Lecture 11: Sampling

Foundations of Digital Signal Processing

Outline

- Sampling
- Sampling in Time = ??? in Frequency
- The Nyquist-Shannon Sampling Theorem
- Continuous-time Reconstruction / Interpolation
- Aliasing

News

- **No Homework This Week**
  - Yay!
- **Homework #5**
  - Due *next week*
  - Submit via canvas
  - Short-ish assignment
- **Coding Problem #3**
  - Due *next week*
  - Submit via canvas

Lecture 11: Sampling

Foundations of Digital Signal Processing

Outline

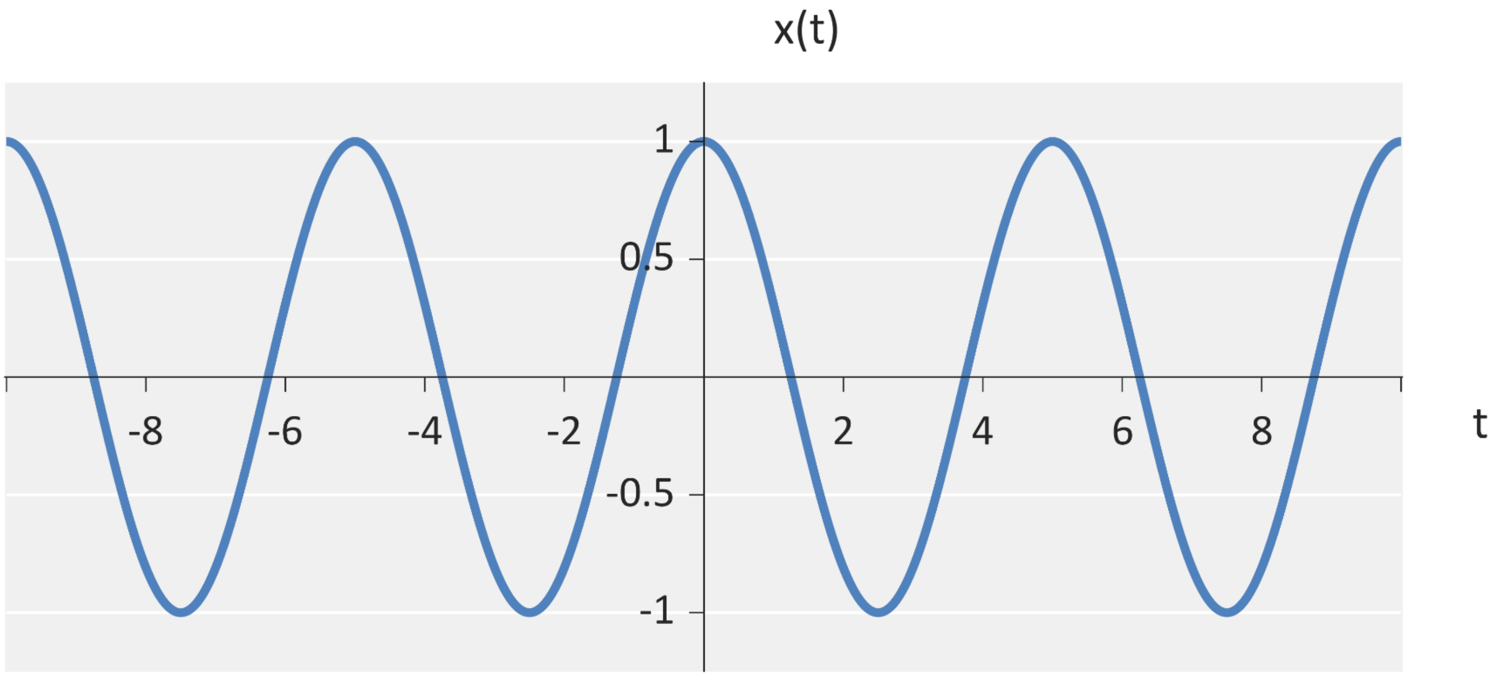
- **Sampling**
- Sampling in Time = ??? in Frequency
- The Nyquist-Shannon Sampling Theorem
- Continuous-time Reconstruction / Interpolation
- Aliasing

Sampling

- **Question:** What is sampling?

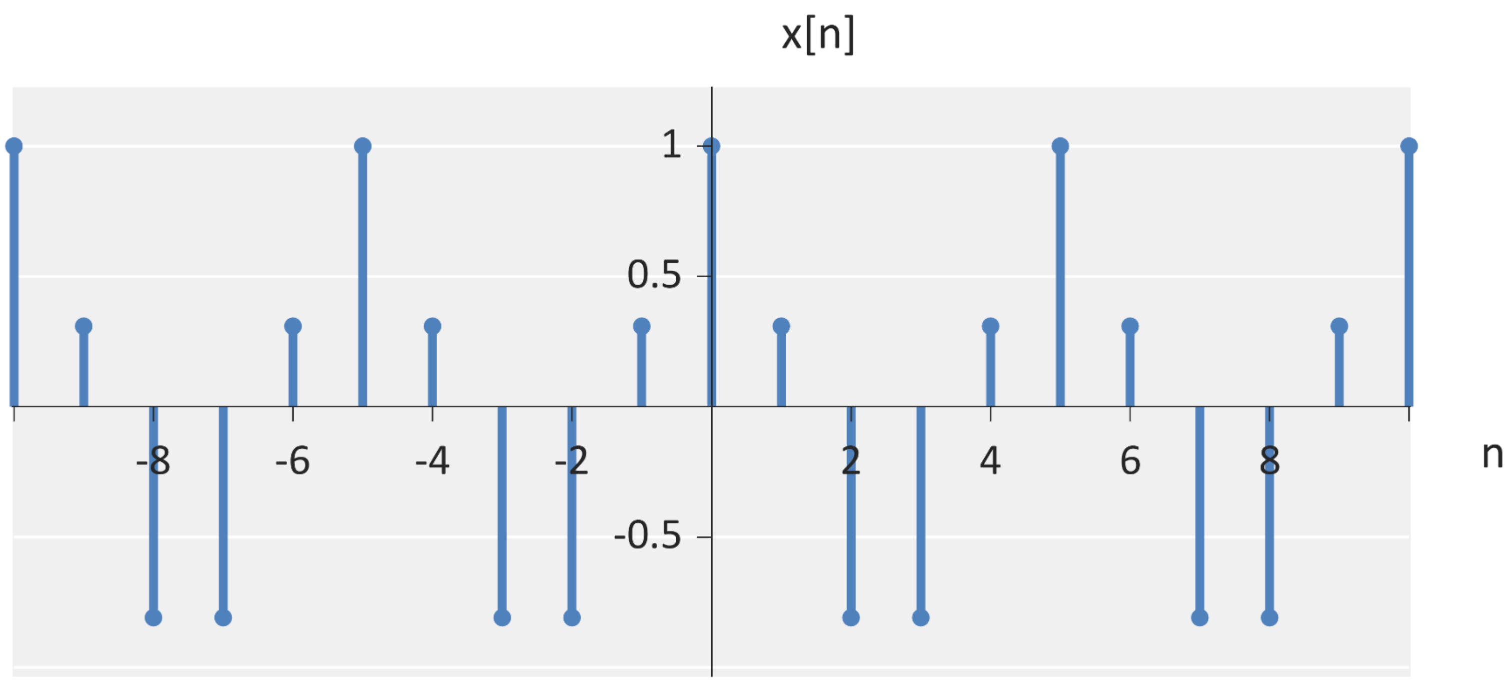
Fourier Transform

- **What is the sampling of**
  - $x(t) = \cos\left(\frac{2\pi}{5}t\right)$



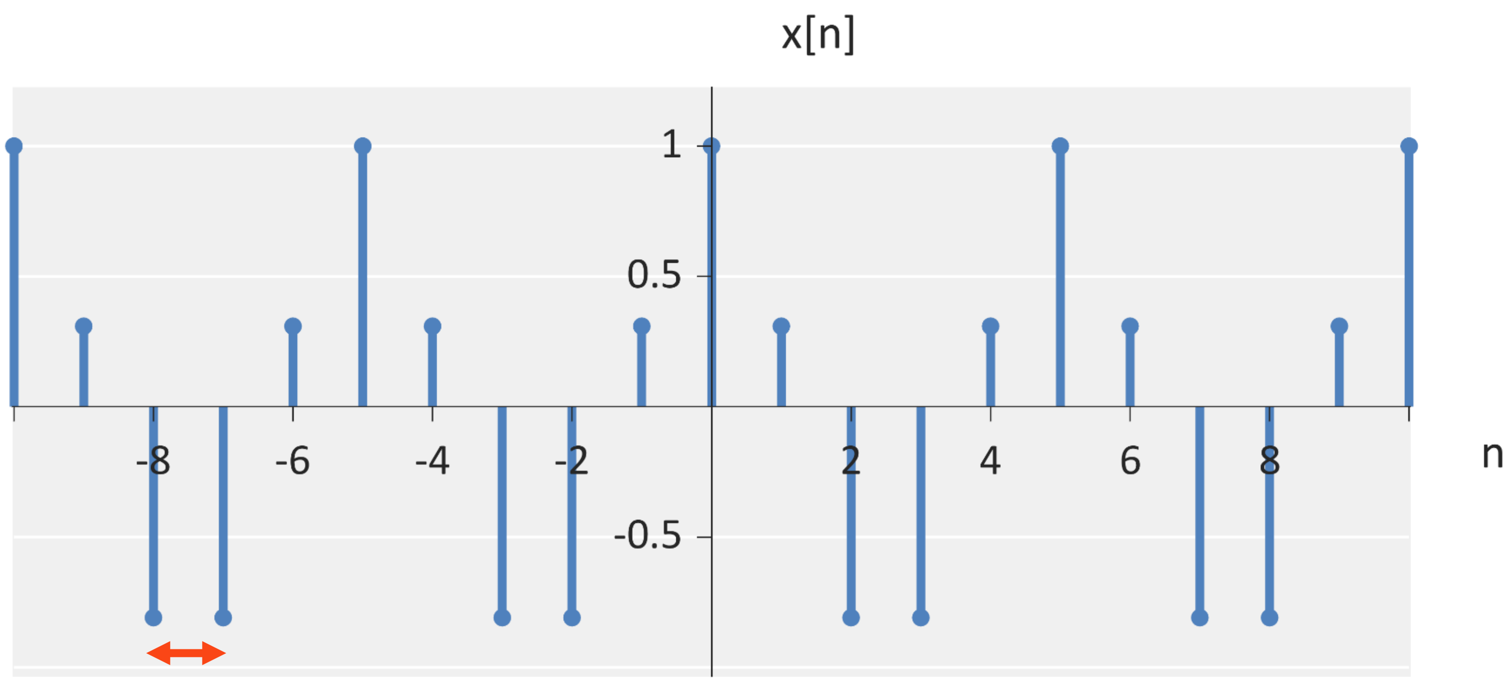
Fourier Transform

- **What is the sampling of**
  - $x[n] = \cos\left(\frac{2\pi}{5} n\right)$



Fourier Transform

- **What is the sampling of**
  - $x[n] = \cos\left(\frac{2\pi}{5}n\right)$



T_s is the sampling period

$f_s = \frac{1}{T_s}$ or $\Omega_s = \frac{2\pi}{T_s}$ is the sampling rate

Sampling

- **Question:** Can I preserve all information when I sample?

Lecture 11: Sampling

Foundations of Digital Signal Processing

Outline

- Sampling
- **Sampling in Time = ??? in Frequency**
- The Nyquist-Shannon Sampling Theorem
- Continuous-time Reconstruction / Interpolation
- Aliasing

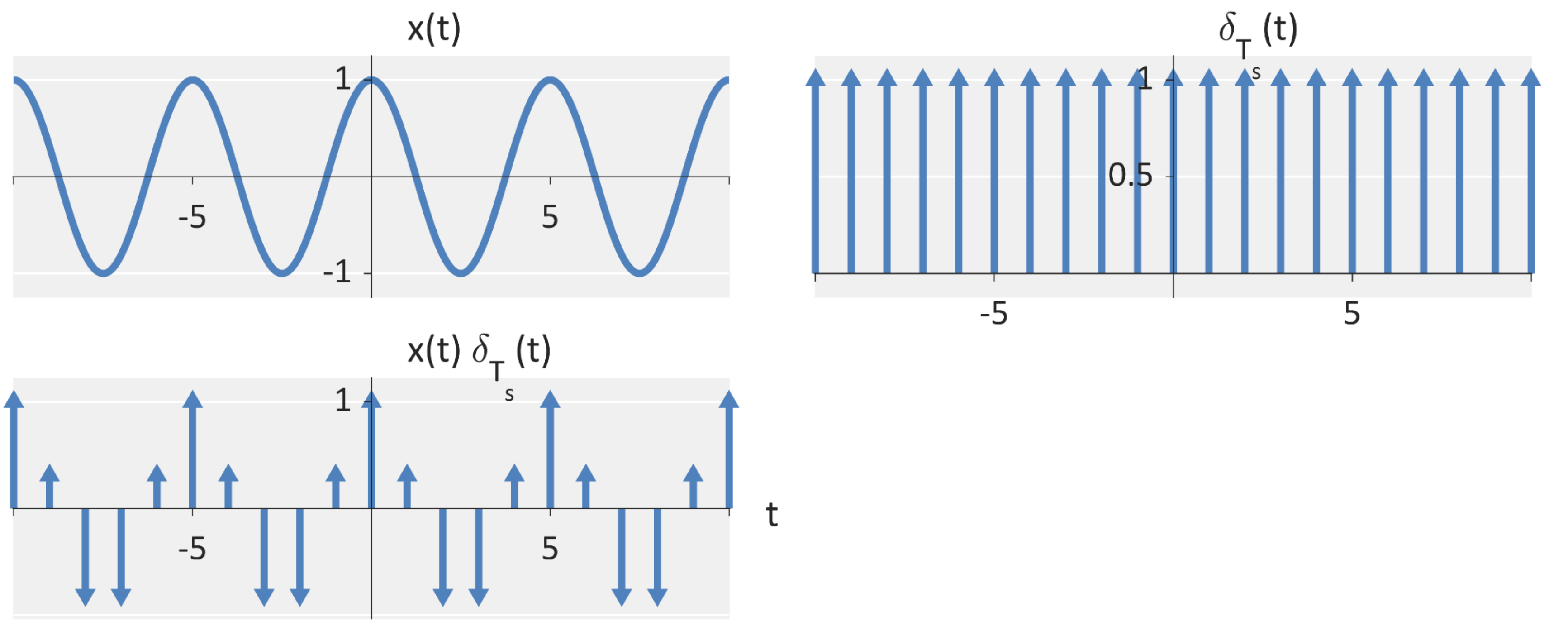
Sampling

- **Question:** What happens in frequency when I sample?

- **Sampling:** Multiplying by train of pulses

Sampling

- **Question:** What happens in frequency when I sample?
- **Sampling:** Multiplying by train of pulses



Sampling

- **Question:** What happens in frequency when I sample?
- **Sampling:** Multiplying by train of pulses
  - **Definition of a pulse train:**

$$\delta_{T_s}(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT_s)$$

Sampling

- **Question:** What happens in frequency when I sample?
- **Sampling:** Multiplying by train of pulses
  - **Sampled signal:**

$$x_s(t) = x(t)\delta_{T_s}(t) = x(t)\sum_{n=-\infty}^{\infty}\delta(t - nT_s)$$

$$= \sum_{n=-\infty}^{\infty} x(t)\delta(t - nT_s)$$

$$= \sum_{n=-\infty}^{\infty} x(nT_s)\delta(t - nT_s)$$

Sampling

- **Question:** What happens in frequency when I sample?

