Lecture 22: Design of FIR / IIR Filters

Foundations of Digital Signal Processing

Outline

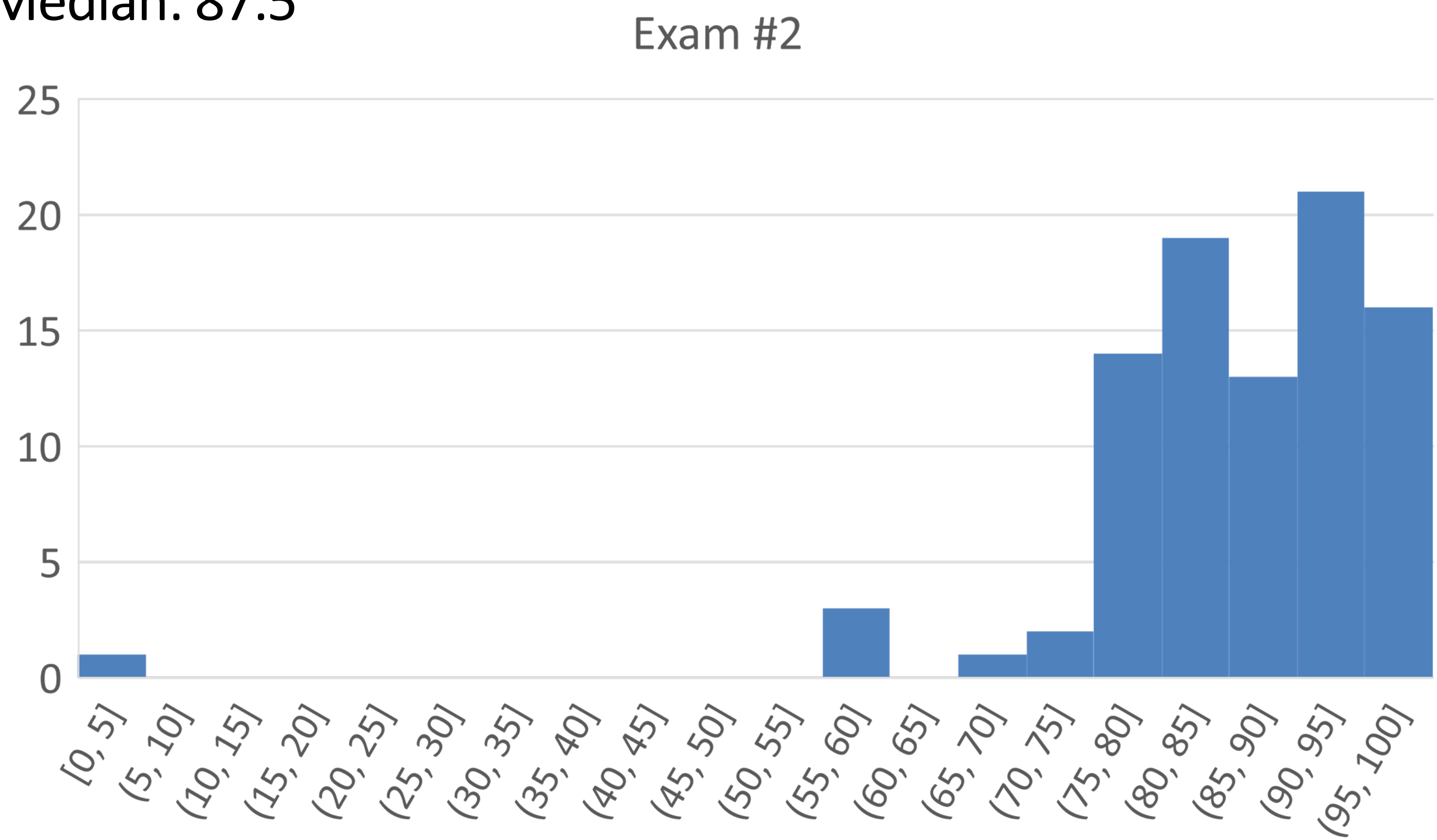
- Designing FIR Filters with Windows
- Designing FIR Filters with Frequency Selection
- Designing FIR Filters with Equi-ripples
- Designing IIR Filters with Discrete Differentiation
- Designing IIR Filters with Impulse Invariance
- Designing IIR Filters with the Bilinear Transform
- Related Analog Filters

News

- **Homework #9**
  - Due *on Thursday*
  - Submit via canvas
- **Coding Assignment #6**
  - Due *on next Monday*
  - Submit via canvas

News

- **Exam #2 – Great Job!**
  - Mean: 86.3
  - Median: 87.5



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Causality and Linear Phase

- **Question: Consider a length-M symmetric, causal filter. What condition must be satisfied?**

Causality and Linear Phase

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- $x[n] = \pm x[-n + (N-1)] = \pm x[N-1-n]$
  - **Positive:** Even symmetry
  - **Negative:** Odd symmetry

Causality and Linear Phase

- **Question: Consider a length-M symmetric, causal filter. What is the phase response?**

Causality and Linear Phase

- **Question: Consider a length-M symmetric, causal filter. What is the phase response? Assume M is even.**

- **Even Symmetry**

$$X(z) = a_0 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_1 z^{-(M-2)} + a_0 z^{-(M-1)}$$

- **Odd Symmetry**

$$X(z) = a_0 + a_1 z^{-1} + a_2 z^{-2} + \cdots - a_1 z^{-(M-2)} - a_0 z^{-(M-1)}$$

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$$G(\omega) = |X(\omega)| e^{j\Theta(\omega)}$$

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$$= z^{-\frac{(M-1)}{2}} \sum_{k=0}^{M/2-1} a_k \left[z^{\frac{(M-1)}{2}-k} + z^{-\left[\frac{(M-1)}{2}-k\right]} \right]$$

Causality and Linear Phase

- **Question: Consider a length-M symmetric, causal filter. What is the phase response? Assume M is even.**

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- **Notice that**

$$X(z) = z^{-(M-1)} X(z^{-1})$$

Causality and Linear Phase

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- **Pole-zero plot property?**

$$X(z) = z^{-(M-1)} X(z^{-1})$$

Causality and Linear Phase

- **Question:** **Consider a length-M symmetric, causal filter. What is the phase response?** **Assume M is even.**

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$$X(z) = a_0 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_1 z^{-(M-2)} + a_0 z^{-(M-1)}$$

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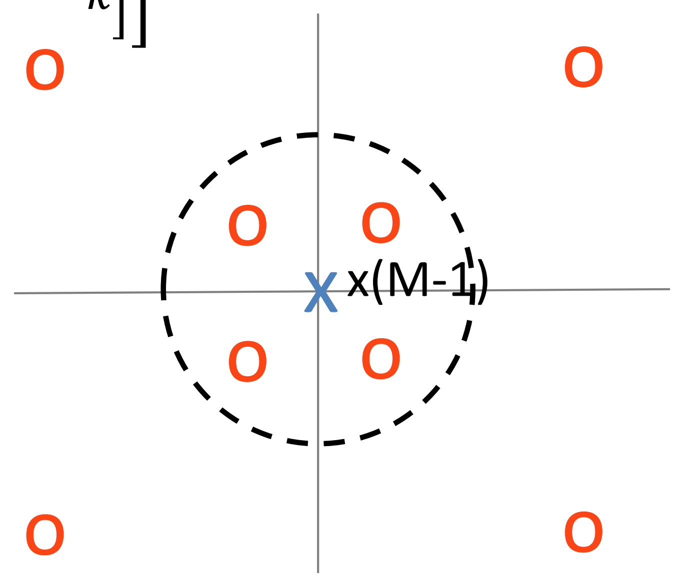
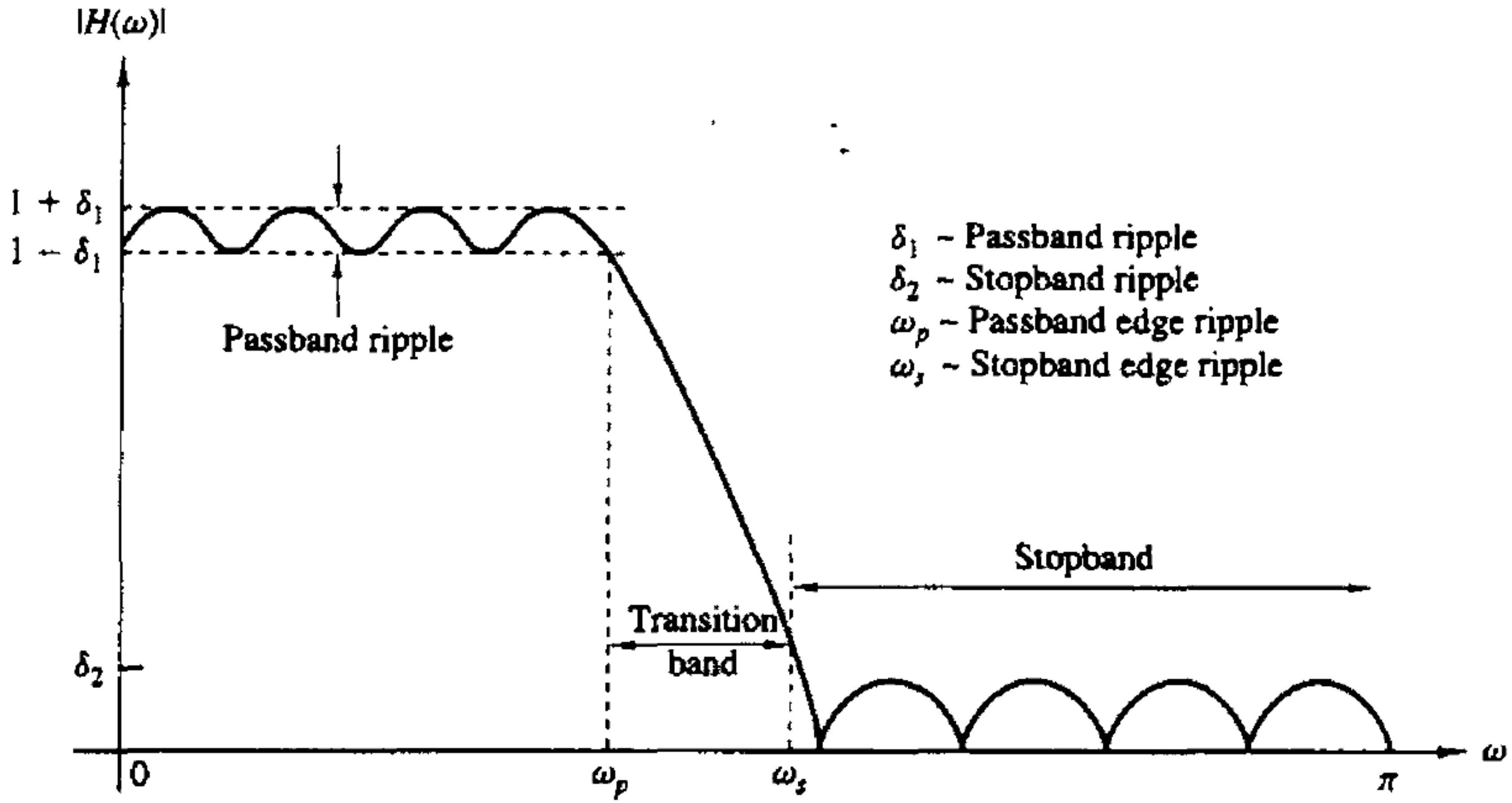


- **Pole-zero plot property?**

$$X(z) = z^{-(M-1)} X(z^{-1})$$

Causality

- **Question: How do we describe causal filter magnitude?**



Causality

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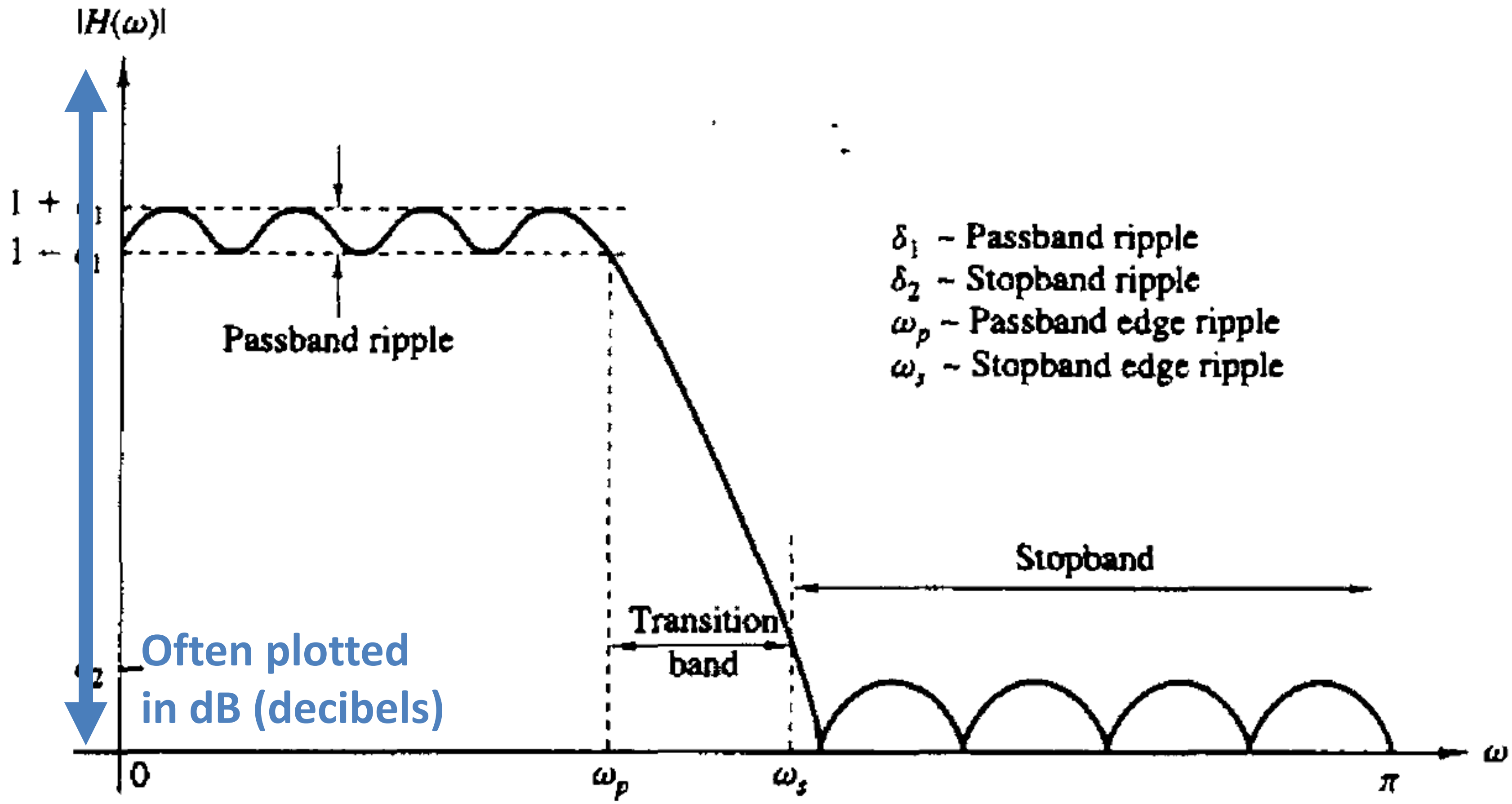

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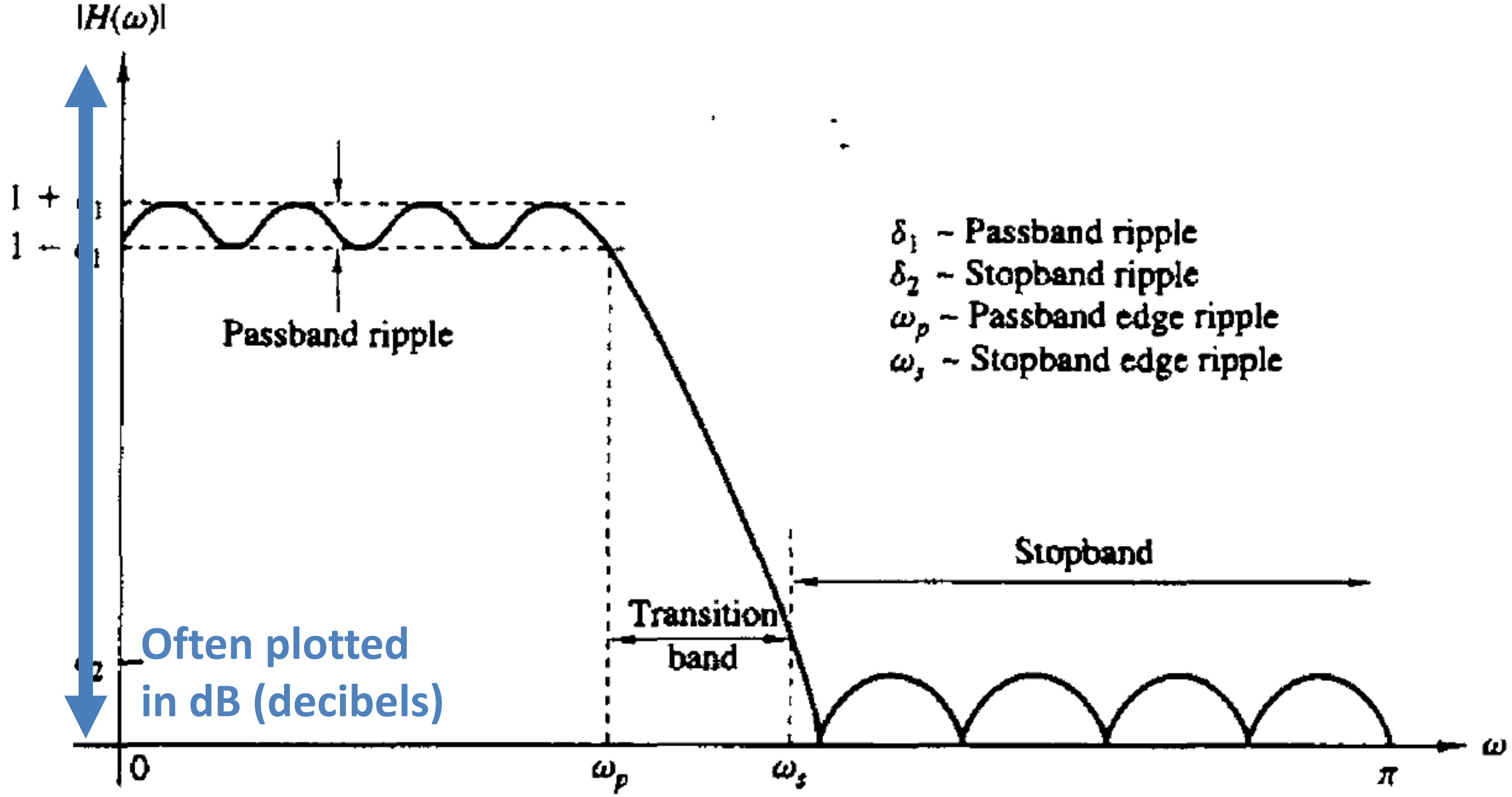


