ALGORITHMS FOR IMAGING SPECTROSCOPY
MATH BACKGROUND ASSESSMENT
SPRING 2017

Name: ____

Happy New Year!

The students enrolled in this course are from a variety of backgrounds. Therefore, please work out the problems and answer the questions on this math background assessment to the best of your ability so that I can understand your levels of mathematical knowledge pertinent to the class. Your results will not be used to calculate any grade in this class. If it turns out that some students have less background than required, then I will offer extra study sessions.

Note: All vectors in the questions below are column vectors unless otherwise indicated.

(1) **Eigenvalues and Eigenvectors**

(a) Use matrix-vector multiplication to choose with of the two vectors $\mathbf{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\mathbf{y} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ are eigenvectors of the matrix

$$\mathbf{A} = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$$

(b) What is the eigenvalue associated with the eigenvector?

(2) **Least Squares**. Suppose $\mathbf{A}$ is an m x n matrix and that $m > n$. How do you find the vector $\mathbf{x}_s$ that gives the least squares solution to the linear system of equations $\mathbf{Ax} = \mathbf{b}$, that is, that minimizes the squared error $(\mathbf{Ax} - \mathbf{b})^t (\mathbf{Ax} - \mathbf{b}) = \|\mathbf{Ax} - \mathbf{b}\|_2^2$.

(3) **Angles between vectors**. Suppose $\mathbf{x}$ and $\mathbf{y}$ are two vectors. Show how to use the *inner product* $\mathbf{x}^t\mathbf{y}$ to compute the cosine of the angle between $\mathbf{x}$ and $\mathbf{y}$

(4) **Gaussian Distributions**. A multi-variate Gaussian distribution is defined using the formula

$$f\left(x|\boldsymbol{\mu},\mathbf{C}\right)=\frac{1}{2\pi|\mathbf{C}|^{\frac{B}{2}}}e^{\left(-\frac{(\mathbf{x}-\boldsymbol{\mu})\mathbf{C}^{-1}(\mathbf{x}-\boldsymbol{\mu})}{2}\right)}$$

For a given sample $X=\{\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_N\}$ from a Gaussian distribution, one can estimate the covariance matrix using the formula

$$\bar{\mathbf{C}}=\frac{1}{N-1}\sum_{n=1}^{N}\left(\mathbf{x}_n-\bar{\boldsymbol{\mu}}\right)\left(\mathbf{x}_n-\bar{\boldsymbol{\mu}}\right)^t$$

where $\bar{\boldsymbol{\mu}}$ is the sample mean. Compute the estimate of the covariance using the sample

$$X=\{\begin{pmatrix}1\\1\end{pmatrix},\begin{pmatrix}1\\-1\end{pmatrix},\begin{pmatrix}-1\\1\end{pmatrix},\begin{pmatrix}-1\\-1\end{pmatrix}\}$$

(5) **Maximum Likelihood**. Assume you want to estimate the probability of Heads, p_h, by flipping a particular coin so you flip the coin 5 times. You observe that it lands Heads every time. What is the Maximum Likelihood estimate of p_h^{ML} of p_h based on this experiment? (Hint: Count).

(6) **Bayes Rule**.

(a) State Bayes Rule

(b) You still want to estimate p_h but you don't like the Maximum Likelihood answer so you are going to use the same 5 coin flips from before and try to use Bayes rule. You need a prior so you assume the prior probability density of heads is given by the following *Beta distribution*:

$$\beta\left(p_h\right) = K p_h \left(1 - p_h\right)$$

where K is a constant. This Beta distribution is shown in Figure 1.

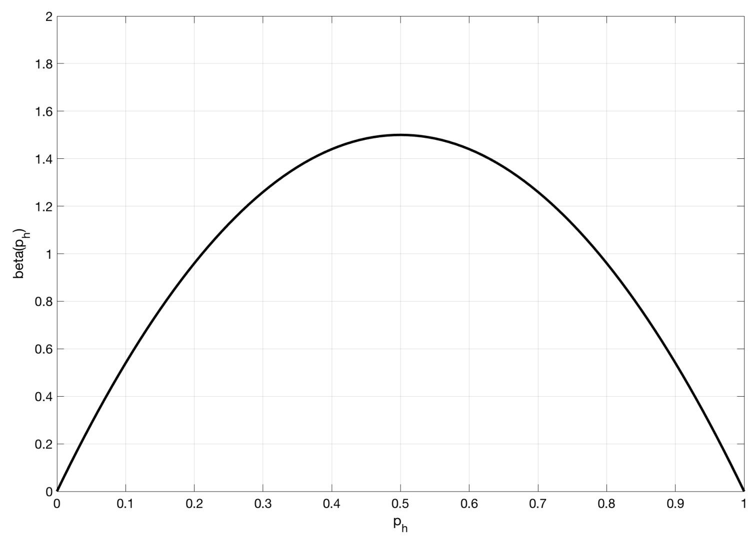

FIGURE 1. A Beta distribution.

The Maximum *A-Posteriori* (MAP) estimate of p_h, p_h^{MAP}, is calculated by finding the value of p_h that maximizes the value of the posterior over all possible values. The MAP is proportional to the product of the Bernoulli and Beta distributions $B\left(N, p_h\right)\beta\left(p_h\right)$ (referred to as the MAP function). In the definition of the Bernoulli, N is the number of trials, p_h is the unknown probability of heads, and k is the number of coin flips resulting in Heads. The definition, and particular form for these coin flips, is:

$$B\left(N, p_h\right) = \binom{N}{k} p_h^k \left(1 - p_h\right)^{N-k} = p_h^5$$

Plot your best estimate of the shape of the MAP function in this case and use it to roughly estimate the MAP estimate of p_h.

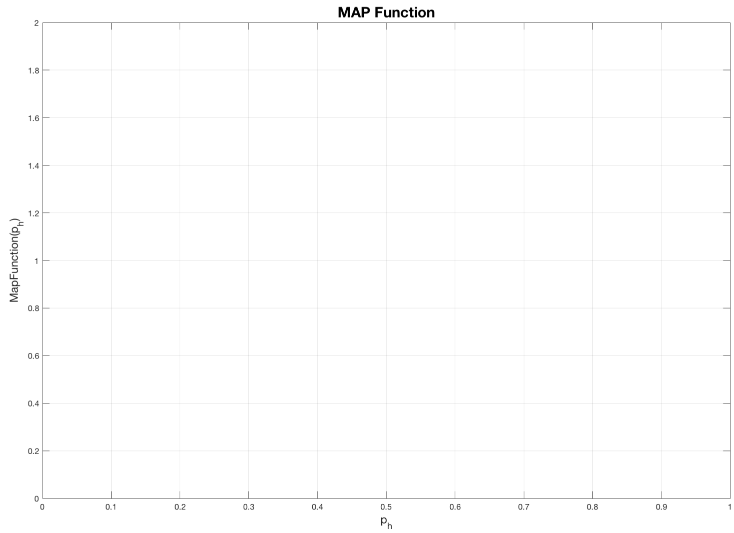

FIGURE 2. Plot the MAP Function here.

(a) **InformationTheory**.

(i) Compute the entropy (using log_2) of the distribution:

$$p_1 = \frac{1}{4}, p_2 = \frac{1}{2}, p_3 = \frac{1}{4}$$

(ii) Suppose $\{p_1, p_2, ..., p_M\}$ is a finite probability distribution, which means that

$$0 \leq p_m \leq 1 \quad \text{and} \quad \sum_{m=1}^{M} p_m = 1.$$

What values of $\{p_1, p_2, ..., p_M\}$ maximize the entropy?