Lecture 13: Practical Fourier Transforms

Foundations of Digital Signal Processing

Outline

- The Discrete Fourier Transform (DFT)
- Circular Convolution
- The DTFT and the DFT: The Relationship
- The Fast Fourier Transform

News

- **Homework #5**
  - Due *today*
  - Submit via canvas
- **Coding Problem #4**
  - Due *today*
  - Submit via canvas

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Foundations of Digital Signal Processing

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- **The Discrete Fourier Transform (DFT)**
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Deriving Transforms

- **Consider the Inverse Discrete-Time Fourier Transform….**
  - What happens if we sample $\mathrm{X}(\omega)$?

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(\omega) e^{+j\omega \mathrm{n}} \, d\omega$$

Deriving Transforms

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$$= \frac{1}{2\pi} \int_{2\pi} X(\omega) \left[2\pi \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_s) \right] e^{+j\omega \mathrm{n}} \, d\omega$$

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$$k\omega_s \geq 0 \qquad k\omega_s < 2\pi$$

$$k \geq 0 \qquad k < \frac{2\pi}{\omega_s}$$

Deriving Transforms

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$$= \sum_{k=0}^{\frac{2\pi}{\omega_s}-1} X(k\omega)e^{+jk\omega_s \mathrm{n}}$$

Let $2\pi/\omega_s = N$

$\omega_s = 2\pi/N$

Deriving Transforms

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$$= \sum_{k=0}^{N-1} X(k\omega) e^{+j\frac{2\pi k}{N}\mathrm{n}}$$

Let $2\pi/\omega_s = K$

$\omega_s = 2\pi/K$

Deriving Transforms

- **Consider the Inverse Discrete-Time Fourier Transform….**
  - What happens if we sample $\mathrm{X}(\omega)$?

$$x[n] = \sum_{k=0}^{N-1} X[k] e^{+j\frac{2\pi}{N}\mathrm{kn}}$$

  - The DTFT becomes the Discrete-Time Fourier Series

The Discrete Fourier Transform

- **The Discrete-Time Fourier Series**
  - Analysis Equations

$$X[k] = \frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi}{K}kn}$$

  - Synthesis Equations

$$x[n] = \sum_{k=0}^{N-1} X[k]e^{+j\frac{2\pi}{N}kn}$$

The Discrete Fourier Transform

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- **Question:** What are the properties of the DFT?
  - How does this work?

The Discrete Fourier Transform

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- **Question:** What are the properties of the DFT?
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The Discrete Fourier Transform

- **Properties of the Discrete Fourier Transform**
  - If $x[n]$ is real
    - ◊ $\text{real}(X[k])$ is even
    - ◊ $\text{imag}(X[k])$ is odd
    - ◊ $|X[k]|$ is even
    - ◊ $\angle X[k]$ is odd

The Discrete Fourier Transform

- **Properties of the Discrete Fourier Transform**
  - If $x[n]$ is real and odd
    - $\text{real}(X[k]) = 0$
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The Discrete Fourier Transform

- **Circular Convolution**
  - Multiplication property (DFT)

$$x[n]y[n] \leftrightarrow \frac{1}{N} X[k] \circledast Y[k]$$

  - Convolution property (DFT)

$$x[n] \circledast y[n] \leftrightarrow X[k]Y[k]$$

The Discrete Fourier Transform

- **Circular Convolution**
  - Multiplication property (DTFT)

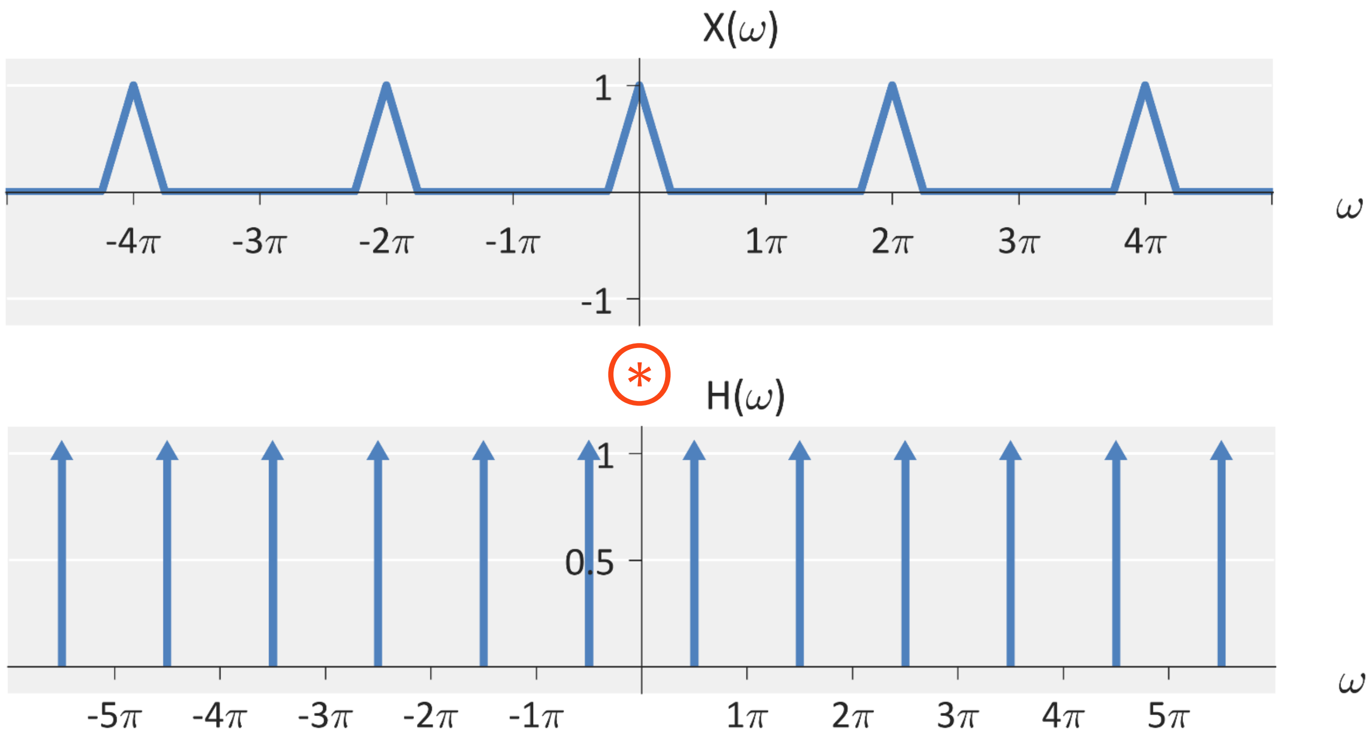
$$x[n]y[n] \leftrightarrow \frac{1}{2\pi} X(\omega) \circledast Y(\omega)$$

  - Convolution property (DTFT)

$$x[n] * y[n] \leftrightarrow X(\omega)Y(\omega)$$

Circular Convolution

- **What is Circular Convolution?**
  - Convolution for periodic signals



Circular Convolution

- **What is Circular Convolution?**
  - Convolution for periodic signals
  - Convolve
    - One period of one signal
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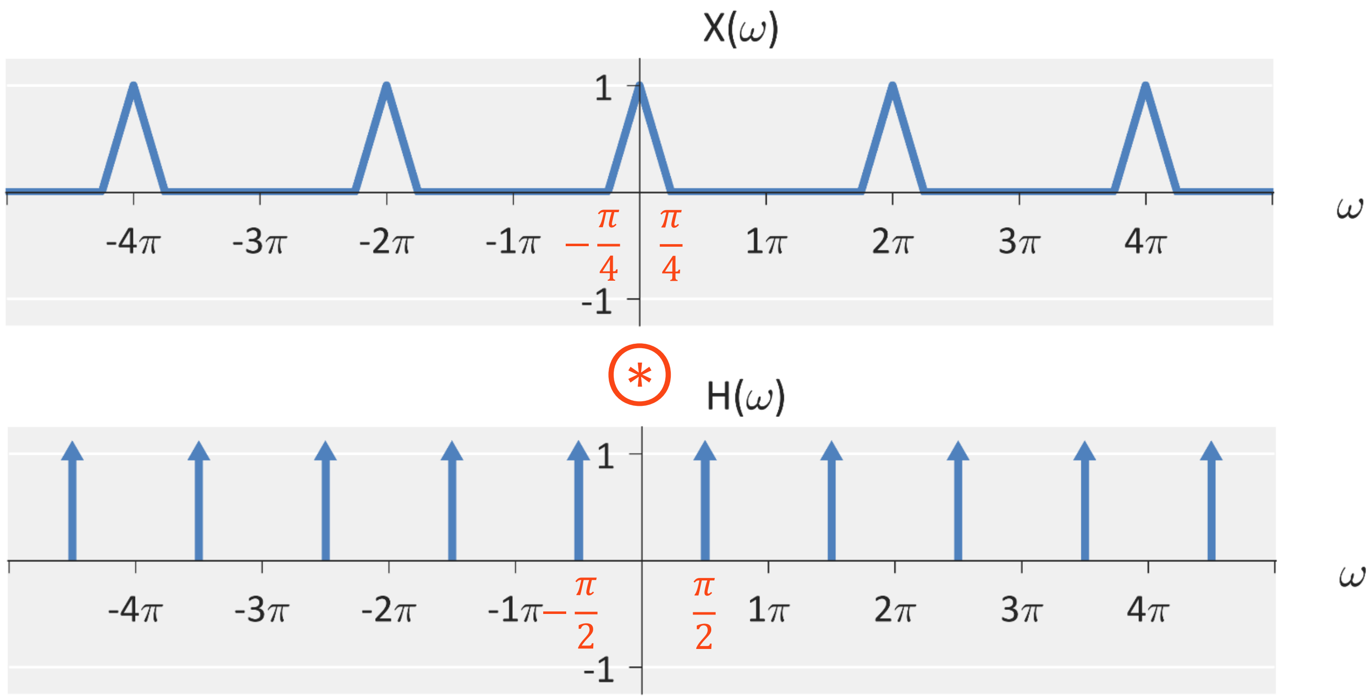