Lecture 21: Design of FIR Filters

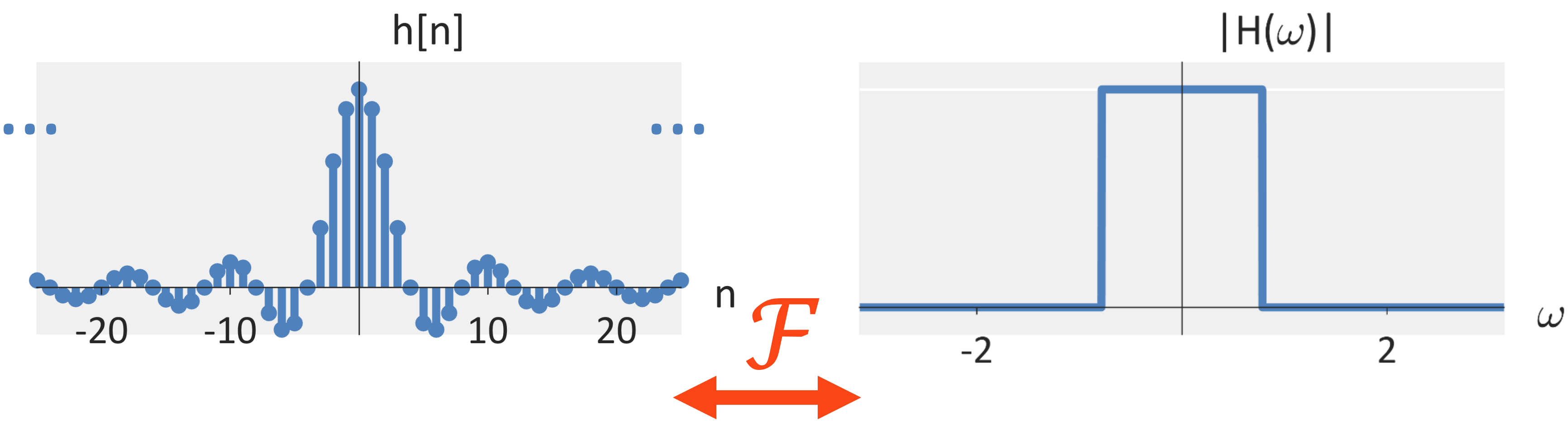
Foundations of Digital Signal Processing

Outline

- Review Downsampling & Upsampling
- Causality in Filters
- **Designing FIR Filters with Windows**
- Designing FIR Filters with Frequency Selection
- Designing FIR Filters with Equi-ripples

Designing with Windows

- **Question: How can I design an FIR filter from an ideal filter?**



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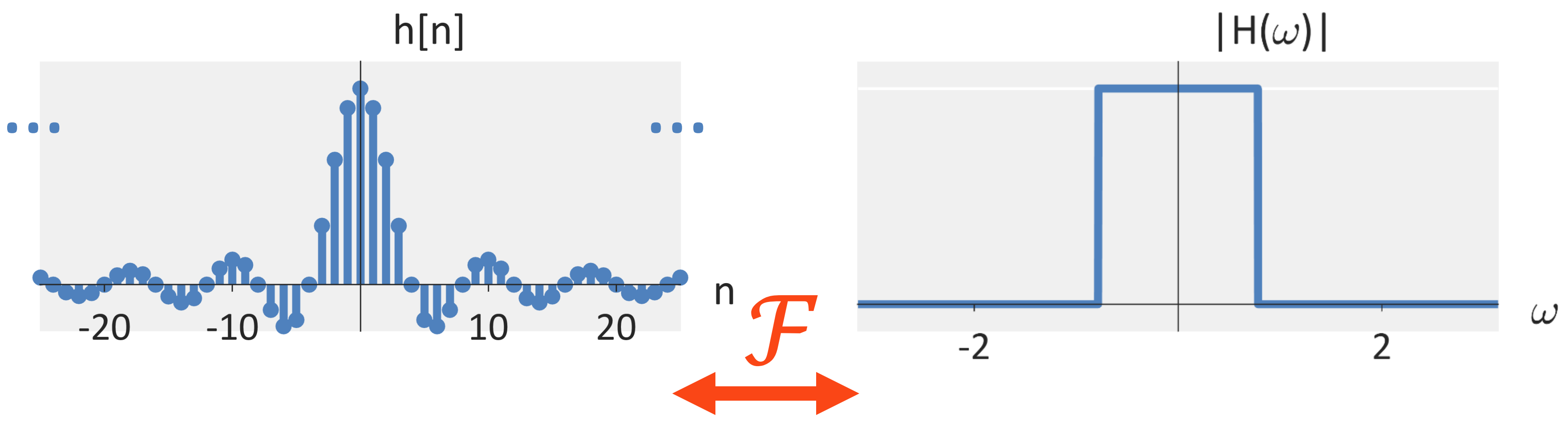

Non-causal
Infinite Response

Ideal filter

Designing with Windows

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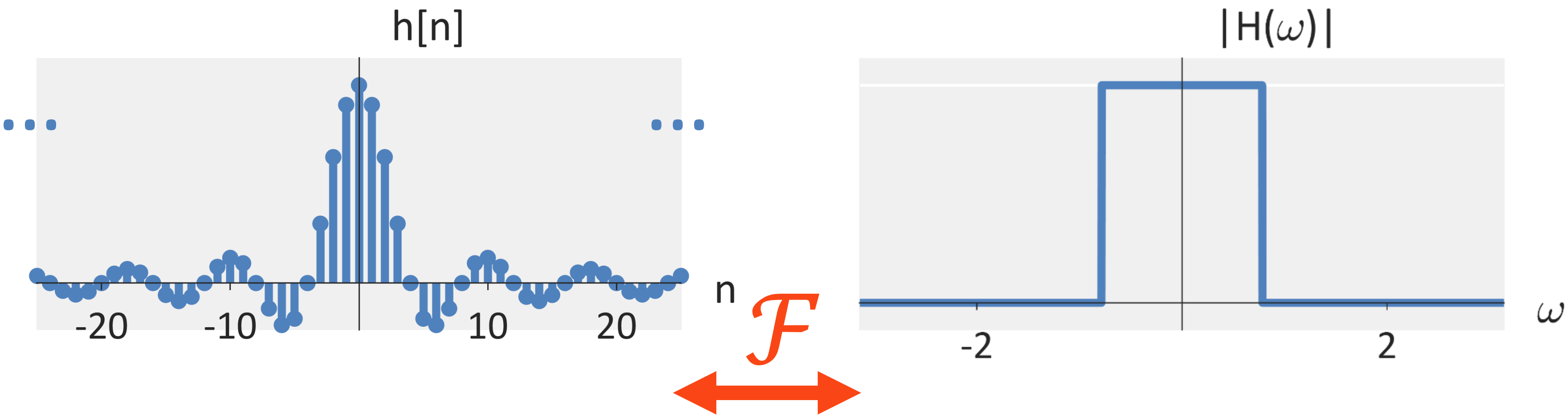

Non-causal
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- **Answer: Window the response!**

Designing with Windows

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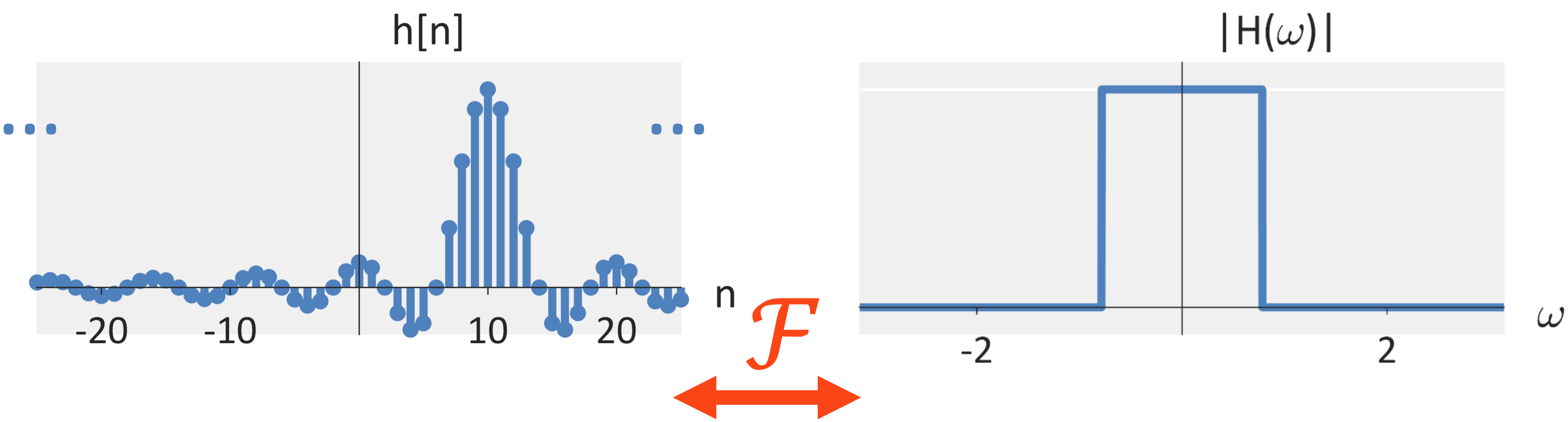

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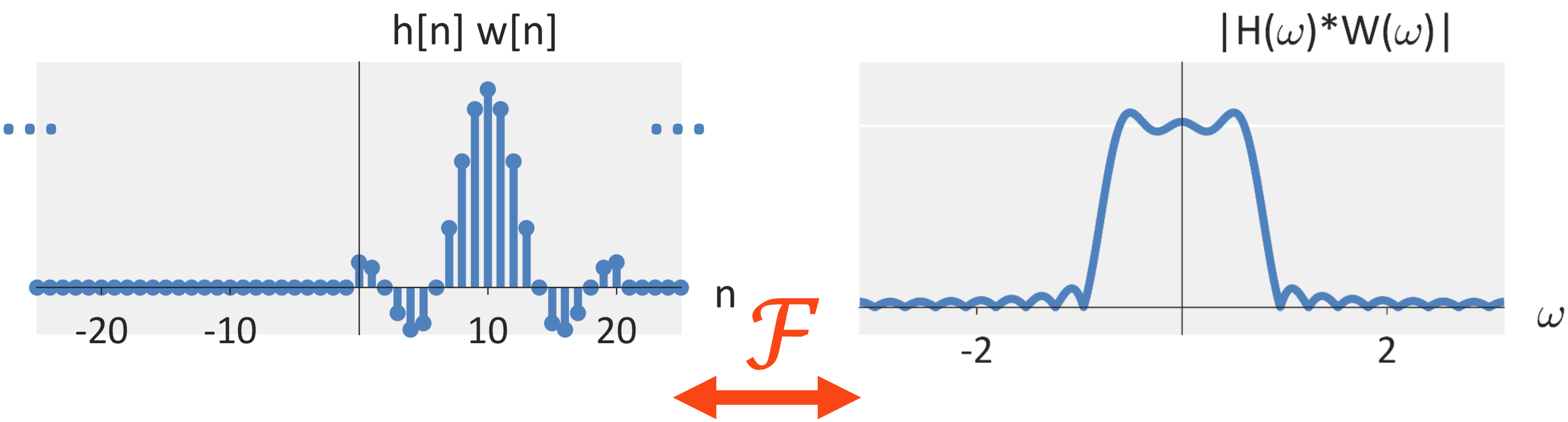

Non-causal
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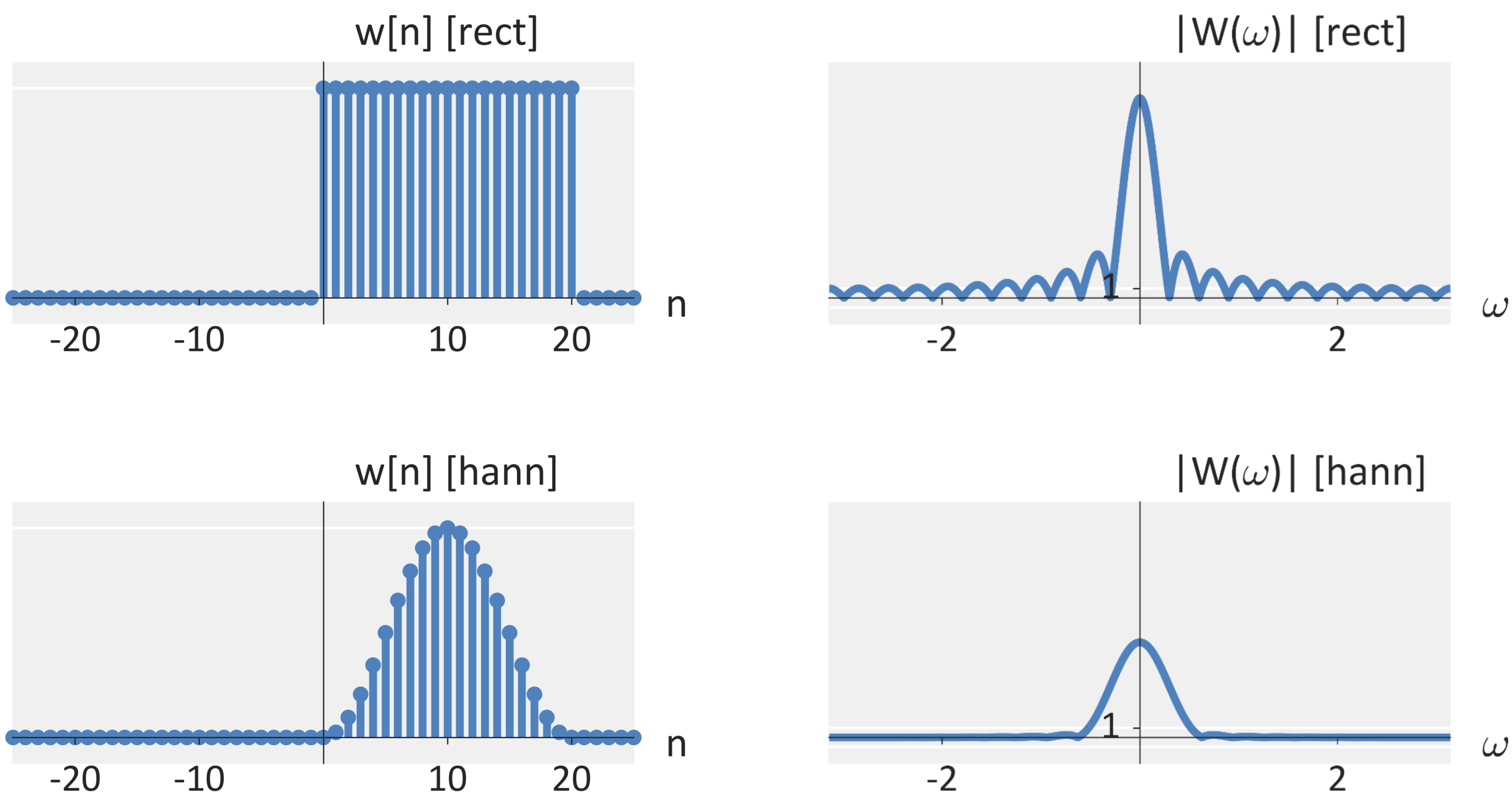
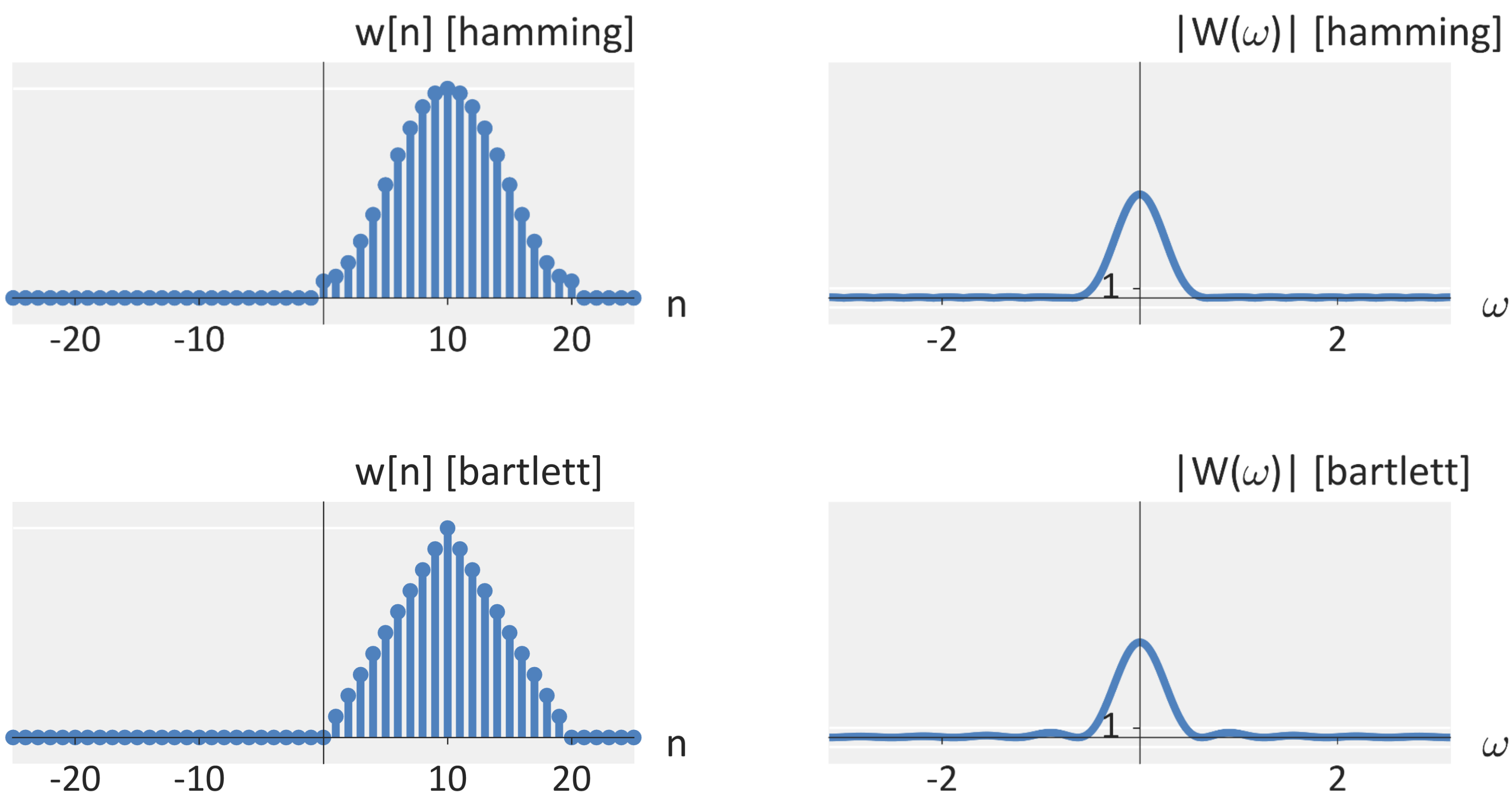
Different Filters



