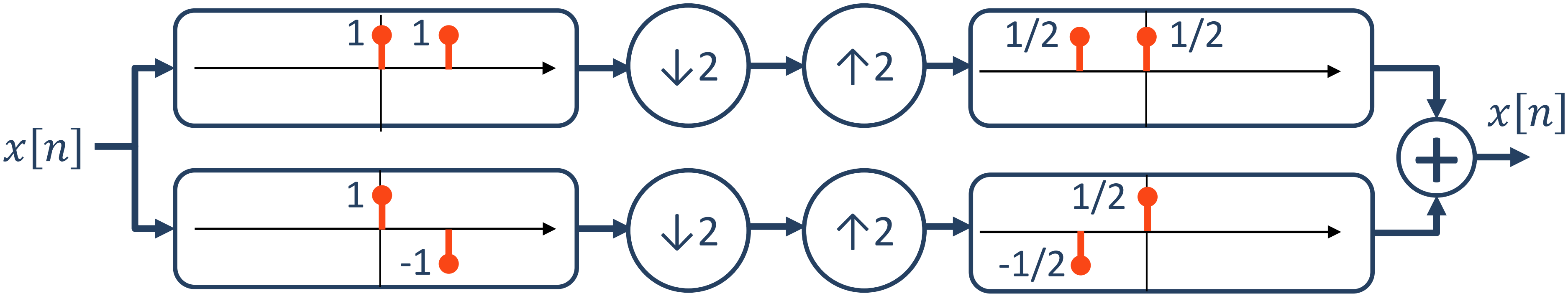
- **Example: The M=2 point DFT and IDFT**



- $h_0[n] = \delta[n] + \delta[n-1]$
  - $H(\omega) = e^{-j\frac{2\pi}{2}\omega(0)} + e^{-j\frac{2\pi}{2}\omega(1)}$
- $h_1[n] = \delta[n] - \delta[n-1]$
  - $H(\omega) = e^{-j\frac{2\pi}{2}\omega(0)} - e^{-j\frac{2\pi}{2}\omega(1)}$

Filter Banks

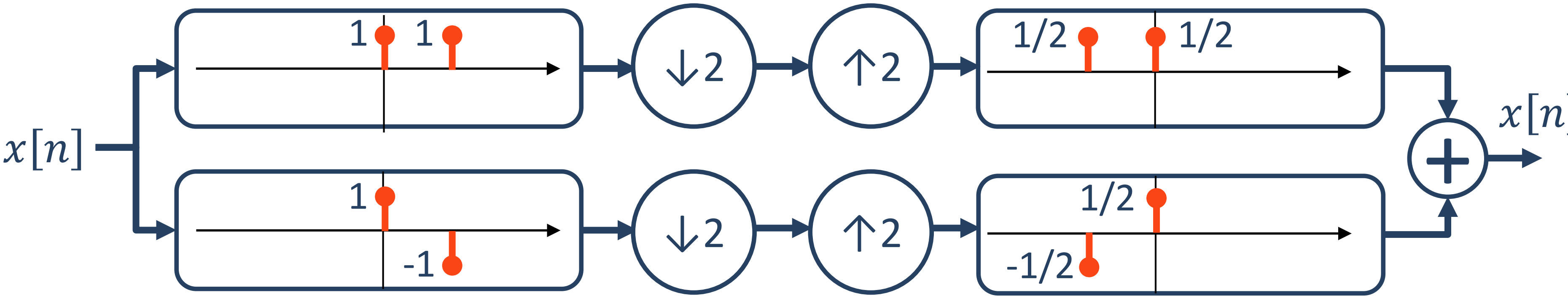
- **Example: The M=2 point DFT and IDFT**



- $h_0[n] = \delta[n] + \delta[n-1]$
  - ◊ $H_0(\omega) = 1 + e^{-j\pi\omega} = 2\cos(\omega/2)\, e^{-j\frac{\pi}{2}\omega}$
- $h_1[n] = \delta[n] - \delta[n-1]$
  - ◊ $H_1(\omega) = 1 - e^{-j\pi\omega} = 2j\sin(\omega/2)\, e^{-j\frac{\pi}{2}\omega}$