- **Sampling:** Multiplying by train of pulses
  - **Sampled signal:**

$$x_s(t) = x(t)\delta_{T_s}(t) = x(t)\sum_{n=-\infty}^{\infty}\delta(t - nT_s)$$

$$= \sum_{n=-\infty}^{\infty} x(t)\delta(t - nT_s)$$

$$= \sum_{n=-\infty}^{\infty} x[n]\delta(t - nT_s)$$

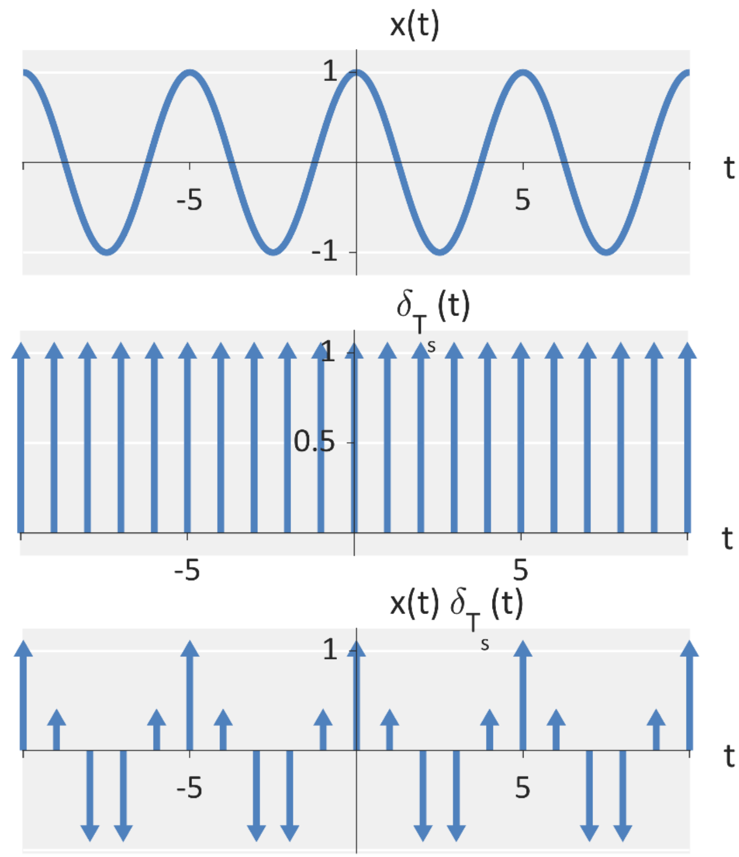
Sampling

- **Question:** What happens in frequency when I sample?
- **Sampling:** Multiplying by train of pulses
  - **Sampled signal:**

$$x_s(t) = x(t)\delta_{T_s}(t)$$

$$= \sum_{n=-\infty}^{\infty} x(kT_s)\delta(t - nT_s)$$

$$x(nT_s) = x[n]$$



Sampling

- **Question: So what happens in the frequency domain?**

$$x_s(t) = x(t)\delta_{T_s}(t)$$

Sampling

- **Question: So what happens in the frequency domain?**

$$x_s(t) = x(t)\delta_{T_s}(t)$$

$$X_s(\Omega) = \frac{1}{2\pi}\left[X(\Omega) * \mathcal{F}\{\delta_{T_s}(t)\}\right]$$

Sampling

- **Question:** **So what happens in the frequency domain?**

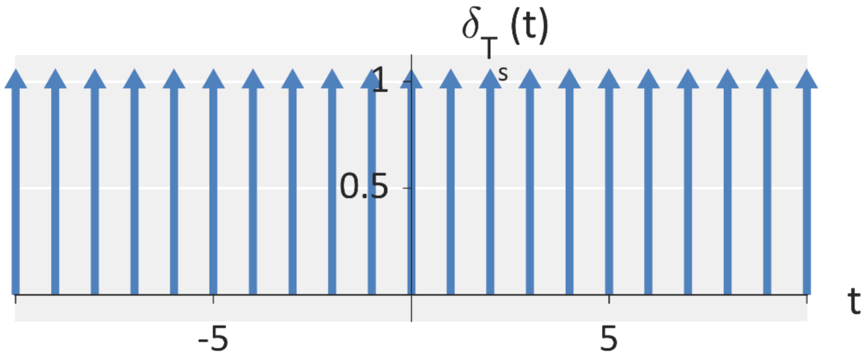
$$x_s(t) = x(t)\delta_{T_s}(t)$$

$$X_s(\Omega) = \frac{1}{2\pi}\left[X(\Omega) * \mathcal{F}\{\delta_{T_s}(t)\}\right]$$

- **Question:** **What is the Fourier transform of a pulse train???**

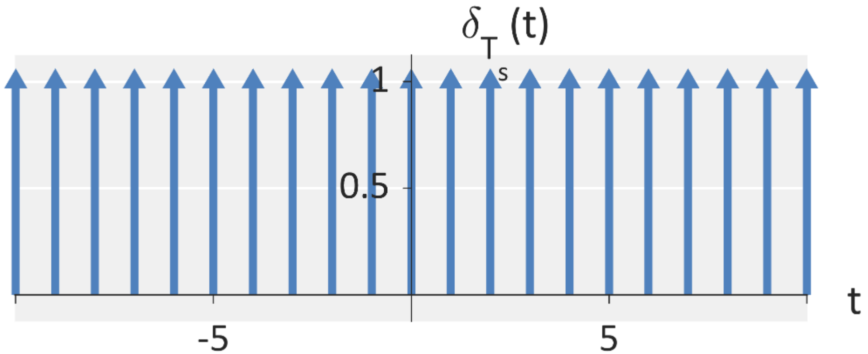
Sampling

- **How do we compute $\mathcal{F}\{\delta_{T_s}(t)\}$?**
  - Consider the Fourier Series…
  - First, what is the fundamental angular frequency?



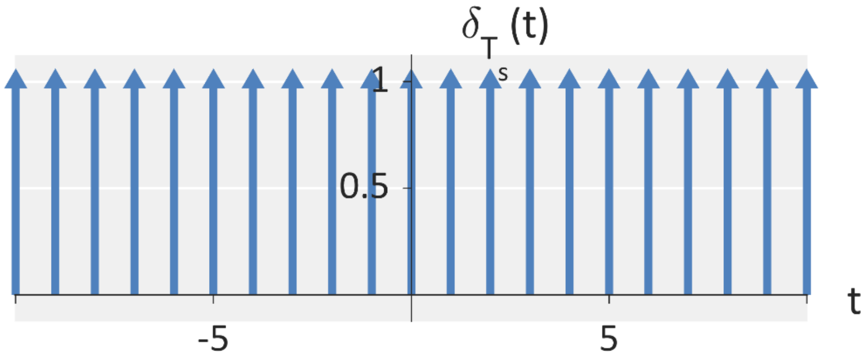
Sampling

- **How do we compute $\mathcal{F}\{\delta_{T_s}(t)\}$?**
  - Consider the Fourier Series…
  - First, what is the fundamental angular frequency?
    - $\Omega_s = \frac{2\pi}{T_s}$



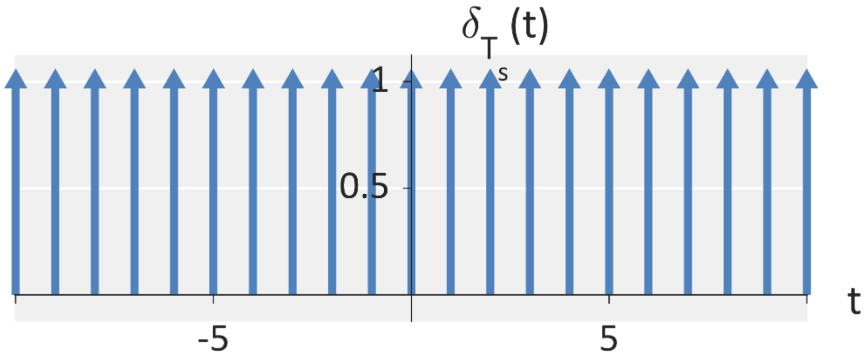
Sampling

- **How do we compute $\mathcal{F}\{\delta_{T_s}(t)\}$?**
  - Consider the Fourier Series…
  - Second, how do I express any one period of the signal?



Sampling

- **How do we compute $\mathcal{F}\{\delta_{T_s}(t)\}$?**
  - Consider the Fourier Series…
  - Second, how do I express any one period of the signal?
    - One period of $x(t) = \delta(t)$



Sampling

- **How do we compute $\mathcal{F}\{\delta_{T_s}(t)\}$?**
  - Consider the Fourier Series…
  - Second, solve the Fourier series equation.

$$c_k = \frac{1}{T}\int_T x(t)e^{-jk\Omega_0 t}\,dt$$

Sampling

- **How do we compute $\mathcal{F}\left\{\delta_{T_s}(t)\right\}$?**
  - Consider the Fourier Series…
  - Second, solve the Fourier series equation.

$$c_k = \frac{1}{T}\int_T x(t)e^{-jk\Omega_0 t}\,dt$$

Fundamental Period

Integrate over one period

Harmonic

Fundamental Frequency

Sampling

- **How do we compute $\mathcal{F}\left\{\delta_{T_s}(t)\right\}$?**
  - Consider the Fourier Series…
  - Second, solve the Fourier series equation.

$$c_k = \frac{1}{T}\int_T x(t)e^{-jk\Omega_0 t}\,dt$$

$$= \frac{1}{T_s}\int_{T_s} \delta(t)e^{-jk\Omega_s t}\,dt$$

Sampling

- **How do we compute $\mathcal{F}\{\delta_{T_s}(t)\}$?**
  - Consider the Fourier Series…
  - Second, solve the Fourier series equation.

$$c_k = \frac{1}{T}\int_T x(t)e^{-jk\Omega_0 t}\,dt$$

$$= \frac{1}{T_s}\int_{T_s} \delta(t)e^{-jk\Omega_s t}\,dt$$

$$= \frac{1}{T_s}\int_{T_s} e^{-jk\Omega_s(0)}\,dt$$

Sampling

- **How do we compute $\mathcal{F}\{\delta_{T_s}(t)\}$?**
  - Consider the Fourier Series…
  - Second, solve the Fourier series equation.

$$c_k = \frac{1}{T}\int_T x(t)e^{-jk\Omega_0 t}\,dt$$

$$= \frac{1}{T_s}\int_{T_s} \delta(t)e^{-jk\Omega_s t}\,dt$$

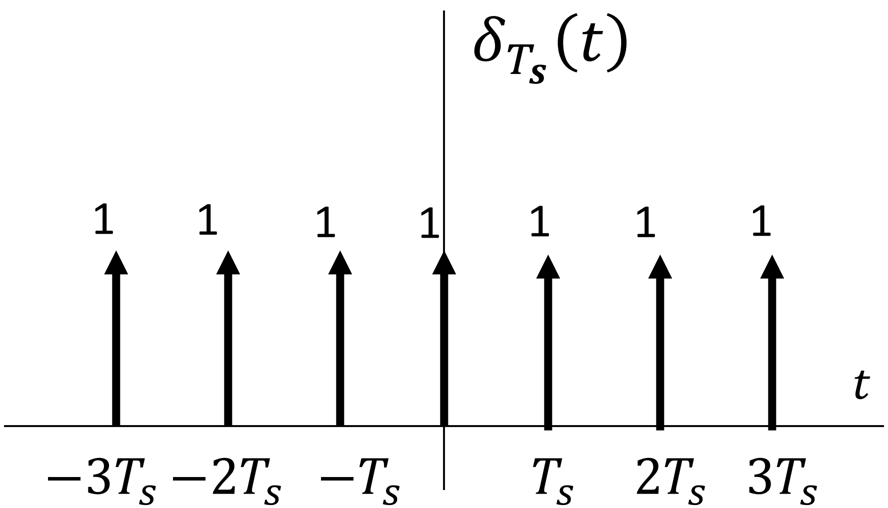
$$= \frac{1}{T_s}\int_{T_s} e^{-jk\Omega_s(0)}\,dt$$

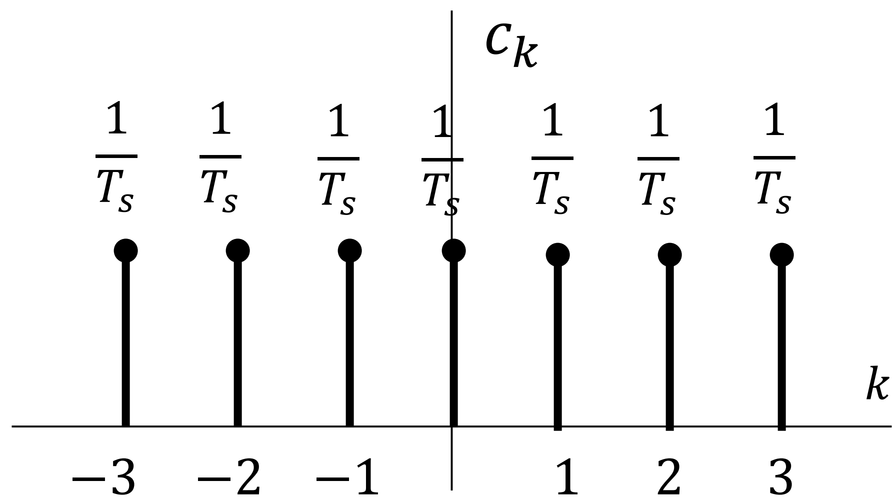
$$c_k = \frac{1}{T_s} = f_s = \frac{\Omega_s}{2\pi} \quad \text{for all } k$$

Sampling

- **How do we compute $\mathcal{F}\{\delta_{T_s}(t)\}$?**
  - Consider the Fourier Series.

$$\delta_{T_s}(t) \longleftrightarrow c_k = c[k] = \frac{1}{T_s}$$

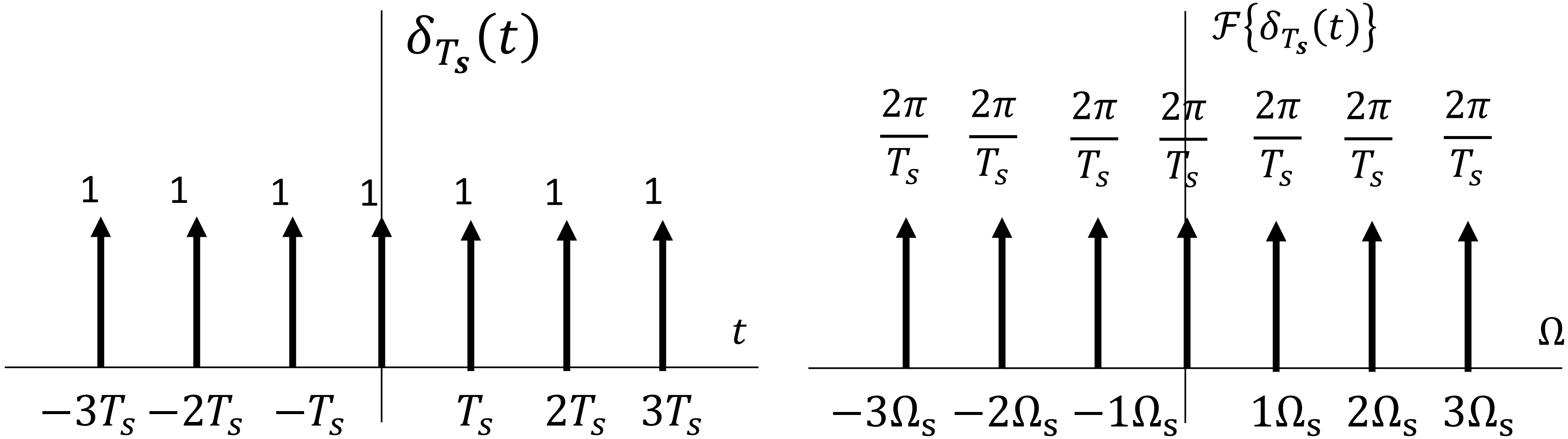




Sampling

- **How do we compute $\mathcal{F}\{\delta_{T_s}(t)\}$?**
  - Consider the Fourier Series.