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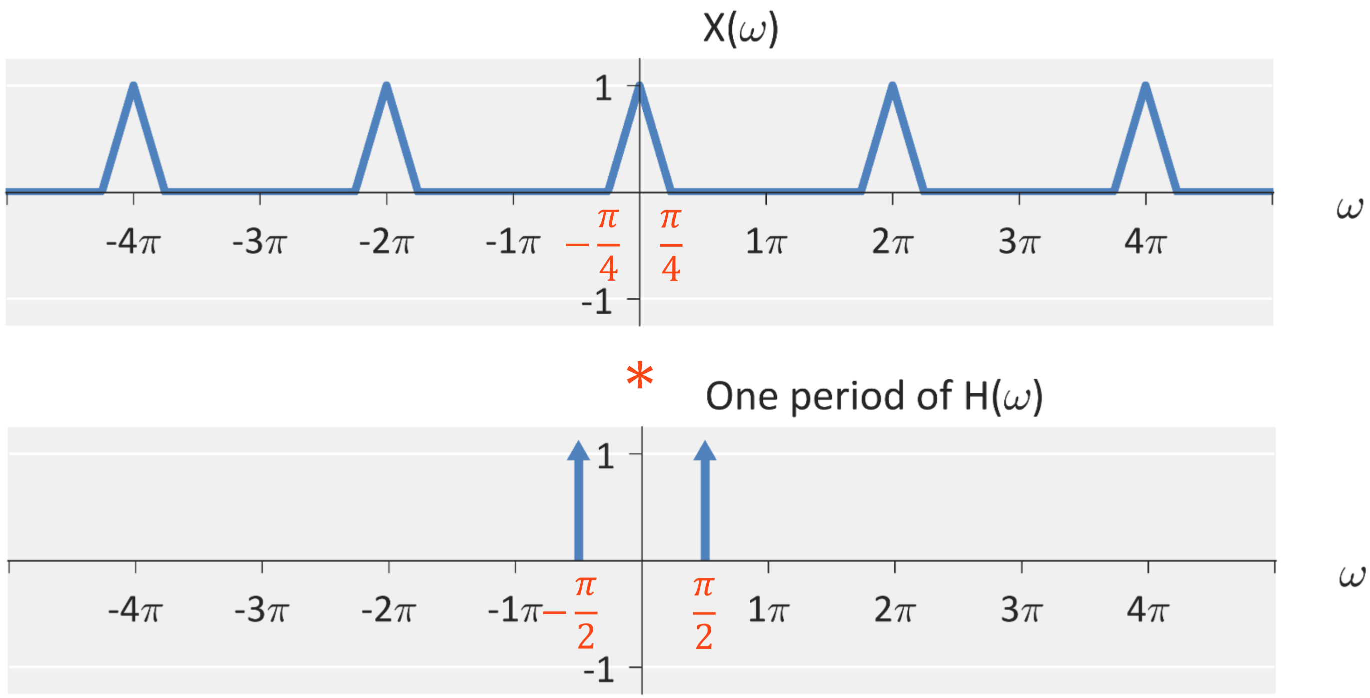

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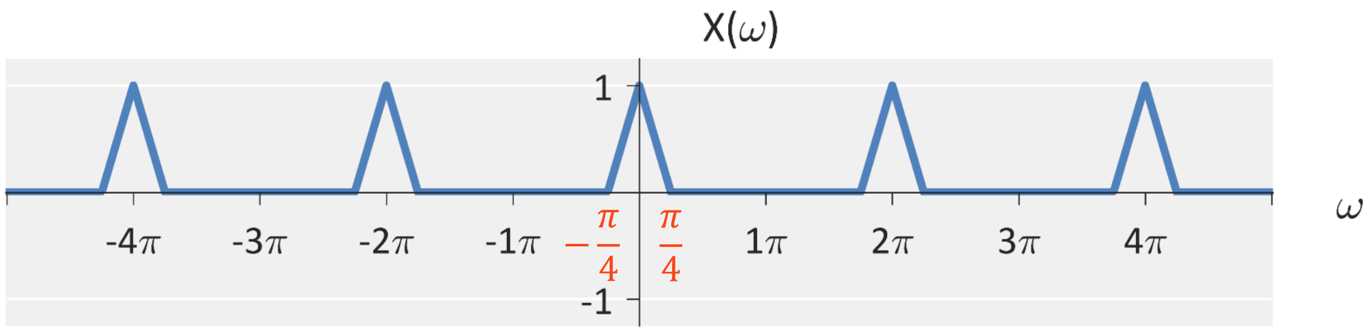

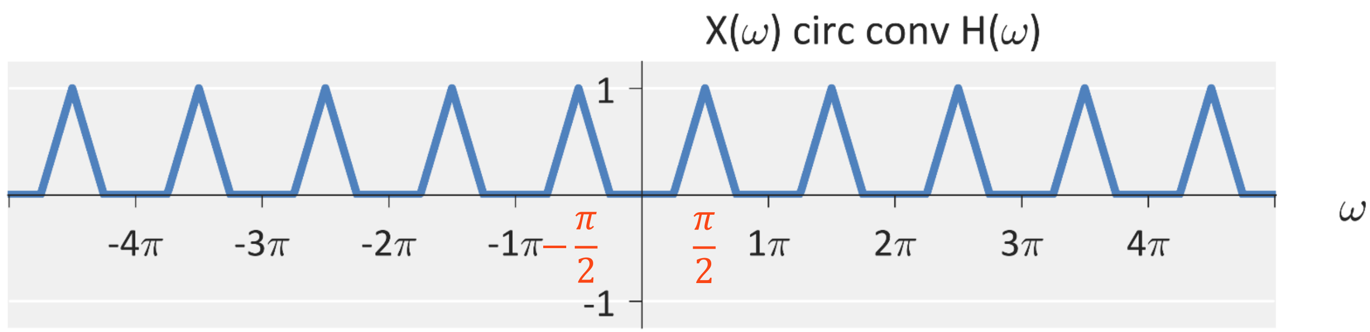

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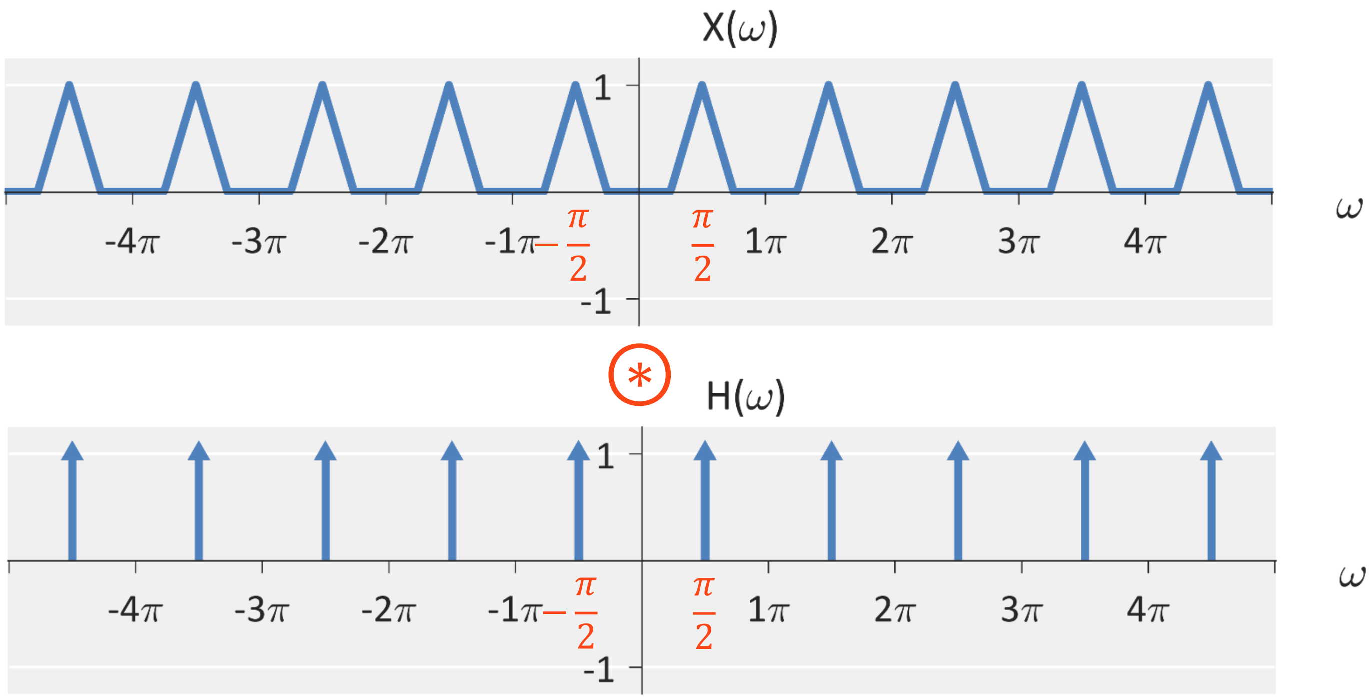

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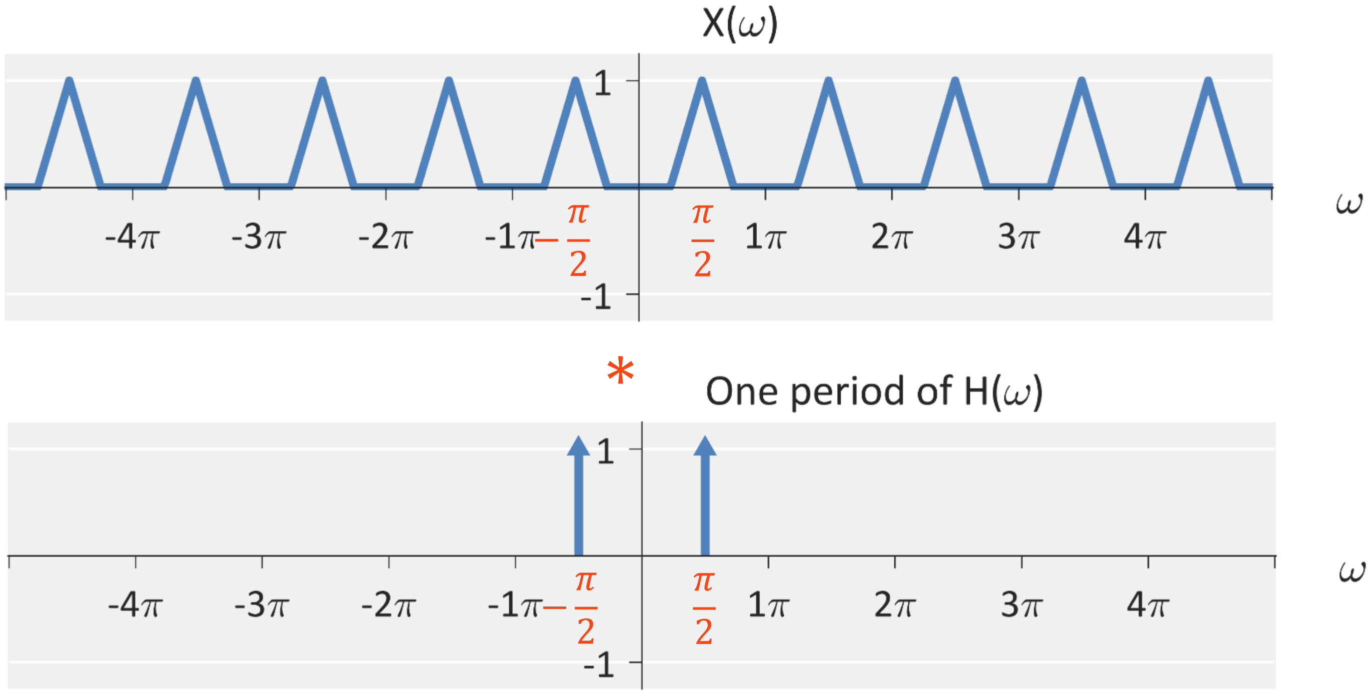

