Lecture 6: The Z-Transform and the Discrete-time Fourier Transform

Foundations of Digital Signal Processing

Outline

- The Z-Transform
- The Properties of the Z-Transform
- Poles, Zeros, and Region of Convergence
- The Discrete-time Fourier Transform (DTFT)
- The Properties of the Discrete-time Fourier Transform (DTFT)

News

- **Homework #3**
  - Due *Thursday* by 11:59 PM
  - Submit via canvas

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- **The Z-Transform**
- The Properties of the Z-Transform
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The Z-Transform

- **The Bilateral Z-Transform**

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

The Z-Transform

- **The Bilateral Z-Transform**

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

- **The Inverse Z-Transform**

$$x[n] = \frac{1}{2\pi j} \oint_C X(z) z^{n-1}\, dz$$

The Z-Transform

- **The Bilateral Z-Transform**

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

A complex number

$$X\left(z = Re^{j\phi}\right) = \sum_{n=-\infty}^{\infty} x[n]\left(Re^{j\phi}\right)^{-n}$$

The Z-Transform

The Bilateral Z-Transform

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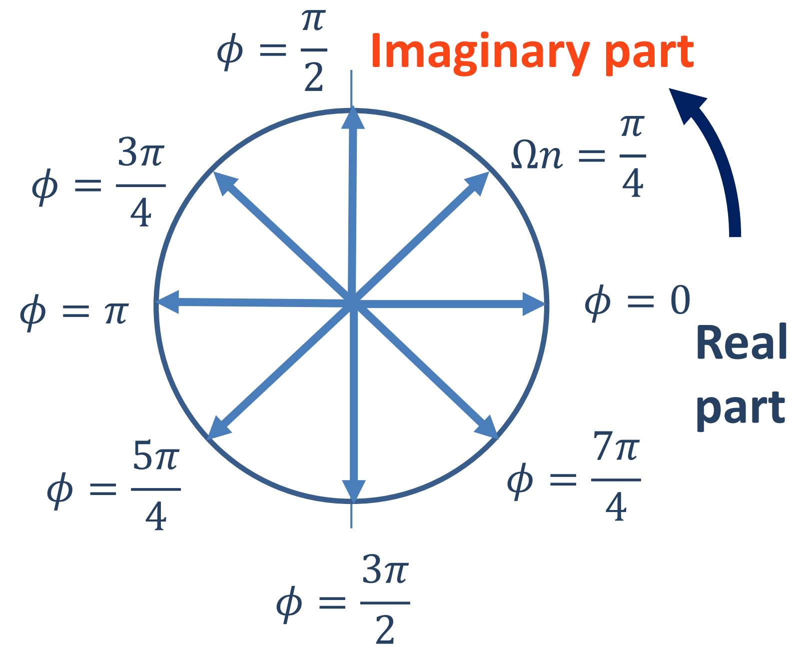
A complex number

$$X\left(z = Re^{j\phi}\right) = \sum_{n=-\infty}^{\infty} x[n]\left(Re^{j\phi}\right)^{-n}$$

Decay

$$X\left(z = Re^{j\phi}\right) = \sum_{n=-\infty}^{\infty} x[n]R^{-n}e^{-j\phi n}$$

Sinusoids



The Z-Transform

- **Question: How do I interpret this?????**

Decay

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$$X\left(z = Re^{j\phi}\right) = \sum_{n=-\infty}^{\infty} x[n]z^{*}[n]$$

Complex conjugate

The Z-Transform

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Inner project?!?!?!

Complex conjugate

The Z-Transform

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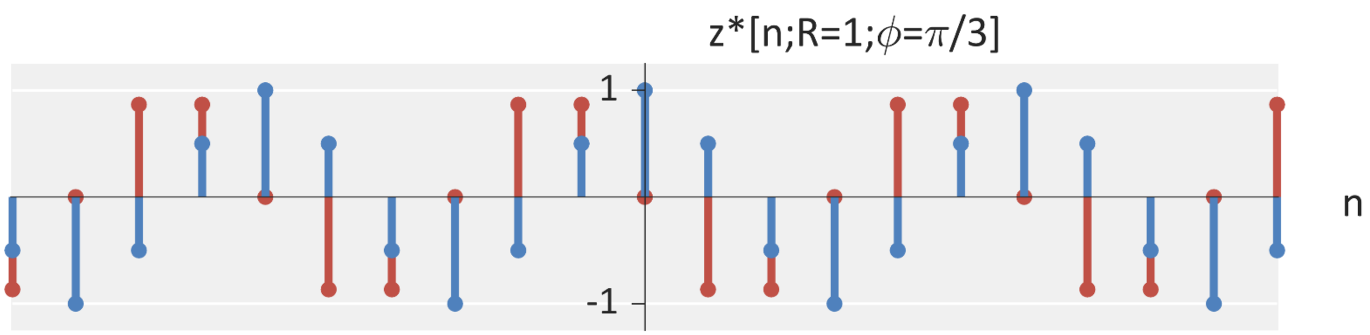

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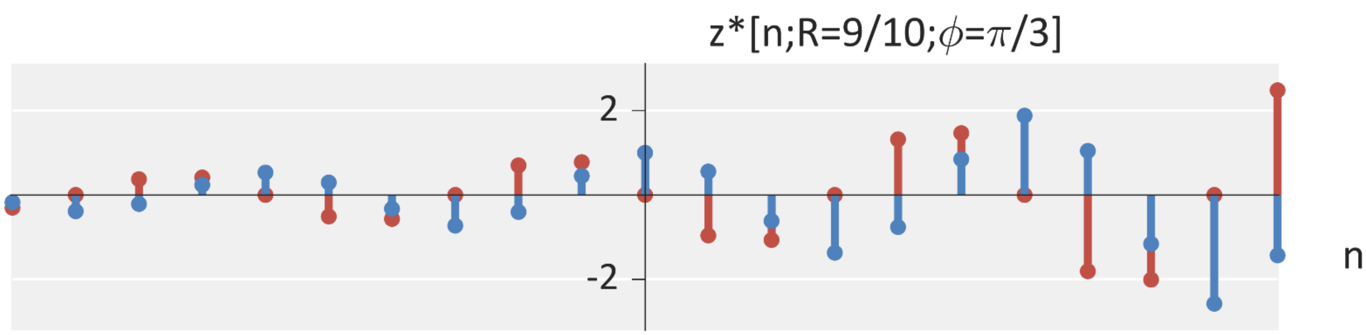

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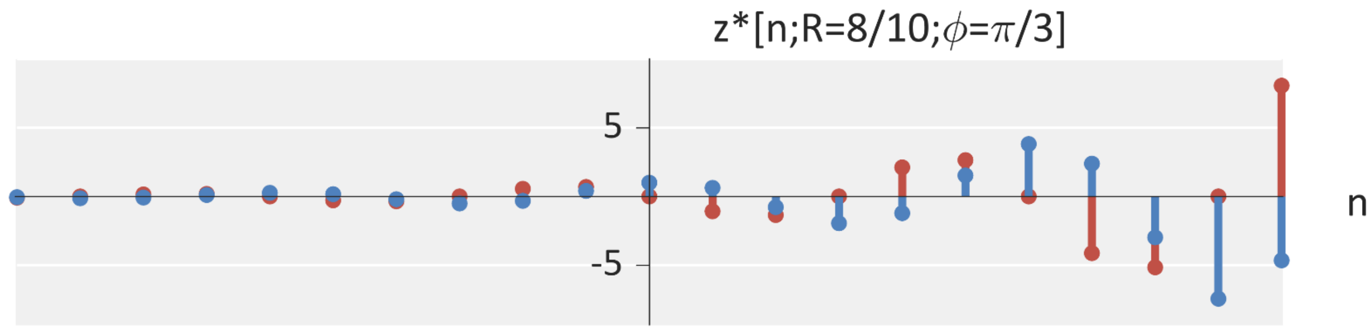

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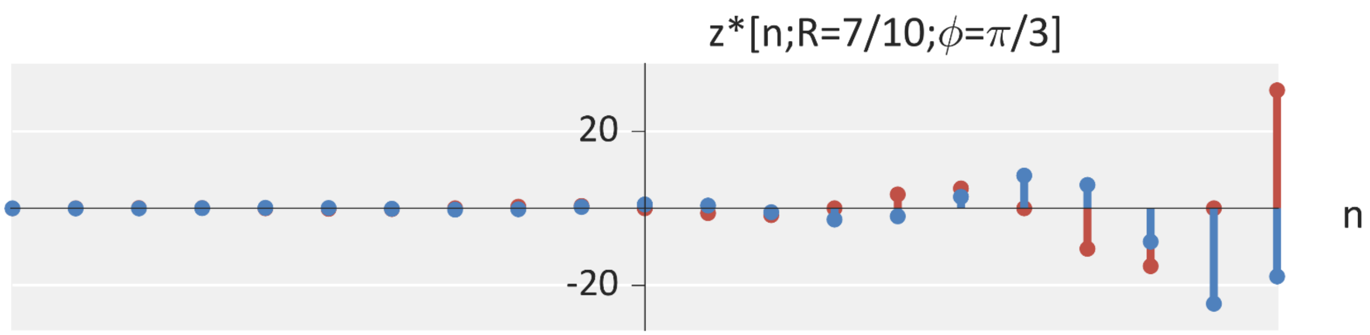

