Lecture 21: Design of FIR Filters

Foundations of Digital Signal Processing

Outline

- Review Downsampling & Upsampling
- Causality in Filters
- Designing FIR Filters with Windows
- Designing FIR Filters with Frequency Selection
- Designing FIR Filters with Equi-ripples

News

- **I am back**
- **No Homework this Week!**
  - Yay!
- **Coding Problem #5**
  - Due *<u>on Monday</u>*
  - Submit via canvas
- **Exam Grading**
  - This Friday

Engineering with Signals and Systems

- **Murata Cheerleaders**
  - Control, signal processing, and more

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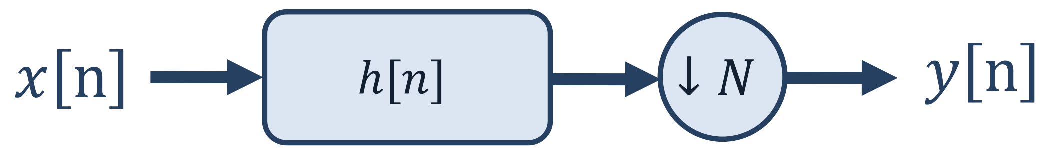
Downsampling

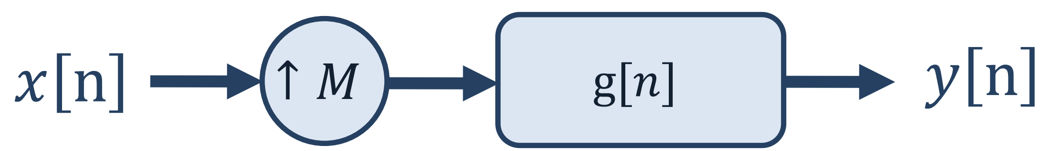
Upsampling and Downsampling

Downsampling with Anti-Aliasing



Low pass anti-aliasing filter
Cutoff: $\Omega = \pi/N$

Upsampling with Interpolation



Low pass interpolation filter
Cutoff: $\Omega = \pi/M$

Upsampling and Downsampling

- **Downsampling**

$$x[\mathrm{n}] \longrightarrow \downarrow N \longrightarrow y[\mathrm{n}]$$

  - Stretch each signal period by N AROUND every 2π
  - Reduce amplitude by N

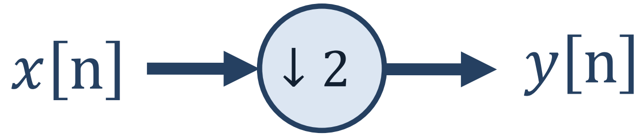
- **Upsampling**

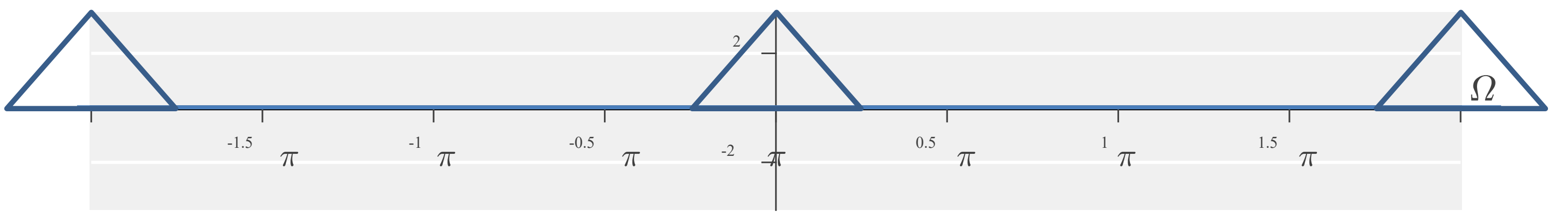
$$x[\mathrm{n}] \longrightarrow \uparrow M \longrightarrow y[\mathrm{n}]$$

  - Shrink EVERYTHING by M
  - Keep amplitude unchanged

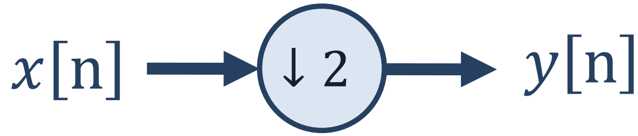
Upsampling and Downsampling

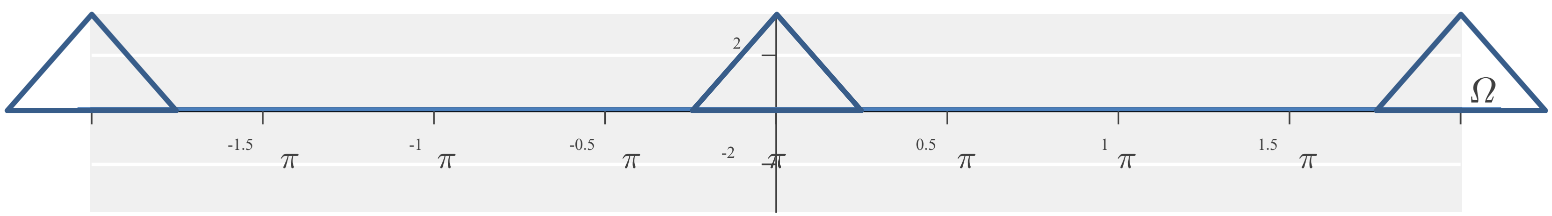
- **Downsampling with Anti-Aliasing**

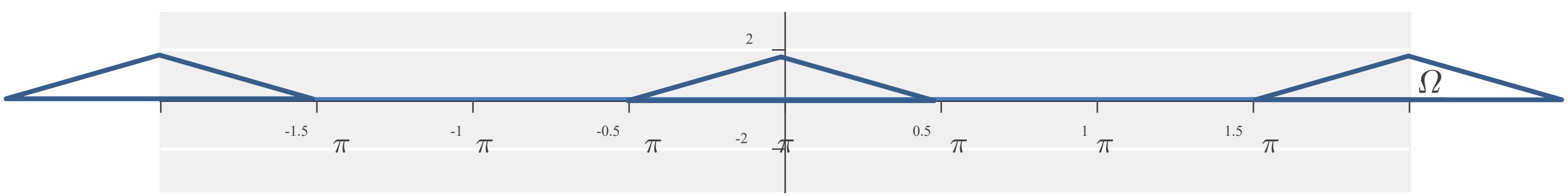




Downsampling with Anti-Aliasing



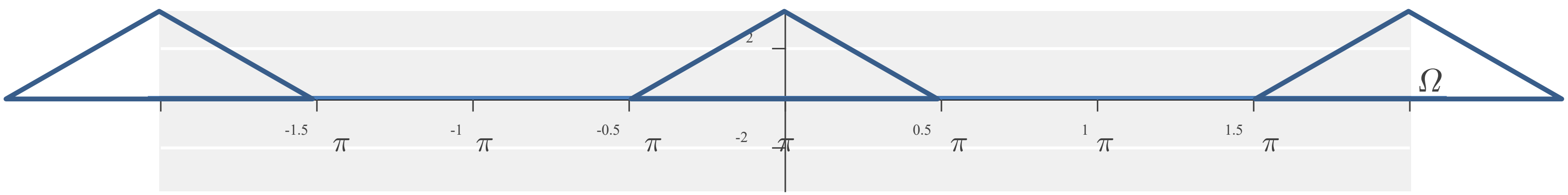


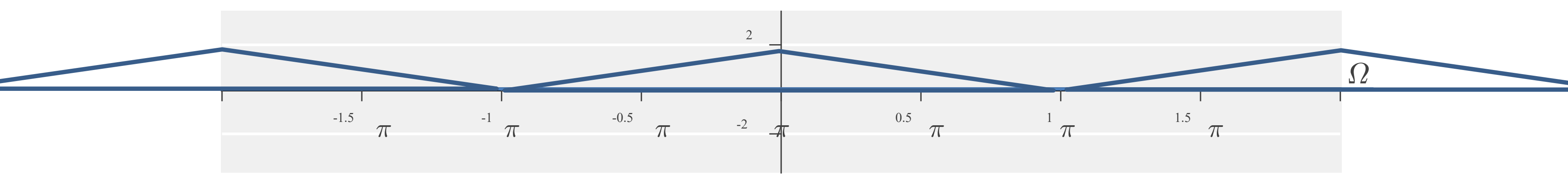


Upsampling and Downsampling

- **Downsampling with Anti-Aliasing**

$$x[n] \rightarrow \boxed{\downarrow 2} \rightarrow y[n]$$





Downsampling with Anti-Aliasing

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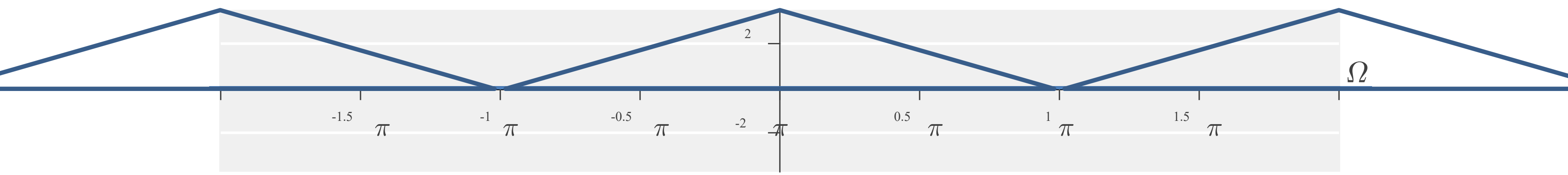
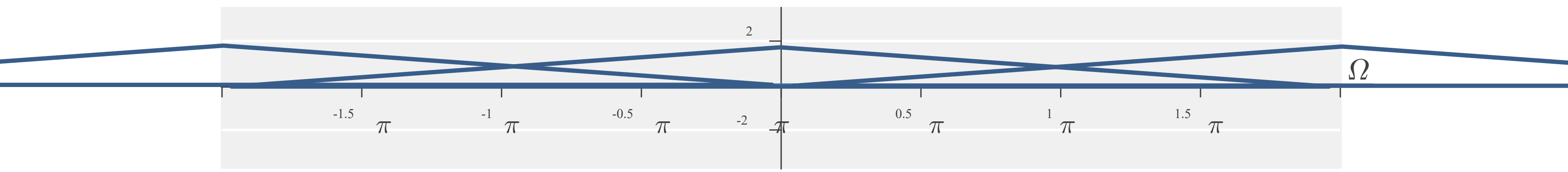
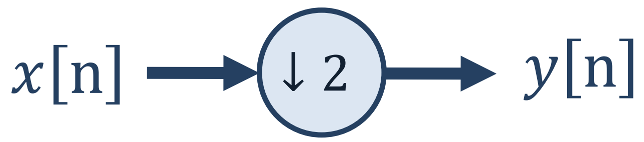
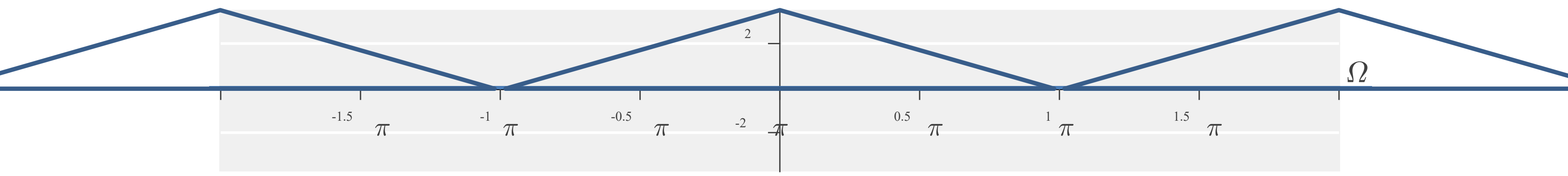
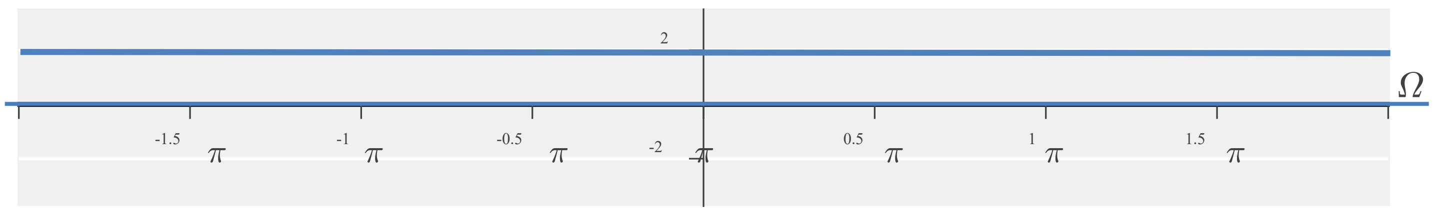