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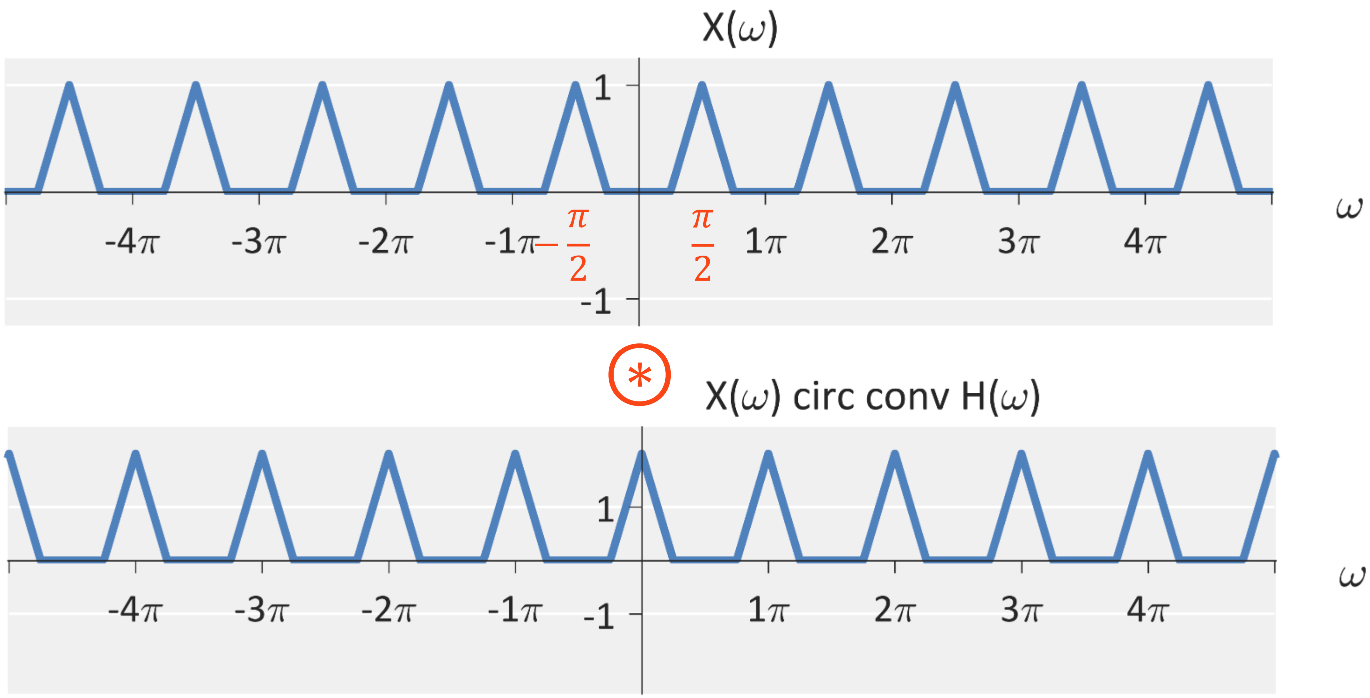

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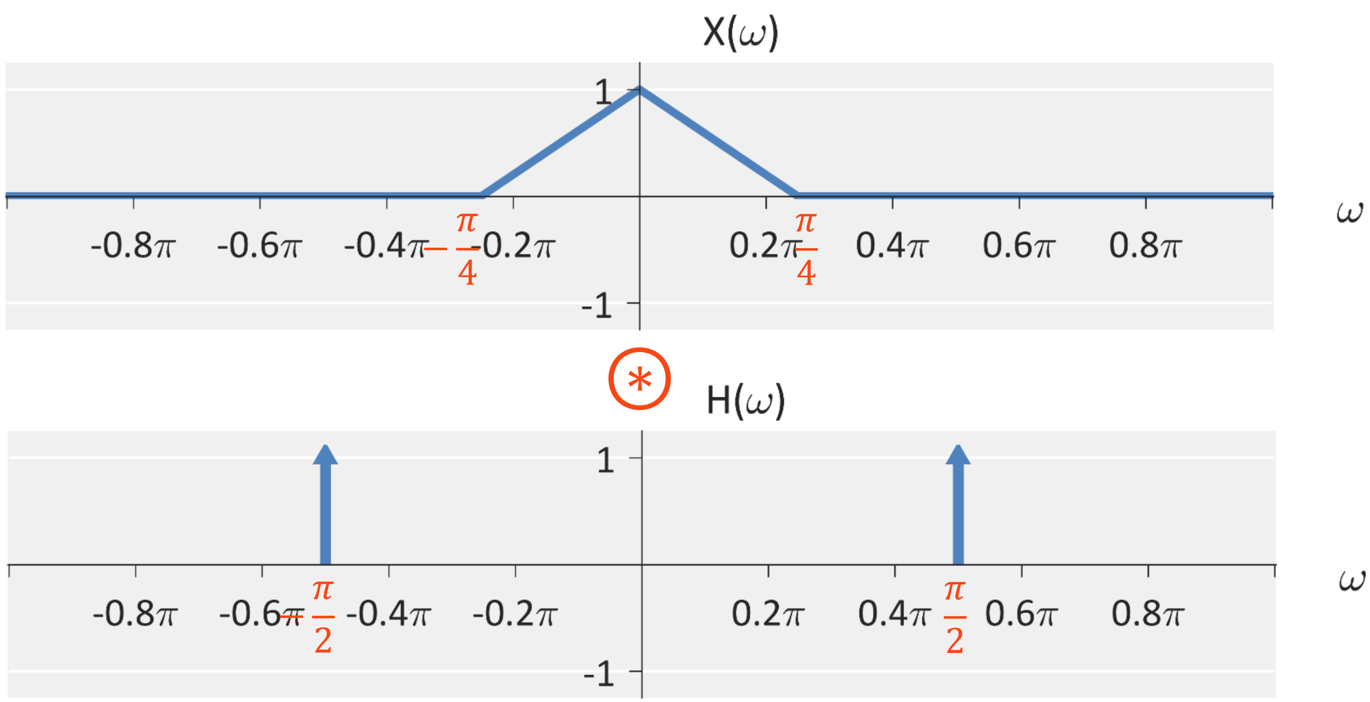

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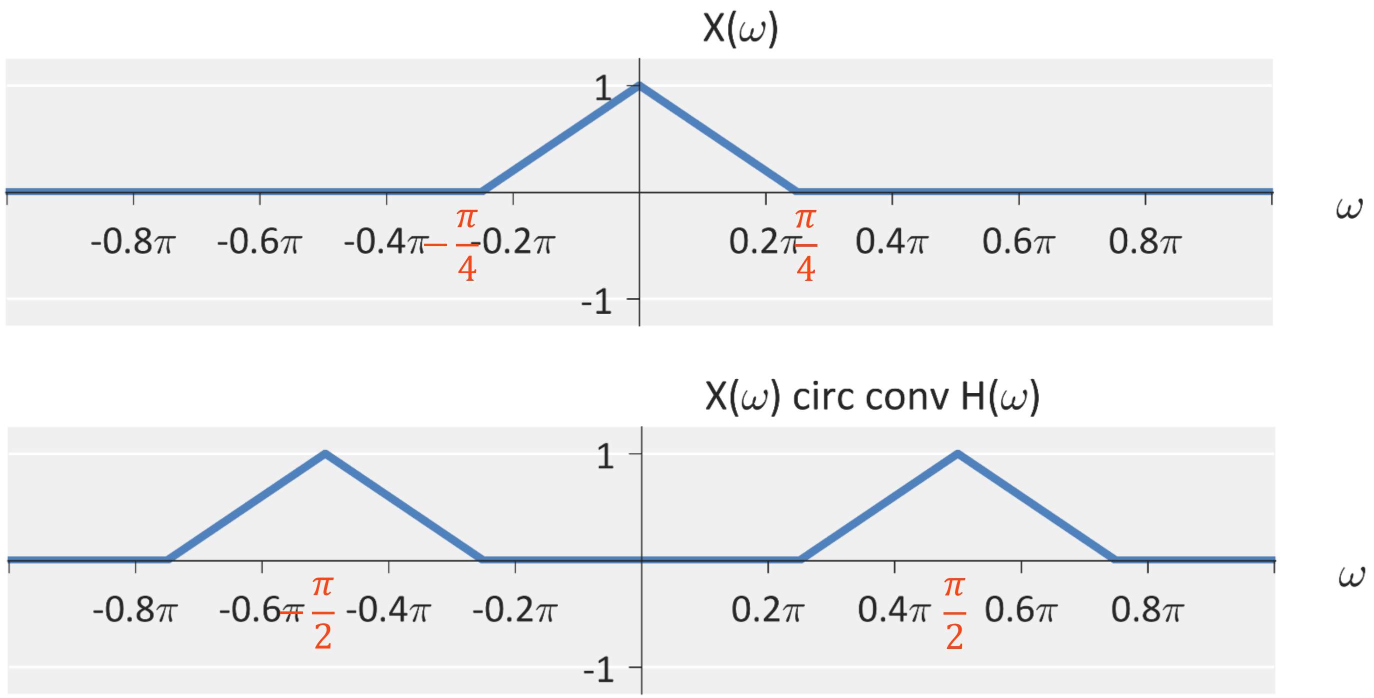

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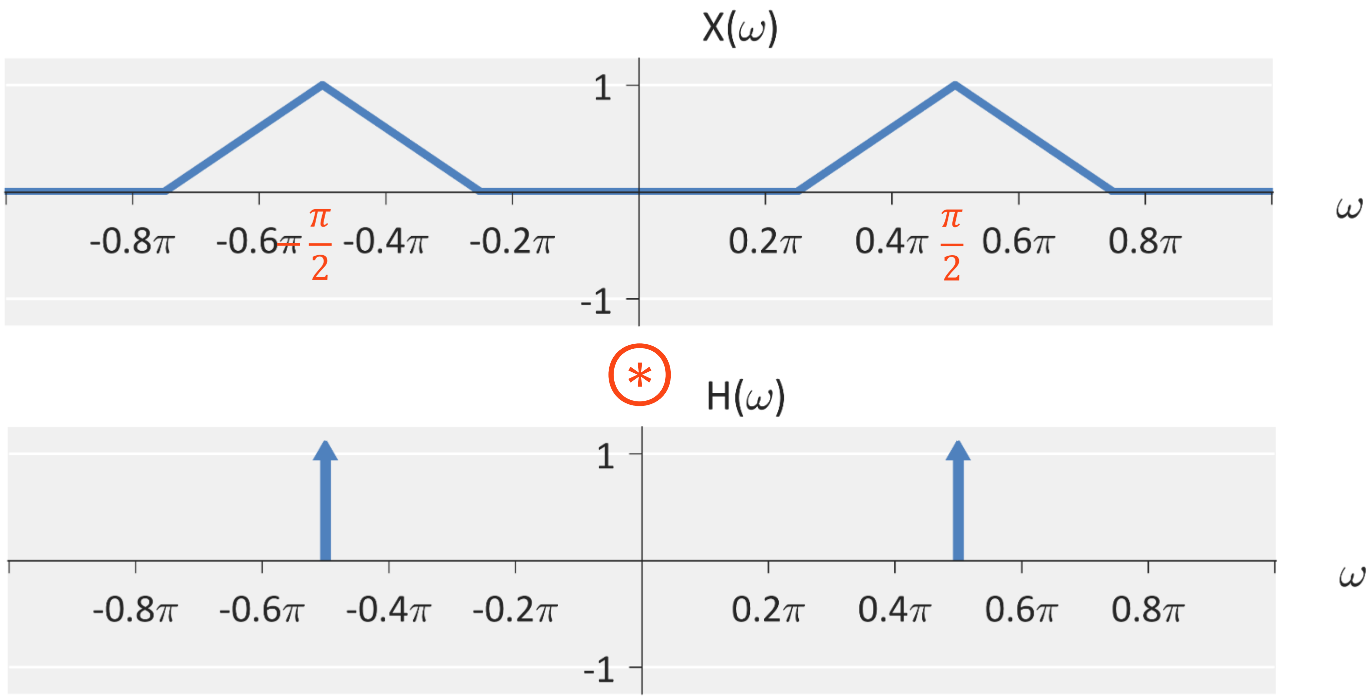

