Lecture 15: Filter Implementation

Foundations of Digital Signal Processing

Outline

- Reviewing magnitude responses
- Different types of filters
- Designing the phase response
- Implementation of FIR Filters

News

- **Homework #6**
  - Due ***FRIDAY***
  - Submit via canvas
- **Coding Problem #4**
  - Due *next week*
  - Submit via canvas
- **Exam #1 Solutions**
  - Now online

Lecture 15: Filter Implementation

Foundations of Digital Signal Processing

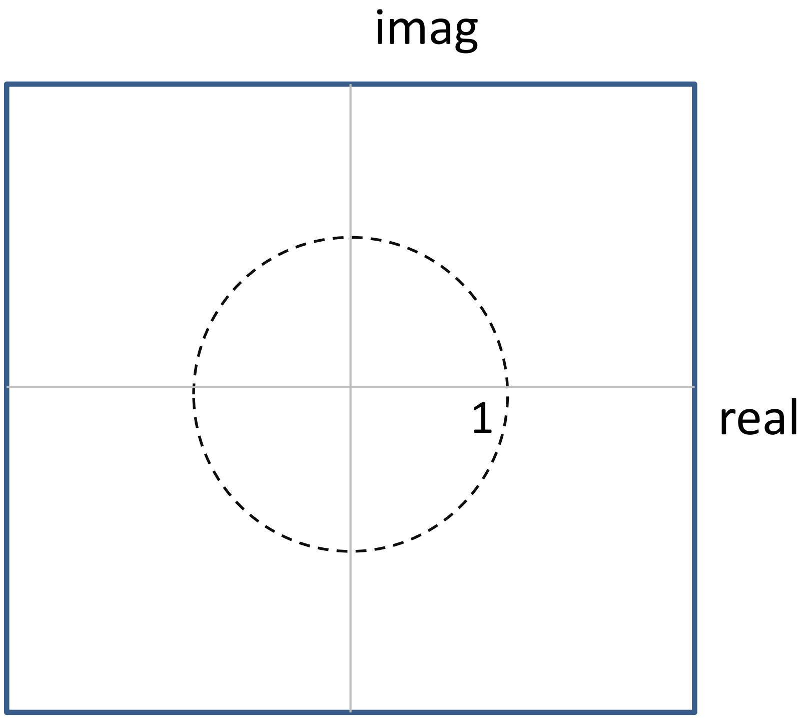
Outline

- **Reviewing magnitude responses**
- Different types of filters
- Designing the phase response
- Implementation of FIR Filters

Designing the Magnitude Response

- **Question: What happens when we move poles and zeros around for a filter?**

$$H(z) = \frac{\prod_{k=1}^{M}(1 - z_k z^{-1})}{\prod_{k=1}^{N}(1 - p_k z^{-1})}$$



Designing the Magnitude Response

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What is the $H(z)$ corresponding to this?

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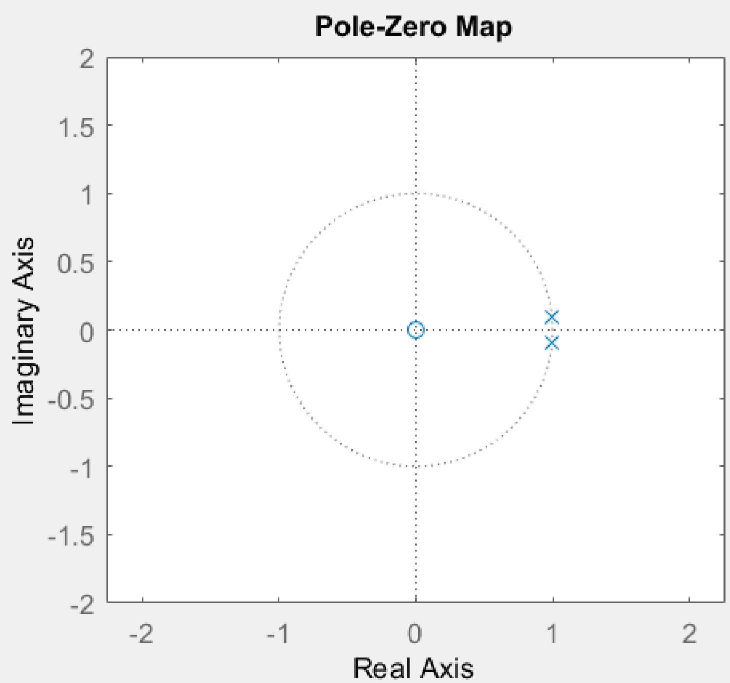
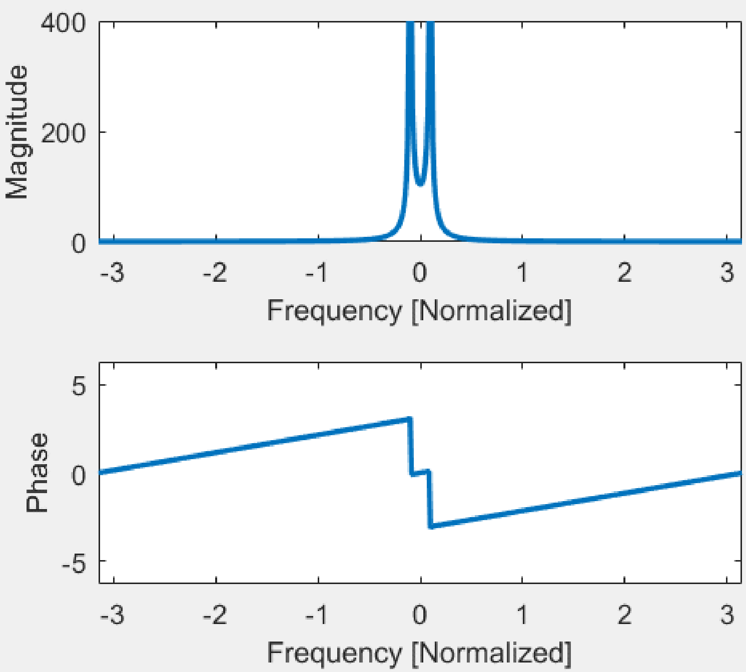


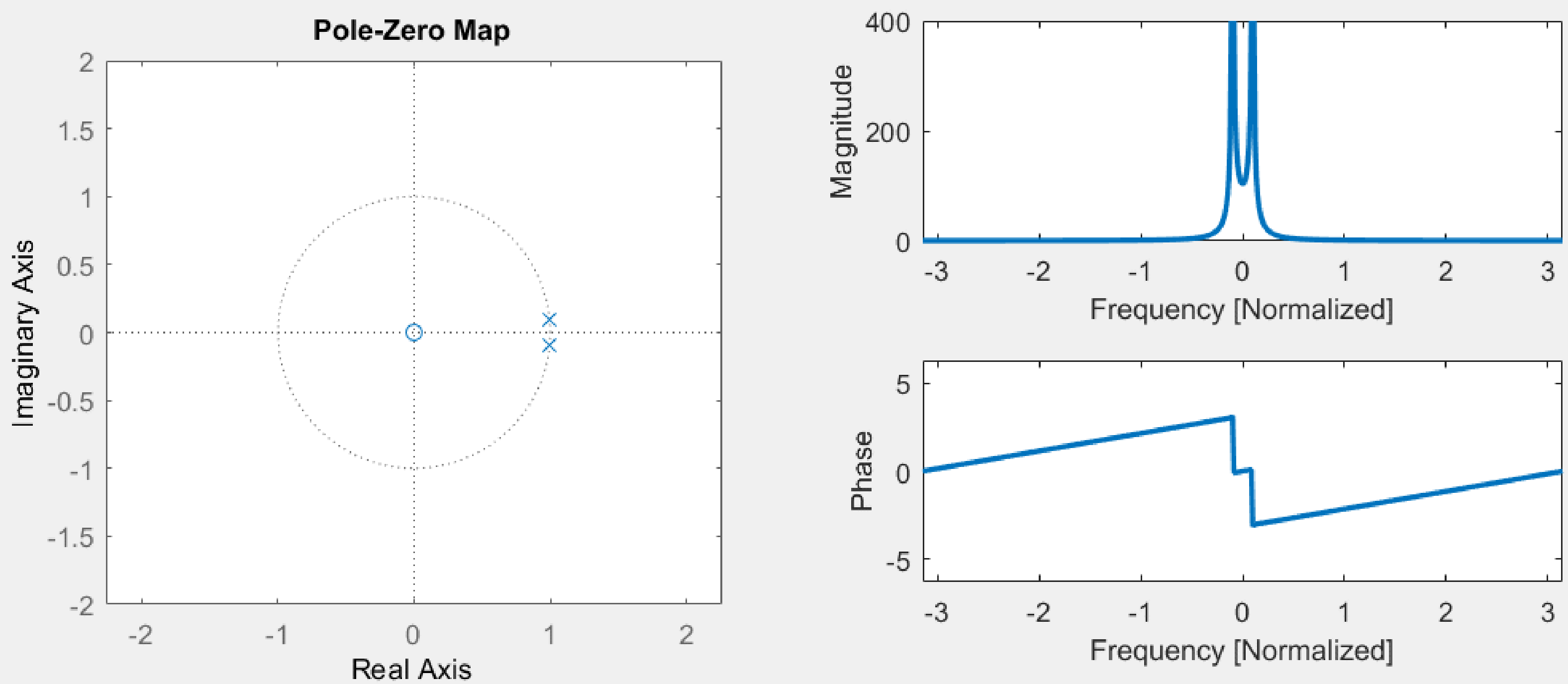


Designing the Magnitude Response

- **Question: What happens when we move poles and zeros around for a filter?**

What is the $H(z)$ corresponding to this?

