Lecture 8 : Fourier Theory

Foundations of Digital Signal Processing

Outline

- The Discrete-time Fourier Transform (DTFT) Review
- The Properties of the Discrete-time Fourier Transform (DTFT)
- General Fourier Theory
- The Fourier Transform
- The Fourier Series
- The Discrete Fourier Series (The Discrete Fourier Transform)

News

- **Homework #4**
  - Due *Thursday* by 11:59 PM
  - Submit via canvas
- **Coding Problem #2**
  - Due *Thursday* by 11:59 PM
  - Submit via canvas

News

- **Exam #1**
  - September 25^{th} (1 week away)
  - Will cover all material up to today… such as
    - Signal properties
    - System properties
    - LTI Systems
    - Difference equations
    - Discrete-time convolution
    - The Z-transform and its properties
    - The Discrete-time Fourier Transform and its properties
    - Etc.

Lecture 8 : Fourier Theory

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- **The Discrete-time Fourier Transform (DTFT) Review**
- The Properties of the Discrete-time Fourier Transform (DTFT)
- General Representation / Fourier Theory
- The Fourier Series
- The Fourier Transform
- Laplace Transform
- The Fourier Relationships

The Discrete-Time Fourier Transform

- **The Discrete-Time Fourier Transform (DTFT)**

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

- **The Inverse Discrete-Time Fourier Transform**

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(\omega)e^{j\omega n}\, d\omega$$

The Discrete-Time Fourier Transform

- **The Discrete-Time Fourier Transform (DTFT)**

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

- **Question: How do I interpret this DTFT?**

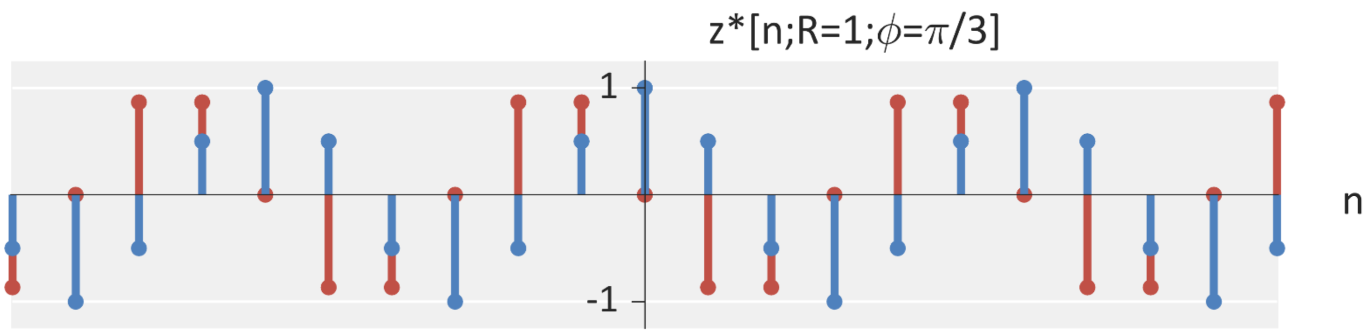
The Discrete-Time Fourier Transform

- **The Discrete-Time Fourier Transform (DTFT)**

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

Inner product of signal and sinusoids!

- **Question: How do I interpret this DTFT?**



The Discrete-Time Fourier Transform

- **The Discrete-Time Fourier Transform (DTFT) Table**
  - http://smartdata.ece.ufl.edu/eee5502/eee5502_DiscreteTransforms.pdf

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The Discrete-Time Fourier Transform

- **Example:** Compute and sketch the DTFT of $x[n] = \cos\left(\frac{\pi n}{3}\right)$

- **What is interesting about this answer?**

The Discrete-Time Fourier Transform

- **Example:** Compute and sketch the DTFT of $x[n] = \cos\left(\frac{\pi n}{3}\right)$

- **What is interesting about this answer?**

$$\mathrm{X}(\omega) = 2\pi \sum_{k=-\infty}^{\infty} \left[\delta\left(\omega - \frac{\pi}{3} - 2\pi k\right) + \delta\left(\omega + \frac{\pi}{3} - 2\pi k\right)\right]$$

The Discrete-Time Fourier Transform

- **Example:** Compute and sketch the DTFT of $x[n] = \cos\left(\frac{\pi n}{3}\right)$

- **What is interesting about this answer?**

$$X(\omega) = \pi \sum_{k=-\infty}^{\infty} \left[\delta\left(\omega - \frac{\pi}{3} - 2\pi k\right) + \delta\left(\omega + \frac{\pi}{3} - 2\pi k\right)\right]$$

The Discrete-Time Fourier Transform

- **Example:** Sketch the magnitude of

$$X(\omega) = \frac{1}{1-(1/2)e^{-j\omega}}$$

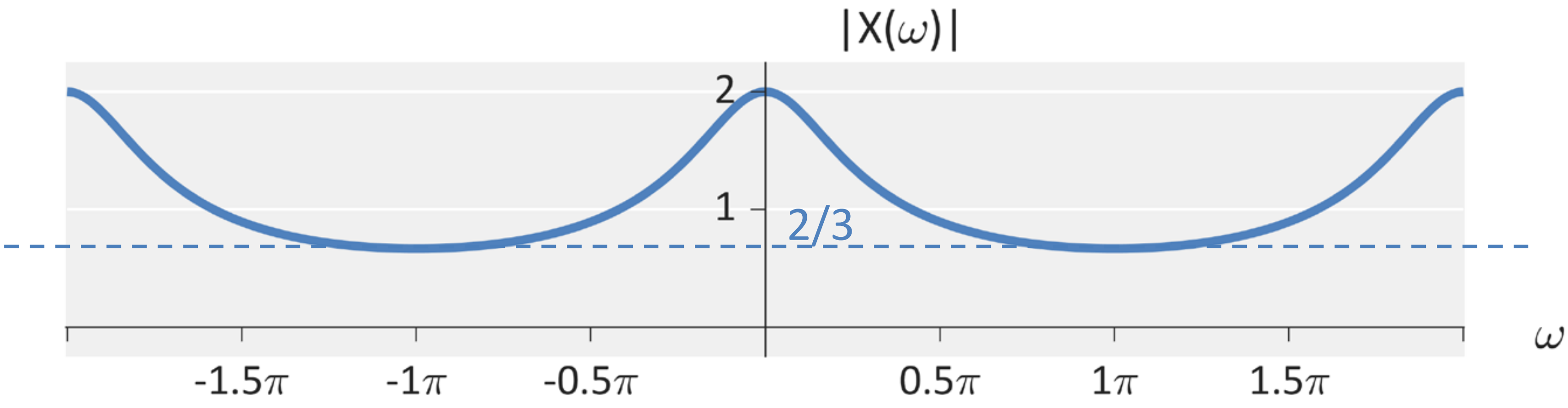
The Discrete-Time Fourier Transform

- **Example:** Sketch the magnitude of

$$X(\omega) = \frac{1}{1 - (1/2)e^{-j\omega}}$$

Solution:

$$|X(\omega)| = \frac{1}{\sqrt{5/4 - \cos(\omega)}}$$



The Discrete-Time Fourier Transform

- **Example:** Compute the DTFT of

$$x[n] = \delta[n] + 2\delta[n-1] + \delta[n-2]$$

The Discrete-Time Fourier Transform

- **Example:** Compute the magnitude and phase of the DTFT of

$$x[n] = \delta[n] + 2\delta[n-1] + \delta[n-2]$$

The Discrete-Time Fourier Transform

- **Example:** Compute the magnitude and phase of the DTFT of

$$x[n] = \delta[n] + 2\delta[n-1] + \delta[n-2]$$

$$X(\omega) = e^{-j\omega}\left[e^{+j\omega} + 2 + e^{-j\omega}\right] = 2e^{-j\omega}[1 + \cos(\omega)]$$

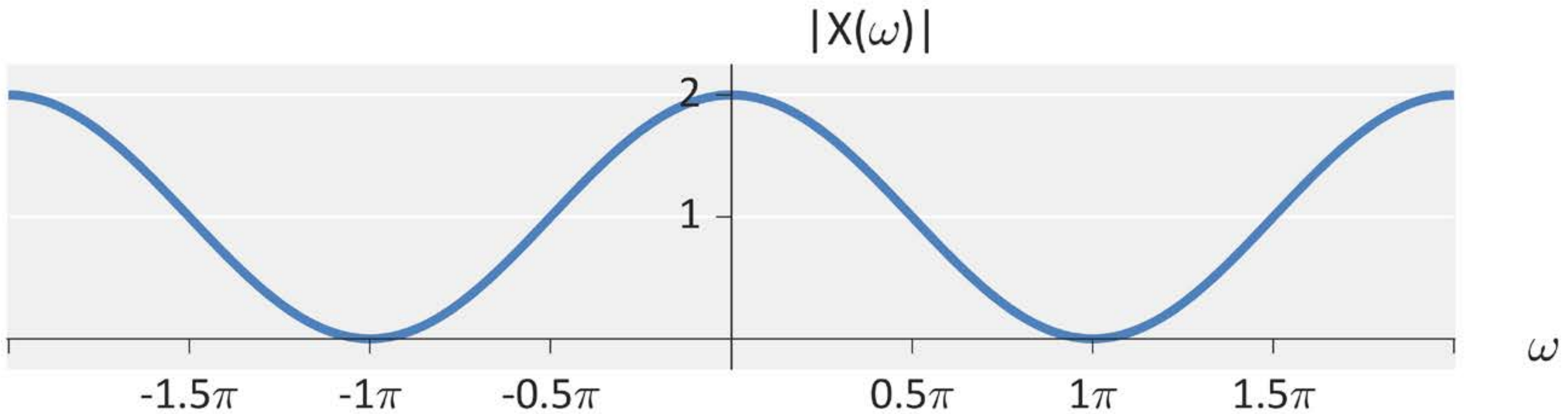
$$|X(\omega)| = 2(1 + \cos(\omega))$$
$$\angle X(\omega) = e^{-j\omega}$$

The Discrete-Time Fourier Transform

- **Example:** Compute the magnitude and phase of the DTFT of

$$x[n] = \delta[n] + 2\delta[n-1] + \delta[n-2]$$

$$|X(\omega)| = 2(1 + \cos(\omega))$$

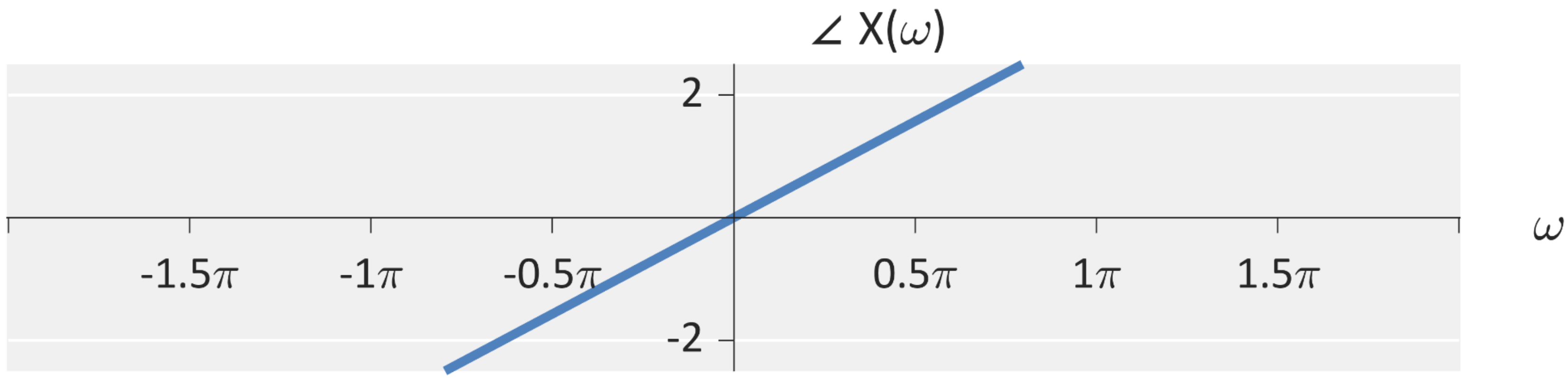


The Discrete-Time Fourier Transform

- **Example:** Compute the magnitude and phase of the DTFT of

$$x[n] = \delta[n] + 2\delta[n-1] + \delta[n-2]$$

$$|X(\omega)| = 1 + \cos(\omega)$$



Lecture 8 : Fourier Theory

Foundations of Digital Signal Processing

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- The Discrete-time Fourier Transform (DTFT) Review
- **The Properties of the Discrete-time Fourier Transform (DTFT)**
- General Representation / Fourier Theory
- The Fourier Series
- The Fourier Transform
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The Discrete-Time Fourier Transform

- **The Discrete-Time Fourier Transform (DTFT) Properties**
  - http://smartdata.ece.ufl.edu/eee5502/eee5502_DiscreteTransforms.pdf

The Discrete-Time Fourier Transform

- **Question: Why is the convolution property important?**

The Discrete-Time Fourier Transform

- **Example:** Compute the DTFT of $\mathrm{x}[n] = \left(\frac{1}{2}\right)^{n-2} u[n-1]$

The Discrete-Time Fourier Transform

- **Example:** Compute the inverse DTFT of $\mathrm{X}(\omega) = \frac{0.5e^{-j\omega}}{1-0.5e^{-j\omega}}$

The Discrete-Time Fourier Transform

- **Example:** Compute the DTFT of $x[n] = \left(\frac{1}{2}\right)^n u[n] * \left(\frac{1}{3}\right)^n u[n]$

The Discrete-Time Fourier Transform

- **Example:** Compute the DTFT of $x[n] = \left(\frac{1}{2}\right)^n u[n] * \left(\frac{1}{3}\right)^n u[n]$

- **Solution:**

- $\mathrm{X}(\omega) = \dfrac{1}{\left(1-(1/2)e^{-j\omega}\right)\left(1-(1/3)e^{-j\omega}\right)}$

The Discrete-Time Fourier Transform

- **Example:** Compute the DTFT of $x[n] = \left(\frac{1}{2}\right)^{-n-1} u[-n-1]$

The Discrete-Time Fourier Transform

- **Example:** Let $x[n] = 2(u[n] - u[n-4])$
- **Compute the power of $X(\omega)$**

Lecture 8 : Fourier Theory

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Representations and Bases

- **Representations and Bases**
  - We often care about how to be “represent” data
  - That is, how we can decompose one signal into other signals?

$$x[n] = \sum_k a_k \phi_k[n]$$

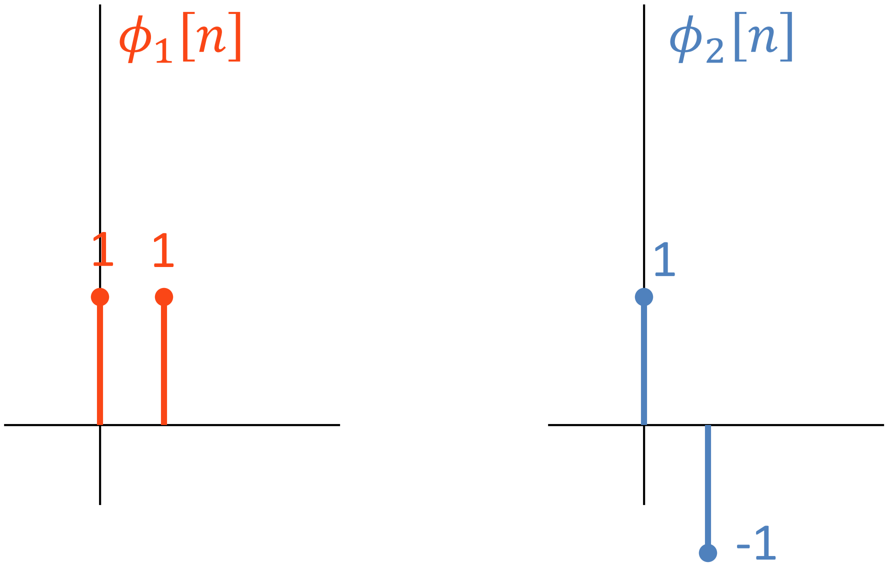
My signal

Basis

Representations and Bases

- **Representations and Bases**
  - We often care about how to be “represent” data
  - That is, how we can decompose one signal into other signals?

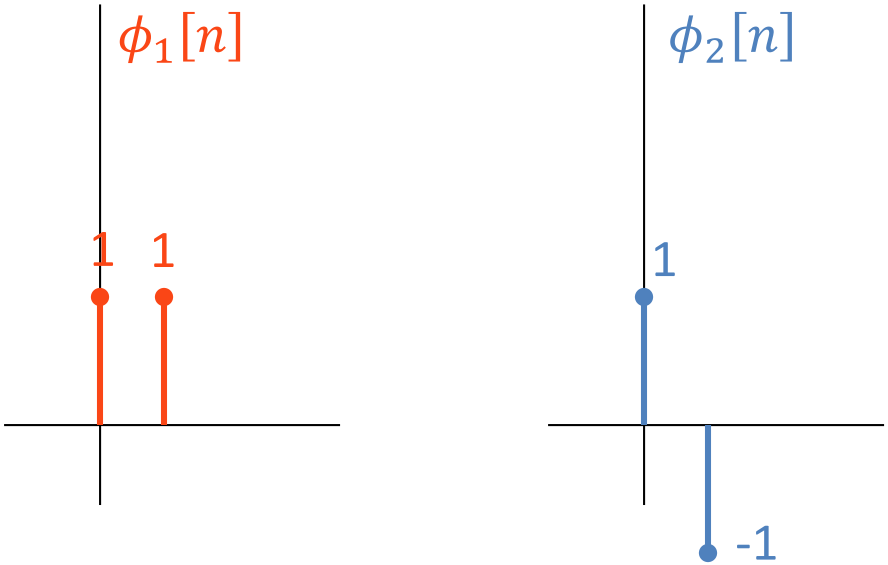
$$x[n] = \sum_k a_k \phi_k[n]$$



Representations and Bases

- **Representations and Bases**
  - We often care about how to be "represent" data
  - That is, how we can decompose one signal into other signals?