Filter Banks

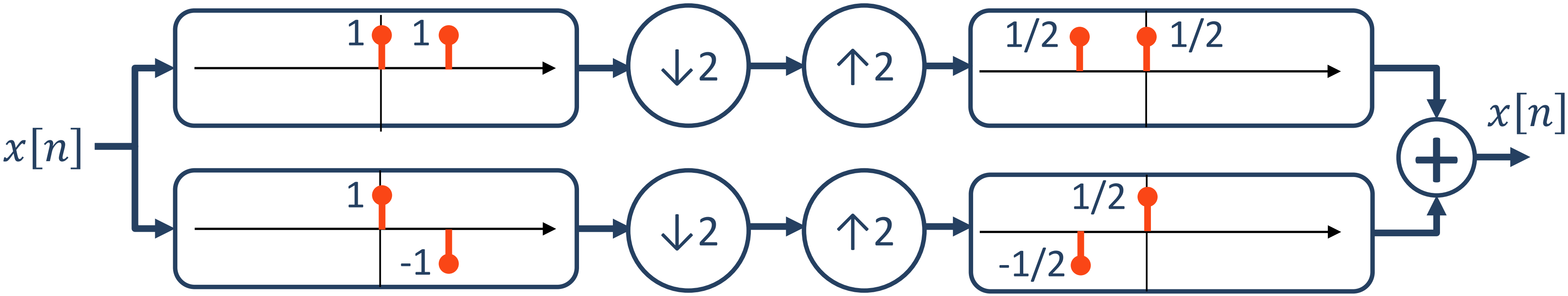
- **Example: The M=2 point DFT and IDFT**



- $h_0[n] = \delta[n] + \delta[n-1]$
  - $|H_0(\omega)| = 2|\cos(\omega/2)|$
- $h_1[n] = \delta[n] - \delta[n-1]$
  - $|H_1(\omega)| = 2|\sin(\omega/2)|$

Filter Banks

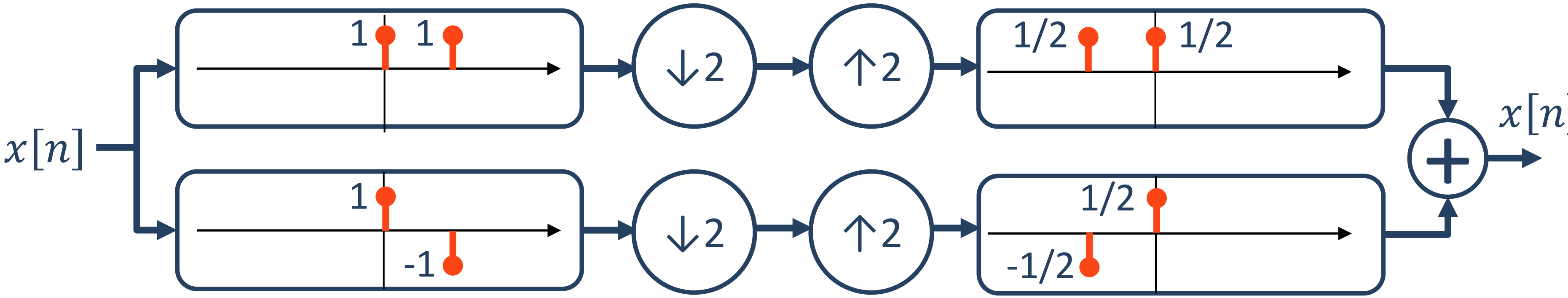
- **Example: The M=2 point DFT and IDFT**

