Nth power of a square matrix and the Binet Formula for Fibonacci sequence

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Given $A = \begin{pmatrix} 4 & -12 \\ -12 & 11 \end{pmatrix}$. We begin to investigate how to find A^n .

(1) The story begins in finding the eigenvalue(s) and eigenvector(s) of A .

A real number λ is said to be an eigenvalue of a matrix A if there exists a non-zero column vector **v** such that $A\mathbf{v} = \lambda\mathbf{v}$ or $(A - \lambda I)\mathbf{v} = 0$

(a) Eigenvalues

$$A = \begin{pmatrix} 4 & -12 \\ -12 & 11 \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} x \\ y \end{pmatrix}, \quad (A - \lambda I)\mathbf{v} = \begin{pmatrix} 4-\lambda & -12 \\ -12 & 11-\lambda \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0$$

Now, $(A - \lambda I)\mathbf{v} = 0$ has non-zero solution, $|A - \lambda I| = 0$

$$\begin{vmatrix} 4-\lambda & -12 \\ -12 & 11-\lambda \end{vmatrix} = 0$$

$$(4 - \lambda)(11 - \lambda) - 144 = 0$$

$$\lambda^2 - 15\lambda - 100 = 0$$

$$(\lambda - 20)(\lambda + 5) = 0$$

$$\therefore \quad \lambda_1 = 20 \quad \text{or} \quad \lambda_2 = -5 \quad , \text{and these are the eigenvalues.}$$

(b) Eigenvectors

We usually would like to find the **unit** eigenvector corresponding to each eigenvalue. The process is called normalization.

For $\lambda_1 = 20$,

$$\begin{cases} \begin{pmatrix} 4 & -12 \\ -12 & 11 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = 20\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \\ x_1{}^2 + y_1{}^2 = 1 \end{cases} \Leftrightarrow \begin{cases} 4x_1 - 12y_1 = 20x_1 \\ -12x_1 + 11y_1 = 20y_1 \\ x_1{}^2 + y_1{}^2 = 1 \end{cases}$$

$$\begin{cases} 4x_1 + 3y_1 = 0 \\ x_1{}^2 + y_1{}^2 = 1 \end{cases}$$

$$\therefore \quad \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 3/5 \\ -4/5 \end{pmatrix} \quad \text{, which is a unit eigenvector.}$$

For $\lambda_2 = -5$,

$$\begin{cases} \begin{pmatrix} 4 & -12 \\ -12 & 11 \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = -5\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \\ x_2{}^2 + y_2{}^2 = 1 \end{cases} \Leftrightarrow \begin{cases} 4x_2 - 12y_2 = -5x_2 \\ -12x_2 + 11y_2 = -5y_2 \\ x_2{}^2 + y_2{}^2 = 1 \end{cases}$$

$$\begin{cases} 3x_1 - 4y_1 = 0 \\ x_2{}^2 + y_2{}^2 = 1 \end{cases}$$

$$\therefore \quad \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 4/5 \\ 3/5 \end{pmatrix} \quad \text{, which is another unit eigenvector.}$$

(c) Diagonalization of matrix

Let $P = \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix} = \begin{pmatrix} 3/5 & 4/5 \\ -4/5 & 3/5 \end{pmatrix}$

Consider another matrix:

$$B = P^{-1}AP = \begin{pmatrix} 3/5 & 4/5 \\ -4/5 & 3/5 \end{pmatrix}^{-1} \begin{pmatrix} 4 & -12 \\ -12 & 11 \end{pmatrix} \begin{pmatrix} 3/5 & 4/5 \\ -4/5 & 3/5 \end{pmatrix}$$

$$= \begin{pmatrix} 3/5 & -4/5 \\ 4/5 & 3/5 \end{pmatrix} \begin{pmatrix} 4 & -12 \\ -12 & 11 \end{pmatrix} \begin{pmatrix} 3/5 & 4/5 \\ -4/5 & 3/5 \end{pmatrix}$$

$$= \begin{pmatrix} 20 & 0 \\ 0 & -5 \end{pmatrix}$$

Note that **B** is a diagonal matrix with eigenvalues as entries in the main diagonal. The nth power of a diagonal matrix is much easier to find than the original matrix.

$$B^n = \begin{pmatrix} 20 & 0 \\ 0 & -5 \end{pmatrix}^n = \begin{pmatrix} 20^n & 0 \\ 0 & (-5)^n \end{pmatrix}$$

Note:

The diagonalization of a matrix may not be a simple subject since $|A - \lambda I| = 0$ may have equal roots or even complex roots. Although most matrices are not diagonal, they can be diagonalized. Not all square matrices can be diagonalised.A thorough study of diagonalization of a matrix is not discussed here.