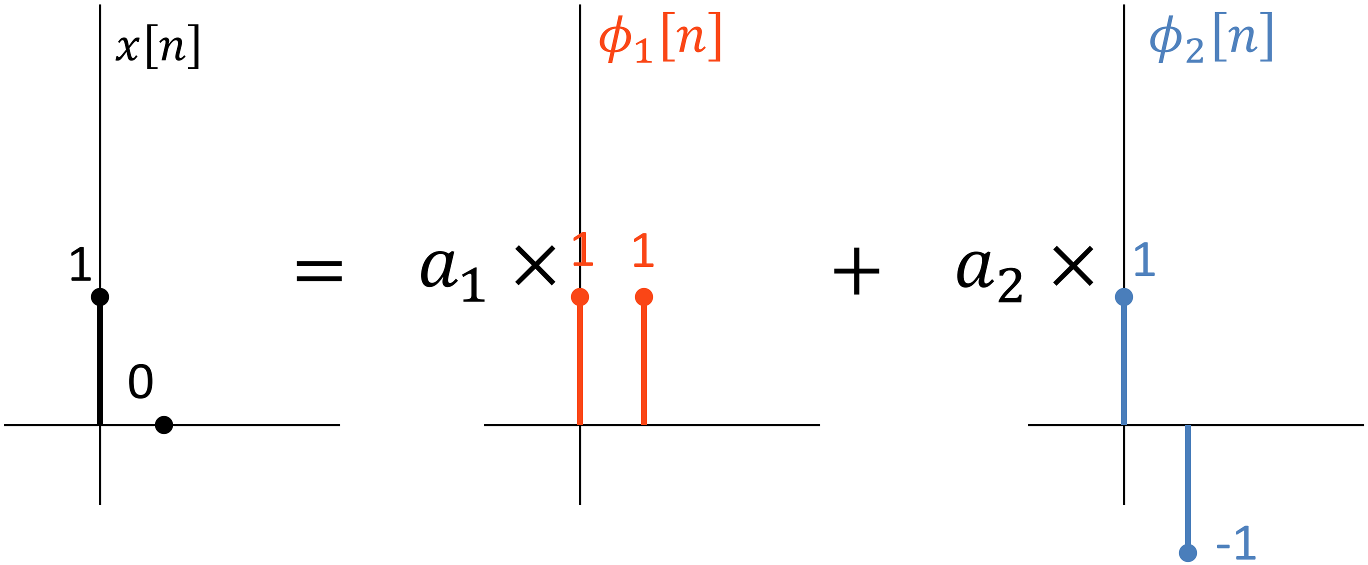
$$x[n] = a_1\phi_1[n] + a_2\phi_2[n]$$



Representations and Bases

- **Representations and Bases**
  - We often care about how to be “represent” data
  - **Question:** What values of a_1 and a_2 give us $x[n]$.

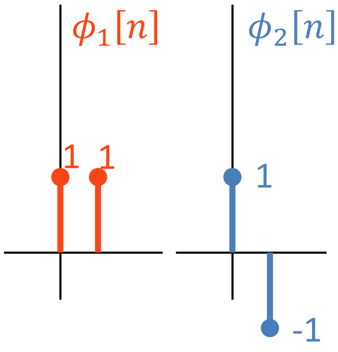
$$x[n] = a_1\phi_1[n] + a_2\phi_2[n]$$



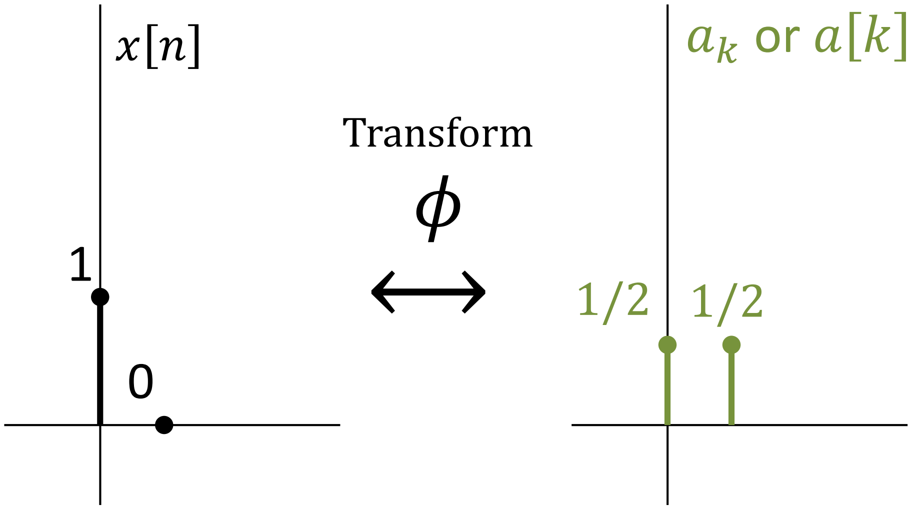
Representations and Bases

Representations and Bases

- We often care about how to be "represent" data

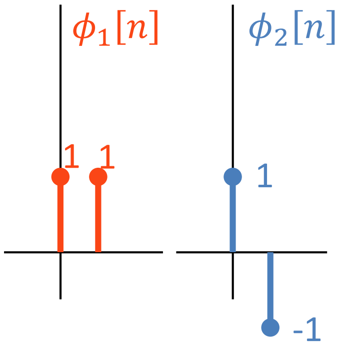


$$x[n] = a_1\phi_1[n] + a_2\phi_2[n]$$

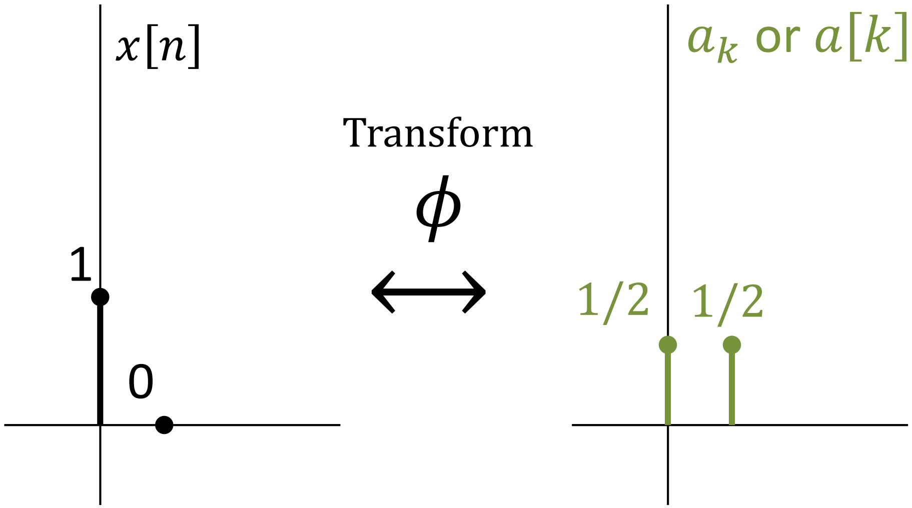


Representations and Bases

- **Representations and Bases**
  - We often care about how to be "represent" data
  - **Forward transform**

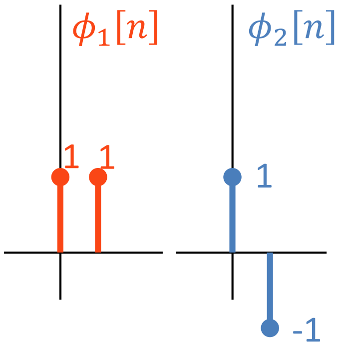


$$x[n] = a[1]\phi_1[n] + a[2]\phi_2[n]$$

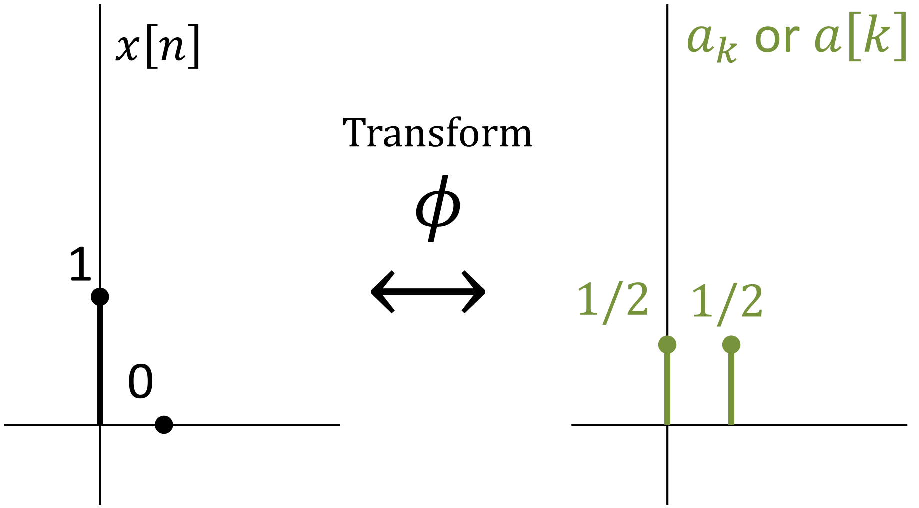


Representations and Bases

- We often care about how to be "represent" data
- **Inverse transform**

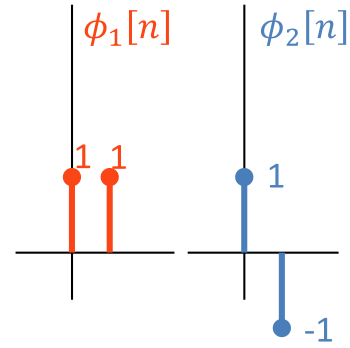


$$a[k] = \frac{1}{2}\left[a[1]\phi_1[k] + a[2]\phi_2[k]\right]$$



Representations and Bases

- **Representations and Bases**
  - We often care about how to be “represent” data
  - **Inverse transform**



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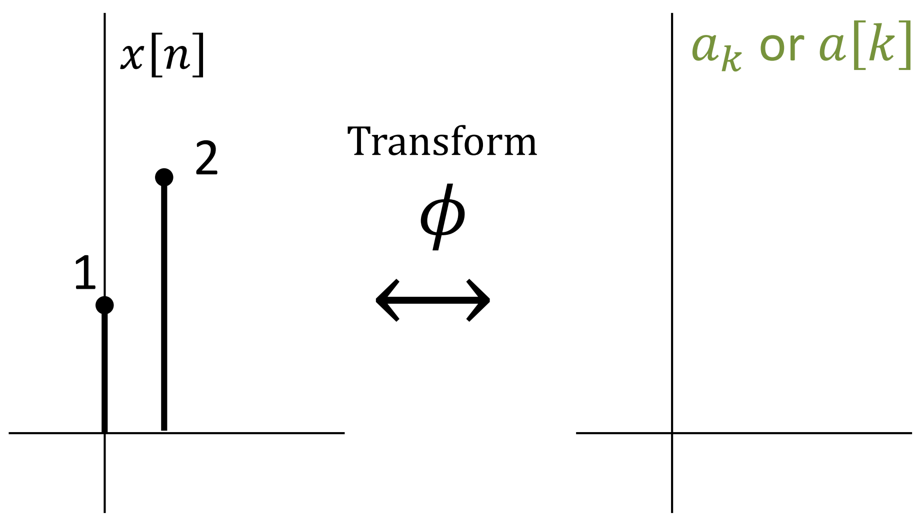
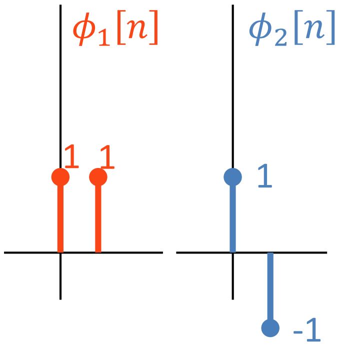


Representations and Bases

Representations and Bases

- We often care about how to be "represent" data
- **Inverse transform**



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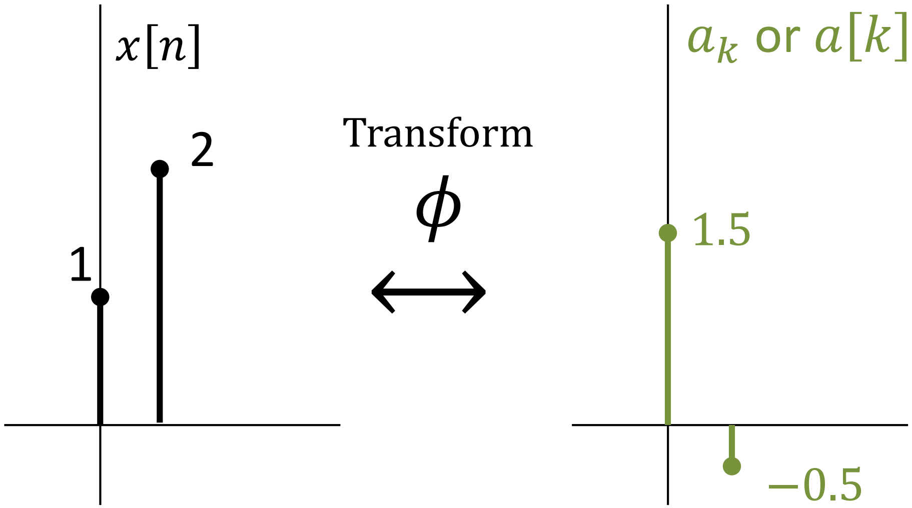
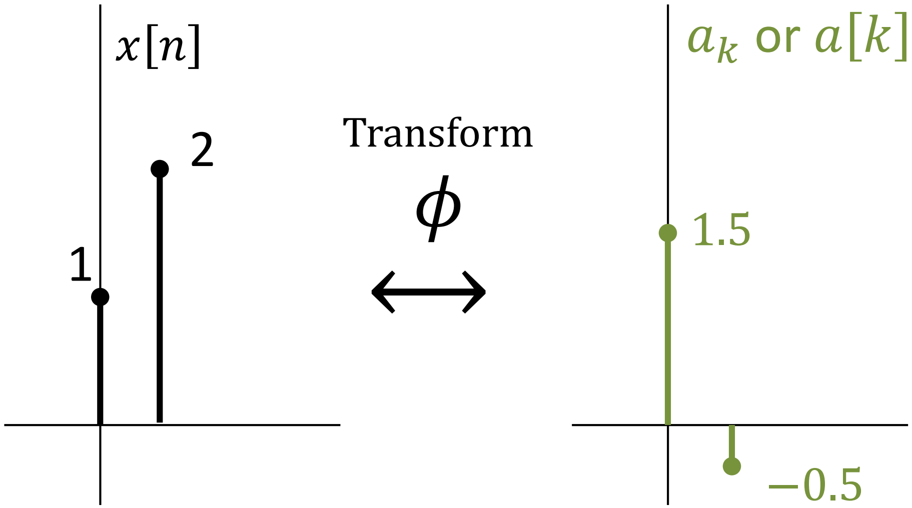
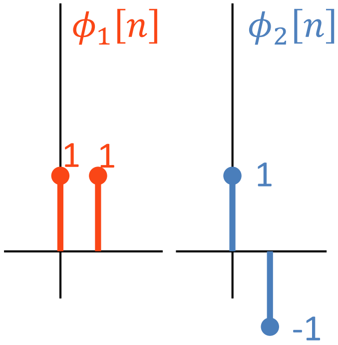


- **Representations and Bases**
  - We often care about how to be “represent” data
  - **Question:** Why is the forward and inverse almost the same?

Answer
Orthogonality:

$$\sum_{n=0}^{N-1} \phi_1[n]\phi_2[n] = 0$$



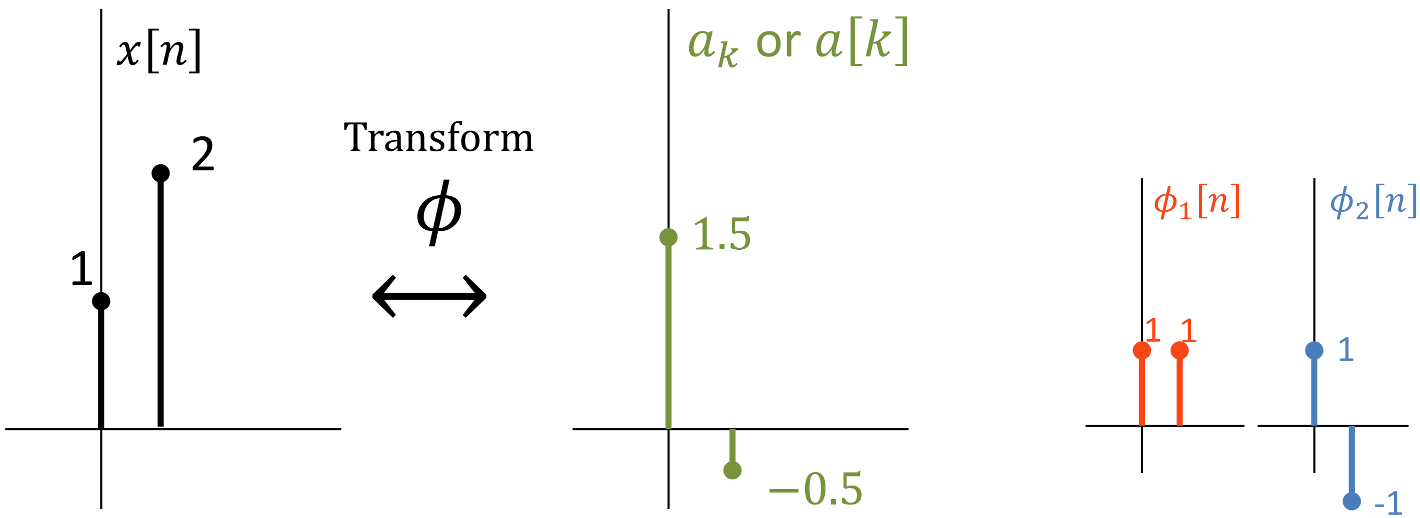


Representations and Bases

- We often care about how to be “represent” data
- **Question:** Why is the forward and inverse almost the same?