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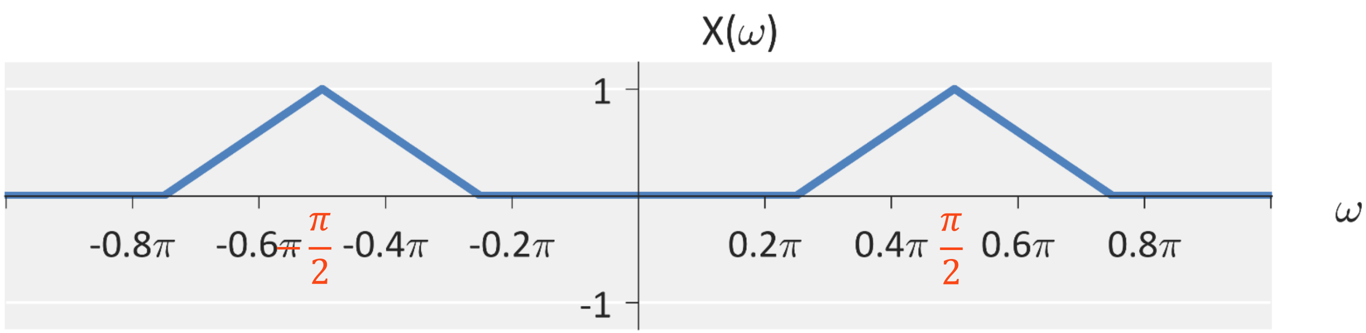

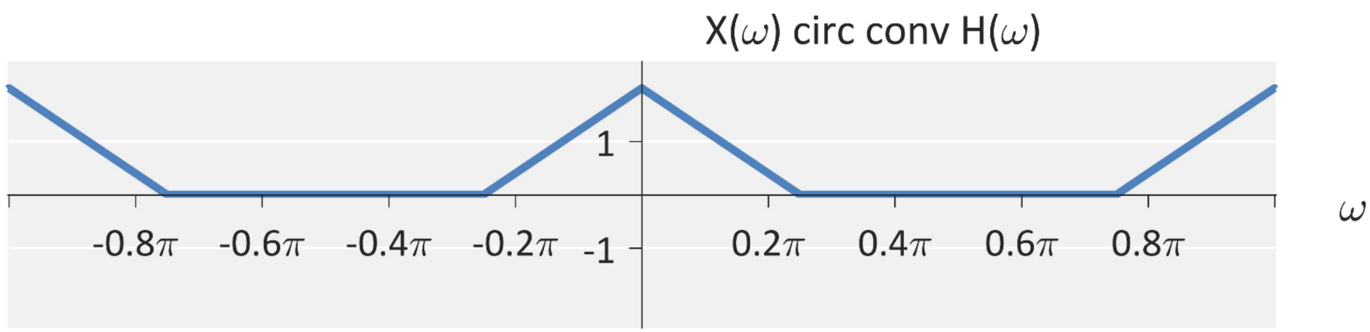

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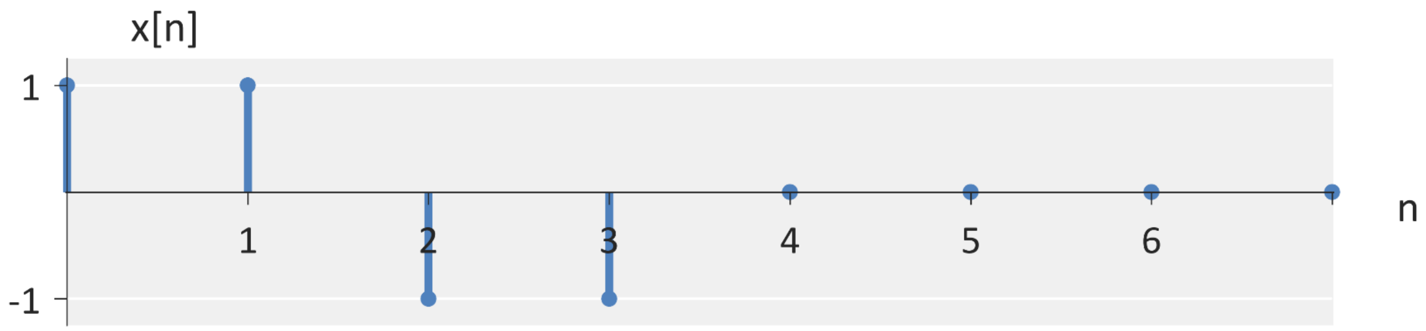

$$\circledast$$

$$h[n] = \delta[n-6]$$

Assume these are *finite* length-8 signals

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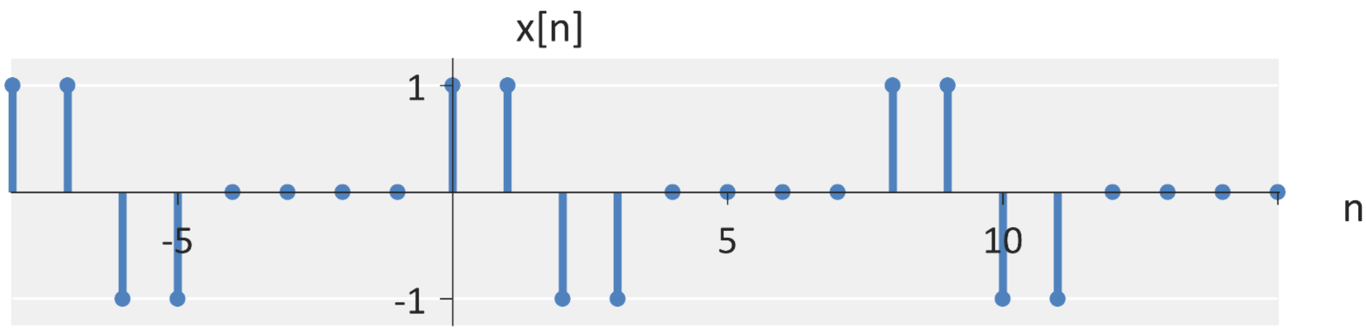

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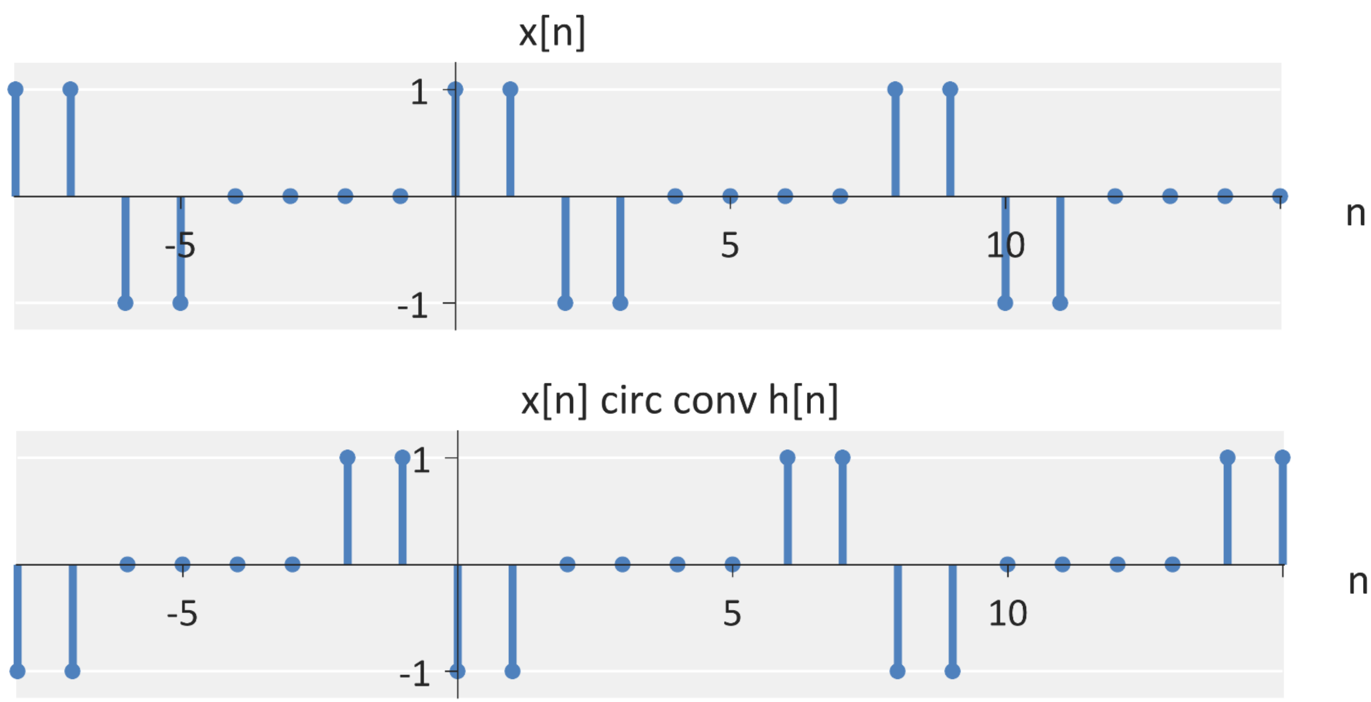

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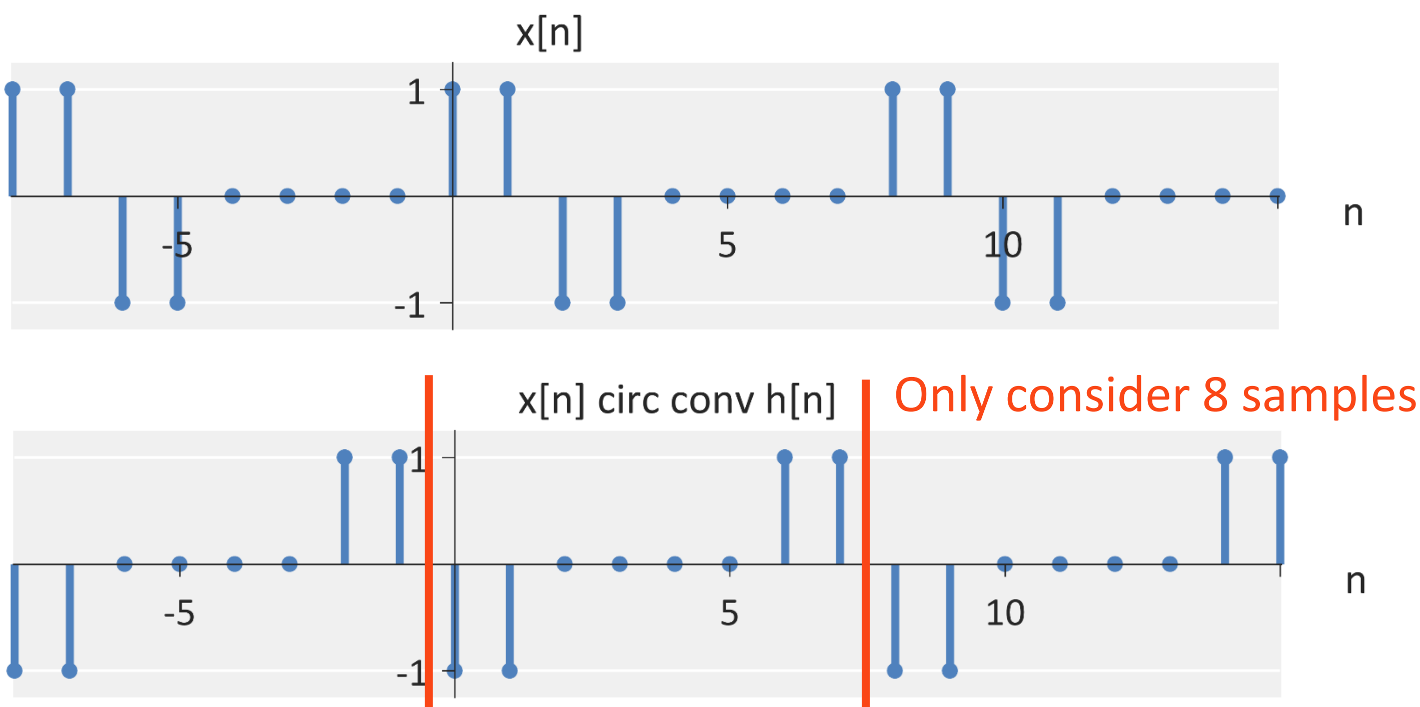