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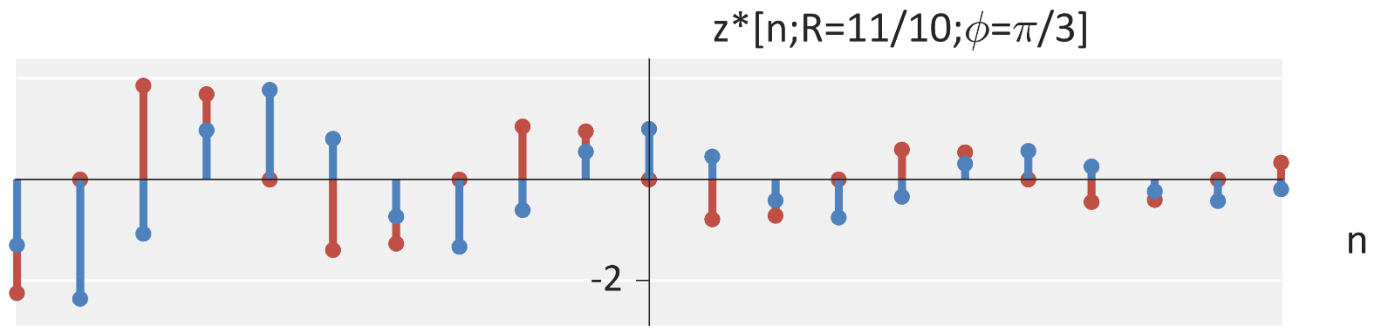

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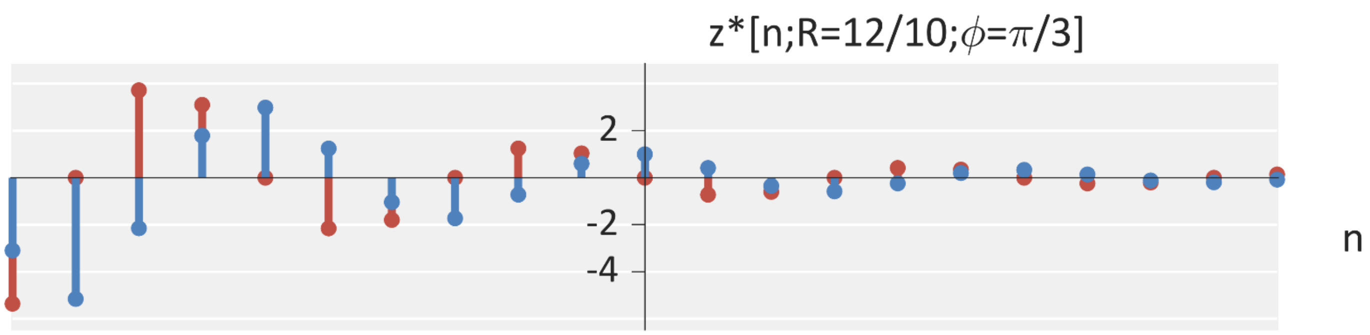

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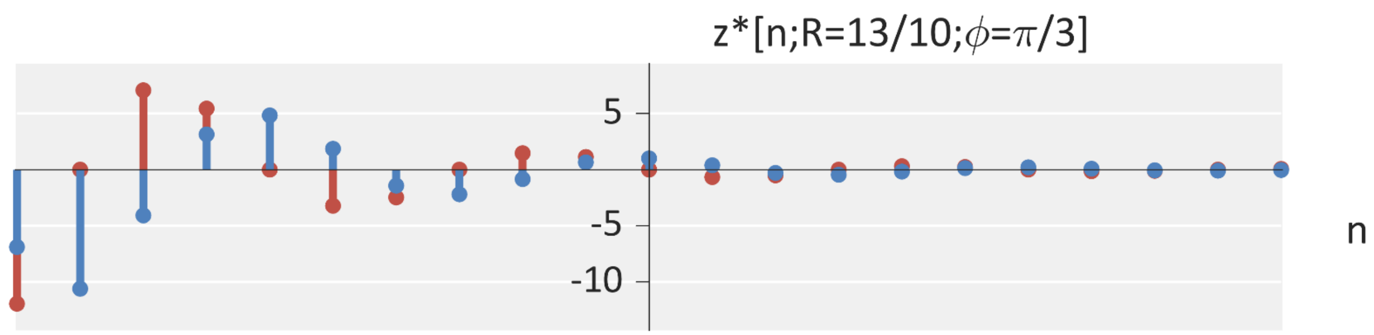

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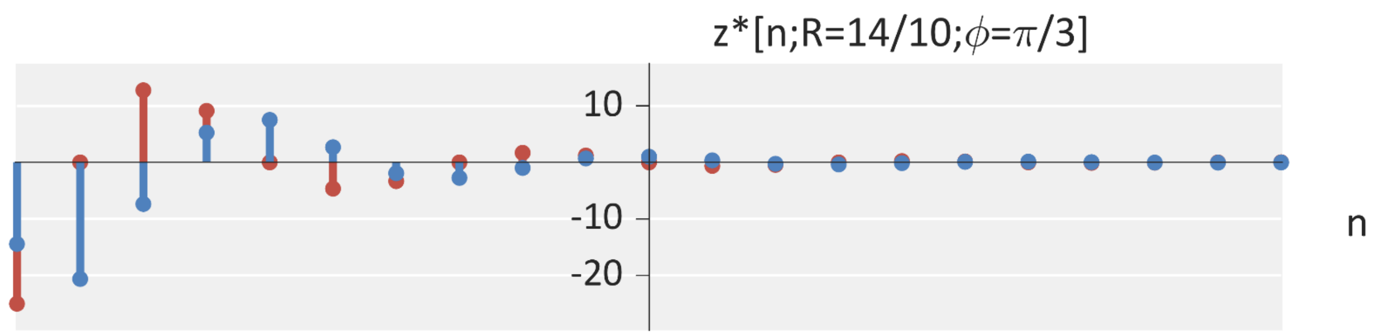

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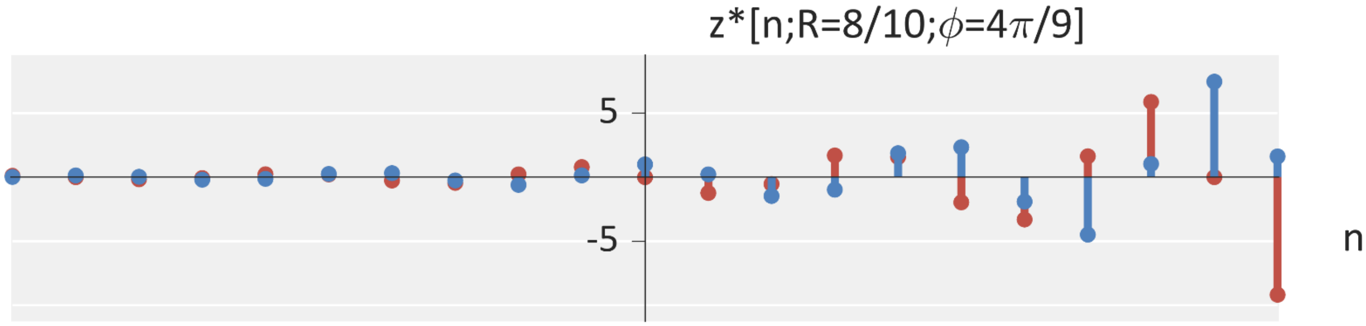

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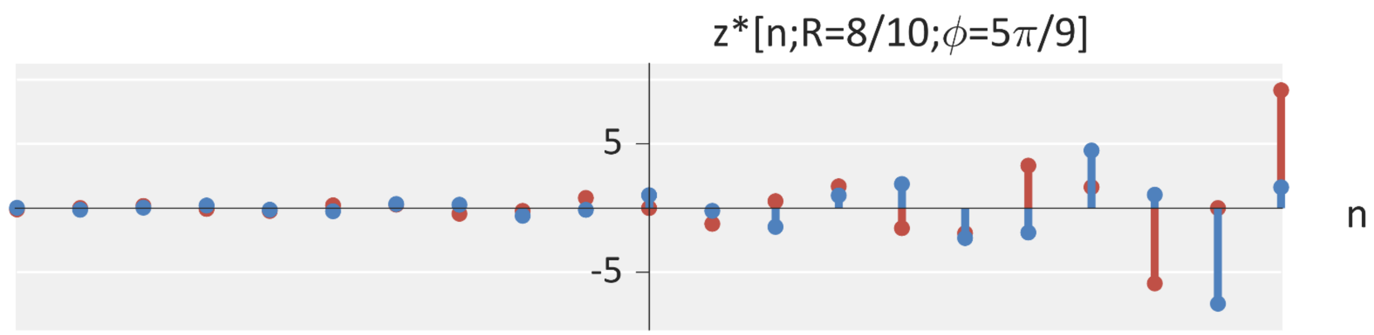