Designing with Windows

- **Different Filters**



Designing with Windows

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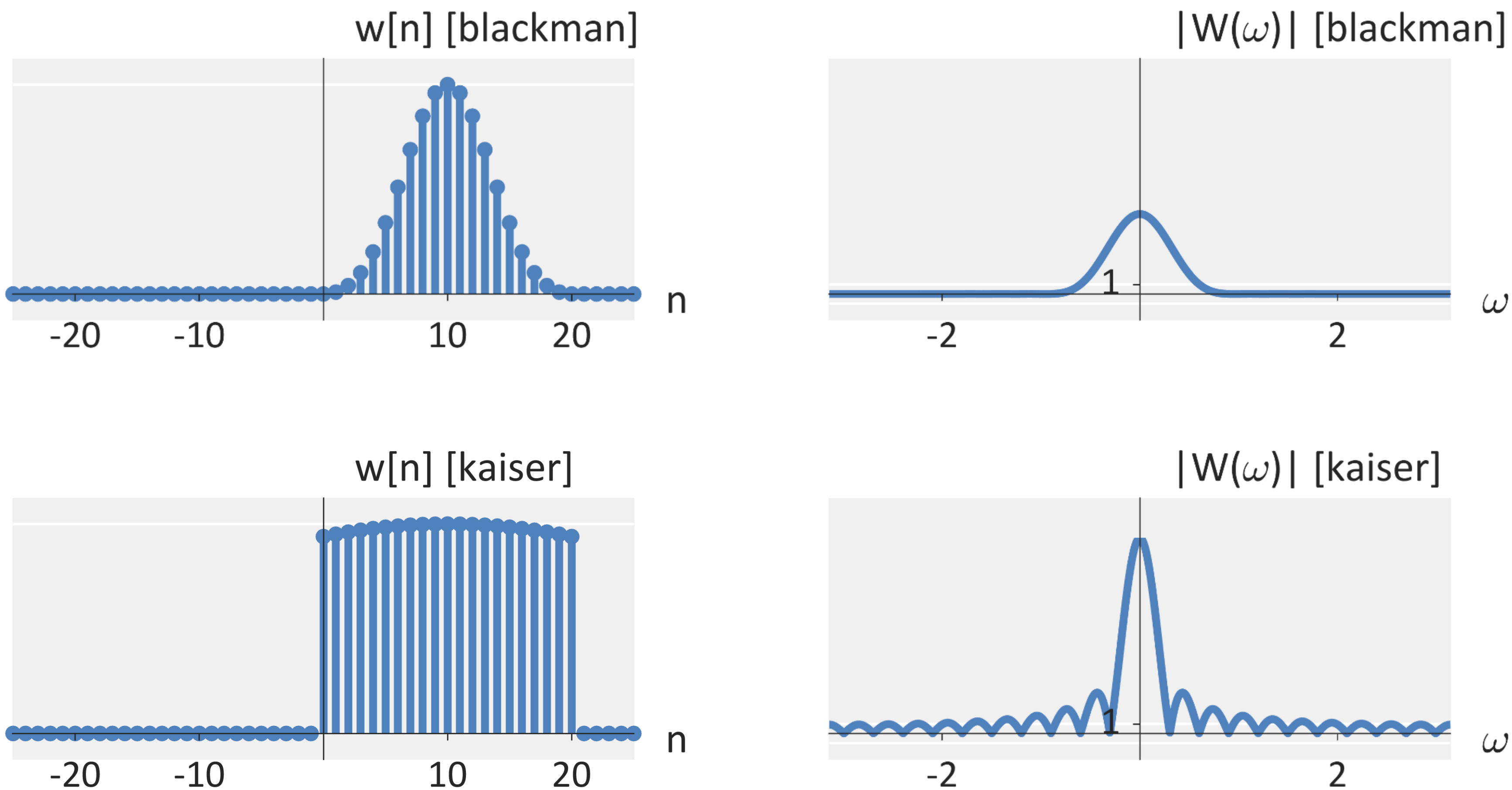
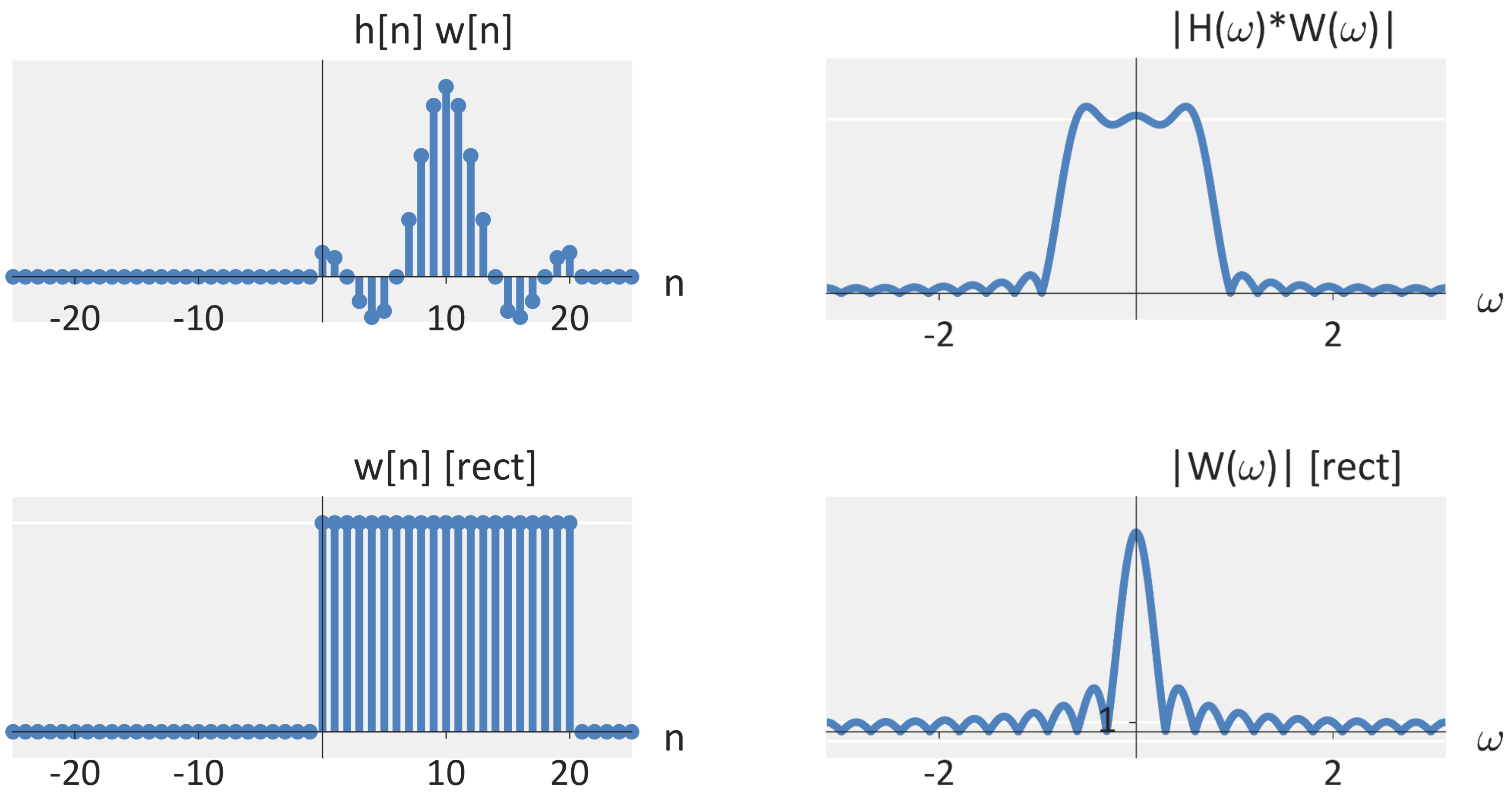


Designing with Windows

- **Windowing the sinc impulse response**



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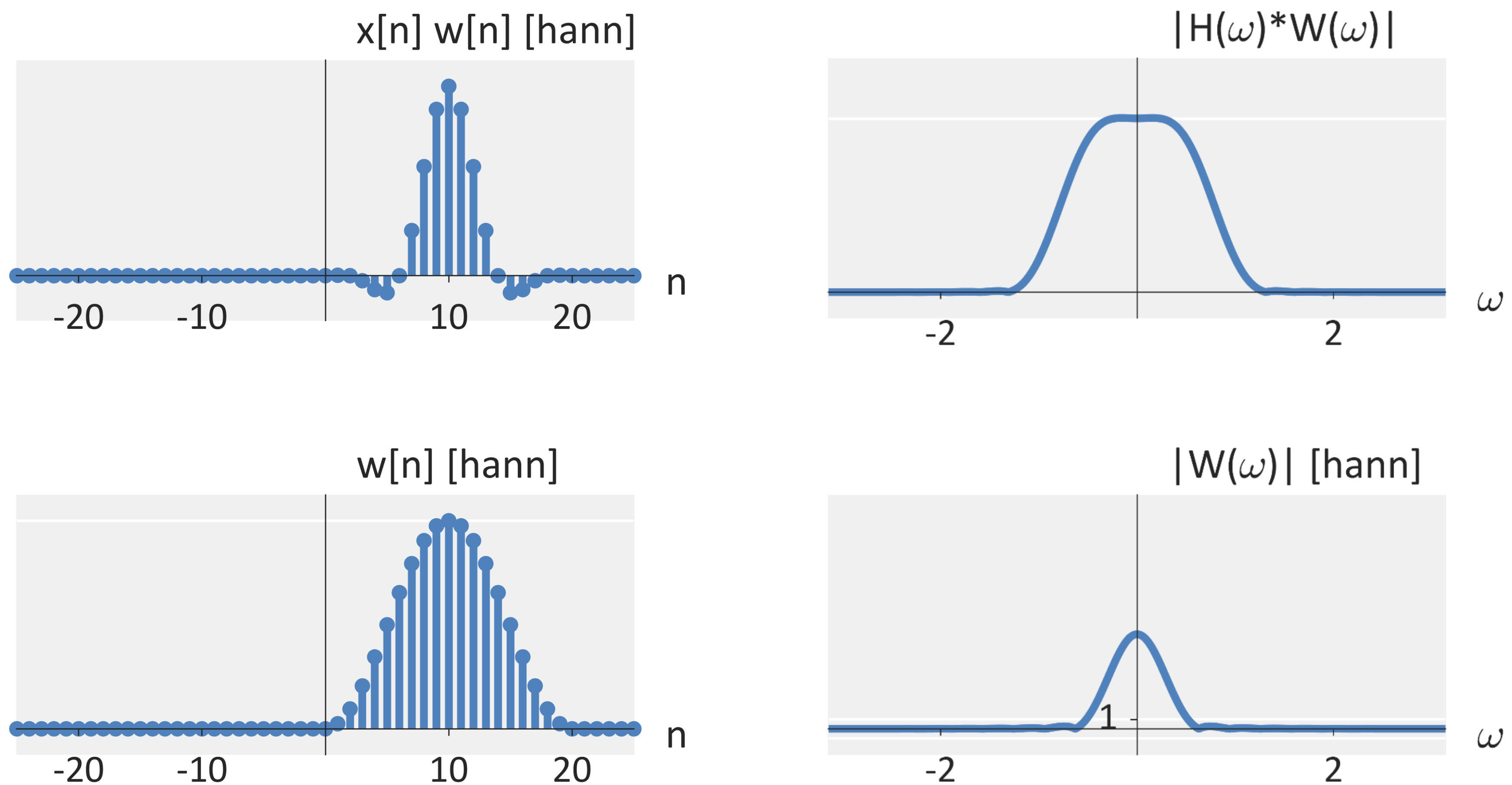

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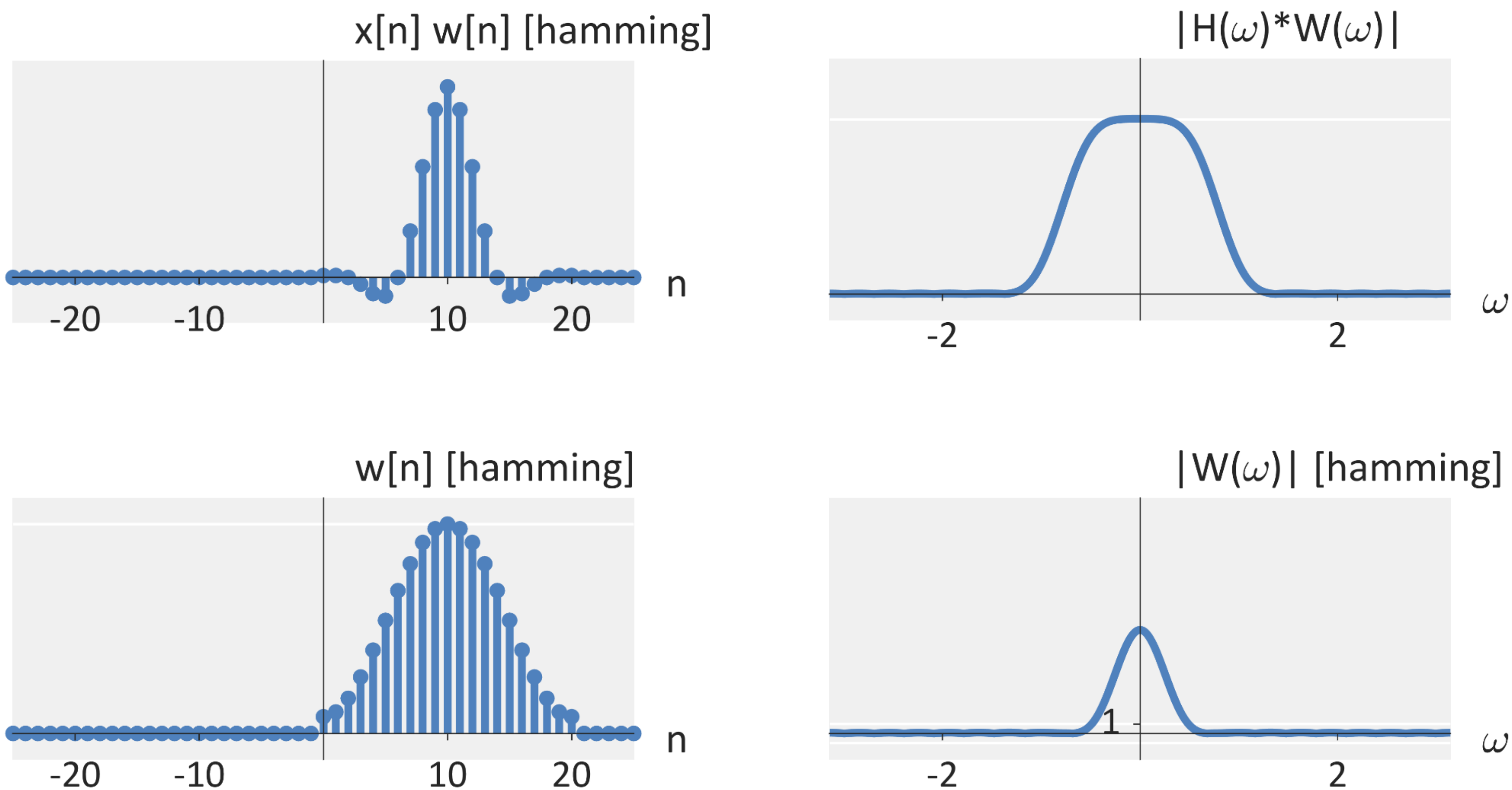

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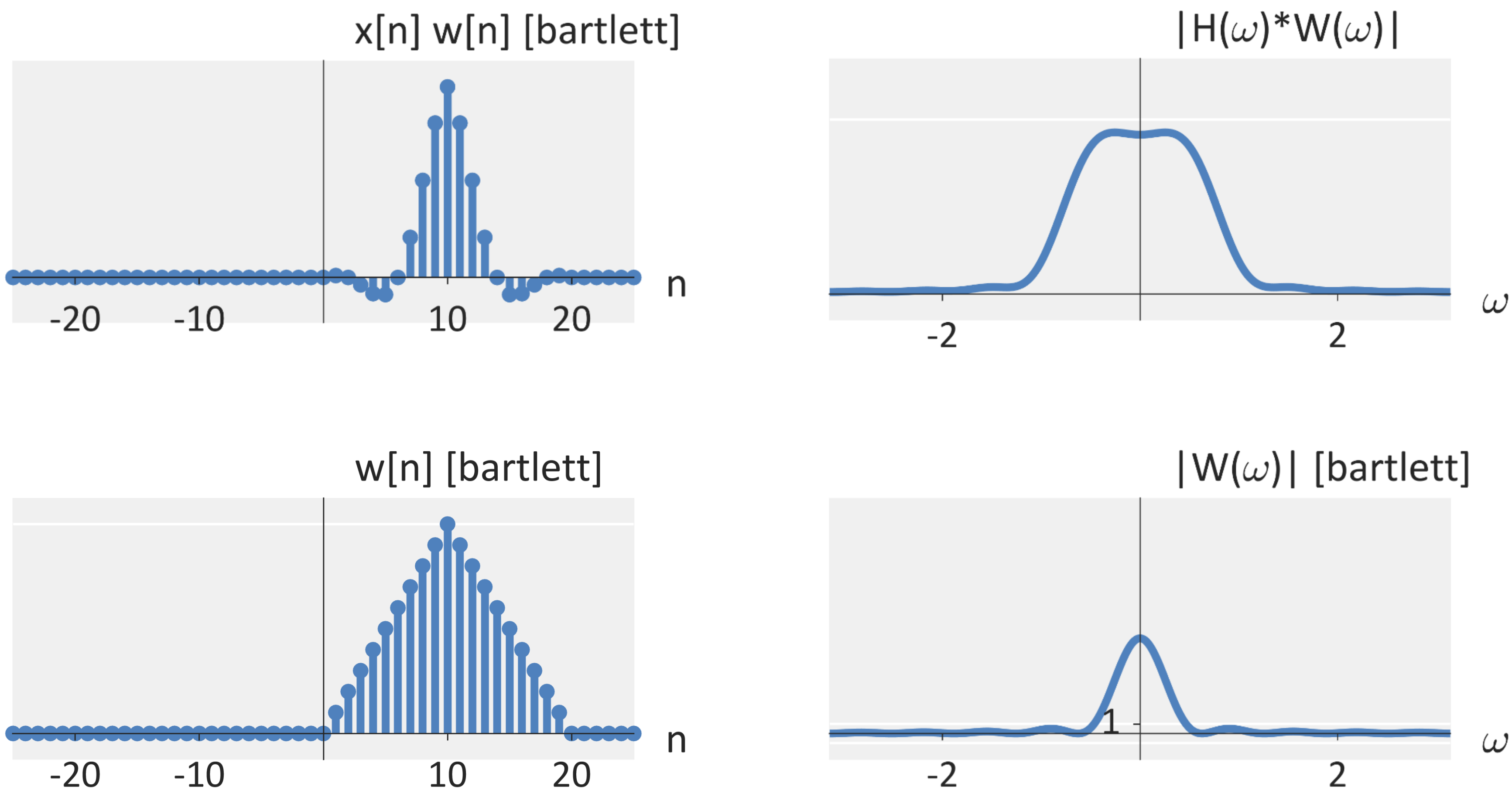