$$H(z) = \frac{1}{(1 - e^{+j\phi}z^{-1})(1 - e^{-j\phi}z^{-1})}$$

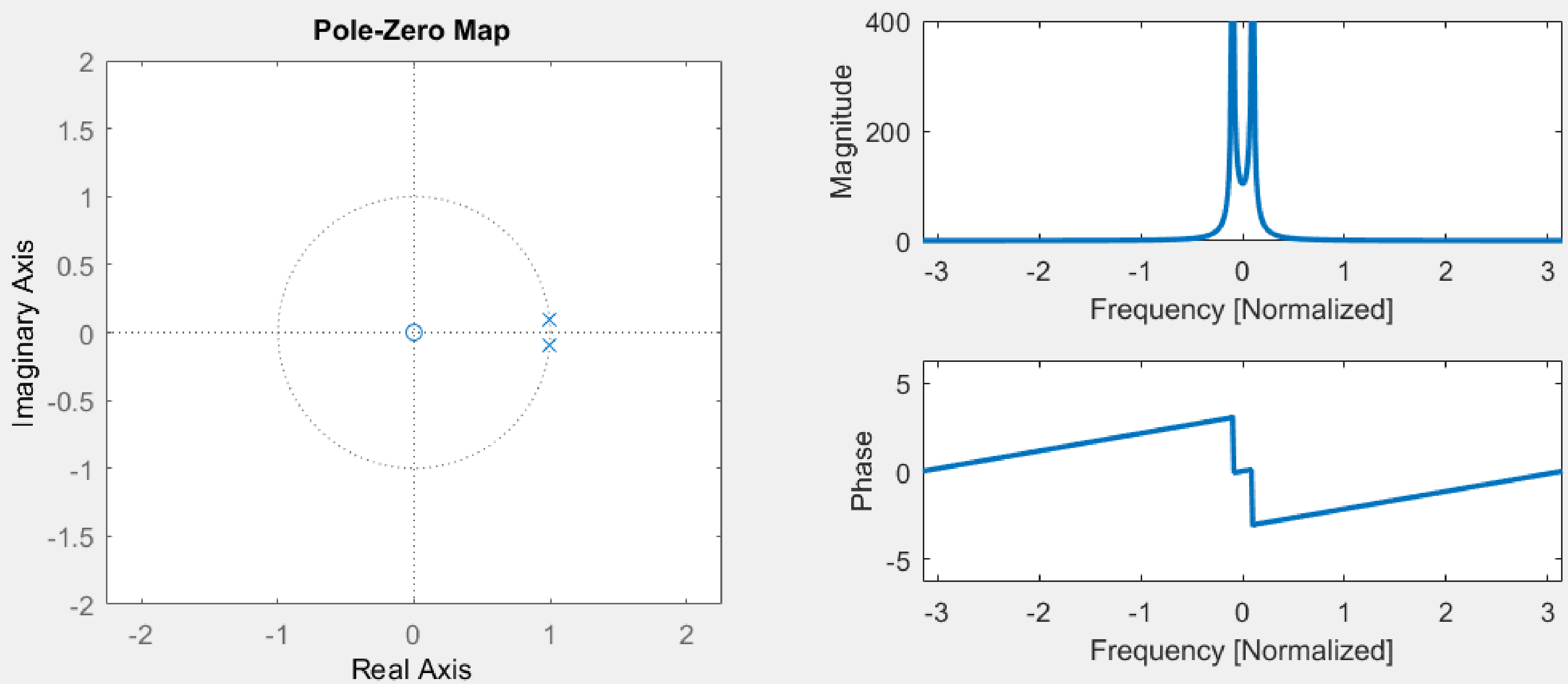


Designing the Magnitude Response

- **Question: What happens when we move poles and zeros around for a filter?**

What is the $H(z)$ corresponding to this?

$$H(z) = \frac{1}{1 - 2\cos(\phi)\, z^{-1} + z^{-2}}$$



Designing the Magnitude Response

- **Question: What happens when we move poles and zeros around for a filter?**

What is the $|H(\omega)|$ corresponding to this?

$$
\begin{aligned}
|H(\omega)| &= \left|\frac{1}{e^{+j\omega} - 2\cos(\phi) + e^{-j\omega}}\right| \\
&= \left|\frac{1}{e^{+j\omega} + e^{-j\omega} - 2\cos(\phi)}\right| \\
&= \left|\frac{1}{2\cos(\omega) - 2\cos(\phi)}\right| \\
&= \left|\frac{1/2}{\cos(\omega) - \cos(\phi)}\right|
\end{aligned}
$$

Designing the Magnitude Response

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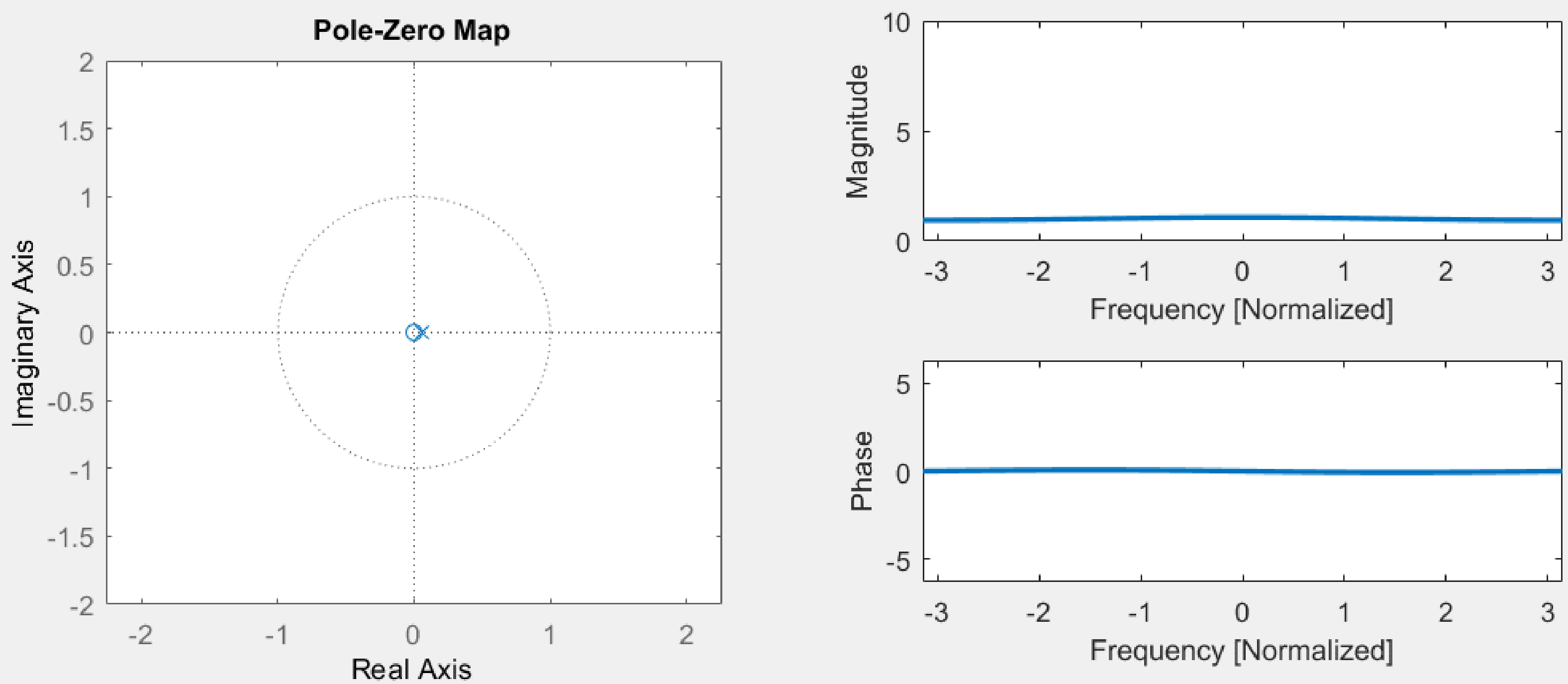
$$H(z) = \frac{\prod_{k=1}^{M}(1 - z_k z^{-1})}{\prod_{k=1}^{N}(1 - p_k z^{-1})}$$

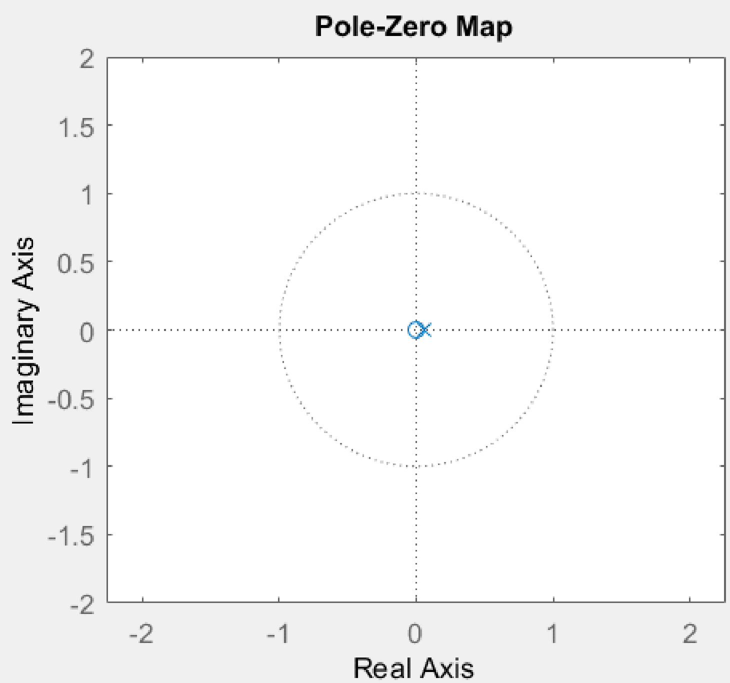


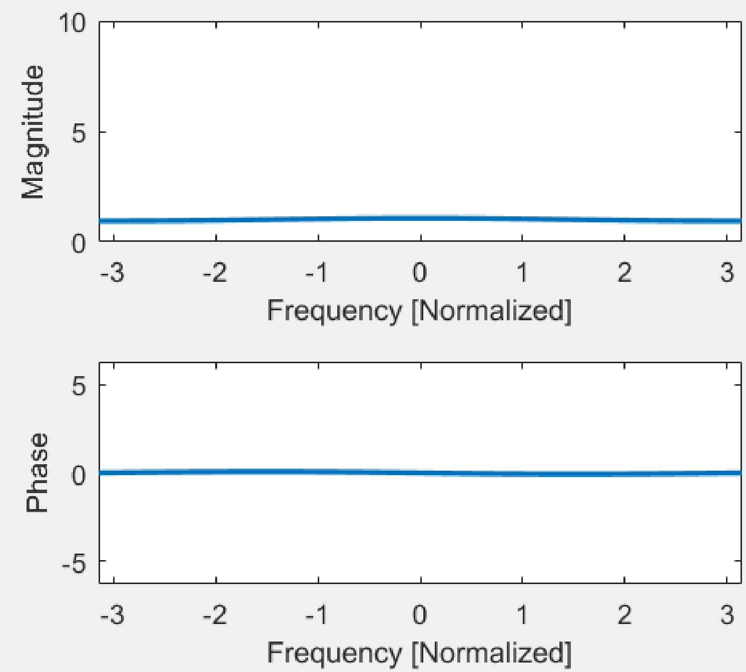
Designing the Magnitude Response

- **Question: What happens when we move poles and zeros around for a filter?**

What is the $H(z)$ corresponding to this?