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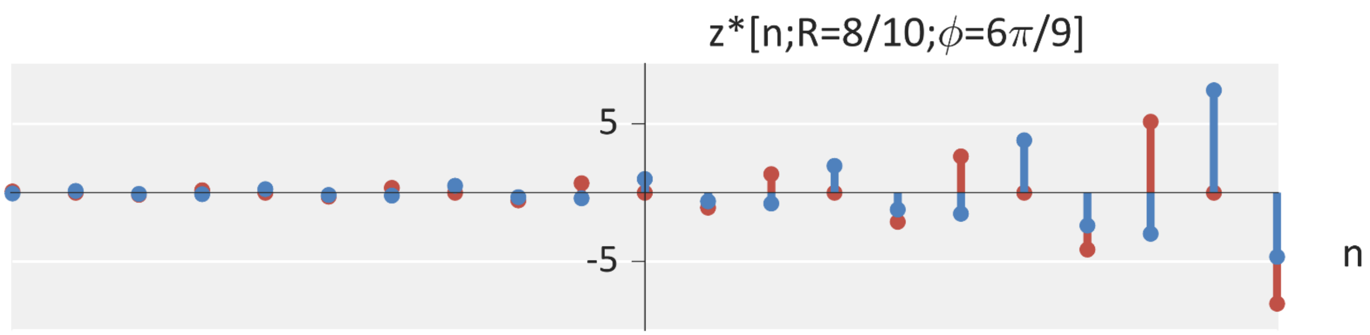

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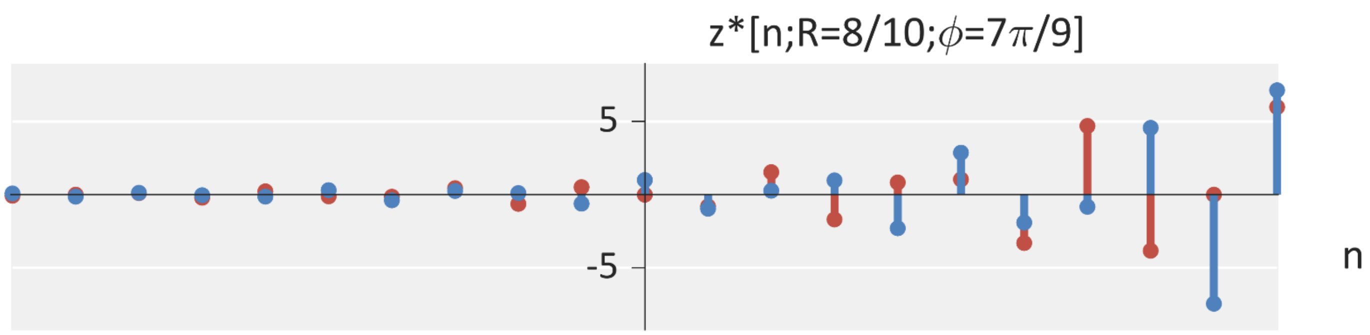

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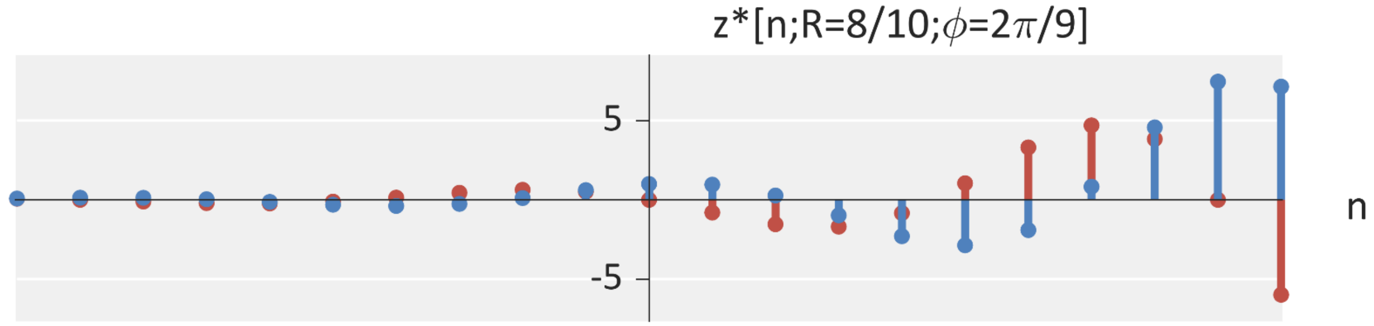

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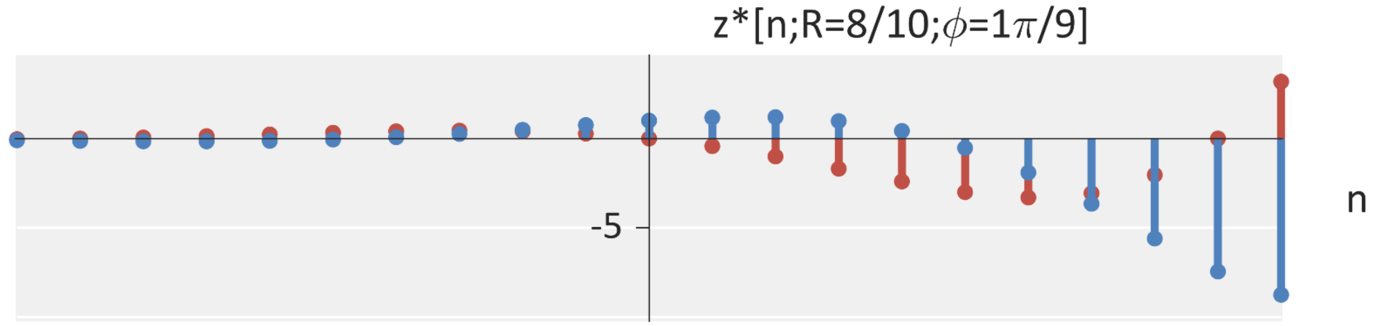

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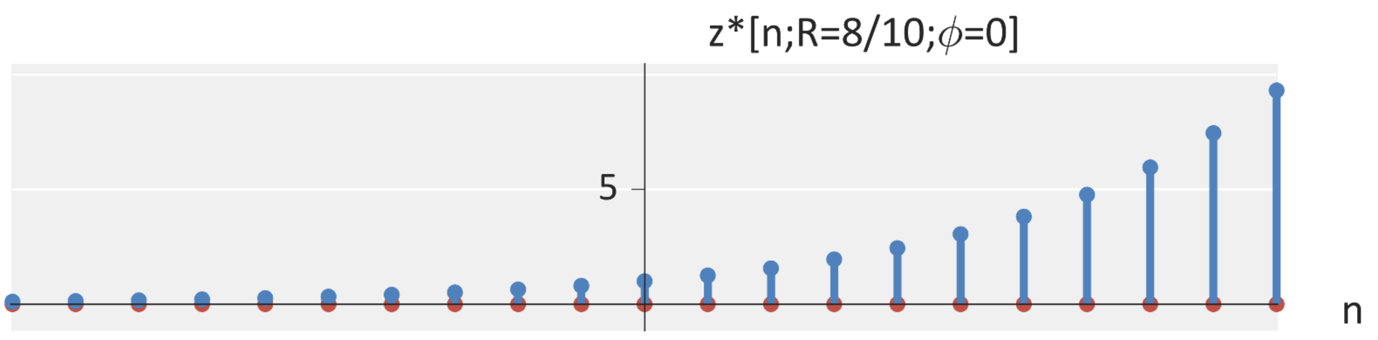

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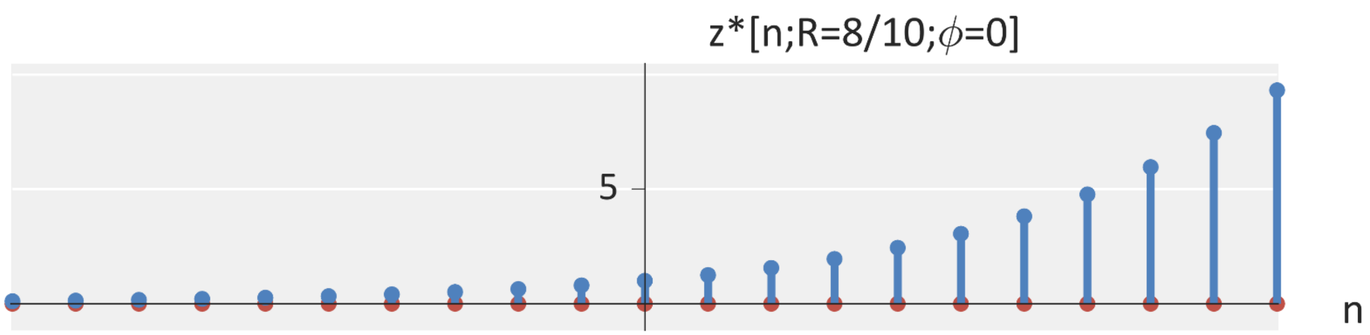