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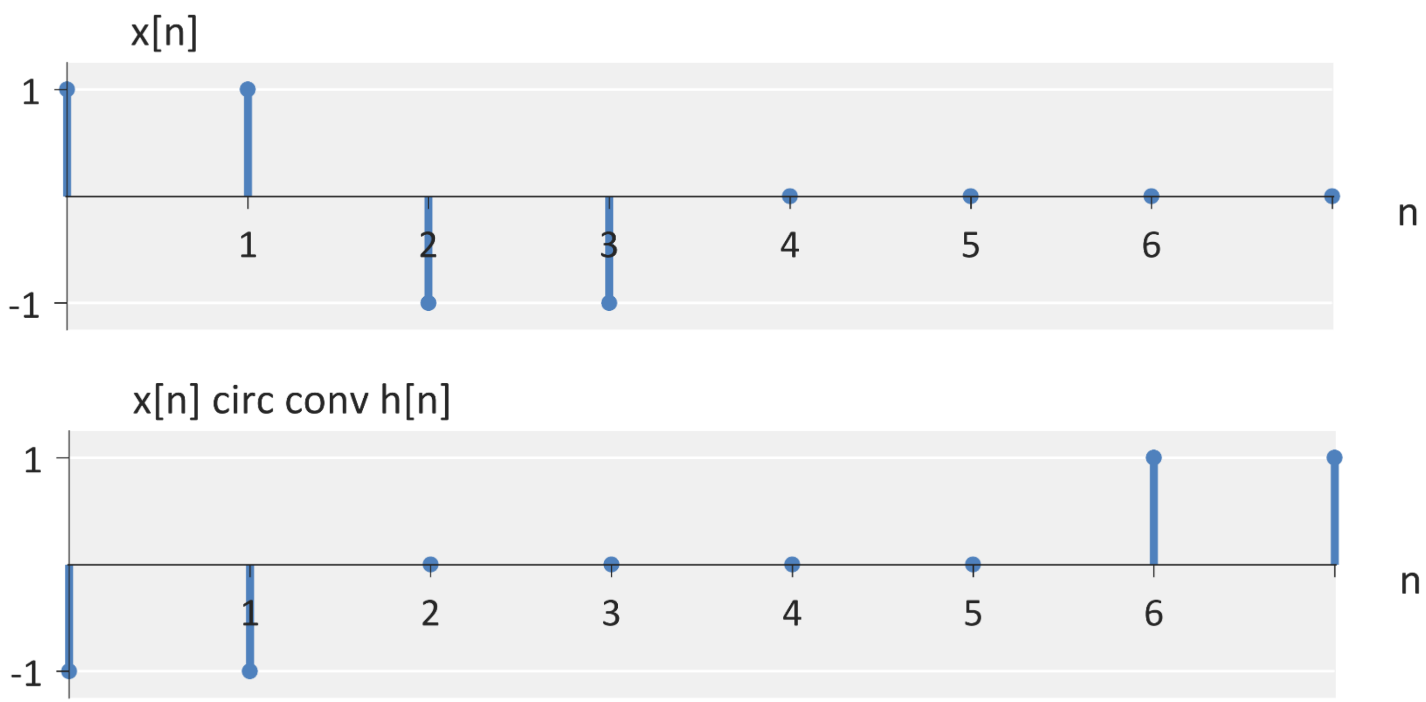

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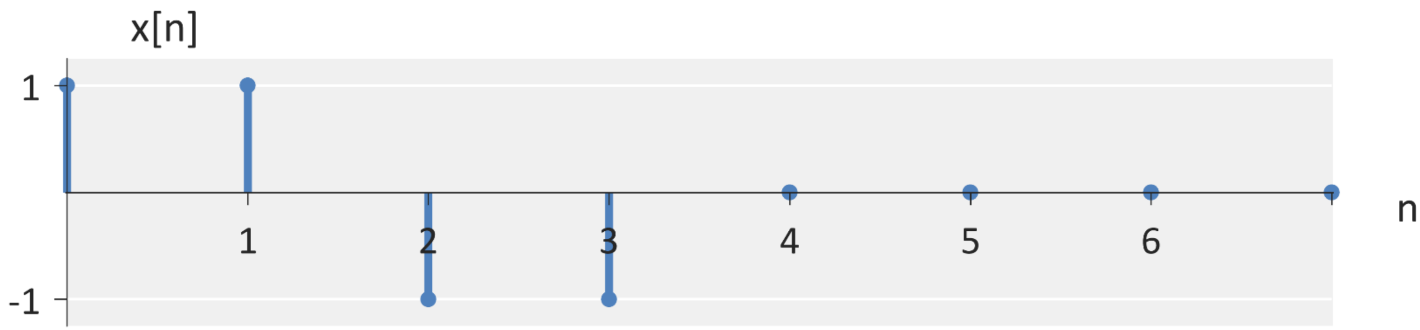

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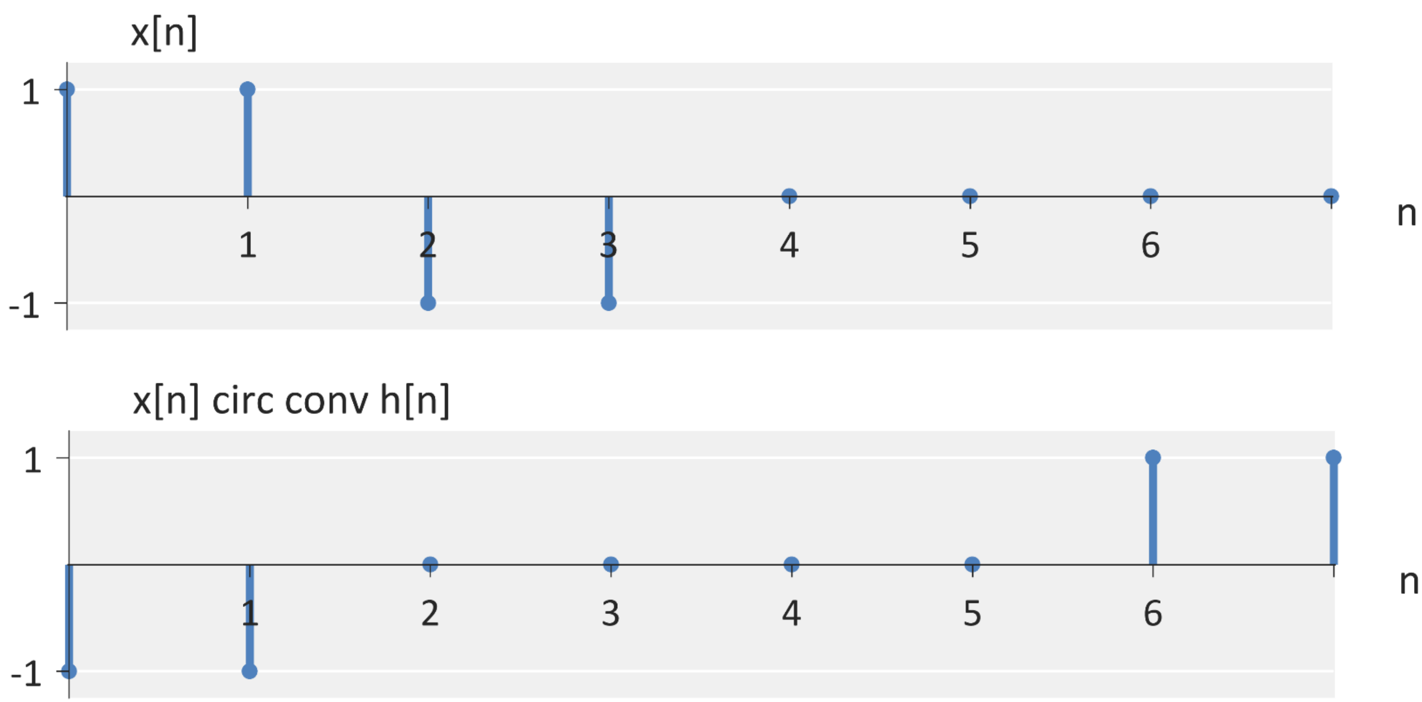

Assume these are *finite* length-8 signals

Circular Convolution

- **Example: Compute the circular convolution**
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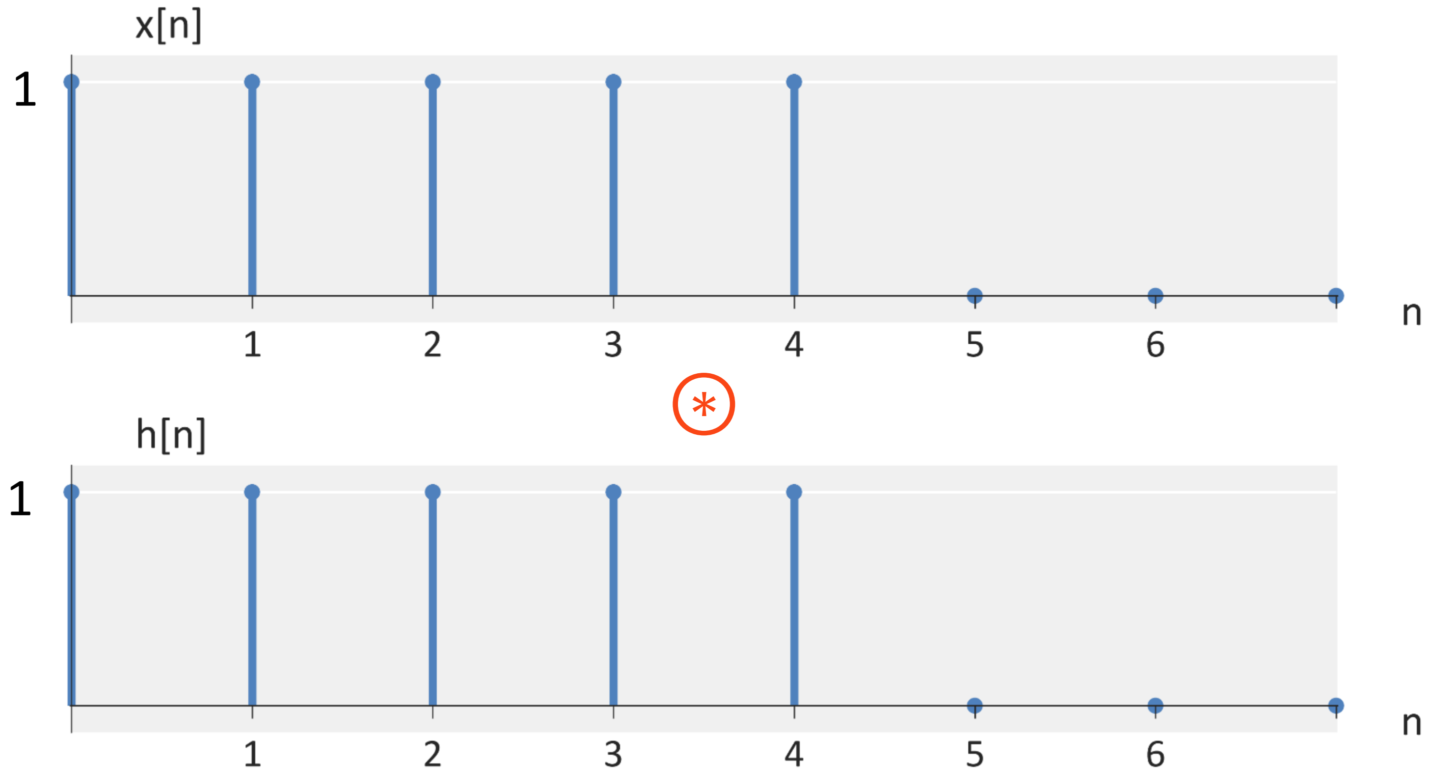