$$H(z) = \frac{1}{(1 - az^{-1})}$$





Designing the Magnitude Response

- **Question: What happens when we move poles and zeros around for a filter?**

$$|H(\omega)| = \left|\frac{1}{(1 - ae^{-j\omega})}\right|$$

$$= \frac{1}{|1 - ae^{-j\omega}|}$$

$$= \frac{1}{\sqrt{(1 - a\cos(\omega))^2 + a^2\sin^2(\omega)}}$$

$$= \frac{1}{\sqrt{1 - 2\cos(\omega) + a^2\cos^2(\omega) + a^2\sin^2(\omega)}}$$

$$= \frac{1}{\sqrt{(1 + a^2) - 2\cos(\omega)}}$$

Designing the Magnitude Response

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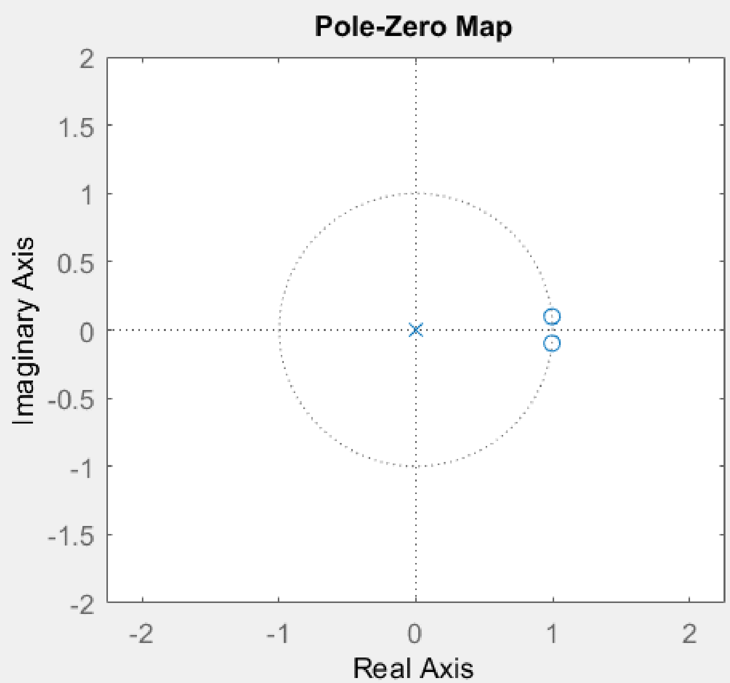
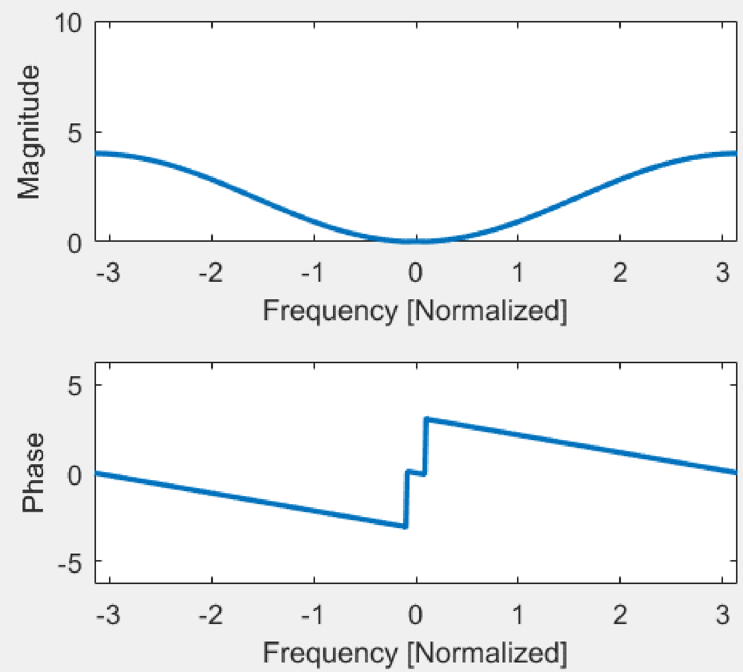
$$H(z) = \frac{\prod_{k=1}^{M}(1 - z_k z^{-1})}{\prod_{k=1}^{N}(1 - p_k z^{-1})}$$





Designing the Magnitude Response

- **Question: What happens when we move poles and zeros around for a filter?**

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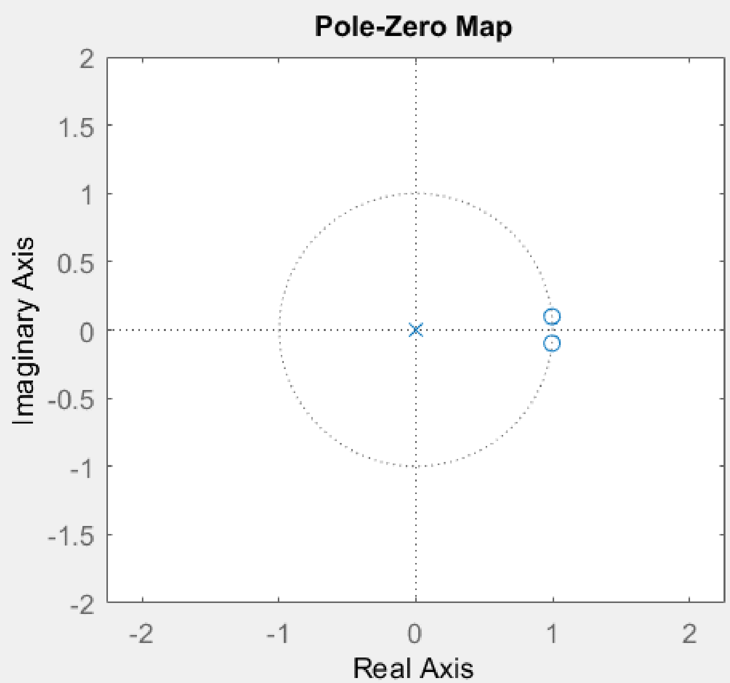
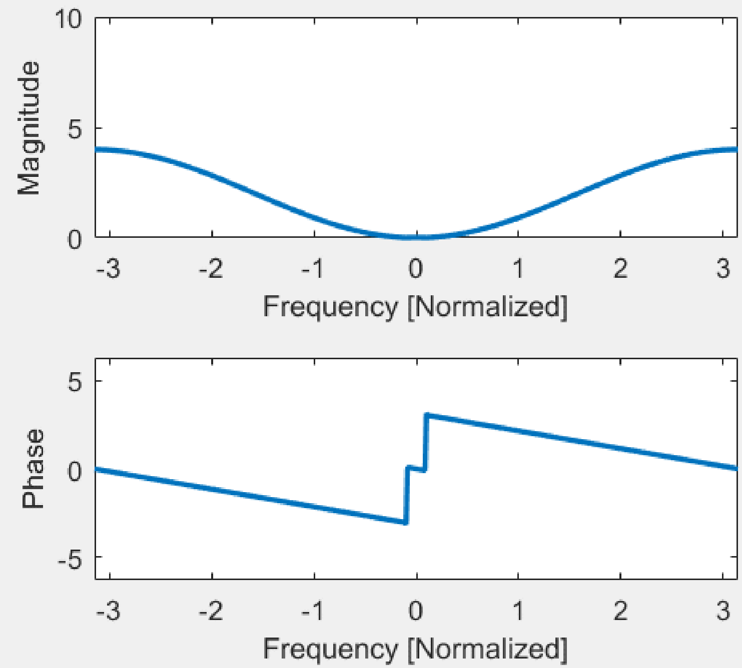
$$H(z) = \left(1 - e^{+j\phi} z^{-1}\right)\left(1 - e^{-j\phi} z^{-1}\right)$$