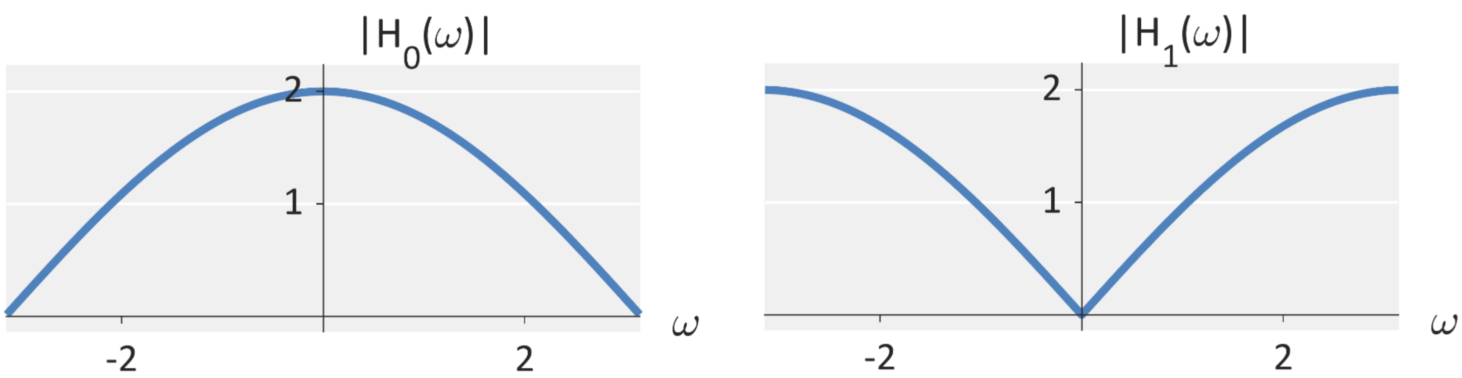


  - $h_0[n] = \delta[n] + \delta[n-1]$
    - $|H_0(\omega)| = 2|\cos(\omega/2)|$
    - $|G_0(\omega)| = |\cos(\omega/2)|$
  - $h_1[n] = \delta[n] - \delta[n-1]$
    - $|H_1(\omega)| = 2|\sin(\omega/2)|$
    - $|G_1(\omega)| = |\sin(\omega/2)|$

Filter Banks

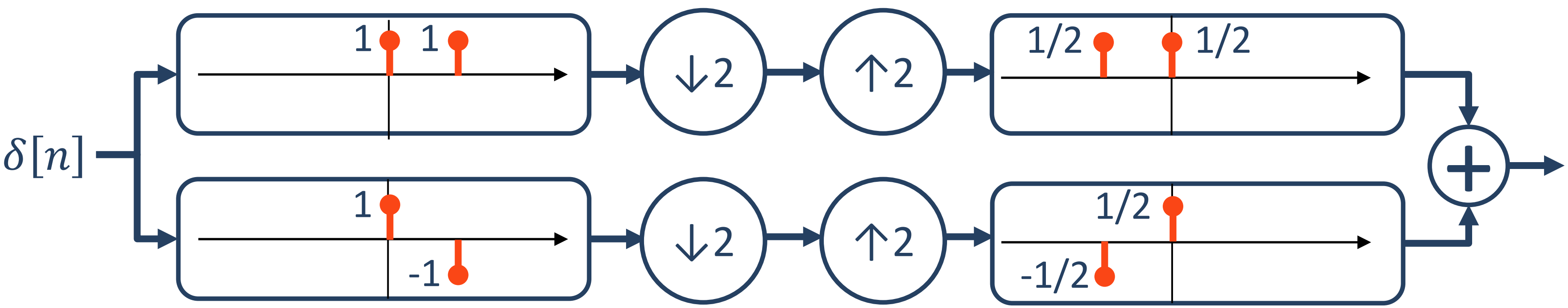
- **Example: The M=2 point DFT and IDFT**