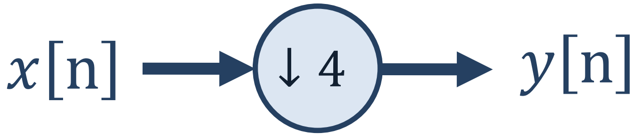
- **Downsampling with Anti-Aliasing**

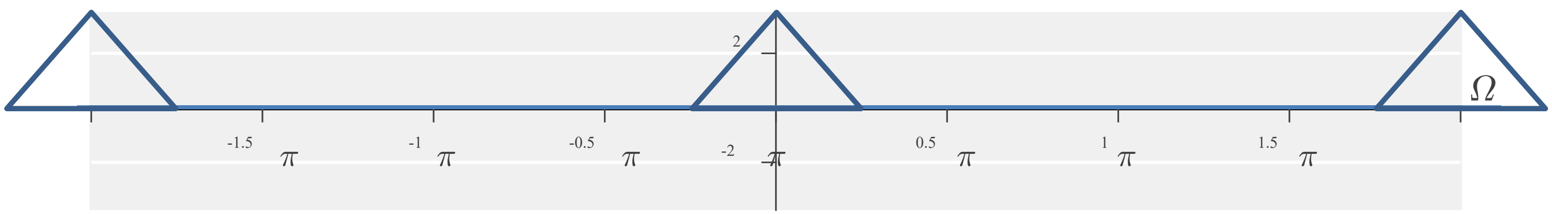


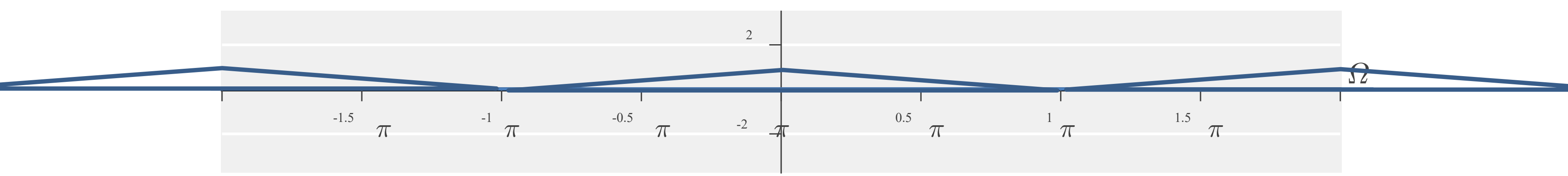




Downsampling with Anti-Aliasing

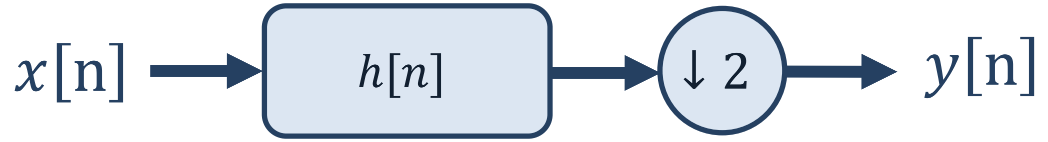




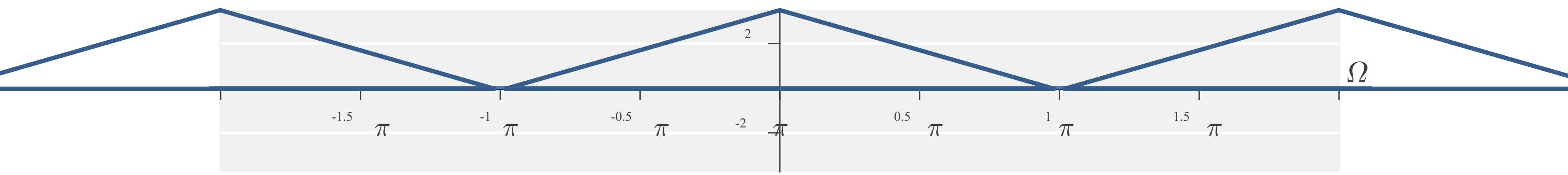


Upsampling and Downsampling

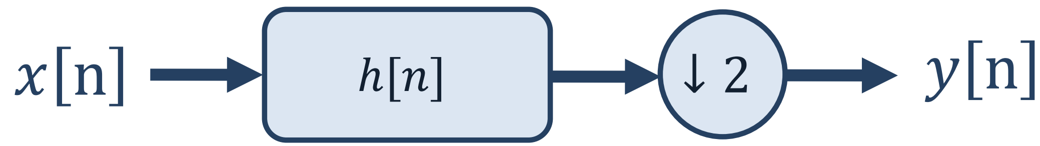
- **Downsampling with Anti-Aliasing**



Low pass anti-aliasing filter
Cutoff: $\Omega = \pi/2$, Gain: 1



Downsampling with Anti-Aliasing



Low pass anti-aliasing filter
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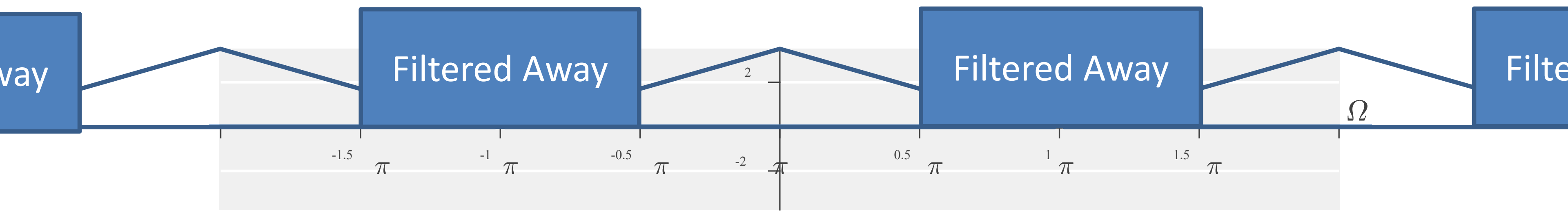
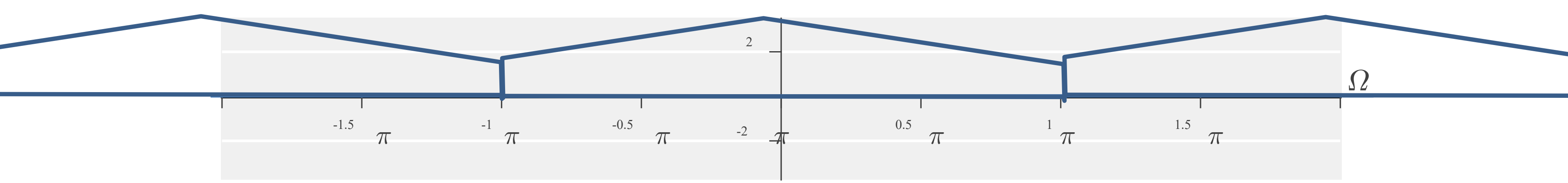
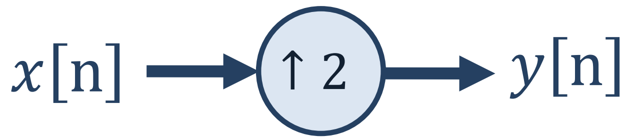
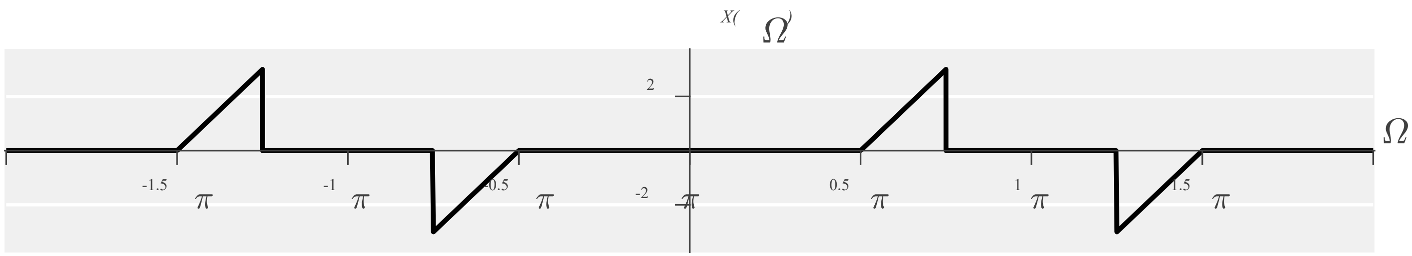


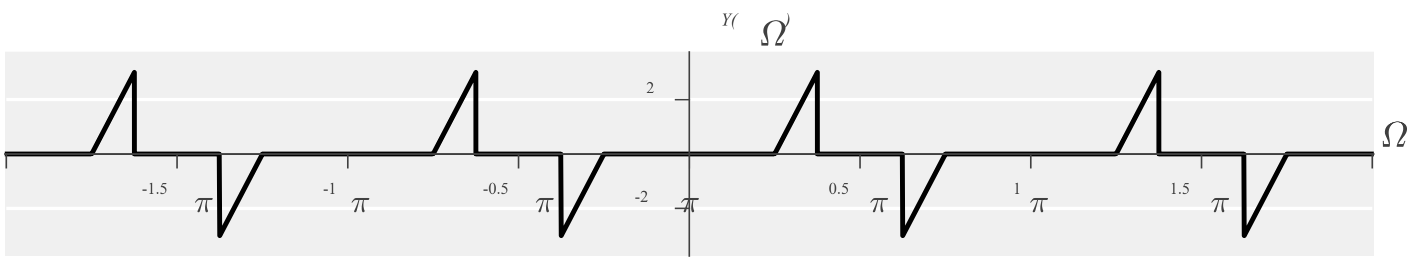


Upsampling

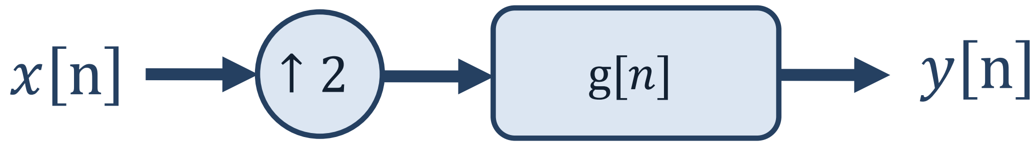
- **Upsampling with Interpolation**



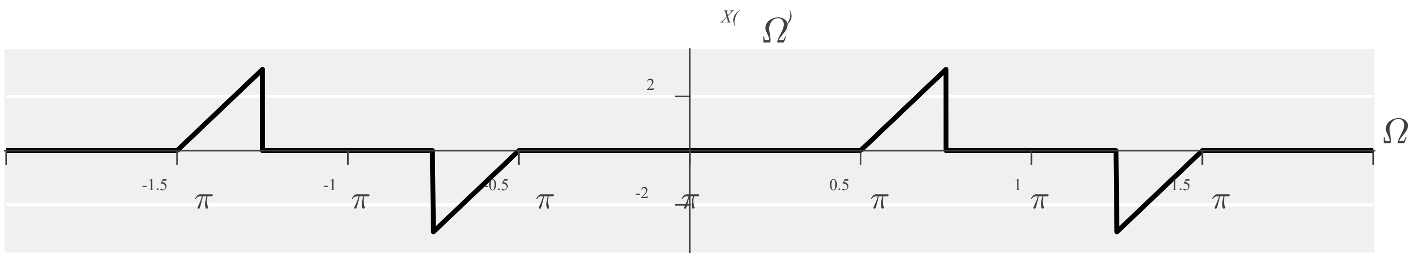




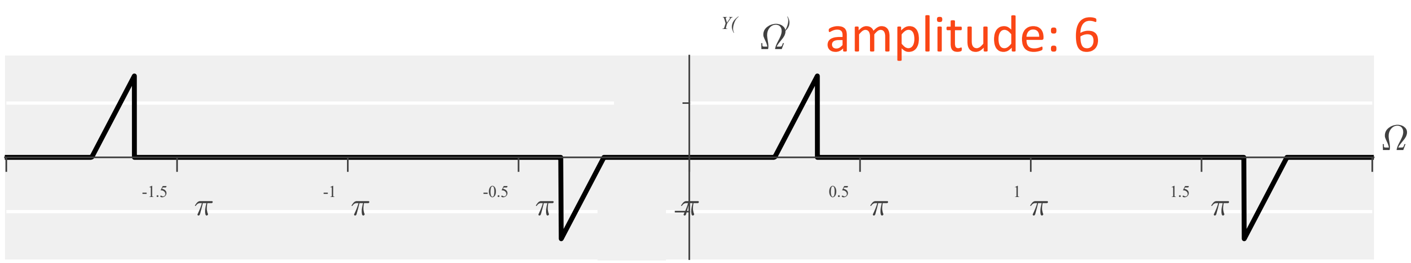
Upsampling with Interpolation



Low pass interpolation filter
Cutoff: $\Omega = \pi/2$, Gain: M=2



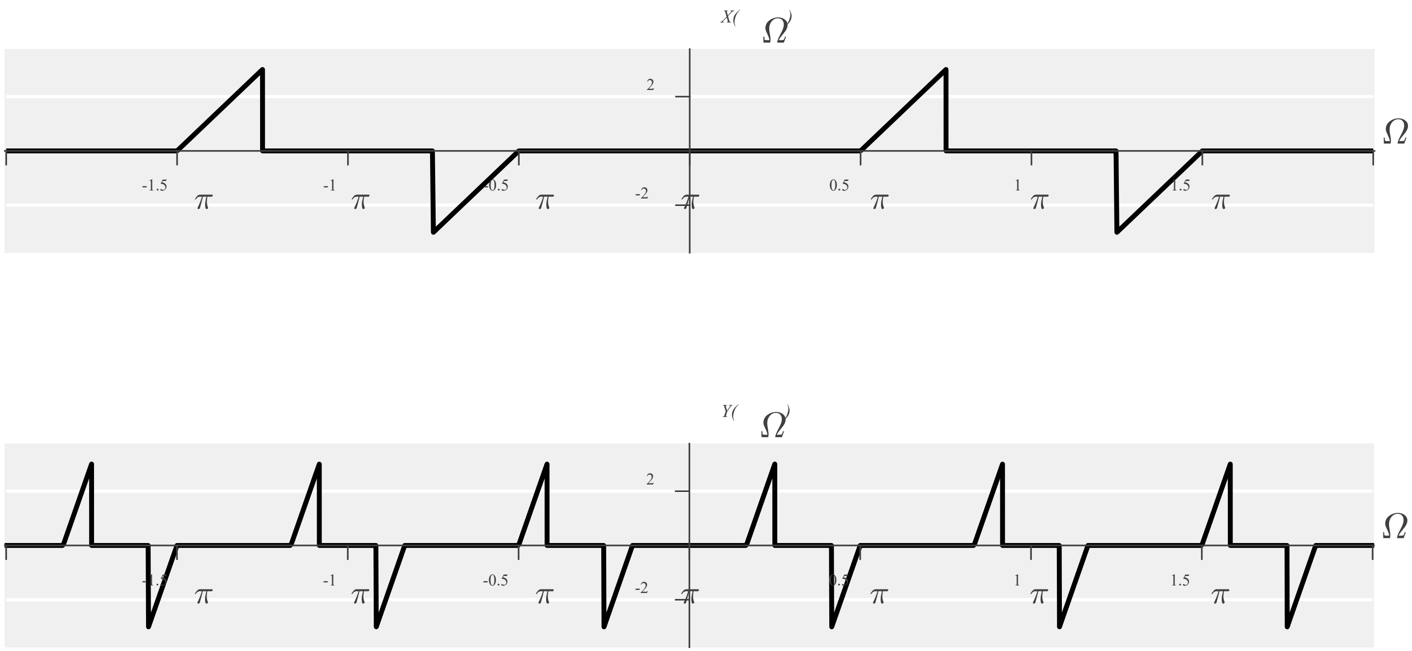
New Maximum amplitude: 6



Upsampling and Downsampling

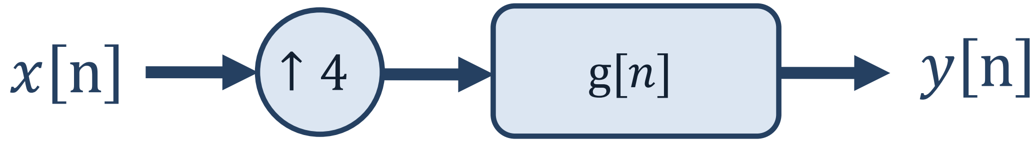
Upsampling with Interpolation

$$x[n] \longrightarrow \boxed{\uparrow 4} \longrightarrow y[n]$$

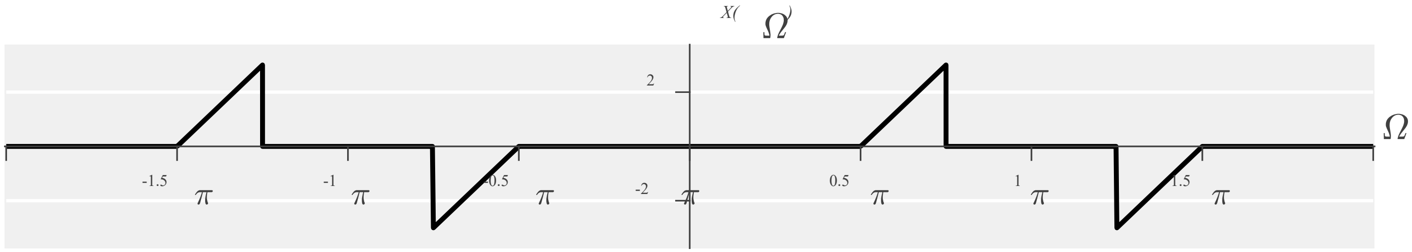


Upsampling and Downsampling

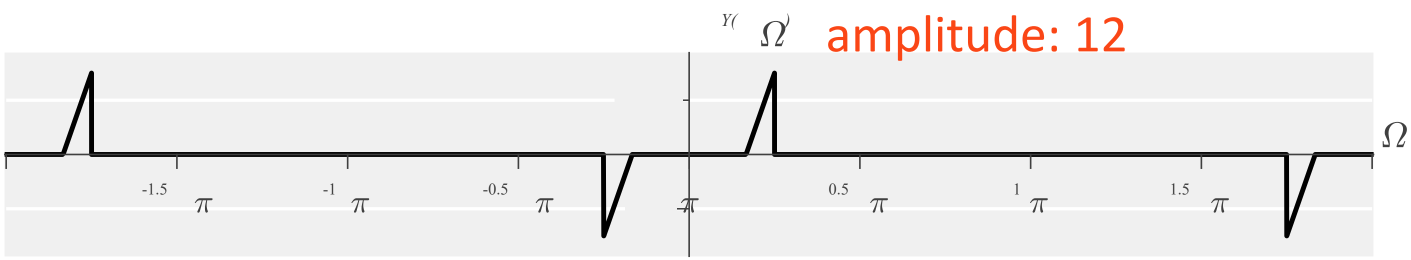
Upsampling with Interpolation



Low pass interpolation filter
Cutoff: $\Omega = \pi/4$, Gain: M=4

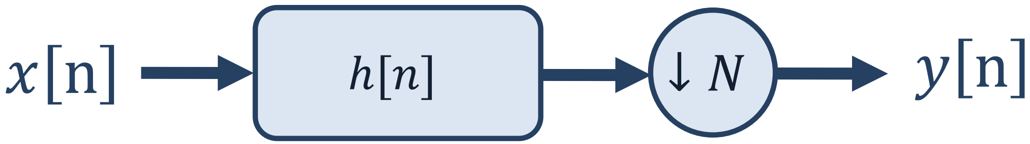


New Maximum amplitude: 12



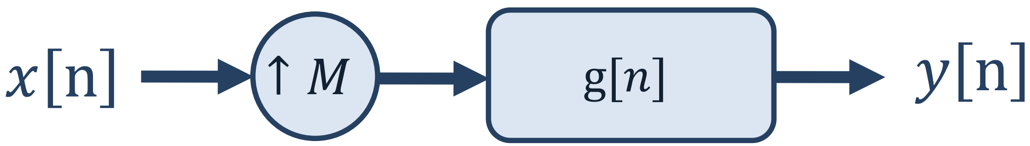
Upsampling and Downsampling

- **Downsampling with Anti-Aliasing**



Low pass anti-aliasing filter
Cutoff: $\Omega = \pi/N$, Gain: 1

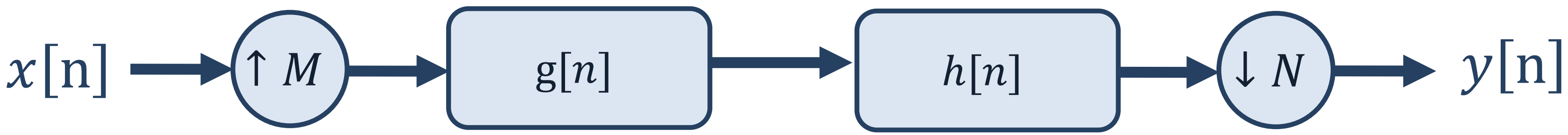
- **Upsampling with Interpolation**



Low pass interpolation filter
Cutoff: $\Omega = \pi/M$, Gain: M

Upsampling and Downsampling

- **Upsampling Followed By Downsampling (Resampling)**



Low pass interpolation filter
Cutoff: $\Omega = \pi/M$, Gain: M

Low pass anti-aliasing filter
Cutoff: $\Omega = \pi/N$, Gain: 1

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Review

- **Question:** **What filter properties considered thus far?**

Review

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Causality

- **Question: What is causality in the frequency domain?**

- $x[n] = h[n]u[n]$

$$X(\omega) = \frac{1}{2\pi}[H(\omega) * U(\omega)]$$
$$= \frac{1}{2\pi}\left[H(\omega) \circledast \left(\frac{1}{1-e^{-j\omega}} + \pi \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k)\right)\right]$$
$$= \frac{1}{2\pi}\left[H(\omega) \circledast \left(\frac{1}{1-e^{-j\omega}}\right)\right] + \frac{1}{2}H(\omega)$$

Causality

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$$= \frac{1}{2\pi}\left[H(\omega) \circledast \left(\frac{1}{1-e^{-j\omega}}\right)\right] + \frac{1}{2}H(\omega)$$

$$\frac{1}{e^{-j\omega/2}[e^{+j\omega/2} - e^{-j\omega/2}]} = \frac{\frac{1}{2j}}{e^{-\frac{j\omega}{2}}\left(\frac{1}{2j}[e^{+j\omega/2} - e^{-j\omega/2}]\right)}$$