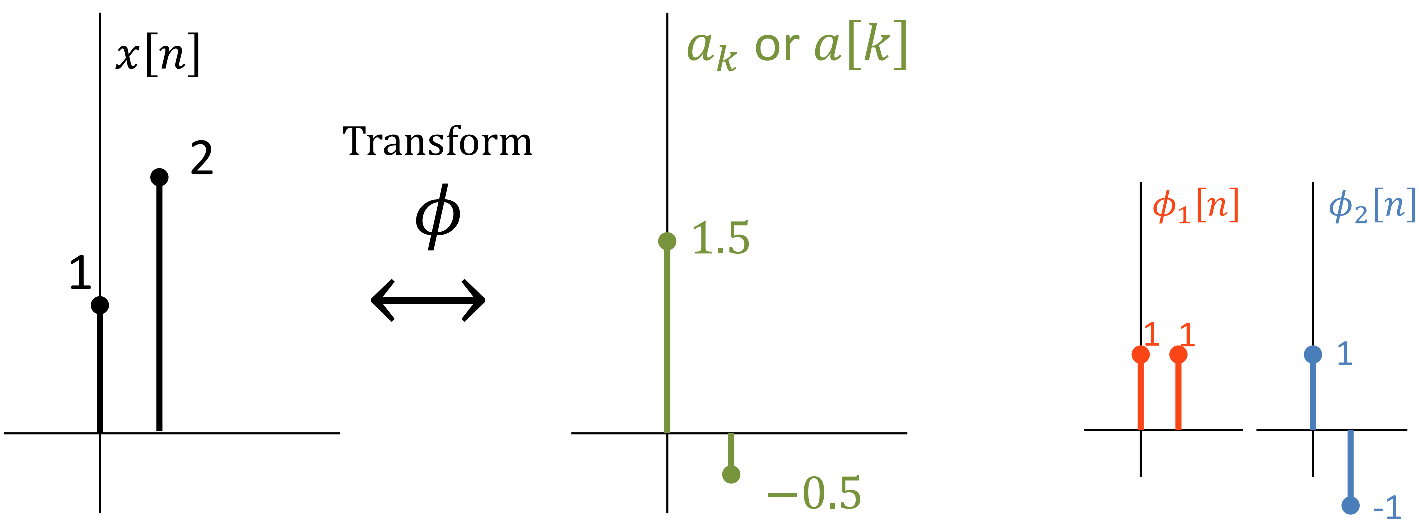
More general Orthogonality:

$$\sum_{n=0}^{N-1} \phi_i[n]\phi_j[n] = 0 \quad \text{For any } i \neq j$$



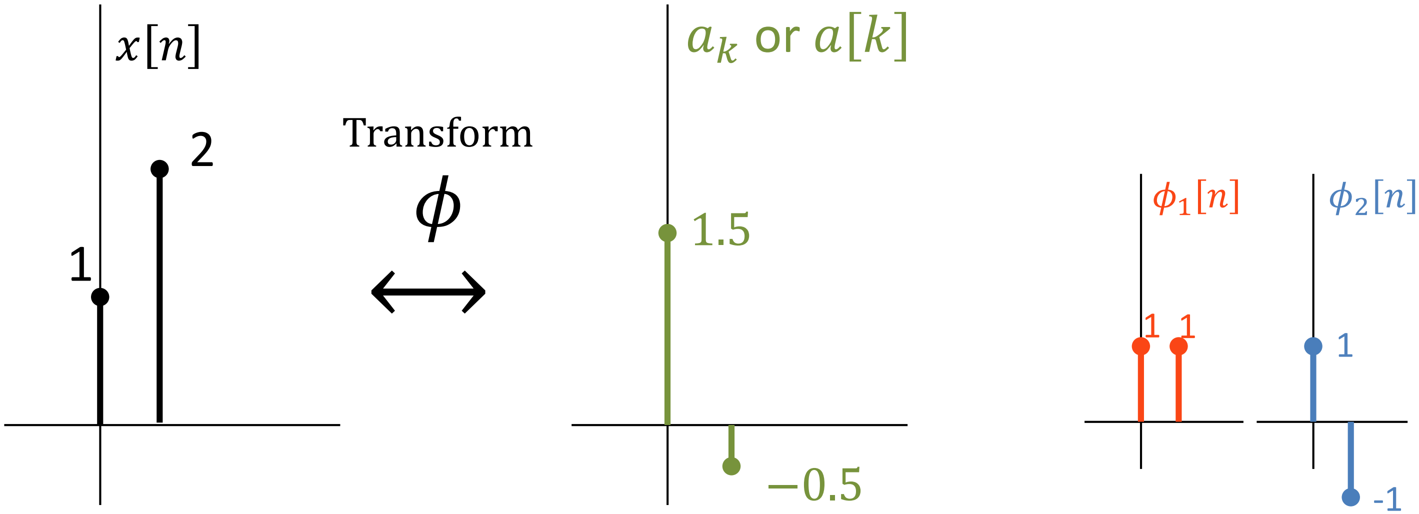
Representations and Bases

- **Representations and Bases**
  - We often care about how to be “represent” data
  - **Question:** Why do I care about transforms and representations?



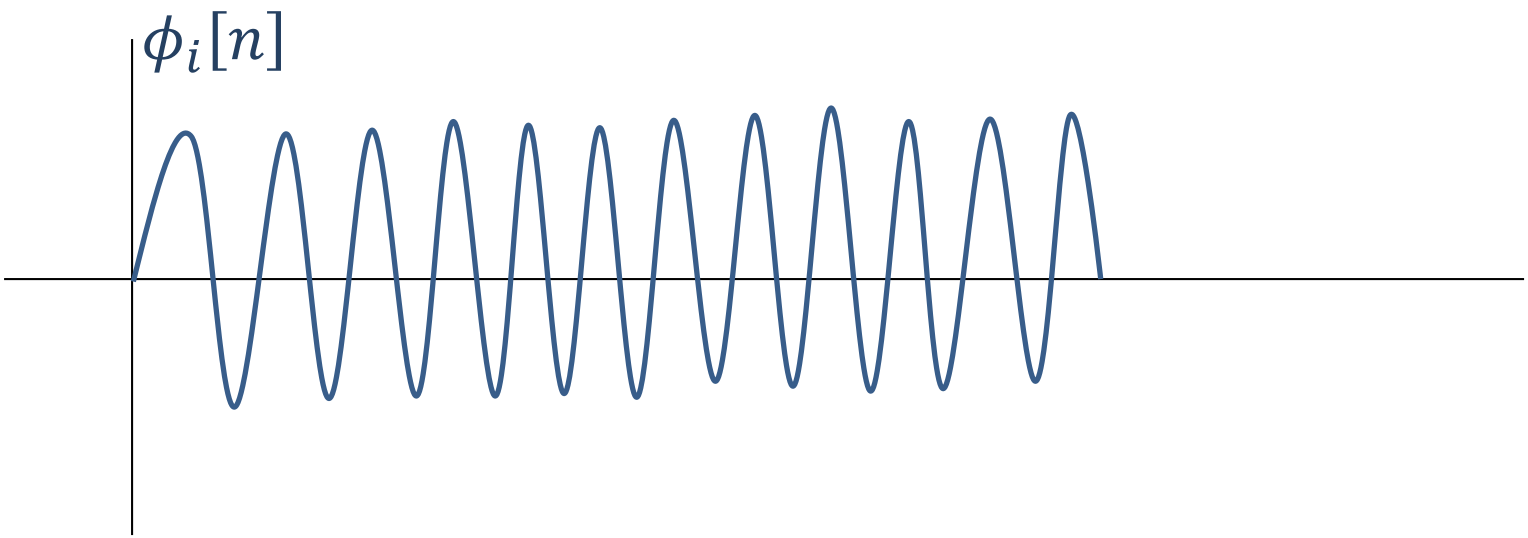
Representations and Bases

- **Representations and Bases**
  - We often care about how to be "represent" data
  - **Question:** Why do I care about transforms and representations?
    - ◊ Compression
    - ◊ Better to understand what it going on in the signal?
    - ◊ Better processing



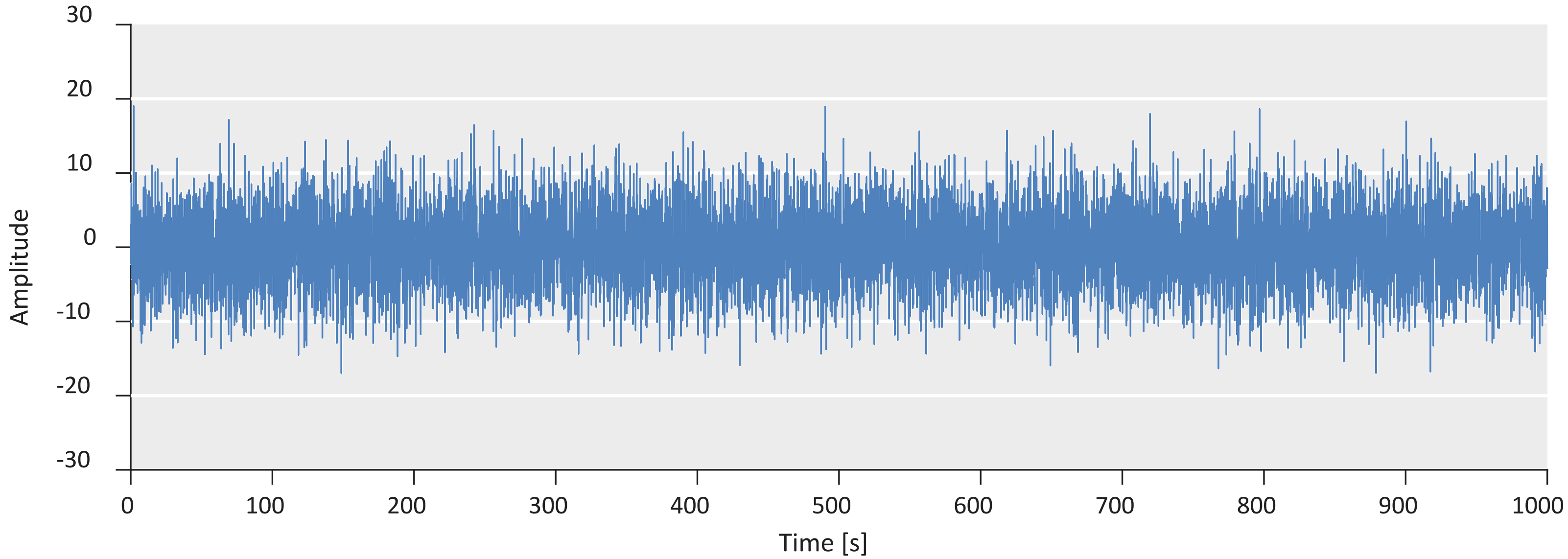
Representations and Bases

- **Example**
  - Fourier representation – why might we care about this?



Representations and Bases

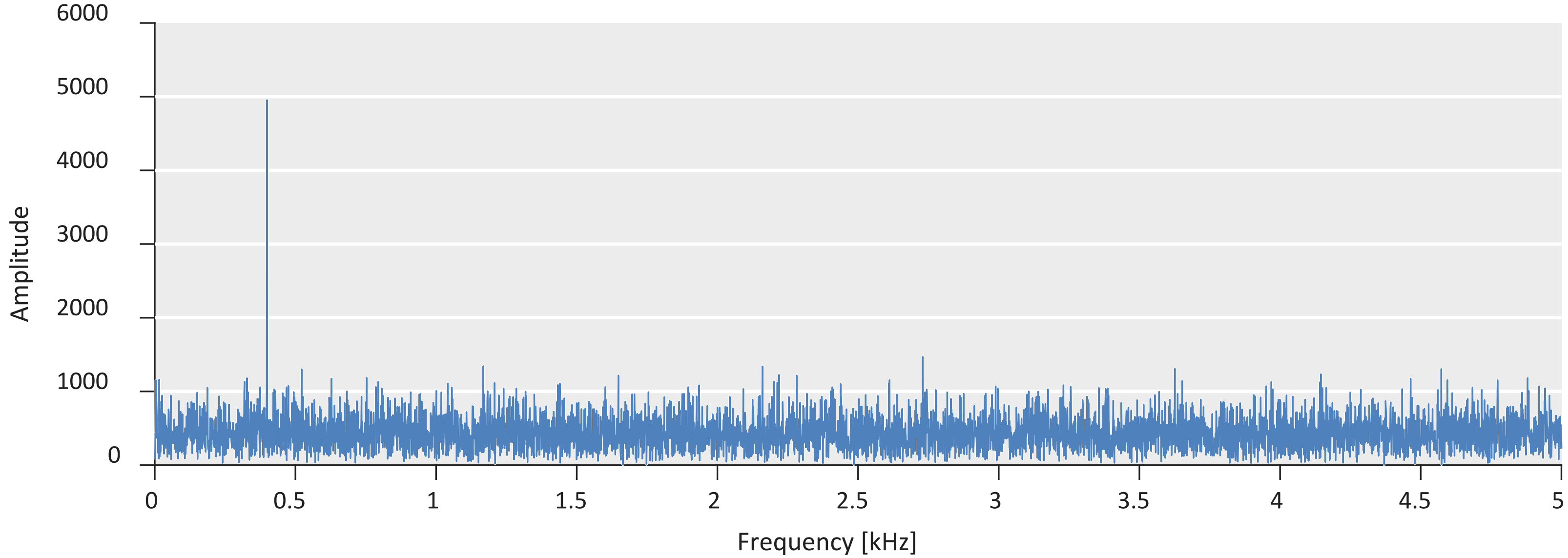
- **Fourier Representation Example**
  - What is this?



Representations and Bases

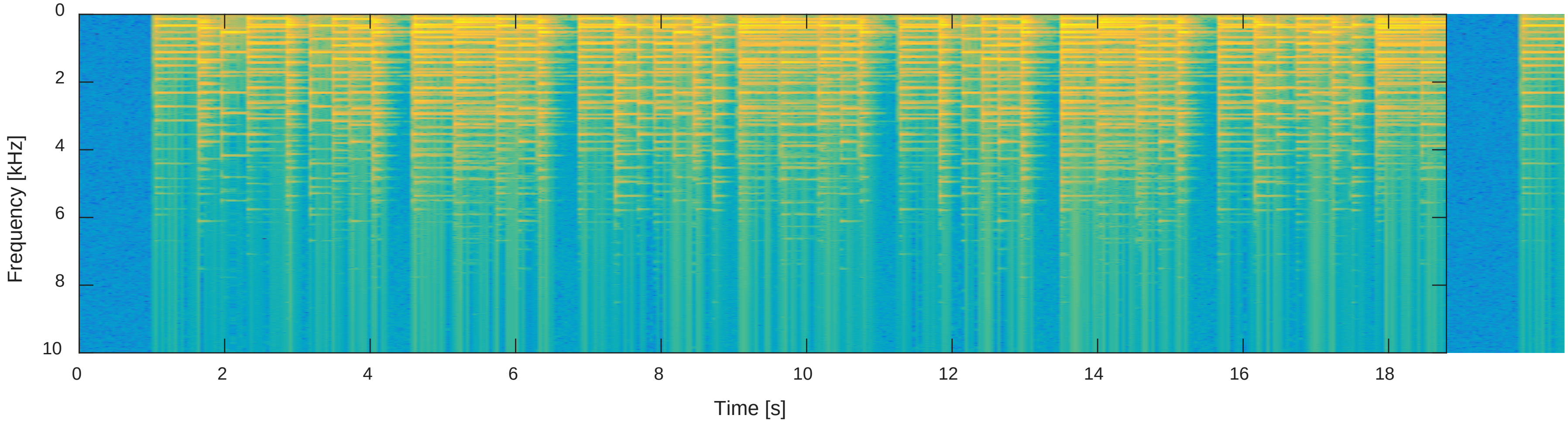
Fourier Representation Example

- It’s a sinusoid!