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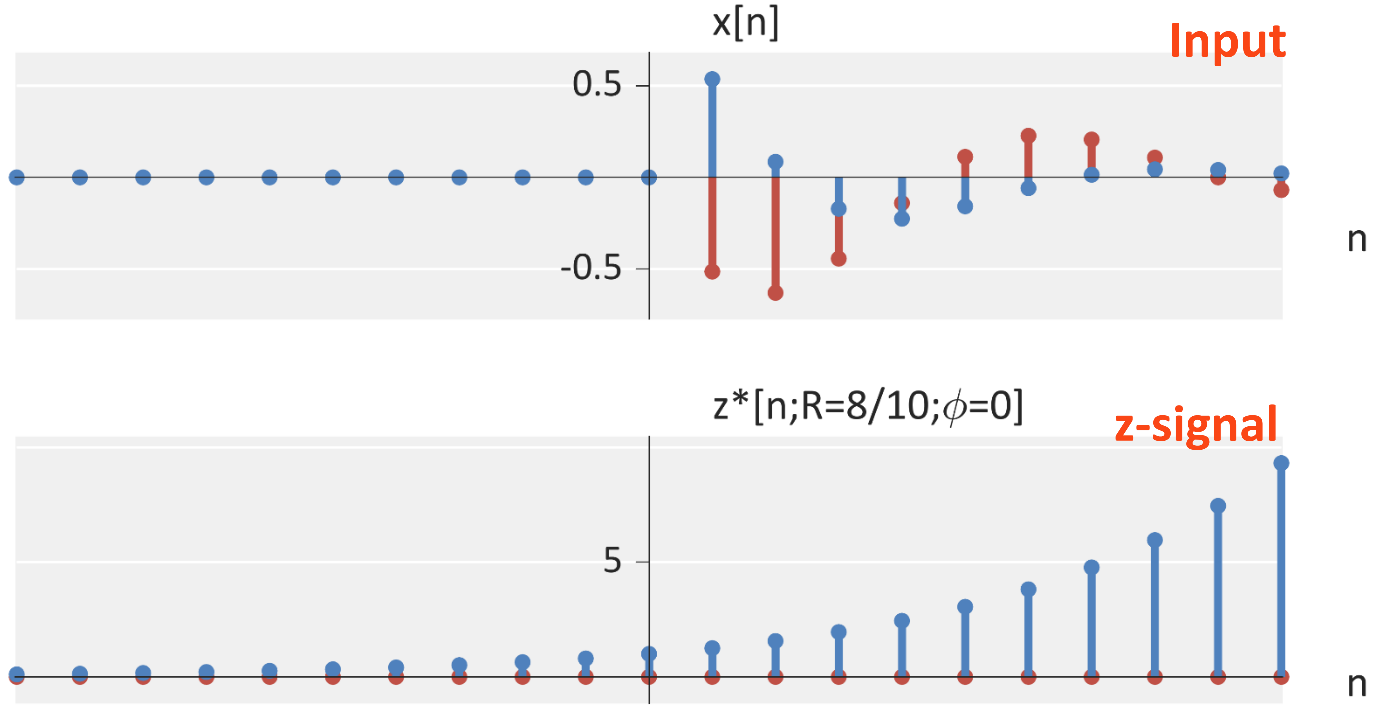

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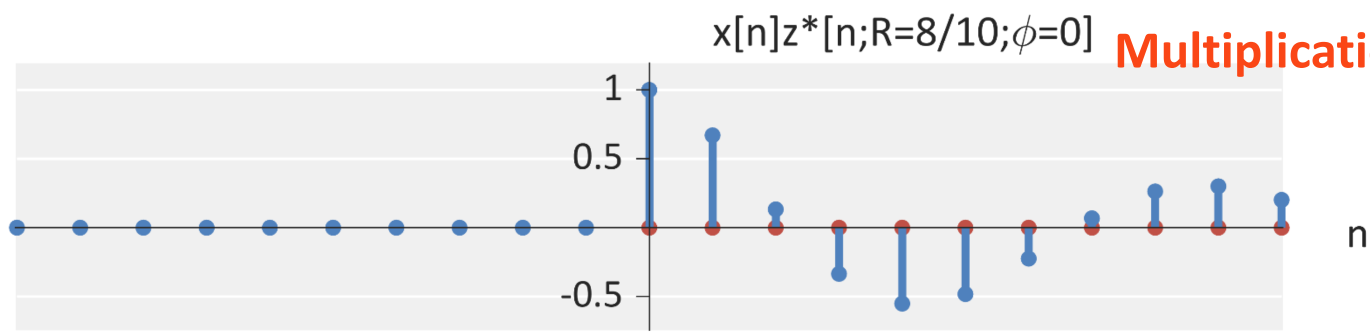


Multiplication

The Z-Transform

- **Example Problem: Compute the Z-transform of**

  $x[n] = \delta[n - 78]$

  $$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

The Z-Transform

- **Example Problem: Compute the Z-transform of** $x[n] = 10\delta[n] + 12\delta[n-1] - 5\delta[n-2] + 8\delta[n-3]$

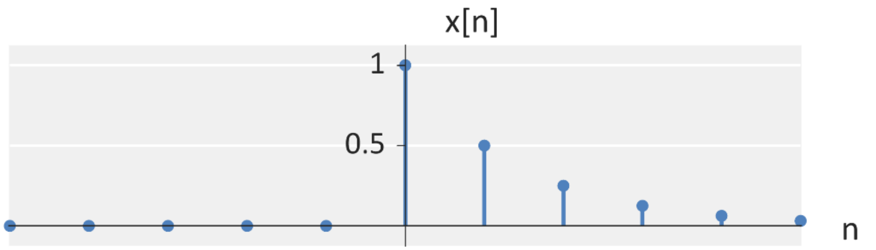
$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

The Z-Transform

- **Example Problem: Compute the Z-transform of**

$$x[n] = 2^{-n}u[n] = (1/2)^n u[n]$$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$



The Z-Transform

- **Example Problem: Compute the Z-transform of**

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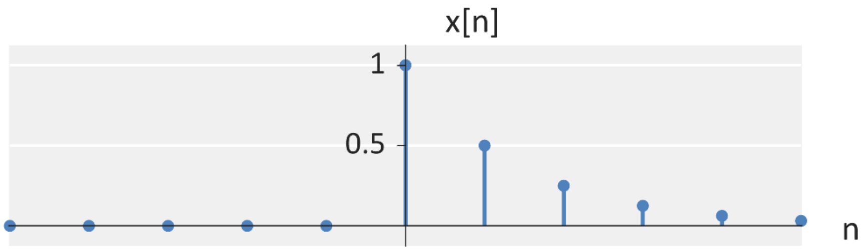


$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

- **Solution:**

Geometric Series

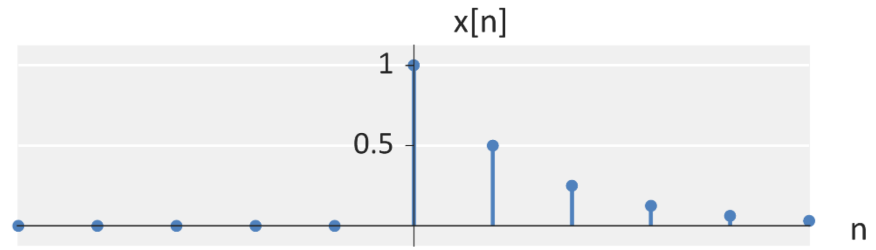
  - $X(z) = \sum_{n=-\infty}^{\infty} 2^{-n}u[n]z^{-n}$
  - $X(z) = \sum_{n=0}^{\infty}(2z)^{-n} = \sum_{n=0}^{\infty}\left((1/2)z^{-1}\right)^n$
  - $X(z) = \frac{1}{1-\left(\frac{1}{2}\right)z^{-1}}$

The Z-Transform

- **Example Problem: Compute the Z-transform of**

  $x[n] = 2^{-n}u[n] = (1/2)^n u[n]$

  $$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$



- **Solution:**

  Requirement?

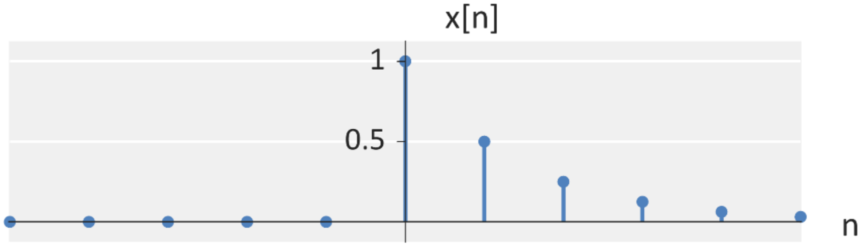
  - $X(z) = \sum_{n=-\infty}^{\infty} 2^{-n}u[n]z^{-n}$
  - $X(z) = \sum_{n=0}^{\infty} (2z)^{-n} = \sum_{n=0}^{\infty} \left((1/2)z^{-1}\right)^n$
  - $X(z) = \frac{1}{1-\left(\frac{1}{2}\right)z^{-1}}$

The Z-Transform

- **Example Problem: Compute the Z-transform of**

$x[n] = 2^{-n}u[n] = (1/2)^n u[n]$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$



Requirement?