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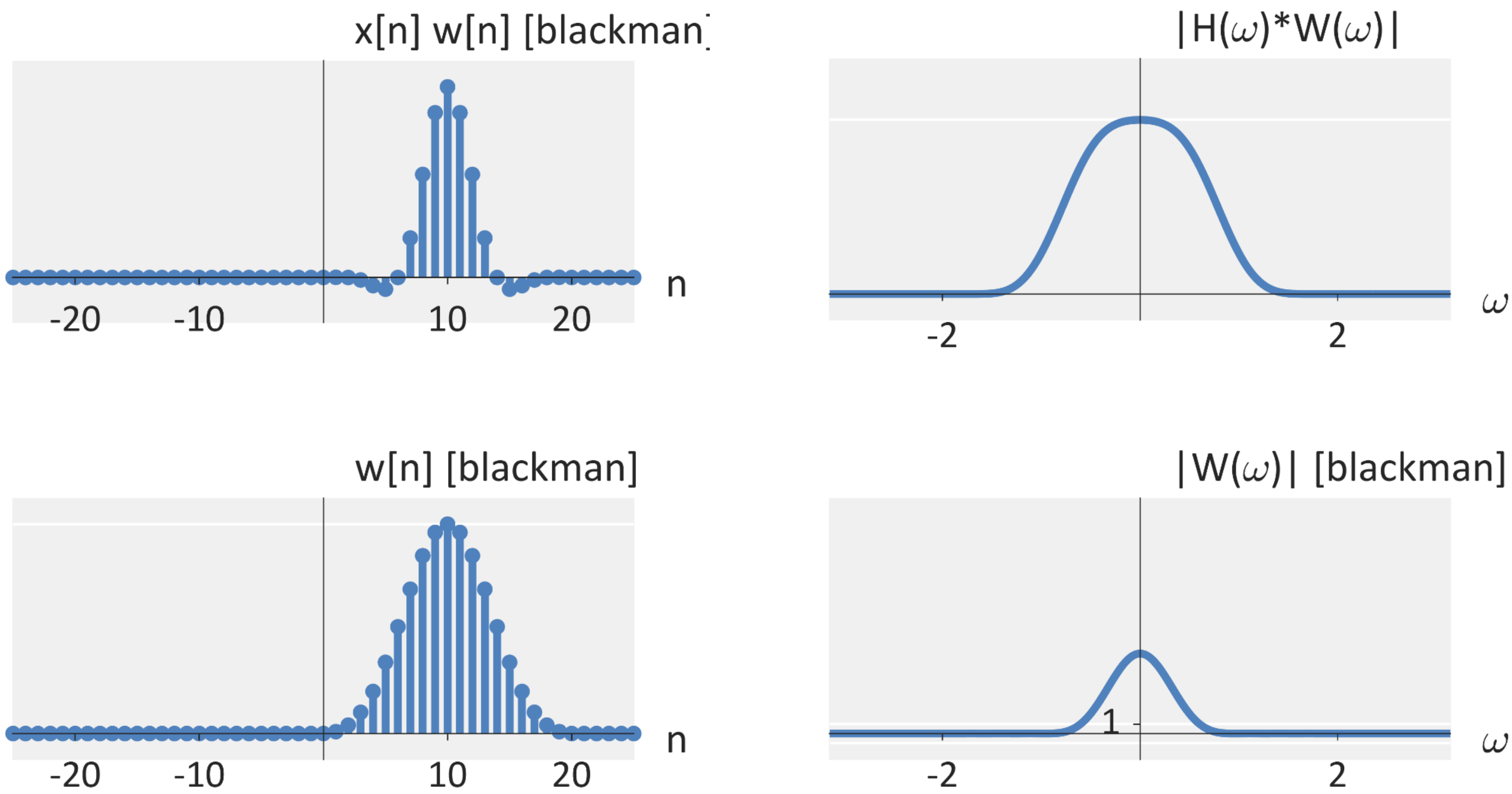

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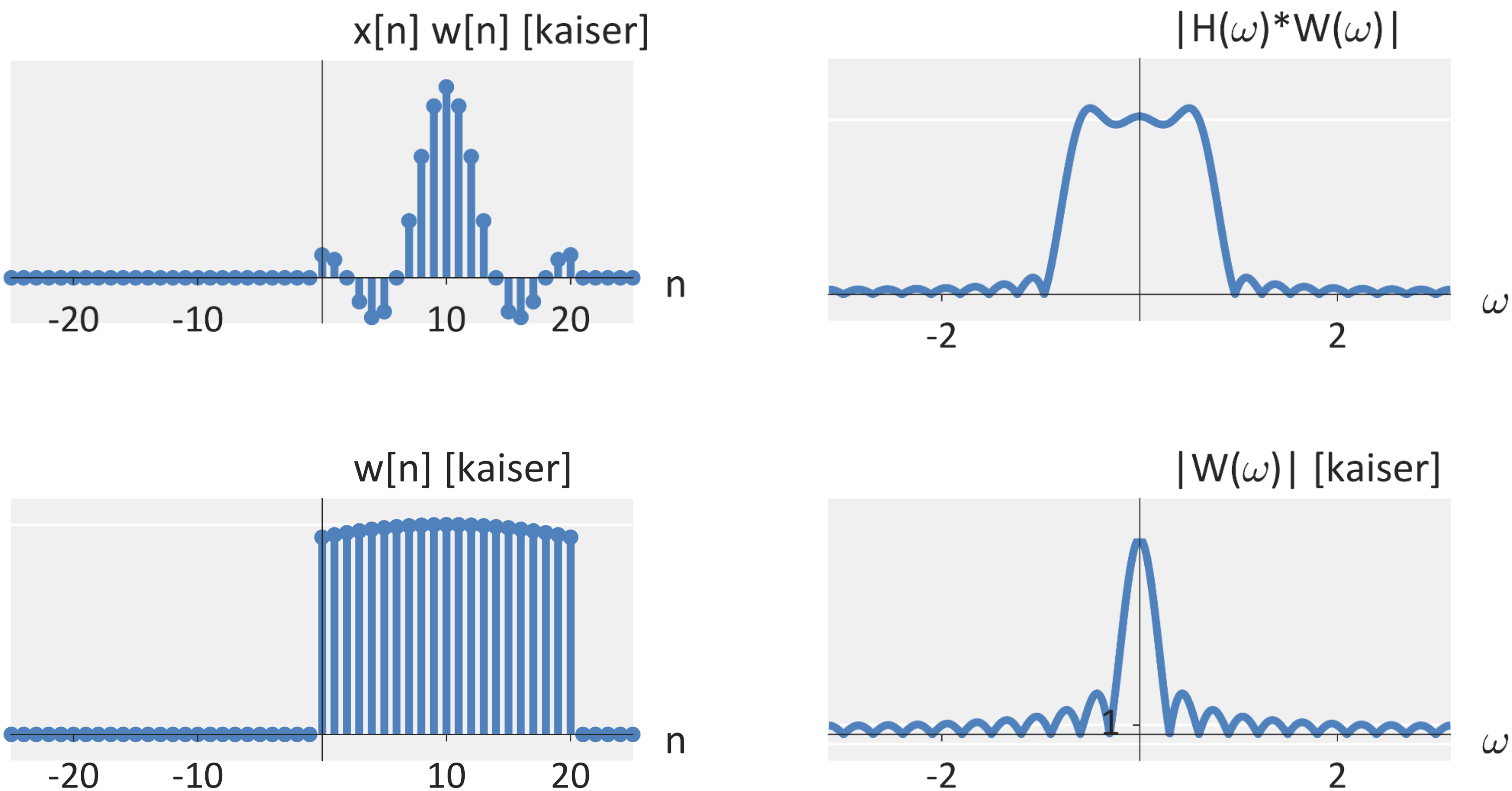

Designing with Windows

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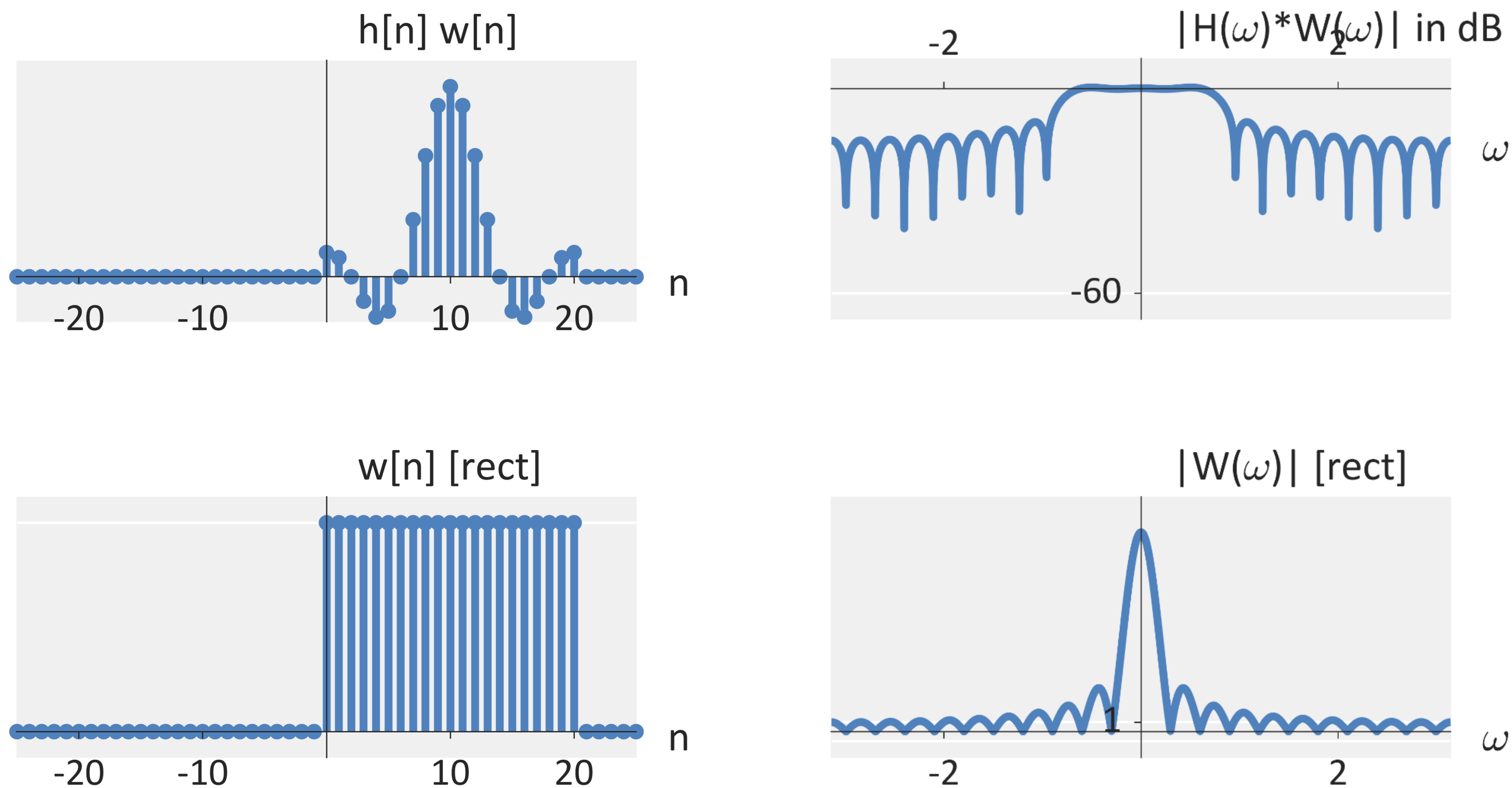

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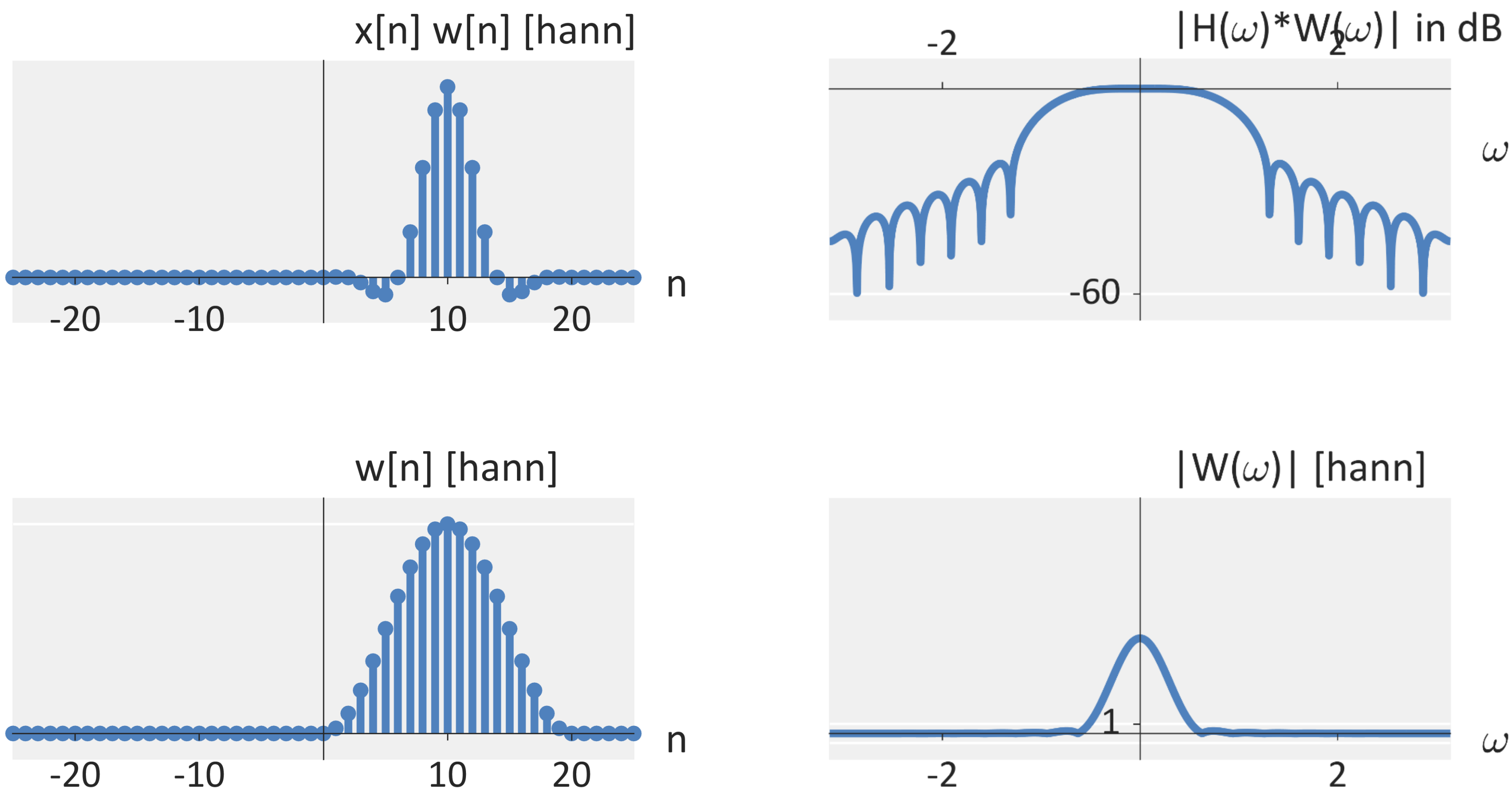

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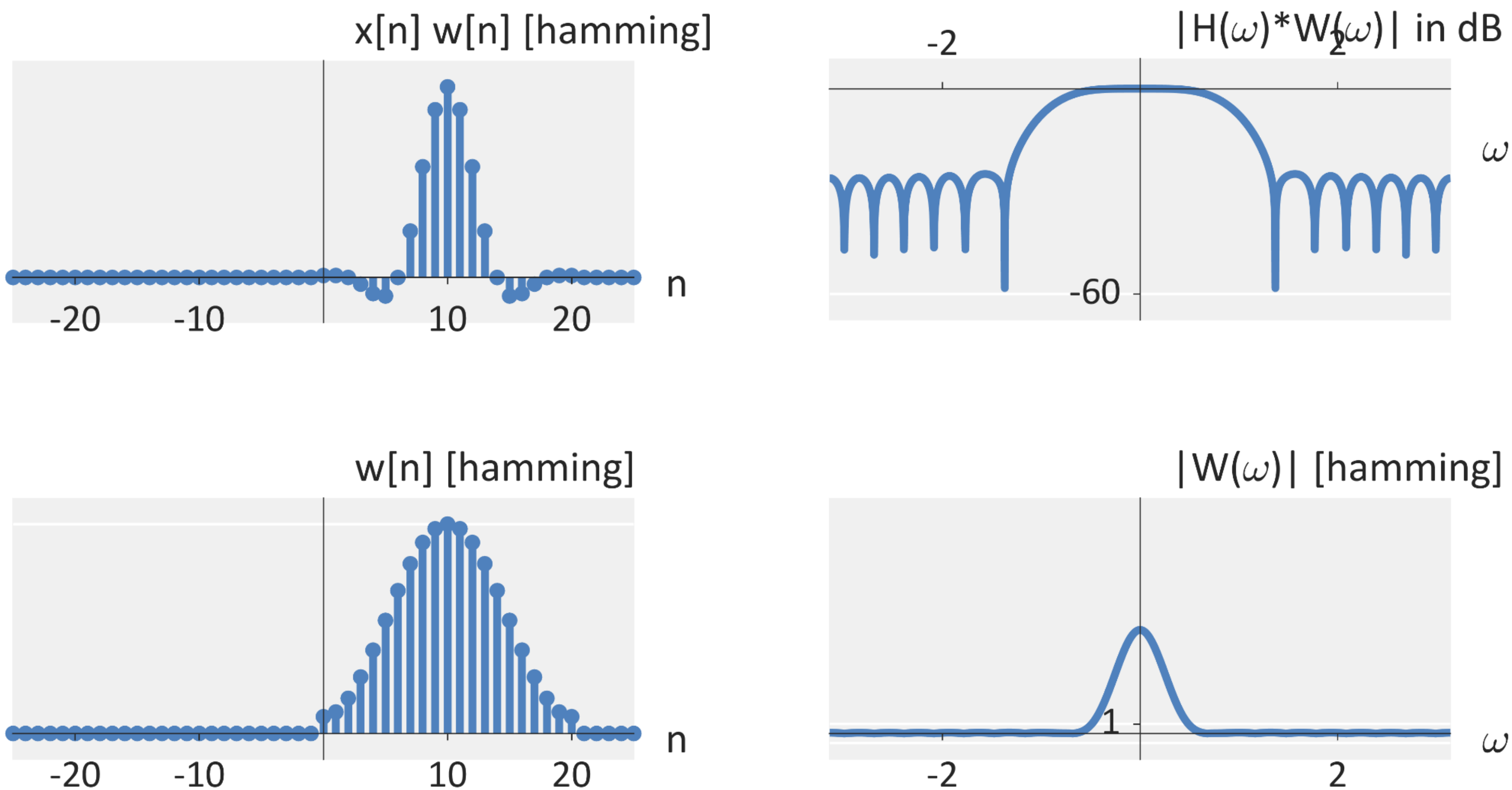