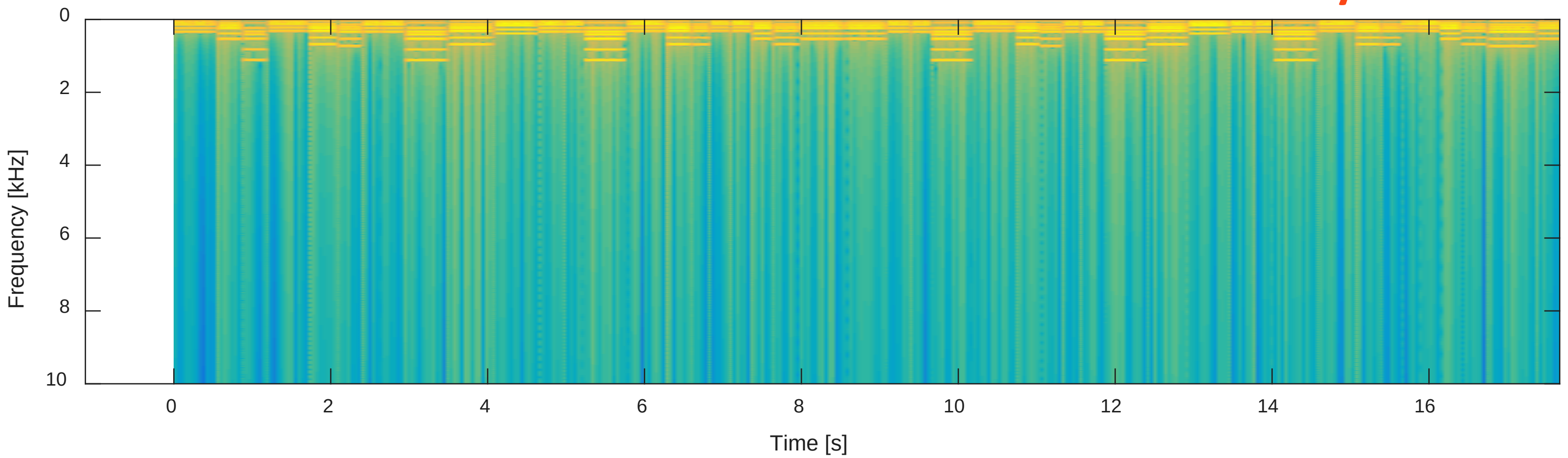


Time-Frequency Representation Example

Circus



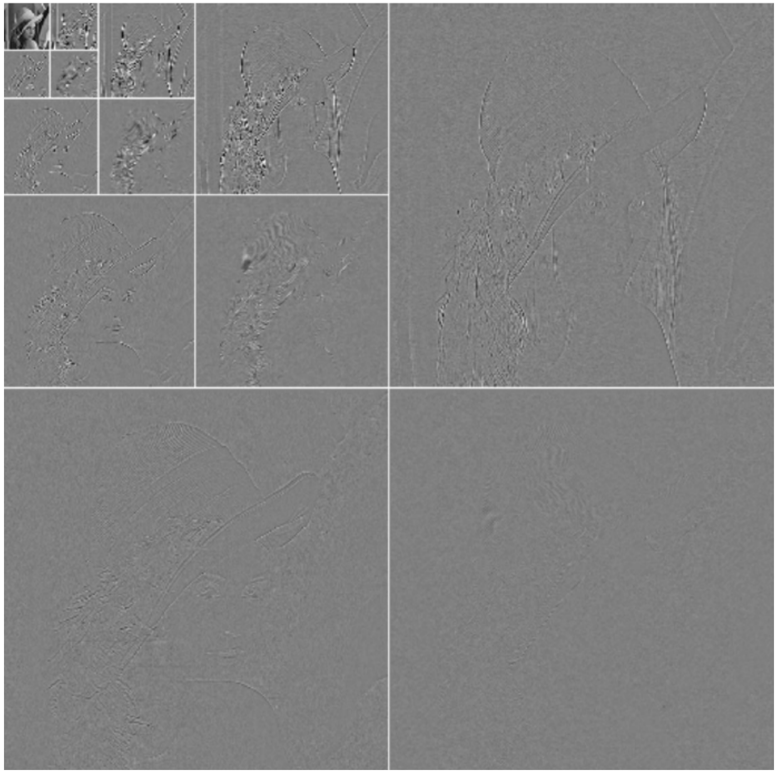
Circus - synthesized



Representations and Bases

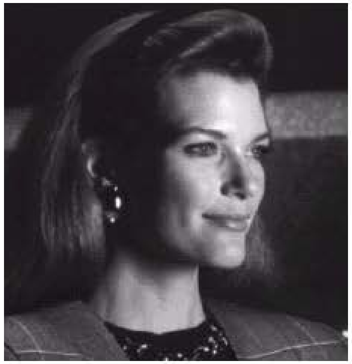
Wavelet Example

These pictures have the same amount of information!!



From: http://web.stanford.edu/class/ee398a/handouts/lectures/09-SubbandCoding.pdf

Representations and Bases



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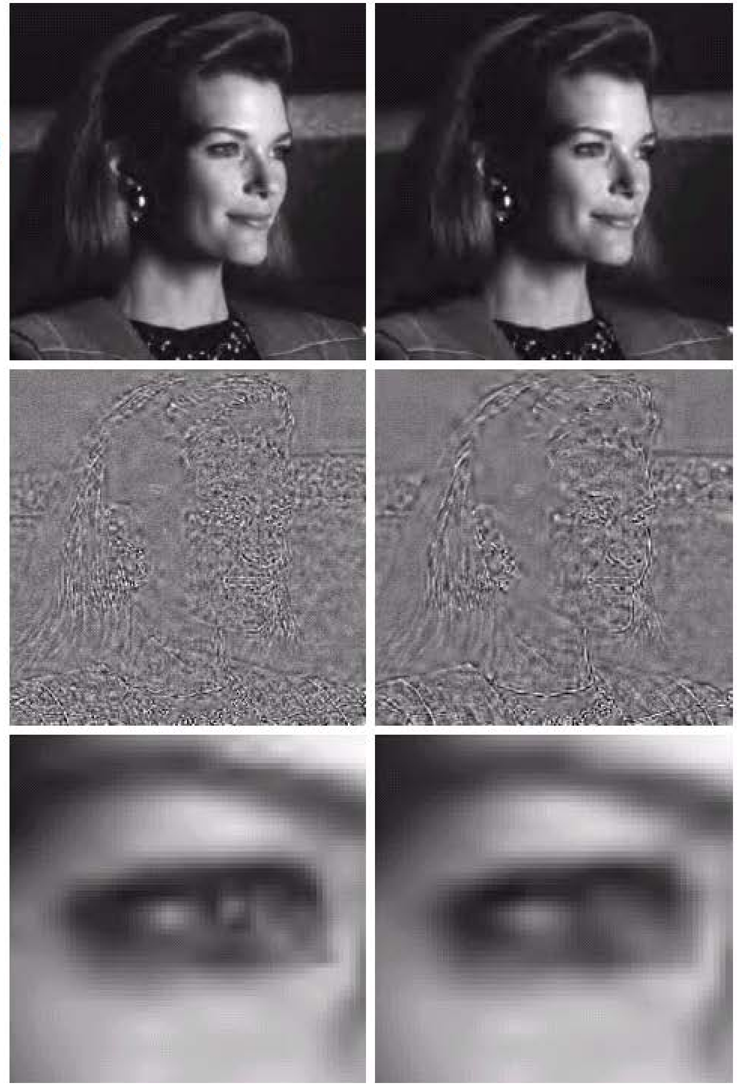
0.048 bpp

Error images

enlarged



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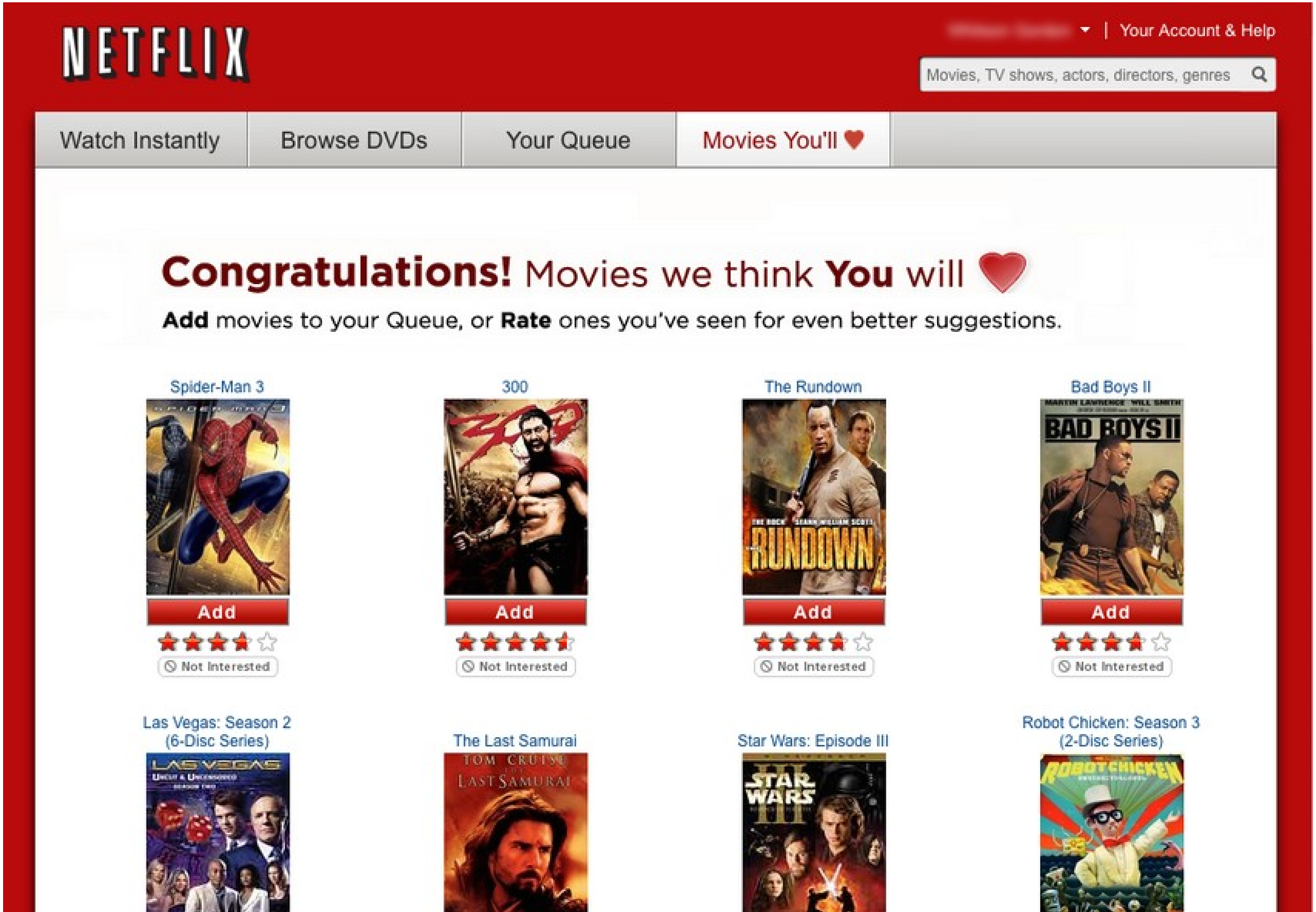
[Gonzalez, Woods, 2001]

Representations and Bases – Empirical Bases

Representations and Bases – Empirical Bases

Representations and Bases – Empirical Bases



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The Fourier Series

- **Cosines and Sines**

My signal

Basis

$$x(t) = \sum_k a_k \phi_k(t)$$

- **Synthesis Equation (Inverse transform)**

$$x(t) = a_0 + \sum_{k=1}^{\infty} a_k \cos(k\omega_0 t) + b_k \sin(k\omega_0 t)$$

Fundamental Frequency

The Fourier Series

- **Cosines and Sines**

My signal

Basis

$$x(t) = \sum_k a_k \phi_k(t)$$

The Fourier Series

- **Cosines and Sines**

My signal

Basis

$$x(t) = \sum_k a_k \phi_k(t)$$

- **Analysis Equations**

$$a_0 = \frac{1}{T_0} \int_{T_0} x(t) dt$$

$$a_k = \frac{2}{T_0} \int_{T_0} x(t) \cos(k\omega_0 t)\, dt \qquad k \geq 1$$

$$b_k = \frac{2}{T_0} \int_{T_0} x(t) \sin(k\omega_0 t)\, dt \qquad k \geq 1$$

The Fourier Series

- **Synthesis Equation**

$$x(t) = a_0 + \sum_{k=1}^{\infty} a_k \cos(k\omega_0 t) + b_k \sin(k\omega_0 t)$$

- **Analysis Equations**

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The Fourier Series

- **Synthesis Equation**

$$x(t) = a_0 + \sum_{k=1}^{\infty} a_k \cos(k\omega_0 t) + b_k \sin(k\omega_0 t)$$

- **Analysis Equations**

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- **Question: Is orthogonality found here??**

The Fourier Series

- **Synthesis Equation**

$$x(t) = a_0 + \sum_{k=1}^{\infty} a_k \cos(k\omega_0 t) + b_k \sin(k\omega_0 t)$$

- **Analysis Equations**

$$a_0 = \frac{1}{T_0}\int_{T_0} x(t)dt$$

$$a_k = \frac{2}{T_0}\int_{T_0} x(t)\cos(k\omega_0 t)\,dt \qquad k \geq 1$$

$$b_k = \frac{2}{T_0}\int_{T_0} x(t)\sin(k\omega_0 t)\,dt \qquad k \geq 1$$

- **Question: Is orthogonality found here?? Yes!**

The Fourier Series

- **Testing Orthogonality**

$$\frac{1}{T_0}\int_{T_0} \cos(k_1\omega_0 t)\cos(k_2\omega_0 t)\,dt$$

$$= \frac{1}{2T_0}\int_{T_0} \cos\big((k_1-k_2)\omega_0 t\big) + \cos\big((k_1+k_2)\omega_0 t\big)\,dt$$

$$= \begin{cases} 0 & \text{if} \quad k_1 \neq k_2 \\ 1/2 & \text{if} \quad k_1 = k_2 \end{cases}$$ **Orthogonal!**

$$\frac{1}{T_0}\int_{T_0} \cos(k_1\omega_0 t)\sin(k_2\omega_0 t)\,dt$$

$$= \frac{1}{2T_0}\int_{T_0} \sin\big((k_1-k_2)\omega_0 t\big) + \sin\big((k_1+k_2)\omega_0 t\big)\,dt$$

$$= 0$$ **Orthogonal!**

The Fourier Series

- **Testing Orthogonality**

$$\frac{1}{T_0}\int_{T_0} \sin(k_1\omega_0 t)\sin(k_2\omega_0 t)\,dt$$

$$= \frac{1}{2T_0}\int_{T_0} \cos\big((k_1-k_2)\omega_0 t\big) - \cos\big((k_1+k_2)\omega_0 t\big)\,dt$$

$$= \begin{cases} 0 & \text{if} \quad k_1 \neq k_2 \\ 1/2 & \text{if} \quad k_1 = k_2 \end{cases}$$ **Orthogonal!**

$$\frac{1}{T_0}\int_{T_0} \cos(k_1\omega_0 t)\sin(k_2\omega_0 t)\,dt$$

$$= \frac{1}{2T_0}\int_{T_0} \sin\big((k_1-k_2)\omega_0 t\big) + \sin\big((k_1+k_2)\omega_0 t\big)\,dt$$

$$= 0$$ **Orthogonal!**

The Fourier Series

- **Example: Find the Fourier Coefficients of…**

$$x(t) = \cos(2t) + 4\sin(3t)$$

- **Easy Way (if sum of sinusoids)**
  - **Step 1:** Determine the fundamental frequency ω_0
  - **Step 2:** Determine your cosine / sine harmonics of the fundamental
  - **Step 3:** Determine your cosine / sine amplitudes for those harmonics

The Fourier Series

- **Example: Find the Fourier Coefficients of…**

$$x(t) = \cos(2t) + 4\sin(3t)$$

- **Easy Way (if sum of sinusoids)**
  - **Step 1:** Determine the fundamental frequency ω_0 [$\omega_0 = 1$]
  - **Step 2:** Determine your cosine / sine harmonics (k) of the fundamental [$k = 2, 3$]
  - **Step 3:** Determine your cosine / sine amplitudes for those harmonics
    - $a_2 = 1$
    - $b_3 = 4$
    - All other coefficients are zero

The Fourier Series

- **Example: Find the Fourier Coefficients of…**

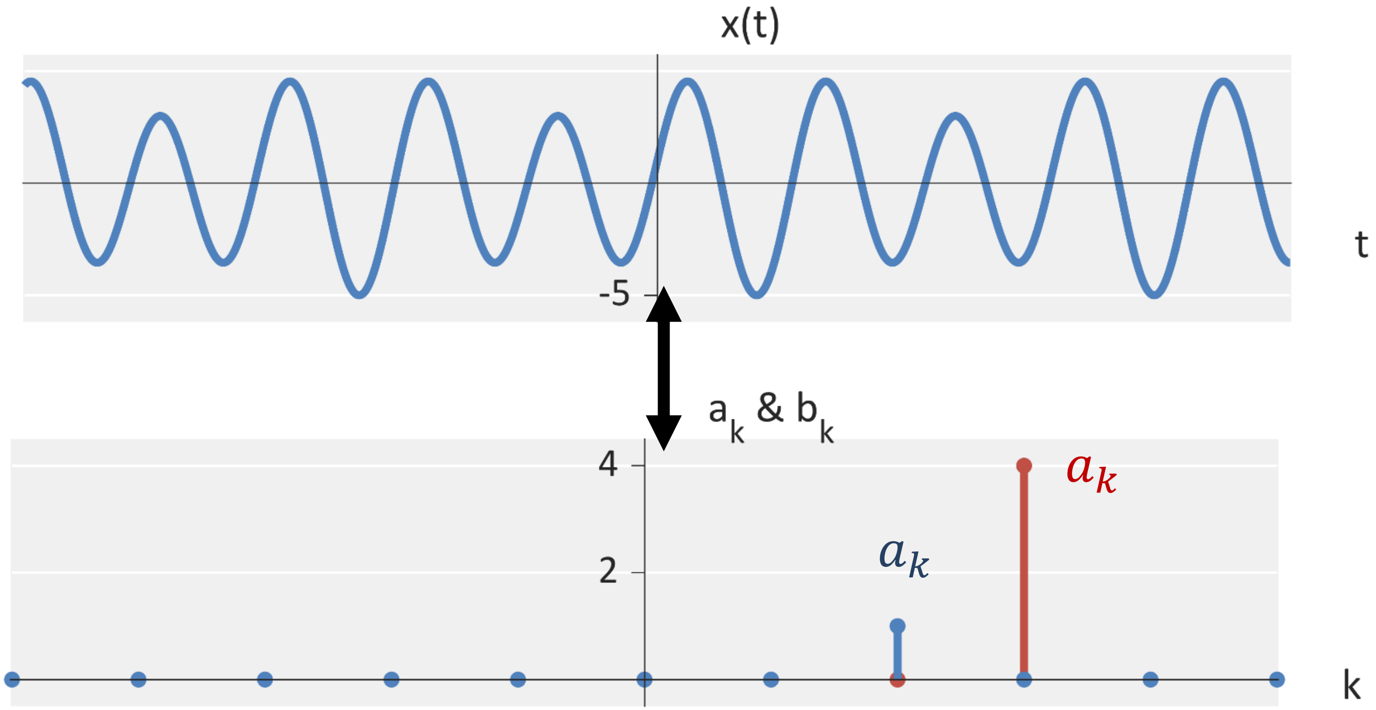
$$x(t) = \cos(2t) + 4\sin(3t) \longleftrightarrow$$