(d) Finding the nth power of the original matrix A

Since $B = P^{-1}AP$, we have $A = PBP^{-1}$

$A^n = (PBP^{-1})^n = (PBP^{-1})(PBP^{-1}) \dots (PBP^{-1}) = PB^nP^{-1}$

$$A^n = \begin{pmatrix} 3/5 & 4/5 \\ -4/5 & 3/5 \end{pmatrix} \begin{pmatrix} 20^n & 0 \\ 0 & (-5)^n \end{pmatrix} \begin{pmatrix} 3/5 & 4/5 \\ -4/5 & 3/5 \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} 3/5 & 4/5 \\ -4/5 & 3/5 \end{pmatrix} \begin{pmatrix} 20^n & 0 \\ 0 & (-5)^n \end{pmatrix} \begin{pmatrix} 3/5 & -4/5 \\ 4/5 & 3/5 \end{pmatrix}$$

$$= \frac{1}{25} \begin{pmatrix} 3 & 4 \\ -4 & 3 \end{pmatrix} \begin{pmatrix} 20^n & 0 \\ 0 & (-5)^n \end{pmatrix} \begin{pmatrix} 3 & -4 \\ 4 & 3 \end{pmatrix}$$

$$= \frac{1}{25} \begin{pmatrix} 3 & 4 \\ -4 & 3 \end{pmatrix} \begin{pmatrix} 20^n & 0 \\ 0 & (-5)^n \end{pmatrix} \begin{pmatrix} 3 & -4 \\ 4 & 3 \end{pmatrix} = \frac{1}{25} \begin{pmatrix} 3(20^n) & 4(-5)^n \\ -4(20^n) & 3(-5)^n \end{pmatrix} \begin{pmatrix} 3 & -4 \\ 4 & 3 \end{pmatrix}$$

$$= \frac{1}{25} \begin{pmatrix} 9(20^n) + 12(-5)^n & -12(20^n) + 12(-5)^n \\ -12(20^n) + 12(-5)^n & -16(20^n) + 9(-5)^n \end{pmatrix}$$
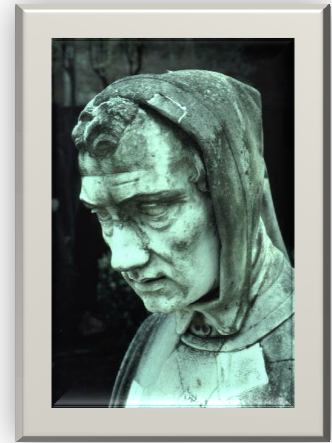
(2) (a) Fibonacci sequence

The sequence 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, …,

defined by F(1) = 1, F(2) = 1, and

   F(n) = F(n-1) + F(n-2) for n = 3, 4, 5, …

is named the Fibonacci sequence.

Fibonacci, also known as Leonardo of Pisa, was born in Pisa, home of the famous leaning tower (inclined at an angle of 16.5 degrees to the vertical).   Little is known of his life, and no portrait exists. However, a statue of   Fibonacci was erected by the citizens of Pisa. A picture, seen here, of the head of the Fibonacci statue, is some fifteen feet above ground. Since then, the statue has been moved to its present (and original) location at Camposanto Monumentale (at Piazza dei Miracoli, where the Leaning Tower stands.)

(b) Binet's formula for finding the nth term of Fibonacci sequence

The general term of the Fibonacci sequence was derived   by Jacques Binet in 1843, although the result was known to Euler,   Daniel Bernoulli, and de Moivre more than a century earlier.