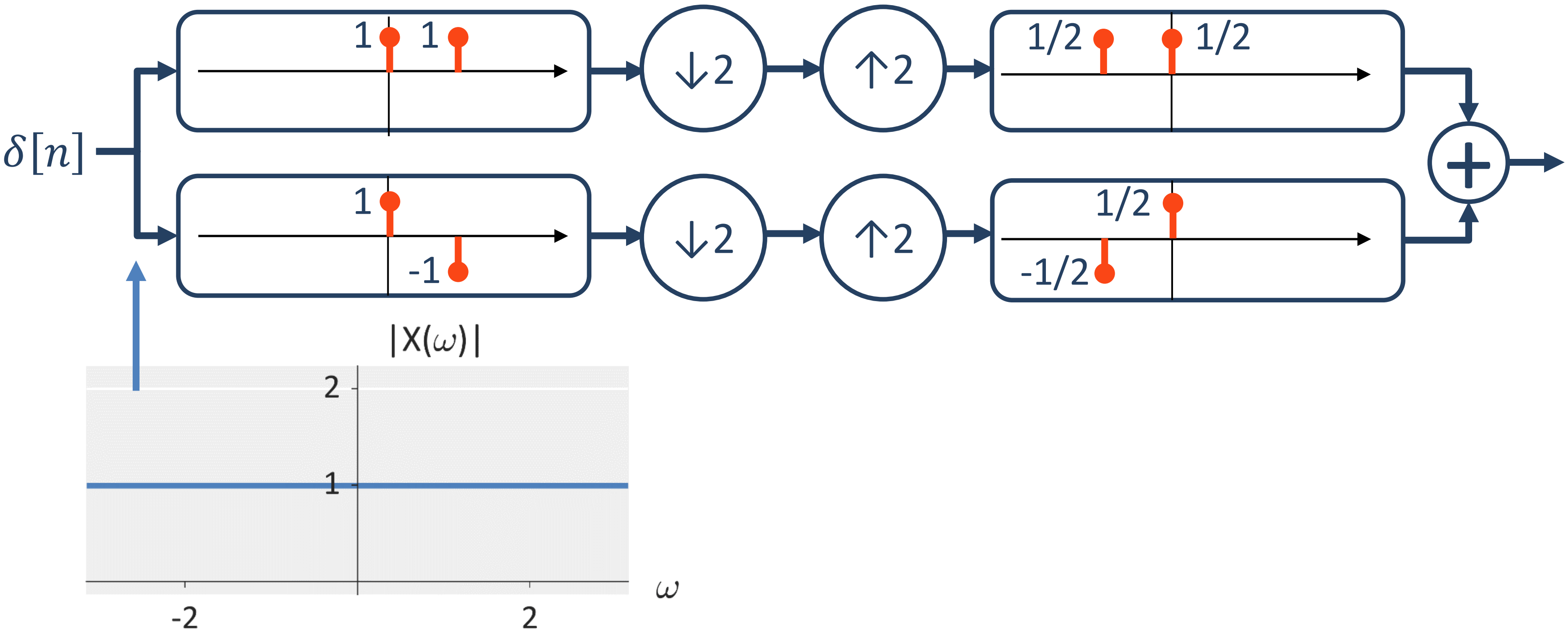
Filter Banks

- **Example: Plot of the intermediate frequency magnitudes**



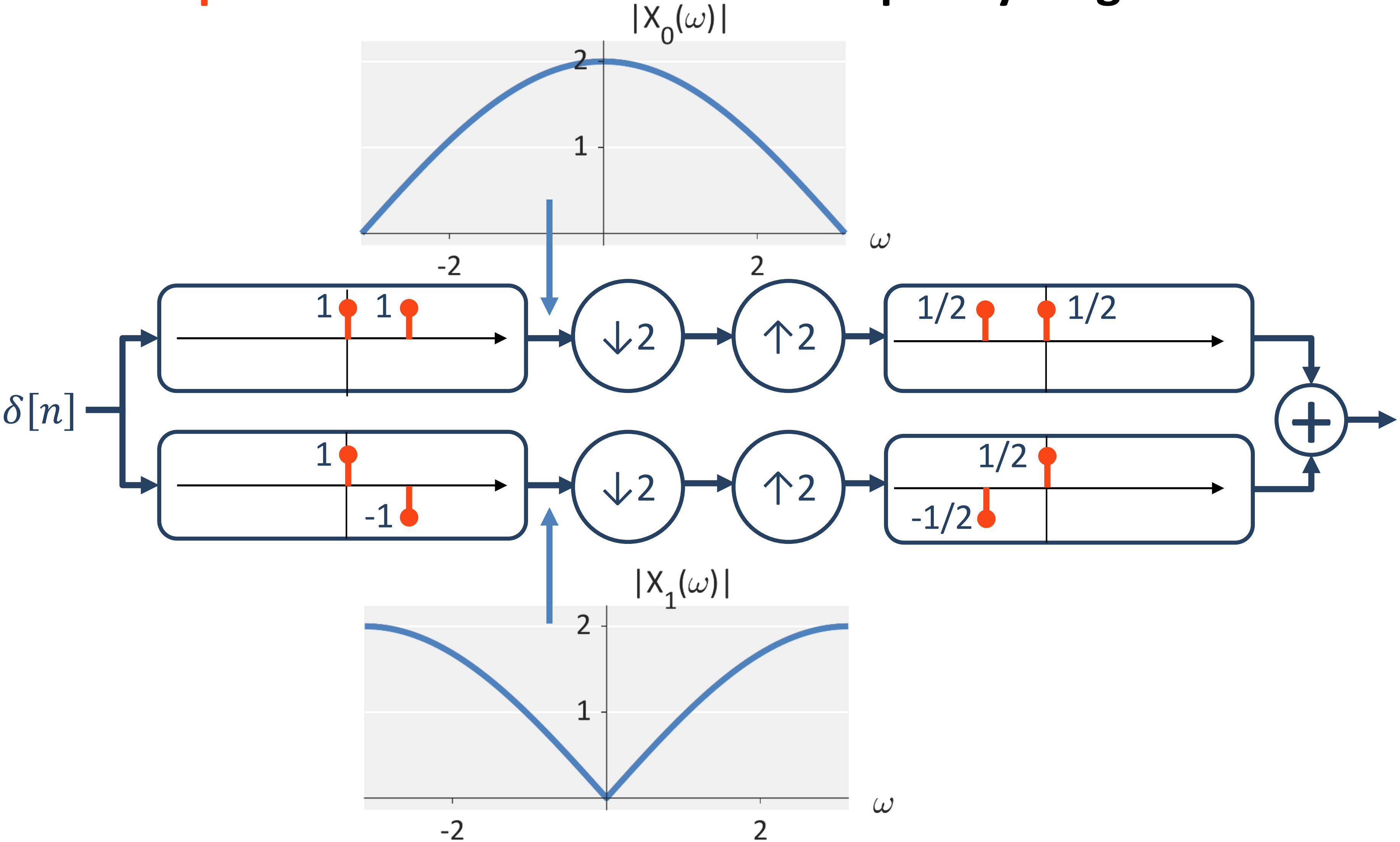
Filter Banks