$a_2 = 1$
$b_3 = 4$
All other
coefficients are zero

The Fourier Series

- **Example: Find the Fourier Coefficients of…**

$$x(t) = \cos(2t) + 4\sin(3t) \longleftrightarrow \begin{aligned} a[k] &= \delta[k-2] \\ b[k] &= 4\delta[k-3] \\ \omega_0 &= 1 \end{aligned}$$



The Fourier Series

- **Example: Find the Fourier Coefficients of…**

$$a_2 = 1$$
$$b_3 = 4$$

- **Confirm**

$$x(t) = a_0 + \sum_{k=1}^{\infty} a_k \cos(k\omega_0 t) + b_k \sin(k\omega_0 t)$$

$$x(t) = 0 + (1)\cos\big((2)\omega_0 t\big) + (4)\cos\big((3)\omega_0 t\big)$$

$$\Big\downarrow \; \omega_0 = 1$$

$$\boxed{x(t) = \cos(2t) + 4\cos(3t)}$$

The Fourier Series

- **Find the Fourier Coefficients of…**

$$x(t) = \cos(2t) + 4\sin(3t)$$

- **Hard Way**
  - **Step 1:** Find fundamental frequency ω_0
  - **Step 2:** Solve for a_0
  - **Step 3:** Solve for a_k
  - **Step 4:** Solve for b_k

The Fourier Series

- **Find the Fourier Coefficients of…**

$$x(t) = \cos(2t) + 4\sin(3t)$$

- **Hard Way**
  - **Step 1:** Find fundamental frequency ω_0
    - $\omega_0 = 1$
  - **Step 2:** Solve for a_0
  - **Step 3:** Solve for a_k

The Fourier Series

- **Find the Fourier Coefficients of…**

  $x(t) = \cos(2t) + 4\sin(3t)$

- **Hard Way**
  - **Step 1:** Find fundamental frequency ω_0
  - **Step 2:** Solve for a_0
    - $a_0 = \frac{1}{T_0}\int_{T_0} x(t)dt = \frac{1}{2\pi}\int_0^{2\pi} \cos(2t) + 4\sin(3t)\,dt = 0$
  - **Step 3:** Solve for a_k

The Fourier Series

- **Find the Fourier Coefficients of…**

$$x(t) = \cos(2t) + 4\sin(3t)$$

- **Hard Way**
  - **Step 1:** Find fundamental frequency ω_0
  - **Step 2:** Solve for a_0
  - **Step 3:** Solve for a_k

$$a_k = \frac{2}{T_0}\int_{T_0} [\cos(2t) + 4\sin(3t)]\cos(k\omega_0 t)\, dt$$
$$= \begin{cases} 1 & \text{if} \quad k = 2 \\ 0 & \text{if} \quad \text{otherwise} \end{cases}$$

The Fourier Series

- **Find the Fourier Coefficients of…**

$$x(t) = \cos(2t) + 4\sin(3t)$$

- **Hard Way**
  - **Step 1:** Find fundamental frequency ω_0
  - **Step 2:** Solve for a_0
  - **Step 4:** Solve for b_k

$$b_k = \frac{2}{T_0}\int_{T_0} [\cos(2t) + 4\sin(3t)] \sin(k\omega_0 t)\, dt$$

$$= \begin{cases} 4 & \text{if} \quad k = 3 \\ 0 & \text{if} \quad \text{otherwise} \end{cases}$$

The Continuous-Time Fourier Series
Complex Exponentials

The Fourier Series

- **Synthesis Equation**

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$

- **Analysis Equation**

$$c_k = \frac{1}{T_0} \int_{T_0} x(t) e^{-jk\omega_0 t} dt$$

- **Testing Orthogonality**

$$\frac{1}{T_0}\int_{T_0} e^{jk_1\omega_0 t}e^{-jk_2\omega_0 t}dt$$

$$= \frac{1}{T_0}\int_{T_0} e^{j(k_1-k_2)\omega_0 t}dt$$

$$= \begin{cases} 0 & \text{if} \quad k_1 \neq k_2 \\ 1 & \text{if} \quad k_1 = k_2 \end{cases}$$

The Fourier Series

- **Example: Find the Fourier Coefficients of…**

$$x(t) = \cos(2t) + 4\sin(3t)$$

- **Easy Way (if sum of sinusoids)**
  - **Step 1:** Determine the fundamental frequency ω_0
  - **Step 2:** Determine your harmonics of the fundamental
  - **Step 3:** Determine your amplitudes for those harmonics (+ and -)

The Fourier Series

- **Example: Find the Fourier Coefficients of…**

$$x(t) = \cos(2t) + 4\sin(3t)$$

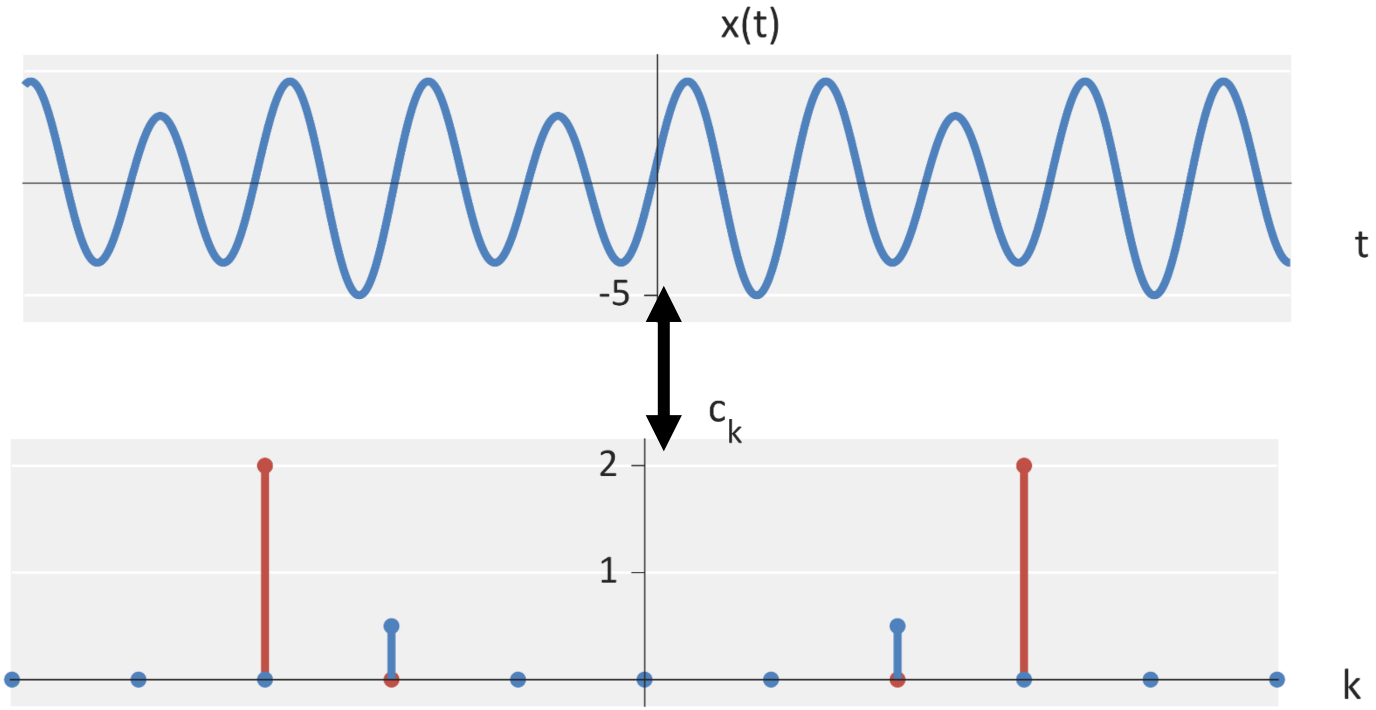
- **Easy Way (if sum of sinusoids)**
  - **Step 1:** Determine the fundamental frequency ω_0 [$\omega_0 = 1$]
  - **Step 2:** Determine your harmonics of the fundamental [$k = 2, 3$]
  - **Step 3:** Determine your amplitudes for those harmonics (+ and -)
    - ◊ $c_2 = 1/2, c_{-2} = 1/2$
    - ◊ $c_3 = -2j, c_{-3} = 2j$
    - ◊ All other coefficients are zero

The Fourier Series

- **Example: Find the Fourier Coefficients of…**

$x(t) = \cos(2t) + 4\sin(3t) \longleftrightarrow$

$c_2 = 1/2, c_{-2} = 1/2$
$c_3 = -2j, c_{-3} = 2j$
All other coefficients



The Fourier Series

- **Find the Fourier Coefficients of…**

$$c_2 = 1/2, c_{-2} = 1/2$$
$$c_3 = -2j, c_{-3} = 2j$$

- **Confirm**

$$x(t) = \frac{1}{2}e^{2\omega_0 t} + \frac{1}{2}e^{-2\omega_0 t} - 2je^{3\omega_0 t} + 2je^{-3\omega_0 t}$$

$$x(t) = \frac{1}{2}e^{2\omega_0 t} + \frac{1}{2}e^{-2\omega_0 t} - 2je^{-3\omega_0 t} + 2je^{3\omega_0 t}$$

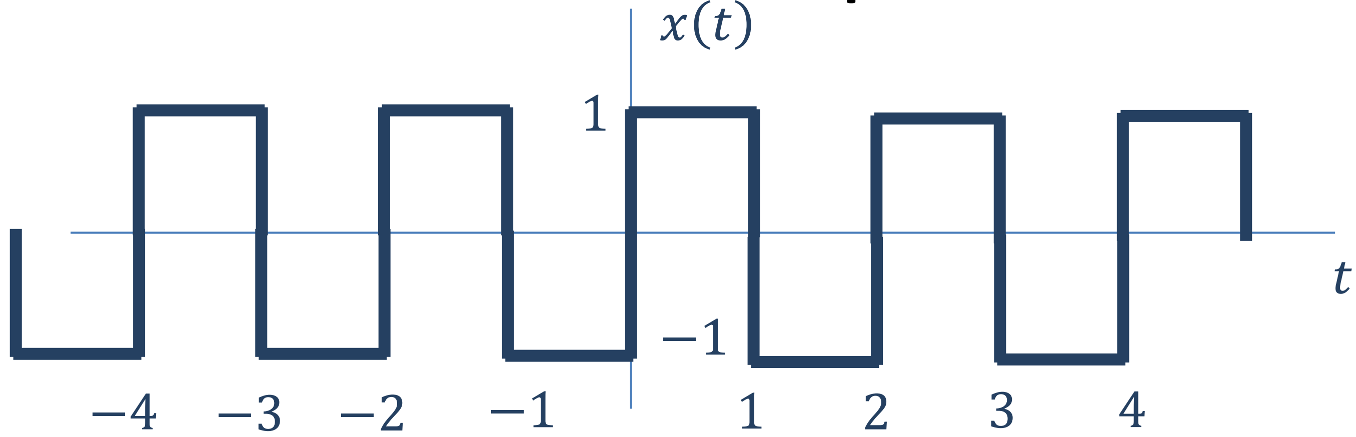
$$x(t) = \frac{1}{2}(e^{2\omega_0 t} + e^{-2\omega_0 t}) + 2j(-e^{3\omega_0 t} + e^{-3\omega_0 t})$$

$$x(t) = \frac{1}{2}(e^{2\omega_0 t} + e^{-2\omega_0 t}) + \frac{2}{j}(e^{3\omega_0 t} - e^{-3\omega_0 t})$$

$$\boxed{x(t) = \cos(2\omega_0 t) + 4\cos(3\omega_0 t)}$$

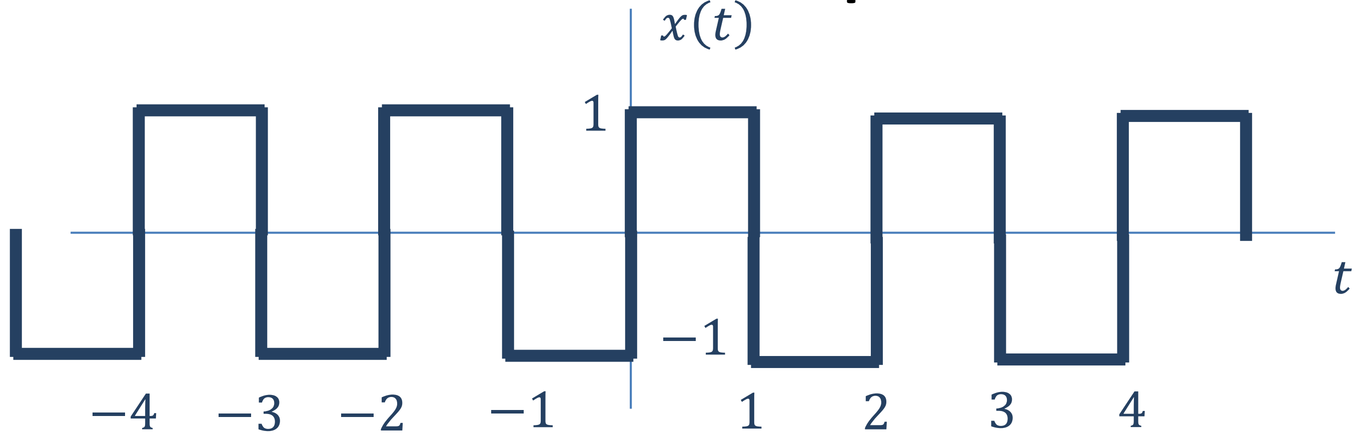
Fourier Series

- **Compute the Fourier Series of an odd Square Wave**



Fourier Series

- **Compute the Fourier Series of an odd Square Wave**



$$\frac{1}{T_0}\int_{T_0} x(t)e^{-j\omega_0 kt}dt = \frac{1}{2}\int_0^1 e^{-j\omega_0 kt}dt - \frac{1}{2}\int_1^2 e^{-j\omega_0 kt}dt = \frac{-1/2}{j\omega_0 k}e^{-j\omega_0 kt}\Big|_0^1 + \frac{1}{j\omega_0 k}e^{-j\omega_0 kt}\Big|_1^2$$

$$= \frac{-1/2}{j\omega_0 k}\left(e^{-j\omega_0 k} - 1\right) + \frac{1/2}{j\omega_0 k}\left(e^{-j2\omega_0 k} - e^{-j\omega_0 k}\right)$$

$$= \frac{1/2}{j\pi k}\left(1 - e^{-j\pi k}\right) + \frac{1/2}{j\pi k}\left(e^{-j2\pi k} - e^{-j\pi k}\right)$$

$$= \begin{cases} \frac{1/2}{j\pi k}(1-1) + \frac{1/2}{j\pi k}(1-1) = 0 & \text{If } k \text{ is even} \\ \frac{1/2}{j\pi k}(1-(-1)) + \frac{1/2}{j\pi k}(1-(-1)) = \frac{2}{j\pi k} & \text{If } k \text{ is odd} \end{cases}$$

$$T_0 = 2$$

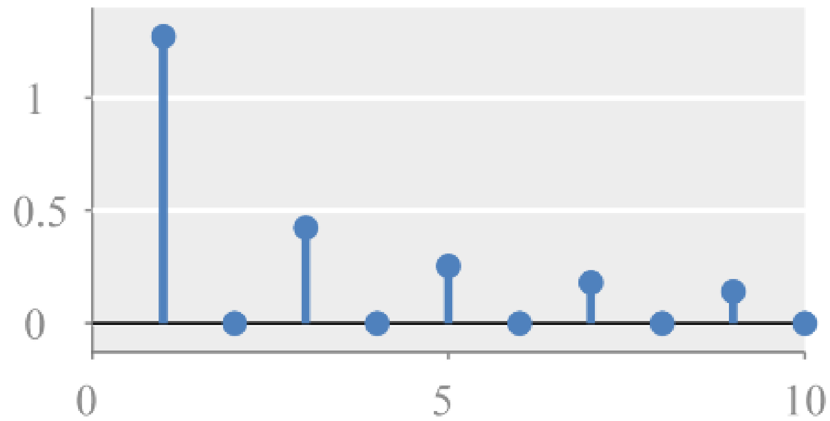
$$\omega_0 = \frac{2\pi}{T_0} = \pi$$

Fourier Series

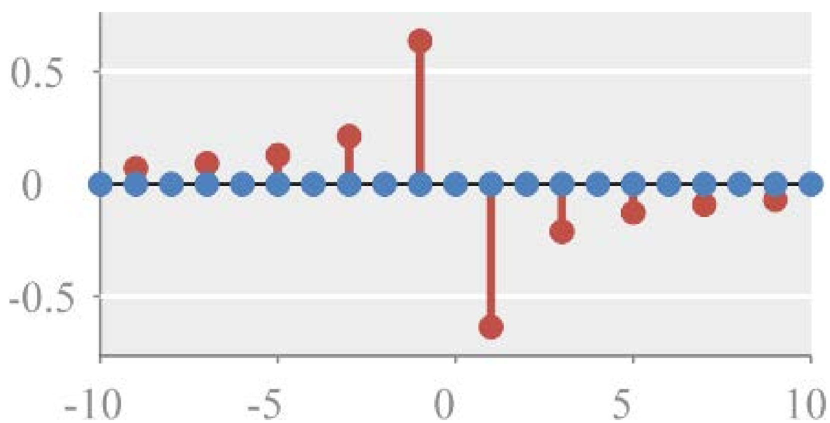
Fourier Series of a Square Wave

$$a_k = 0$$

$$b_k = \begin{cases} \dfrac{4}{\pi n} & \text{if} \quad n > 0 \text{ is odd} \\ 0 & \text{if} \quad n > 0 \text{ is even} \end{cases}$$