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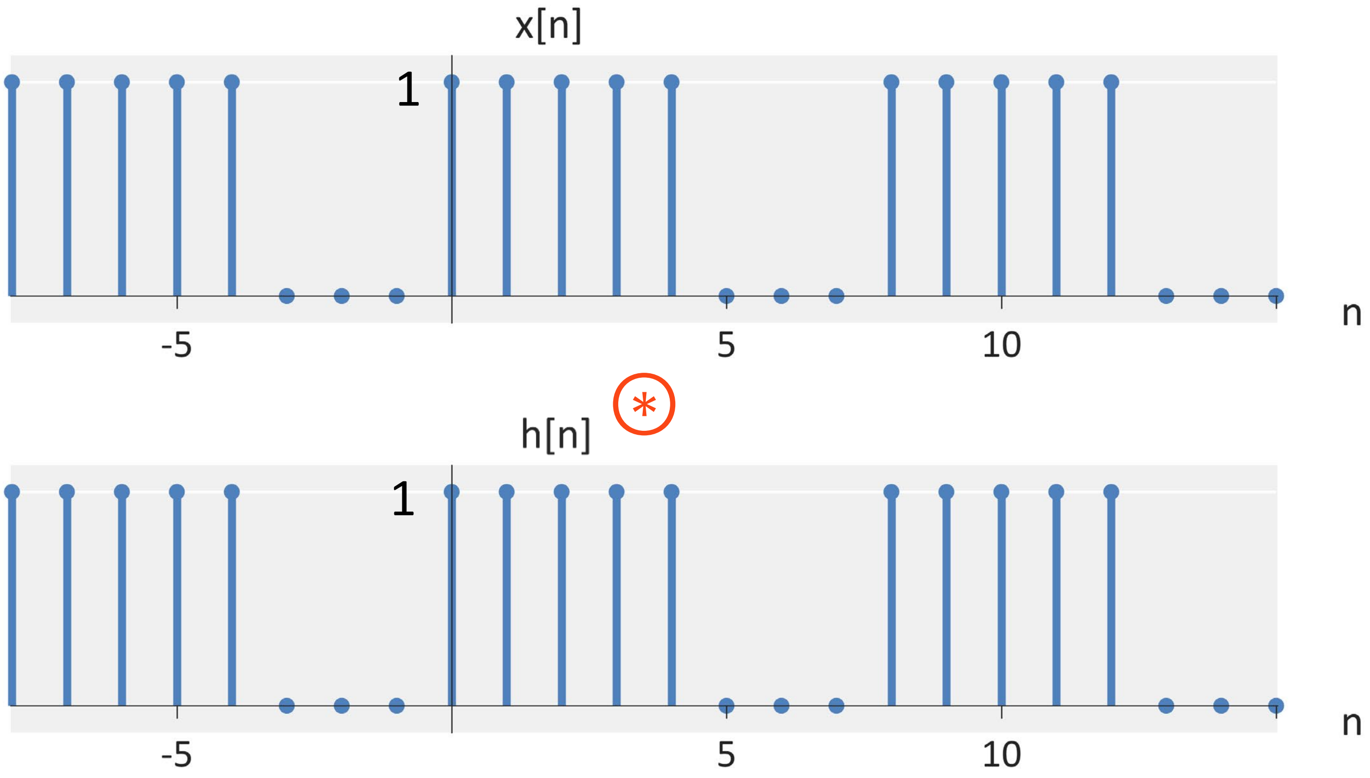

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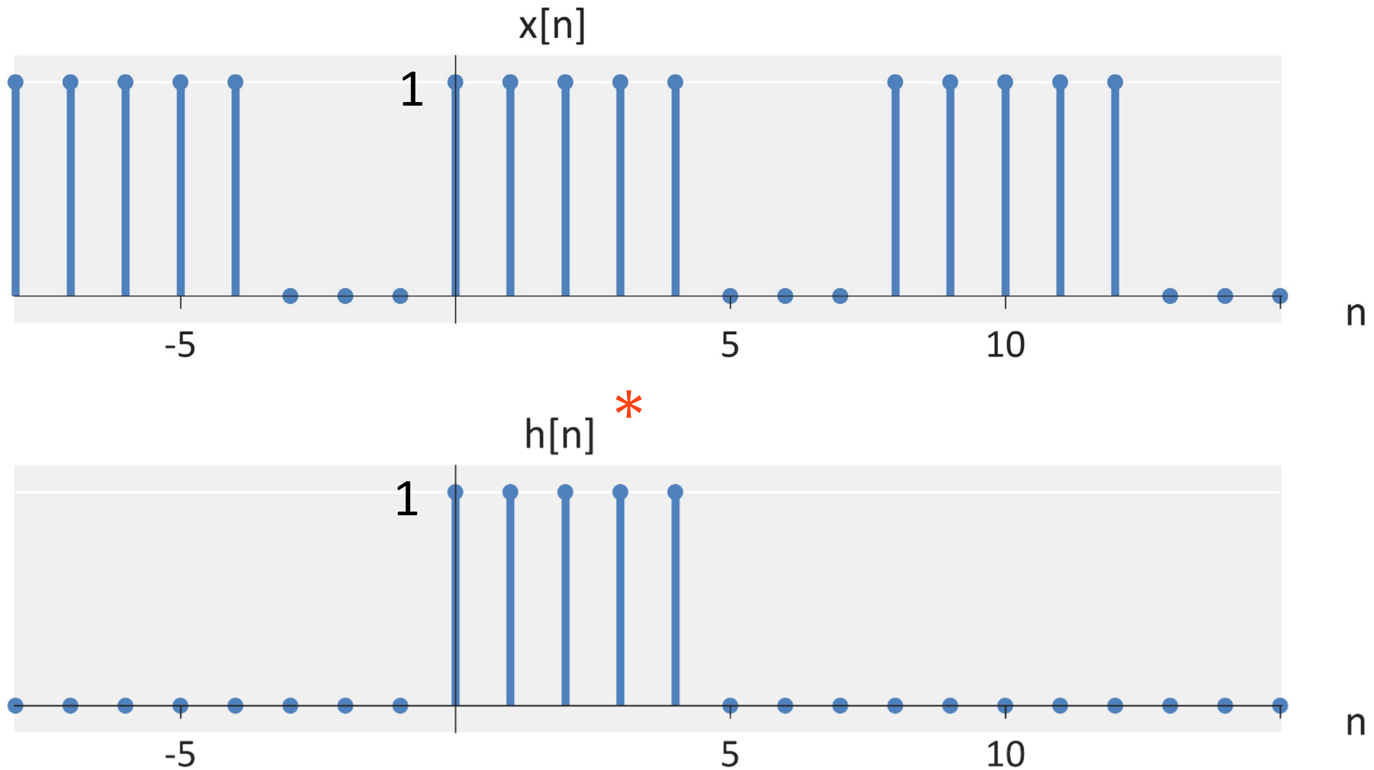

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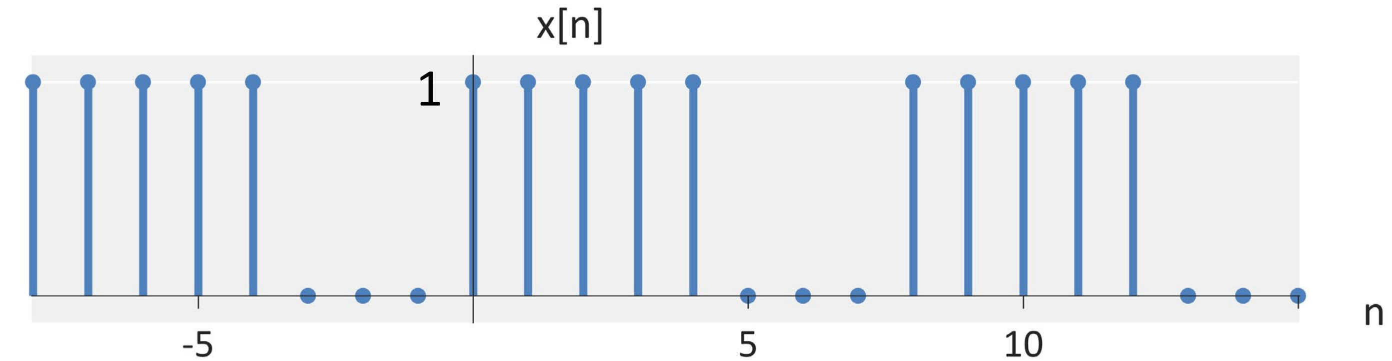

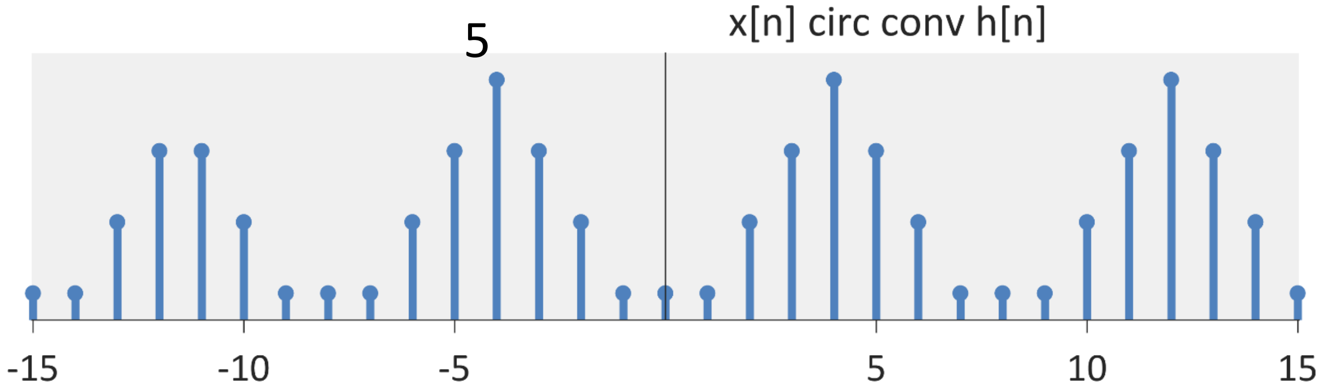

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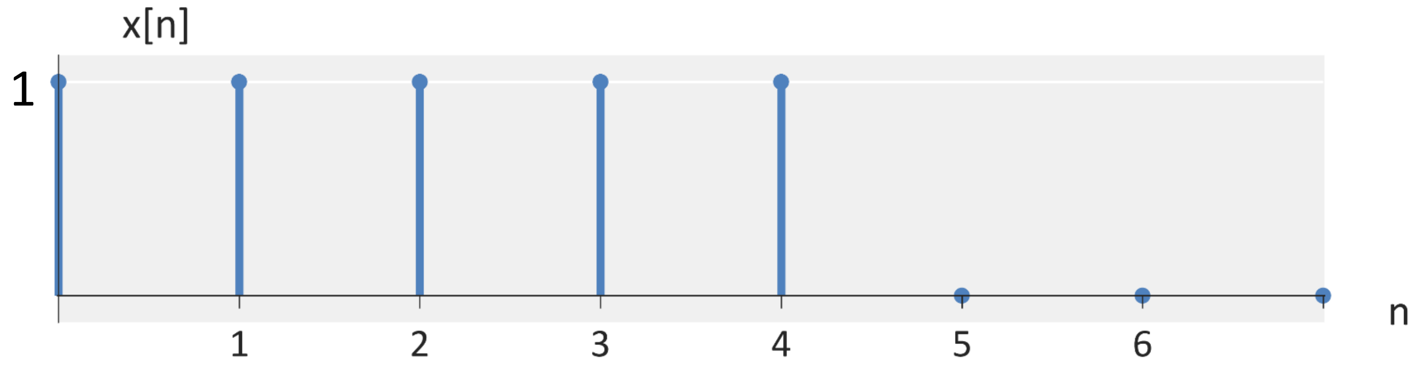