- **Example:** **Plot of the intermediate frequency magnitudes**



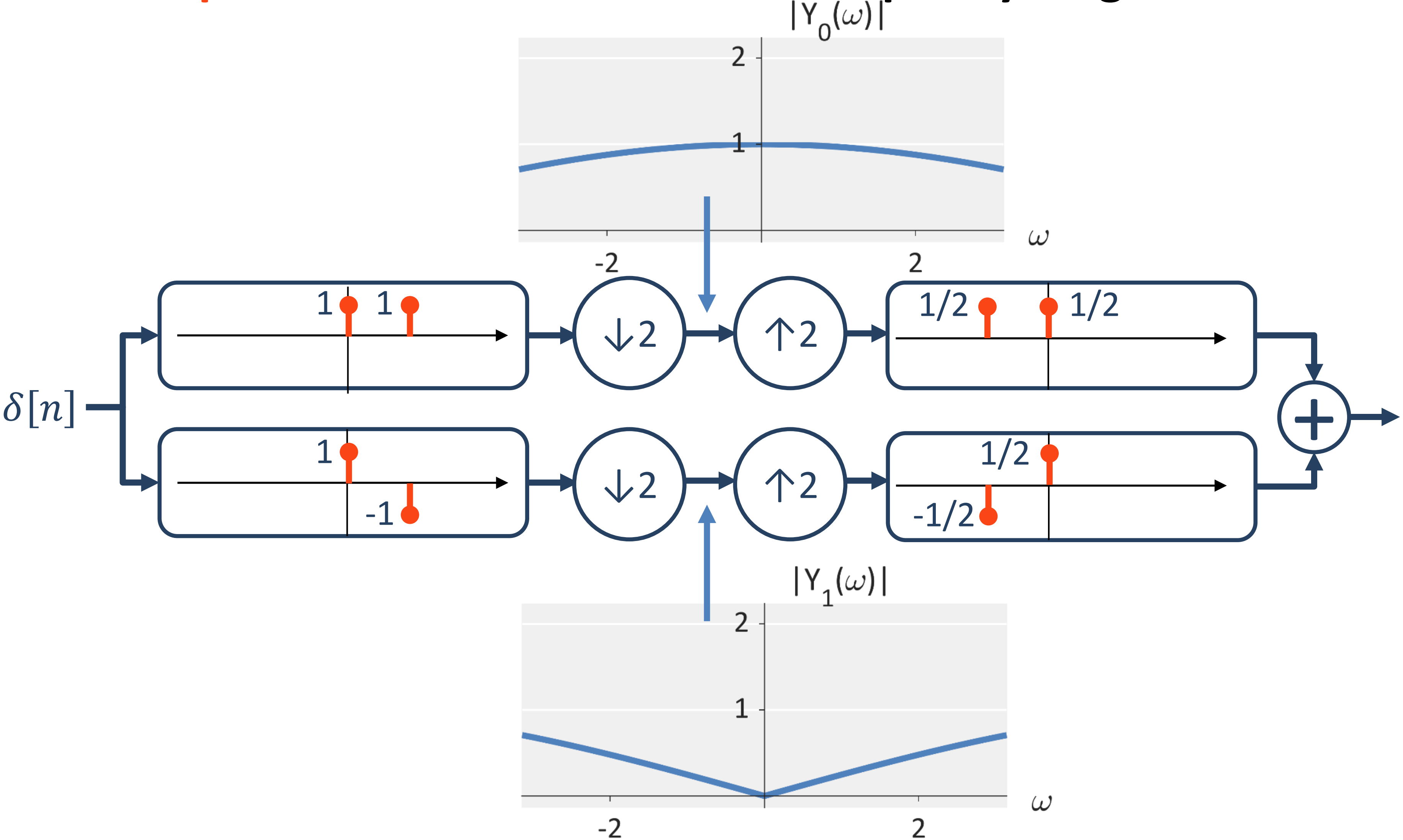
Filter Banks

- **Example: Plot of the intermediate frequency magnitudes**