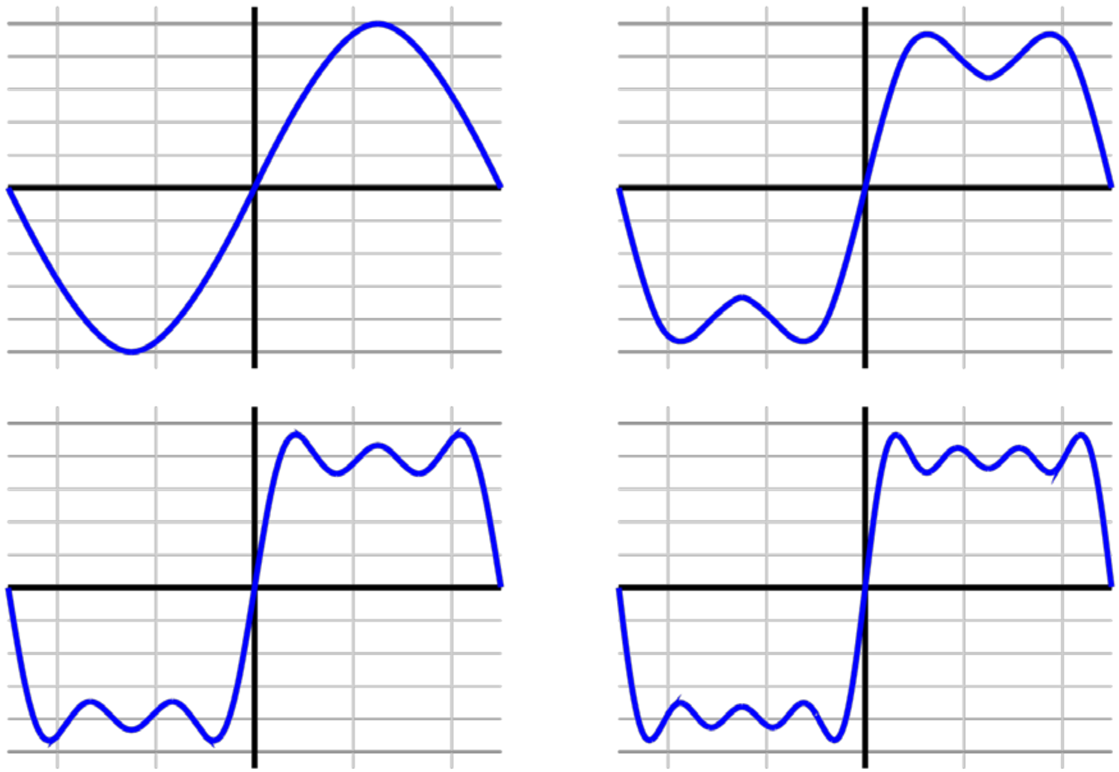
$$c_k = \begin{cases} \dfrac{2}{j\pi n} & \text{if} \quad n \text{ is odd} \\ 0 & \text{if} \quad n \text{ is even} \end{cases}$$



- **Constructing a square wave**

First Harmonic

First + Second Harmonic



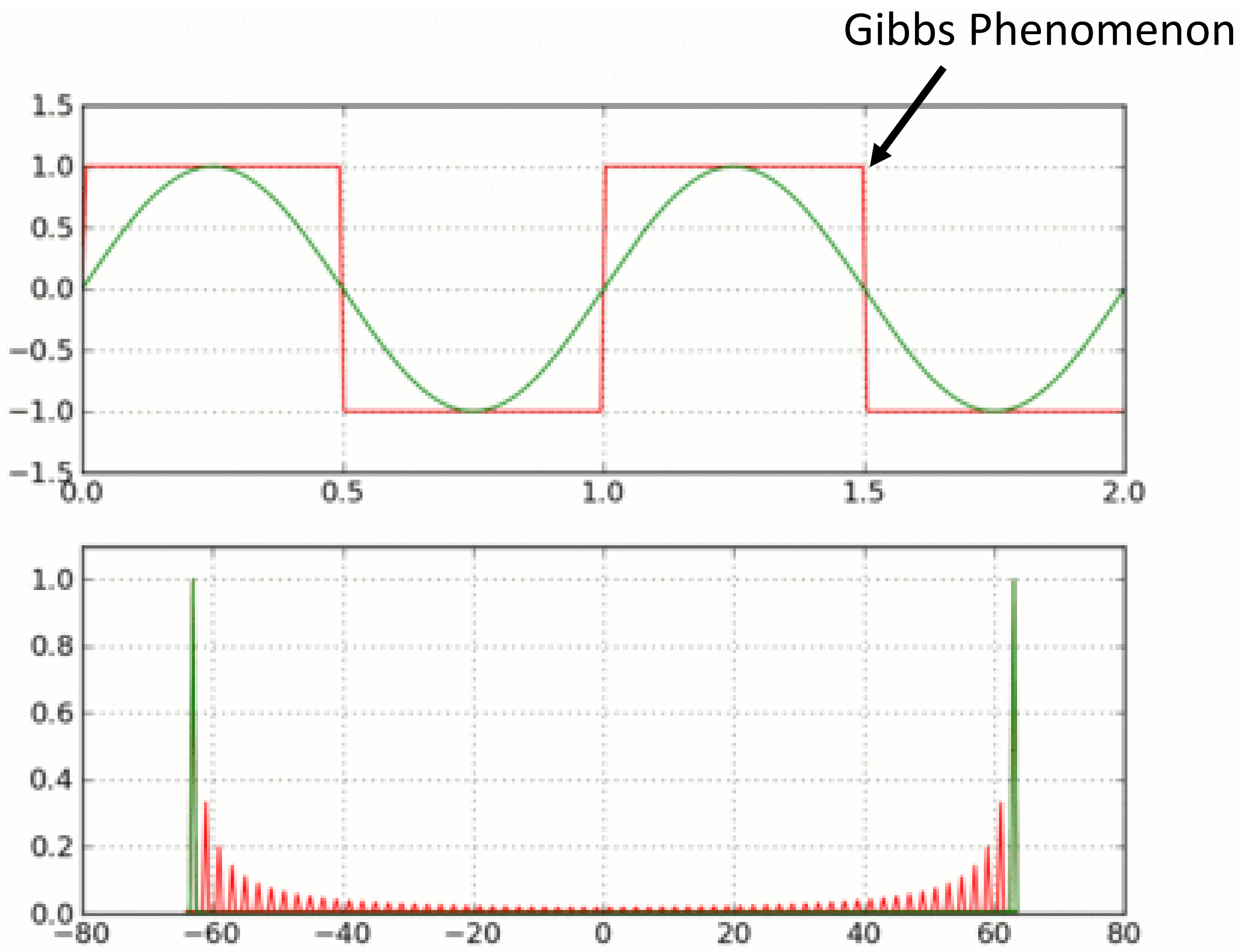
First + Second + Third Harmonic

First + Second + Third + Fourth Harmonic

http://www.upv.es/~rfuster/xpicture/classroom.html

Fourier Series

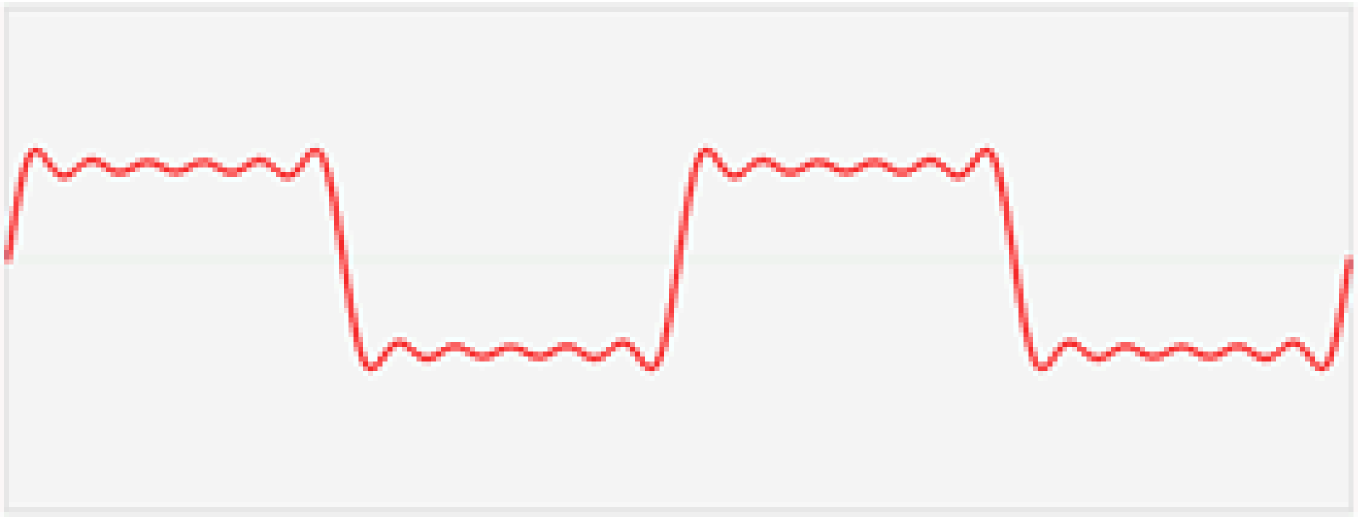
- **Constructing a square wave**



https://en.wikipedia.org/wiki/Gibbs_phenomenon

Fourier Series

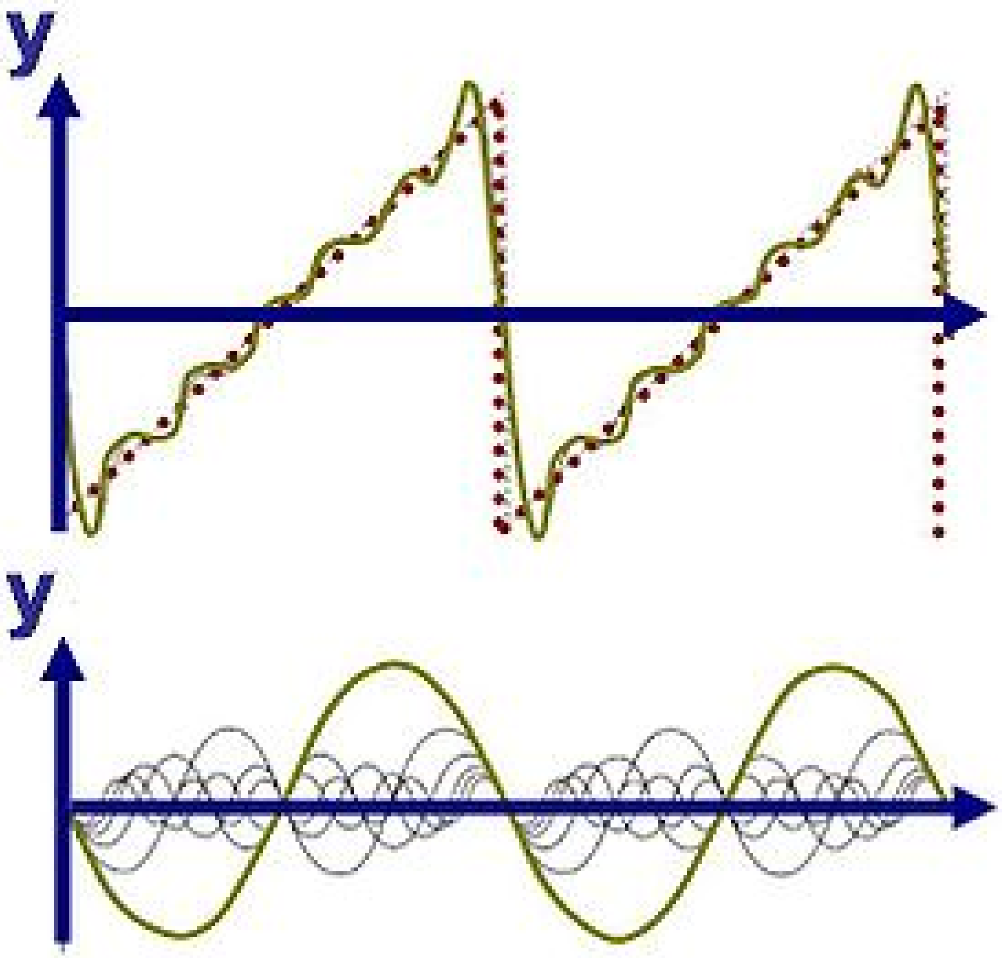
- **Constructing a square wave**



https://en.wikipedia.org/wiki/Fourier_series

Fourier Series

- **Constructing a ramp**



Reconstruction

Each Fourier
Component

https://en.wikipedia.org/wiki/Convergence_of_Fourier_series

Lecture 8 : Fourier Theory

Foundations of Digital Signal Processing

Outline

The Fourier Transform

- **Fourier Series Synthesis Equation**

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$

- **Fourier Series Analysis Equation**

$$c_k = \frac{1}{T_0} \int_{T_0} x(t) e^{-jk\omega_0 t} dt$$

The Fourier Transform

- **Fourier Series Synthesis Equation**

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$

Let the fundamental period $T_0 \to \infty$
Or the fundamental period $\omega_0 \to 0$
Or $k\omega_0 \to \Omega$

The Fourier Transform

- **Fourier Series Synthesis Equation**

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$

Let the fundamental period $T_0 \to \infty$
Or the fundamental period $\omega_0 \to 0$
Or $k\omega_0 \to \Omega$
Or $c_k \to \mathrm{X}(\Omega)$

- **So this turns into the Fourier Transform Synthesis Equation**

$$x(t) = \int_{-\infty}^{\infty} X(\Omega) e^{\mathrm{j}\Omega t} d\Omega$$

The Fourier Transform

- **Fourier Transform Synthesis Equation**

$$x(t) = \int_{-\infty}^{\infty} X(\Omega) e^{j\Omega t} d\Omega$$

- **Fourier Transform Analysis Equation**

$$X(\Omega) = \int_{-\infty}^{\infty} x(t) e^{-j\Omega t} dt$$

The Fourier Transform

- **Fourier Transform Analysis Equation**

$$X(\Omega) = \int_{-\infty}^{\infty} x(t)e^{-j\Omega t}dt$$

- **Example: Compute the Fourier Transform of**

$$\mathrm{x}(t) = e^{-2t}u(t)$$

The Fourier Transform

- **Fourier Transform Analysis Equation**

$$X(\Omega) = \int_{-\infty}^{\infty} x(t) e^{-\mathrm{j}\Omega t} dt$$

- **Example: Compute the Fourier Transform of**

$$\mathrm{x}(t) = e^{-2t} u(t)$$

$$X(\Omega) = \int_{-\infty}^{\infty} e^{-2t} u(t) e^{-\mathrm{j}\Omega t} dt = \int_{0}^{\infty} e^{-(\mathrm{j}\Omega+2)t} dt$$

$$= \frac{-1}{\mathrm{j}\Omega + 2} e^{-(\mathrm{j}\Omega+2)t} \Big|_{0}^{\infty} = \frac{1}{2 + \mathrm{j}\Omega}$$

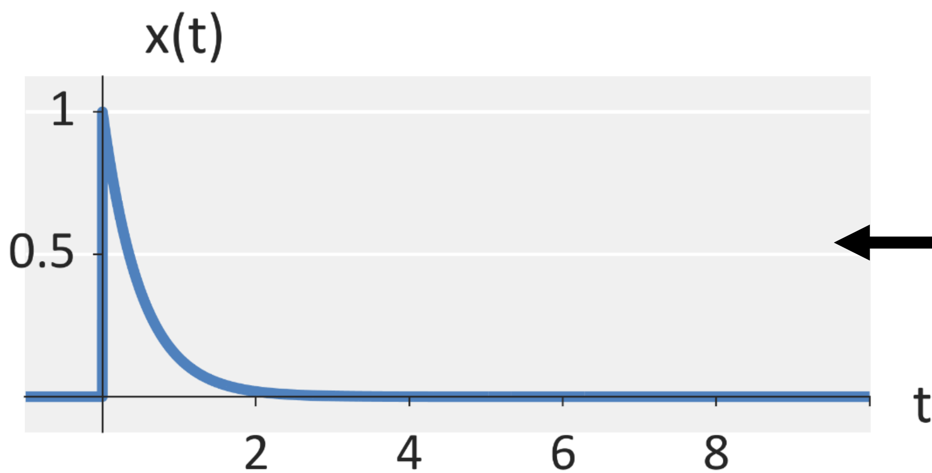
The Fourier Transform

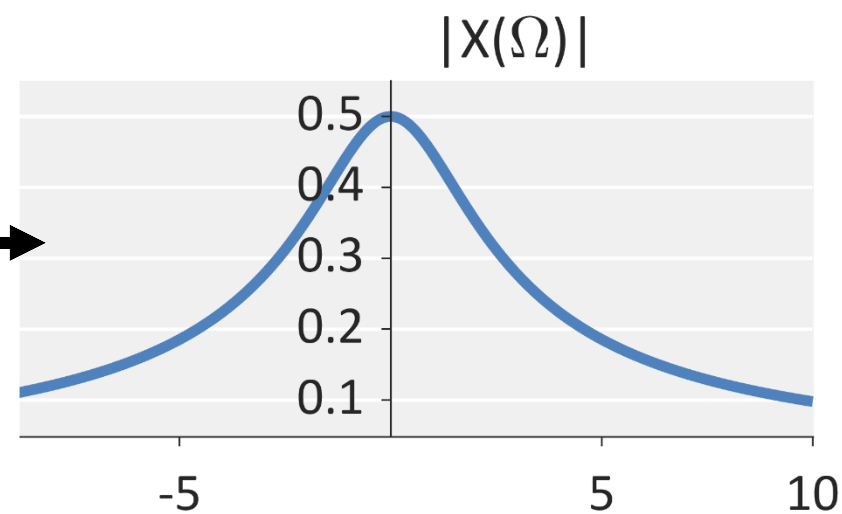
- **Fourier Transform Analysis Equation**

$$X(\Omega) = \int_{-\infty}^{\infty} x(t) e^{-j\Omega t} dt$$

- **Example:** **Compute the Fourier Transform of**

$$\mathrm{x}(t) = e^{-2t} u(t) \quad \longleftrightarrow \quad X(\Omega) = \frac{1}{2 + j\Omega}$$