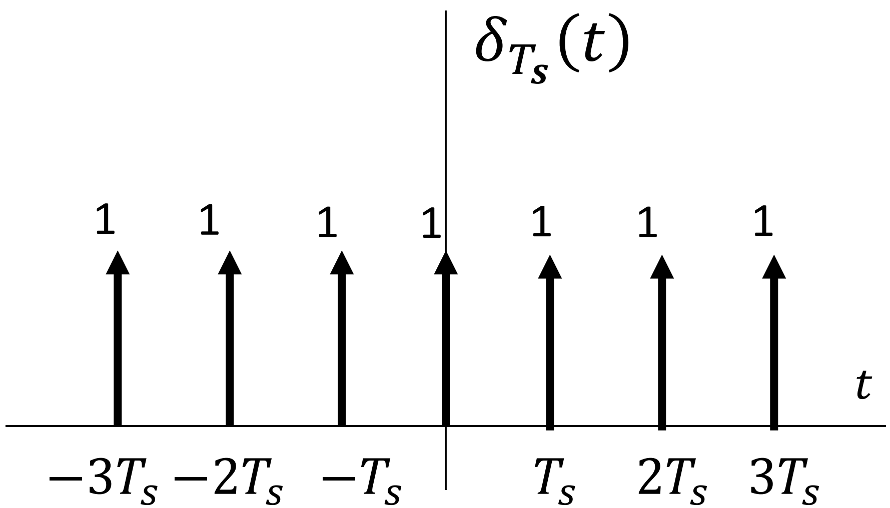
$$\delta_{T_s}(t) \longleftrightarrow \frac{2\pi}{T_s}\delta_{\Omega_s}(\Omega)$$

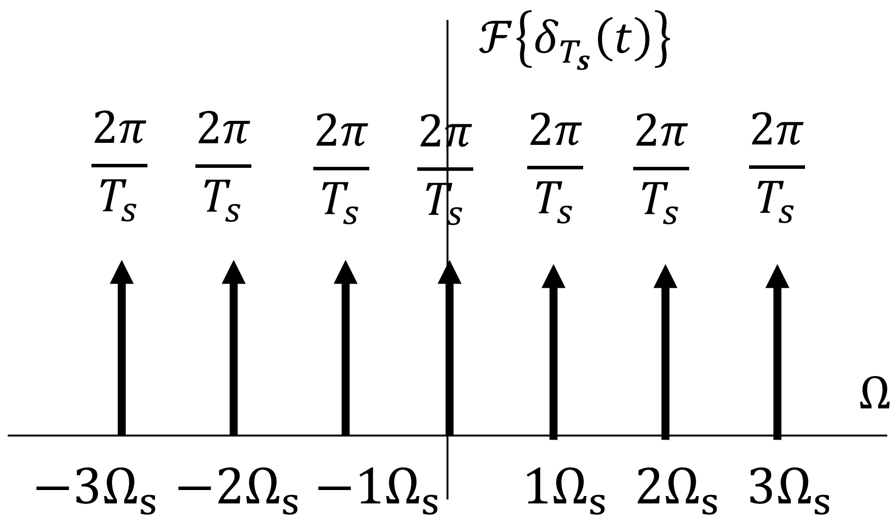


Sampling

- **How do we compute $\mathcal{F}\{\delta_{T_s}(t)\}$?**
  - Consider the Fourier Series.

$$\delta_{T_s}(t) \longleftrightarrow \Omega_s \delta_{\Omega_s}(\Omega)$$





Sampling

- **Question: So what happens in the frequency domain?**

$$x_s(t) = x(t)\delta_{T_s}(t)$$

$$X_s(\Omega) = \frac{1}{2\pi}\left[X(\Omega) * \mathcal{F}\{\delta_{T_s}(t)\}\right]$$

Sampling

- **Question: So what happens in the frequency domain?**

$$x_s(t) = x(t)\delta_{T_s}(t)$$

$$X_s(\Omega) = \frac{1}{2\pi}\left[X(\Omega) * \mathcal{F}\{\delta_{T_s}(t)\}\right]$$

$$X_s(\Omega) = \frac{1}{2\pi}\left[X(\Omega) * \Omega_s\delta_{\Omega_s}(\Omega)\right] = \frac{\Omega_s}{2\pi}\left[X(\Omega) * \delta_{\Omega_s}(\Omega)\right]$$

$$X_s(\Omega) = \frac{\Omega_s}{2\pi}\left[X(\Omega) * \sum_{k=-\infty}^{\infty} \delta(\Omega - k\Omega_s)\right]$$

$$X_s(\Omega) = \frac{1}{T_s}\left[X(\Omega) * \sum_{k=-\infty}^{\infty} \delta(\Omega - k\Omega_s)\right]$$

Sampling

- **Question: So what happens in the frequency domain?**

$$x_s(t) = x(t)\delta_{T_s}(t)$$

$$X_s(\Omega) = \frac{1}{2\pi}\left[X(\Omega) * \mathcal{F}\{\delta_{T_s}(t)\}\right]$$

$$X_s(\Omega) = \frac{1}{2\pi}\left[X(\Omega) * \Omega_s\delta_{\Omega_s}(\Omega)\right] = \frac{\Omega_s}{2\pi}\left[X(\Omega) * \delta_{\Omega_s}(\Omega)\right]$$

$$X_s(\Omega) = \frac{\Omega_s}{2\pi}\left[X(\Omega) * \sum_{k=-\infty}^{\infty} \delta(\Omega - k\Omega_s)\right]$$

$$X_s(\Omega) = \frac{1}{T_s}\sum_{k=-\infty}^{\infty} X(\Omega - k\Omega_s)$$

Lecture 11: Sampling

Foundations of Digital Signal Processing

Outline

- Sampling
- Sampling in Time = ??? in Frequency
- **The Nyquist-Shannon Sampling Theorem**
- Continuous-time Reconstruction / Interpolation
- Aliasing

Sampling

- **Question: What is happening here?**

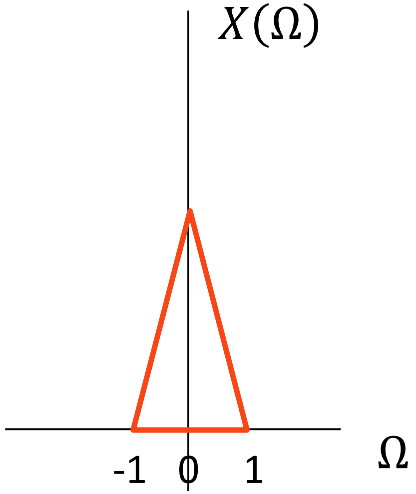
$$X_s(\Omega) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X(\Omega - k\Omega_s)$$

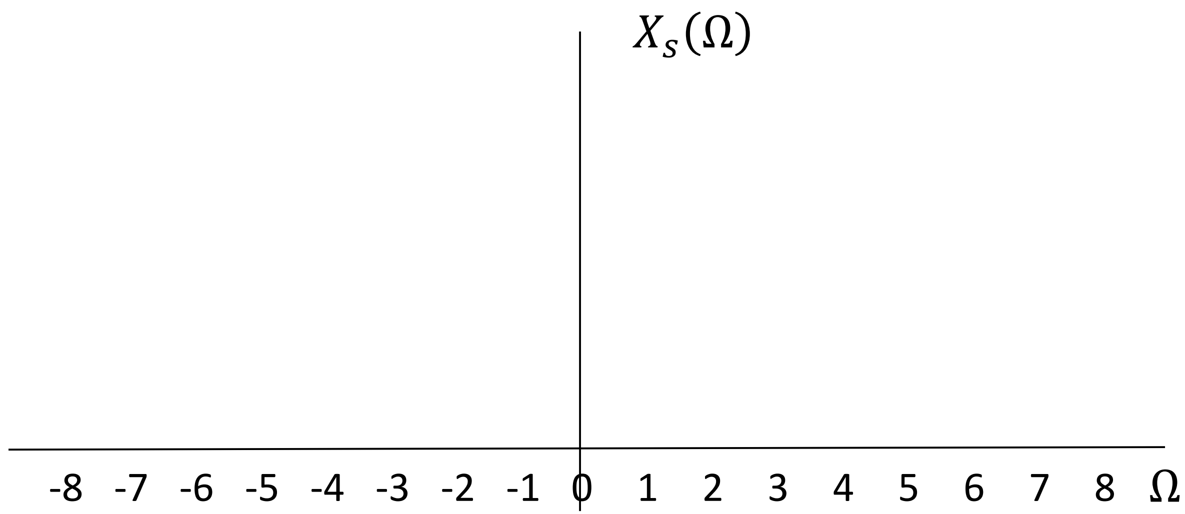
Sampling

- **Question: What is happening here?**

$$X_s(\Omega) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X(\Omega - k\Omega_s)$$

- **Sample at a rate of** $\Omega_s = 4$



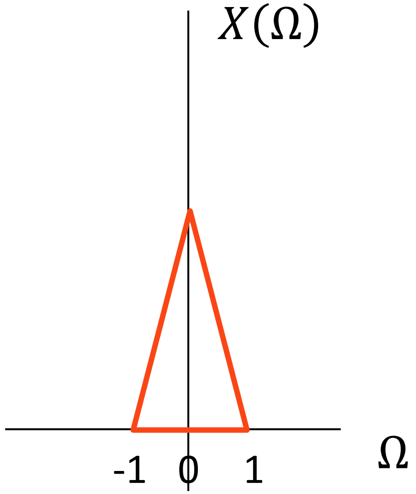


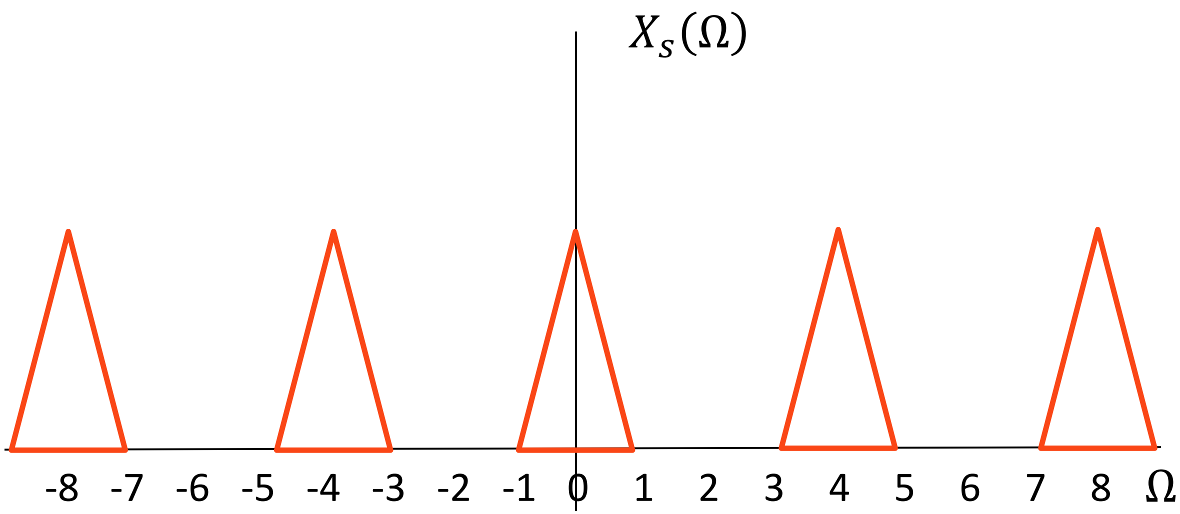
Sampling

- **Question: What is happening here?**

$$X_s(\Omega) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X(\Omega - k\Omega_s)$$

- **Sample at a rate of** $\Omega_s = 4$



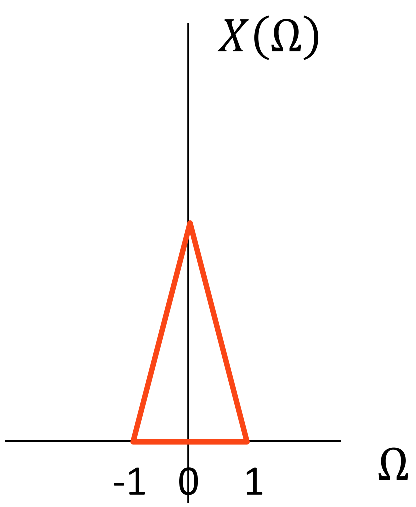


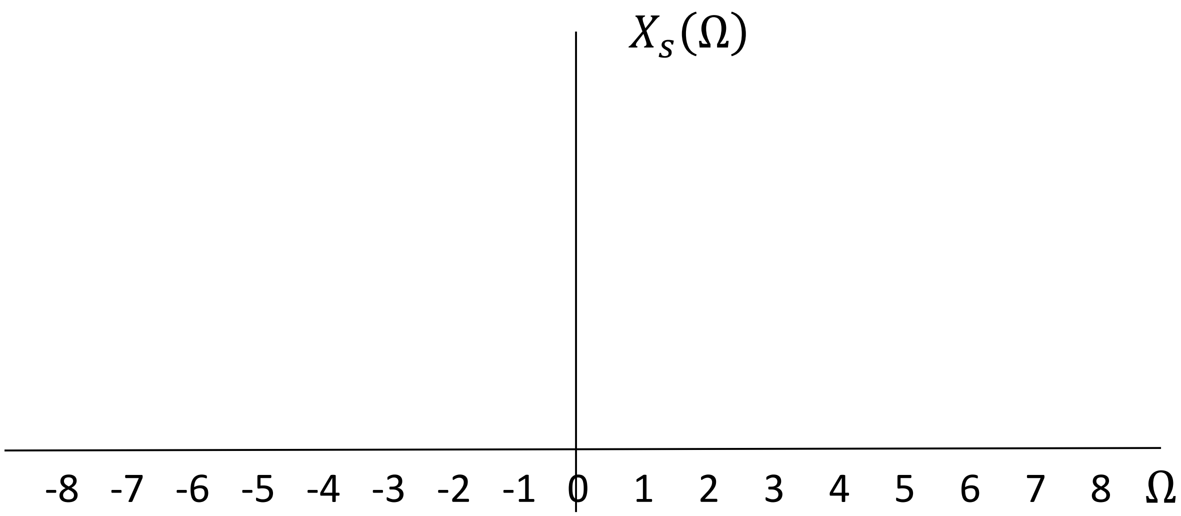
Sampling

- **Question: What is happening here?**

$$X_s(\Omega) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X(\Omega - k\Omega_s)$$

- **Sample at a rate of** $\Omega_s = 2$



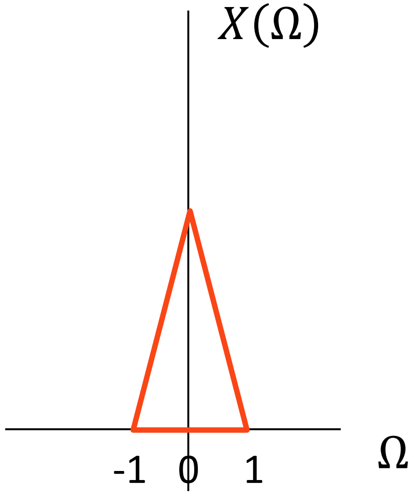


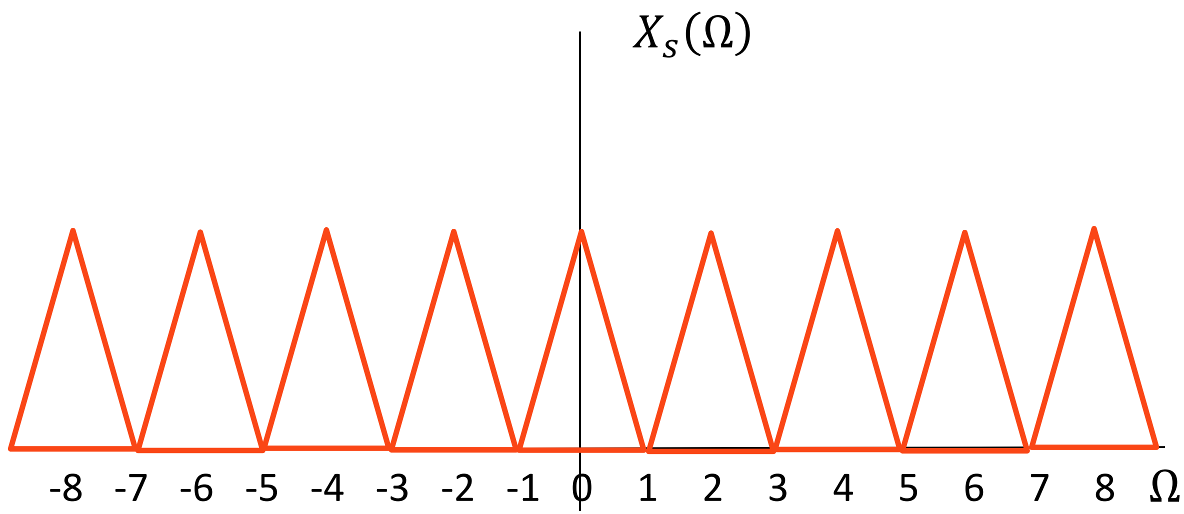
Sampling

- **Question: What is happening here?**

$$X_s(\Omega) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X(\Omega - k\Omega_s)$$

- **Sample at a rate of $\Omega_s = 2$**



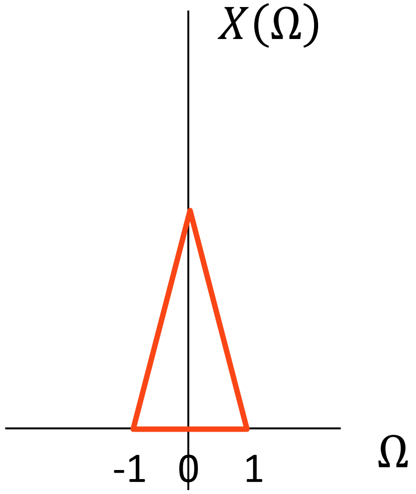


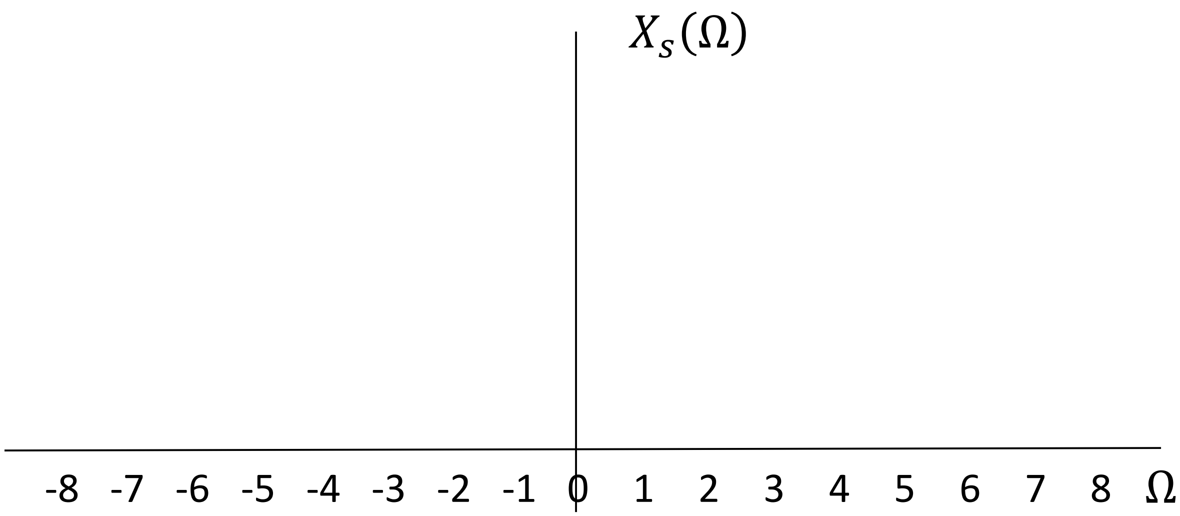
Sampling

- **Question: What is happening here?**

$$X_s(\Omega) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X(\Omega - k\Omega_s)$$