$|(1/2)z^{-1}| < 1$

$|z^{-1}| < 2$

$|z| > 1/2$

- **Solution:**
  - $X(z) = \sum_{n=-\infty}^{\infty} 2^{-n}u[n]z^{-n}$
  - $X(z) = \sum_{n=0}^{\infty}(2z)^{-n} = \sum_{n=0}^{\infty}\left((1/2)z^{-1}\right)^n$
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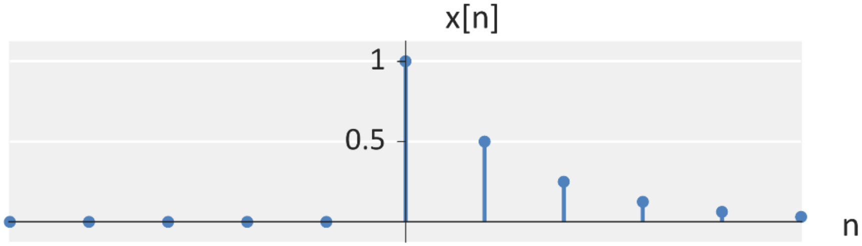
The Z-Transform

- **Example Problem: Compute the Z-transform of**

$$x[n] = 2^{-n}u[n] = (1/2)^n u[n]$$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$



Region of Convergence:

$$|z| > \frac{1}{2}$$

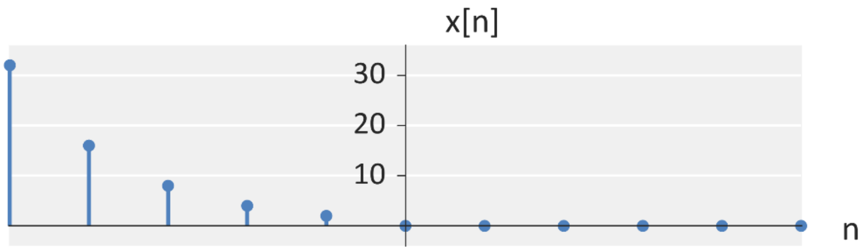
- **Solution:**
  - $X(z) = \sum_{n=-\infty}^{\infty} 2^{-n}u[n]z^{-n}$
  - $X(z) = \sum_{n=0}^{\infty}(2z)^{-n} = \sum_{n=0}^{\infty}\left((1/2)z^{-1}\right)^n$
  - $X(z) = \dfrac{1}{1-\left(\frac{1}{2}\right)z^{-1}}$

The Z-Transform

- **Example Problem: Compute the Z-transform of**

$$x[n] = -(1/2)^n u[-n-1]$$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

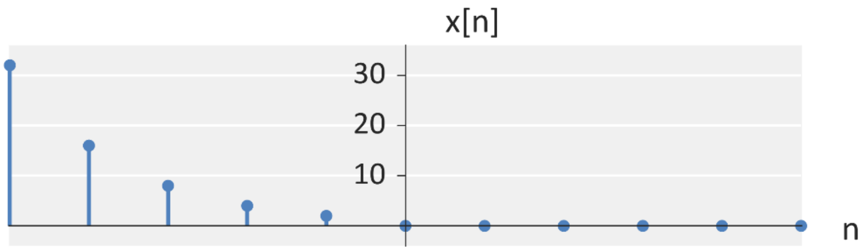


The Z-Transform

- **Example Problem: Compute the Z-transform of**

$x[n] = -(1/2)^n u[-n-1]$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$



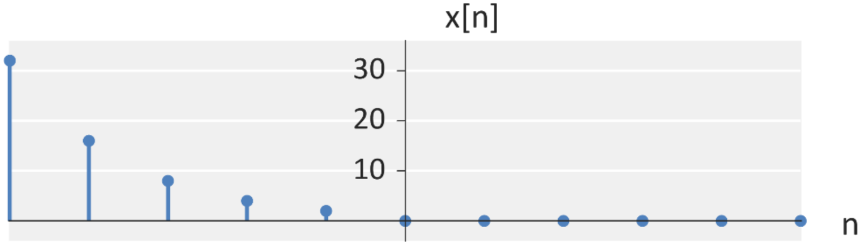
- **Solution:**
  - $X(z) = \sum_{n=-\infty}^{\infty} -(1/2)^n u[-n-1] z^{-n}$
  - $X(z) = \sum_{n=-\infty}^{-1} -\left((1/2)z^{-1}\right)^n$
  - $X(z) = \sum_{n=1}^{\infty} -\left((1/2)z^{-1}\right)^{-n} = \sum_{n=1}^{\infty} -(2z)^n$
  - $X(z) = \frac{-2z}{1-2z} = \frac{z}{z-\frac{1}{2}} = \frac{1}{1-\left(\frac{1}{2}\right)z^{-1}}$ **Same as before!**

The Z-Transform

- **Example Problem: Compute the Z-transform of**

$$x[n] = -(1/2)^n u[-n-1]$$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$



- **Solution:**
  - $X(z) = \sum_{n=-\infty}^{\infty} -(1/2)^n u[-n-1] z^{-n}$
  - $X(z) = \sum_{n=-\infty}^{-1} -\left((1/2)z^{-1}\right)^n$
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  - $X(z) = \frac{-2z}{1-2z} = \frac{z}{z-\frac{1}{2}} = \frac{1}{1-\left(\frac{1}{2}\right)z^{-1}}$

Requirement?

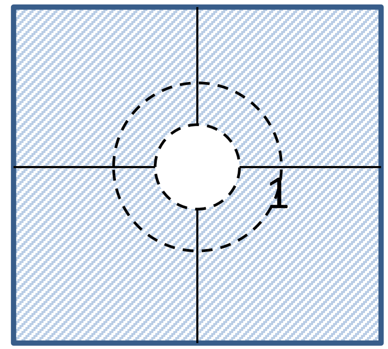
$|2z| < 1$

$|z| < \frac{1}{2}$

Same as before!

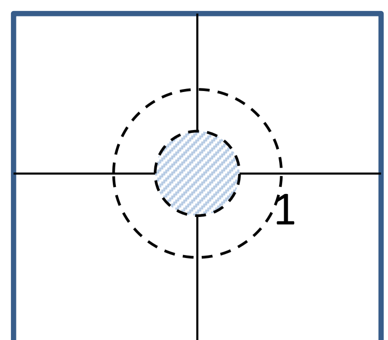
The Z-Transform

- **Causal Transform**
  - Time-domain: $x[n] = 2^{-n}u[n] = (1/2)^n u[n]$
  - Z-domain: $X(z) = \frac{1}{1-(1/2)z^{-1}}$



Region of Convergence $|z| > \frac{1}{2}$

- **Anti-Causal Transform**
  - Time-domain: $x[n] = -(1/2)^n u[-n-1]$
  - Z-domain: $X(z) = \frac{1}{1-(1/2)z^{-1}}$



Region of Convergence $|z| < \frac{1}{2}$

Z-Transform Table

- **Find online**
  - http://smartdata.ece.ufl.edu/eee5502/eee5502_DiscreteTransforms.pdf

The Z-Transform

- **Example: Compute the Z-transform of**
  - $y[n] = n\left(-\frac{1}{3}\right)^n u[n]$

The Z-Transform

- **Example: Compute the Z-transform of**
  - $y[n] = n\left(-\frac{1}{3}\right)^n u[n]$

- **Solution:**
  - $na^n u[n] \overset{\mathcal{Z}}{\leftrightarrow} \frac{az^{-1}}{(1-az^{-1})^2}$
  - $Y(z) = \frac{(-1/3)z^{-1}}{(1+(1/3)z^{-1})^2}$

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- **The Properties of the Z-Transform**
- Poles, Zeros, and Region of Convergence
- The Discrete-time Fourier Transform (DTFT)
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Properties of the Z-Transform

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Properties of the Z-Transform

- **Example: Compute the Z-transform of**
  - $y[n] = (1/2)^{n-5}u[n-5]$

Properties of the Z-Transform

- **Example: Compute the Z-transform of**
  - $y[n] = (1/2)^n u[n] * u[n]$

Properties of the Z-Transform

- **Example: Compute the Z-transform of**
  - $y[n] = (1/2)^{-n} u[-n]$

Properties of the Z-Transform

- **Example: Compute the Z-transform of**
  - $y[n] = x[n-5]$

Properties of the Z-Transform

- **Example: Compute the Z-transform of**
  - $y[n] = x[n-5]$

Properties of the Z-Transform

- **Example: Compute the Z-transform of**
  - $y[n] - (1.1)y[n-1] = x[n]$

Properties of the Z-Transform

- **Example: Compute the Z-transform of**
  - $y[n] - (1/2)y[n-1] = x[n-5] + 10x[n-10]$

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- **Poles, Zeros, and Region of Convergence**
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Difference Equations

- **General form for an LTI system is:**

$$\sum_{m=-\infty}^{\infty} a[m]y[n-m] = \sum_{m=-\infty}^{\infty} b[m]x[n-m]$$

$$a[n] * y[n] = b[n] * x[n]$$

Difference Equations

- **General form for an LTI system is:**

$$\sum_{m=-\infty}^{\infty} a[m]y[n-m] = \sum_{m=-\infty}^{\infty} b[m]x[n-m]$$

$$a[n] * y[n] = b[n] * x[n]$$

- **Apply the Z-Transform to both sides**

$$A(z)Y(z) = B(z)X(z)$$

$$\frac{Y(z)}{X(z)} = H(z) = \frac{B(z)}{A(z)}$$

Difference Equations

- **General form for an LTI system is:**

$$\sum_{m=-\infty}^{\infty} a[m]y[n-m] = \sum_{m=-\infty}^{\infty} b[m]x[n-m]$$

- **Apply the Z-Transform to both sides**

$$\sum_{m=-\infty}^{\infty} a[m]Y(z)z^{-m} = \sum_{m=-\infty}^{\infty} b[m]X(z)z^{-m}$$

$$\frac{Y(z)}{X(z)} = H(z) = \frac{\sum_{m=-\infty}^{\infty} b[m]z^{-m}}{\sum_{m=-\infty}^{\infty} a[m]z^{-m}}$$