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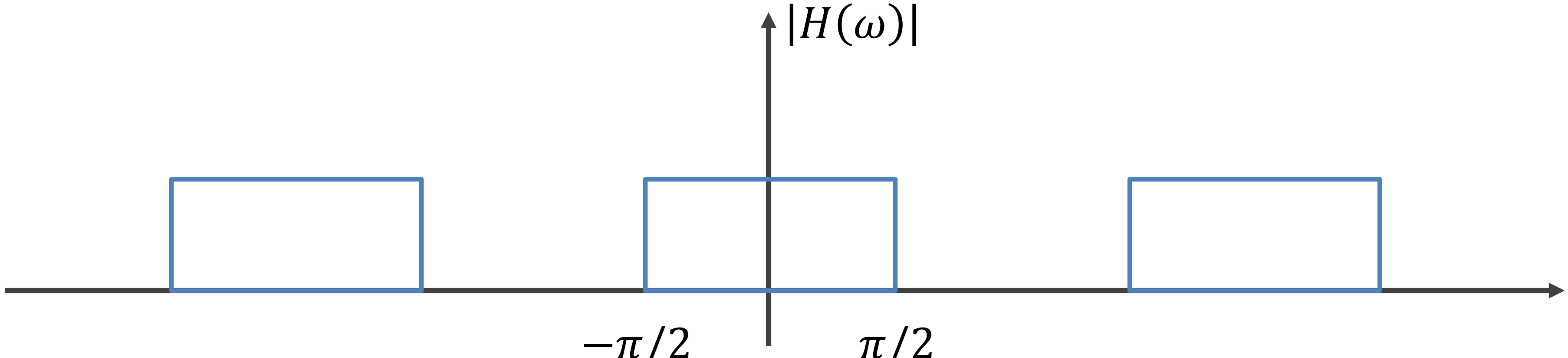
Causality

- **Question: What are the consequences of causality?**
  1. The frequency response $H(\omega)$ cannot be zero, except at a finite set of points in frequency
  2. The magnitude $|H(\omega)|$ cannot be constant in any finite range of frequency and the transition from passband to stopband cannot be infinitely sharp
  3. The real and imaginary parts of $H(\omega)$ are interdependent



Causality

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  2. The magnitude $|H(\omega)|$ cannot be constant in any finite range of frequency and the transition from passband to stopband cannot be infinitely sharp
  3. The real and imaginary parts of $H(\omega)$ are interdependent

- **So, there are no ideal filters**

Causality and Linear Phase

- **Question: Okay… Then what filter properties do we want?**

Causality and Linear Phase

- **Question: Consider a length-M symmetric, causal filter. What condition must be satisfied?**

Causality and Linear Phase

- **Question: Consider a length-M symmetric, causal filter. What condition must be satisfied?**

- $x[n] = \pm x[-n + (N - 1)] = \pm x[N - 1 - n]$
  - **Positive:** Even symmetry
  - **Negative:** Odd symmetry

Causality and Linear Phase

- **Question: Consider a length-M symmetric, causal filter. What is the phase response?**

Causality and Linear Phase

- **Question: Consider a length-M symmetric, causal filter. What is the phase response? Assume M is even.**

- **Even Symmetry**

$$X(z) = a_0 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_1 z^{-(M-2)} + a_0 z^{-(M-1)}$$

- **Odd Symmetry**

$$X(z) = a_0 + a_1 z^{-1} + a_2 z^{-2} + \cdots - a_1 z^{-(M-2)} - a_0 z^{-(M-1)}$$

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$$X(z) = a_0 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_1 z^{-(M-2)} + a_0 z^{-(M-1)}$$
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$$G(\omega) = |X(\omega)| e^{j\Theta(\omega)} \qquad \Theta(\omega) = \begin{cases} 0 & \text{for} \quad G(\omega) > 0 \\ \pi & \text{for} \quad G(\omega) < 0 \end{cases}$$

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Causality and Linear Phase

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$$= z^{-\frac{(M-1)}{2}} \sum_{k=0}^{M/2-1} a_k \left[z^{\frac{(M-1)}{2}-k} + z^{-\left[\frac{(M-1)}{2}-k\right]} \right]$$

Causality and Linear Phase

- **Question: Consider a length-M symmetric, causal filter. What is the phase response? Assume M is even.**

- **Even Symmetry**

$$X(z) = a_0 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_1 z^{-(M-2)} + a_0 z^{-(M-1)}$$

$$= z^{-\frac{(M-1)}{2}} \left[a_0 z^{\frac{(M-1)}{2}} + a_1 z^{\frac{(M-1)}{2}-1} + \cdots + a_1 z^{1-\frac{(M-1)}{2}} + a_0 z^{-\frac{(M-1)}{2}} \right]$$

$$= z^{-\frac{(M-1)}{2}} \sum_{k=0}^{M/2-1} a_k \left[z^{\frac{(M-1)}{2}-k} + z^{-\left[\frac{(M-1)}{2}-k\right]} \right]$$

- **Notice that**

$$X(z) = z^{-(M-1)} X(z^{-1})$$

Causality and Linear Phase

- **Question: Consider a length-M symmetric, causal filter. What is the phase response? Assume M is even.**

- **Even Symmetry**

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- **Pole-zero plot property?**

$$X(z) = z^{-(M-1)} X(z^{-1})$$

Causality and Linear Phase

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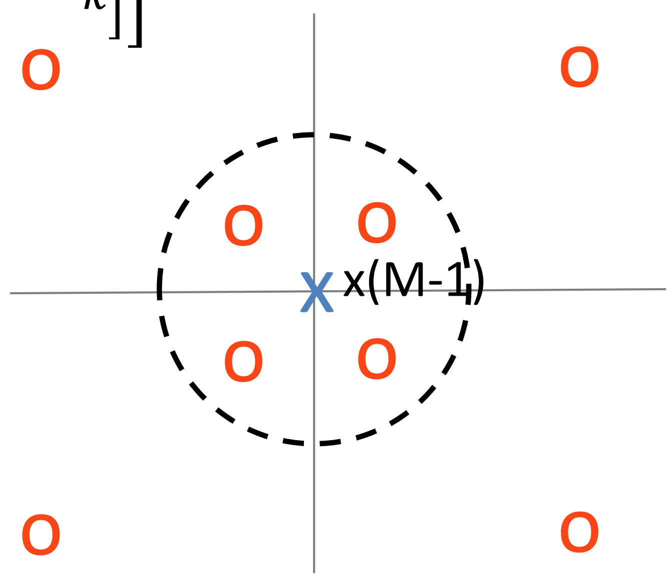
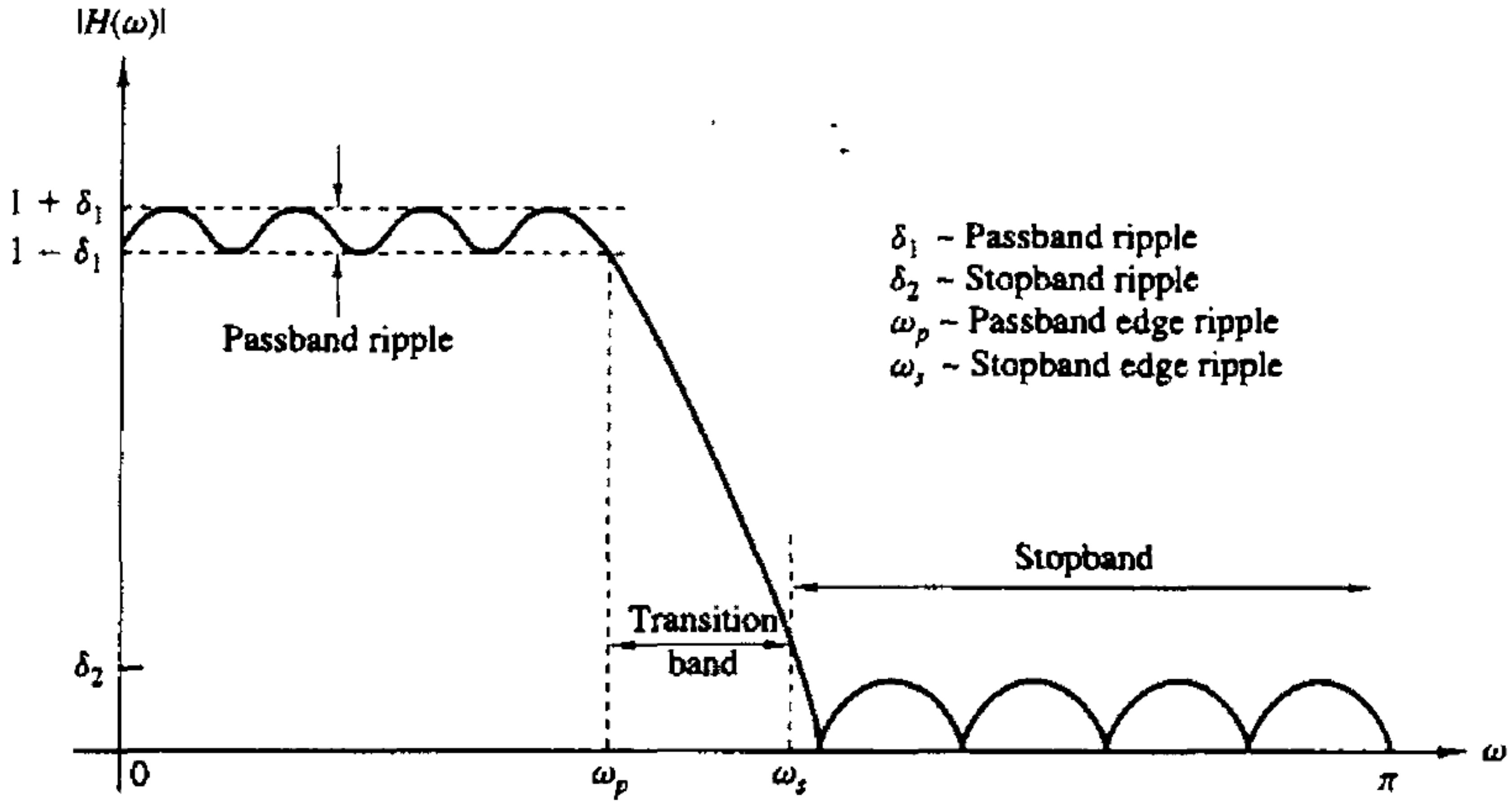


- **Pole-zero plot property?**

$$X(z) = z^{-(M-1)} X(z^{-1})$$

Causality

- **Question:** **How do we describe causal filter magnitude?**



Causality

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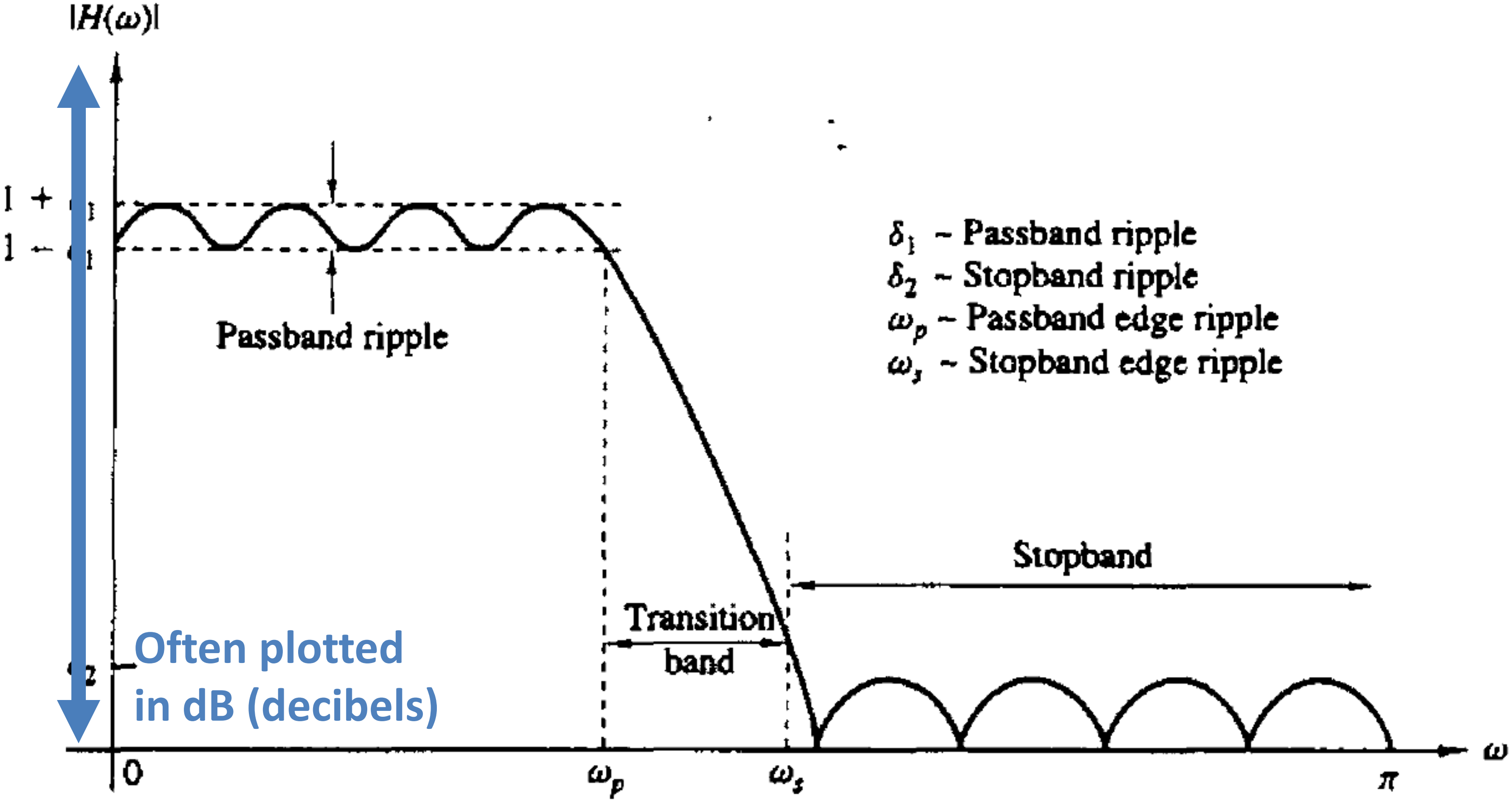

Causality

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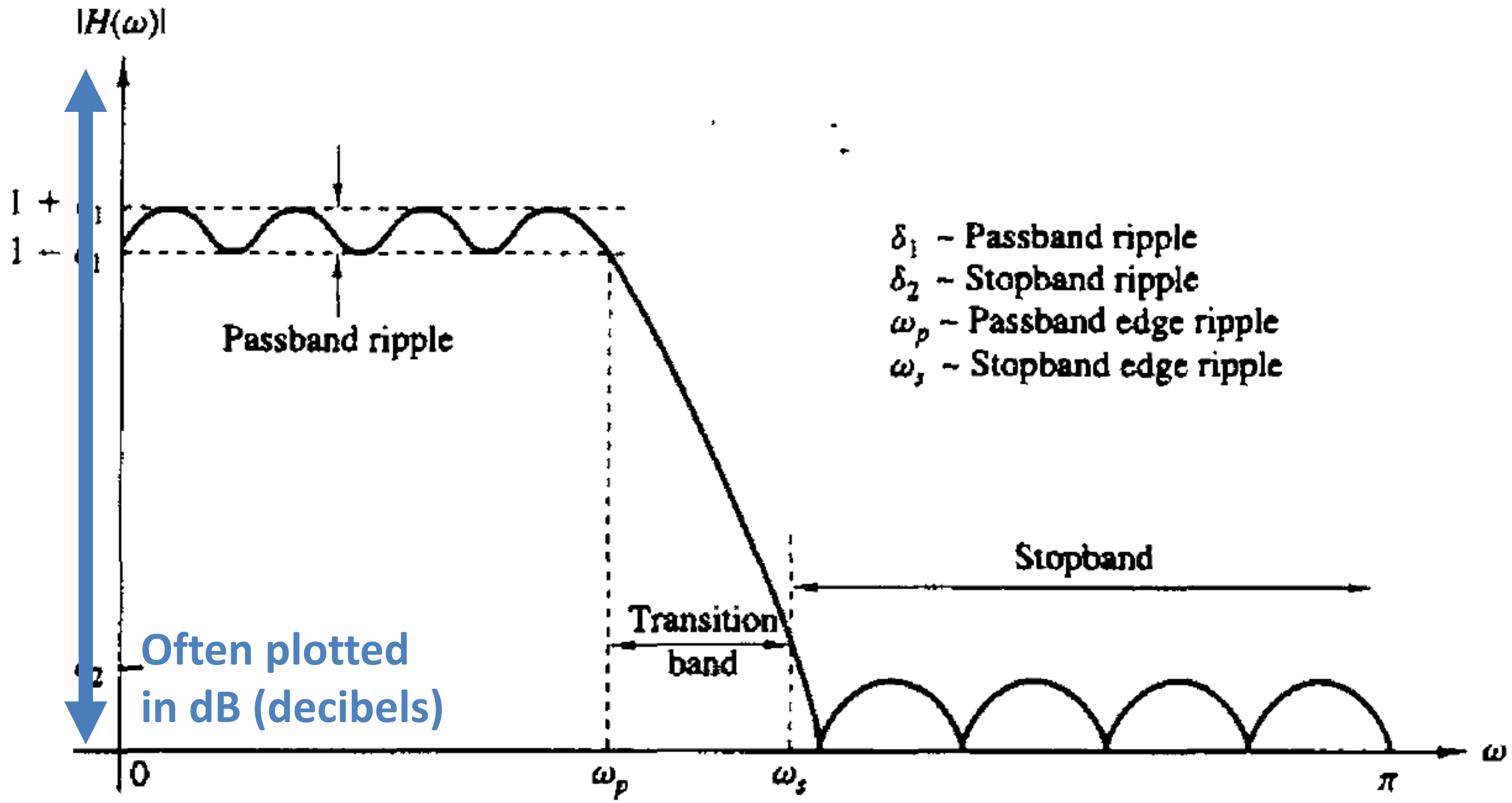