- **Sample at a rate of $\Omega_s = 1$**



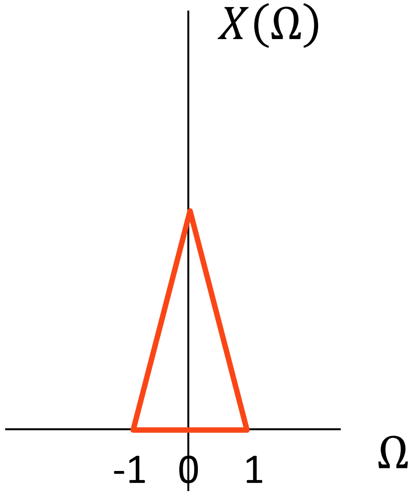


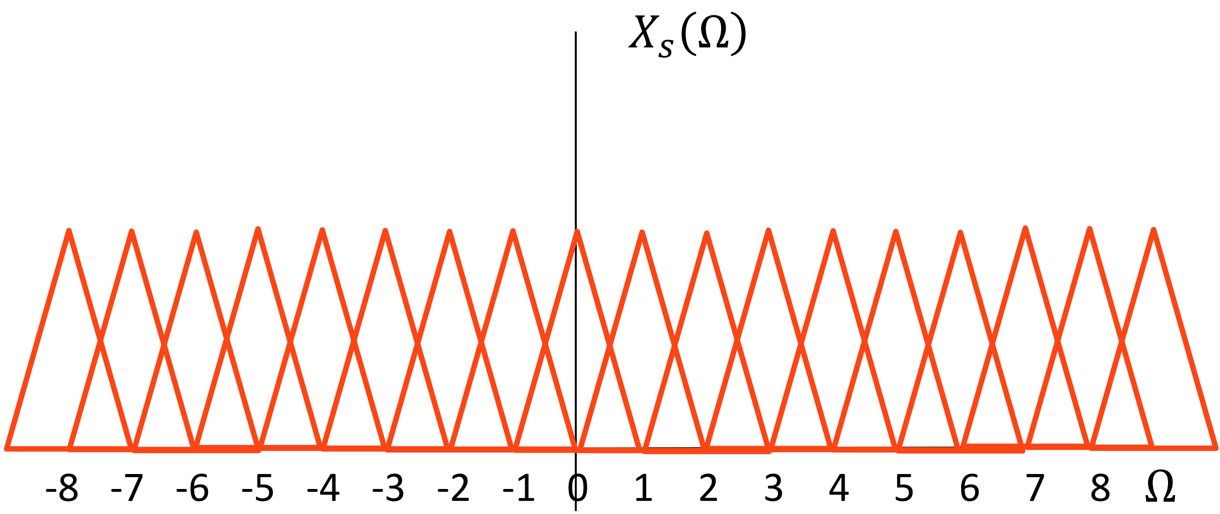
Sampling

- **Question: What is happening here?**

$$X_s(\Omega) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X(\Omega - k\Omega_s)$$

- **Sample at a rate of $\Omega_s = 2$**



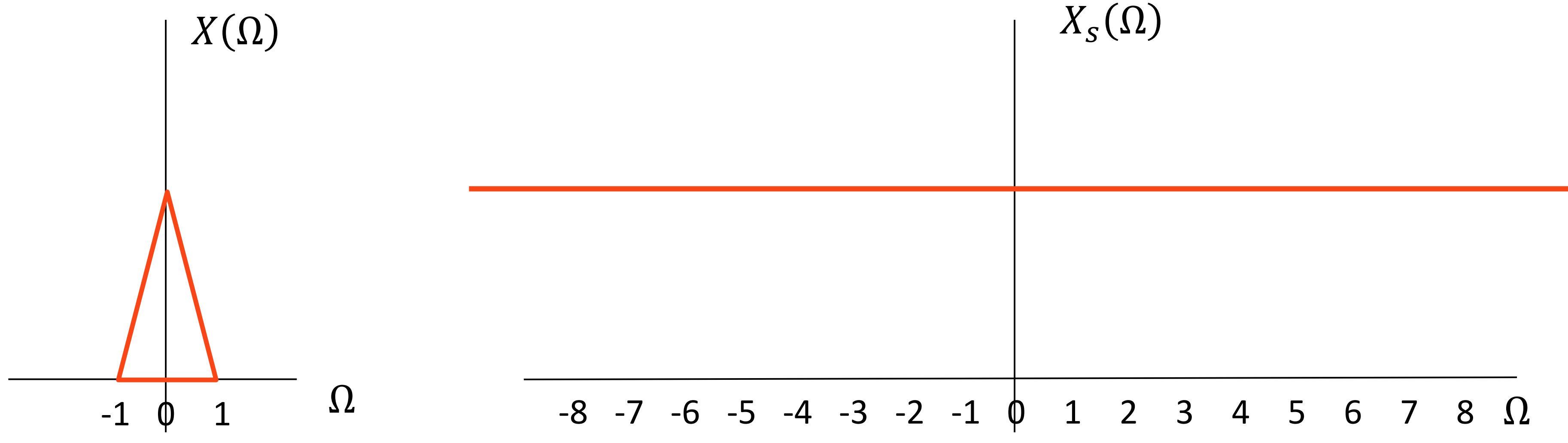


Sampling

- **Question: What is happening here?**

$$X_s(\Omega) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X(\Omega - k\Omega_s)$$

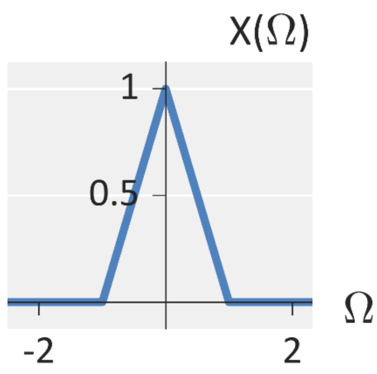
- **Sample at a rate of** $\Omega_s = 2$

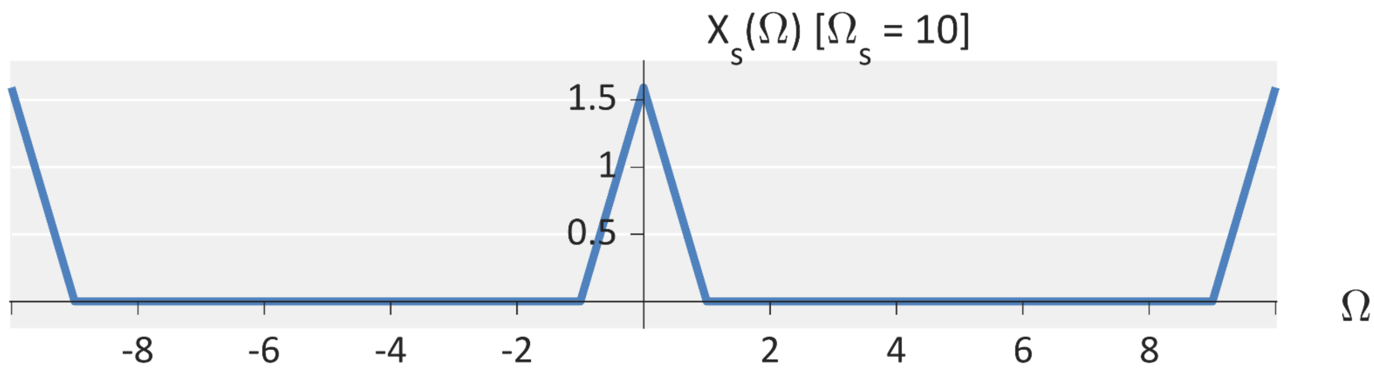


Sampling

- **Question: What is happening here?**

$$X_s(\Omega) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X(\Omega - k\Omega_s)$$

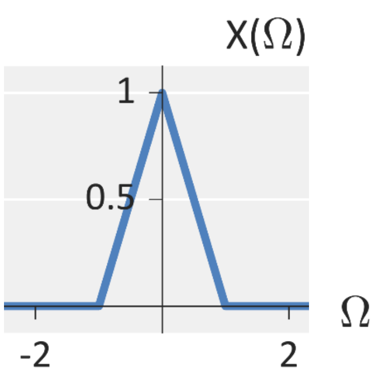


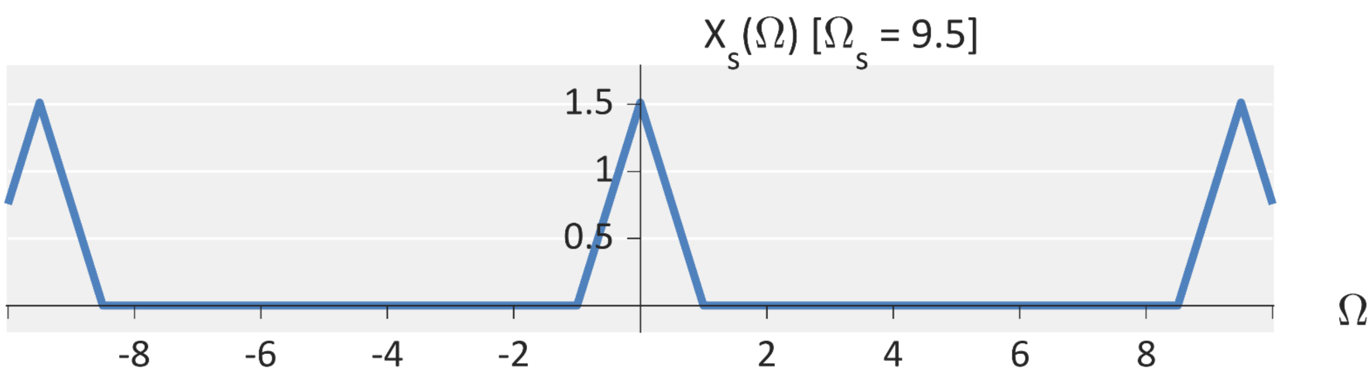


Sampling

- **Question: What is happening here?**

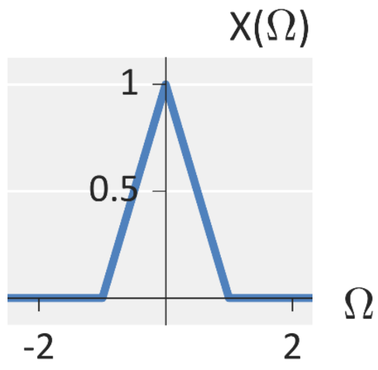
$$X_s(\Omega) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X(\Omega - k\Omega_s)$$

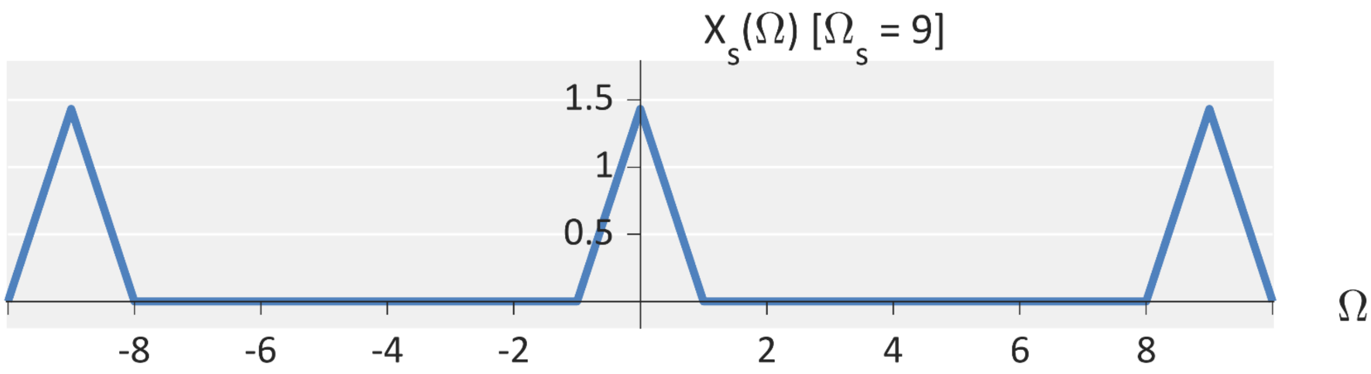




- **Question: What is happening here?**

$$X_s(\Omega) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X(\Omega - k\Omega_s)$$

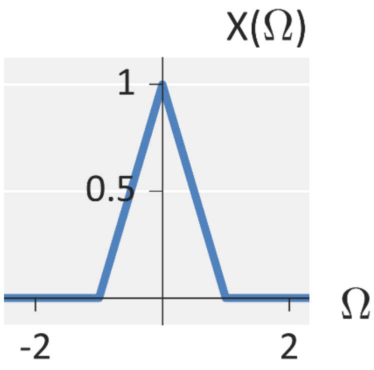


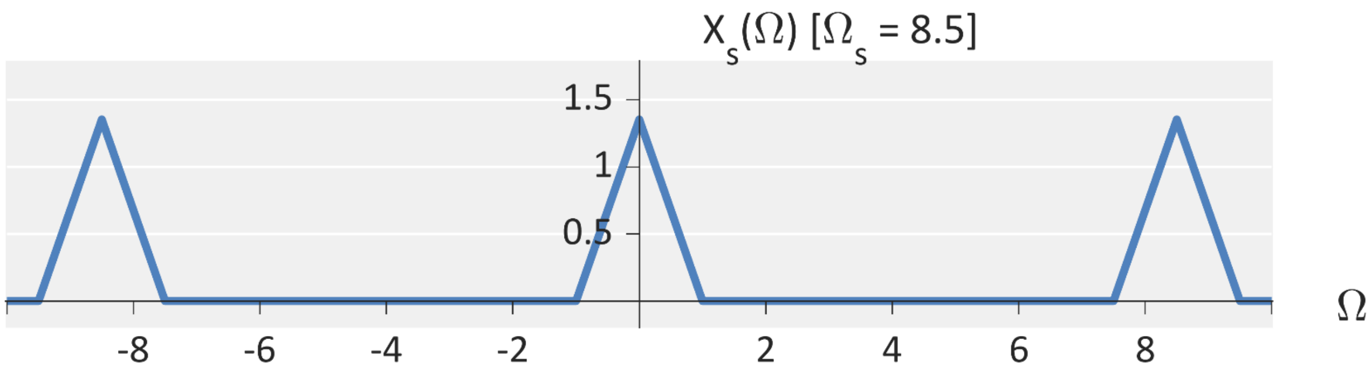


Sampling

- **Question: What is happening here?**

$$X_s(\Omega) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X(\Omega - k\Omega_s)$$

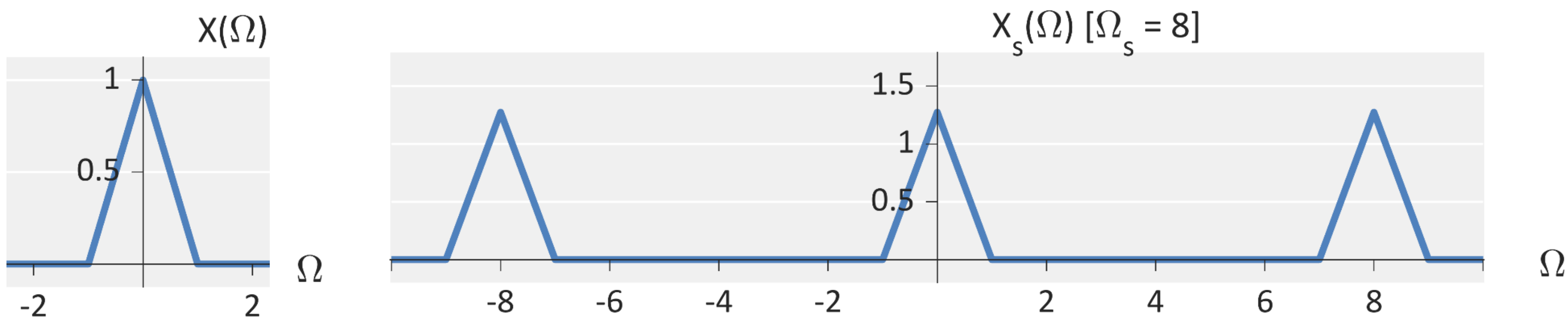




Sampling

- **Question: What is happening here?**

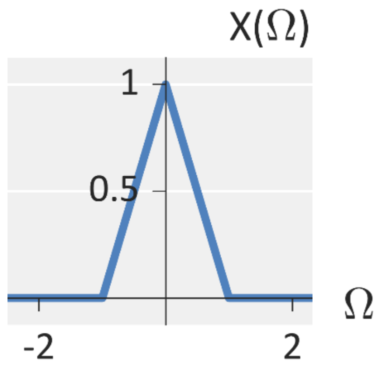
$$X_s(\Omega) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X(\Omega - k\Omega_s)$$