Designing the Magnitude Response

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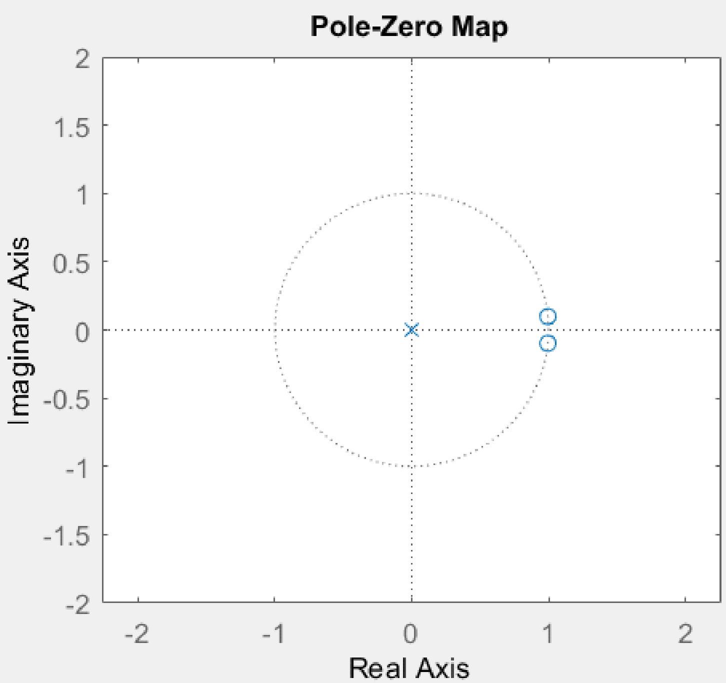
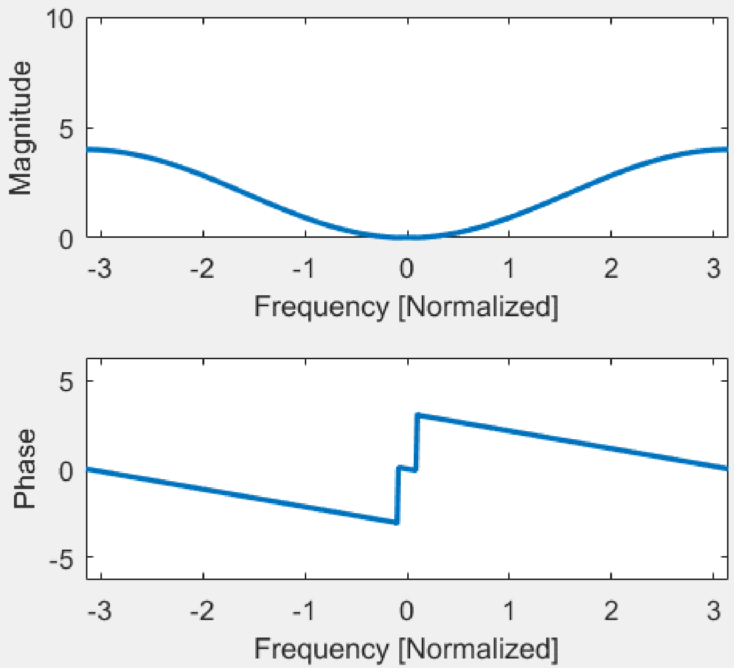
What is the $H(z)$ corresponding to this?

$$H(z) = 1 - 2\cos(\phi)\, z^{-1} + z^{-2}$$





Designing the Magnitude Response

- **Question: What happens when we move poles and zeros around for a filter?**

What is the $|H(\omega)|$ corresponding to this?

$$|H(\omega)| = 2|\cos(\omega) - \cos(\phi)|$$

Designing the Magnitude Response

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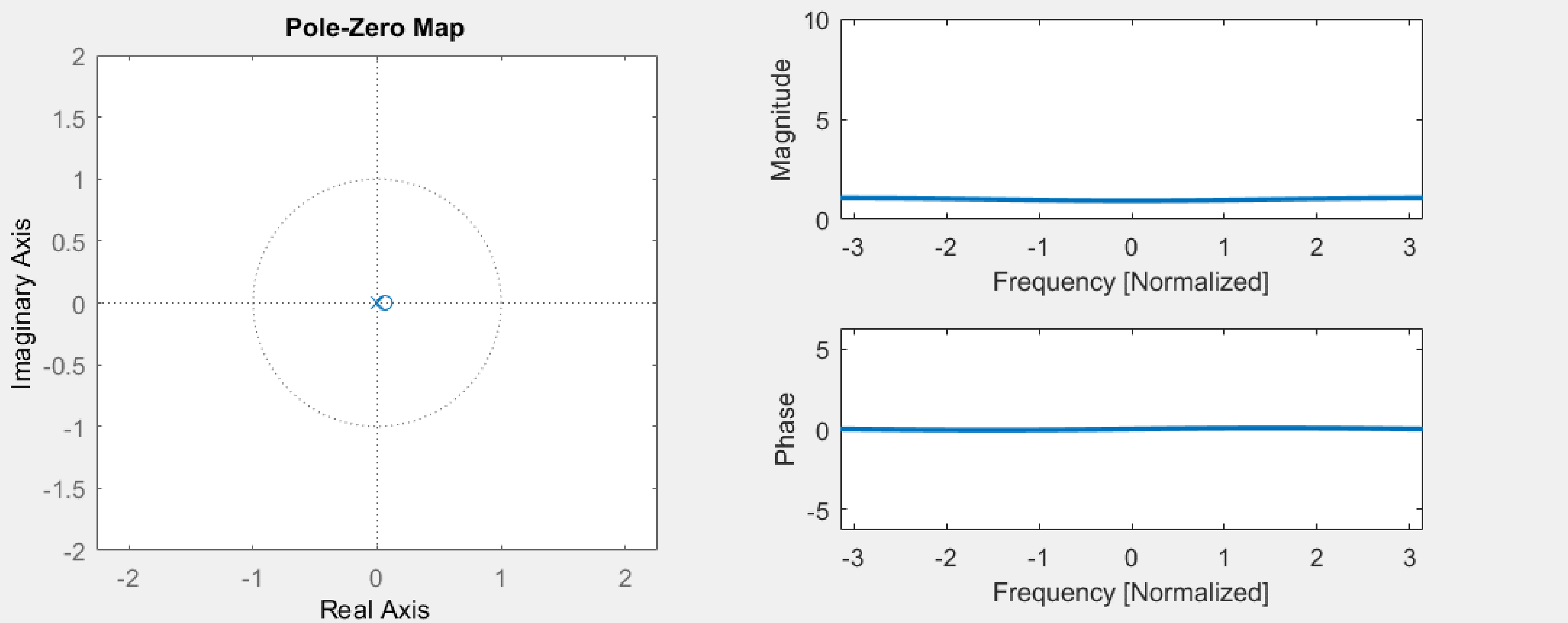
What is the $H(z)$ corresponding to this?

$$H(z) = \frac{\prod_{k=1}^{M}(1 - z_k z^{-1})}{\prod_{k=1}^{N}(1 - p_k z^{-1})}$$



Designing the Magnitude Response

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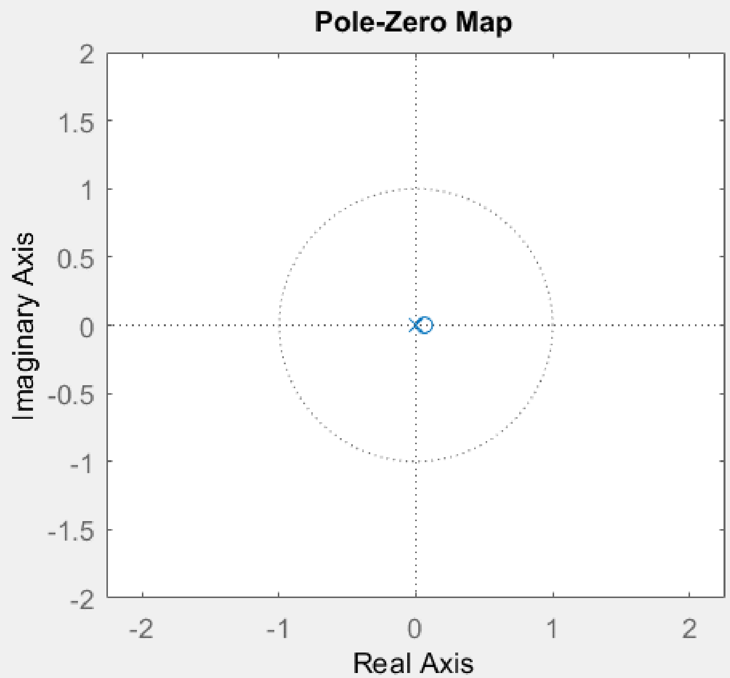
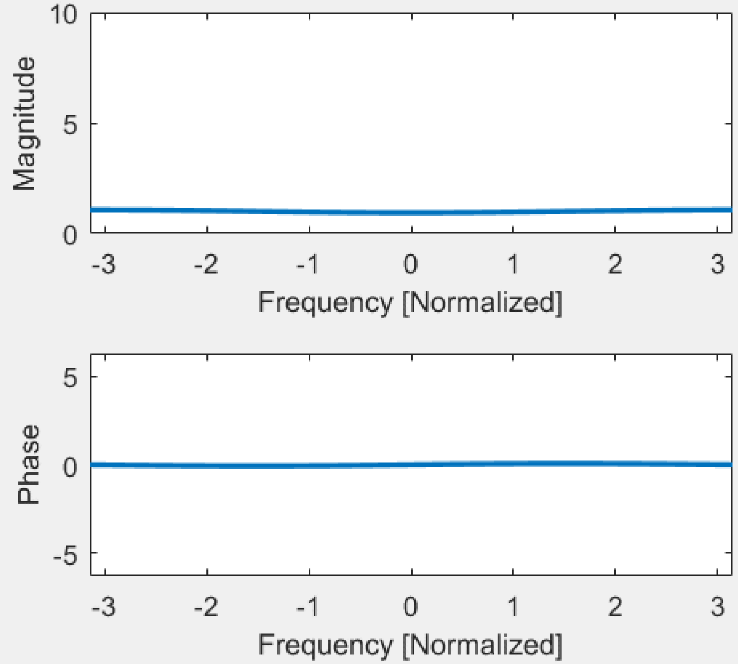
What is the $H(z)$ corresponding to this?

$$H(z) = 1 - ae^{-j\omega}$$





Designing the Magnitude Response

- **Question: What happens when we move poles and zeros around for a filter?**

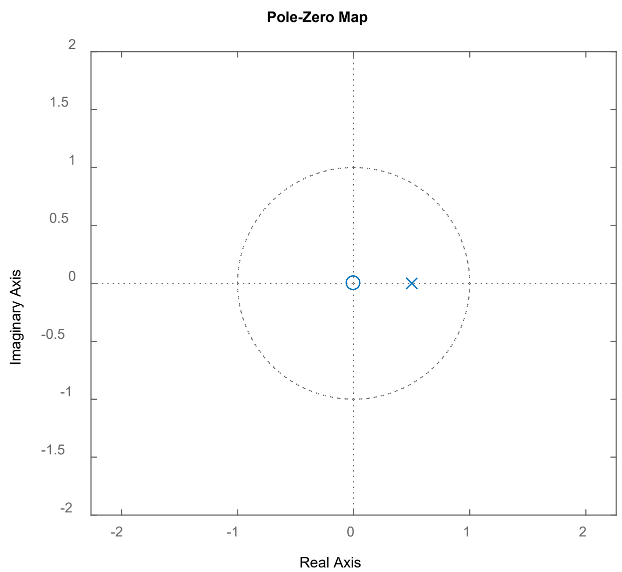
What is the $|H(\omega)|$ corresponding to this?