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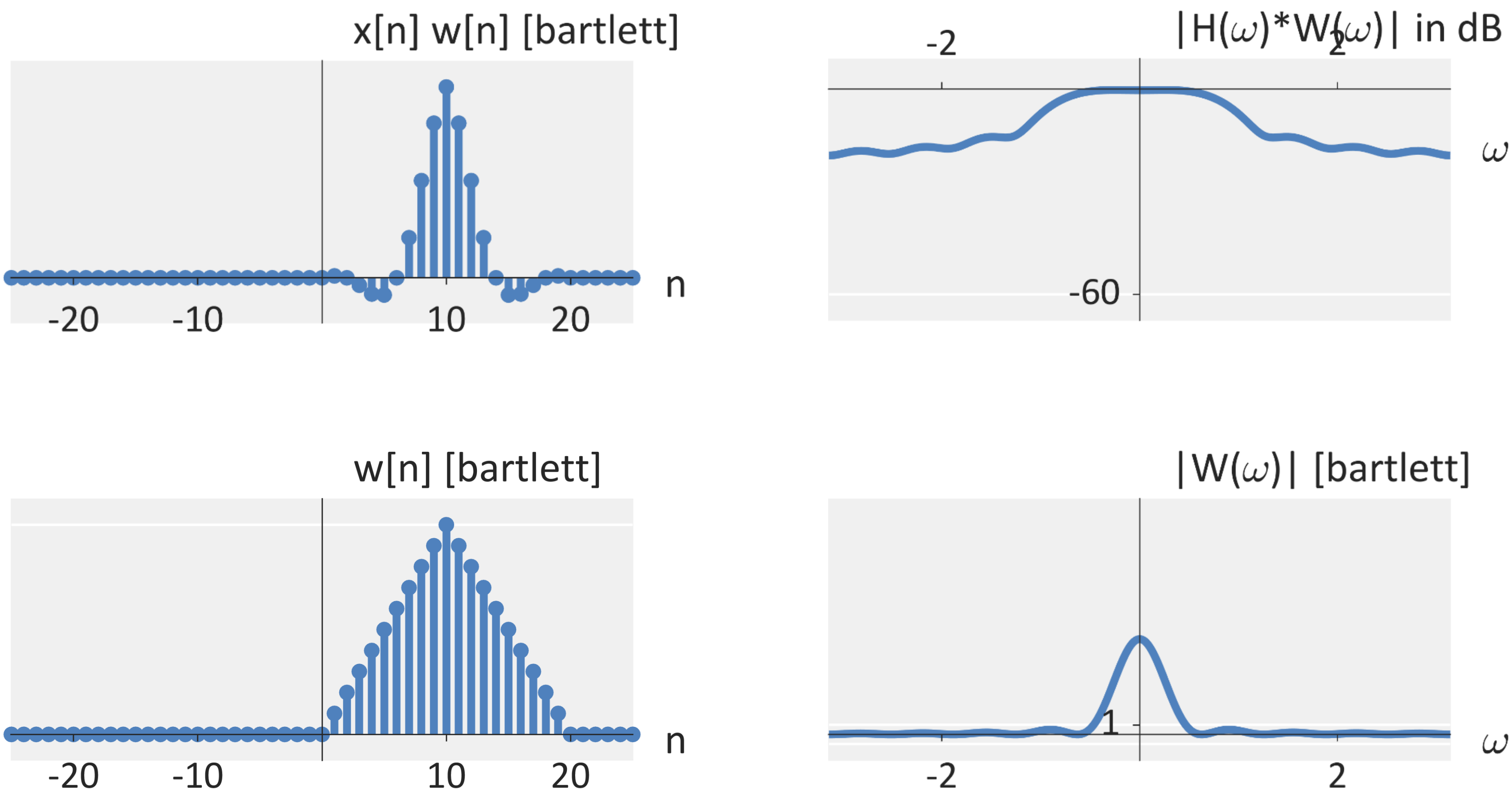

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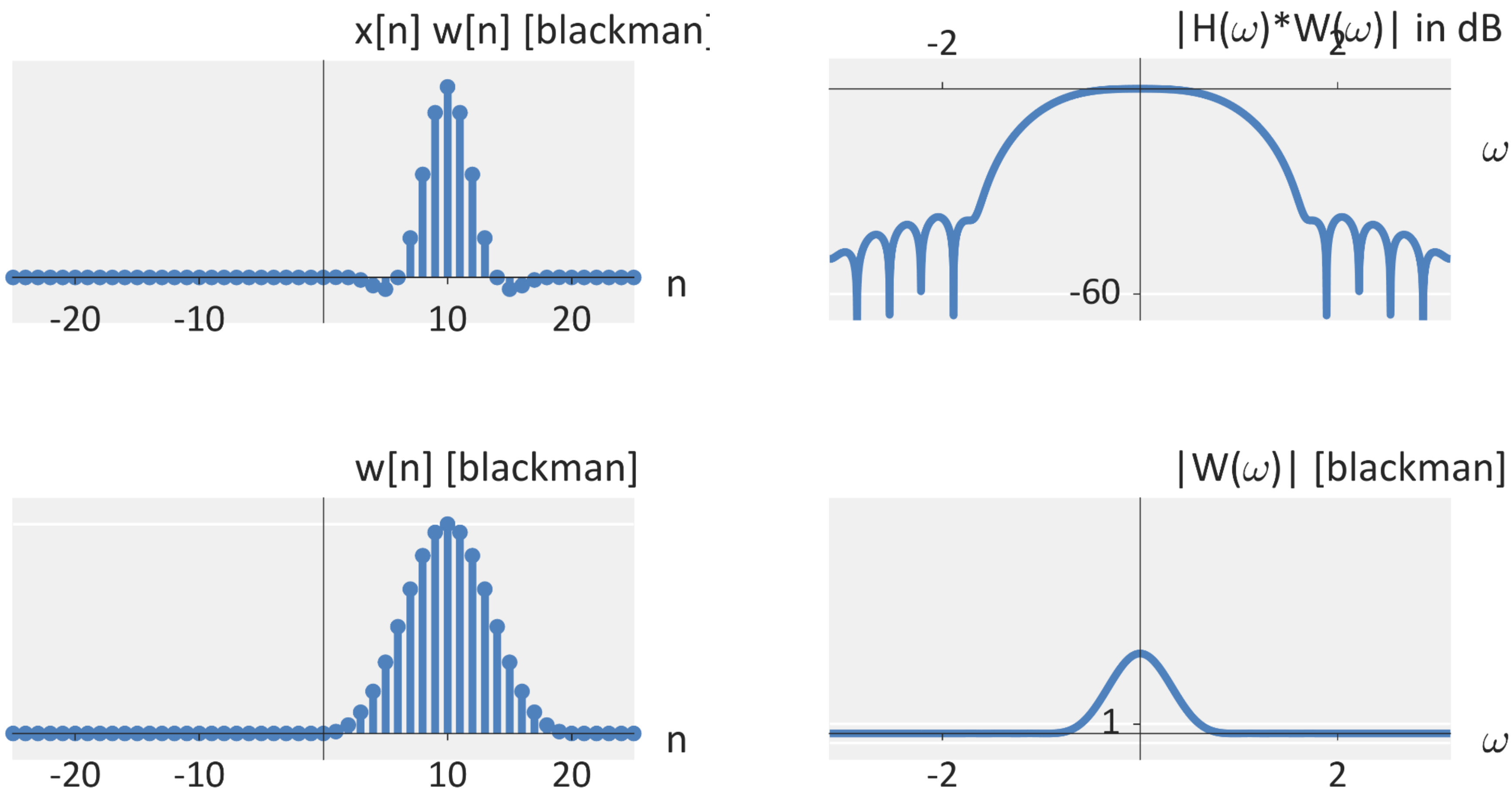

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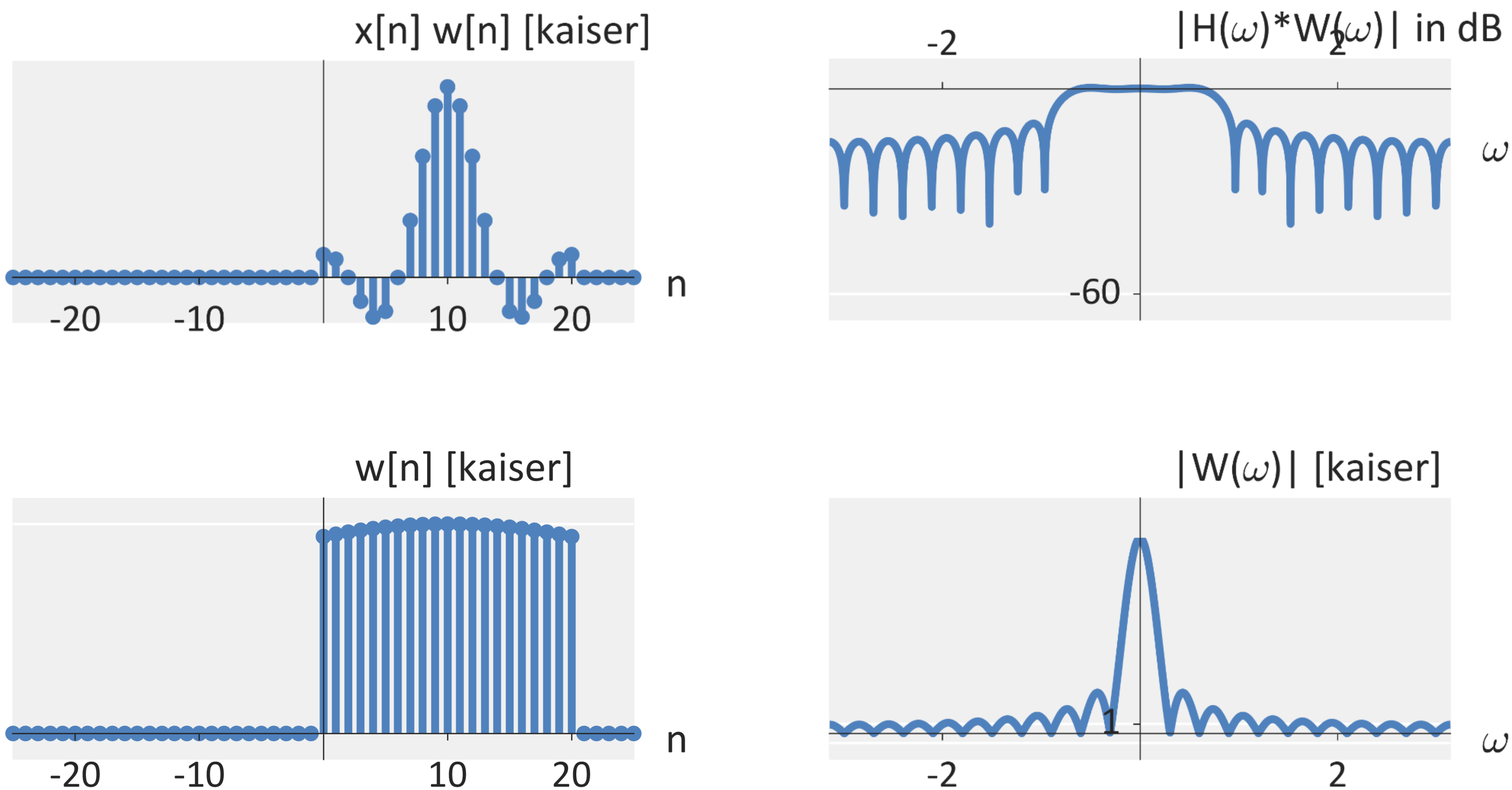

Lecture 22: Design of FIR / IIR Filters

Foundations of Digital Signal Processing

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- **Designing FIR Filters with Frequency Sampling**
- Designing FIR Filters with Equi-ripples
- Designing IIR Filters with Discrete Differentiation
- Designing IIR Filters with Impulse Invariance
- Designing IIR Filters with the Bilinear Transform
- Related Analog Filters

Design with Frequency Sampling

- **Option 2: Work backwards with constraints**

Consider the DFT:

$$h[n] = \sum_{k=0}^{N-1} H[k] e^{j\frac{2\pi}{N}nk} \qquad \text{such that} \qquad H[k] = H[N-k]$$

Design with Frequency Sampling

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$$h[n] = H[0] + \sum_{k=1}^{(N-1)/2} H[k] e^{j\frac{2\pi}{N}nk} + \sum_{k=(N+1)/2}^{N-1} H[k] e^{j\frac{2\pi}{N}nk}$$

Design with Frequency Sampling

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Design with Frequency Sampling

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Consider the DFT:

$$h[n] = \sum_{k=0}^{N-1} H[k] e^{j\frac{2\pi}{N}nk} \qquad \text{such that} \qquad H[k] = H[N-k]$$

$$h[n] = H[0] + \sum_{k=1}^{(N-1)/2} H[k] e^{j\frac{2\pi}{N}nk} + \sum_{k=1}^{(N-1)/2} H[k] e^{j\frac{2\pi}{N}n(N-k)}$$

$$h[n] = H[0] + 2 \sum_{k=1}^{(N-1)/2} H[k] \cos\left(\frac{2\pi}{N}nk\right)$$

Design with Frequency Sampling

- **An inverse DFT that forces time-symmetry**

$$h[n] = H[0] + 2\sum_{k=1}^{(N-1)/2} H[k]\cos\left(\frac{2\pi}{N}nk\right)$$

Design with Frequency Sampling

- **An inverse DFT that forces time-symmetry**

$$h[n] = H[0] + 2\sum_{k=1}^{(N-1)/2} H[k]\cos\left(\frac{2\pi}{N}nk\right)$$

- **Example: Consider the desired 9-sample frequency response with the first half defined by** [1 1 0 0]
  - Compute the frequency sampled filter

Design with Frequency Sampling

- **An inverse DFT that forces time-symmetry**

$$h[n] = H[0] + 2\sum_{k=1}^{(N-1)/2} H[k]\cos\left(\frac{2\pi}{N}nk\right)$$

- **Example: Consider the desired 9-sample frequency response with the first half defined by** [1 1 0 0]

  - Compute the frequency sampled filter

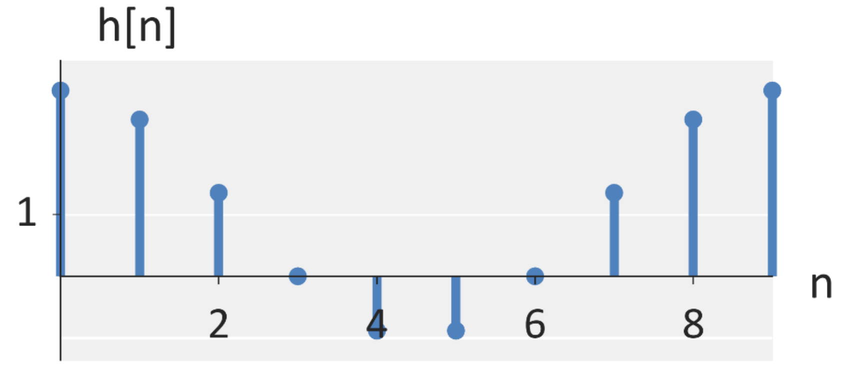
$$h[n] = 1 + 2\cos\big((2\pi/9)n\big)$$

Design with Frequency Sampling

- **Example: Consider the desired 9-sample frequency response with the first half defined by** [1 1 0 0]

  - Compute the frequency sampled filter

$$h[n] = 1 + 2\cos\big((2\pi/9)n\big)$$



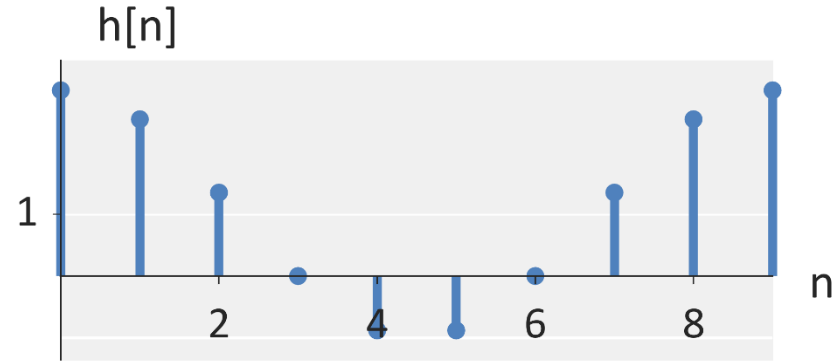
Design with Frequency Sampling

- **Example: Consider the desired 9-sample frequency response with the first half defined by** [1 1 0 0]

  - Compute the frequency sampled filter

$$h[n] = 1 + 2\cos\left((2\pi/9)n\right)$$

  - In practice, this should be circularly shifted so that the maximum is centered.