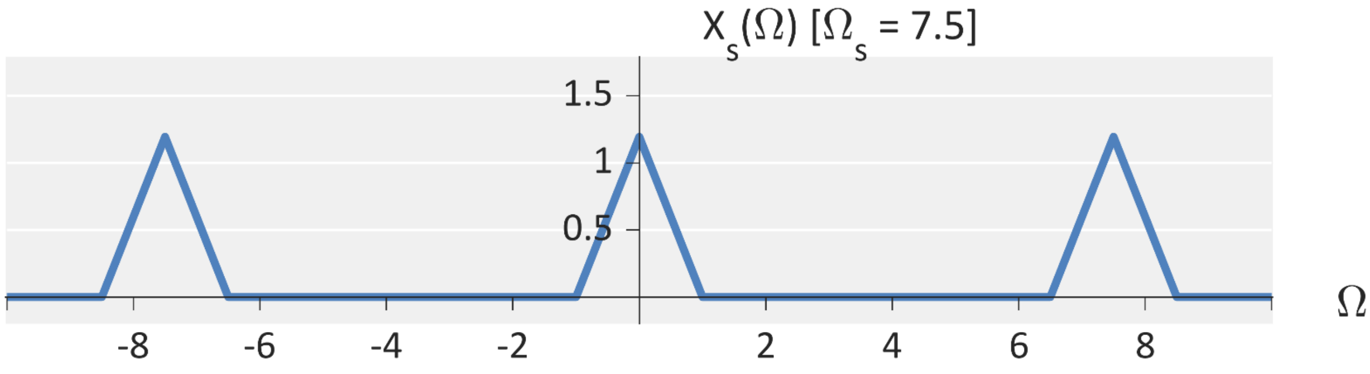


Sampling

■ **Question: What is happening here?**

$$X_s(\Omega) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X(\Omega - k\Omega_s)$$

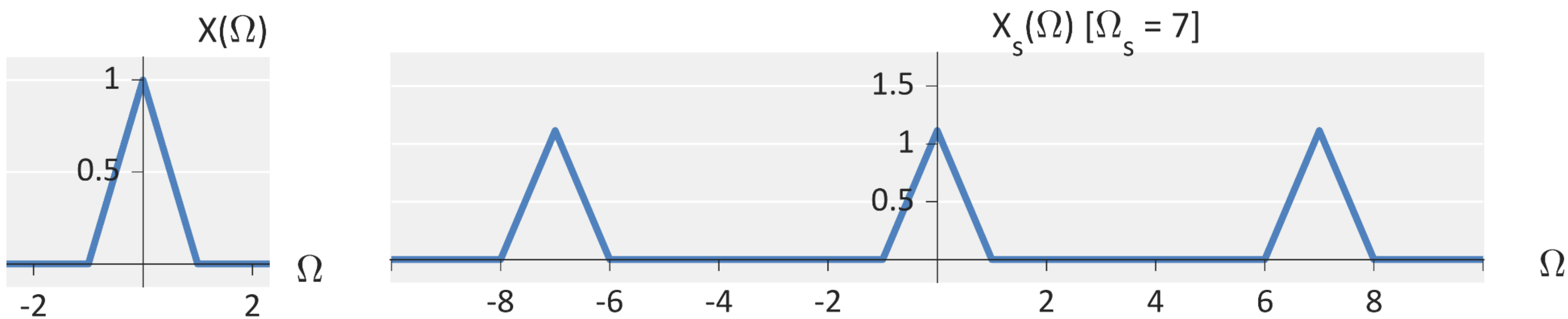




Sampling

- **Question: What is happening here?**

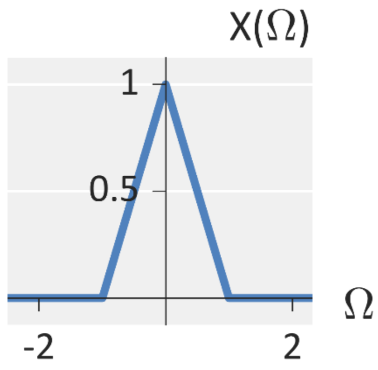
$$X_s(\Omega) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X(\Omega - k\Omega_s)$$

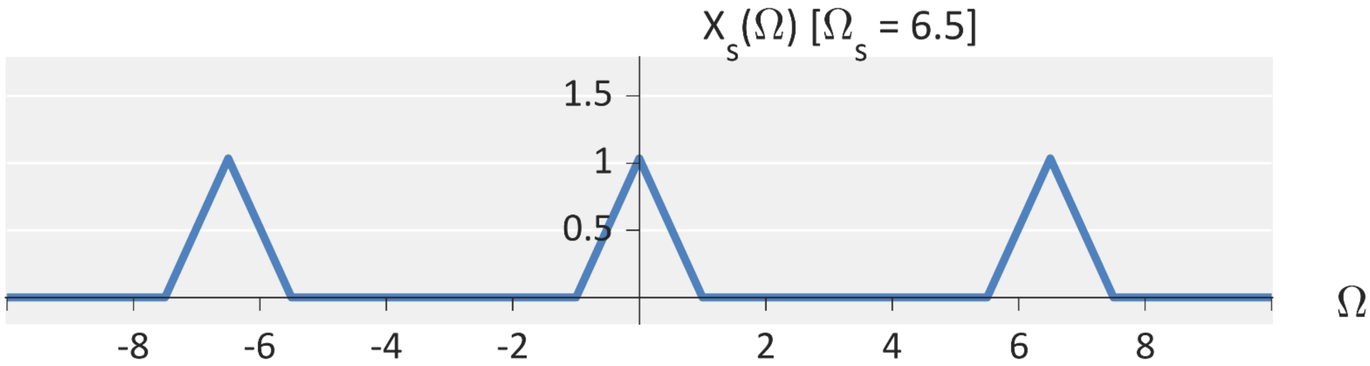


Sampling

- **Question: What is happening here?**

$$X_s(\Omega) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X(\Omega - k\Omega_s)$$

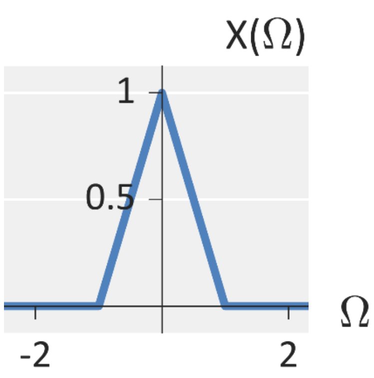


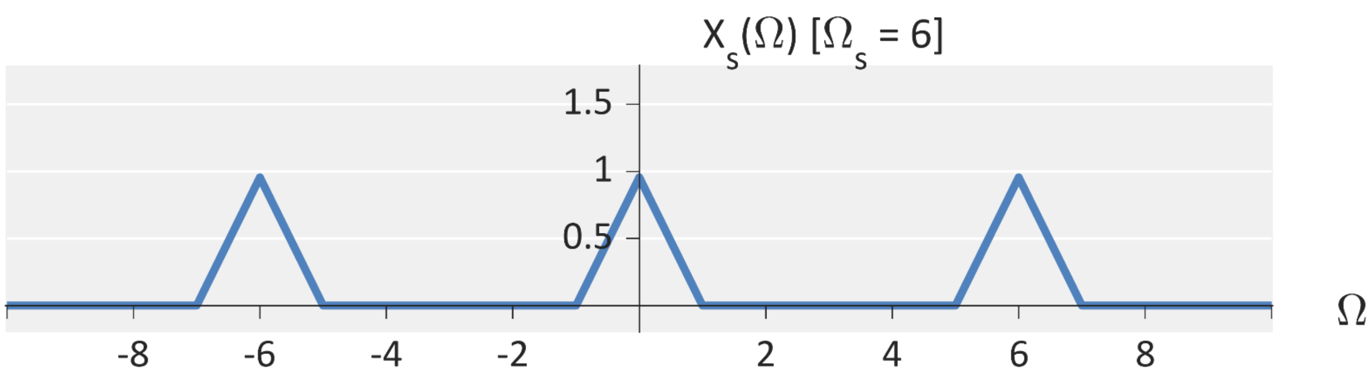


Sampling

- **Question: What is happening here?**

$$X_s(\Omega) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X(\Omega - k\Omega_s)$$

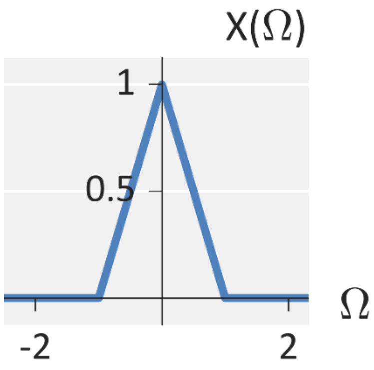


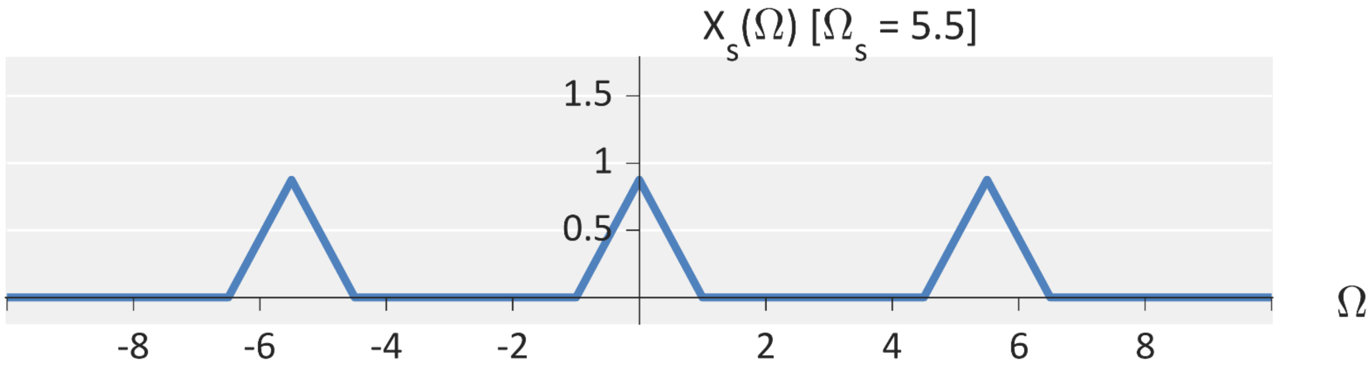


Sampling

- **Question: What is happening here?**

$$X_s(\Omega) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X(\Omega - k\Omega_s)$$

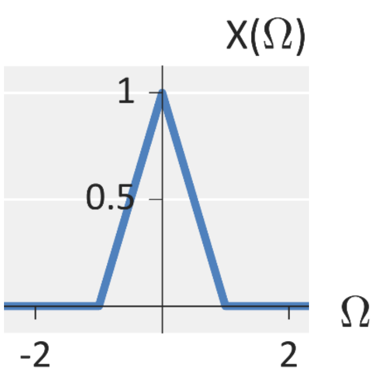


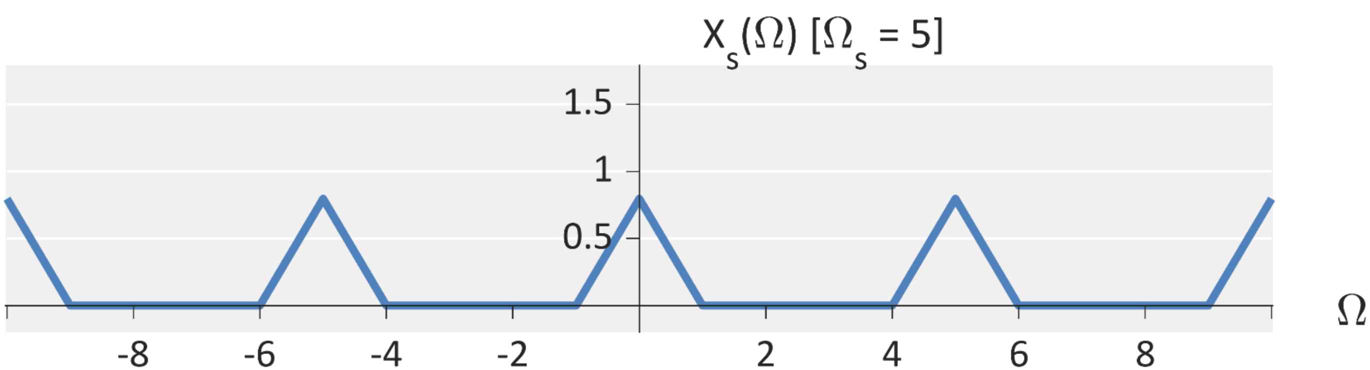


Sampling

- **Question: What is happening here?**

$$X_s(\Omega) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X(\Omega - k\Omega_s)$$

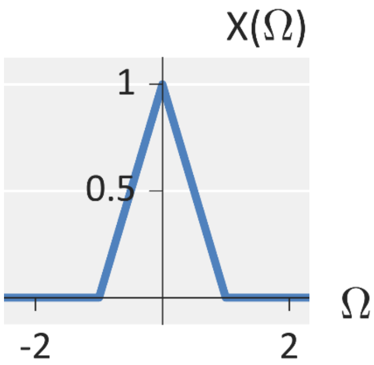


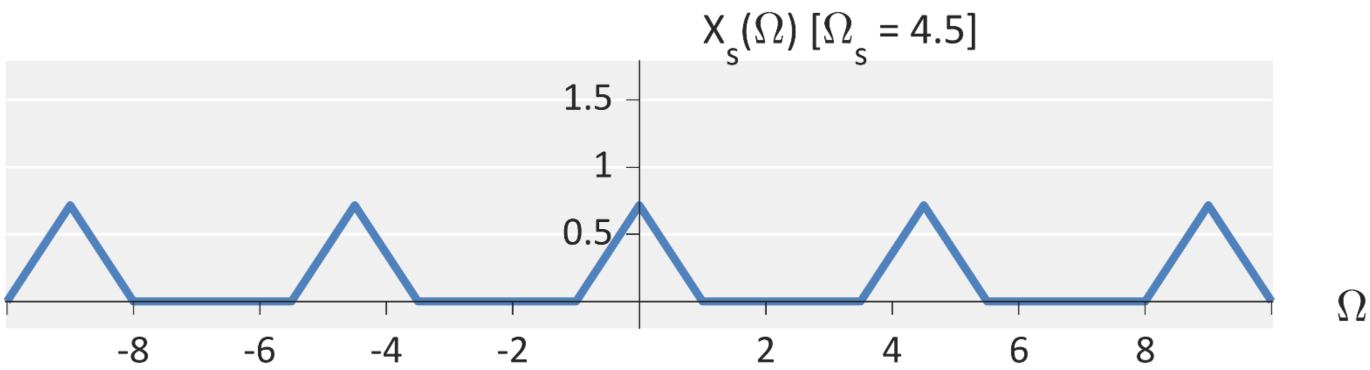


Sampling

- **Question: What is happening here?**

$$X_s(\Omega) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X(\Omega - k\Omega_s)$$

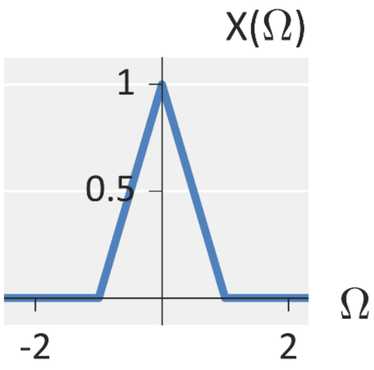


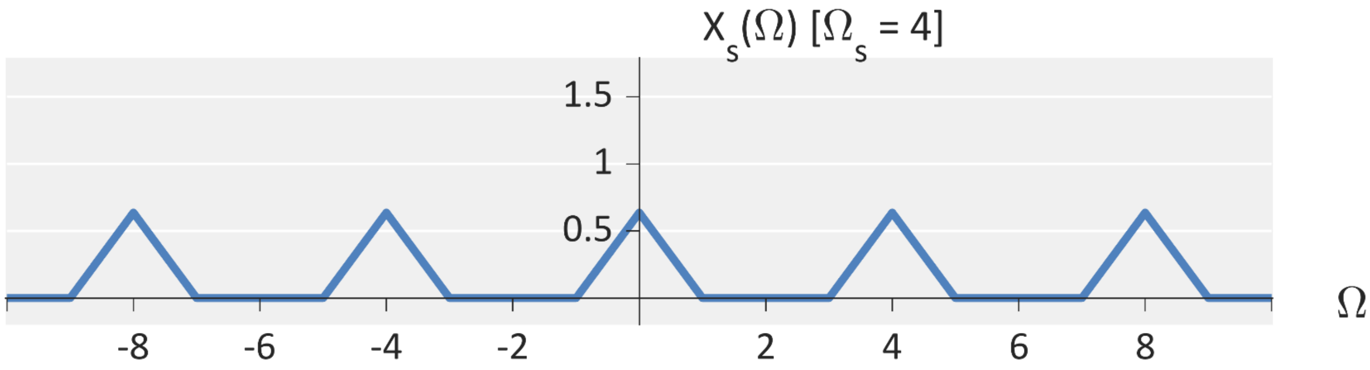


Sampling

- **Question: What is happening here?**

$$X_s(\Omega) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X(\Omega - k\Omega_s)$$