Lecture 21: Design of FIR Filters

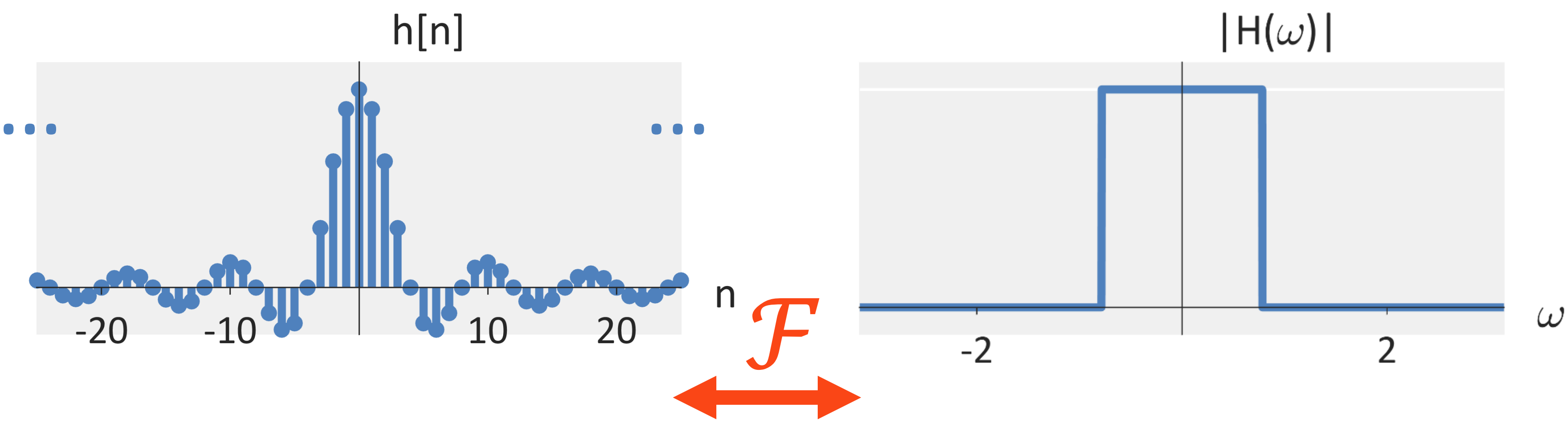
Foundations of Digital Signal Processing

Outline

- Review Downsampling & Upsampling
- Causality in Filters
- **Designing FIR Filters with Windows**
- Designing FIR Filters with Frequency Selection
- Designing FIR Filters with Equi-ripples

Designing with Windows

- **Question: How can I design an FIR filter from an ideal filter?**



Designing with Windows

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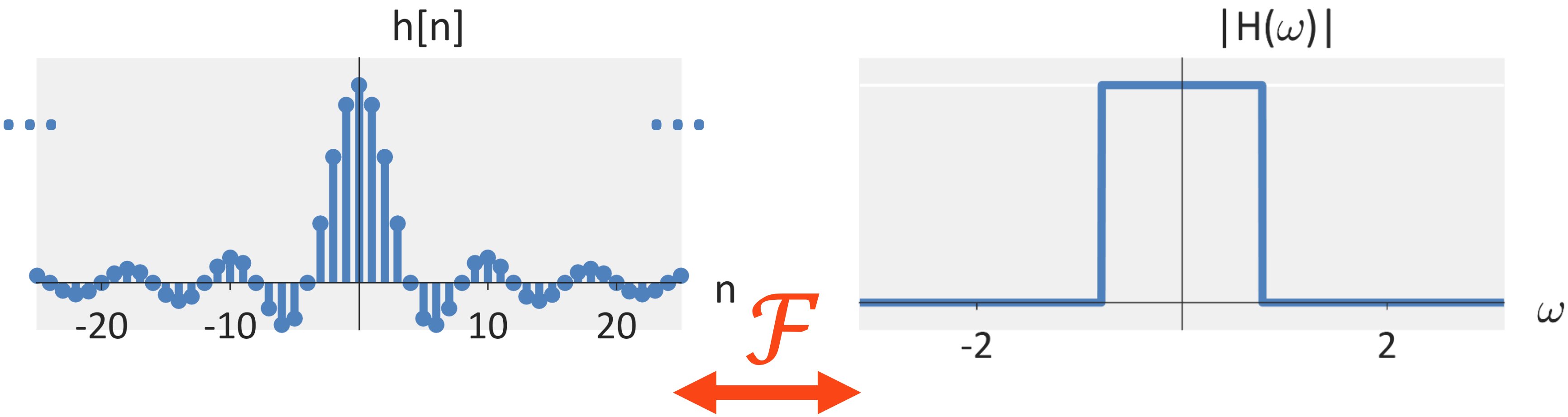

Non-causal
Infinite Response

Ideal filter

Designing with Windows

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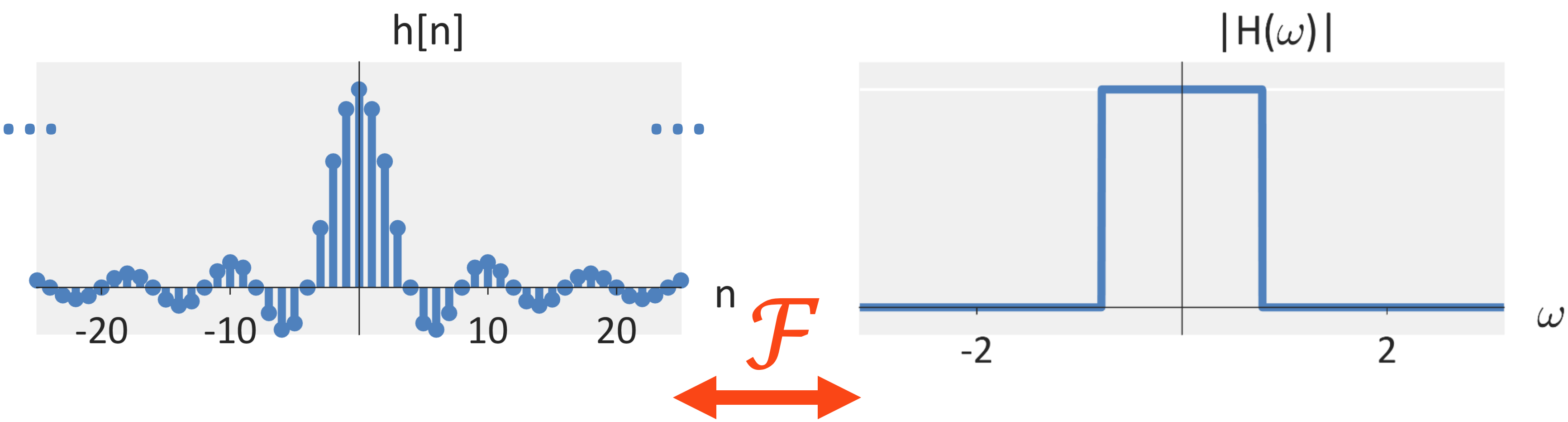

Non-causal
Infinite Response

Ideal filter

- **Answer: Window the response!**

Designing with Windows

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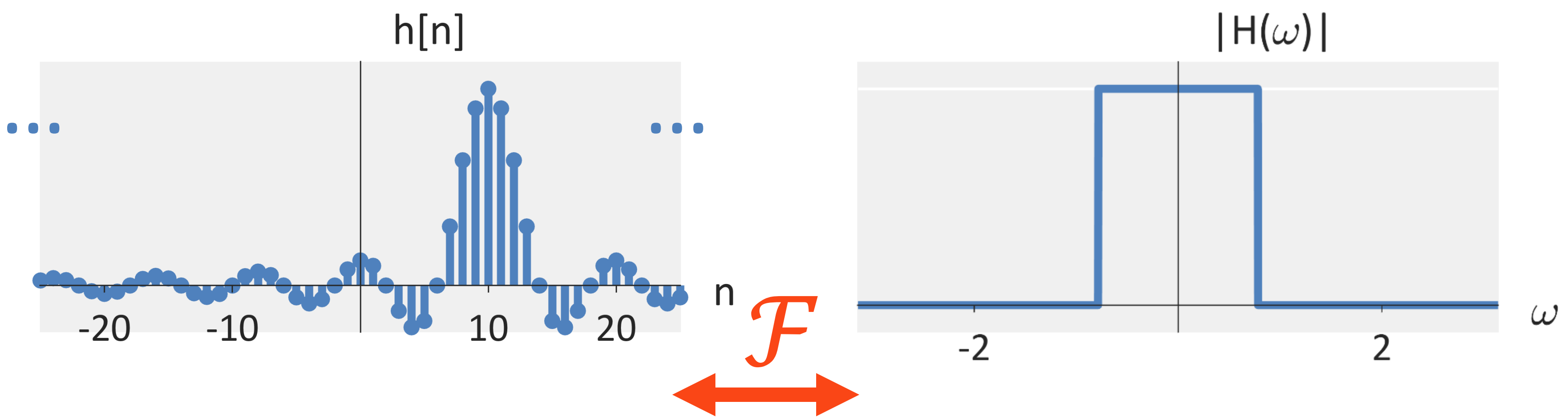

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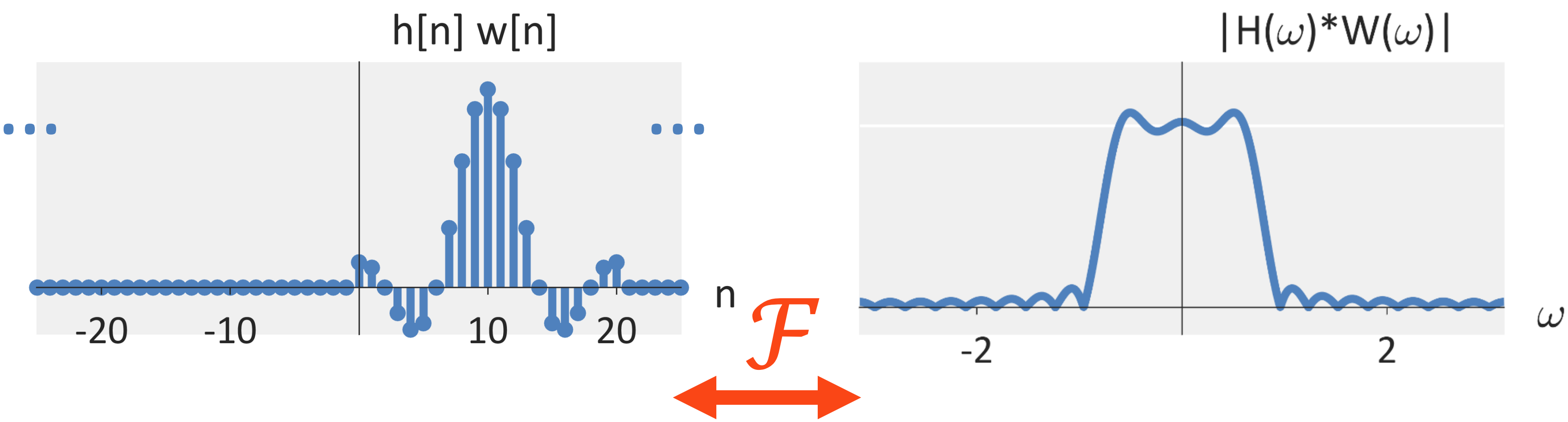