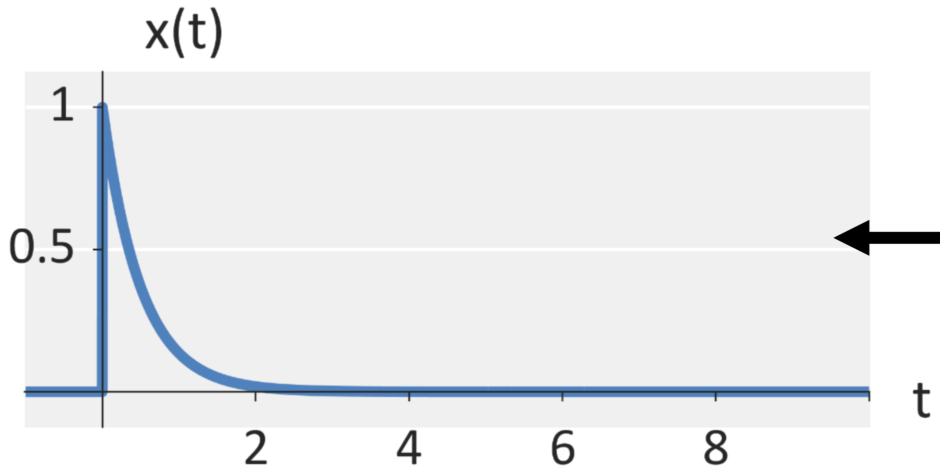
The Fourier Transform

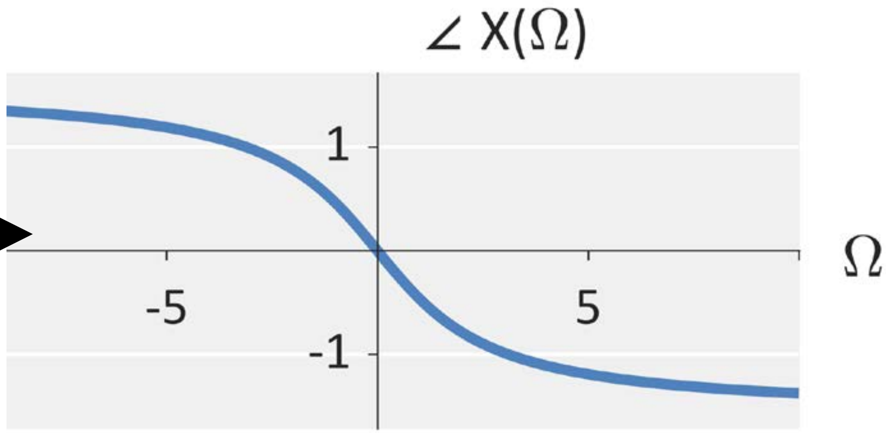
- **Fourier Transform Analysis Equation**

$$X(\Omega) = \int_{-\infty}^{\infty} x(t)e^{-\mathrm{j}\Omega t}dt$$

- **Example: Compute the Fourier Transform of**

$$\mathrm{x}(t) = e^{-2t}u(t) \quad \longleftrightarrow \quad X(\Omega) = \frac{1}{2 + \mathrm{j}\Omega}$$





The Fourier Transform

- **Fourier Transform Analysis Equation**

$$X(\Omega) = \int_{-\infty}^{\infty} x(t) e^{-\mathrm{j}\Omega t} dt$$

- **Example:** **Compute the Fourier Transform of**

$$\mathrm{x}(t) = u\left(t + \frac{1}{2}\right) - u\left(t - \frac{1}{2}\right)$$

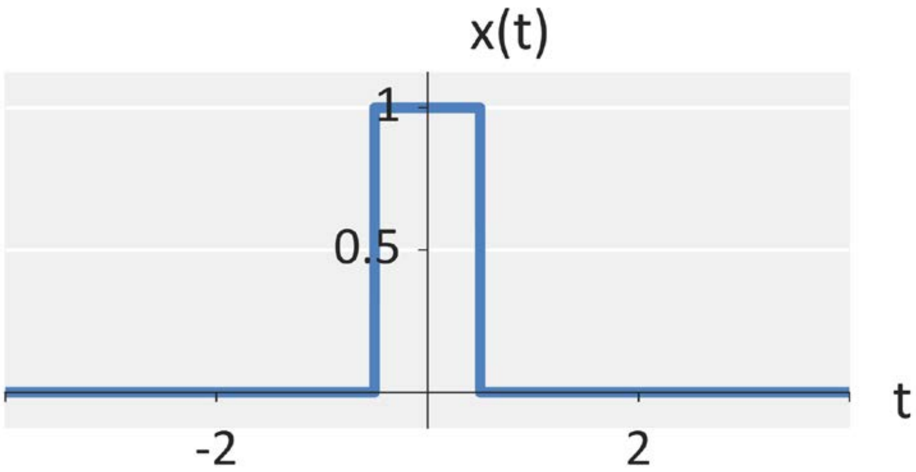
The Fourier Transform

- **Fourier Transform Analysis Equation**

$$X(\Omega) = \int_{-\infty}^{\infty} x(t) e^{-\mathrm{j}\Omega t} dt$$

- **Example:** **Compute the Fourier Transform of**

$$\mathrm{x}(t) = u\left(t + \frac{1}{2}\right) - u\left(t - \frac{1}{2}\right)$$

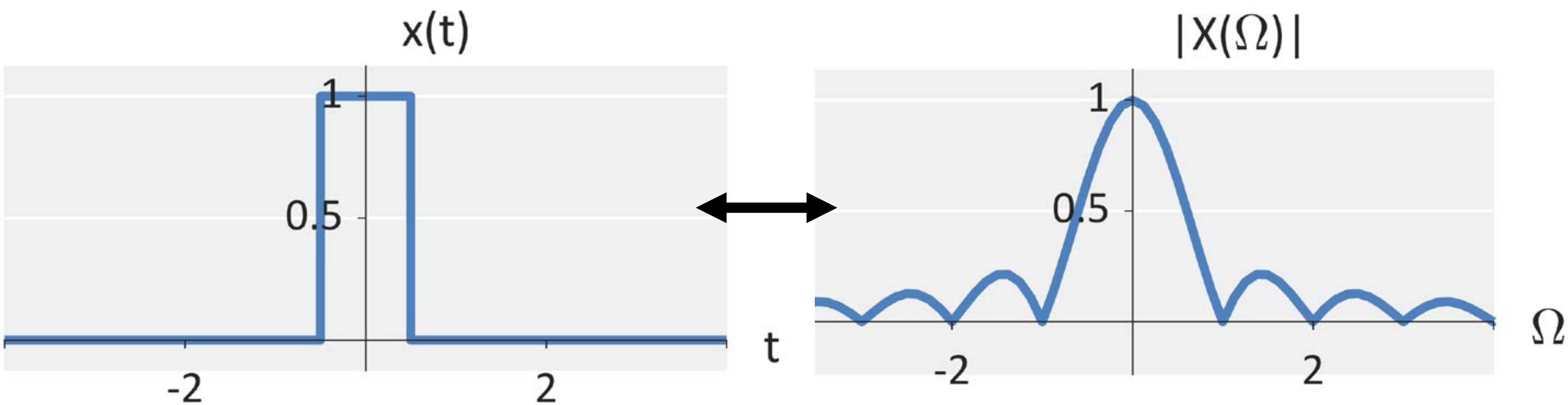


- **Fourier Transform Analysis Equation**

$$X(\Omega) = \int_{-\infty}^{\infty} x(t) e^{-j\Omega t} dt$$

- **Example:** **Compute the Fourier Transform of**

$$\mathrm{x}(t) = u\left(t + \frac{1}{2}\right) - u\left(t - \frac{1}{2}\right) \longleftrightarrow \mathrm{x}(t) = \mathrm{sinc}(\Omega/2)$$



The Fourier Transform

Lecture 8 : Fourier Theory

Foundations of Digital Signal Processing

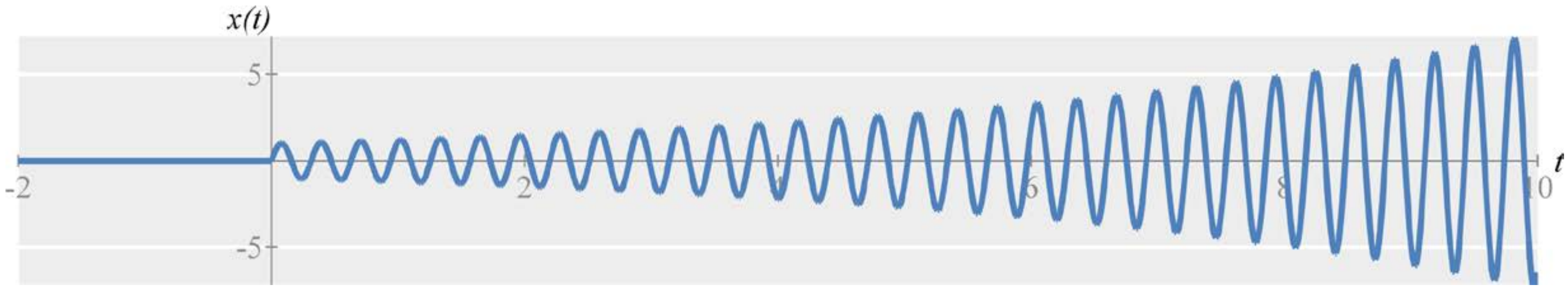
Outline

- The Discrete-time Fourier Transform (DTFT) Review
- The Properties of the Discrete-time Fourier Transform (DTFT)
- General Representation / Fourier Theory
- The Fourier Series
- The Fourier Transform
- **Laplace Transform**
- The Fourier Relationships

Laplace Transform

- **The Bilateral Laplace Transform**
  - Consider the signal

$$x(t) = e^{2t}\cos(t)$$

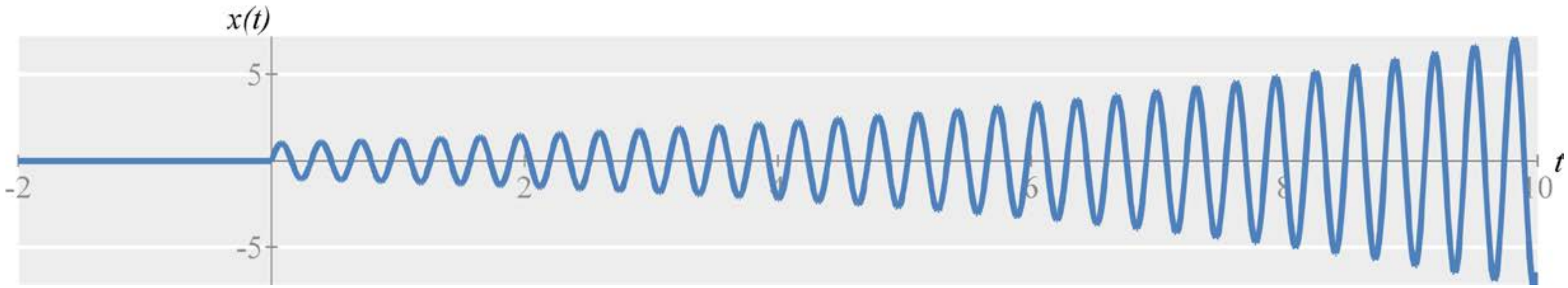


Does this have a Fourier Transform?

Laplace Transform

- **The Bilateral Laplace Transform**
  - Consider the signal

$$x(t) = e^{2t}\cos(t)$$



However, let's multiply the signal by a exponential

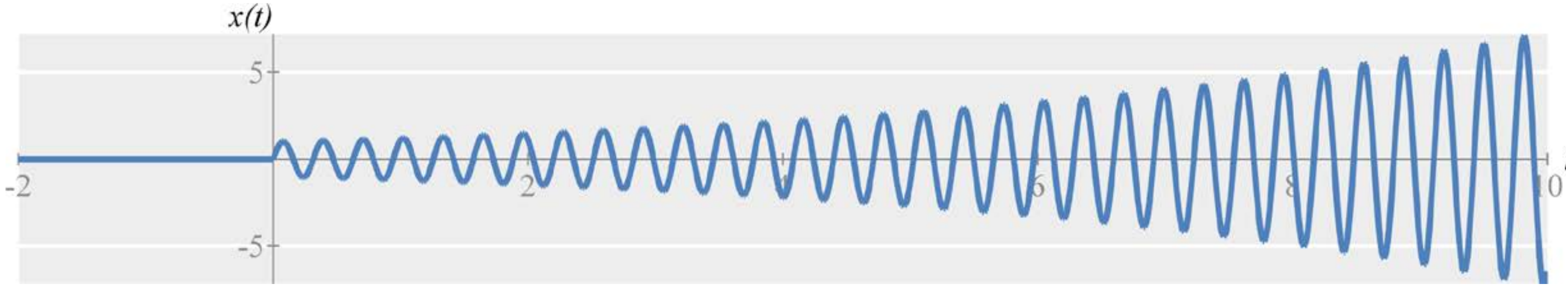
$$x(t)e^{-\sigma t} = e^{2t}e^{-\sigma t}\cos(t) = e^{(2-\sigma)t}\cos(t)$$

For what values of σ does the Fourier transform exist? $\sigma > 2$

Laplace Transform

- **The Bilateral Laplace Transform**
  - Consider the signal

$$x(t) = e^{2t}\cos(t)$$



We want to incorporate this to achieve convergence at any time.

Laplace Transform

- **The Fourier Transform**

$$X(\Omega) = \int_{-\infty}^{\infty} x(t) e^{-j\Omega t}\, dt$$

$$x(t) = \int_{\infty}^{\infty} X(\Omega) e^{-j\Omega \mathrm{t}}\, d\Omega$$

where $s = \sigma + j\omega$. s is complex.

Laplace Transform

- **The Bilateral Laplace Transform**

$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-\sigma t} e^{-j\Omega t}\, dt$$

$$x(t) = \int_{-c-j\infty}^{c+j\infty} X(s) e^{-\sigma t} e^{-j\Omega t}\, ds$$

where $s = \sigma + j\omega$. s is complex.

Laplace Transform

- **The Bilateral Laplace Transform**

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}\,dt$$

$$x(t) = \int_{-c-j\infty}^{c+j\infty} X(s)e^{st}\,ds$$

where $s = \sigma + j\omega$. s is complex.

Laplace Transform

- **The Bilateral Laplace Transform**
  - Laplace transform of $x(t) = e^{at}u(t)$

$$X(s) = \int_{-\infty}^{\infty} [e^{at}u(t)]e^{-\sigma t}e^{-j\omega}\,dt$$

Causal

When does this converge? $\sigma > a$

  - Laplace transform of $x(t) = e^{-at}u(-t)$

$$X(s) = \int_{-\infty}^{\infty} [e^{-at}u(-t)]e^{-\sigma t}e^{-j\omega}\,dt$$

Anti-Causal

When does this converge? $\sigma < -a$

Laplace Transform

- **The Bilateral Laplace Transform**
  - Laplace transform of $x_1(t) = e^{at}u(t)$

$$X_1(\omega) = \frac{1}{s-a}$$

When does this converge? $\sigma > a$

  - Laplace transform of $x_2(t) = e^{-at}u(-t)$

$$X_2(s) = X(-s) = \frac{1}{-s-a} = \frac{-1}{s+a}$$

When does this converge? $\sigma < -a$

Laplace Transform

- **The Bilateral Laplace Transform**
  - Laplace transform of $x_1(t) = e^{at}u(t)$

$$X_1(\omega) = \frac{1}{s-a}$$

When does this converge? $\sigma > a$

  - Laplace transform of $x_3(t) = -e^{+at}u(-t)$

$$X_3(-s) = \mathcal{L}\{-e^{-at}u(t)\} = \frac{-1}{s+a}$$

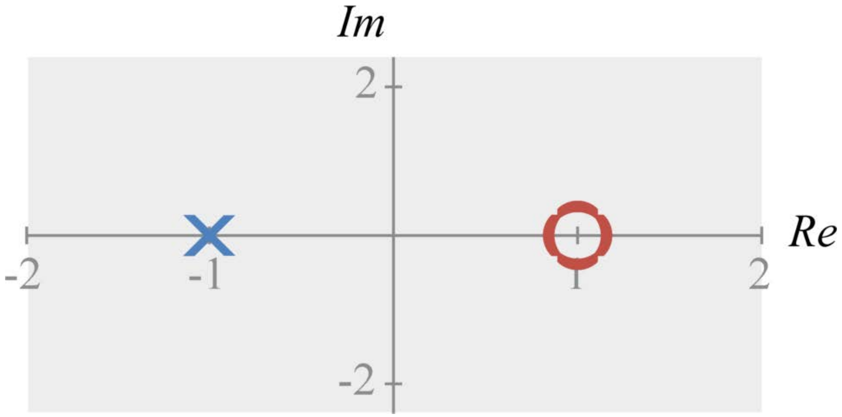
$$X_3(s) = \frac{-1}{-s+a} = \frac{-1}{s-a}$$