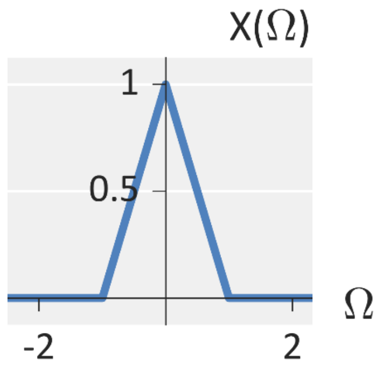


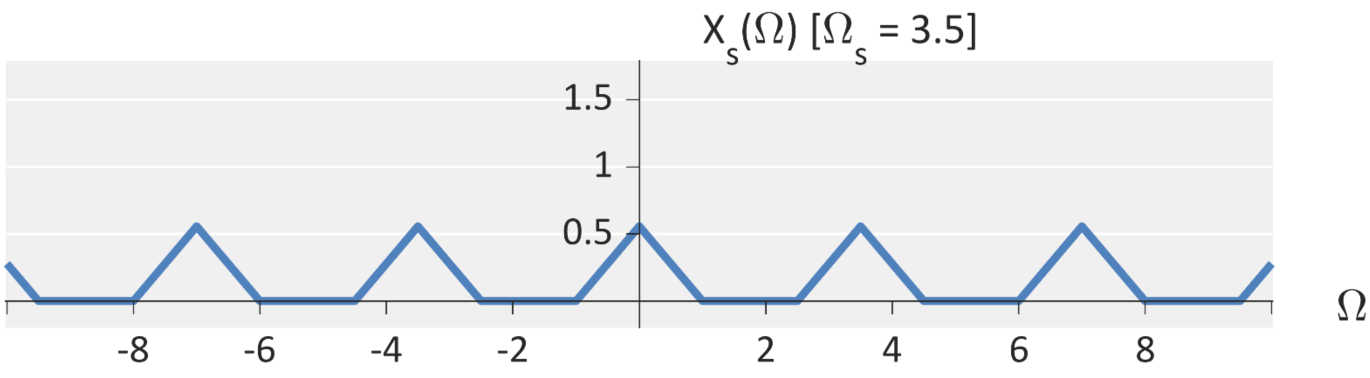


Sampling

- **Question: What is happening here?**

$$X_s(\Omega) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X(\Omega - k\Omega_s)$$

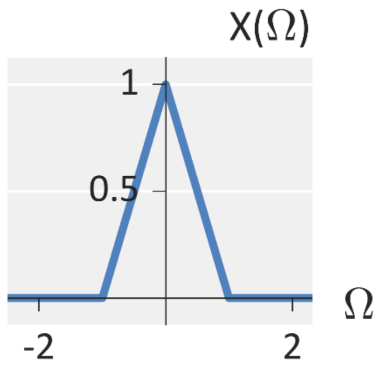


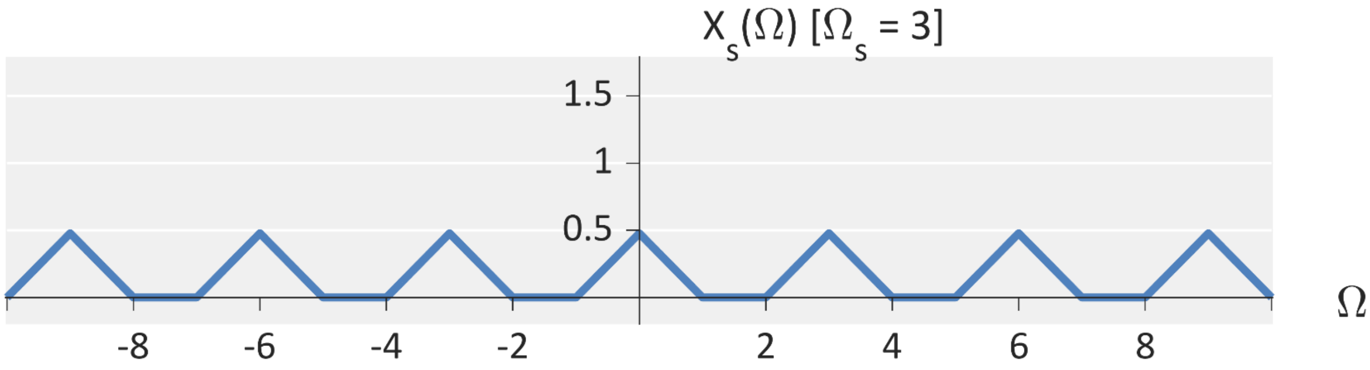


Sampling

- **Question: What is happening here?**

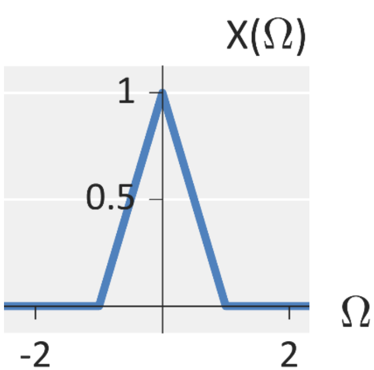
$$X_s(\Omega) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X(\Omega - k\Omega_s)$$

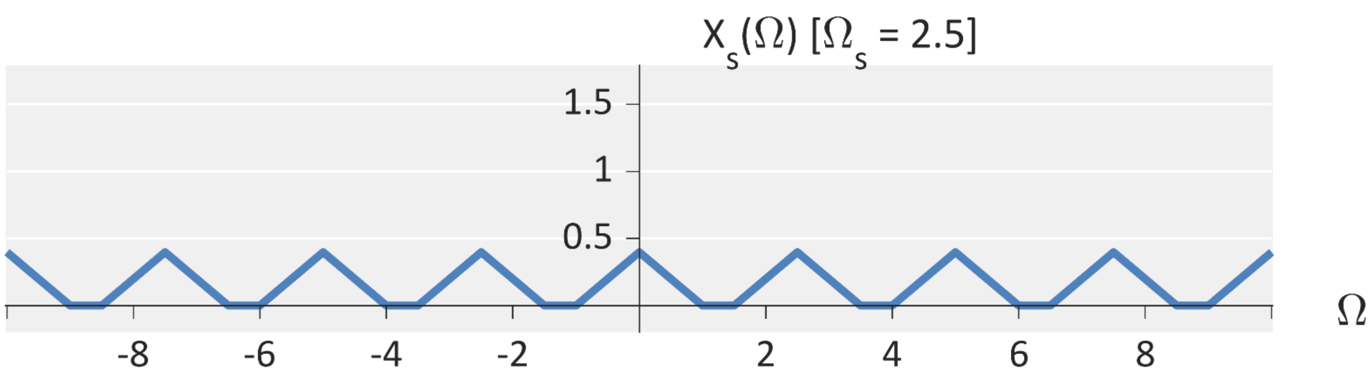




- **Question:** **What is happening here?**

$$X_s(\Omega) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X(\Omega - k\Omega_s)$$

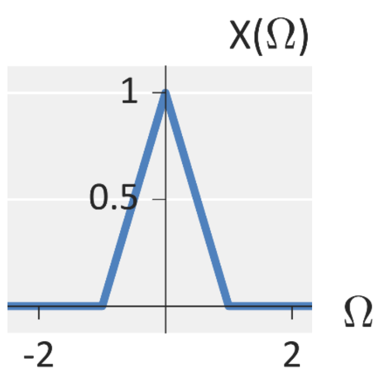


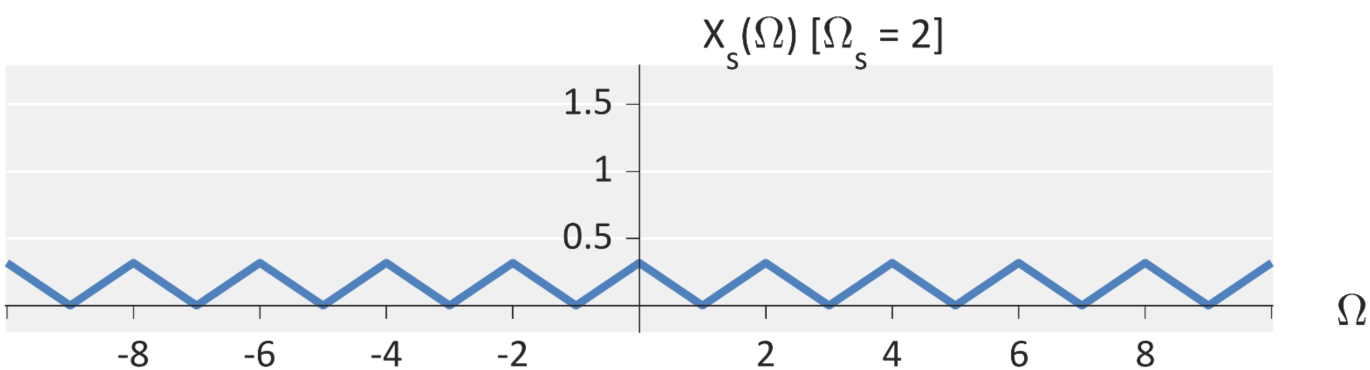


Sampling

- **Question:** **What is happening here?**

$$X_s(\Omega) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X(\Omega - k\Omega_s)$$

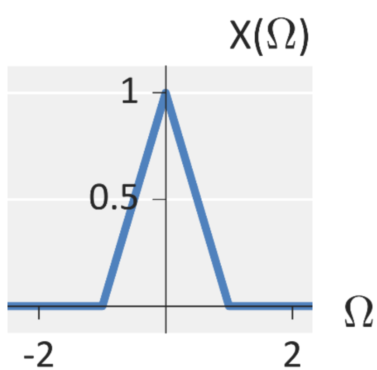


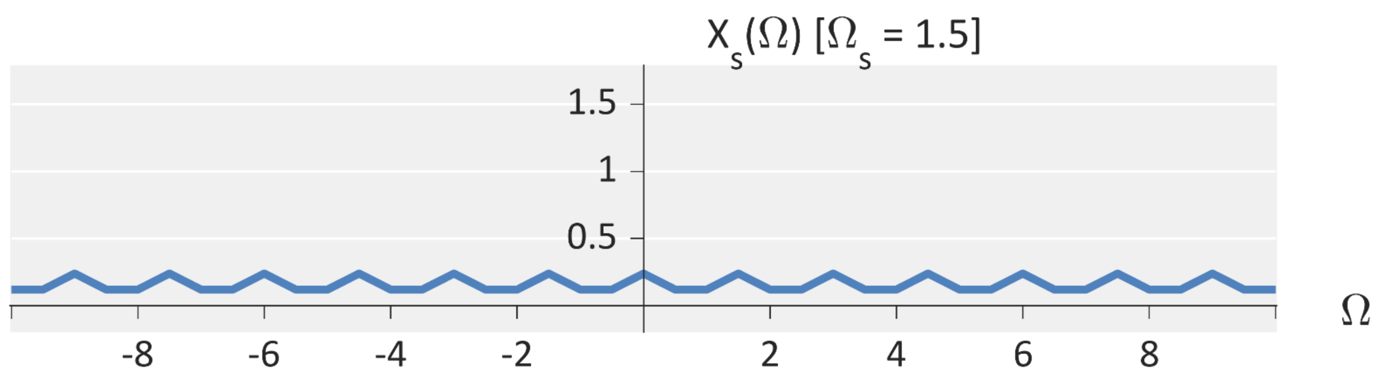


Sampling

- **Question: What is happening here?**

$$X_s(\Omega) = \frac{1}{T_s}\sum_{k=-\infty}^{\infty} X(\Omega - k\Omega_s)$$

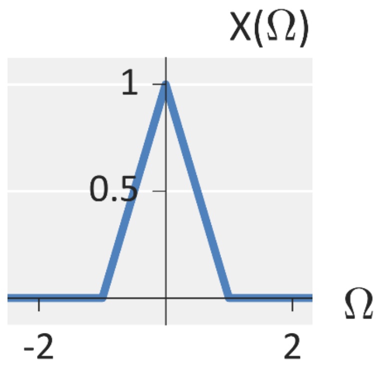


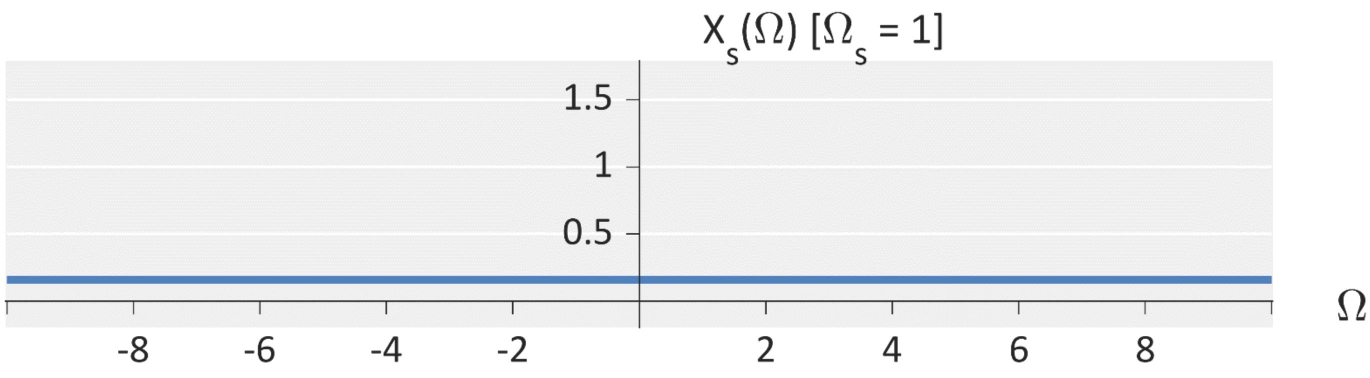


Sampling

- **Question: What is happening here?**

$$X_s(\Omega) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X(\Omega - k\Omega_s)$$

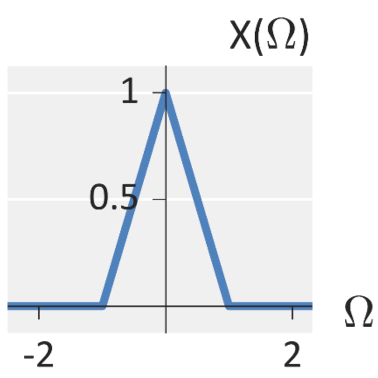


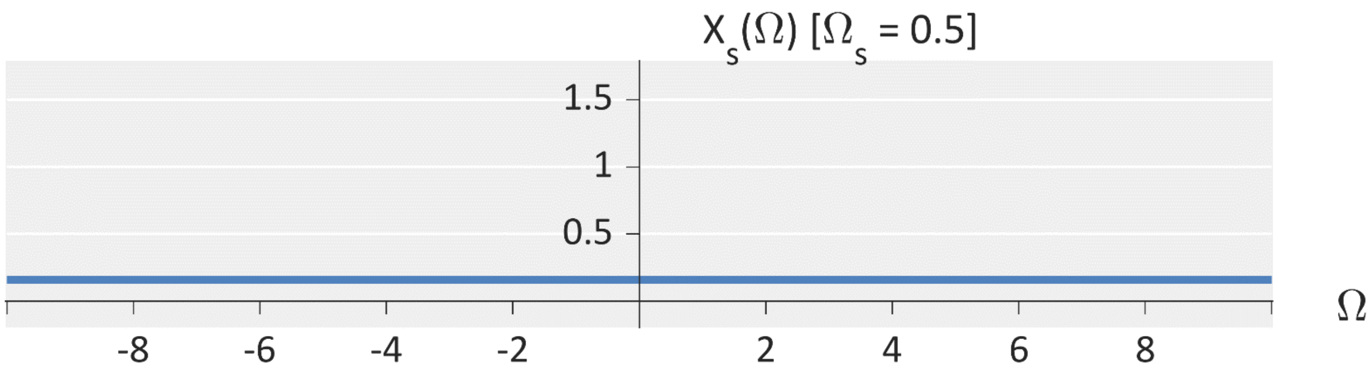


Sampling

- **Question: What is happening here?**

$$X_s(\Omega) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X(\Omega - k\Omega_s)$$