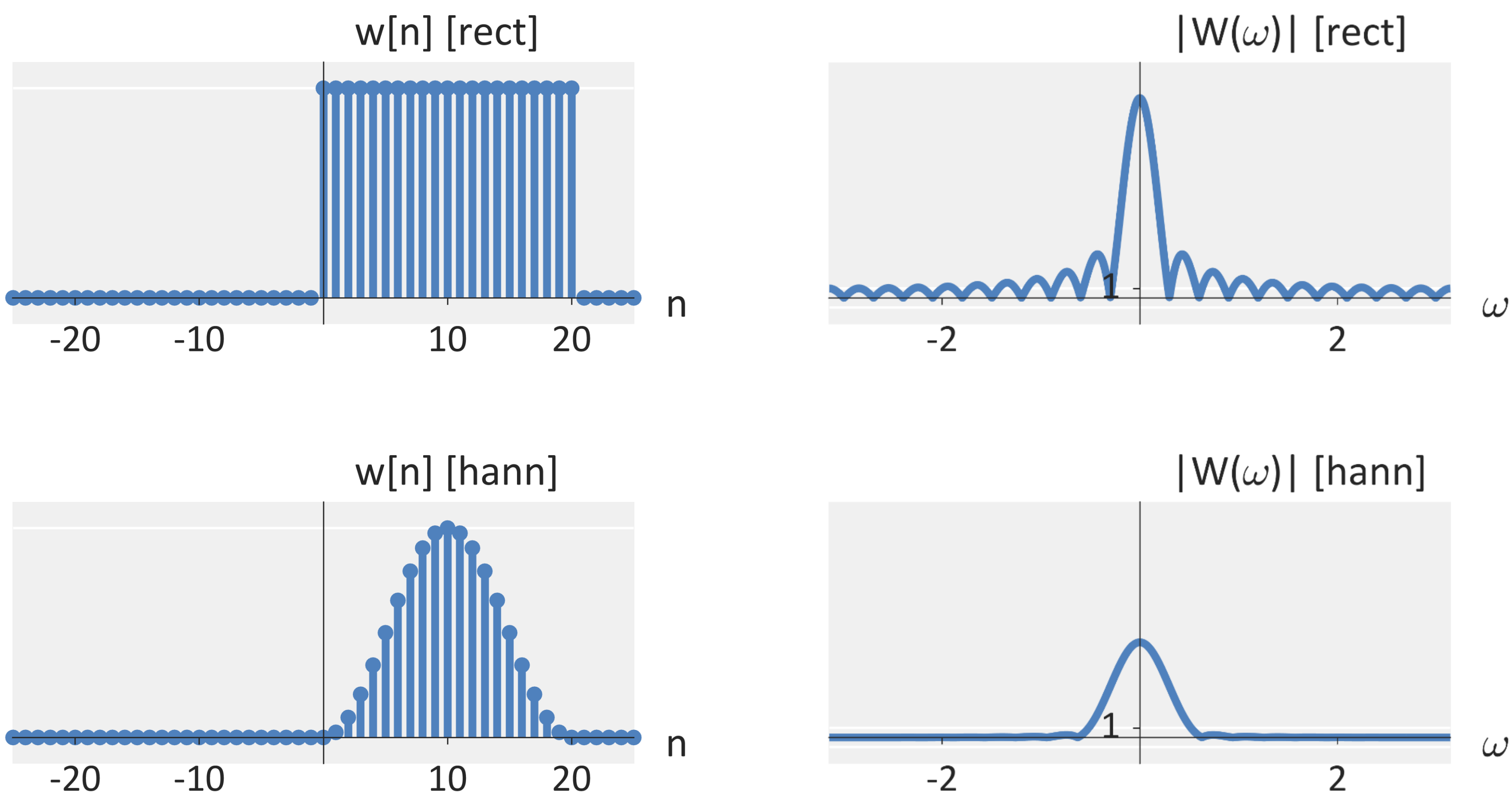
Non-causal
Infinite Response

Ideal filter

- **Answer: Window the response!**

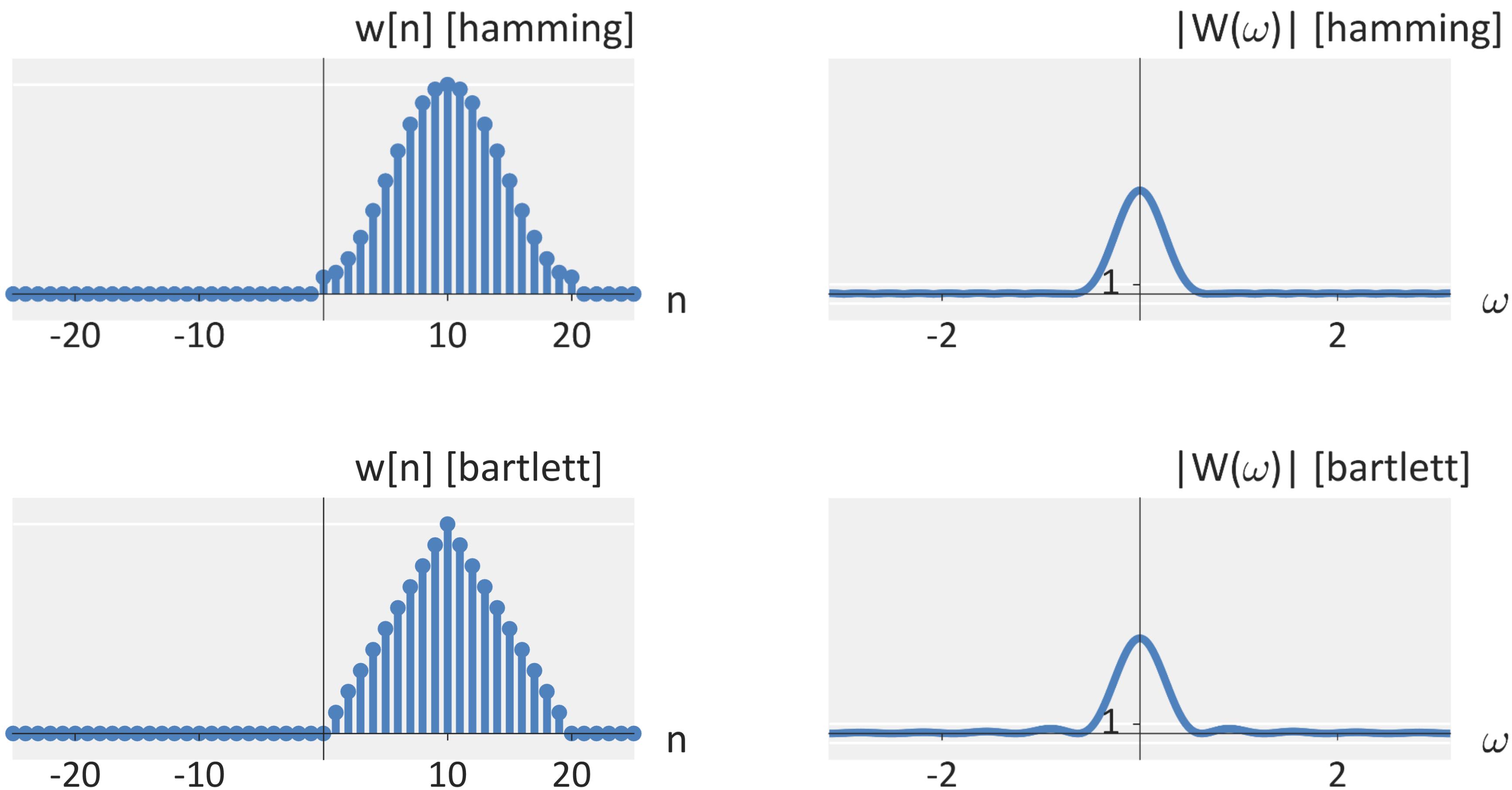
Designing with Windows

- **Different Filters**



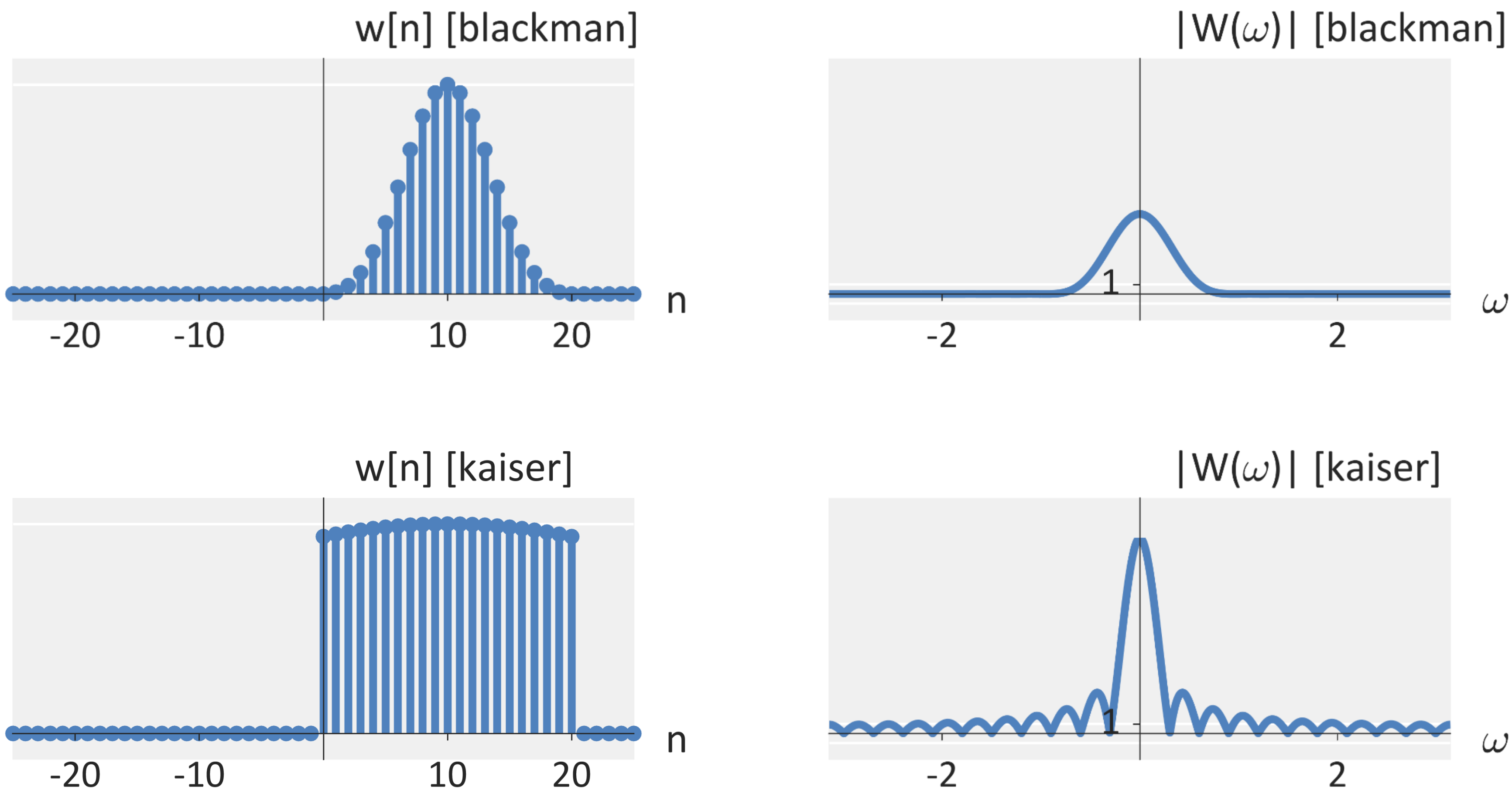
Designing with Windows

Different Filters



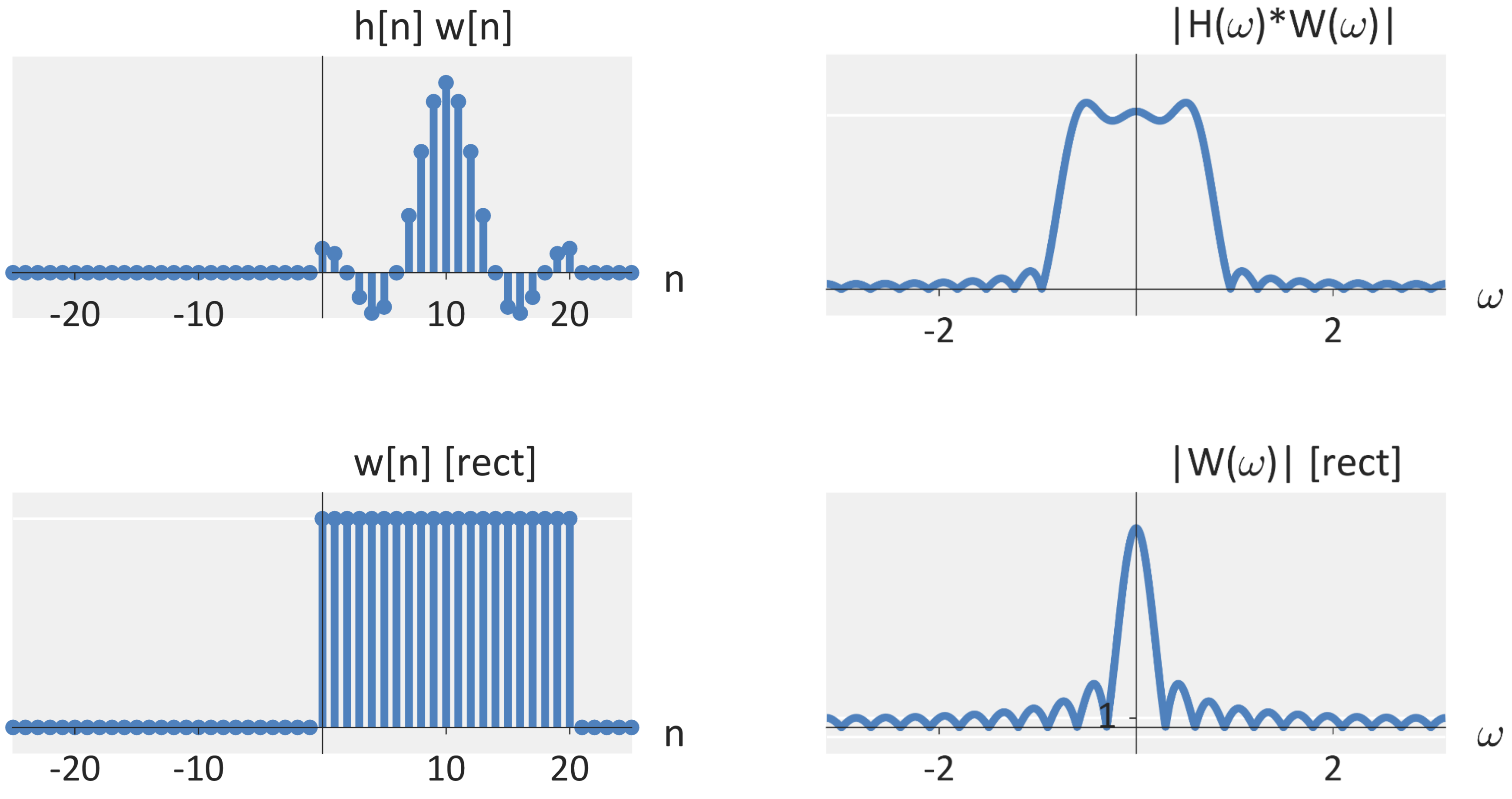
Designing with Windows

- **Different Filters**



Designing with Windows

- **Windowing the sinc impulse response**



Designing with Windows

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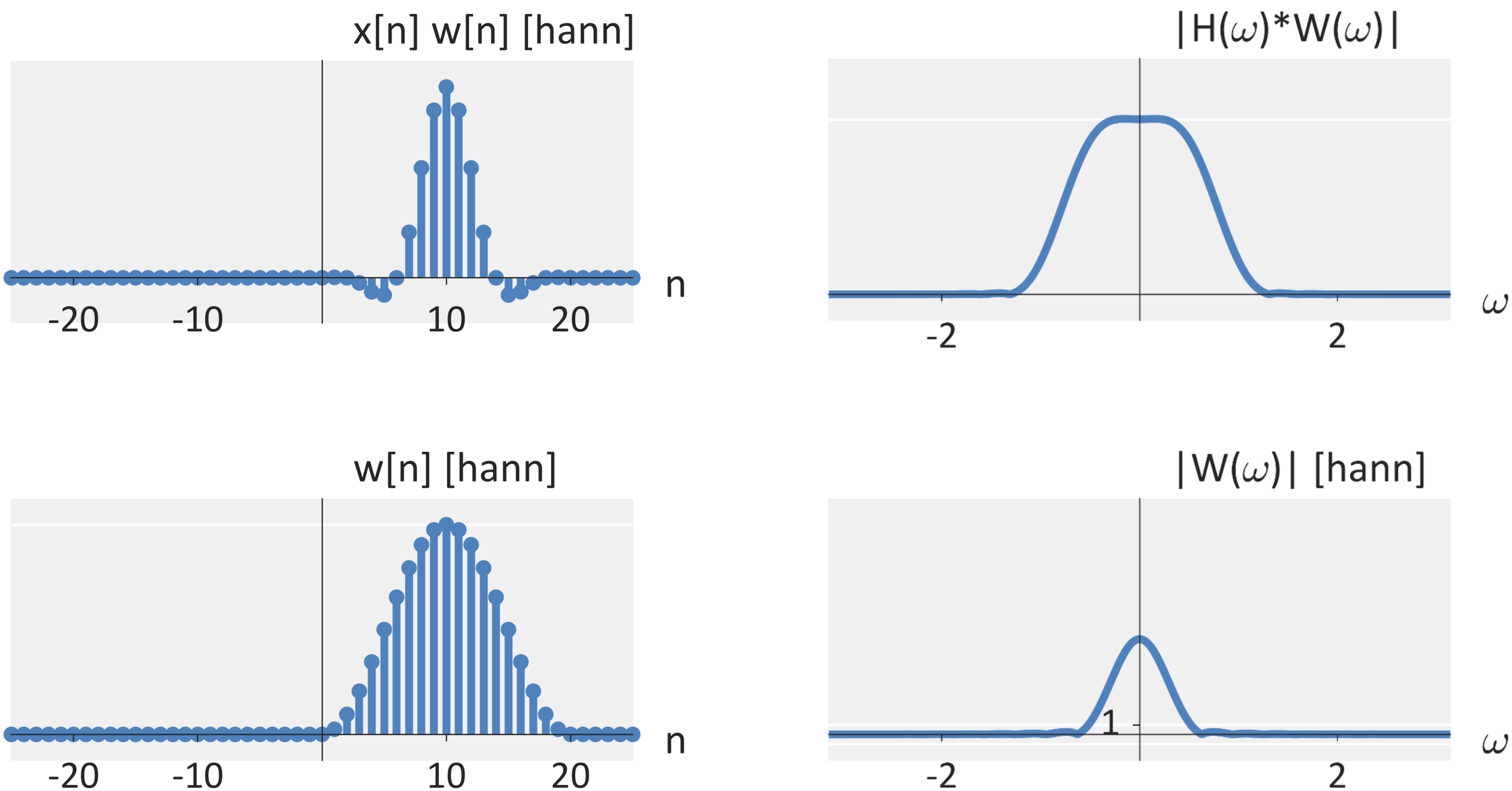

Designing with Windows

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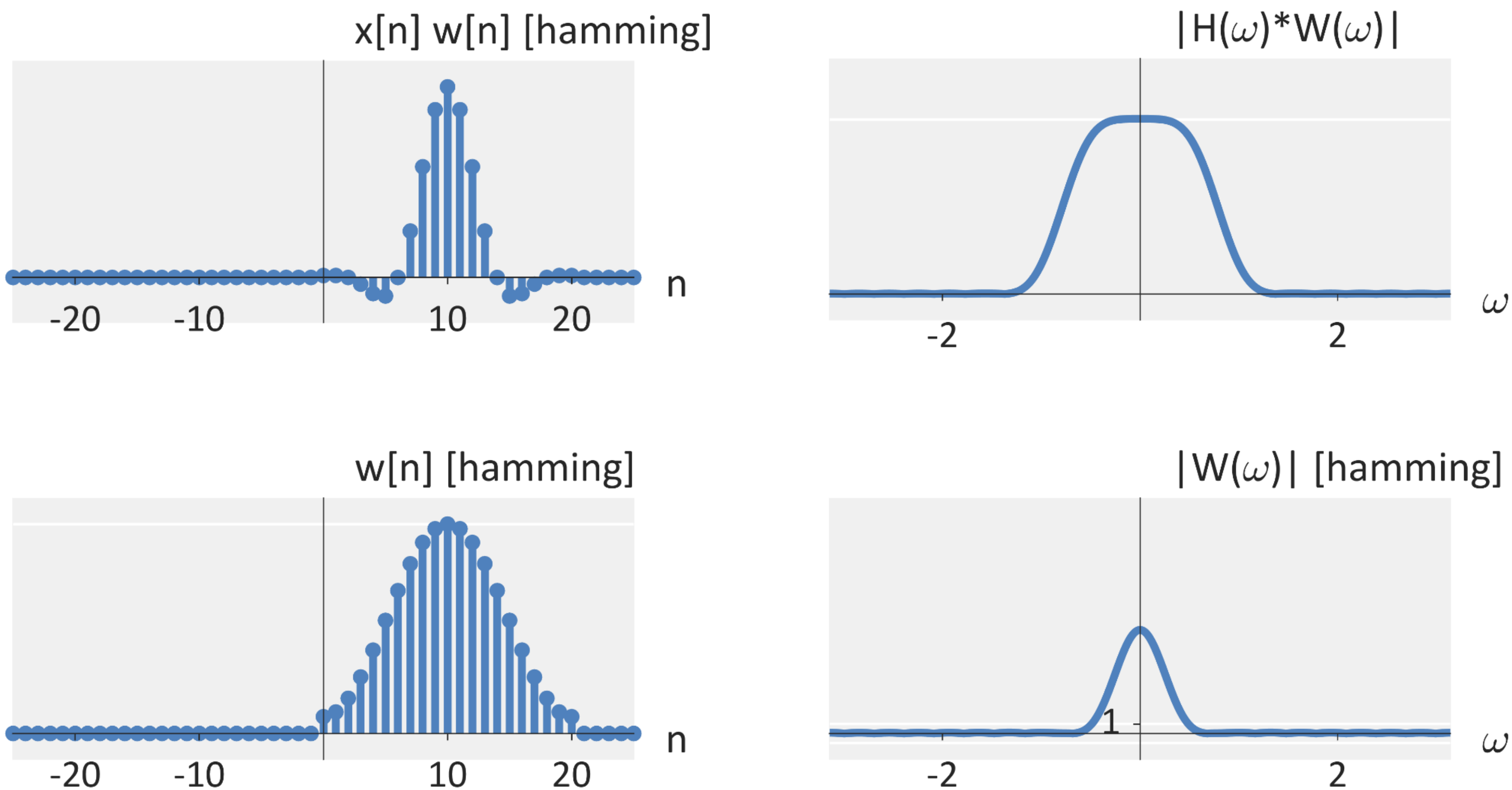

Designing with Windows

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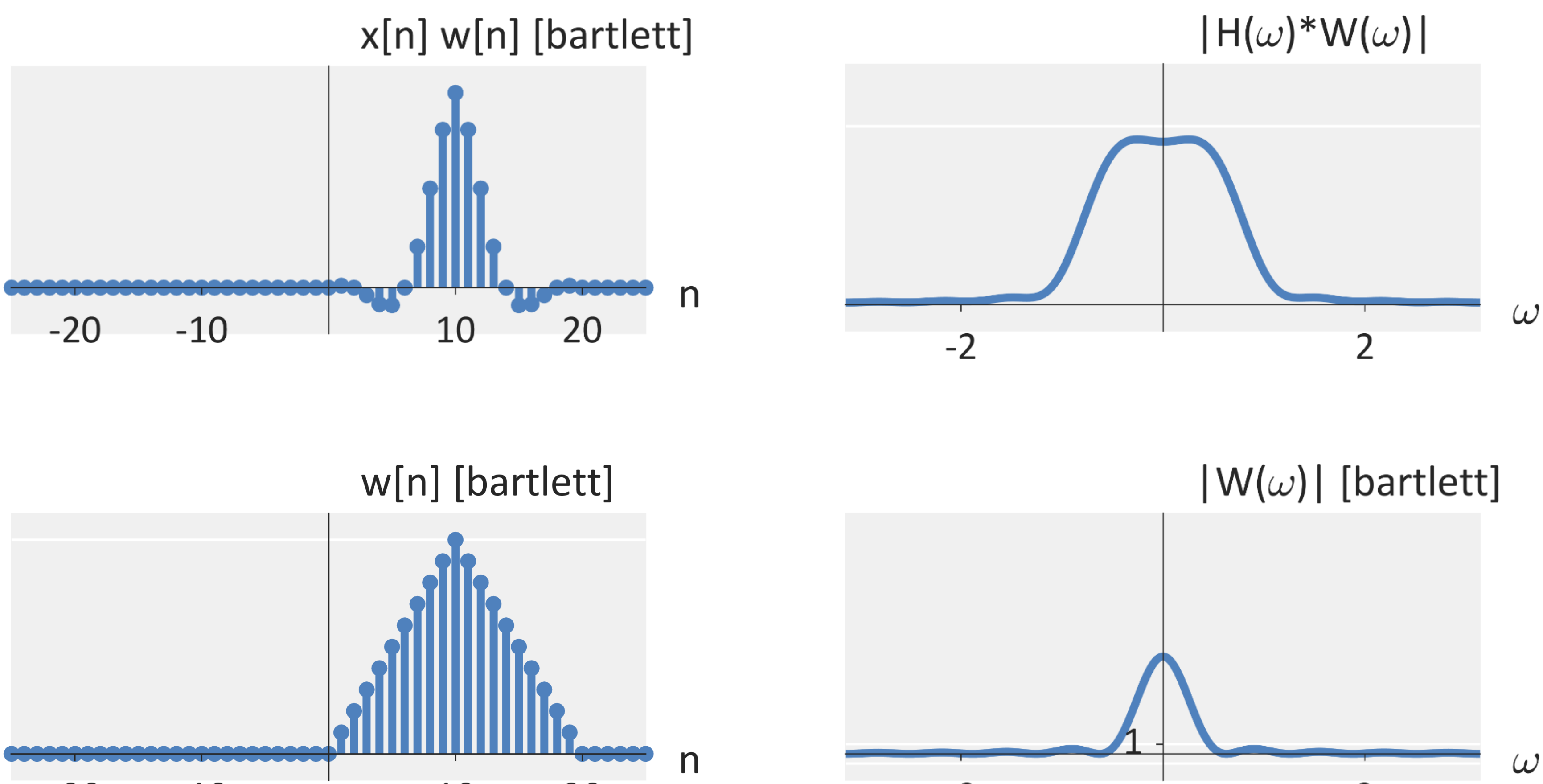