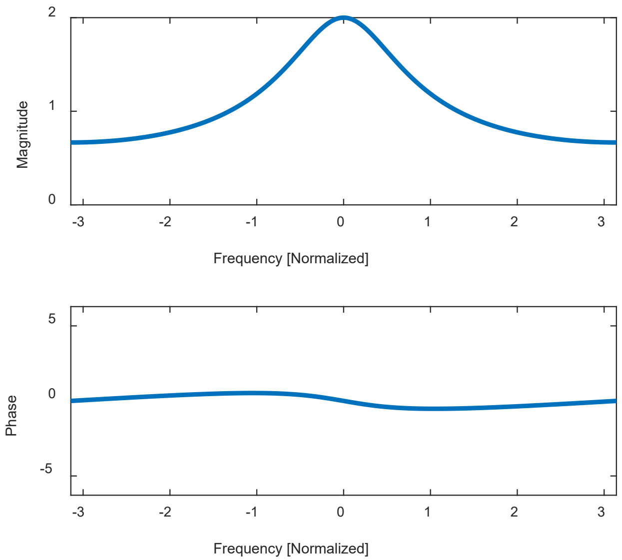
$$|H(\omega)| = \sqrt{(1 + a^2) - 2\cos(\omega)}$$

Designing the Magnitude Response

- **Question: How can I make the high frequencies closer to zero?**

$$H(z) = \frac{1}{(1 - (1/2)z^{-1})}$$



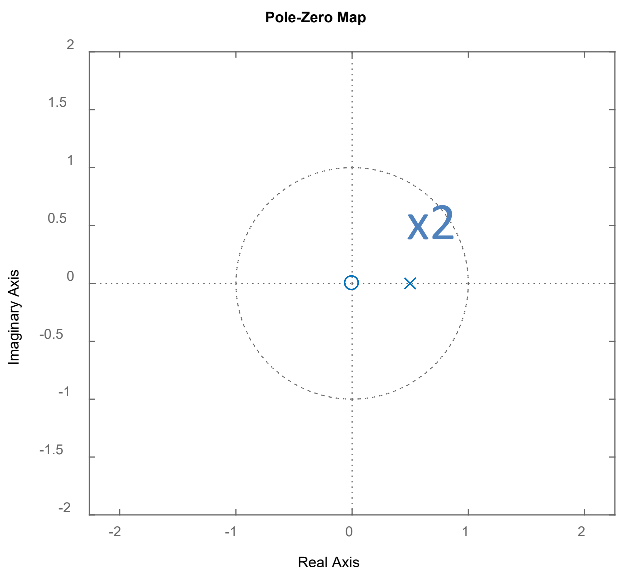


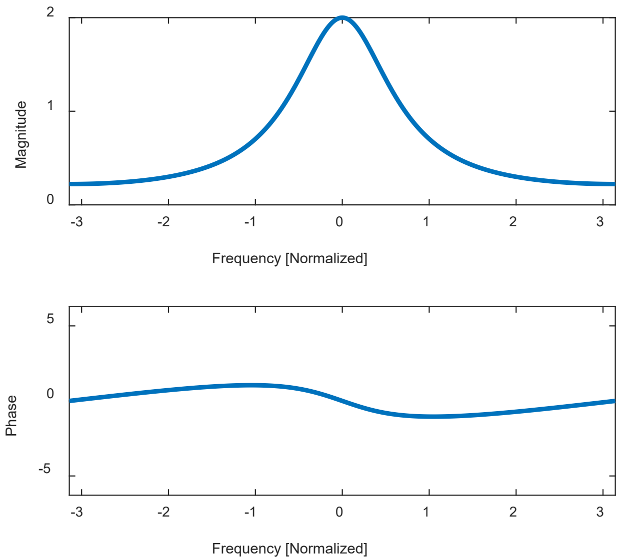
Designing the Magnitude Response

- **Question: How can I make the high frequencies closer to zero?**

Option:
Add poles

$$H(z) = \frac{1/2}{(1-(1/2)z^{-1})^2}$$



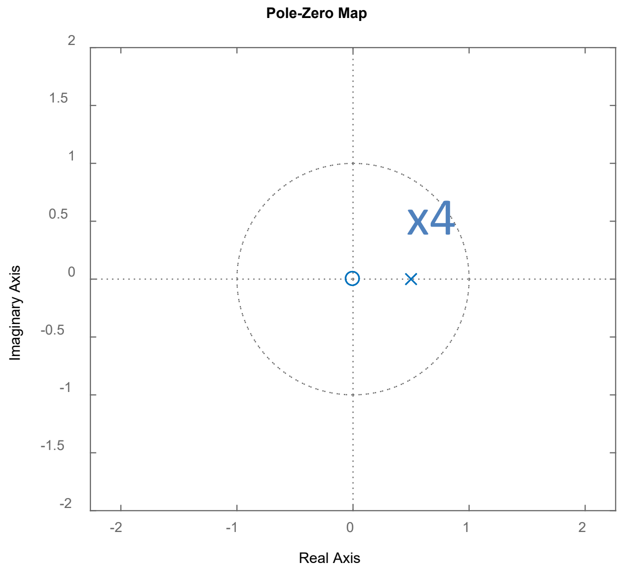


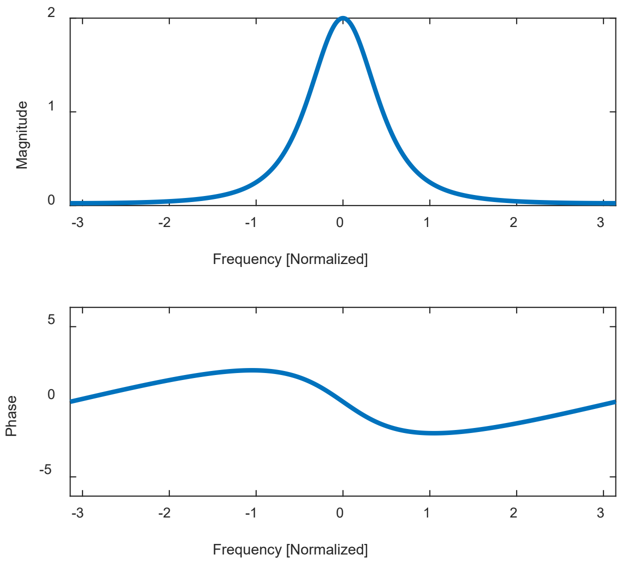
Designing the Magnitude Response

- **Question: How can I make the high frequencies closer to zero?**

Option:
Add poles

$$H(z) = \frac{1/8}{(1-(1/2)z^{-1})^4}$$





Designing the Magnitude Response

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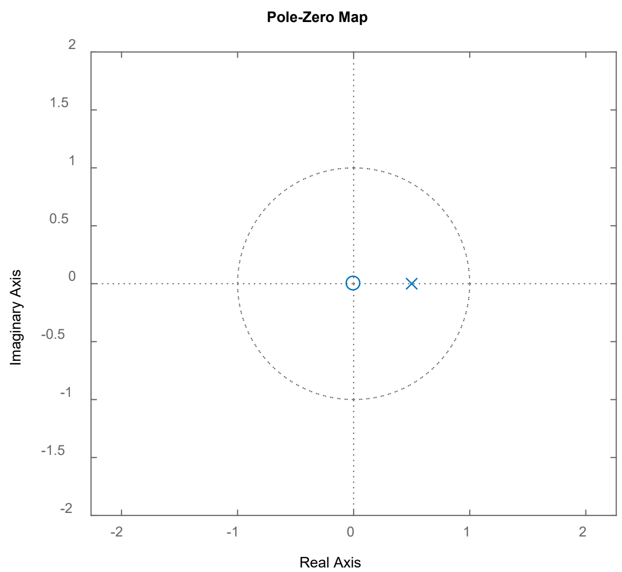