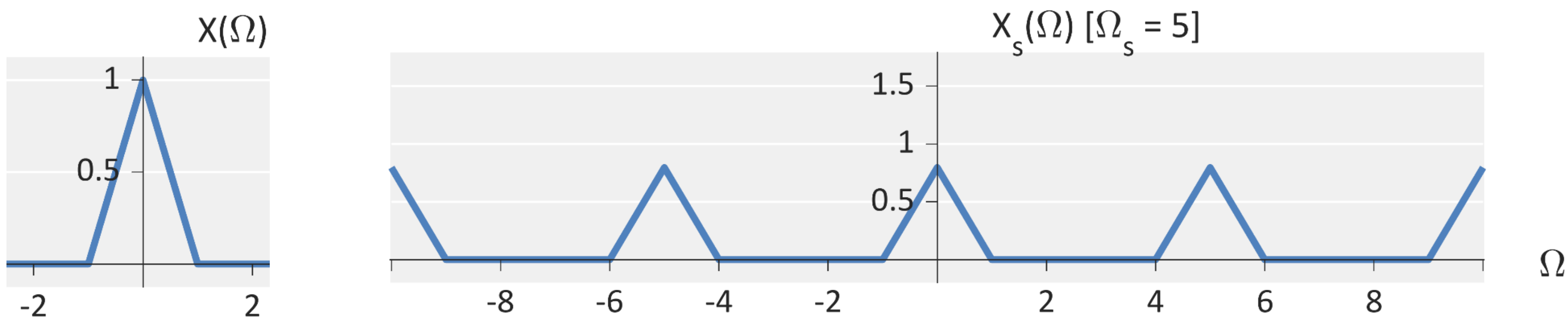


Sampling

- **Question: Can I preserve all information when I sample?**

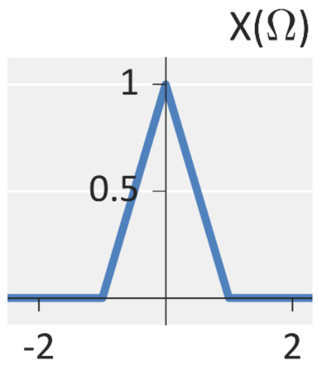
Sampling

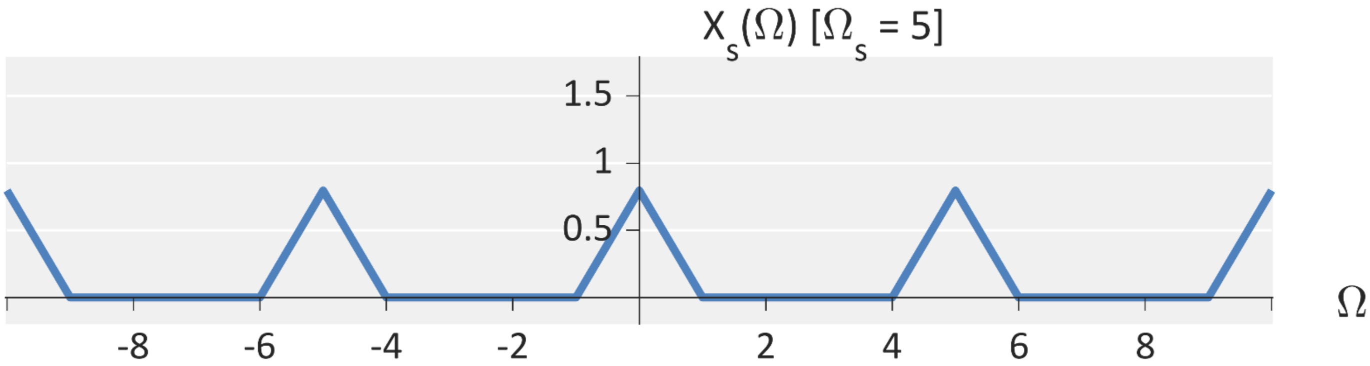
- **Question: Can I preserve all information when I sample?**
  - Yes!



Sampling

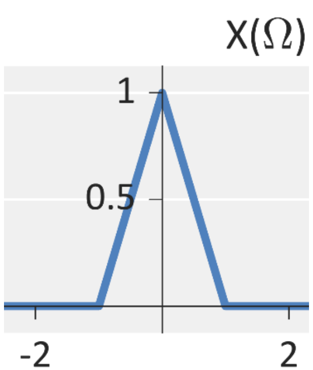
- **Question: How fast do I sample to preserve information?**

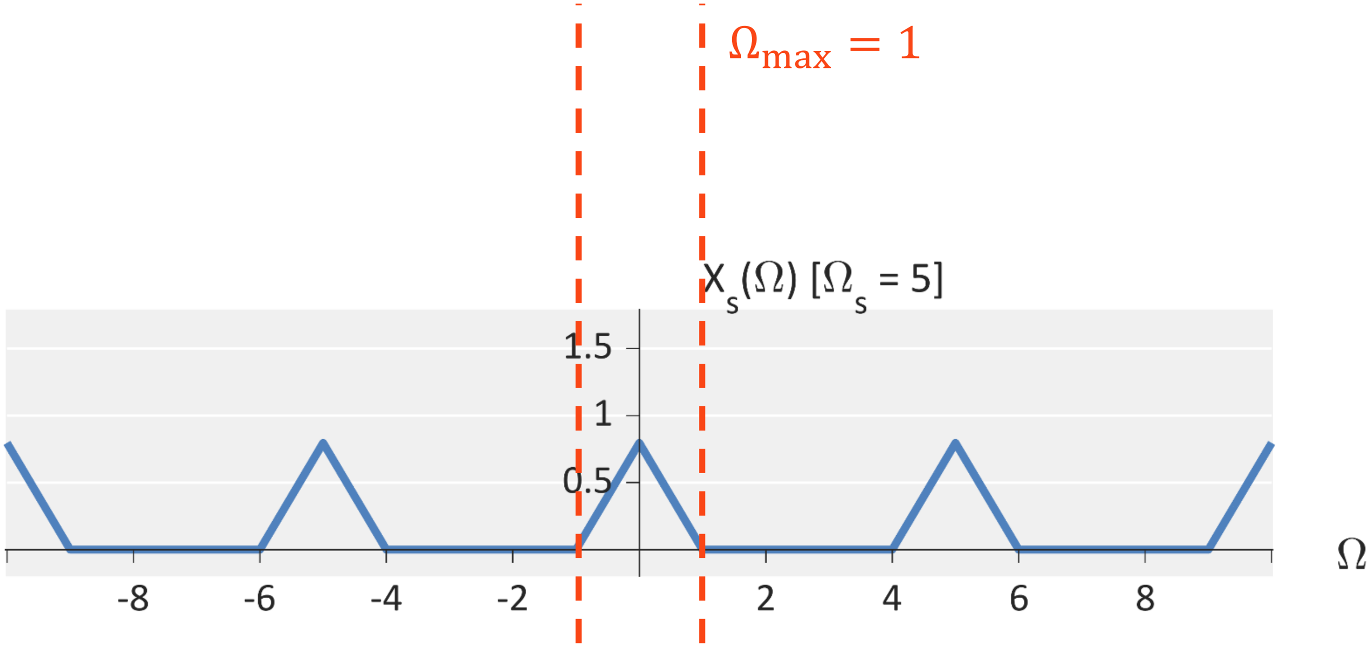




Sampling

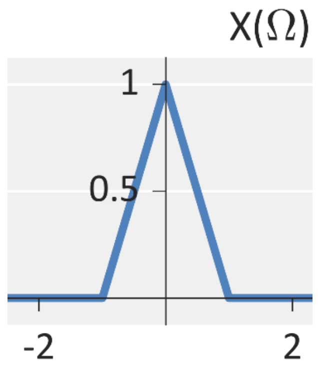
- **Question:** **How fast do I sample to preserve information?**

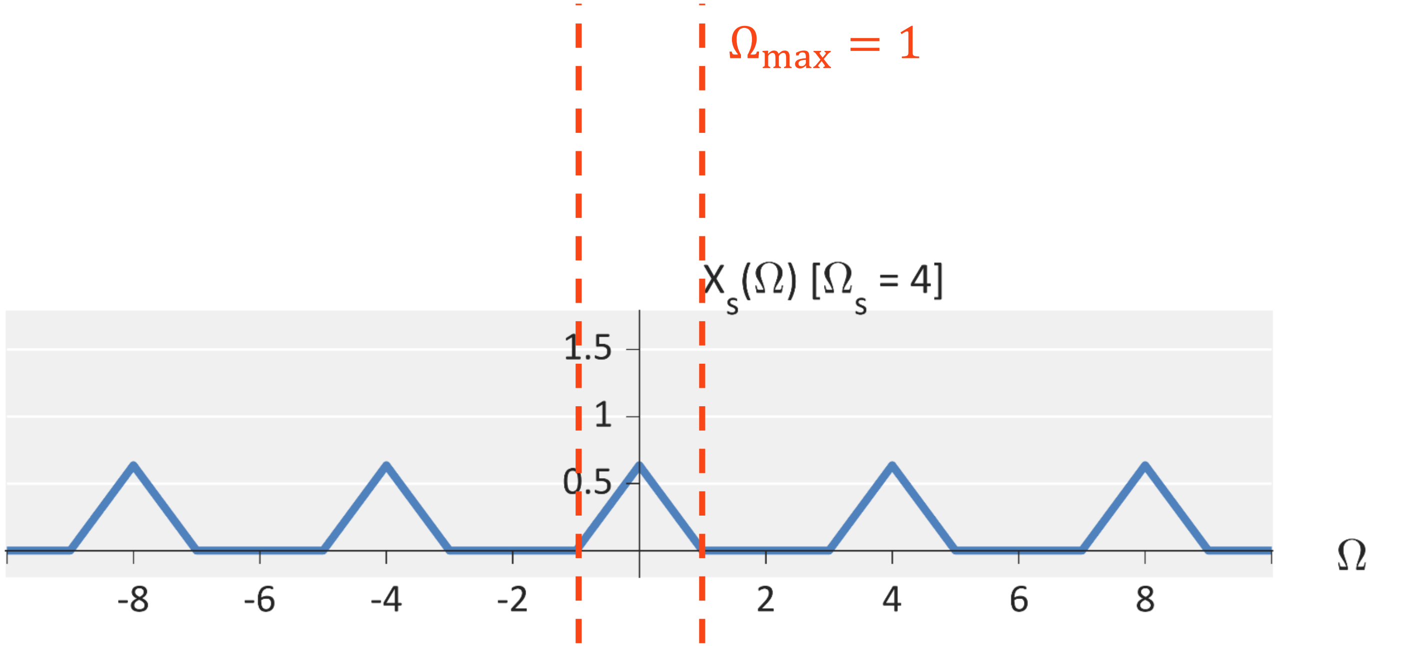




Sampling

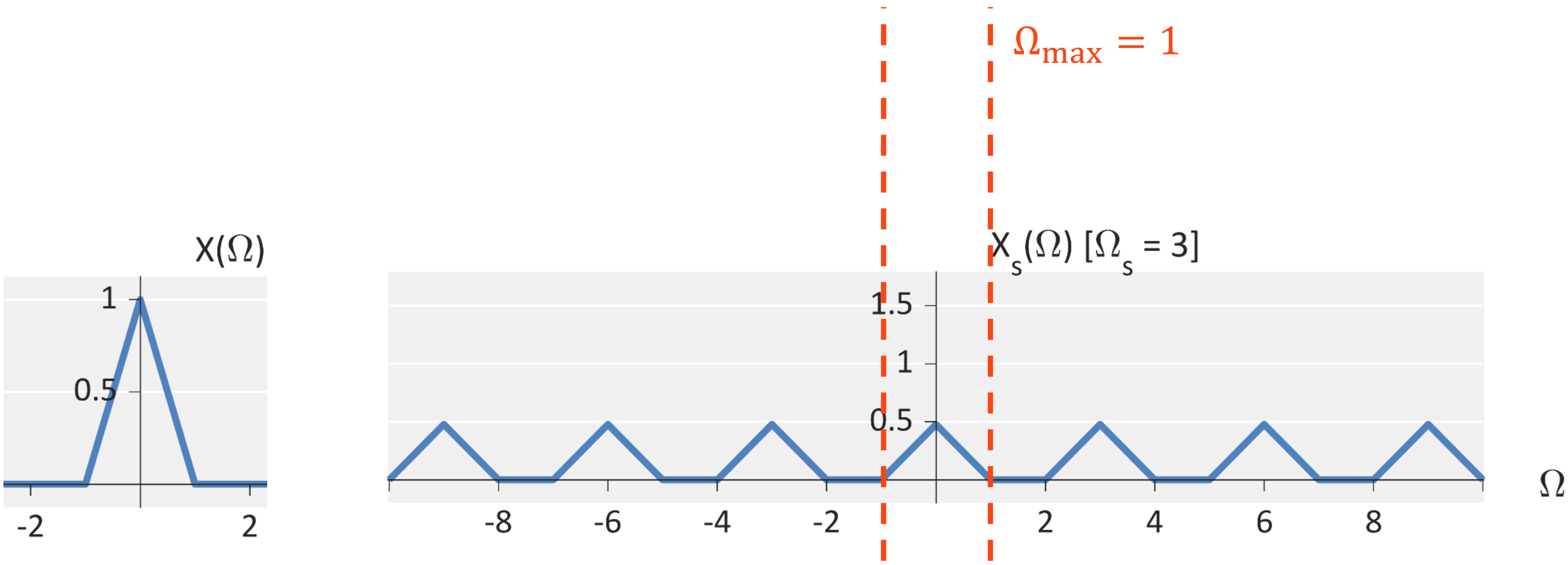
- **Question:** **How fast do I sample to preserve information?**





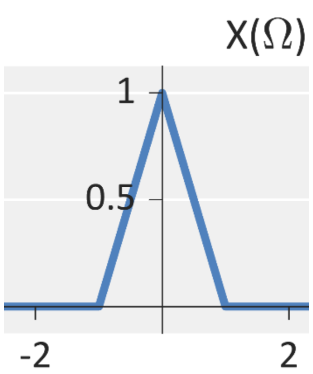
Sampling

- **Question:** **How fast do I sample to preserve information?**

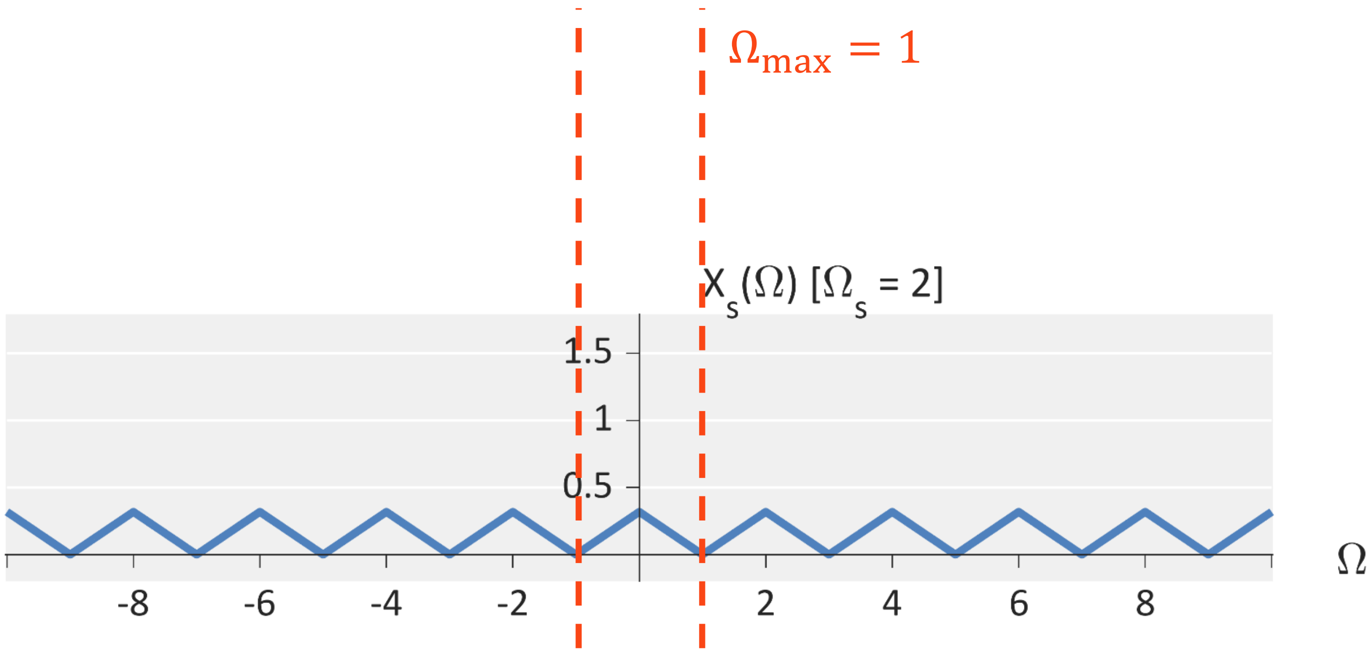


Sampling

- **Question:** **How fast do I sample to preserve information?**



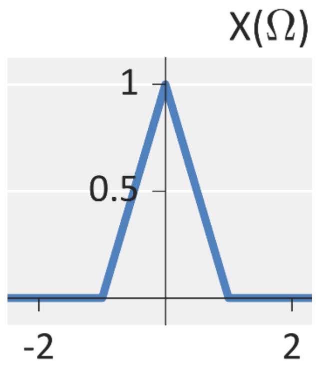
$\Omega_{\mathrm{max}} = 1$

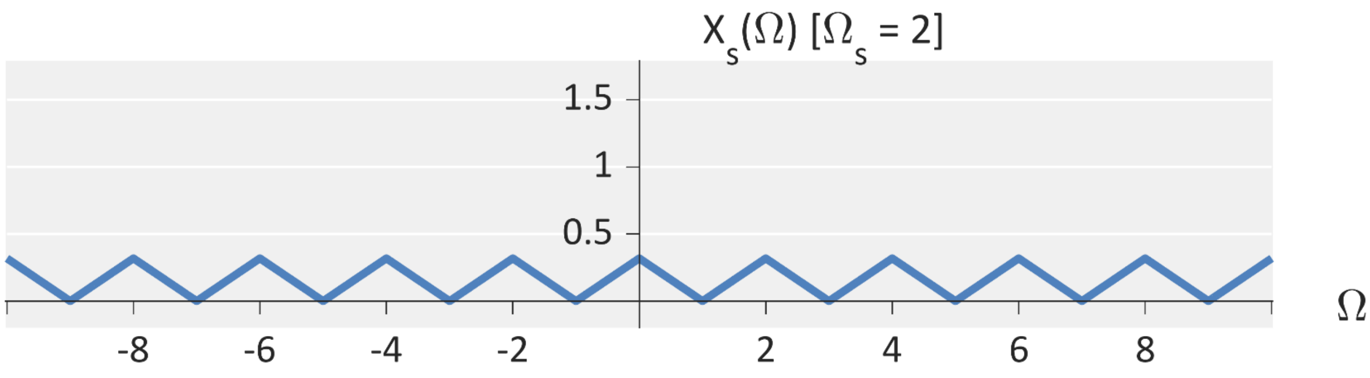


Sampling

- **Question: How fast do I sample to preserve information?**
  - We need to sample twice as fast as the maximum frequency