Designing with Windows

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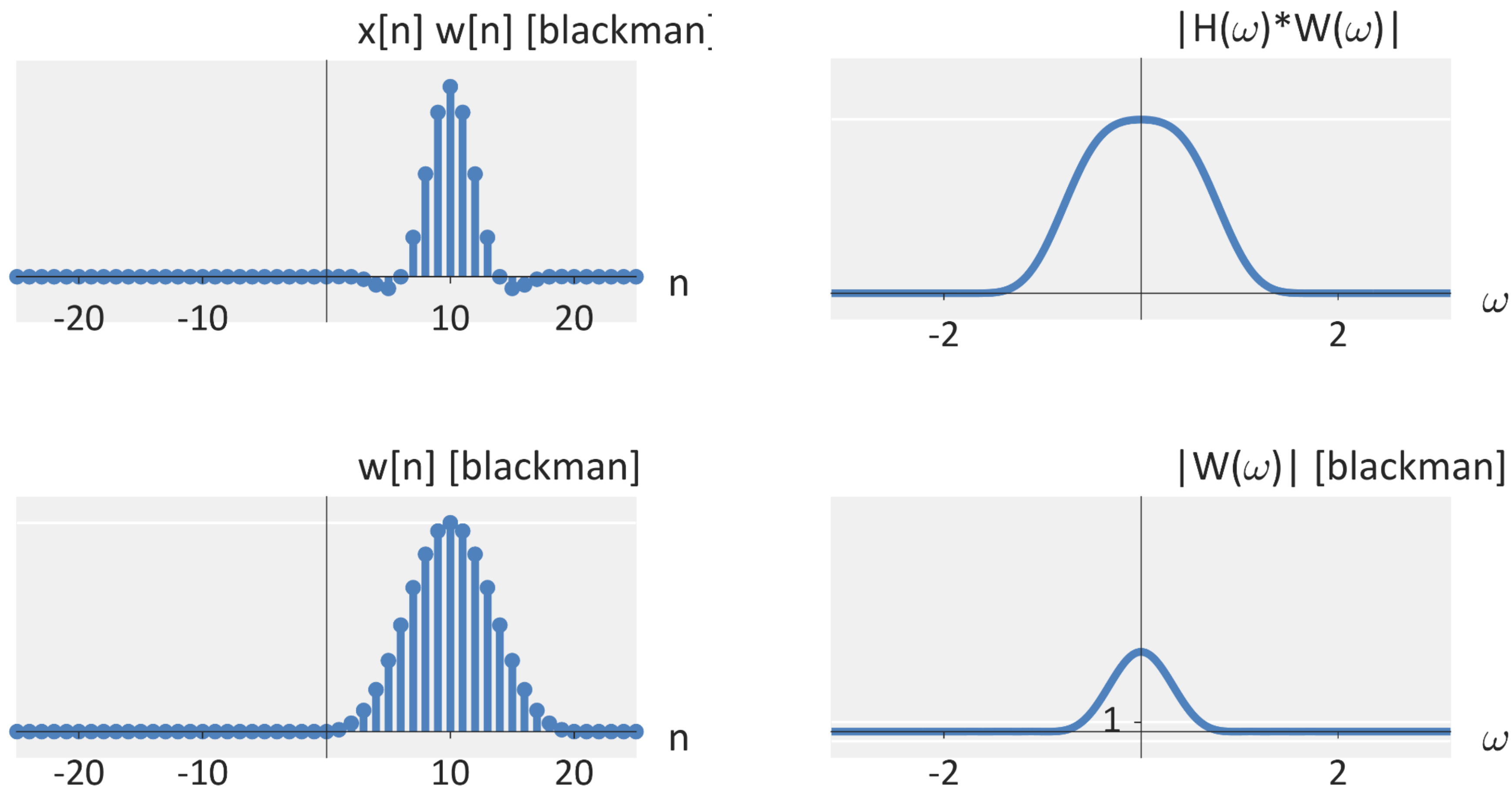

Designing with Windows

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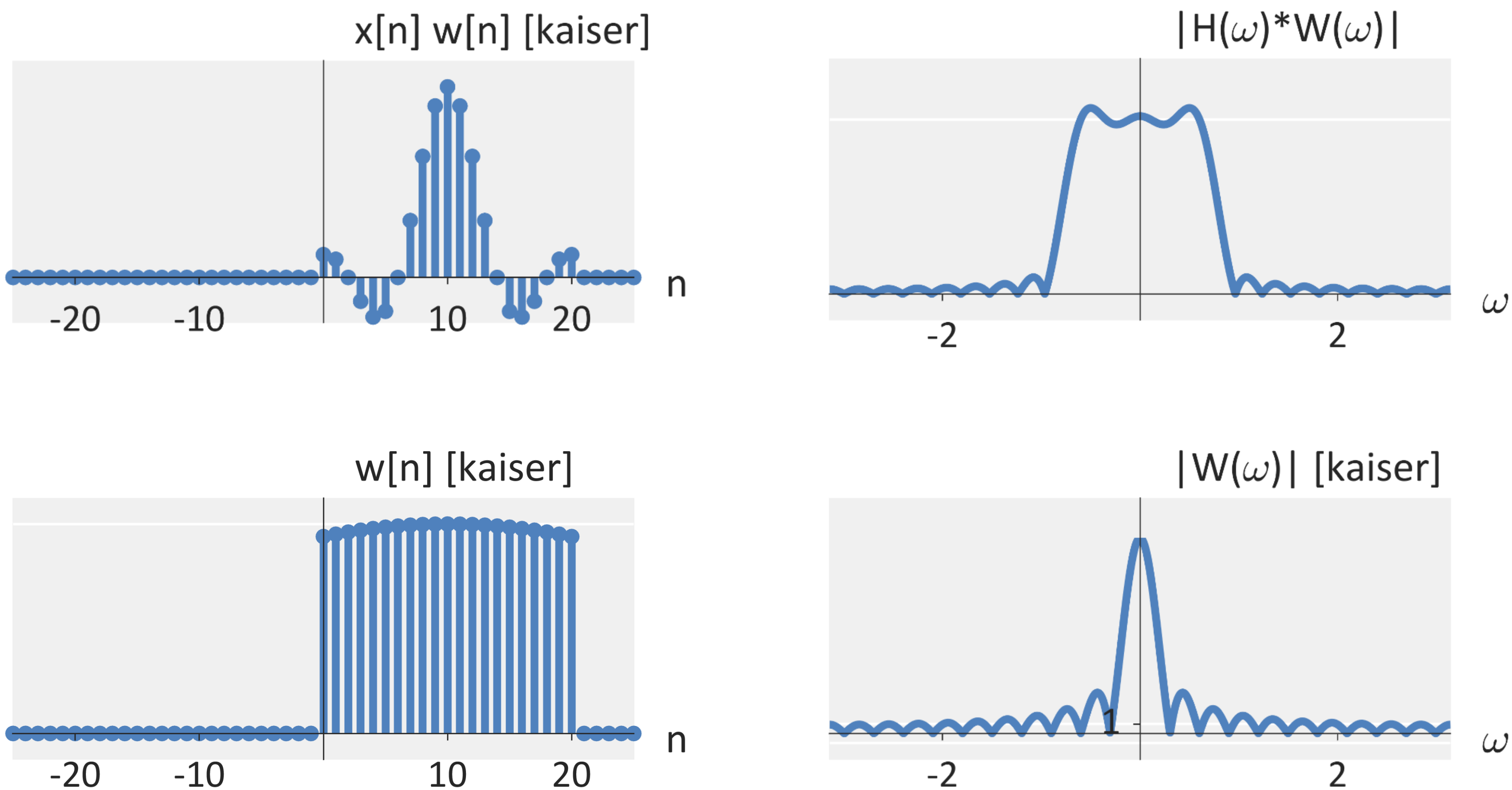

Designing with Windows

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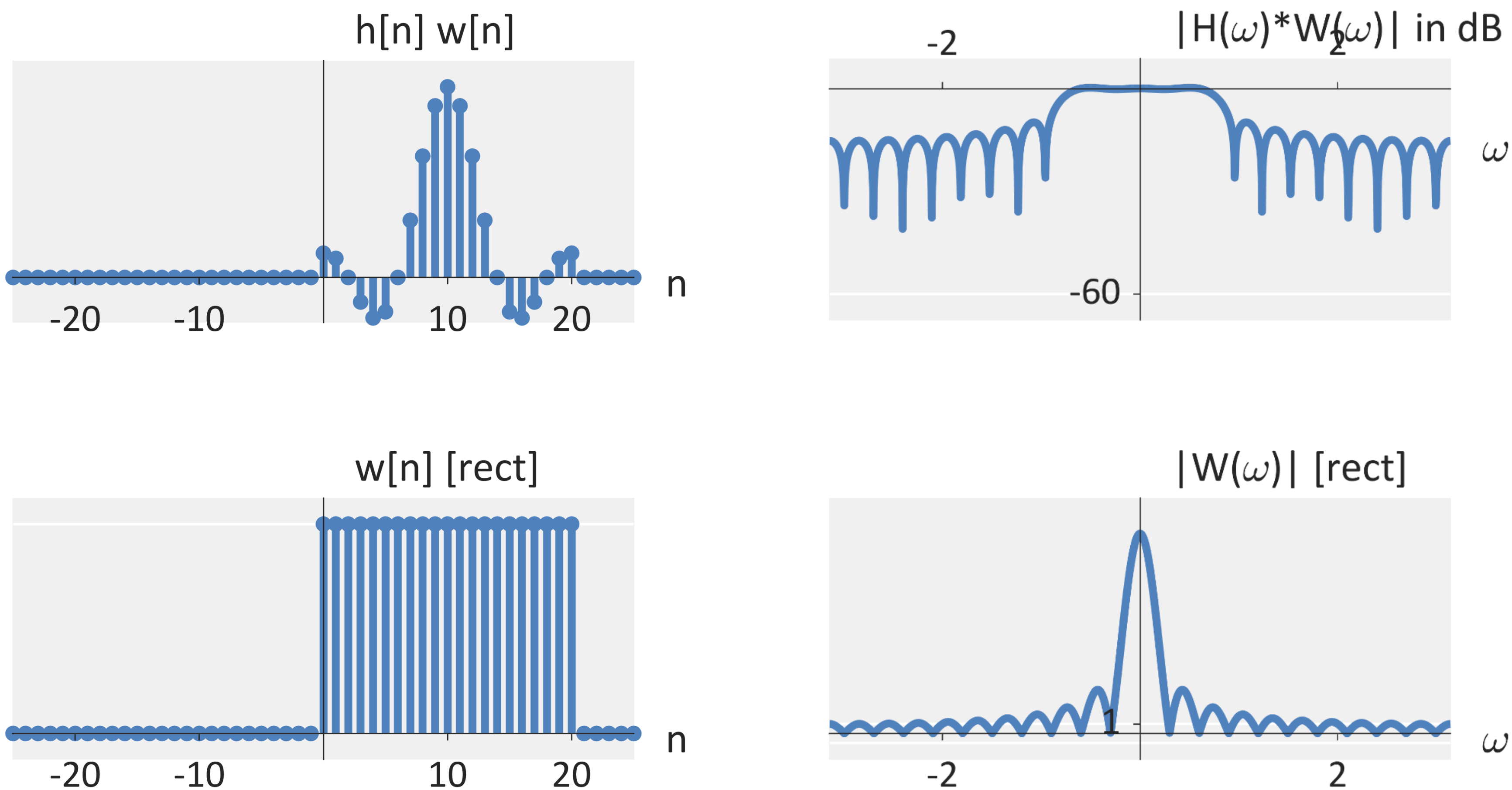

Designing with Windows

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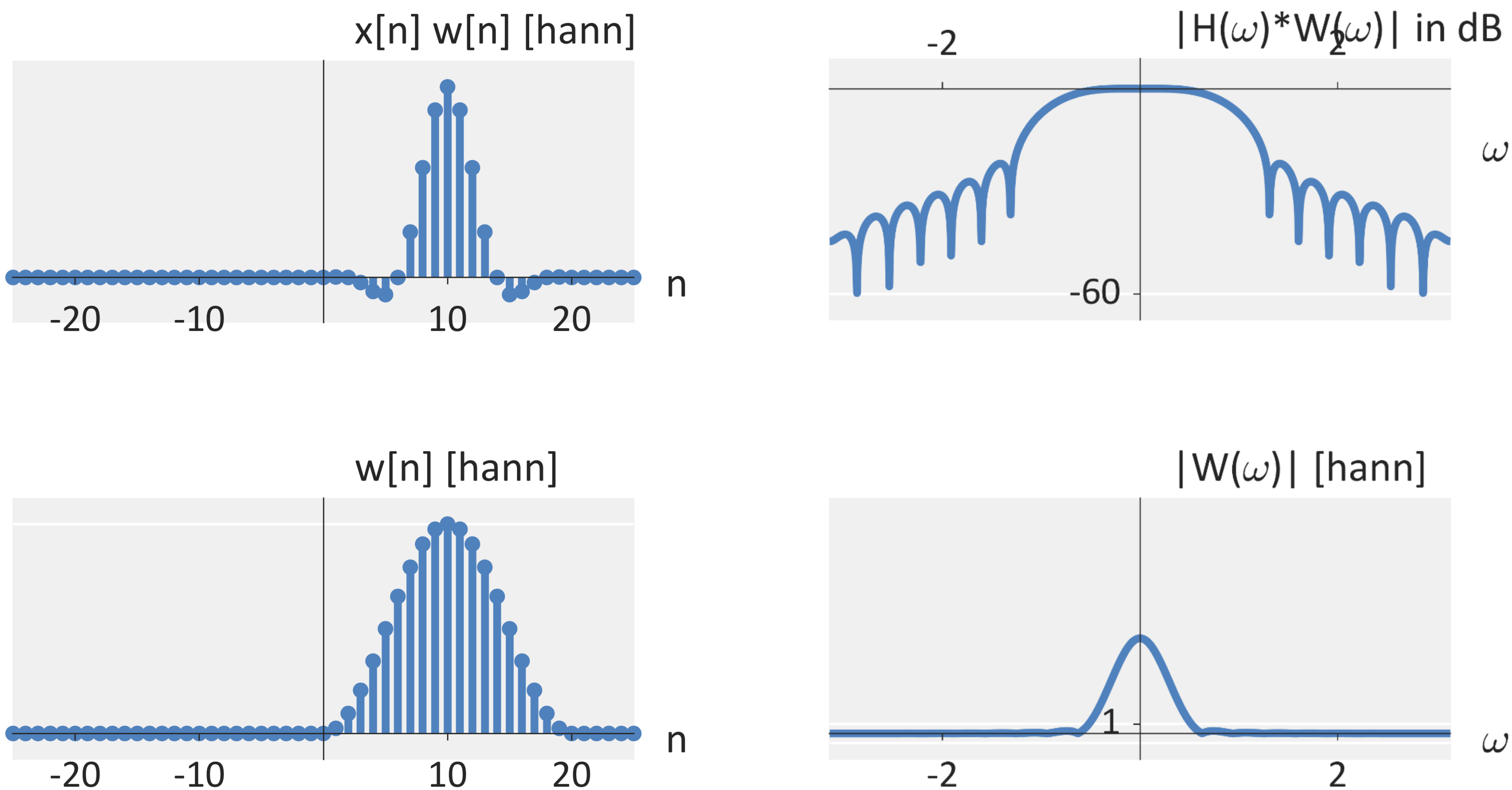