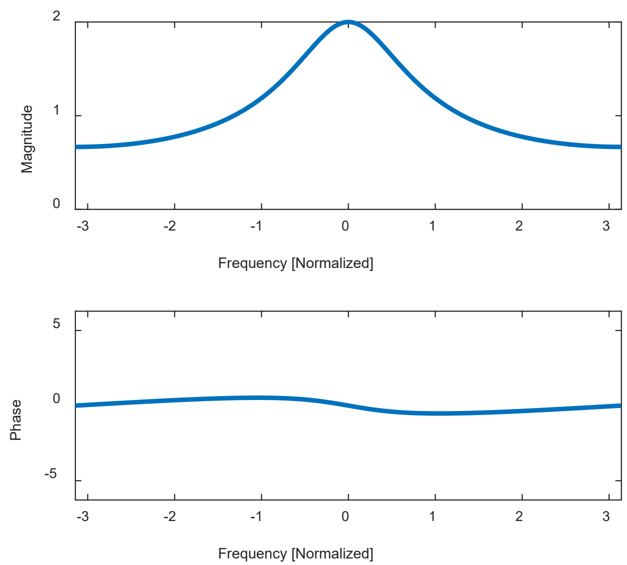

Designing the Magnitude Response

- **Question: How can I make the high frequencies closer to zero?**

Option:
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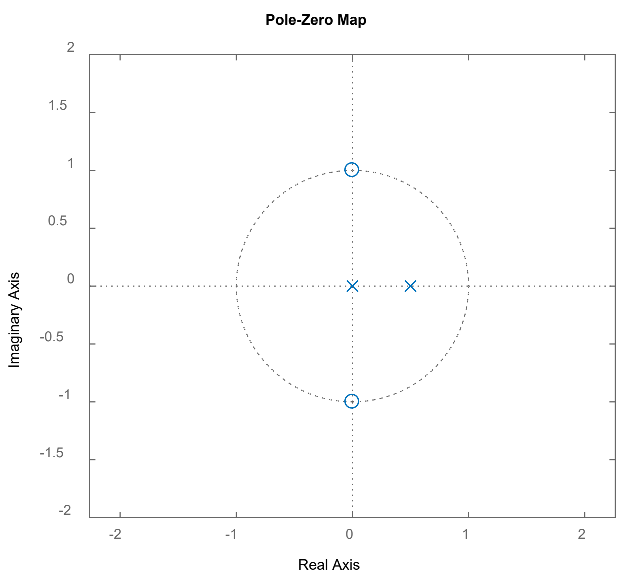
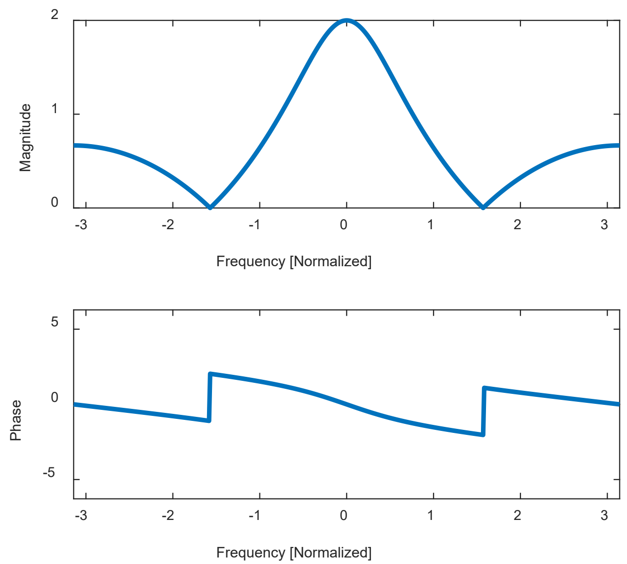
$$H(z) = \frac{(1 - jz^{-1})(1 + jz^{-1})}{2(1 - (1/2)z^{-1})}$$



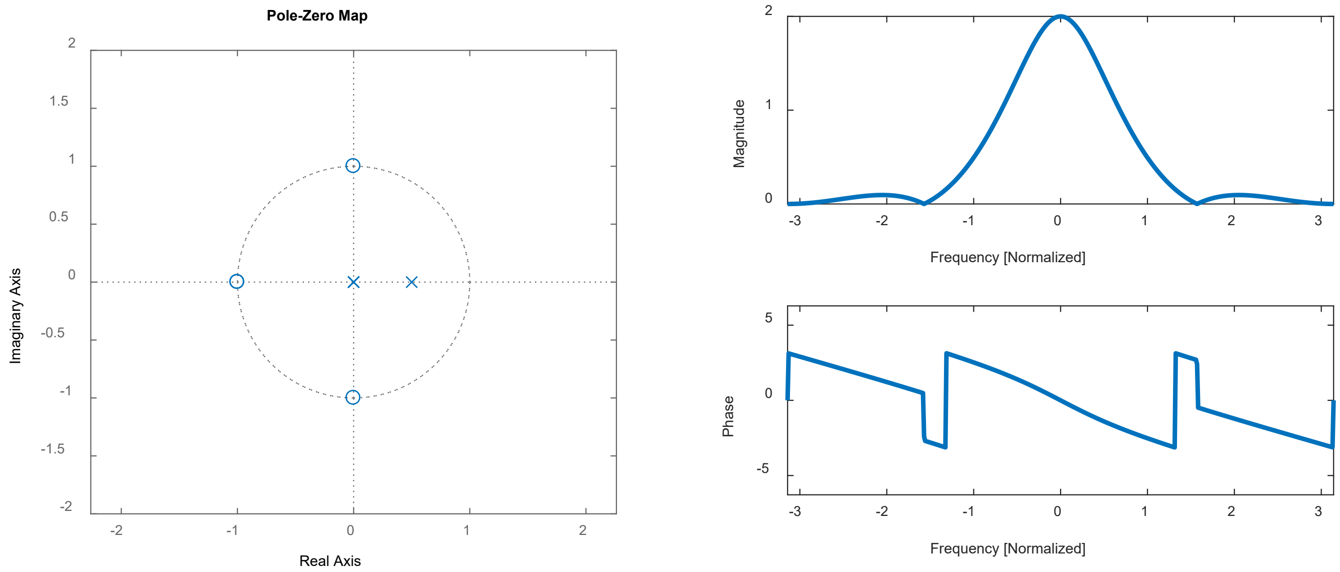


Designing the Magnitude Response

- **Question: How can I make the high frequencies closer to zero?**

Option:
Add zeros

$$H(z) = \frac{(1 - jz^{-1})(1 + jz^{-1})(1 + z^{-1})}{8(1 - (1/2)z^{-1})}$$



Designing the Magnitude Response

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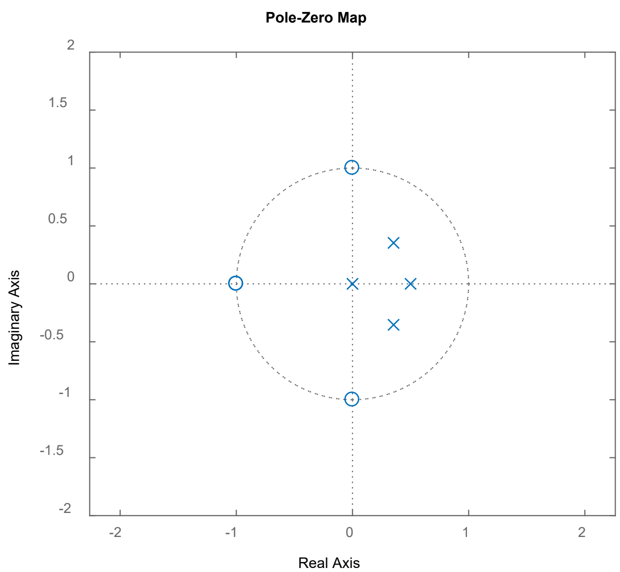

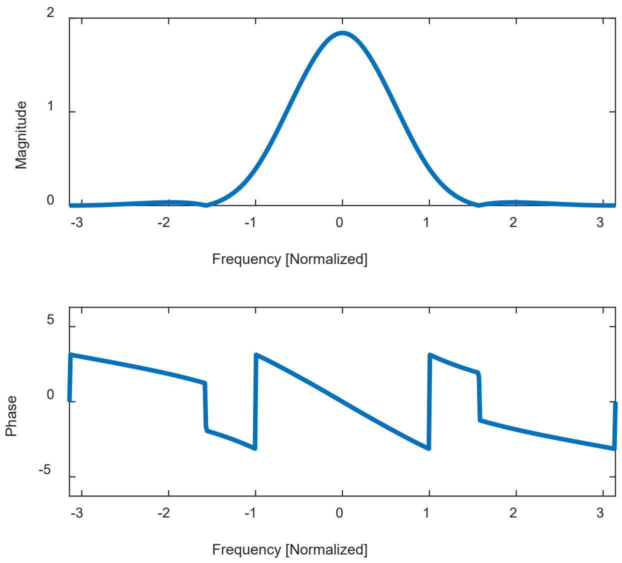

Designing the Magnitude Response

- **Example: Consider the FIR filter described by:**
  - $y[n] = x[n] - x[n-6]$

- **Determine the pole-zero plot of the filter.**

Designing the Magnitude Response

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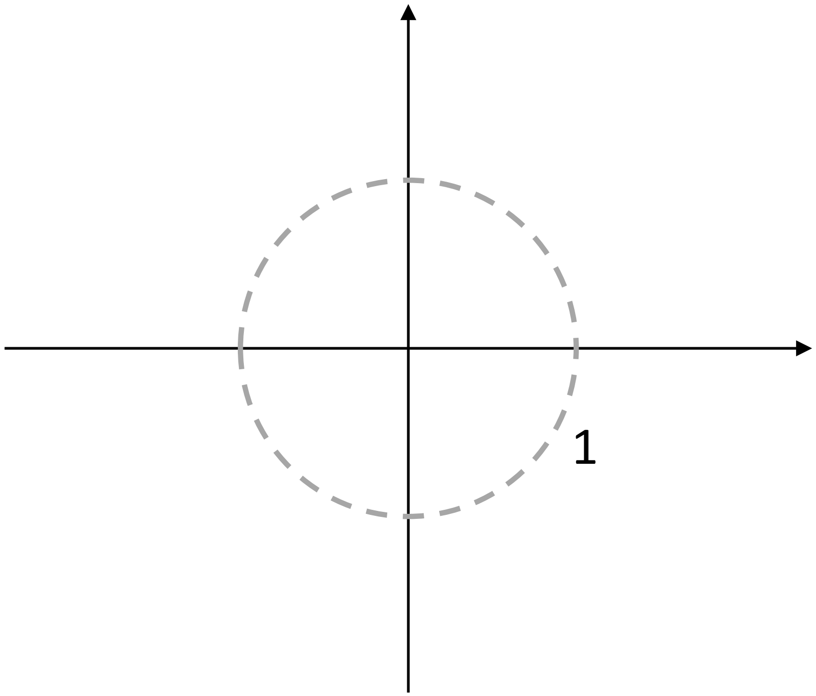
- **Determine the pole-zero plot of the filter.**

$$H(z) = z^{-1} - z^{-6}$$

$$= \frac{z^6 - 1}{z^6}$$



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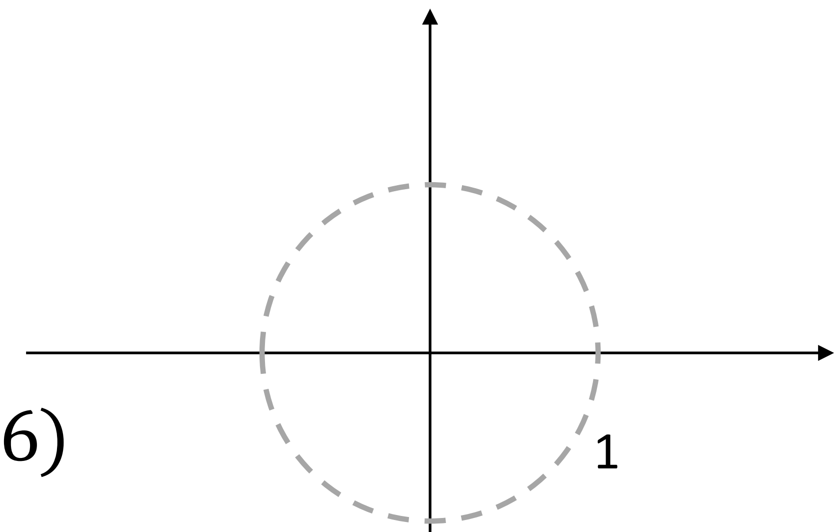