Difference Equations

- **General form for an LTI system is:**

$$\sum_{m=0}^{\infty} a[m]y[n-m] = \sum_{m=0}^{\infty} b[m]x[n-m]$$

- **Apply the Z-Transform to both sides (when causal)**

$$\sum_{m=0}^{\infty} a[m]Y(z)z^{-m} = \sum_{m=0}^{\infty} b[m]X(z)z^{-m}$$

$$\frac{Y(z)}{X(z)} = H(z) = \frac{\sum_{m=0}^{\infty} b[m]z^{-m}}{\sum_{m=0}^{\infty} a[m]z^{-m}}$$

Difference Equations

- **General form for an LTI system is:**

$$\sum_{m=0}^{\infty} a[m]y[n-m] = \sum_{m=0}^{\infty} b[m]x[n-m]$$

- **Apply the Z-Transform to both sides (when causal)**

$$\sum_{m=0}^{\infty} a[m]Y(z)z^{-m} = \sum_{m=0}^{\infty} b[m]X(z)z^{-m}$$

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$$\frac{Y(z)}{X(z)} = H(z) = \frac{\prod_{k=1}^{M}(1 - z_k z^{-1})}{\prod_{k=1}^{N}(1 - p_k z^{-1})}$$

Difference Equations

- **General form for an LTI system is:**

$$\sum_{m=0}^{\infty} a[m]y[n-m] = \sum_{m=0}^{\infty} b[m]x[n-m]$$

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$$\frac{Y(z)}{X(z)} = H(z) = Gz^{-M+N}\frac{\prod_{k=1}^{M}(z-z_k)}{\prod_{k=1}^{N}(z-p_k)}$$

Difference Equations

- **This can be expressed as**

$$H(z) = Gz^{-M+N} \frac{\prod_{k=1}^{M}(z - z_k)}{\prod_{k=1}^{N}(z - p_k)}$$

- **Zeros occur where**
  - $\prod_{k=1}^{M}(z - z_k) = 0, H(z) = 0$
- **Poles occur where**
  - $\prod_{k=1}^{N}(z - p_k) = 0, H(z) = \infty$

The Z-Transform

- **Consider the Z-transform signal**
  - $X(z) = \frac{1}{1-\left(\frac{1}{2}\right)z^{-1}}$

- **Determine the poles and zeros.**

The Z-Transform

- **Consider the Z-transform signal**
  - $X(z) = \frac{1}{1-\left(\frac{1}{2}\right)z^{-1}}$

- **Determine the poles and zeros.**
  - $X(z) = \frac{1}{1-\left(\frac{1}{2}\right)z^{-1}} = \frac{z}{z-\left(\frac{1}{2}\right)}$
  - Zeros: $z = 0$
  - Poles: $z = 1/2$

The Z-Transform

- **Consider the Z-transform signal**
  - $X(z) = \frac{1}{1-\left(\frac{1}{2}\right)z^{-1}} + \frac{1}{1-2z^{-1}}$
  - Determine the poles and zeros.

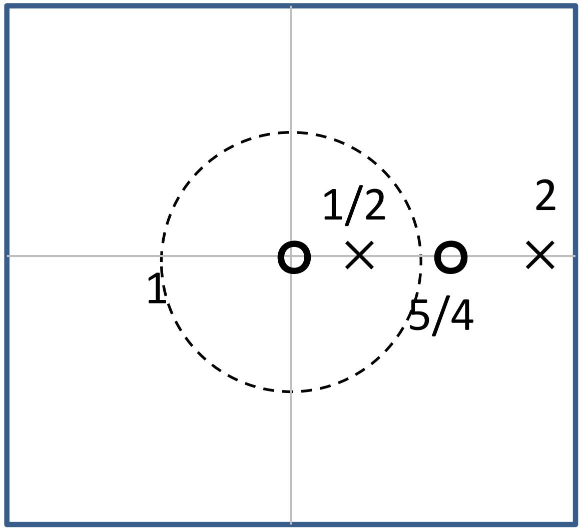
The Z-Transform

- **Consider the Z-transform signal**
  - $X(z) = \frac{1}{1-\left(\frac{1}{2}\right)z^{-1}} + \frac{1}{1-2z^{-1}}$
- **Determine the poles and zeros.**
- **Solution:**
  - $X(z) = \frac{z}{z-1/2} + \frac{z}{z-2}$
  - $X(z) = \frac{z(z-2)+z(z-1/2)}{(z-1/2)(z-2)} = \frac{2z^2-(5/2)z}{(z-1/2)(z-2)} = \frac{2z(z-5/4)}{(z-1/2)(z-2)}$
  - **Zeros:** $z = 5/4,0$
  - **Poles:** $z = 1/2,2$

The Z-Transform

- **Consider the Z-transform signal**
  - $X(z) = \frac{1}{1-\left(\frac{1}{2}\right)z^{-1}} + \frac{1}{1-2z^{-1}}$
  - Determine the poles and zeros.

- **Solution:**
  - $X(z) = \frac{2z(z-5/4)}{(z-1/2)(z-2)}$
  - **Zeros:** $z = 5/4$
  - **Poles:** $z = 1/2, 2$



The Z-Transform

- **Rules for the Region of Convergence**
  - If we have multiple ROCs
    - ◊ The true ROC is the intersection of all ROCs
    - ◊ The ROCs all begin and end at a pole, the origin, or infinity

- **Consider the Z-transform signal**
  - $X(z) = \frac{1}{1-\left(\frac{1}{2}\right)z^{-1}} + \frac{1}{1-2z^{-1}}$

The Z-Transform

- **Rules for the Region of Convergence**
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- **Consider the Z-transform signal**
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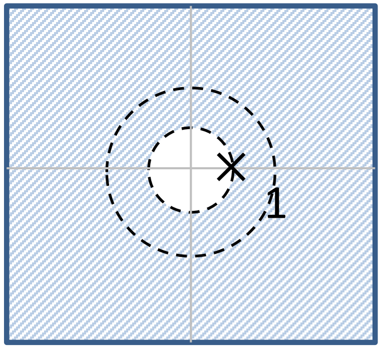

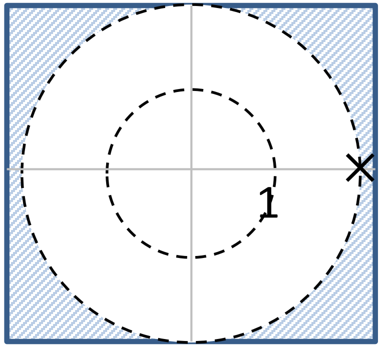

The Z-Transform

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  - If we have multiple ROCs
    - The true ROC is the intersection of all ROCs
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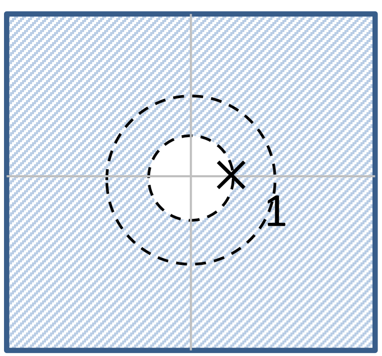
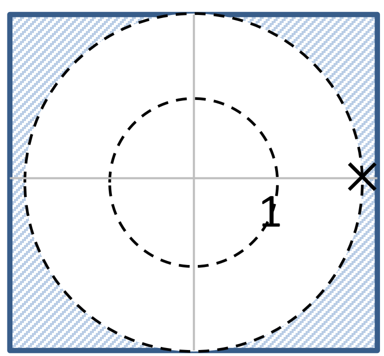
- **Consider the Z-transform signal**
  - $X(z) = \frac{1}{1-\left(\frac{1}{2}\right)z^{-1}} + \frac{1}{1-2z^{-1}}$



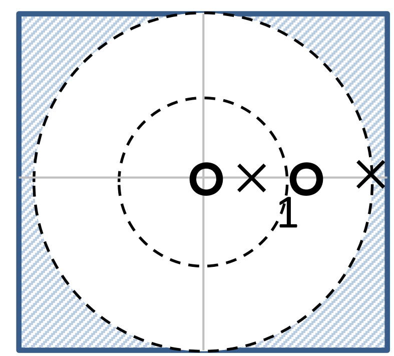
ROC:
$|z| > \frac{1}{2}$



ROC:
$|z| > 2$

Total ROC

ROC:
$|z| > 2$



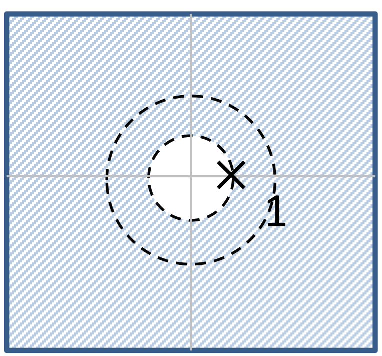
Rules for the Region of Convergence

- If we have multiple ROCs
  - ◊ The true ROC is the intersection of all ROCs
  - ◊ The ROCs all begin and end at a pole, the origin, or infinity