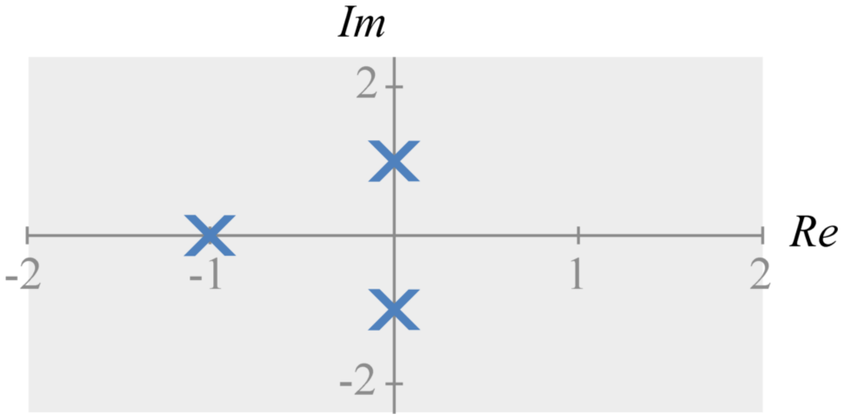
When does this converge? $\sigma < a$

Laplace Transform

- **Poles and Zeros**

$$X(s) = \frac{\prod_{z=1}^{Z}(s - a_z)}{\prod_{p=1}^{P}(s - b_p)}$$

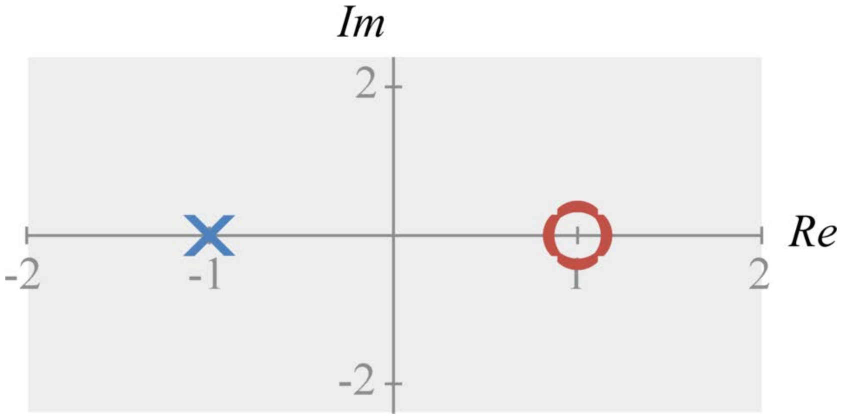


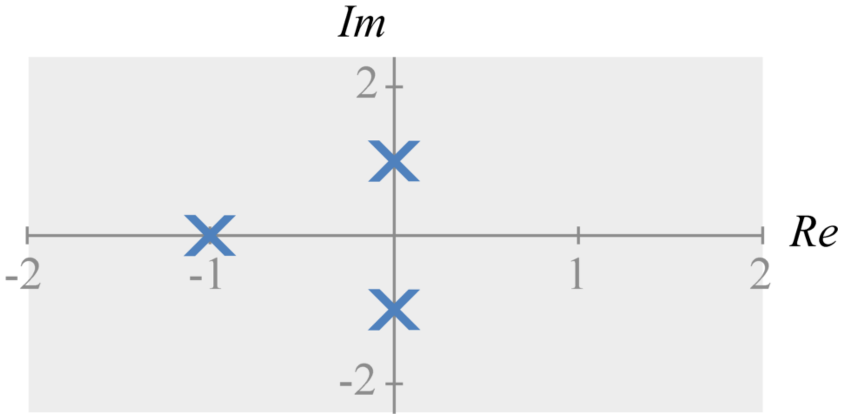


Laplace Transform

Region of Convergence

$$X(s) = \frac{\prod_{z=1}^{Z}(s - a_z)}{\prod_{p=1}^{P}(s - b_p)}$$



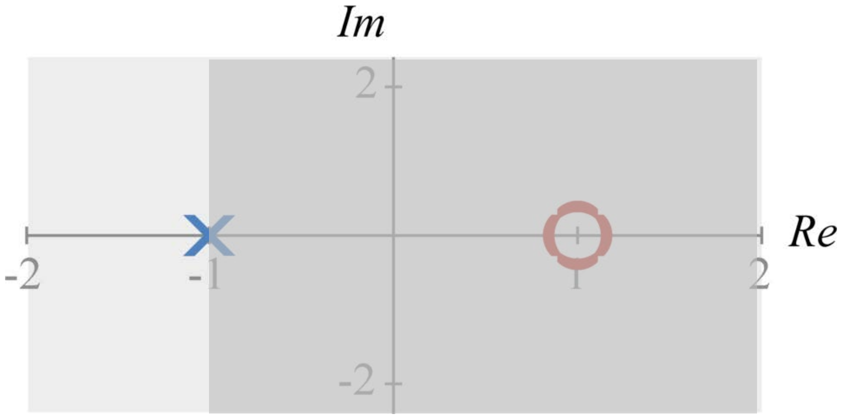


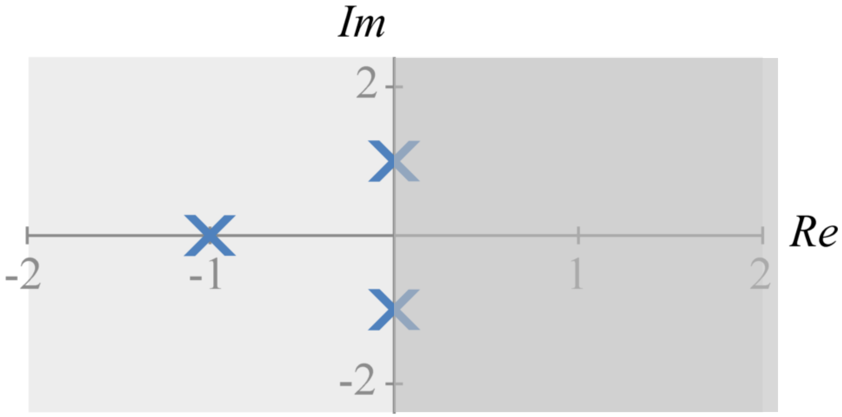
Laplace Transform

Region of Convergence

$$X(s) = \frac{\prod_{z=1}^{Z}(s - a_z)}{\prod_{p=1}^{P}(s - b_p)}$$

If a causal system:

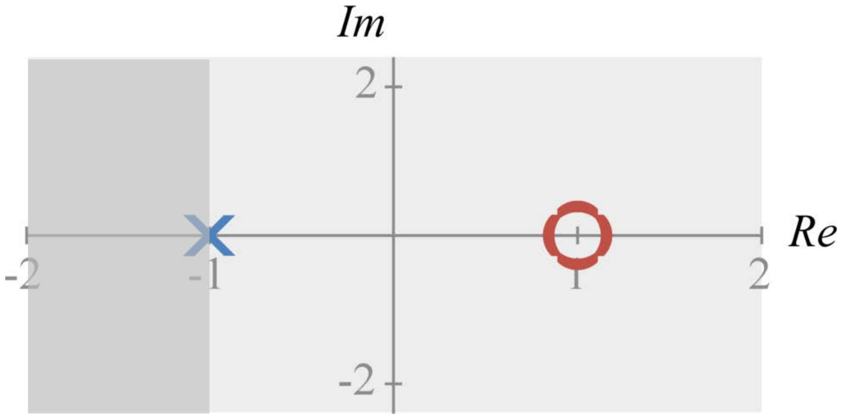


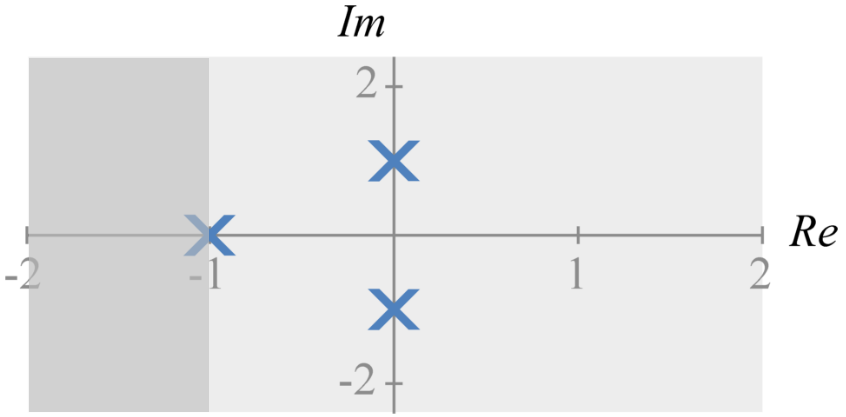


Region of Convergence

$$X(s) = \frac{\prod_{z=1}^{Z}(s - a_z)}{\prod_{p=1}^{P}(s - b_p)}$$

If an anti-causal system:

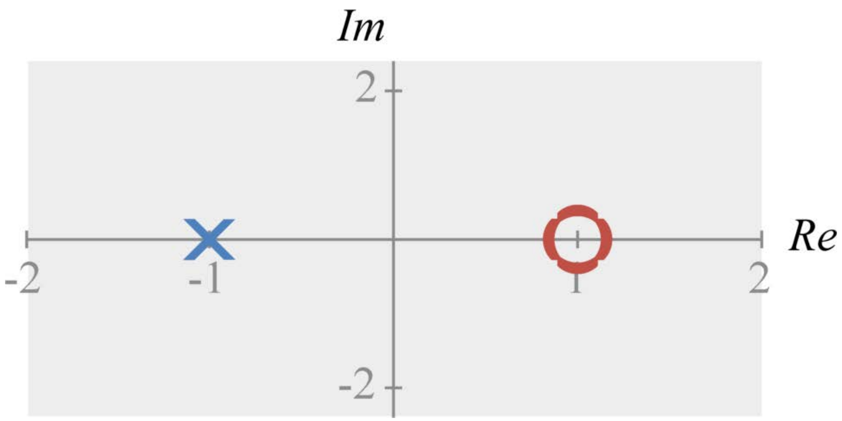


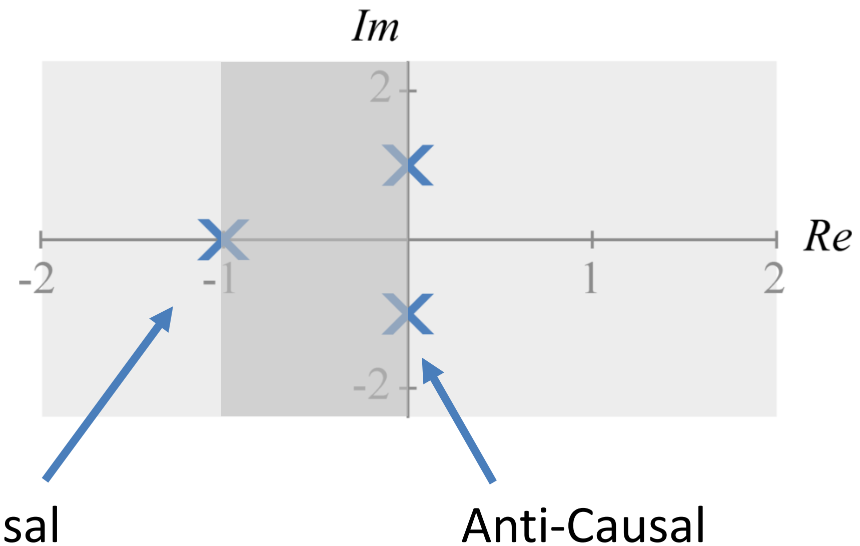


Region of Convergence

$$X(s) = \frac{\prod_{z=1}^{Z}(s - a_z)}{\prod_{p=1}^{P}(s - b_p)}$$

If an anti-causal & causal system:



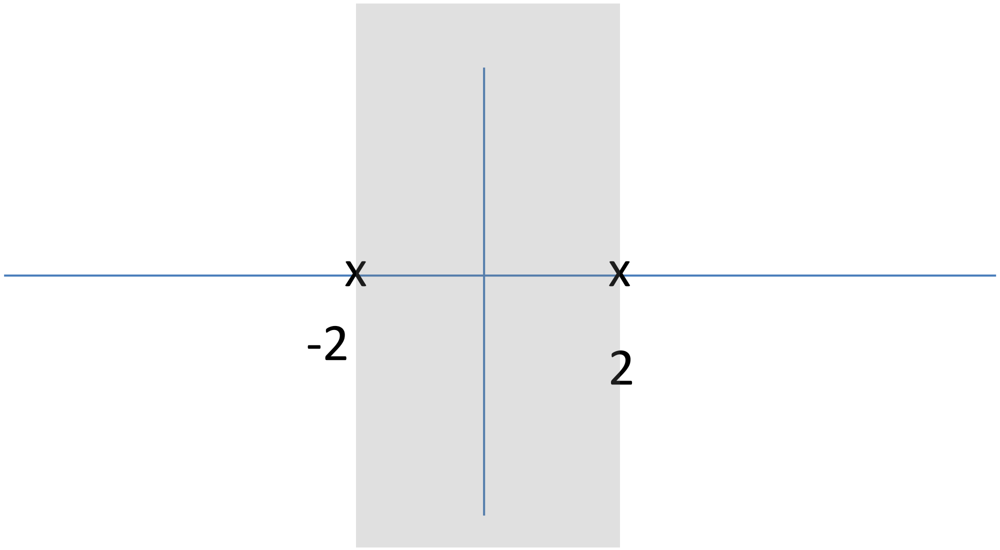


Laplace Transform

- **Example: Laplace Transform of**

$$x(t) = e^{-2t}u(t) * [e^{2t}u(-t)]$$

$$X(s) = \left(\frac{1}{s+2}\right)\left(\frac{1}{-s+2}\right)$$



Lecture 8 : Fourier Theory

Foundations of Digital Signal Processing

Outline

- The Discrete-time Fourier Transform (DTFT) Review
- The Properties of the Discrete-time Fourier Transform (DTFT)
- General Representation / Fourier Theory
- The Fourier Series
- The Fourier Transform
- Laplace Transform
- **The Fourier Relationships**

Fourier Theory

- **For digital signal processing, just remember:**

Everything's connected!

Fourier Theory

	Continuous Time	Discrete Time
Fourier Series	The Fourier Series	The Discrete Fourier Series
Fourier Transform	The Fourier Transform	The Discrete-Time Fourier Transform

Fourier Theory

	Continuous Time	Discrete Time
Fourier Series	The Fourier Series In time: **Continuous** In frequency: **Discrete**	The Discrete Fourier Series In time: **Discrete** In frequency: **Discrete**
Fourier Transform	The Fourier Transform In time: **Continuous** In frequency: **Continuous**	The Discrete-Time Fourier Transform In time: **Discrete** In frequency: **Continuous**

Fourier Theory

	Continuous Time	Discrete Time
Fourier Series	The Fourier Series In time: **Continuous** In frequency: **Discrete**	The Discrete Fourier Series In time: **Discrete** In frequency: **Discrete**
Fourier Transform	The Fourier Transform In time: **Continuous**, **aperiodic** In frequency: **Continuous**	The Discrete-Time Fourier Transform In time: **Discrete**, **aperiodic** In frequency: **Continuous**

Fourier Theory

	Continuous Time	Discrete Time
Fourier Series	The Fourier Series In time: **Continuous**, **periodic** In frequency: **Discrete**	The Discrete Fourier Series In time: **Discrete**, **periodic** In frequency: **Discrete**
Fourier Transform	The Fourier Transform In time: **Continuous**, **aperiodic** In frequency: **Continuous**	The Discrete-Time Fourier Transform In time: **Discrete**, **aperiodic** In frequency: **Continuous**

Fourier Theory

	Continuous Time	Discrete Time
Fourier Series	The Fourier Series In time: **Continuous**, **periodic** In frequency: **Discrete**	The Discrete Fourier Series In time: **Discrete**, **periodic** In frequency: **Discrete**
Fourier Transform	The Fourier Transform In time: **Continuous**, **aperiodic** In frequency: **Continuous**	The Discrete-Time Fourier Transform In time: **Discrete**, **aperiodic** In frequency: **Continuous**

Fourier Theory

	Continuous Time	Discrete Time
Fourier Series	The Fourier Series In time: **Continuous**, **periodic** In frequency: **Discrete**, **aperiodic**	The Discrete Fourier Series In time: **Discrete**, **periodic** In frequency: **Discrete**
Fourier Transform	The Fourier Transform In time: **Continuous**, **aperiodic** In frequency: **Continuous**, **aperiodic**	The Discrete-Time Fourier Transform In time: **Discrete**, **aperiodic** In frequency: **Continuous**

Fourier Theory

	Continuous Time	Discrete Time
Fourier Series	The Fourier Series In time: **Continuous**, **periodic** In frequency: **Discrete**, **aperiodic**	The Discrete Fourier Series In time: **Discrete**, **periodic** In frequency: **Discrete**, **periodic**
Fourier Transform	The Fourier Transform In time: **Continuous**, **aperiodic** In frequency: **Continuous**, **aperiodic**	The Discrete-Time Fourier Transform In time: **Discrete**, **aperiodic** In frequency: **Continuous**, **periodic**

Fourier Theory

	Continuous Time	Discrete Time
Fourier Series	The Fourier Series In time: **Continuous**, **periodic** In frequency: **Discrete**, **aperiodic**	The Discrete Fourier Series In time: **Discrete**, **periodic** In frequency: **Discrete**, **periodic**
Fourier Transform	The Fourier Transform In time: **Continuous**, **aperiodic** In frequency: **Continuous**, **aperiodic**	The Discrete-Time Fourier Transform In time: **Discrete**, **aperiodic** In frequency: **Continuous**, **periodic**

Repeat in frequency / sample in time

Fourier Theory

	Continuous Time	Discrete Time
Fourier Series	The Fourier Series In time: **Continuous**, **periodic** In frequency: **Discrete**, **aperiodic**	The Discrete Fourier Series In time: **Discrete**, **periodic** In frequency: **Discrete**, **periodic**
Fourier Transform	The Fourier Transform In time: **Continuous**, **aperiodic** In frequency: **Continuous**, **aperiodic**	The Discrete-Time Fourier Transform In time: **Discrete**, **aperiodic** In frequency: **Continuous**, **periodic**

Sample in frequency / repeat in time

Repeat in frequency / sample in time