Poles: $z \ = \ 0,0,0,0,0,0 \ (0x6)$

Zeros: $z \ = e^{j\frac{2\pi}{6}0}, e^{j\frac{2\pi}{6}1}, e^{j\frac{2\pi}{5}2}, e^{j\frac{2\pi}{6}3}, e^{j\frac{2\pi}{6}4}, e^{j\frac{2\pi}{6}5}$

Designing the Magnitude Response

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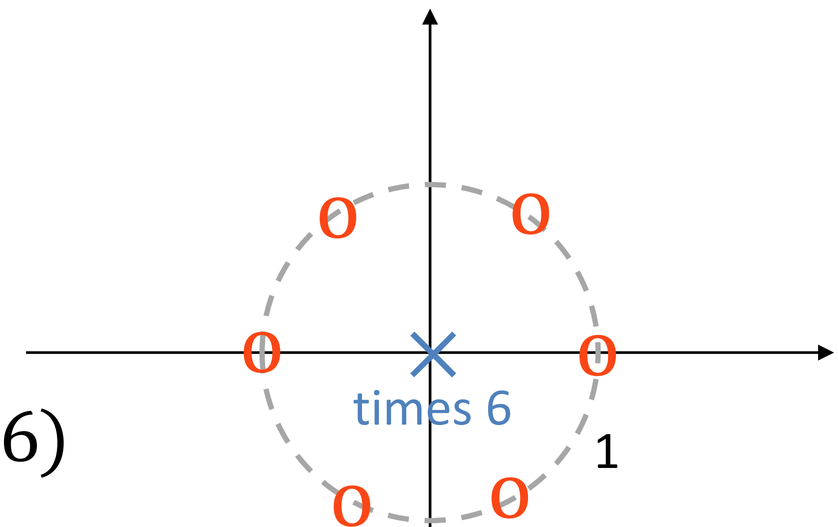
$$= \frac{z^5 - 1}{z^6}$$



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Designing the Magnitude Response

- **Example: Consider the FIR filter described by:**
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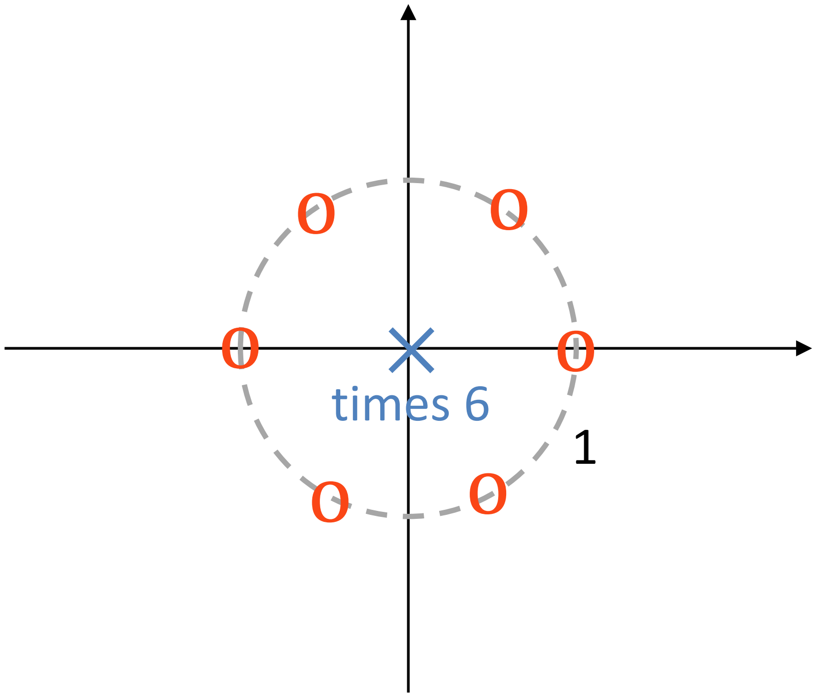
- **Roughly sketch the magnitude response based on pole-zero plot.**



Designing the Magnitude Response

- **Example: Consider the FIR filter described by:**
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- **Sketch the magnitude and phase response of the system.**

Designing the Magnitude Response

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  - $y[n] = x[n] - x[n-6]$
- **Sketch the magnitude and phase response of the system.**

$$H(\omega) = 1 - e^{-j\omega 6}$$

$$|H(\omega)| = \left|1 - e^{-j\omega 6}\right| = \left|1 - \left(\cos(6\omega) - j\sin(6\omega)\right)\right|$$

$$= \sqrt{(1 - \cos(6\omega))^2 + \sin^2(6\omega)}$$

$$= \sqrt{1 - 2\cos(6\omega) + \cos^2(6\omega) + \sin^2(6\omega)}$$

$$= \sqrt{2 - 2\cos(6\omega)}$$

Designing the Magnitude Response

- **Example: Consider the FIR filter described by:**
  - $y[n] = x[n] - x[n-6]$
- **Sketch the magnitude and phase response of the system.**

$$H(\omega) = 1 - e^{-j\omega 6} = e^{-j\omega 3}\left[e^{+j\omega 3} - e^{-j\omega 3}\right]$$

$$= 2je^{-j\omega 3}\left(\frac{1}{2j}\left[e^{+j\omega 3} - e^{-j\omega 3}\right]\right) = 2je^{-j\omega 3}\sin(3\omega)$$

$$|H(\omega)| = |2\sin(3\omega)|$$

Note – trig identify: $\sin^2(3\omega) = \frac{1-\cos(6\omega)}{2}$

Designing the Magnitude Response

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$$|H(\omega)| = |2\sin(3\omega)| = 2\sqrt{\frac{1-\cos(6\omega)}{2}} = \sqrt{2 - 2\cos(6\omega)}$$

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$$\angle H(\omega) = \angle je^{-j\omega 3}\sin(3\omega) = \angle e^{j[(\pi/2)-3\omega]}\sin(3\omega)$$

$$= \begin{cases} \pi/2 - 3\omega & \text{when} \quad \sin(3\omega) \geq 0 \\ -\pi/2 - 3\omega & \text{when} \quad \sin(3\omega) < 0 \end{cases}$$

Designing the Magnitude Response

- **Example: Consider the FIR filter described by:**
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$$H(\omega) = \begin{cases} |2\sin(3\omega)|e^{j[\pi/2 - 3\omega]} & \text{when} \quad \sin(3\omega) \geq 0 \\ |2\sin(3\omega)|e^{j[-\pi/2 - 3\omega]} & \text{when} \quad \sin(3\omega) < 0 \end{cases}$$

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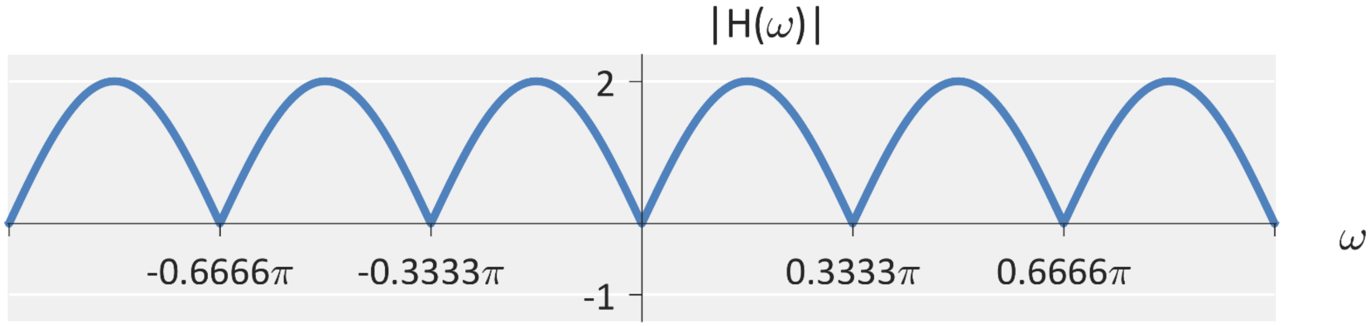