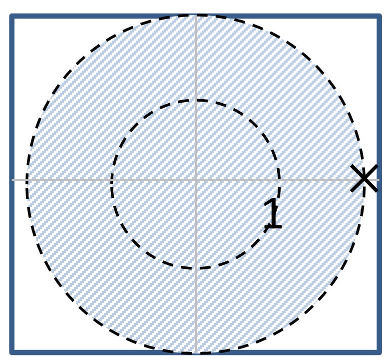
Consider the Z-transform signal

- $X(z) = \dfrac{1}{1-\left(\frac{1}{2}\right)z^{-1}} + \dfrac{1}{1-2z^{-1}}$

ROC:
$\dfrac{1}{2} < |z| < 2$

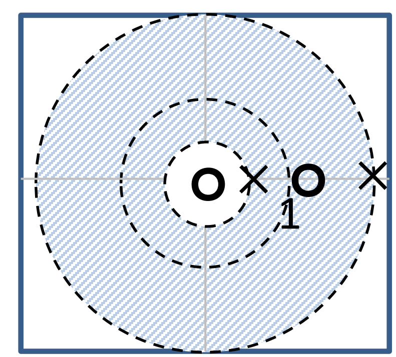


ROC:
$|z| > \dfrac{1}{2}$



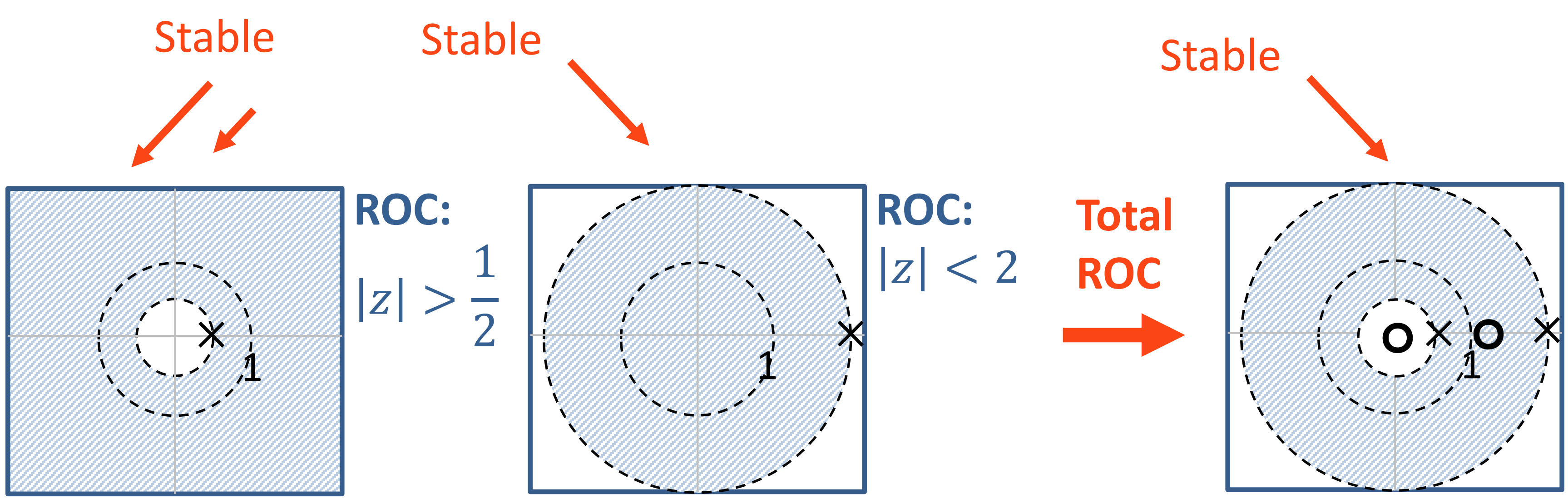
ROC:
$|z| < 2$

Total ROC



The Z-Transform

- **Rules for the Region of Convergence**
  - If we have multiple ROCs
    - The true ROC is the intersection of all ROCs
  - If the ROC includes the unit circle
    - The signal is BIBO stable



The Z-Transform

- **Determine the Z-Transform and ROC for:**
  - $x[n] = (-1/4)^n u[n] - 2(-1/2)^n u[n]$

The Z-Transform

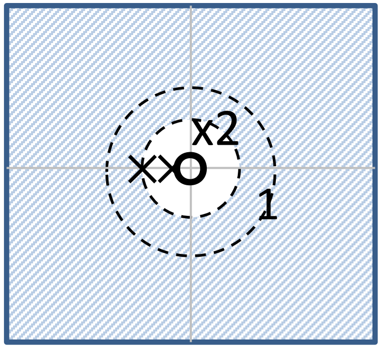
- **Determine the Z-Transform and ROC for:**
  - $x[n] = (-1/4)^n u[n] - 2(-1/2)^n u[n]$

- **Solution: Based on the transform table**
  - $X(z) = \frac{1}{1+(1/4)z^{-1}} - \frac{2}{1+(1/2)z^{-1}}$

The Z-Transform

- **Determine the Z-Transform and ROC for:**
  - $x[n] = (-1/4)^n u[n] - 2(-1/2)^n u[n]$

- **Solution: Based on the transform table**
  - $X(z) = \frac{1}{1-\left(-\frac{1}{4}\right)z^{-1}} - \frac{2}{1-\left(-\frac{1}{2}\right)z^{-1}} = \frac{z}{z+\frac{1}{4}} - \frac{2z}{z+\frac{1}{2}} = \frac{z\left(z+\frac{1}{2}-2z-2\frac{1}{4}\right)}{\left(z+\frac{1}{4}\right)\left(z+\frac{1}{2}\right)}$
  - $X(z) = \frac{-z^2}{\left(z+\frac{1}{4}\right)\left(z+\frac{1}{2}\right)}$



ROC:

$|z| > \frac{1}{2}$

The Z-Transform

- **Determine the Z-Transform and ROC for:**
  - $x[n] = (3/2)^n u[n] - (2)^n u[-n-1]$

The Z-Transform

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  - $x[n] = (3/2)^n u[n] - (2)^n u[-n-1]$

- **Solution: Based on the transform table**
  - $X(z) = \frac{1}{1-\left(\frac{3}{2}\right)z^{-1}} + \frac{1}{1-(2)z^{-1}} = \frac{z(z-2+z-3/2)}{(z-3/2)(z-2)} = \frac{2z(z-7/4)}{(z-3/2)(z-2)}$

The Z-Transform

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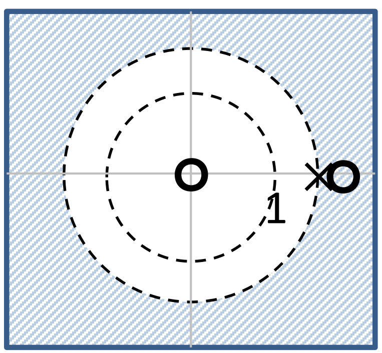
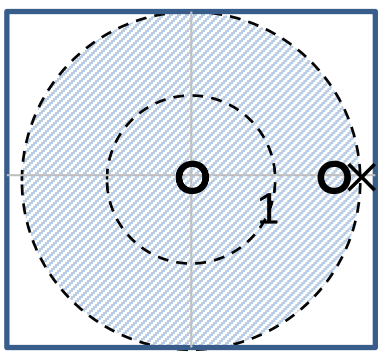


ROC:

$|z| > \frac{3}{2}$



ROC:

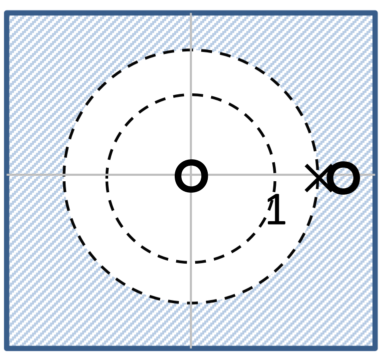
$|z| < 2$

The Z-Transform

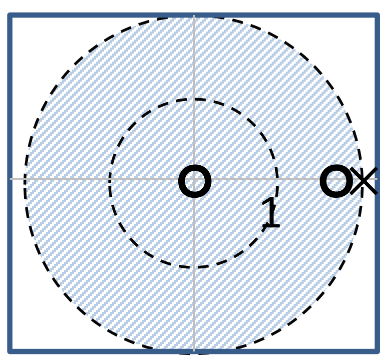
- **Determine the Z-Transform and ROC for:**
  - $x[n] = (3/2)^n u[n] - (2)^n u[-n-1]$

- **Solution:** **Based on the transform table**
  - $X(z) = \dfrac{1}{1-\left(\frac{3}{2}\right)z^{-1}} + \dfrac{1}{1-(2)z^{-1}} = \dfrac{2-(7/2)z^{-1}}{\left(1-\left(\frac{3}{2}\right)z^{-1}\right)(1-(2)z^{-1})}$