Design with Frequency Sampling

- **An inverse DFT that forces time-symmetry**

$$h[n] = H[0] + 2\sum_{k=1}^{(N-1)/2} H[k]\cos\left(\frac{2\pi}{N}\left(n - \frac{N-1}{2}\right)k\right)$$

- **Example: Consider the desired 9-sample frequency response with the first half defined by** [1 1 0 0]

  - Compute the frequency sampled filter

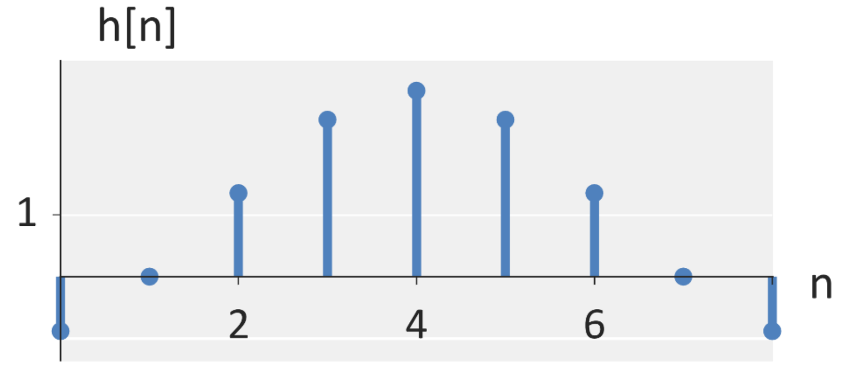
$$h[n] = 1 + 2\cos\big((2\pi/9)n\big)$$

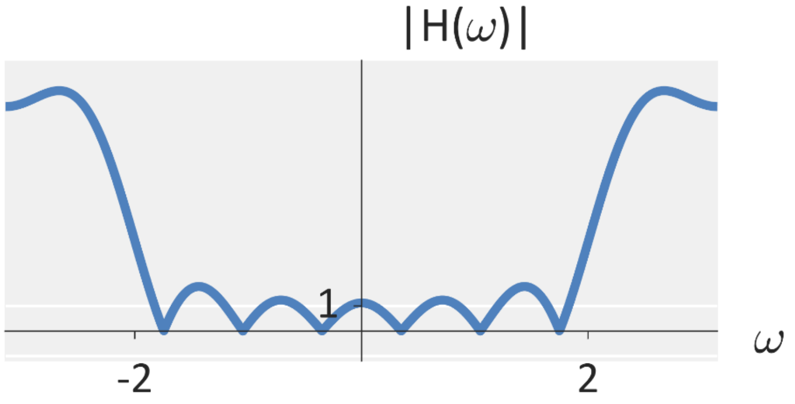
Design with Frequency Sampling

- **Example: Consider the desired 9-sample frequency response with the first half defined by** [1 1 0 0]

  - Compute the frequency sampled filter

$$h[n] = 1 + 2\cos\left((2\pi/9)(n - 8/2)\right)$$





Design with Frequency Sampling

- **Example: Consider the desired 17-sample frequency response with the first half defined by** [1 1 1 1 0 0 0 0]

  - Compute the frequency sampled filter

$$h[n] = 1 + 2\cos\big((2\pi/19)n_c\big) + 2\cos\big((4\pi/19)n_c\big) + 2\cos\big((6\pi/19)n_c\big)$$

$$n_c = n - \frac{16}{2}$$

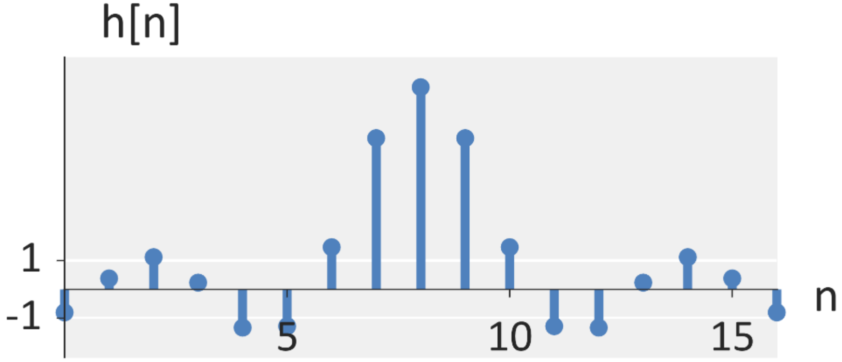
Design with Frequency Sampling

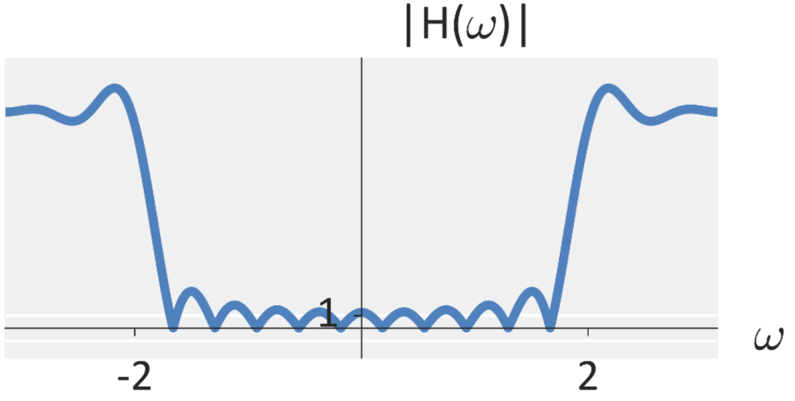
- **Example: Consider the desired 17-sample frequency response with the first half defined by** [1 1 1 1 0 0 0 0]

  - Compute the frequency sampled filter

$$h[n] = 1 + 2\cos\big((2\pi/19)n_c\big) + 2\cos\big((4\pi/19)n_c\big) + 2\cos\big((6\pi/19)n_c\big)$$

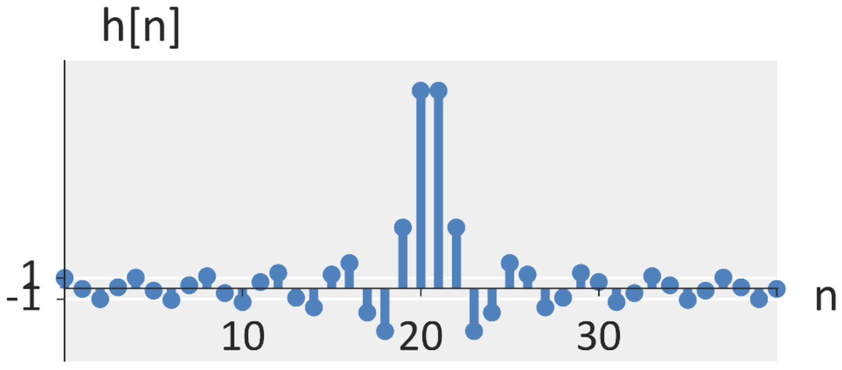
$$n_c = n - \frac{16}{2}$$

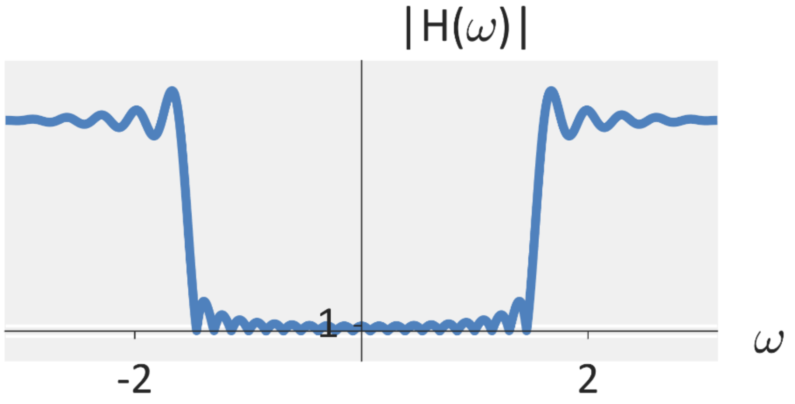




Design with Frequency Sampling

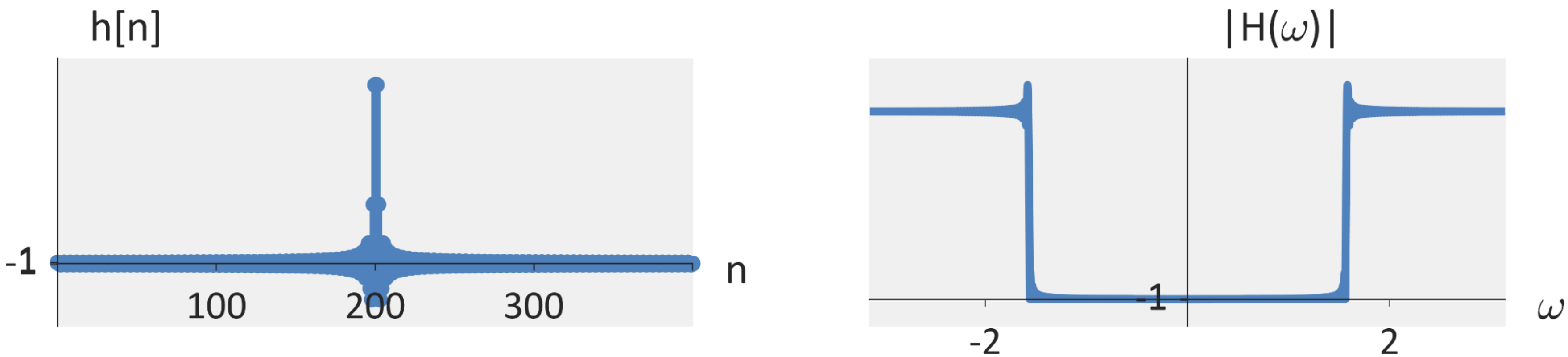
- **Example: Consider the desired 41-sample frequency response with the first 10 values defined by 1**
  - Compute the frequency sampled filter





Design with Frequency Sampling

- **Example: Consider the desired 401-sample frequency response with the first 100 values defined by 1**
  - Compute the frequency sampled filter
  - Note that in practice, this needs to be circularly shifted to the center



Design with Frequency Sampling

- **Note: The definition can be slightly modified**
  - Our definition:

$$
\begin{aligned}
h[n] &= H[0] + 2\sum_{k=1}^{(N-1)/2} H[k]\cos\left(\frac{2\pi}{N}\left(n-\frac{N-1}{2}\right)k\right) \\
&= H[0] + 2\sum_{k=1}^{(N-1)/2} H[k]\cos\left(\frac{2\pi}{N}\left(n-\frac{N}{2}+\frac{1}{2}\right)k\right) \\
&= H[0] + 2\sum_{k=1}^{(N-1)/2} H[k]\cos\left(\frac{2\pi}{N}\left(n+\frac{1}{2}\right)k-\pi k\right) \\
&= H[0] + 2\sum_{k=1}^{(N-1)/2} (-1)^k H[k]\cos\left(\frac{2\pi}{N}\left(n+\frac{1}{2}\right)k\right)
\end{aligned}
$$

Design with Frequency Sampling

- **Final Definition**

$$h[n] = H[0] + 2\sum_{k=1}^{(N-1)/2} (-1)^k H[k] \cos\left(\frac{2\pi}{N}\left(n + \frac{1}{2}\right)k\right)$$

Side note: This is very closely related to the discrete cosine transform

Lecture 22: Design of FIR / IIR Filters

Foundations of Digital Signal Processing

Outline

- Designing FIR Filters with Windows
- Designing FIR Filters with Frequency Sampling
- **Designing FIR Filters with Equi-ripples**
- Designing IIR Filters with Discrete Differentiation
- Designing IIR Filters with Impulse Invariance
- Designing IIR Filters with the Bilinear Transform
- Related Analog Filters

Design with Equi-ripples

Previously derived:

$$X(z) = z^{-\frac{(M-1)}{2}} \sum_{k=0}^{M/2-1} a_k \left[z^{\frac{(M-1)}{2}-k} + z^{-\left[\frac{(M-1)}{2} - k\right]}\right]$$

$$X(\omega) = e^{-j\omega\frac{(M-1)}{2}} \sum_{k=0}^{M/2-1} a_k \left[e^{j\omega\left[\frac{(M-1)}{2}-k\right]} + j\omega^{-j\omega\left[\frac{(M-1)}{2} - k\right]}\right]$$

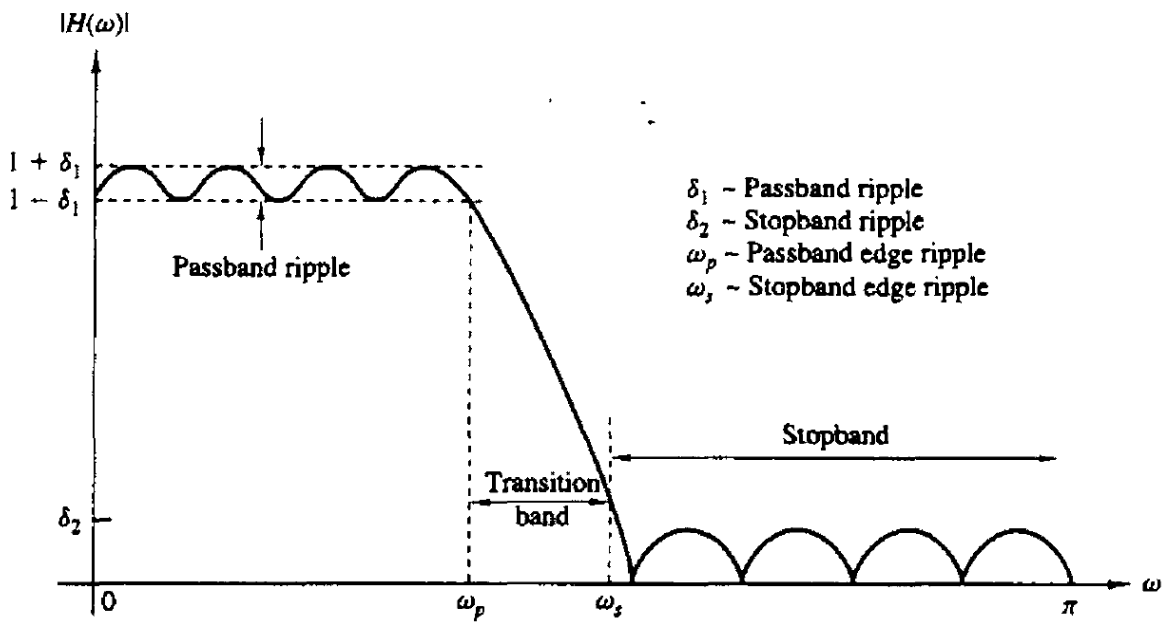
$$= 2e^{-j\omega\frac{(M-1)}{2}} \sum_{k=0}^{M/2-1} a_k \cos\left(\omega\left[\frac{M-1}{2} - k\right]\right)$$

Design with Equi-ripples

- **Equi-ripple design**

$$X(\omega) = 2e^{-j\omega\frac{(M-1)}{2}} \sum_{k=0}^{M/2-1} a_k \cos\left(\omega\left[\frac{M-1}{2} - k\right]\right)$$

- **Goal:** Find the optimal a_ks that satisfies **passband / stopband** ripple constraints.



Design with Equi-ripples

- **Equi-ripple design**

$$\min_{a_k} W(\omega)\left[H_d(\omega) - 2e^{-j\omega\frac{(M-1)}{2}}\sum_{k=0}^{M/2-1} a_k \cos\left(\omega\left[\frac{M-1}{2} - k\right]\right)\right]$$

Desired frequency response

Equals:

$\dfrac{\delta_2}{\delta_1}$ for ω in pass band

1 for ω in stop band

$\delta_2 =$ stopband ripple

$\delta_1 =$ passband ripple

Lecture 22: Design of FIR / IIR Filters

Foundations of Digital Signal Processing

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- Related Analog Filters

IIR Filter Design from Derivatives

- **Designing IIR Filters**
  - No easy ways to design digital IIR filters
  - So let us start from analog filters

IIR Filter Design from Derivatives

- **Designing IIR Filters**
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- **Option 1: Preserve the difference equation!**

IIR Filter Design from Derivatives

- **Question: What is a derivative in discrete-time?**
  - In continuous-time

$$\frac{dx(t)}{dt} \rightarrow sX(s)$$

  - In discrete-time

IIR Filter Design from Derivatives

- **Question: What is a derivative in discrete-time?**
  - In continuous-time

$$\frac{dx(t)}{dt} \to sX(s)$$

  - In discrete-time

$$\frac{dx(t)}{dt} = \lim_{\Delta T \to 0} \frac{x(t) - x(t - \Delta T)}{\Delta T}$$

T=1

$$\left.\frac{dx(t)}{dt}\right|_{t=nT} = \frac{x(nT) - x(nT - T)}{T} = x[n] - x[n-1]$$

IIR Filter Design from Derivatives

- **Question: What is a derivative in discrete-time?**
  - In continuous-time

$$\frac{dx(t)}{dt} \rightarrow sX(s)$$

  - In discrete-time

$$\frac{dx(t)}{dt} = \lim_{\Delta T \rightarrow 0} \frac{x(t) - x(t - \Delta T)}{\Delta T}$$

$$\left.\frac{dx(t)}{dt}\right|_{t=nT} = \frac{x(nT) - x(nT - T)}{T} = \frac{1}{T}\left(x[n] - x[n-1]\right)$$

IIR Filter Design from Derivatives

- **Question: What is a derivative in discrete-time?**
  - In continuous-time

$$\frac{dx(t)}{dt} \rightarrow sX(s)$$

  - In discrete-time

$$\frac{dx(t)}{dt} = \lim_{\Delta T \rightarrow 0} \frac{x(t) - x(t - \Delta T)}{\Delta T}$$

$$\left.\frac{dx(t)}{dt}\right|_{t=nT} = \frac{x(nT) - x(nT - T)}{T} = \frac{1}{T}(x[n] - x[n-1])$$

$$\left.\frac{dx(t)}{dt}\right|_{t=nT} \rightarrow \frac{1}{T}(1 - z^{-1})X(z)$$

IIR Filter Design from Derivatives

- **Question: What is a second-derivative in discrete-time?**
  - In continuous-time

$$\frac{d^2x(t)}{dt^2} \rightarrow s^2X(s)$$

  - In discrete-time

$$\frac{d^2x(t)}{d^2t} = \frac{dx(t)}{dt}\left[\frac{dx(t)}{dt}\right]$$

$$\left.\frac{dx(t)}{dt}\right|_{t=nT} = \frac{x(nT) - x(nT - T)}{T}$$

IIR Filter Design from Derivatives

- **Question: What is a second-derivative in discrete-time?**
  - In continuous-time

$$\frac{d^2x(t)}{dt^2} \rightarrow s^2X(s)$$

  - In discrete-time

$$\frac{d^2x(t)}{d^2t} = \frac{dx(t)}{dt}\left[\frac{dx(t)}{dt}\right]$$

$$\left.\frac{d^2x(t)}{dt^2}\right|_{t=nT} = \frac{[x(nT) - x(nT-T)]/T - [x(nT-T) - x(nT-2T)]/T}{T}$$

IIR Filter Design from Derivatives

- **Question: What is a second-derivative in discrete-time?**
  - In continuous-time

$$\frac{d^2x(t)}{dt^2} \rightarrow s^2X(s)$$

  - In discrete-time

$$\frac{d^2x(t)}{d^2t} = \frac{dx(t)}{dt}\left[\frac{dx(t)}{dt}\right]$$

$$\left.\frac{d^2x(t)}{dt^2}\right|_{t=nT} = \frac{x(nT) - 2x(nT-T) + x(nT-2T)}{T^2} \rightarrow \frac{x[n] - 2x[n-1] + x[n-2]}{T^2}$$