$$F(n) = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$$

There are many different proof for this formula. One way is to use mathematical induction. Here we are going to use the concept of diagonalization of matrix to prove this formula.
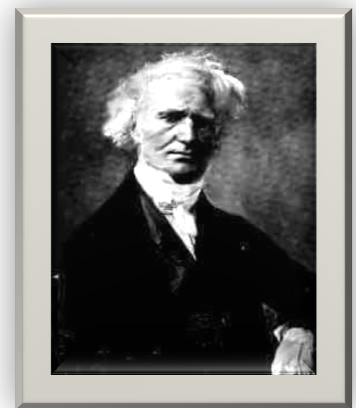
(c) Redefine the problem

$$\begin{pmatrix} F(n+2) \\ F(n+1) \end{pmatrix} = \begin{pmatrix} F(n+1)+F(n) \\ F(n+1) \end{pmatrix} \text{, where } \begin{pmatrix} F(2) \\ F(1) \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} F(n+1) \\ F(n) \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^2 \begin{pmatrix} F(n) \\ F(n-1) \end{pmatrix}$$

$$= \dots$$

$$= \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n \begin{pmatrix} F(2) \\ F(1) \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad\qquad \dots \quad (1)$$

The problem is reduced to finding A^n , where $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$

(d) Diagonalize A

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} x \\ y \end{pmatrix}, \quad (A - \lambda I)\mathbf{v} = \begin{pmatrix} 1 - \lambda & 1 \\ 1 & -\lambda \end{pmatrix} = 0$$

Now, $(A - \lambda I)\mathbf{v} = 0$ has non-zero solution, $|A - \lambda I| = 0$

$$\begin{vmatrix} 1 - \lambda & 1 \\ 1 & -\lambda \end{vmatrix} = 0$$

$$\lambda^2 - \lambda - 1 = 0 \qquad\qquad\qquad\qquad\qquad \dots \quad (2)$$

$\lambda_1 = \frac{1+\sqrt{5}}{2}$ or $\lambda_2 = \frac{1-\sqrt{5}}{2}$, which are the eigenvalues .

$$\therefore \quad \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \lambda_1 \begin{pmatrix} x \\ y \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \lambda_2 \begin{pmatrix} x \\ y \end{pmatrix} \qquad \dots \quad (3)$$

Since the sum of roots of eq (2), $\lambda_1 + \lambda_2 = 1$,

(2) is reduced to $x = \lambda_1 y, \ x = \lambda_2 y$

And $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} \lambda_1 \\ 1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} \lambda_2 \\ 1 \end{pmatrix}$ are the corresponding eigenvectors.

The eigenvectors are not normalized, that is, are not unit vectors, but is alright in working for diagonalization.

Take $P = \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix} = \begin{pmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{pmatrix}$.

$$B = P^{-1}AP = \begin{pmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{pmatrix}$$

$$= \frac{1}{\lambda_1 - \lambda_2}\begin{pmatrix} 1 & -\lambda_2 \\ -1 & \lambda_1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{pmatrix}$$

$$= \frac{1}{\sqrt{5}}\begin{pmatrix} 1 & -\lambda_2 \\ -1 & \lambda_1 \end{pmatrix}\begin{pmatrix} \lambda_1 + 1 & \lambda_2 + 1 \\ \lambda_1 & \lambda_2 \end{pmatrix}$$

$$= \frac{1}{\sqrt{5}}\begin{pmatrix} \lambda_1 + 1 - \lambda_1\lambda_2 & \lambda_2 + 1 - \lambda_2{}^2 \\ -\lambda_1 - 1 + \lambda_1{}^2 & -\lambda_2 - 1 + \lambda_1\lambda_2 \end{pmatrix}$$

$$= \frac{1}{\sqrt{5}}\begin{pmatrix} \lambda_1 + 2 & 0 \\ 0 & -\lambda_2 - 2 \end{pmatrix}$$

, since the product of roots of eq (2) = $\lambda_1\lambda_2 = -1$ and λ_1, λ_2 are roots of (2).

Substitute the values of λ_1, λ_2 ,

$$B = \frac{1}{\sqrt{5}}\begin{pmatrix} \frac{1+\sqrt{5}}{2} + 2 & 0 \\ 0 & -\frac{1-\sqrt{5}}{2} - 2 \end{pmatrix} = \frac{1}{\sqrt{5}}\begin{pmatrix} \frac{5+\sqrt{5}}{2} & 0 \\ 0 & -\frac{5-\sqrt{5}}{2} \end{pmatrix} = \begin{pmatrix} \frac{5+\sqrt{5}}{2\sqrt{5}} & 0 \\ 0 & -\frac{5-\sqrt{5}}{2\sqrt{5}} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{1-\sqrt{5}}{2} \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

Note that B is a diagonal matrix with eigenvalues as entries in the main diagonal.