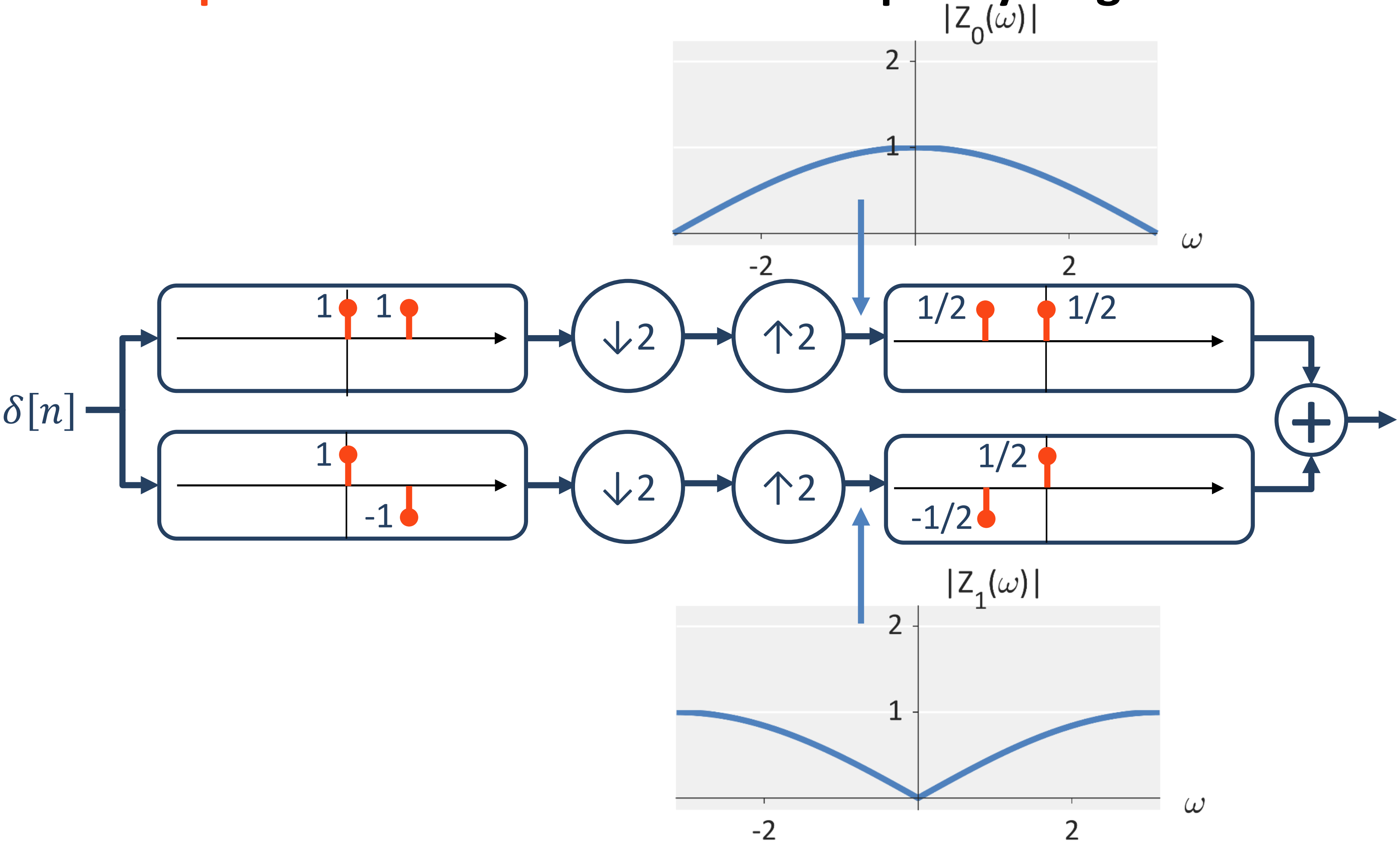
Filter Banks

- **Example:** **Plot of the intermediate frequency magnitudes**



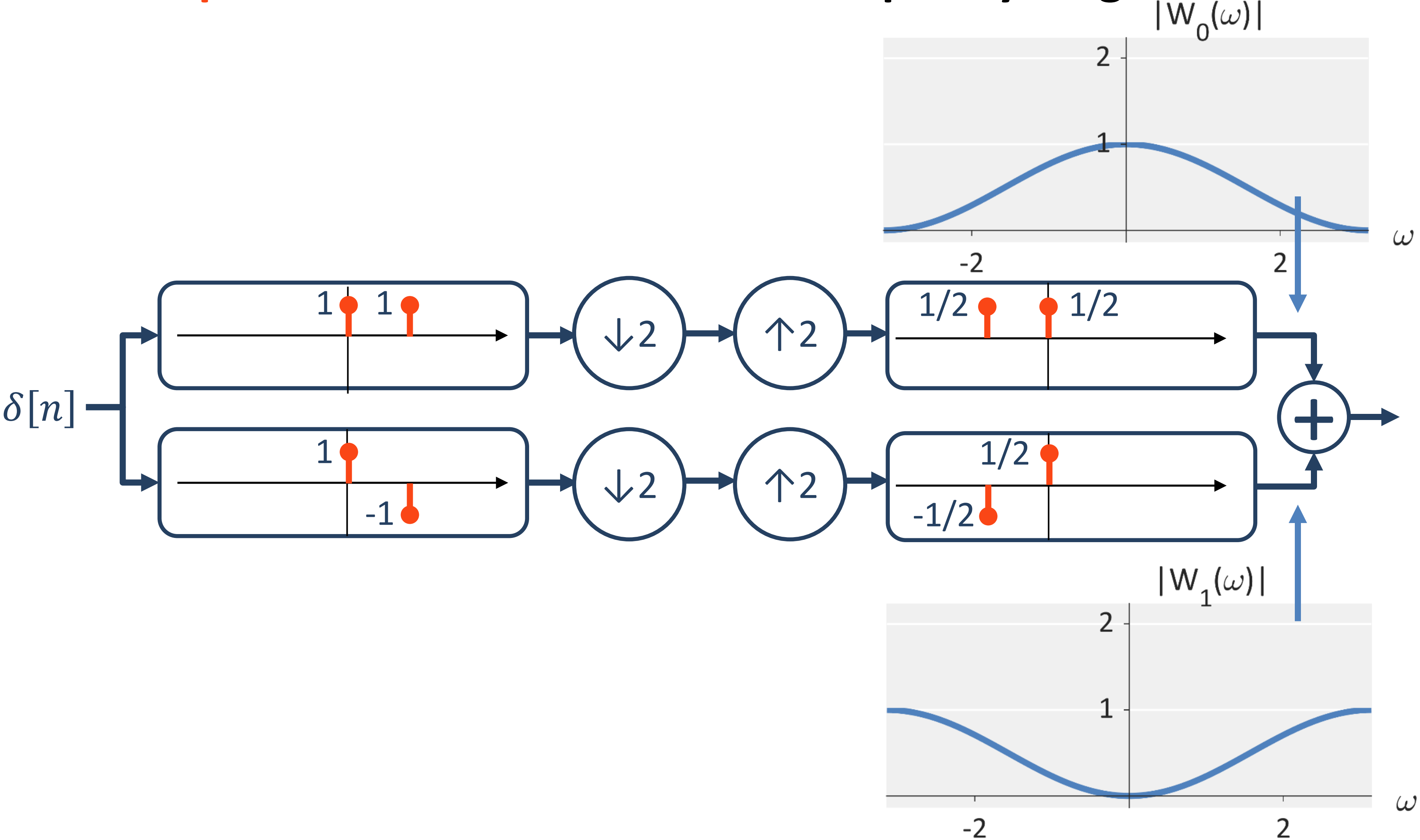