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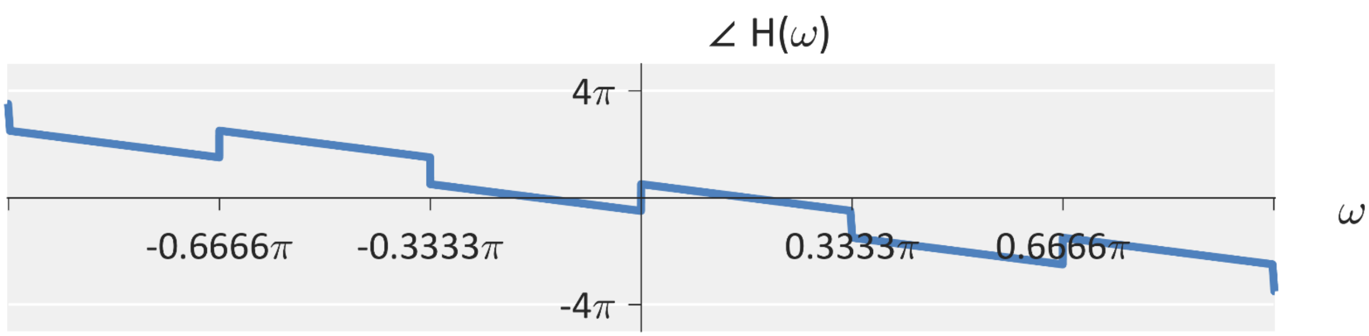
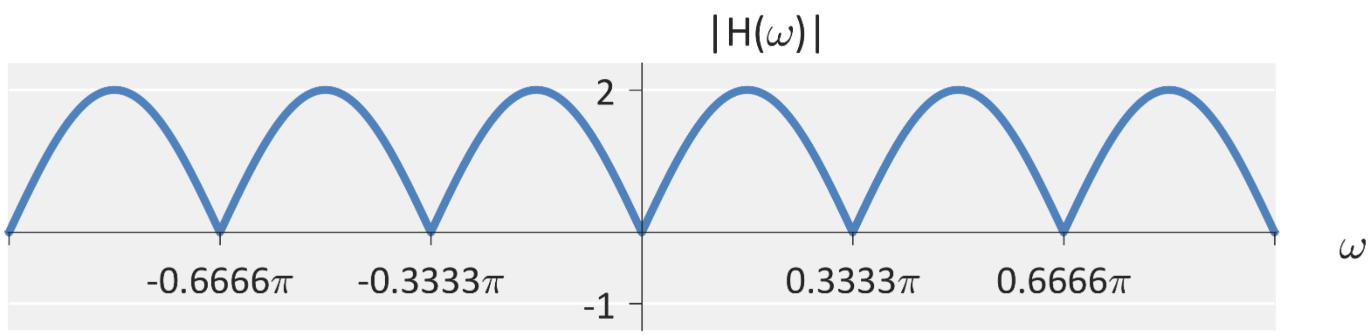


Designing the Magnitude Response

- **Example: Consider the FIR filter described by:**
  - $y[n] = x[n] - x[n-6]$

- **Determine the response to the input**

$$x[n] = \cos\left(\frac{\pi}{9}n\right) + 3\sin\left(\frac{\pi}{3}n + \frac{\pi}{10}\right)$$



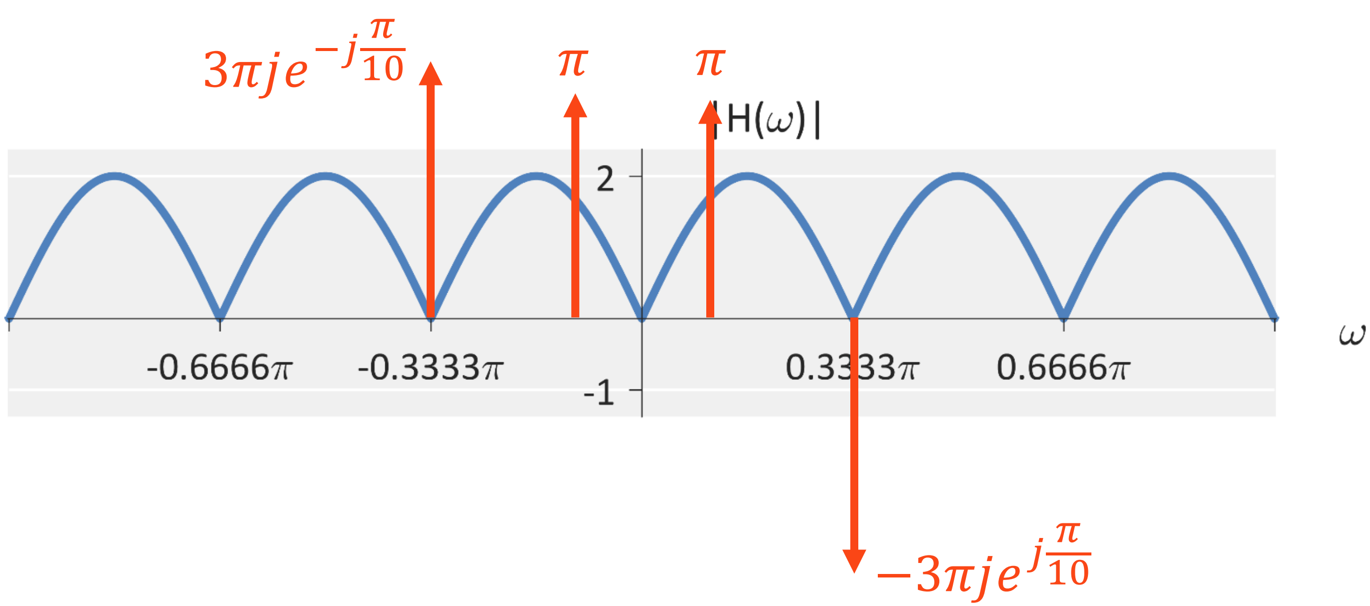
Designing the Magnitude Response

- **Example: Consider the FIR filter described by:**
  - $y[n] = x[n] - x[n-6]$

$$\frac{3}{2}j\left[e^{j\left(\frac{\pi}{3}n+\frac{\pi}{10}\right)} - e^{-j\left(\frac{\pi}{3}n+\frac{\pi}{10}\right)}\right]$$

- **Determine the response to the input**

$$x[n] = \cos\left(\frac{\pi}{9}n\right) + 3\sin\left(\frac{\pi}{3}n + \frac{\pi}{10}\right)$$



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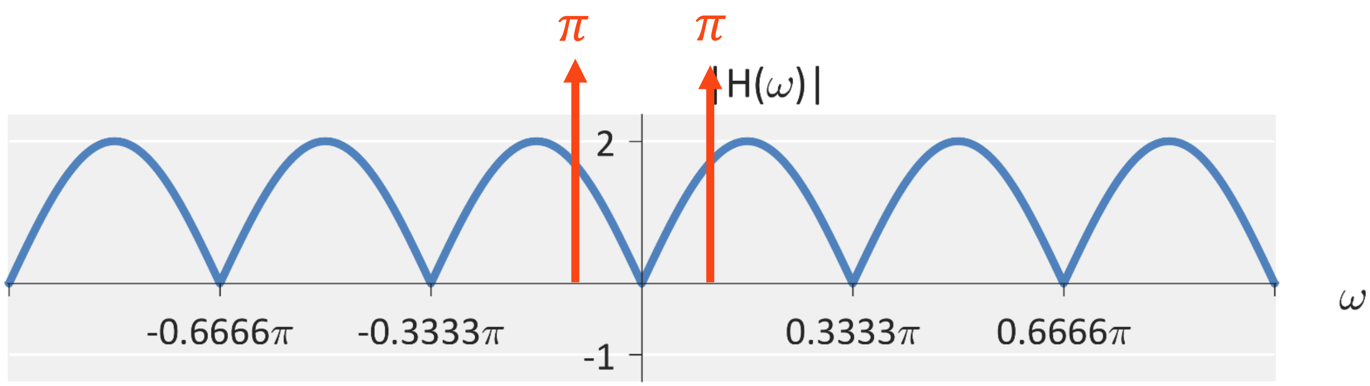

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$$\pi|2\sin(3\pi/9)|e^{j[-\pi/2-3\omega]}$$
$$= \pi|2\sin(\pi/3)|e^{j[-\pi/2+\pi/3]}$$
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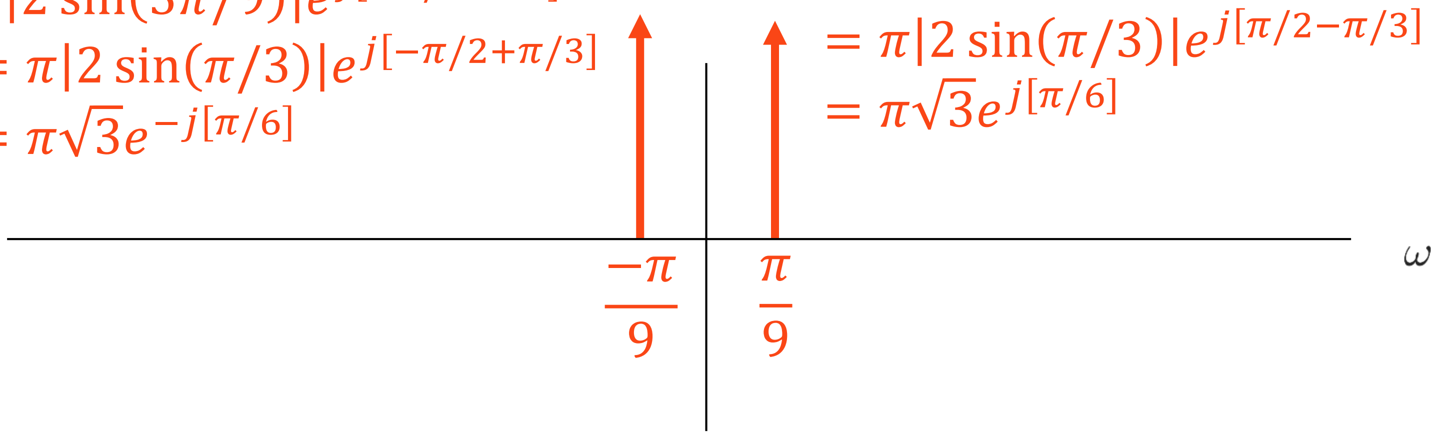

Designing the Magnitude Response

- **Example: Consider the FIR filter described by:**
  - $y[n] = x[n] - x[n-6]$
- **Determine the response to the input**

$$y[n] = \sqrt{3}\cos\left(\frac{\pi}{9}n + \frac{\pi}{6}\right)$$

$$\pi|2\sin(3\pi/9)|e^{j[-\pi/2-3\omega]}$$
$$= \pi|2\sin(\pi/3)|e^{j[-\pi/2+\pi/3]}$$
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