Designing with Windows

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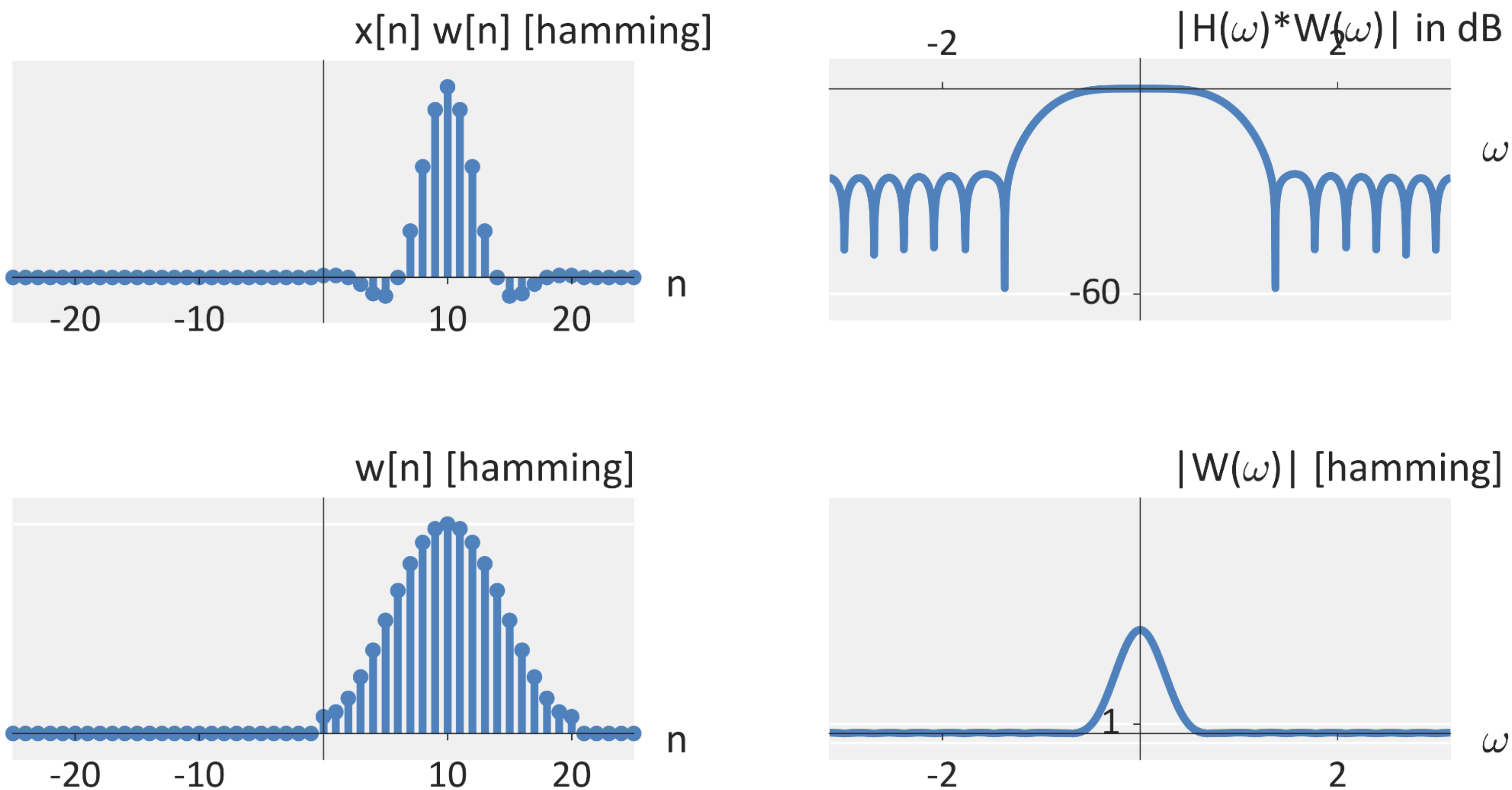

Designing with Windows

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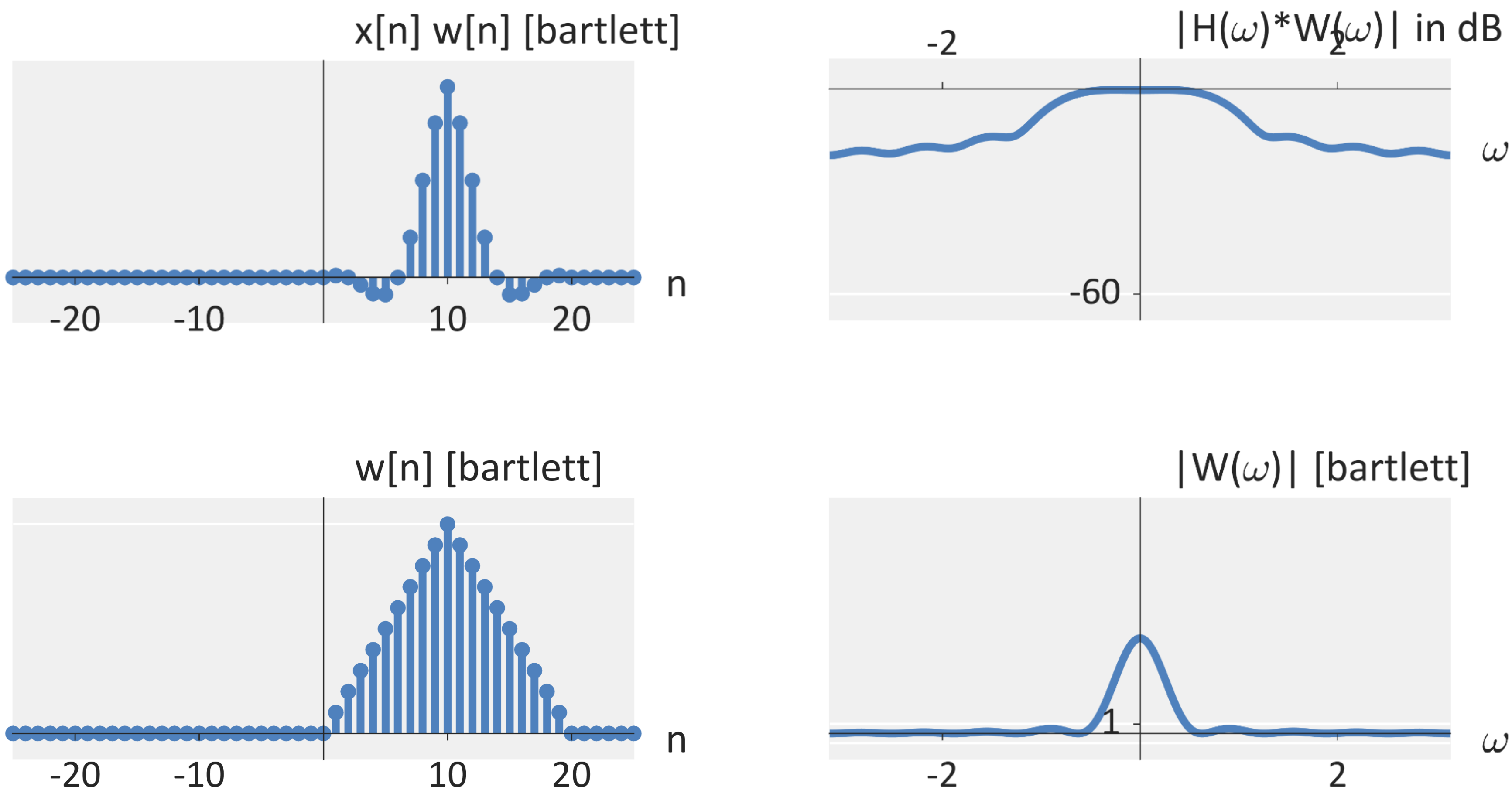

Designing with Windows

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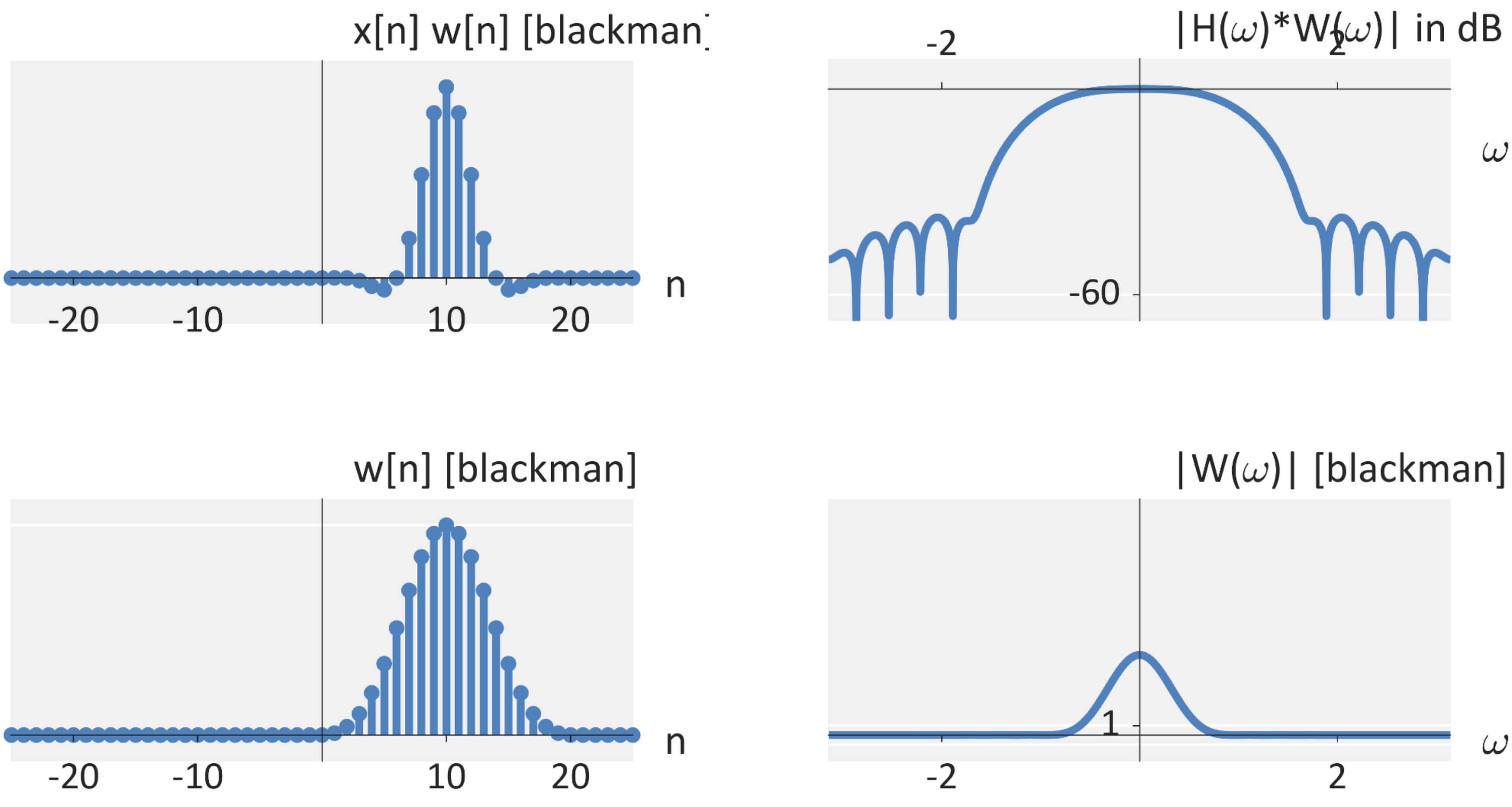

Designing with Windows

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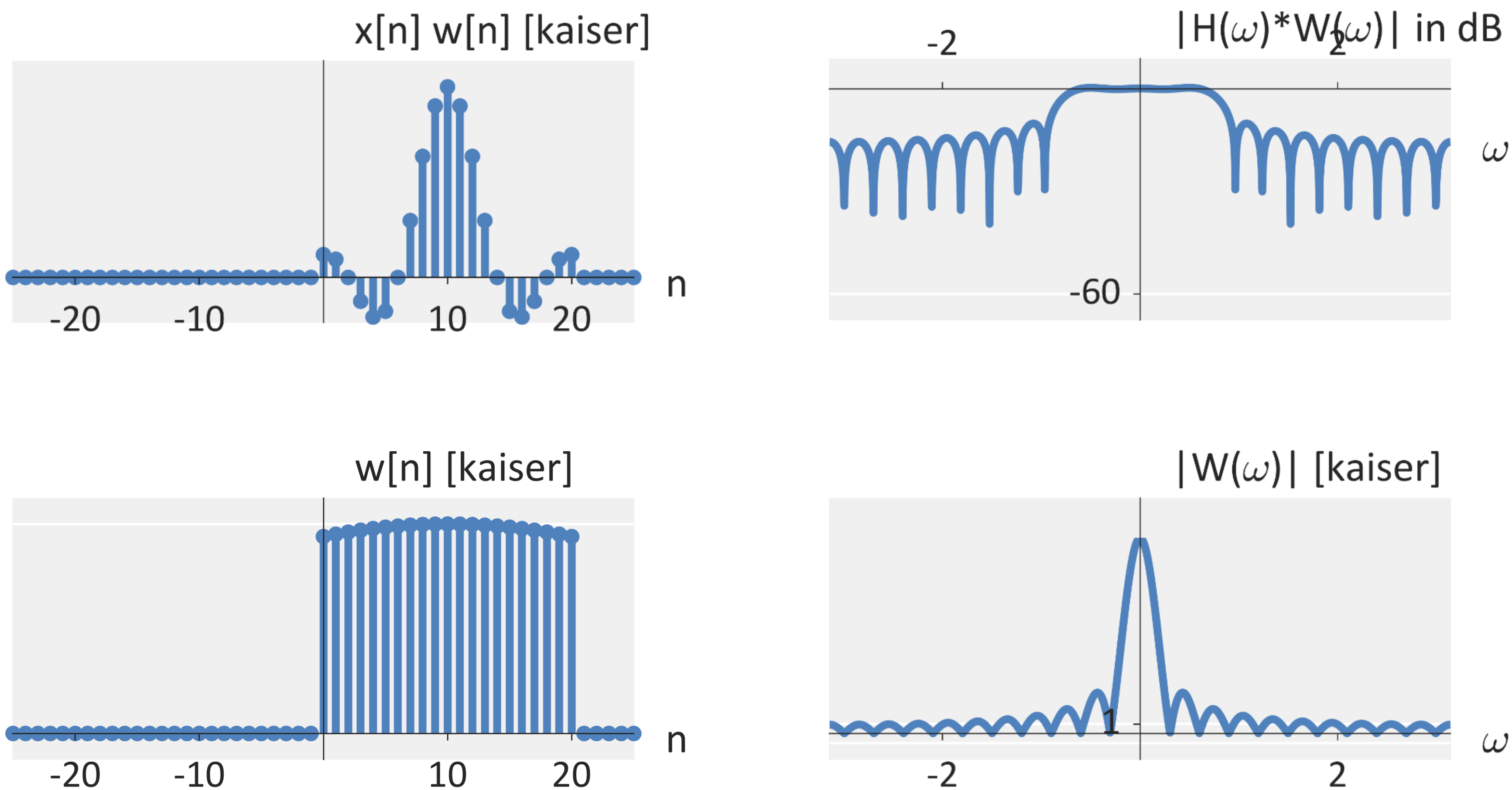

Lecture 21: Design of FIR Filters

Foundations of Digital Signal Processing

Outline

- Review Downsampling & Upsampling
- Causality in Filters
- Designing FIR Filters with Windows
- **Designing FIR Filters with Frequency Selection**
- Designing FIR Filters with Equi-ripples

Design with Frequency Selection

- **Option 2: Work backwards with constraints**

Consider the DFT:

$$h[n] = \sum_{k=0}^{N-1} H[k] e^{j\frac{2\pi}{N}nk} \qquad \text{such that} \qquad H[k] = H[N-k]$$

Design with Frequency Selection

- **Option 2: Work backwards with constraints**

Consider the DFT:

$$h[n] = \sum_{k=0}^{N-1} H[k] e^{j\frac{2\pi}{N}nk} \qquad \text{such that} \qquad H[k] = H[N-k]$$

$$h[n] = H[0] + \sum_{k=1}^{(N-1)/2} H[k] e^{j\frac{2\pi}{N}nk} + \sum_{k=(N+1)/2}^{N-1} H[k] e^{j\frac{2\pi}{N}nk}$$

Design with Frequency Selection

- **Option 2: Work backwards with constraints**

Consider the DFT:

$$h[n] = \sum_{k=0}^{N-1} H[k] e^{j\frac{2\pi}{N}nk} \qquad \text{such that} \qquad H[k] = H[N-k]$$

$$h[n] = H[0] + \sum_{k=1}^{(N-1)/2} H[k] e^{j\frac{2\pi}{N}nk} + \sum_{k=1}^{(N-1)/2} H[k] e^{j\frac{2\pi}{N}n(N-k)}$$

Design with Frequency Selection

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$$h[n] = H[0] + \sum_{k=1}^{(N-1)/2} H[k] e^{j\frac{2\pi}{N}nk} + H[k] e^{-j\frac{2\pi}{N}nk} e^{j2\pi n}$$

Design with Frequency Selection

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Consider the DFT:

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Design with Frequency Selection

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$$h[n] = H[0] + \sum_{k=1}^{(N-1)/2} H[k]\left(e^{j\frac{2\pi}{N}nk} + e^{-j\frac{2\pi}{N}nk}\right)$$

Design with Frequency Selection

- **Option 2: Work backwards with constraints**

Consider the DFT:

$$h[n] = \sum_{k=0}^{N-1} H[k] e^{j\frac{2\pi}{N}nk} \qquad \text{such that} \qquad H[k] = H[N-k]$$

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$$h[n] = H[0] + 2\sum_{k=1}^{(N-1)/2} H[k] \cos\left(\frac{2\pi}{N}nk\right)$$

Design with Frequency Selection

- **An inverse DFT that forces time-symmetry**

$$h[n] = H[0] + 2\sum_{k=1}^{(N-1)/2} H[k]\cos\left(\frac{2\pi}{N}nk\right)$$

Design with Frequency Selection

- **An inverse DFT that forces time-symmetry**

$$h[n] = H[0] + 2\sum_{k=1}^{(N-1)/2} H[k]\cos\left(\frac{2\pi}{N}nk\right)$$

- **Example: Consider the desired 9-sample frequency response with the first half defined by** [1 1 0 0]
  - Compute the frequency sampled filter

Design with Frequency Selection

- **An inverse DFT that forces time-symmetry**

$$h[n] = H[0] + 2 \sum_{k=1}^{(N-1)/2} H[k] \cos\left(\frac{2\pi}{N} nk\right)$$

- **Example: Consider the desired 9-sample frequency response with the first half defined by** [1 1 0 0]

  - Compute the frequency sampled filter

$$h[n] = 1 + 2\cos\big((2\pi/9)n\big)$$

Design with Frequency Selection

- **Example: Consider the desired 9-sample frequency response with the first half defined by** [1 1 0 0]

  - Compute the frequency sampled filter

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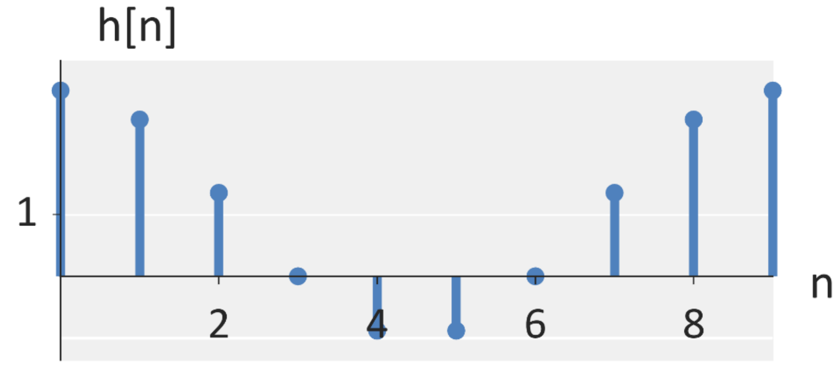


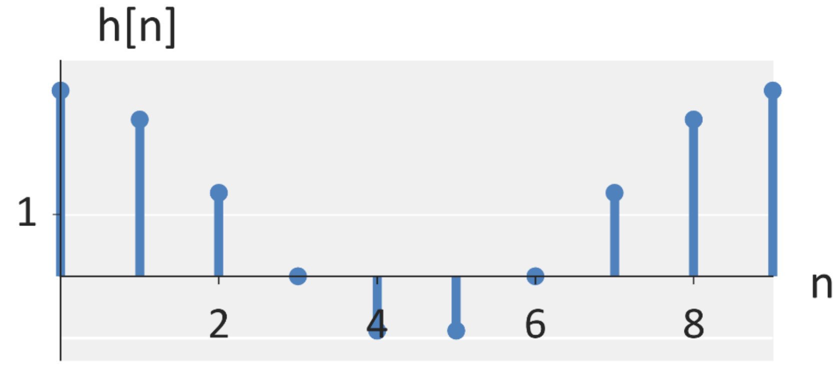
Design with Frequency Selection

- **Example:** **Consider the desired 9-sample frequency response with the first half defined by** [1 1 0 0]

  - Compute the frequency sampled filter

$$h[n] = 1 + 2\cos\big((2\pi/9)n\big)$$

  - In practice, this should be circularly shifted so that the maximum is centered.