IIR Filter Design from Derivatives

- **Question: What is a second-derivative in discrete-time?**
  - In continuous-time

$$\frac{d^2x(t)}{dt^2} \to s^2X(s)$$

  - In discrete-time

$$\frac{d^2x(t)}{d^2t} = \frac{dx(t)}{dt}\left[\frac{dx(t)}{dt}\right]$$

$$\left.\frac{d^2x(t)}{dt^2}\right|_{t=nT} = \frac{x(nT) - 2x(nT-T) + x(nT-2T)}{T^2} \to \frac{x[n] - 2x[n-1] + x[n-2]}{T^2}$$

$$\left.\frac{dx(t)}{dt}\right|_{t=nT} \to \frac{1}{T^2}(1 - 2z^{-1} + z^{-2})X(z) = \frac{1}{T^2}(1 - z^{-1})^2X(z)$$

IIR Filter Design from Derivatives

- **Question: What is a derivative in discrete-time?**
  - Translate continuous-time to discrete-time

$$\frac{d^k x(t)}{dt^k} \rightarrow s^k X(s)$$

$$\left.\frac{d^k x(t)}{dt^k}\right|_{t=nT} \rightarrow \frac{1}{T}(1-z^{-1})^k X(z)$$

IIR Filter Design from Derivatives

- **Question: What is a derivative in discrete-time?**
  - Translate continuous-time to discrete-time

$$\frac{d^k x(t)}{dt^k} \rightarrow s^k X(s)$$

$$\left.\frac{d^k x(t)}{dt^k}\right|_{t=nT} \rightarrow \frac{1}{T}(1 - z^{-1})^k X(z)$$

$$\boxed{s \rightarrow \frac{1}{T}(1 - z^{-1})}$$

IIR Filter Design from Derivatives

- **Example:** $s \rightarrow \frac{1}{T}(1 - z^{-1})$
  - Use the derivative conversion to transform the following biquad filter into the discrete-time domain.

$$H(s) = \frac{1}{(s + 0.1)^2 + 9}$$

IIR Filter Design from Derivatives

- **Example:** $s \rightarrow \frac{1}{T}(1 - z^{-1})$
  - Use the derivative conversion to transform the following bi-quad filter into the discrete-time domain.

$$H(s) = \frac{1}{(s+0.1)^2 + 9}$$

$$H(z) = \frac{1}{\left(\frac{1}{T}(1-z^{-1}) + 0.1\right)^2 + 9} = \frac{T^2}{T^2\left[\left(\frac{1}{T}(1-z^{-1}) + 0.1\right)^2 + 9\right]}$$

$$= \frac{\sqrt{T}}{\left((1-z^{-1}) + 0.1T\right)^2 + 9T^2} = \frac{\sqrt{T}}{\left((1+0.1T) - z^{-1}\right)^2 + 9T^2}$$

IIR Filter Design from Derivatives

- **Example:** $s \rightarrow \frac{1}{T}(1 - z^{-1})$
  - Use the derivative conversion to transform the following bi-quad filter into the discrete-time domain.

$$H(s) = \frac{1}{(s + 0.1)^2 + 9}$$

$$H(z) = \frac{T^2}{\left((1 + 0.1T) - z^{-1}\right)^2 + 9T^2}$$

Finding poles

$$\left((1 + 0.1T) - z^{-1}\right)^2 + 9T^2 = 0$$

$$\left((1 + 0.1T) - z^{-1}\right)^2 = -9T^2$$

$$(1 + 0.1T) - z^{-1} = \pm 3Tj$$

$$z^{-1} = (1 + 0.1T) \mp 3Tj$$

$$z = \frac{10}{1 + (0.1 + 3j)T}$$

IIR Filter Design from Derivatives

- **Example:** $s \to \frac{1}{T}(1 - z^{-1})$
  - Use the derivative conversion to transform the following bi-quad filter into the discrete-time domain.

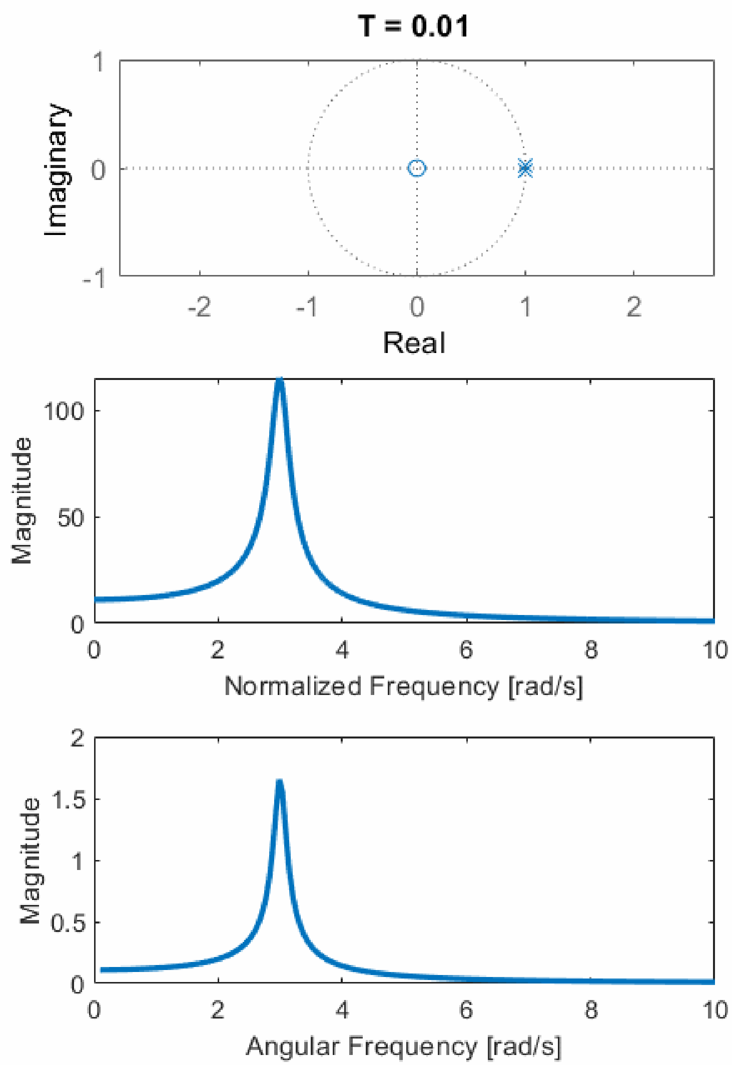
$$z = \frac{10}{1 + (0.1 + 3j)T}$$

Poles

IIR Filter Design from Derivatives

- **Example:** $s \rightarrow \frac{1}{T}(1 - z^{-1})$

$$z = \frac{10}{1 + (0.1 + 3j)T}$$

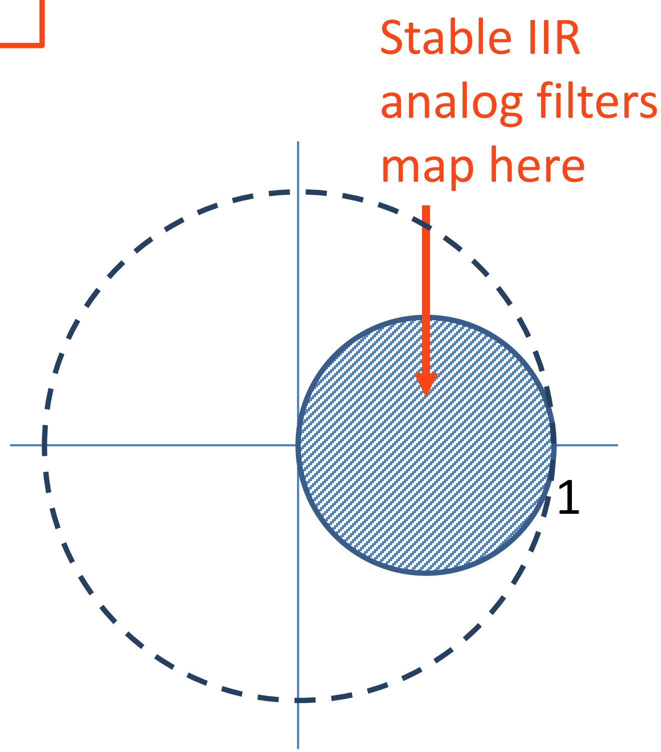


IIR Filter Design from Derivatives

- **Question: What is a derivative in discrete-time?**
  - Translate continuous-time to discrete-time

$$\boxed{s \rightarrow \frac{1}{T}\left(1 - z^{-1}\right)}$$

- **Pros:**
  - Relatively simple
- **Cons:**
  - Very limiting
  - Stable continuous-time poles can only be mapped to low frequencies



Lecture 22: Design of FIR / IIR Filters

Foundations of Digital Signal Processing

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- Designing FIR Filters with Windows
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- **Designing IIR Filters with Impulse Invariance**
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IIR Filter Design by Impulse Invariance

- **Designing IIR Filters**
  - No easy ways to design digital IIR filters
  - So let us start from analog filters

- **Option 2: Preserve the impulse response!**

IIR Filter Design by Impulse Invariance

- **Question: How else can I represent my transfer function?**

$$H(s) = \prod_{k=1}^{K} \frac{1}{s - p_k}$$

IIR Filter Design by Impulse Invariance

- **Question: How else can I represent my transfer function?**

$$H(s) = \prod_{k=1}^{K} \frac{1}{s - p_k} = \sum_{k=1}^{K} c_k e^{p_k t}$$

IIR Filter Design by Impulse Invariance

- **Question: How else can I represent my transfer function?**

$$H(s) = \prod_{k=1}^{K} \frac{1}{s - p_k} = \sum_{k=1}^{K} c_k e^{p_k t}$$

$$h(t) = \sum_{k=1}^{K} c_k e^{p_k t}$$

IIR Filter Design by Impulse Invariance

- **Question: How else can I represent my transfer function?**

$$H(s) = \prod_{k=1}^{K} \frac{1}{s - p_k} = \sum_{k=1}^{K} c_k e^{p_k t}$$

$$h(t) = \sum_{k=1}^{K} c_k e^{p_k t}$$

$$h(nT) = h[n] = \sum_{k=1}^{K} c_k e^{p_k nT} = \sum_{k=1}^{K} c_k [e^{p_k T}]^n$$

IIR Filter Design by Impulse Invariance

- **Question: How else can I represent my transfer function?**

$$H(s) = \prod_{k=1}^{K} \frac{1}{s - p_k} = \sum_{k=1}^{K} c_k e^{p_k t}$$

$$h(t) = \sum_{k=1}^{K} c_k e^{p_k t}$$

$$h(nT) = h[n] = \sum_{k=1}^{K} c_k e^{p_k nT} = \sum_{k=1}^{K} c_k [e^{p_k T}]^n$$

$$H(z) = \sum_{k=1}^{K} \frac{c_k}{1 - e^{p_k T} z^{-1}}$$

IIR Filter Design from Derivatives

- **Example:** $H(z) = \sum_{k=1}^{K} \frac{c_k}{1-e^{p_k T} z^{-1}}$
  - Use impulse invariance to transform the following biquad filter into the discrete-time domain.

$$H(s) = \frac{1}{(s+0.1)^2 + 9}$$

IIR Filter Design from Derivatives

- **Example:** $H(z) = \sum_{k=1}^{K} \frac{c_k}{1-e^{p_k T} z^{-1}}$
  - Use impulse invariance to transform the following biquad filter into the discrete-time domain.

$$H(s) = \frac{1}{(s+0.1)^2 + 9}$$

Poles:

$$(s+0.1)^2 + 9 = 0$$

$$s = \pm 3j - 0.1$$

IIR Filter Design from Derivatives

- **Example:** $H(z) = \sum_{k=1}^{K} \frac{c_k}{1-e^{p_k T} z^{-1}}$
  - Use impulse invariance to transform the following biquad filter into the discrete-time domain.

$$H(s) = \frac{1}{(s+0.1)^2+9} = \frac{1/2}{s+3j+0.1} + \frac{1/2}{s-3j+0.1}$$

Poles:

$$(s+0.1)^2+9=0$$

$$s = \pm 3j - 0.1$$

IIR Filter Design from Derivatives

- **Example:** $H(z) = \sum_{k=1}^{K} \frac{c_k}{1 - e^{p_k T} z^{-1}}$
  - Use impulse invariance to transform the following biquad filter into the discrete-time domain.

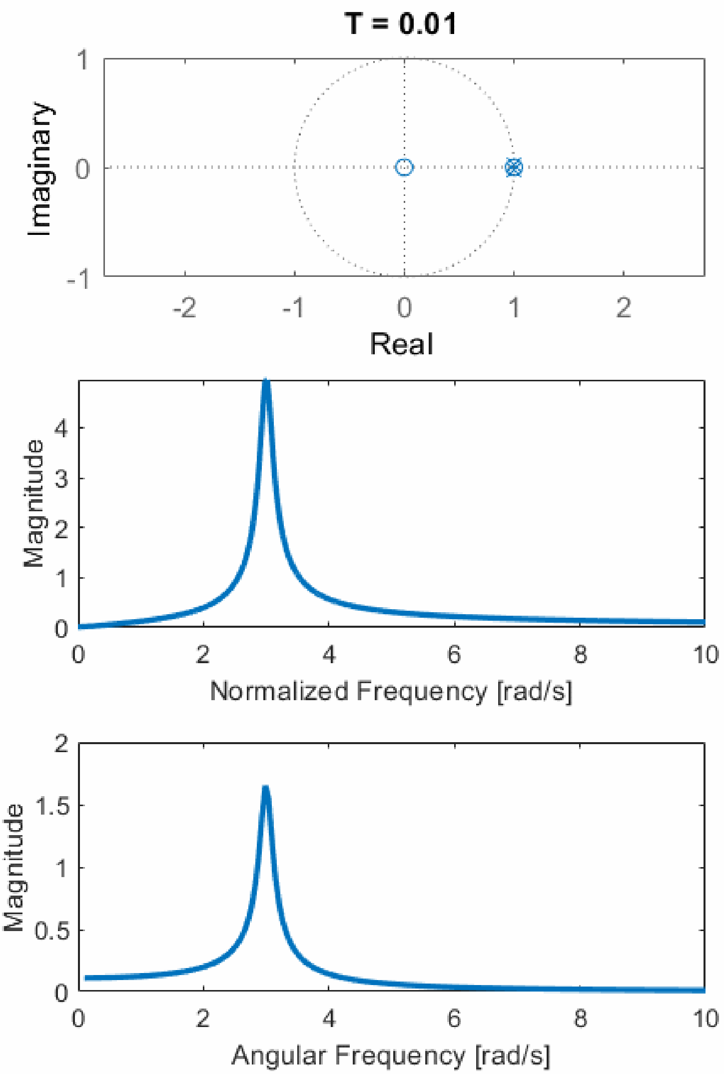
$$H(s) = \frac{1}{(s+0.1)^2 + 9} = \frac{1/2}{s + 3j + 0.1} + \frac{1/2}{s - 3j + 0.1}$$

$$H(z) = \frac{1/2}{1 - e^{(-3j-0.1)T} z^{-1}} + \frac{1/2}{1 - e^{(3j-0.1)T} z^{-1}}$$

IIR Filter Design from Derivatives

- **Example:** $H(z) = \sum_{k=1}^{K} \frac{c_k}{1-e^{p_k T} z^{-1}}$

$$H(z) = \frac{1/2}{1 - e^{(-3j-0.1)T} z^{-1}} + \frac{1/2}{1 - e^{(3j-0.1)T} z^{-1}}$$



Lecture 22: Design of FIR / IIR Filters

Foundations of Digital Signal Processing

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- Designing IIR Filters with Impulse Invariance
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IIR Filter Design by Bilinear Transform

- **Designing IIR Filters**
  - No easy ways to design digital IIR filters
  - So let us start from analog filters

- **Option 3: Preserve the definition of z!**

IIR Filter Design by Bilinear Transform

- **Question: How are z and s related?**
  - From continuous-time to discrete-time

$$s = j\Omega$$

$$e^{st} = e^{snT} = z^n$$

$$z = e^{sT}$$

Taylor Series Expansion / Approximation

  - Building an approximation ($e^x \approx 1 + x$)

$$z = \frac{e^{\frac{sT}{2}}}{e^{-\frac{sT}{2}}} \approx \frac{1 + sT/2}{1 - sT/2}$$

IIR Filter Design by Bilinear Transform

- **Question: How are z and s related?**
  - From continuous-time to discrete-time

$$s = j\Omega$$

$$e^{st} = e^{snT} = z^n$$

$$z = e^{sT} \qquad\qquad s = \frac{1}{T}\ln(z)$$

Bilinear
Expansion / Approximation

  - Building an approximation

$$s \approx \frac{2}{T}\frac{z-1}{z+1}$$

IIR Filter Design by Bilinear Transform

The Bilinear Transform

Continuous-time to discrete-time

$$s \to \frac{2}{T}\frac{1 - z^{-1}}{1 + z^{-1}}$$

Discrete-time to continuous-time

$$z \to \frac{1 + sT/2}{1 - sT/2}$$

IIR Filter Design by Bilinear Transform

- **Example:** $s \rightarrow \frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}, \quad z \rightarrow \frac{1+sT/2}{1-sT/2}$
  - Use the bilinear transform to transform the following biquad filter into the discrete-time domain.

$$H(s) = \frac{1}{(s+0.1)^2 + 9}$$

IIR Filter Design by Bilinear Transform

- **Example:** $s \to \frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}, \quad z \to \frac{1+sT/2}{1-sT/2}$
  - Use the bilinear transform to transform the following biquad filter into the discrete-time domain.

$$H(s) = \frac{1}{(s+0.1)^2+9}$$

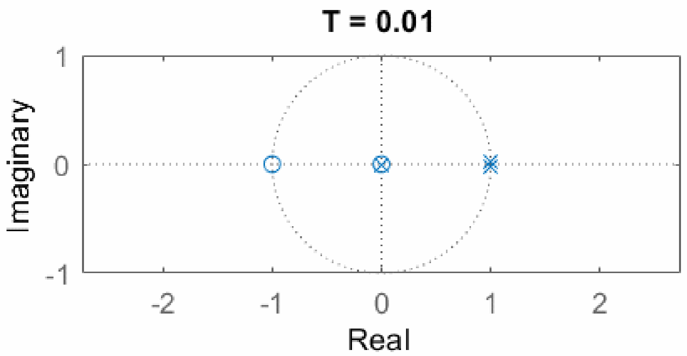
$$H(z) = \frac{1}{\left(\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}+0.1\right)^2+9}$$