$$\Omega_s > 2\Omega_{max}$$



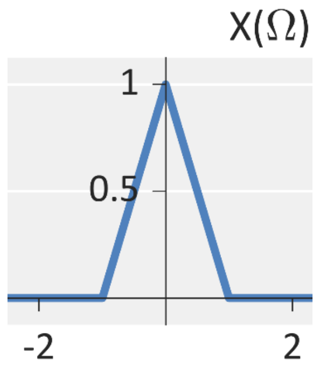


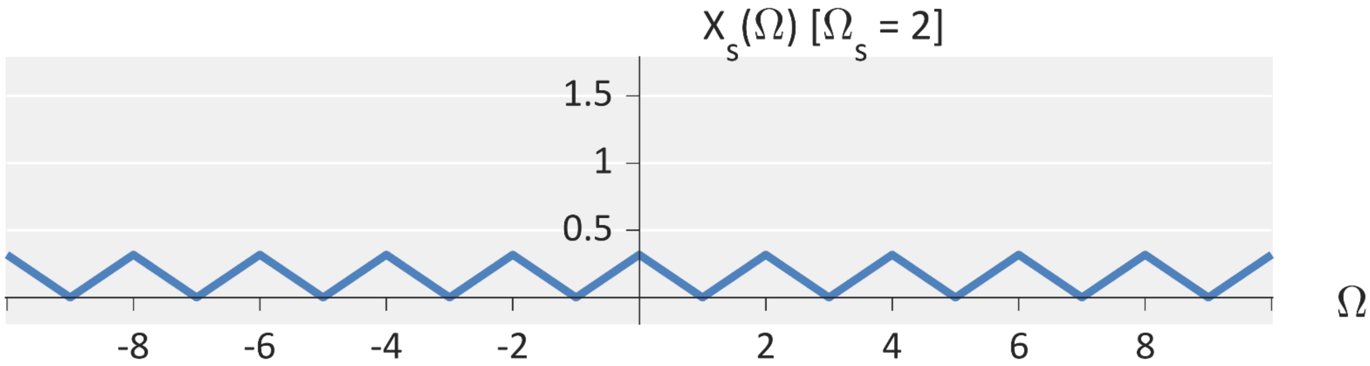
Sampling

- **Question: How fast do I sample to preserve information?**
  - We need to sample twice as fast as the maximum frequency

$$\Omega_s > 2\Omega_{max}$$
$$f_s > 2f_{max}$$

Nyquist-Shannon Sampling Theorem





Lecture 11: Sampling

Foundations of Digital Signal Processing

Outline

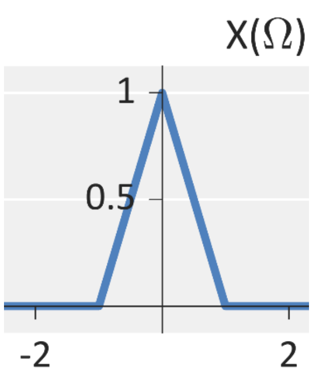
- Sampling
- Sampling in Time = ??? in Frequency
- The Nyquist-Shannon Sampling Theorem
- **Continuous-time Reconstruction / Interpolation**
- Aliasing

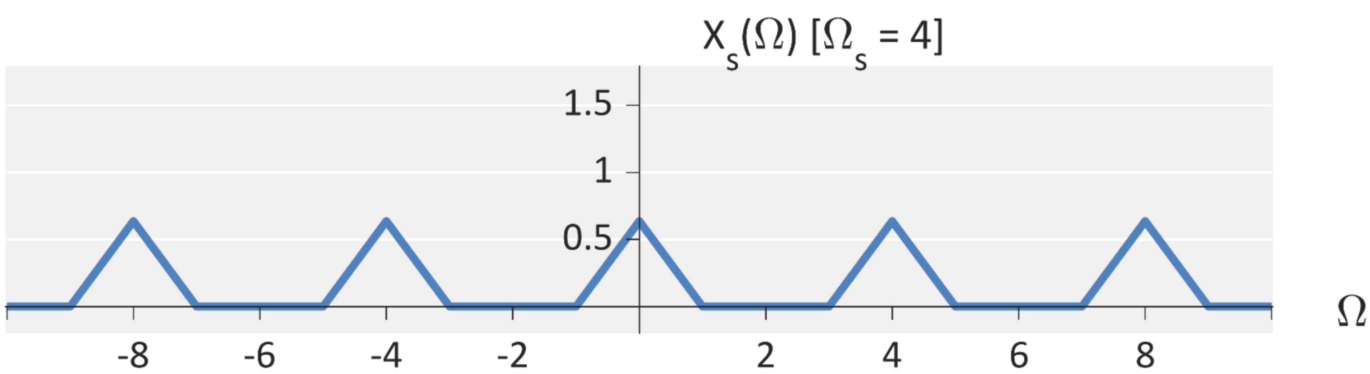
Sampling

- **Question: How do I return to continuous–time?**

$$\Omega_s > 2\Omega_{max}$$
$$f_s > 2f_{max}$$

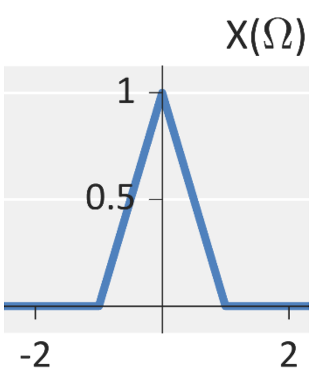
Nyquist-Shannon Sampling Theorem

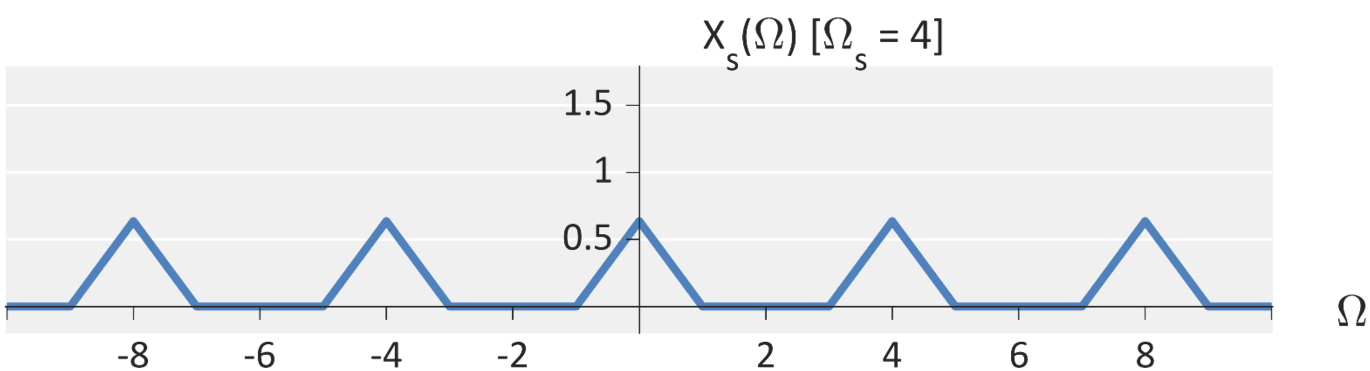




Sampling

- **Question: How do I return to continuous–time?**
  - **Filter:** Low pass filter to keep $\Omega_s/2 \leq \Omega \leq \Omega_s/2$
  - **Amplify:** Amplify signal by T_s

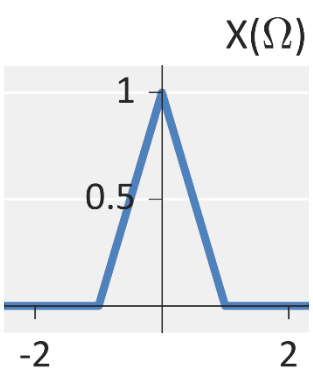


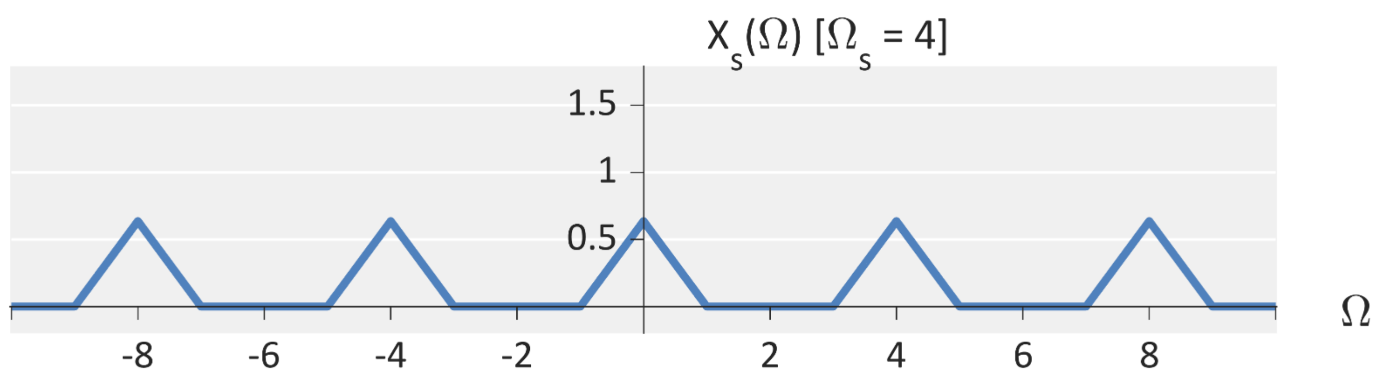


Sampling

- **Question: How do I return to continuous–time?**
  - **Filter:** Low pass filter to keep $\Omega_s/2 \leq \Omega \leq \Omega_s/2$
  - **Amplify:** Amplify signal by T_s
  - **Simplified:** Multiply $X_s(\Omega)$ by

$$T_s[u(\Omega + \Omega_s/2) - u(\Omega - \Omega_s/2)]$$

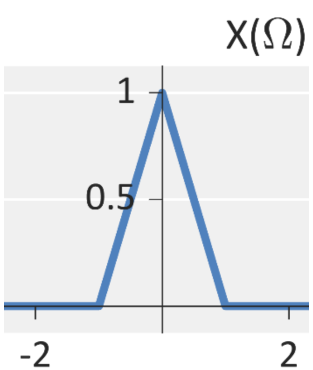




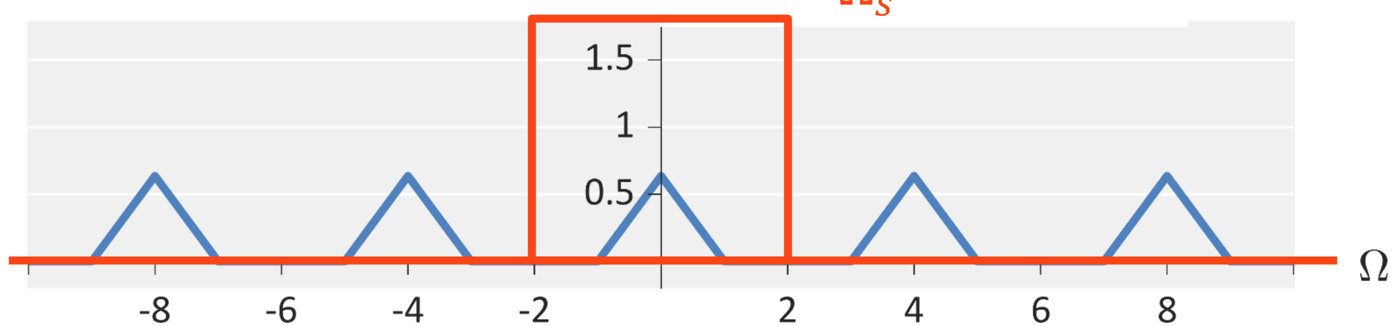
Sampling

- **Question: How do I return to continuous–time?**
  - **Filter:** Low pass filter to keep $\Omega_s/2 \leq \Omega \leq \Omega_s/2$
  - **Amplify:** Amplify signal by T_s
  - **Simplified:** Multiply $X_s(\Omega)$ by

$$T_s[u(\Omega + \Omega_s/2) - u(\Omega - \Omega_s/2)]$$



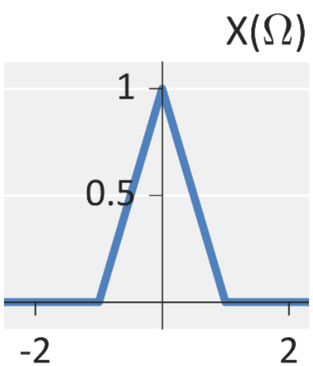
$$T_s = \frac{2\pi}{\Omega_s} = 1.5708$$

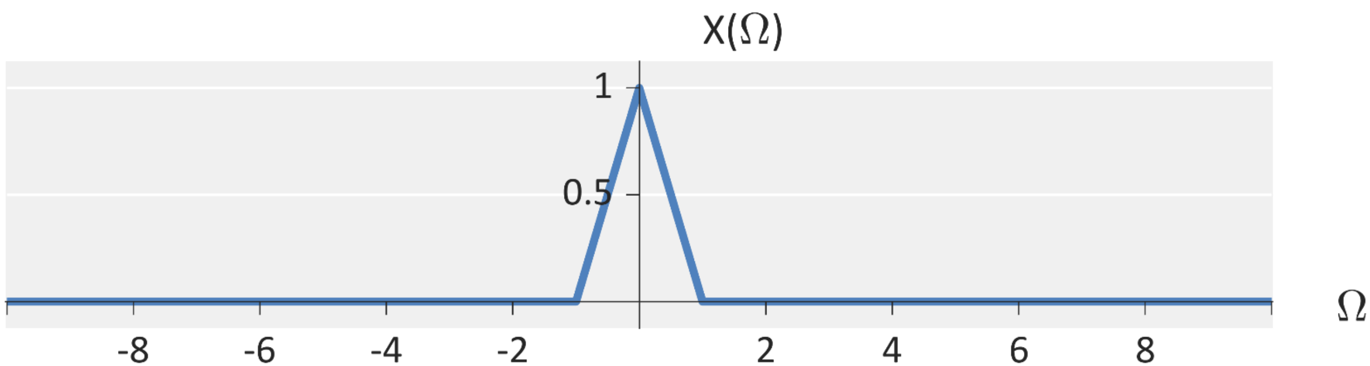


Sampling

- **Question: How do I return to continuous–time?**
  - **Filter:** Low pass filter to keep $\Omega_s/2 \leq \Omega \leq \Omega_s/2$
  - **Amplify:** Amplify signal by T_s
  - **Simplified:** Multiply $X_s(\Omega)$ by

$$T_s[u(\Omega + \Omega_s/2) - u(\Omega - \Omega_s/2)]$$





Lecture 11: Sampling

Foundations of Digital Signal Processing

Outline

- Sampling
- Sampling in Time = ??? in Frequency
- The Nyquist-Shannon Sampling Theorem
- Continuous-time Reconstruction / Interpolation
- Aliasing

Sampling

- **Aliasing occurs when we do not satisfy the sampling theorem**
- **Question: What can happen when there is aliasing?**