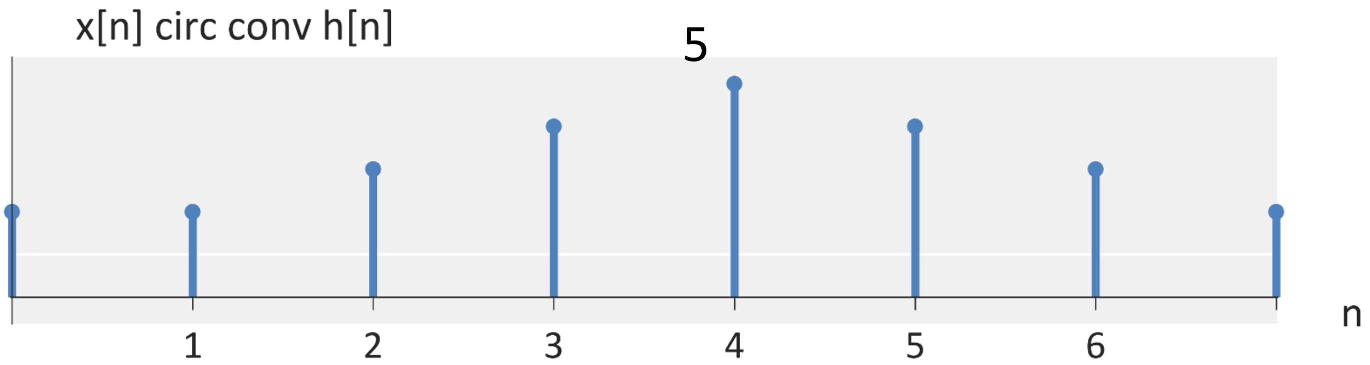

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- **The Z-Transform, the DTFT, and the DFT: The Relationship**
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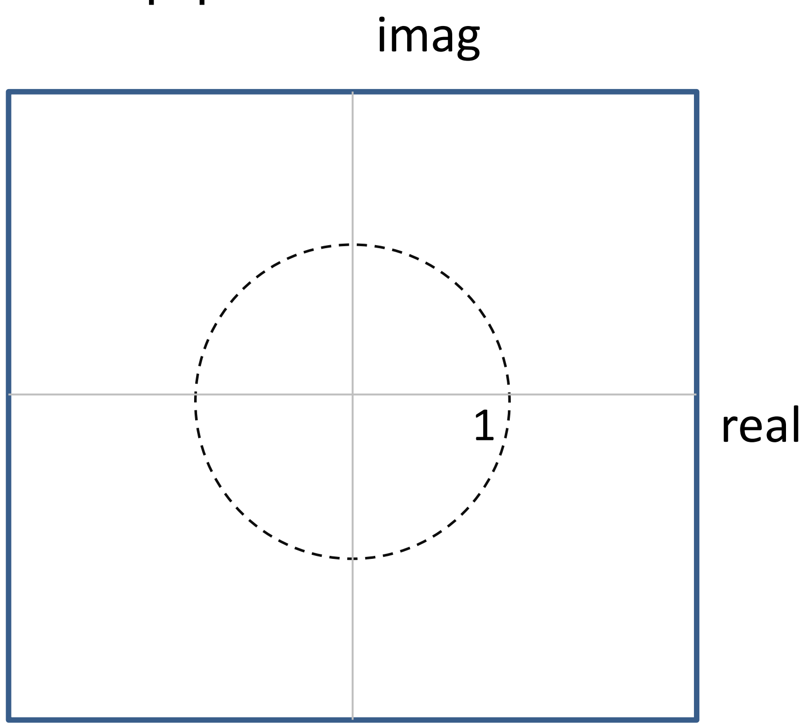
The Fast Fourier Transform

Z-Transform

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

$$z = |r| e^{j\omega}$$



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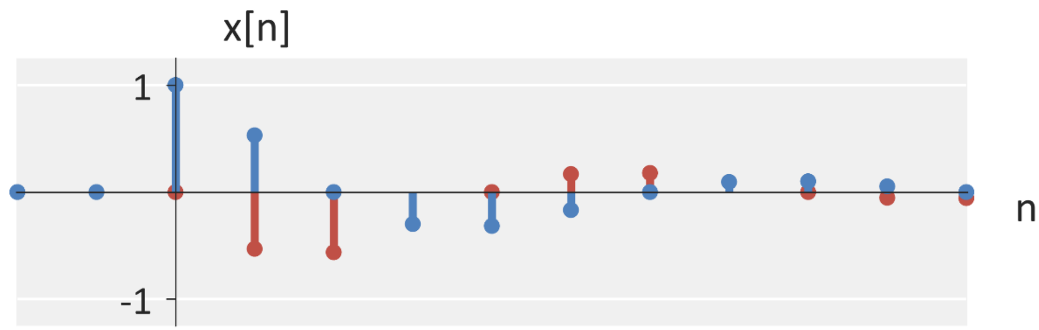

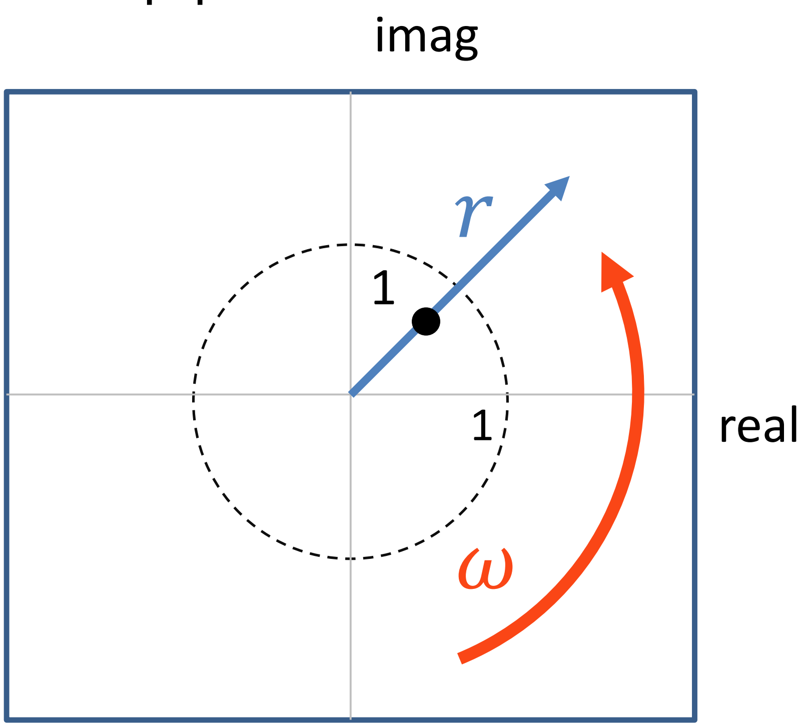

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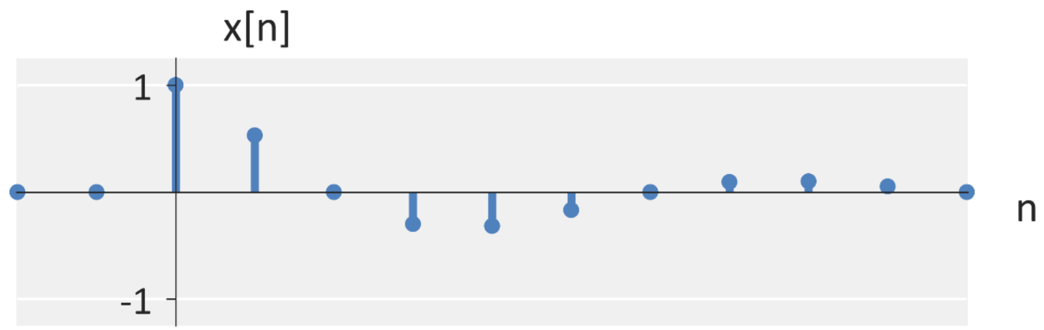

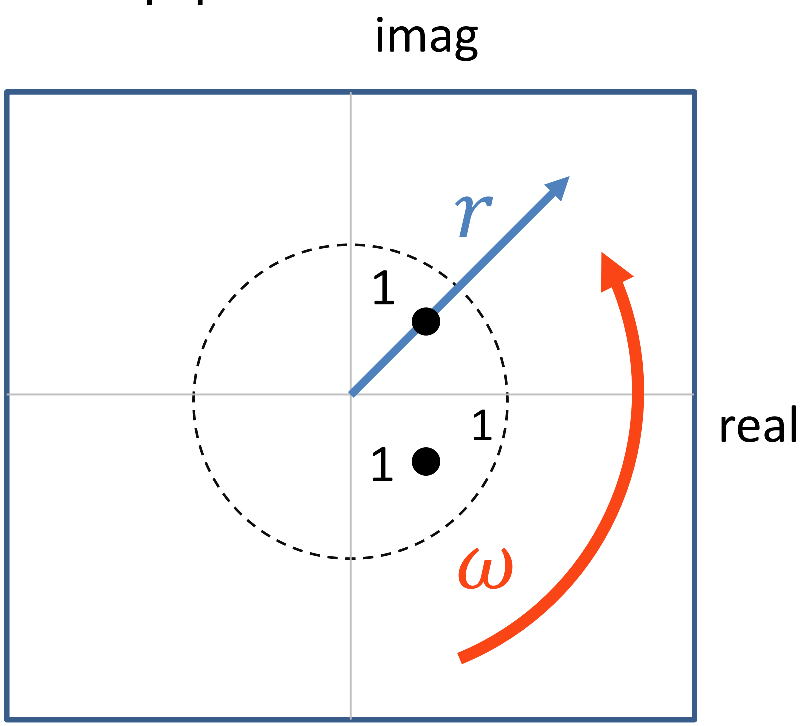

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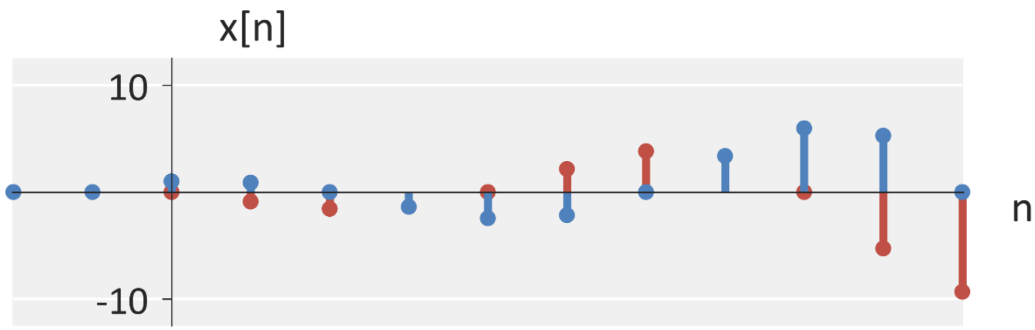