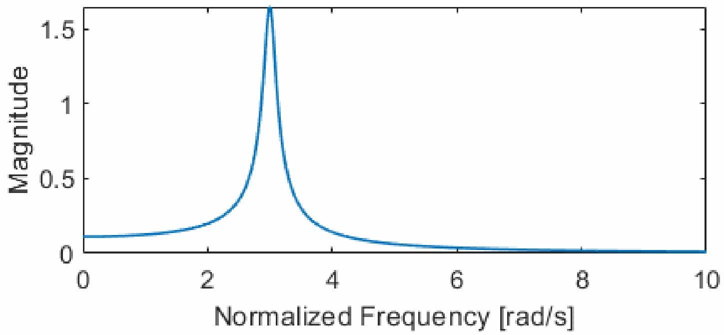
$$= \frac{(1+z^{-1})^2}{\left(\frac{2}{T}(1-z^{-1})+0.1\right)^2+9(1+z^{-1})^2}$$

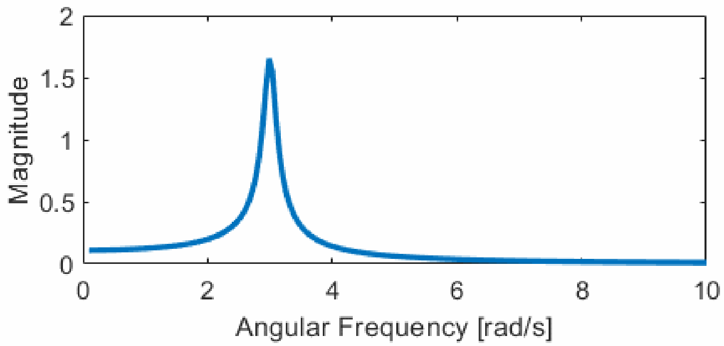
IIR Filter Design by Bilinear Transform

- **Example:** $s \rightarrow \frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}, \quad z \rightarrow \frac{1+sT/2}{1-sT/2}$

$$H(z) = \frac{(1+z^{-1})^2}{\left(\frac{2}{T}(1-z^{-1})+0.1\right)^2 + 9(1+z^{-1})^2}$$







Lecture 22: Design of FIR / IIR Filters

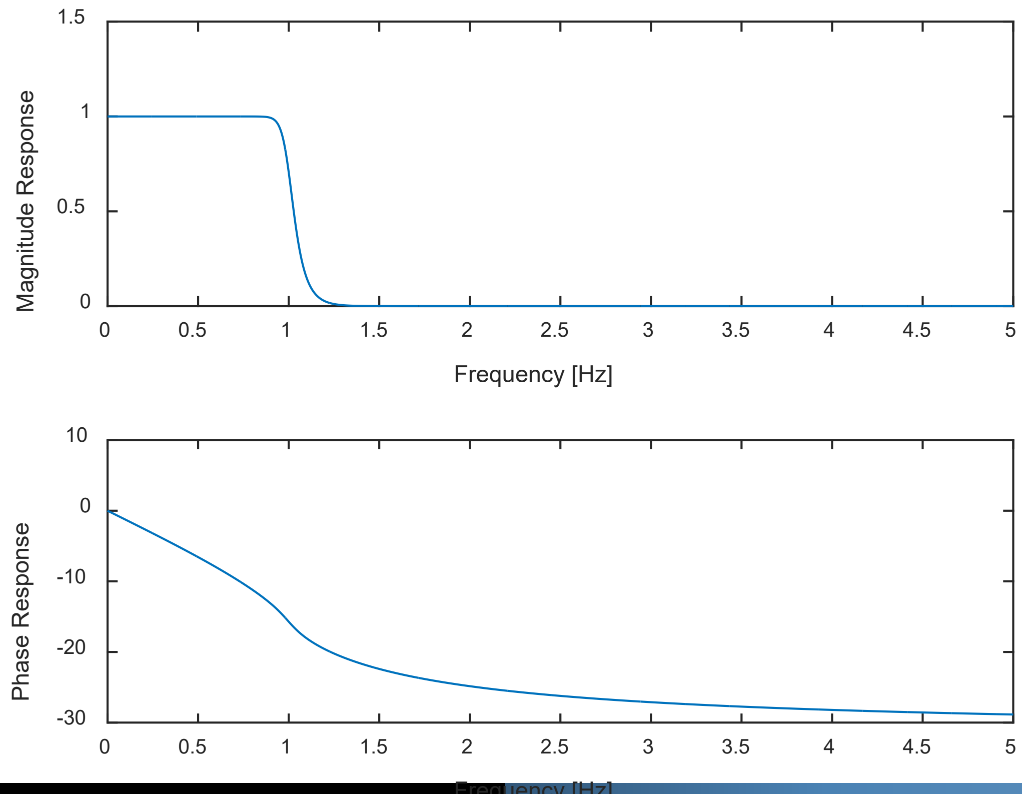
Foundations of Digital Signal Processing

Outline

- Designing FIR Filters with Windows
- Designing FIR Filters with Frequency Sampling
- Designing FIR Filters with Equi-ripples
- Designing IIR Filters with Discrete Differentiation
- Designing IIR Filters with Impulse Invariance
- Designing IIR Filters with the Bilinear Transform
- **Related Analog Filters**

Multi-pole Filters

- **Butterworth:** Maximally flat passband
- **Chebyshev:** Faster cutoff with passband ripple
- **Elliptic:** Fastest cutoff with passband and stopband ripple



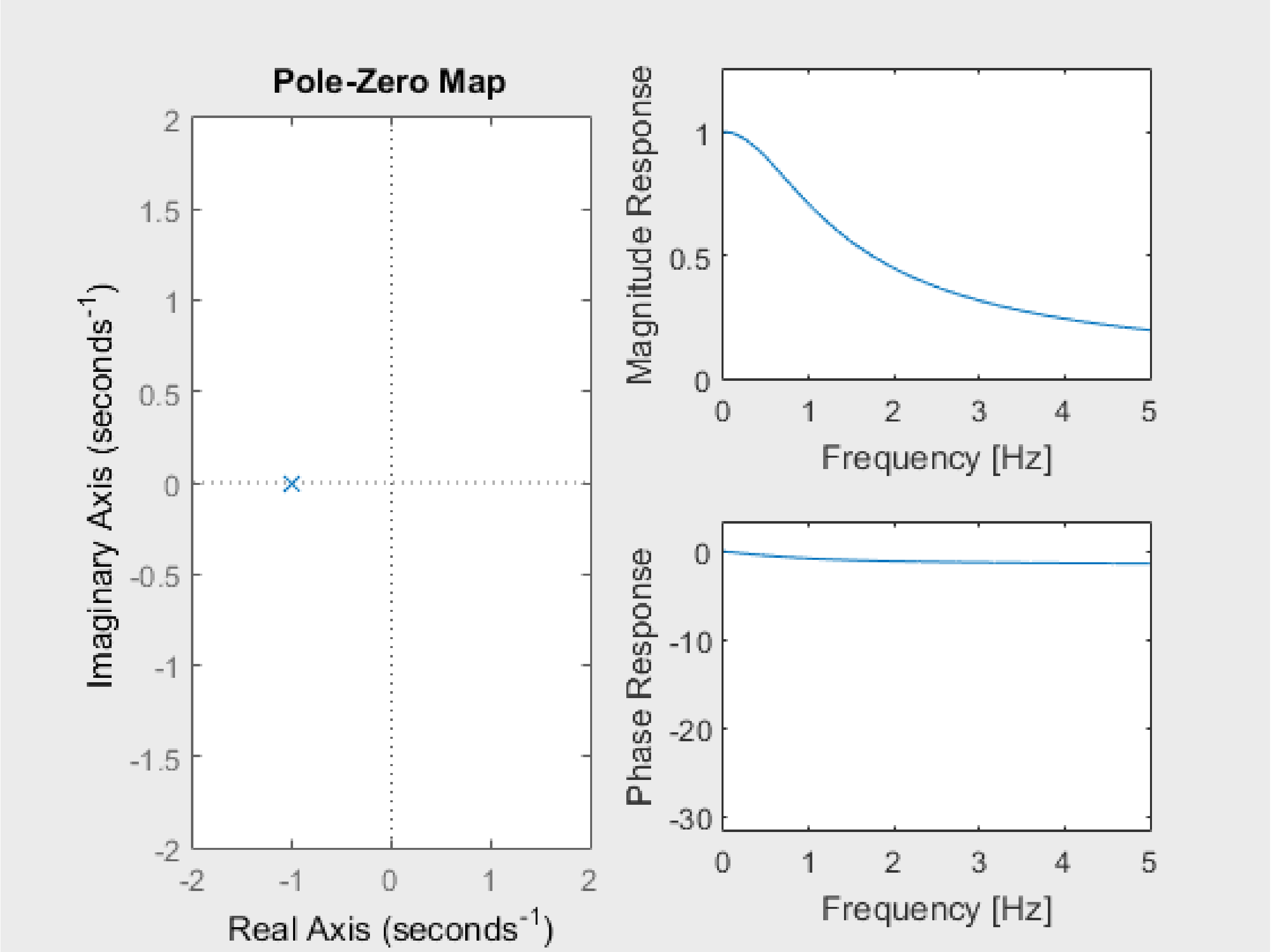
Butterworth Filter

- **Butterworth Filter of order N**

$$|H(j\omega)| = \frac{1}{\sum_{k=1}^{N}(s - s_k)} \qquad s_k = e^{\frac{j(2k+N-1)\pi}{2N}}$$

Butterworth Filter

- **Butterworth Filter of order N**
  - N equally spaced poles on a circle on the left-hand-side of the s-plane



Butterworth Filter

- **Properties of the Butterworth Filter**
  - It is maximally flat at $\omega = 0$
  - It has a cutoff frequency $|H(\omega)| = \frac{1}{\sqrt{2}}$ at $\omega = \omega_c$
  - For large n, it becomes an ideal filter

Chebyshev Filter

- **Chebyshev Filter of order N**

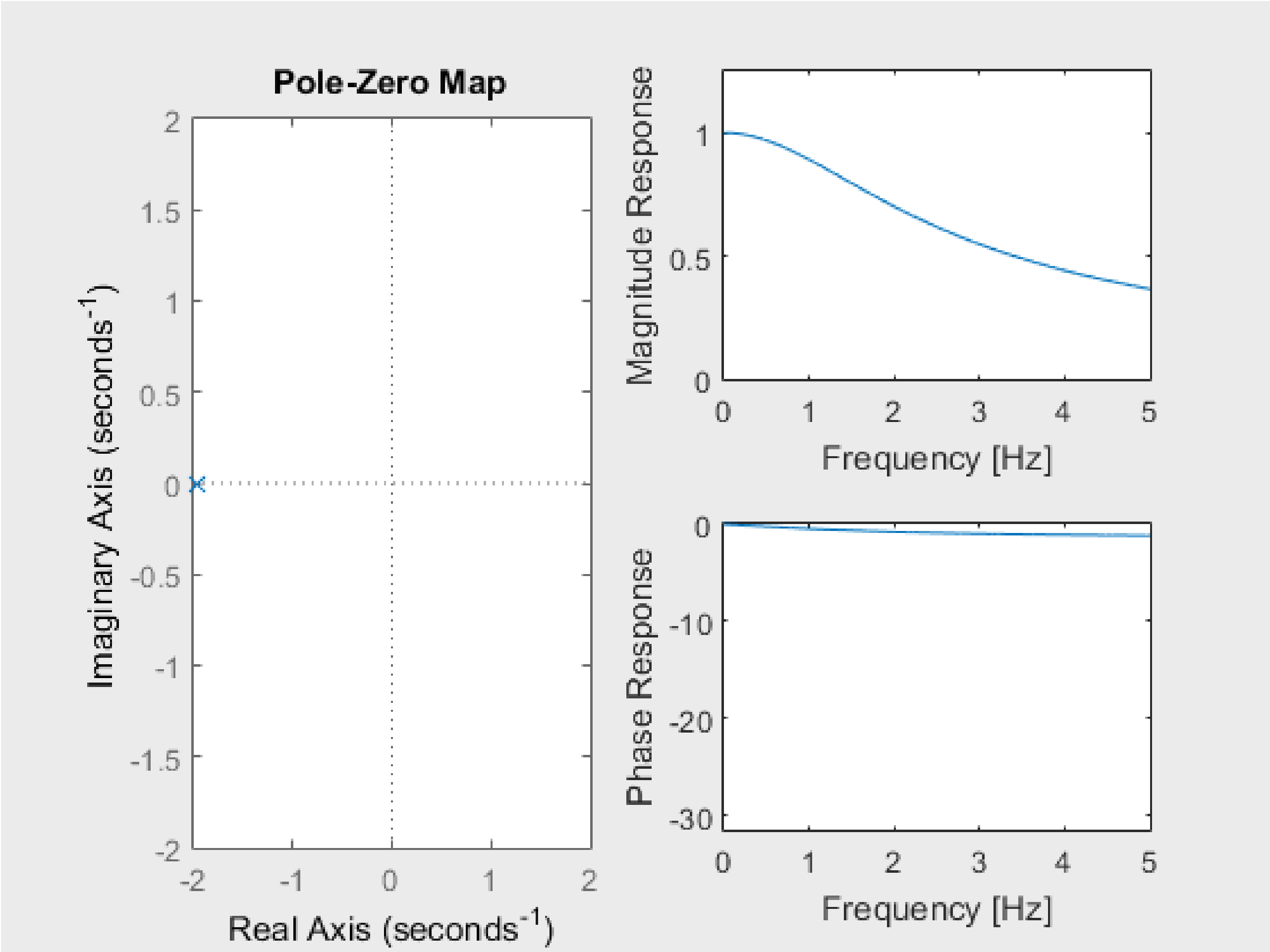
$$|H(j\omega)| = \frac{1}{\sqrt{1 + \epsilon^2 C_n^2(\omega)}}$$

$C_n^2(\omega)$ is an nth-order Chebyshev Polynomial

ϵ^2 controls ripple

Chebyshev Filter

- **Chebyshev Filter of order N**



Chebyshev Filter

- **Properties of the Chebyshev Filter**
  - It has ripples in the passband and is smooth in the stopband.
  - The ratio between the maximum and minimum ripples in the passband is

$$(1+\epsilon^2)-1/2$$

  - If ϵ is reduced (i.e., the ripple size is reduced), then the stopband attenuation is reduced.
  - It has a sharper cut-off than a Butterworth filter, but at the expense of passband rippling

Elliptic Filter

- **Elliptic Filter of order N**

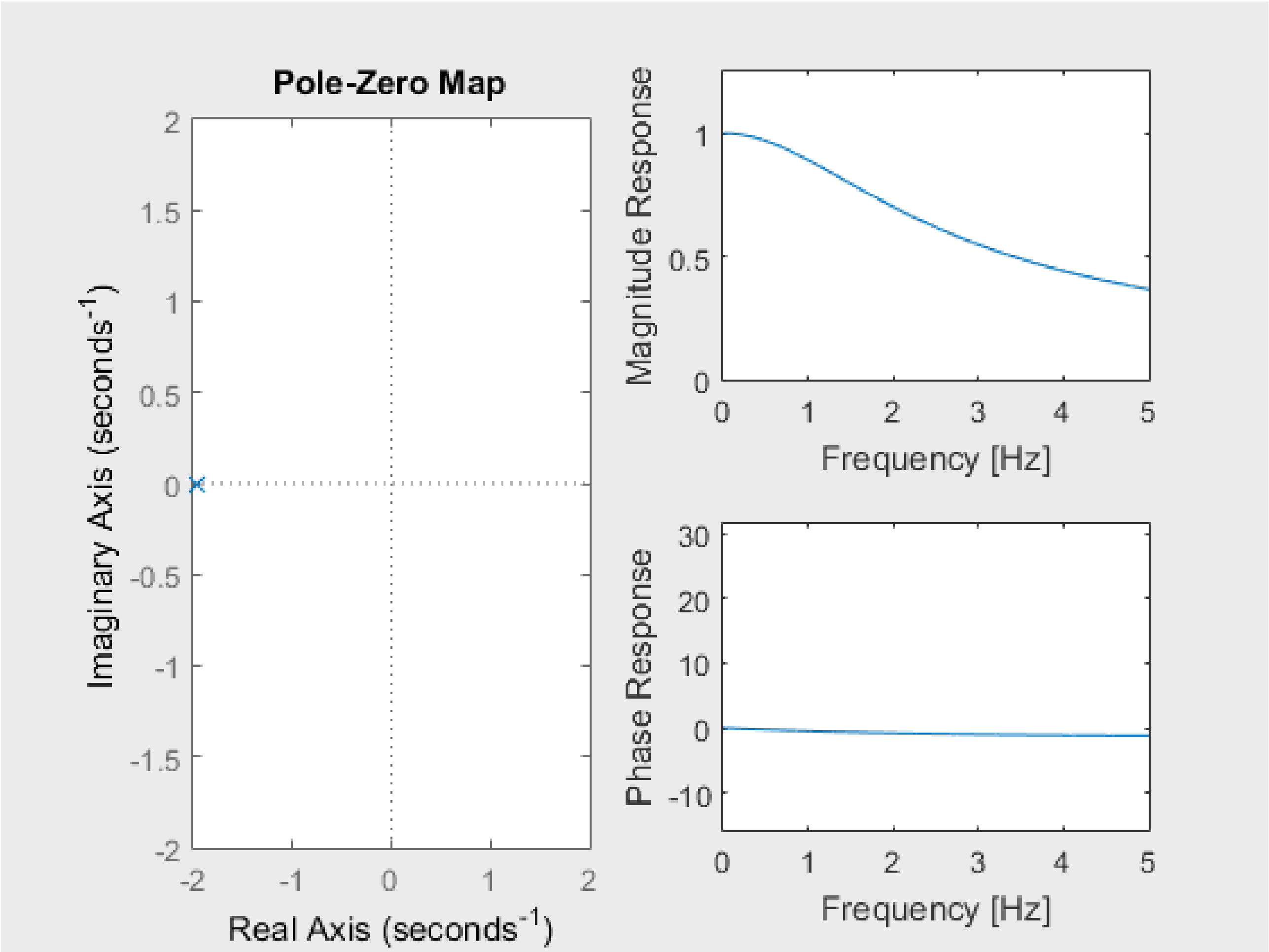
$$|H(j\omega)| = \frac{1}{\sqrt{1 + \epsilon^2 R_n^2(\omega)}}$$

$R_n^2(\omega)$ is an nth-order elliptic function

ϵ^2 controls ripple

Elliptic Filter

- **Elliptic Filter of order N**



Elliptic Filter

- **Properties of the Elliptic Filter**
  - It has ripples in the passband and the stopband
  - The ratio between the maximum and minimum ripples is larger than the Chebyshev filter, but it has an even quicker transition from passband to stopband
  - It has poles and zeros, but they are much more difficult to compute compared with the Butterworth and Chebyshev filters