Design with Frequency Selection

- **An inverse DFT that forces time-symmetry**

$$h[n] = H[0] + 2 \sum_{k=1}^{(N-1)/2} H[k] \cos\left(\frac{2\pi}{N}\left(n - \frac{N-1}{2}\right)k\right)$$

- **Example: Consider the desired 9-sample frequency response with the first half defined by** [1 1 0 0]

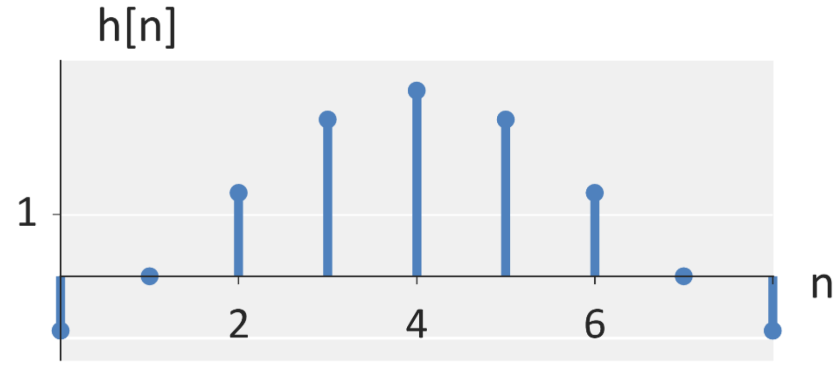
  - Compute the frequency sampled filter

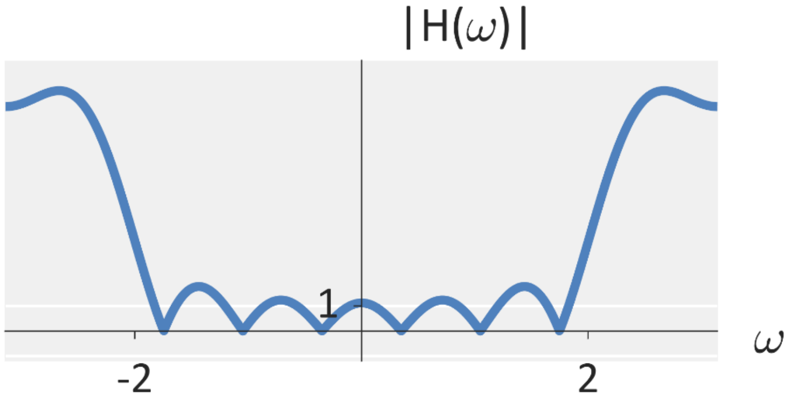
$$h[n] = 1 + 2\cos\big((2\pi/9)n\big)$$

Design with Frequency Selection

- **Example: Consider the desired 9-sample frequency response with the first half defined by** [1 1 0 0]
  - Compute the frequency sampled filter

$$h[n] = 1 + 2\cos\big((2\pi/9)(n - 8/2)\big)$$





Design with Frequency Selection

- **Example: Consider the desired 17-sample frequency response with the first half defined by** [1 1 1 1 0 0 0 0]
  - Compute the frequency sampled filter

$$h[n] = 1 + 2\cos\big((2\pi/19)n_c\big) + 2\cos\big((4\pi/19)n_c\big) + 2\cos\big((6\pi/19)n_c\big)$$

$$n_c = n - \frac{16}{2}$$

Design with Frequency Selection

- **Example: Consider the desired 17-sample frequency response with the first half defined by** [1 1 1 1 0 0 0 0]

  - Compute the frequency sampled filter

$$h[n] = 1 + 2\cos\big((2\pi/19)n_c\big) + 2\cos\big((4\pi/19)n_c\big) + 2\cos\big((6\pi/19)n_c\big)$$

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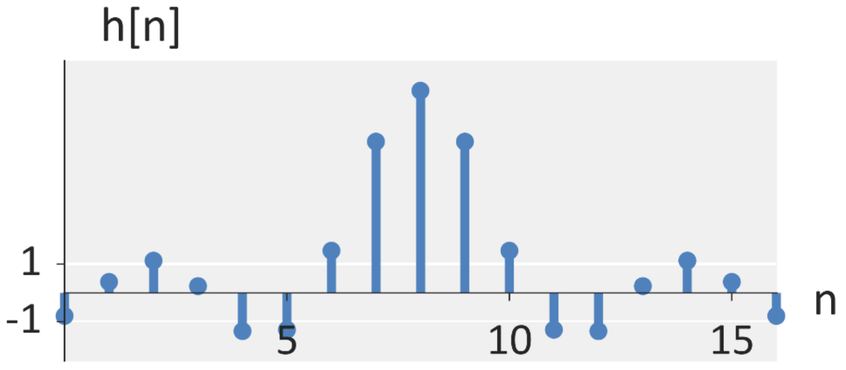
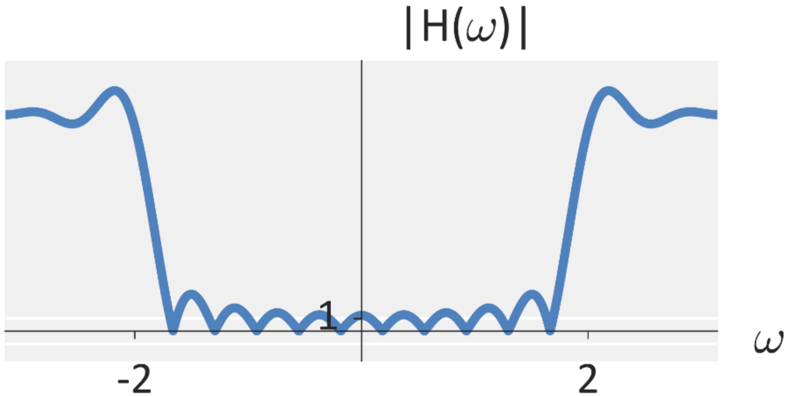
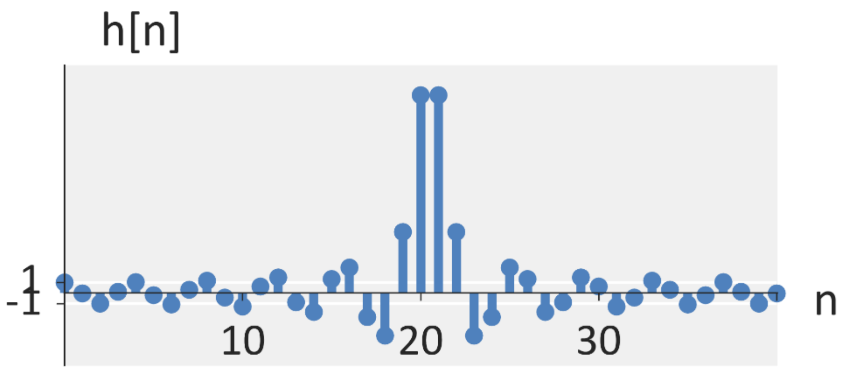
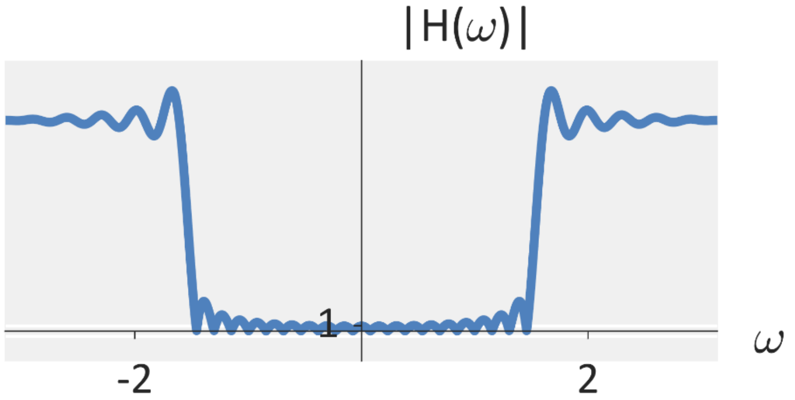




Design with Frequency Selection

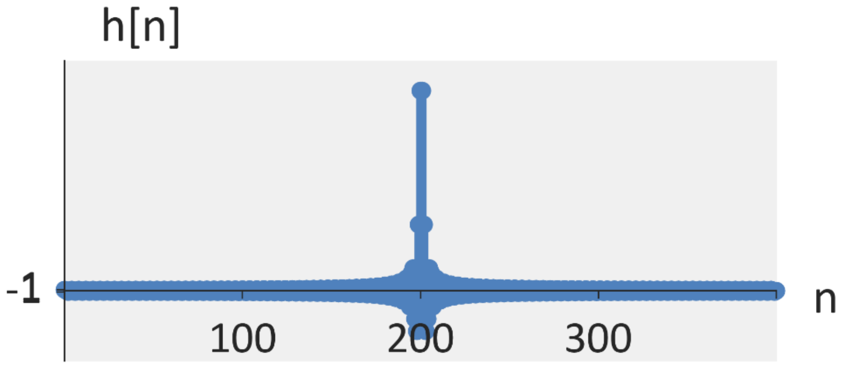
- **Example: Consider the desired 41-sample frequency response with the first 10 values defined by 1**
  - Compute the frequency sampled filter

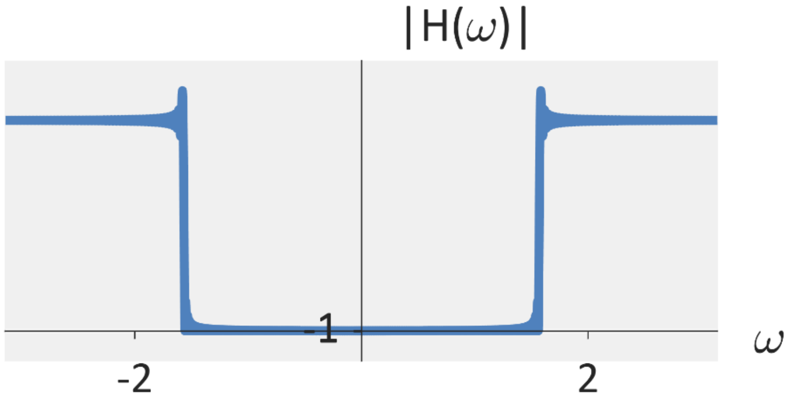




Design with Frequency Selection

- **Example:** **Consider the desired 401-sample frequency response with the first 100 values defined by 1**
  - Compute the frequency sampled filter
  - Note that in practice, this needs to be circularly shifted to the center





Design with Frequency Selection

- **Note: The definition can be slightly modified**
  - Our definition:

$$h[n] = H[0] + 2\sum_{k=1}^{(N-1)/2} H[k]\cos\left(\frac{2\pi}{N}\left(n - \frac{N-1}{2}\right)k\right)$$

$$= H[0] + 2\sum_{k=1}^{(N-1)/2} H[k]\cos\left(\frac{2\pi}{N}\left(n - \frac{N}{2} + \frac{1}{2}\right)k\right)$$

$$= H[0] + 2\sum_{k=1}^{(N-1)/2} H[k]\cos\left(\frac{2\pi}{N}\left(n + \frac{1}{2}\right)k - \pi k\right)$$

$$= H[0] + 2\sum_{k=1}^{(N-1)/2} (-1)^k H[k]\cos\left(\frac{2\pi}{N}\left(n + \frac{1}{2}\right)k\right)$$

Design with Frequency Selection

■ Final Definition

$$h[n] = H[0] + 2 \sum_{k=1}^{(N-1)/2} (-1)^k H[k] \cos\left(\frac{2\pi}{N}\left(n + \frac{1}{2}\right) k\right)$$

Side note: This is very closely related to the discrete cosine transform

Lecture 21: Design of FIR Filters

Foundations of Digital Signal Processing

Outline

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- Designing FIR Filters with Frequency Selection
- **Designing FIR Filters with Equi-ripples**

Design with Equi-ripples

Previously derived:

$$X(z) = z^{-\frac{(M-1)}{2}} \sum_{k=0}^{M/2-1} a_k \left[z^{\frac{(M-1)}{2}-k} + z^{-\left[\frac{(M-1)}{2} - k\right]}\right]$$

$$X(\omega) = e^{-j\omega\frac{(M-1)}{2}} \sum_{k=0}^{M/2-1} a_k \left[e^{j\omega\left[\frac{(M-1)}{2}-k\right]} + j\omega^{-j\omega\left[\frac{(M-1)}{2} - k\right]}\right]$$

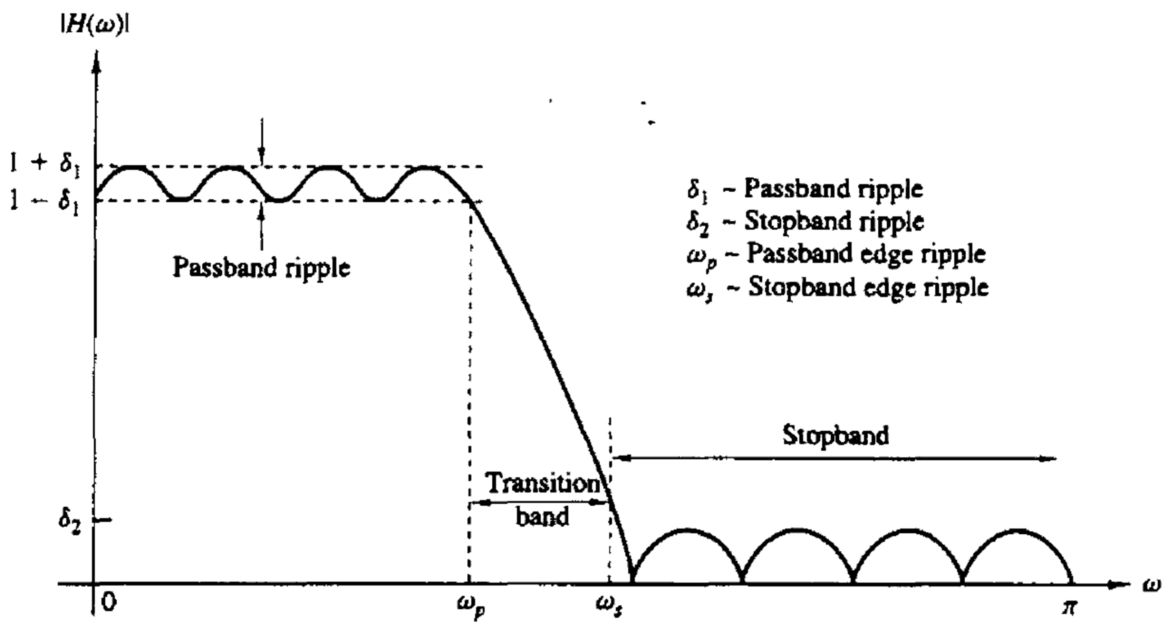
$$= 2e^{-j\omega\frac{(M-1)}{2}} \sum_{k=0}^{M/2-1} a_k \cos\left(\omega\left[\frac{M-1}{2} - k\right]\right)$$

Design with Equi-ripples

- **Equi-ripple design**

$$X(\omega) = 2e^{-j\omega\frac{(M-1)}{2}} \sum_{k=0}^{M/2-1} a_k \cos\left(\omega\left[\frac{M-1}{2} - k\right]\right)$$

- **Goal:** Find the optimal a_ks that satisfies **passband / stopband** ripple constraints.



Design with Equi-ripples

- **Equi-ripple design**

$$\min_{a_k} W(\omega)\left[H_d(\omega) - 2e^{-j\omega\frac{(M-1)}{2}}\sum_{k=0}^{M/2-1} a_k \cos\left(\omega\left[\frac{M-1}{2} - k\right]\right)\right]$$

Desired frequency response

Equals:

$\frac{\delta_2}{\delta_1}$ for ω in pass band

1 for ω in stop band

$\delta_2 =$ stopband ripple

$\delta_1 =$ passband ripple