We easily get $\quad B^n = \begin{pmatrix} \lambda_1{}^n & 0 \\ 0 & \lambda_2{}^n \end{pmatrix}.$

Now,

$$A^n = (PBP^{-1})^n = (PBP^{-1})(PBP^{-1})\dots(PBP^{-1}) = PB^nP^{-1}$$

$$
\begin{aligned}
A^n &= \begin{pmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \lambda_1{}^n & 0 \\ 0 & \lambda_2{}^n \end{pmatrix} \begin{pmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{pmatrix}^{-1} \\
&= \begin{pmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \lambda_1{}^n & 0 \\ 0 & \lambda_2{}^n \end{pmatrix} \frac{1}{\sqrt5}\begin{pmatrix} 1 & -\lambda_2 \\ -1 & \lambda_1 \end{pmatrix} \\
&= \frac{1}{\sqrt5}\begin{pmatrix} \lambda_1{}^{n+1} & \lambda_2{}^{n+1} \\ \lambda_1{}^n & \lambda_2{}^n \end{pmatrix}\begin{pmatrix} 1 & -\lambda_2 \\ -1 & \lambda_1 \end{pmatrix} \\
&= \frac{1}{\sqrt5}\begin{pmatrix} \lambda_1{}^{n+1} - \lambda_2{}^{n+1} & -\lambda_1{}^{n+1}\lambda_2 + \lambda_1\lambda_2{}^{n+1} \\ \lambda_1{}^n - \lambda_2{}^n & -\lambda_1{}^n\lambda_2 + \lambda_1\lambda_2{}^n \end{pmatrix} \\
&= \frac{1}{\sqrt5}\begin{pmatrix} \lambda_1{}^{n+1} - \lambda_2{}^{n+1} & \lambda_1{}^n - \lambda_2{}^n \\ \lambda_1{}^n - \lambda_2{}^n & \lambda_1{}^{n-1} - \lambda_2{}^{n-1} \end{pmatrix} \quad \text{, since } \lambda_1\lambda_2 = -1 \text{ .}
\end{aligned}
$$

Lastly, from (1),

$$\begin{pmatrix} F(n+2) \\ F(n+1) \end{pmatrix}$$

$$
\begin{aligned}
&= \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\
&= \frac{1}{\sqrt5}\begin{pmatrix} \lambda_1{}^{n+1} - \lambda_2{}^{n+1} & \lambda_1{}^n - \lambda_2{}^n \\ \lambda_1{}^n - \lambda_2{}^n & \lambda_1{}^{n-1} - \lambda_2{}^{n-1} \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \\
&= \frac{1}{\sqrt5}\begin{pmatrix} \lambda_1{}^{n+1} - \lambda_2{}^{n+1} + \lambda_1{}^n - \lambda_2{}^n \\ \lambda_1{}^n - \lambda_2{}^n + \lambda_1{}^{n-1} - \lambda_2{}^{n-1} \end{pmatrix} \\
&= \frac{1}{\sqrt5}\begin{pmatrix} \lambda_1{}^n(\lambda_1 + 1) - \lambda_2{}^n(\lambda_2 + 1) \\ \lambda_1{}^{n-1}(\lambda_1 + 1) - \lambda_2{}^{n-1}(\lambda_2 + 1) \end{pmatrix} \\
&= \frac{1}{\sqrt5}\begin{pmatrix} \lambda_1{}^n(\lambda_1{}^2) - \lambda_2{}^n(\lambda_2{}^2) \\ \lambda_1{}^{n-1}(\lambda_1{}^2) - \lambda_2{}^{n-1}(\lambda_2{}^2) \end{pmatrix} \quad \text{, since } \lambda_i{}^2 - \lambda_i - 1 = 0 \text{ , } i = 1,2 \text{ .} \\
&= \frac{1}{\sqrt5}\begin{pmatrix} \lambda_1{}^{n+2} - \lambda_2{}^{n+2} \\ \lambda_1{}^{n+1} - \lambda_2{}^{n+1} \end{pmatrix}
\end{aligned}
$$

In the end of our journey, we get our beautiful Binet's formula for Finonacci sequence:

$$F(n) = \frac{1}{\sqrt5}(\lambda_1{}^n - \lambda_2{}^n) = \frac{1}{\sqrt5}\left[\left(\frac{1+\sqrt5}{2}\right)^n - \left(\frac{1-\sqrt5}{2}\right)^n\right]$$