Filter Banks

- **Example:** **Plot of the intermediate frequency magnitudes**



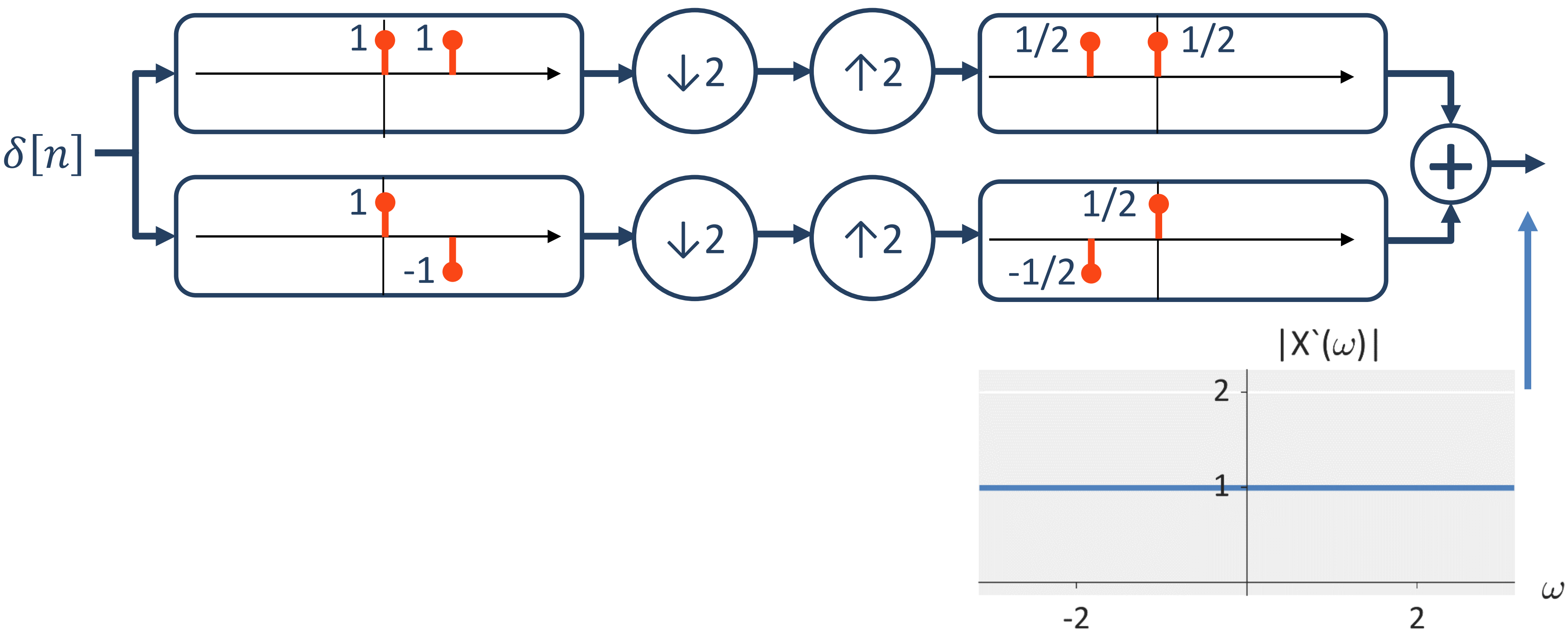
Filter Banks

- **Example:** **Plot of the intermediate frequency magnitudes**



Filter Banks

- **Example: Plot of the intermediate frequency magnitudes**



Filter Banks

- **Question:** **Can we generalize this?**