ROC:
$\frac{3}{2} < |z| < 2$

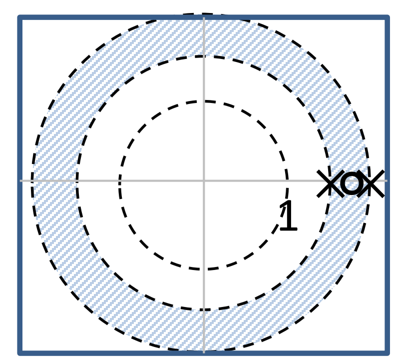


ROC:
$|z| > \frac{1}{4}$



ROC:
$|z| < \frac{3}{2}$

Total ROC



The Z-Transform

- **Determine the Z-Transform and ROC for:**
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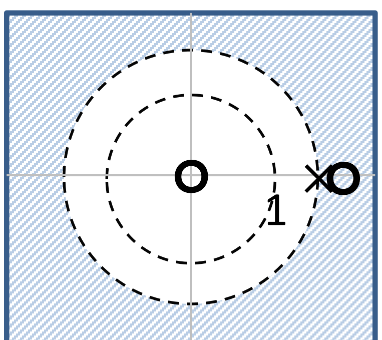
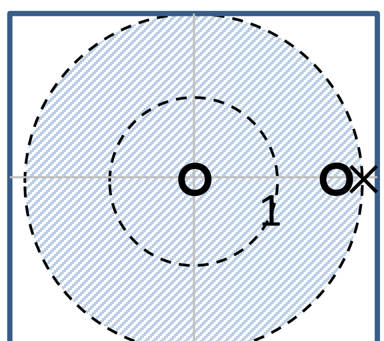
ROC: $\frac{3}{2} < |z| < 2$

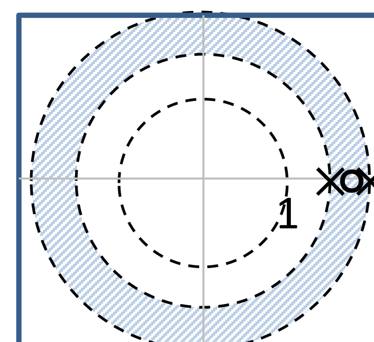


ROC: $|z| > \frac{1}{4}$



ROC: $|z| < \frac{3}{2}$

Total ROC



Not Stable!

The Z-Transform

- **Determine the Z-Transform and ROC for:**
  - $x[n] = \delta[n-1] + 4\delta[n-3]$

The Z-Transform

- **Determine the Z-Transform and ROC for:**
  - $x[n] = \delta[n-1] + 4\delta[n-3]$

- **Solution: Based on the transform table**
  - $X(z) = z^{-1} + 4z^{-3}$
  - $X(z) = \frac{z^2+4}{z^3}$

The Z-Transform

- **Determine the Z-Transform and ROC for:**
  - $x[n] = \delta[n-1] + 4\delta[n-3]$
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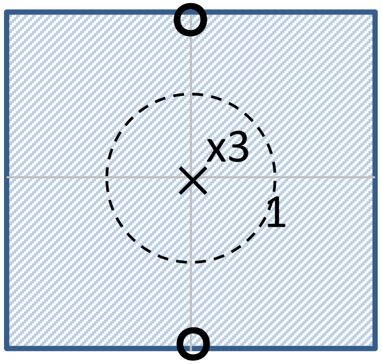


ROC:
$|z| > 0$

Lecture 6: The Z -Transform and the Discrete -time Fourier Transform

Foundations of Digital Signal Processing

Outline

- The Z-Transform
- The Properties of the Z-Transform
- Poles, Zeros, and Region of Convergence
- **The Discrete-time Fourier Transform (DTFT)**
- The Properties of the Discrete-time Fourier Transform (DTFT)

The Discrete-Time Fourier Transform

- **Z-Transform to DTFT**

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

A complex number

$$X\left(z = Re^{j\phi}\right) = \sum_{n=-\infty}^{\infty} x[n]\left(Re^{j\phi}\right)^{-n}$$

The Discrete-Time Fourier Transform

- **Z-Transform to DTFT**

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

A complex number

$$X\left(z = \cancel{R}e^{j\phi}\right) = \sum_{n=-\infty}^{\infty} x[n]\left(\cancel{R}e^{j\phi}\right)^{-n}$$

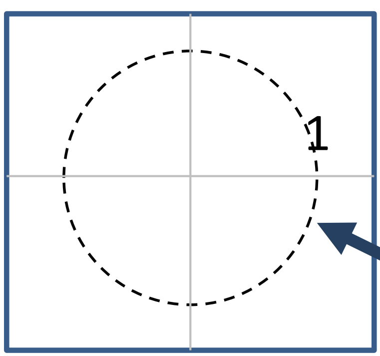
The Discrete-Time Fourier Transform

- **Z-Transform to DTFT**

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

A complex number

$$X\left(z = \cancel{R} e^{j\phi}\right) = \sum_{n=-\infty}^{\infty} x[n] \left(\cancel{R} e^{j\phi}\right)^{-n}$$



Restrict z-transform to the unit circle

The Discrete-Time Fourier Transform

- **Z-Transform to DTFT**

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

$$X\left(z = e^{j\omega}\right) = \sum_{n=-\infty}^{\infty} x[n]\left(e^{j\omega}\right)^{-n}$$

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

The Discrete-Time Fourier Transform

- **The Discrete-Time Fourier Transform (DTFT)**

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

- **The Inverse Discrete-Time Fourier Transform**

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(\omega)e^{j\omega n}\, d\omega$$

The Discrete-Time Fourier Transform

- **The Discrete-Time Fourier Transform (DTFT)**

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

- **Question: How do I interpret this DTFT?**

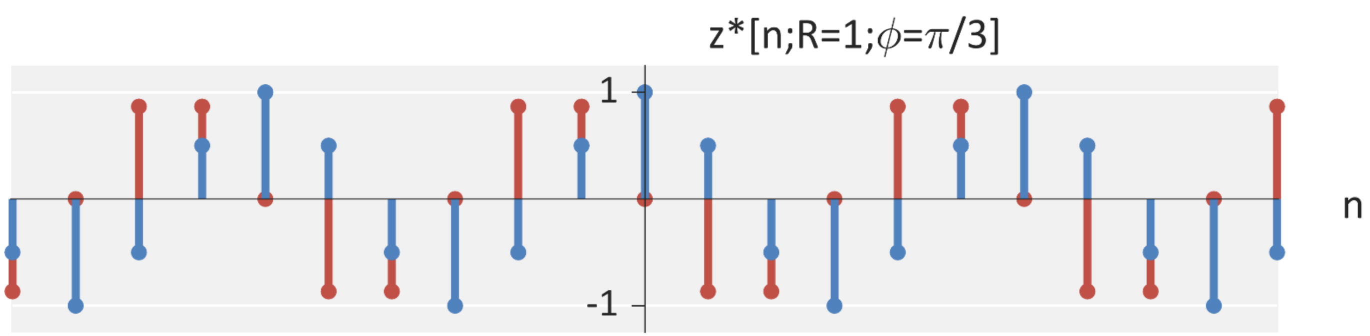
The Discrete-Time Fourier Transform

- **The Discrete-Time Fourier Transform (DTFT)**

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

Inner product of signal and sinusoids!

- **Question: How do I interpret this DTFT?**



The Discrete-Time Fourier Transform

- **The Discrete-Time Fourier Transform (DTFT)**

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

- **Example:** Compute the DTFT of $x[n] = \delta[n]$

The Discrete-Time Fourier Transform

- **The Discrete-Time Fourier Transform (DTFT)**

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

- **Example:** Compute the DTFT of $x[n] = \delta[n]$

$$X(\omega) = \sum_{n=-\infty}^{\infty} \delta[n]e^{-j\omega n} = \sum_{n=-\infty}^{\infty} \delta[n]e^{-j\omega 0} = \sum_{n=-\infty}^{\infty} \delta[n] = 1$$

The Discrete-Time Fourier Transform

- **The Discrete-Time Fourier Transform (DTFT)**

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

- **Example:** Compute the DTFT of $x[n] = 10\,\delta[n-42]$

The Discrete-Time Fourier Transform

- **The Discrete-Time Fourier Transform (DTFT)**

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

- **Example:** Compute the DTFT of $x[n] = 10\,\delta[n-42]$

$$X(\omega) = \sum_{n=-\infty}^{\infty} 10\delta[n-42]e^{-j\omega n} = \sum_{n=-\infty}^{\infty} 10\delta[n-42]e^{-j\omega(42)}$$

$$= 10e^{-j\omega(42)}$$

The Discrete-Time Fourier Transform

- **The Discrete-Time Fourier Transform (DTFT)**

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

- **Example:** Compute the DTFT of $x[n] = a^n u[n]$

The Discrete-Time Fourier Transform

- **The Discrete-Time Fourier Transform (DTFT)**

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

- **Example:** Compute the DTFT of $x[n] = a^n u[n]$

$$X(\omega) = \sum_{n=-\infty}^{\infty} a^n u[n]\, e^{-j\omega n} = \sum_{n=0}^{\infty} \left(ae^{-j\omega}\right)^n = \frac{1}{1 - ae^{-j\omega}}$$

If $|a| < 1$