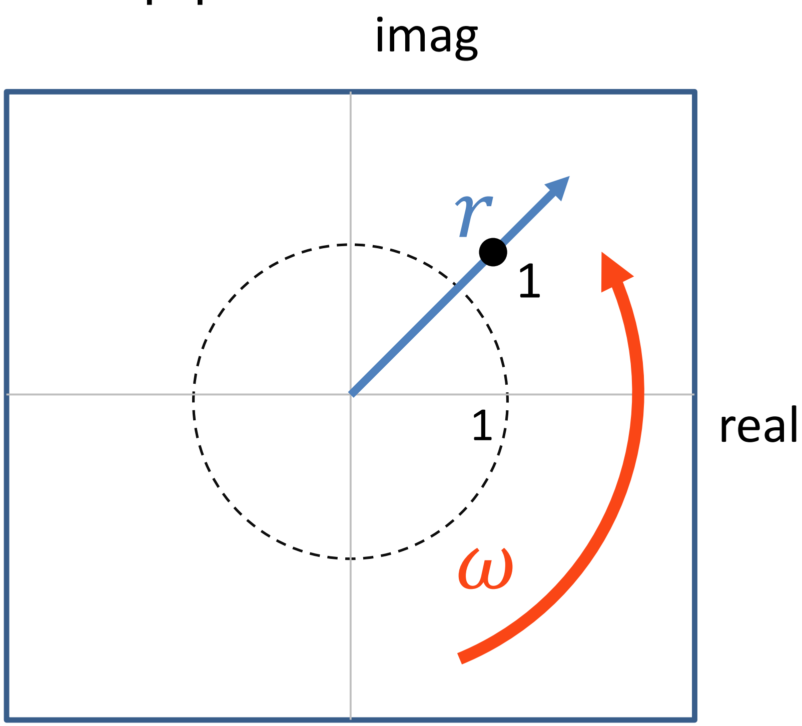

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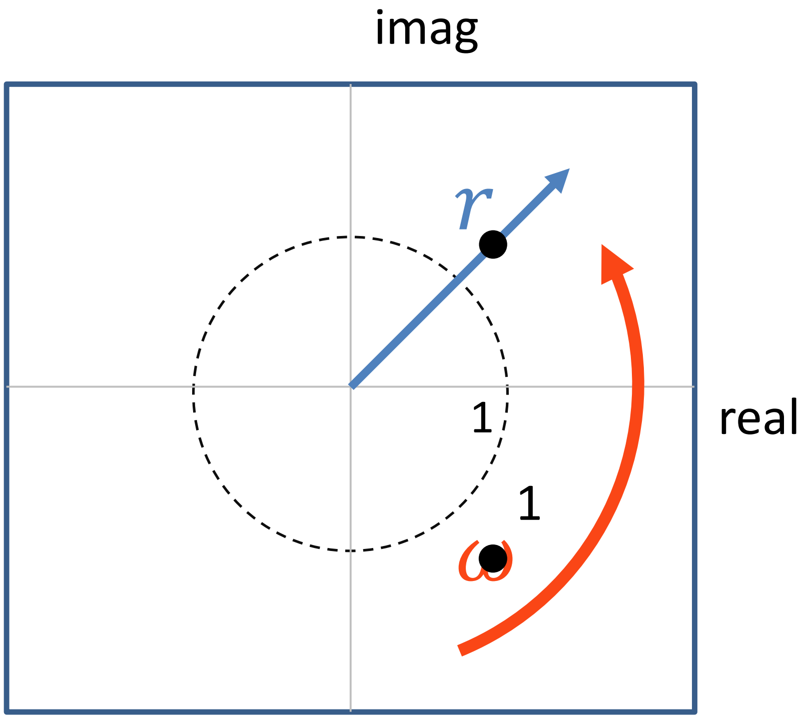

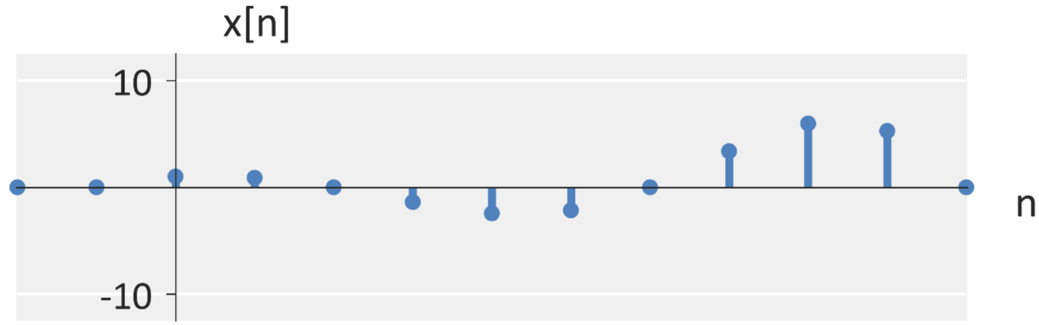

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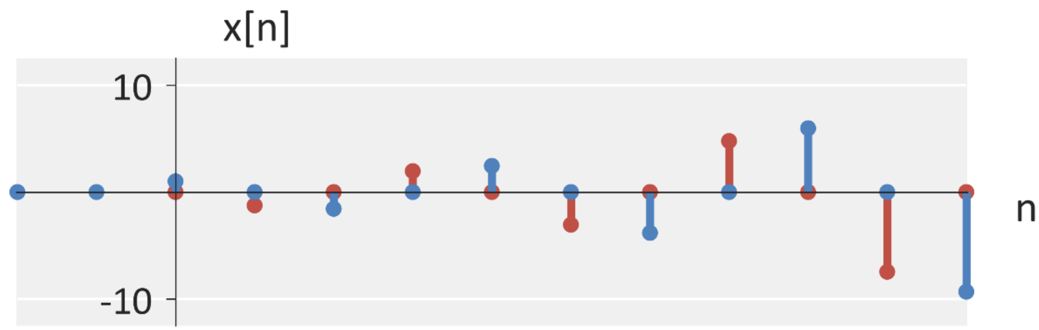

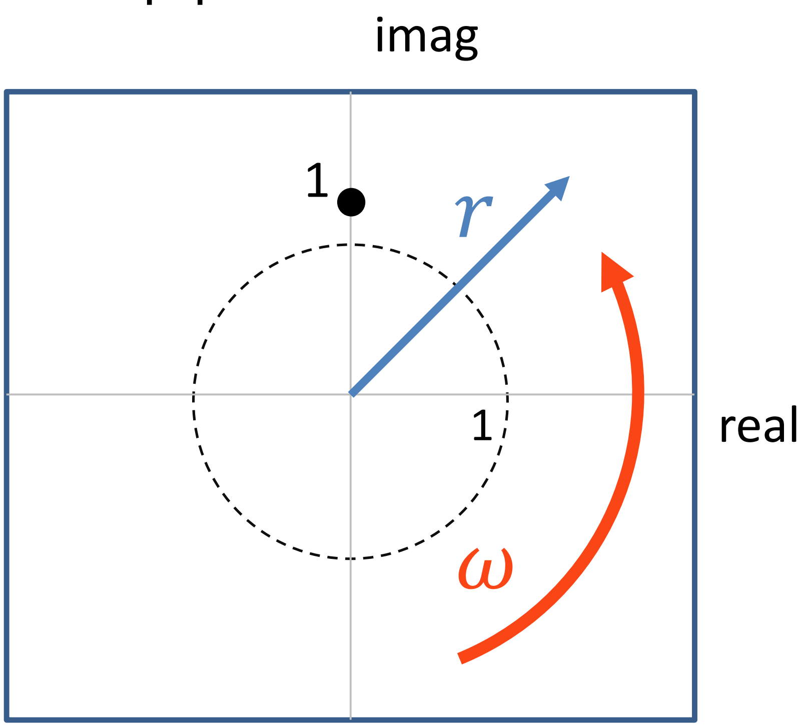

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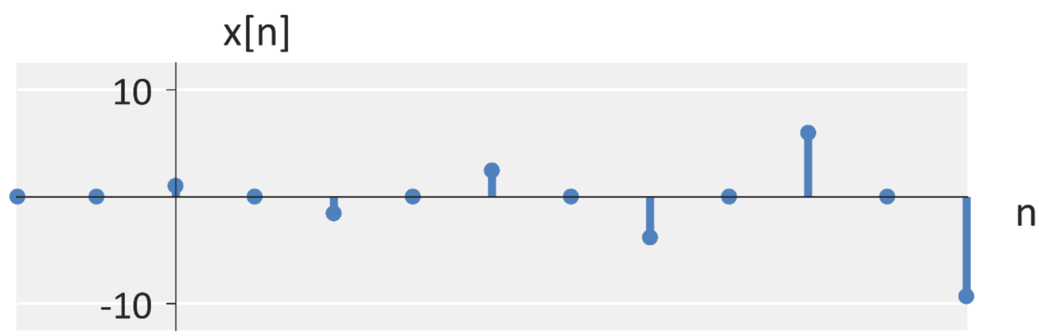

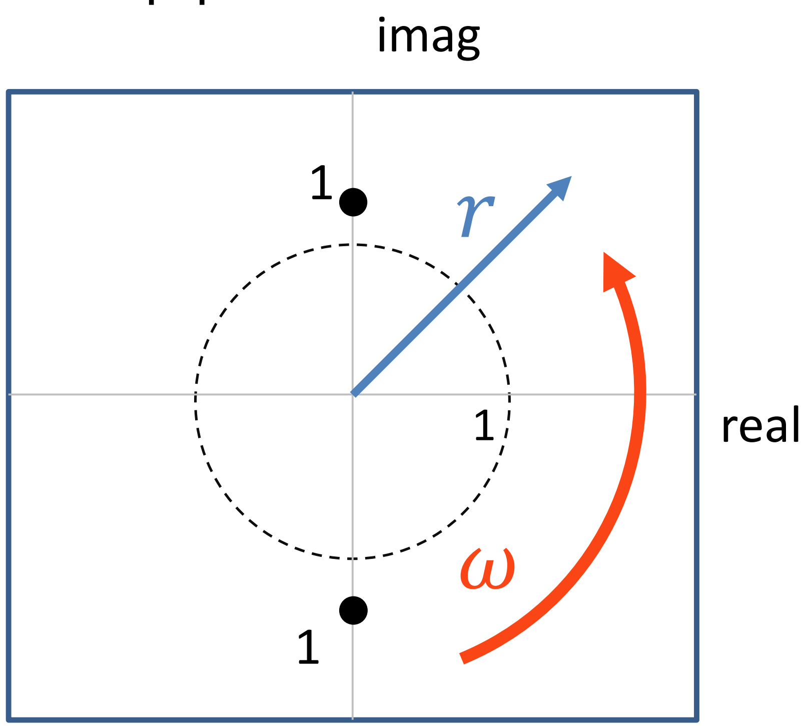

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