EEE-6512: Image Processing and Computer Vision

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Lecture #8: Frequency Domain Processing

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Lecture Outline

- Preliminaries
- Fourier Transform
- Discrete Fourier Transform (DFT)
- Two-Dimensional DFT
- Frequency-Domain Filtering
- Localizing Frequencies in Time
- Discrete Wavelet Transform

Localizing Frequencies In Time

Localizing Frequencies In Time - Gabor Limit

- **Gabor limit**: says that a signal cannot be localized simultaneously in both frequency and time.

- To balance this trade-off between localizing in frequency and localizing in time:
  - The duration of the pulse should be related to the frequency that we are trying to localize.
  - That is, a high-frequency tone should receive a shorter pulse, while a low-frequency tone should receive a longer pulse.

Discrete Wavelet Transform (DWT)

- The key idea of the wavelet transform is to determine the locations of frequencies in a signal in such a way that the frequencies are taken into account when determining their location.

- The wavelet transform also projects the signal onto basis functions, starting with a **mother wavelet**.

$$\psi_{a,b}(x) \equiv \frac{1}{\sqrt{a}}\,\psi\left(\left\lfloor \frac{x-b}{a} \right\rfloor\right)$$

Discrete Wavelet Transform (DWT) (cont'd)

- The **discrete wavelet transform (DWT)** of a 1D discrete signal $g(x)$ is a 2D array of values $G(a,b)$, where each element in the array is the sum of the elementwise product of the signal with the appropriate wavelet function:

$$G(a,b) \equiv \sum_{x} g(x)\psi_{a,b}(x)$$

- If a and b are allowed to take on any integer values, then the transform is **overcomplete**.

Haar Wavelets

- The simplest and oldest type of wavelet is the **Haar wavelet.**
- In the continuous domain, the Haar mother wavelet is two adjacent *boxcar functions* of opposite sign:

$$\psi(x) = \begin{cases} 1 & \text{if } 0 < x < \frac{1}{2} \\ -1 & \text{if } \frac{1}{2} < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

Figure 6.26 Haar basis functions are based on boxcar functions. Shown are the mother (left) and father (right) wavelets.

DWT as Matrix Multiplication

- The discrete wavelet transform (DWT) can be viewed as matrix multiplication.
- For an 8-element signal, for example, the Haar wavelet matrix is

$$\begin{bmatrix} G(0) \\ G(1) \\ G(2) \\ G(3) \\ G(4) \\ G(5) \\ G(6) \\ G(7) \end{bmatrix} = \underbrace{\frac{1}{\sqrt{2}} \begin{bmatrix} 1/2 & 1/2 & 1/2 & 1/2 & 1/2 & 1/2 & 1/2 & 1/2 \\ 1/2 & 1/2 & 1/2 & 1/2 & -1/2 & -1/2 & -1/2 & -1/2 \\ 1/\sqrt{2} & 1/\sqrt{2} & -1/\sqrt{2} & -1/\sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/\sqrt{2} & 1/\sqrt{2} & -1/\sqrt{2} & -1/\sqrt{2} \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}}_{\mathbf{H}_8} \begin{bmatrix} g(0) \\ g(1) \\ g(2) \\ g(3) \\ g(4) \\ g(5) \\ g(6) \\ g(7) \end{bmatrix}$$

Fast Wavelet Transform (FWT)

- **Fast wavelet transform (FWT)**: computes the high- and low-frequency components first, then downsamples the low-passed signal and repeats until the length of the signal is too small to continue.
- The computation at a single resolution is given by:

$$g_{low}(x) = (g \check{\circledast} \phi) \downarrow 2 = \sum_{k} g(2x+k)\phi(k)$$

$$g_{high}(x) = (g \check{\circledast} \psi) \downarrow 2 = \sum_{k} g(2x+k)\psi(k)$$

Inverse Wavelet Transform

- Matrix formulation makes it easy to discover the inverse of DWT.
- The inverse Haar wavelet transform can be rewritten as:

$$g^{(i-1)}(2x) = \frac{1}{2}\left(g_{low}^{(i)}(x) + g_{high}^{(i)}(x)\right)$$

$$g^{(i-1)}(2x+1) = \frac{1}{2}\left(g_{low}^{(i)}(x) - g_{high}^{(i)}(x)\right)$$

Daubechies Wavelets

- A generalization of Haar wavelets are **Daubechies wavelets**—Haar is a special case of Daubechies.
- The key idea behind the Daubechies wavelet is to achieve the highest number of vanishing moments for a defined support width.
- A *vanishing moment* occurs when the moment is zero.
  - The signal bears no resemblance, and therefore the low-order polynomial features of the signal are removed by the wavelet transform, leaving only higher-order features.

2D Wavelet Transform

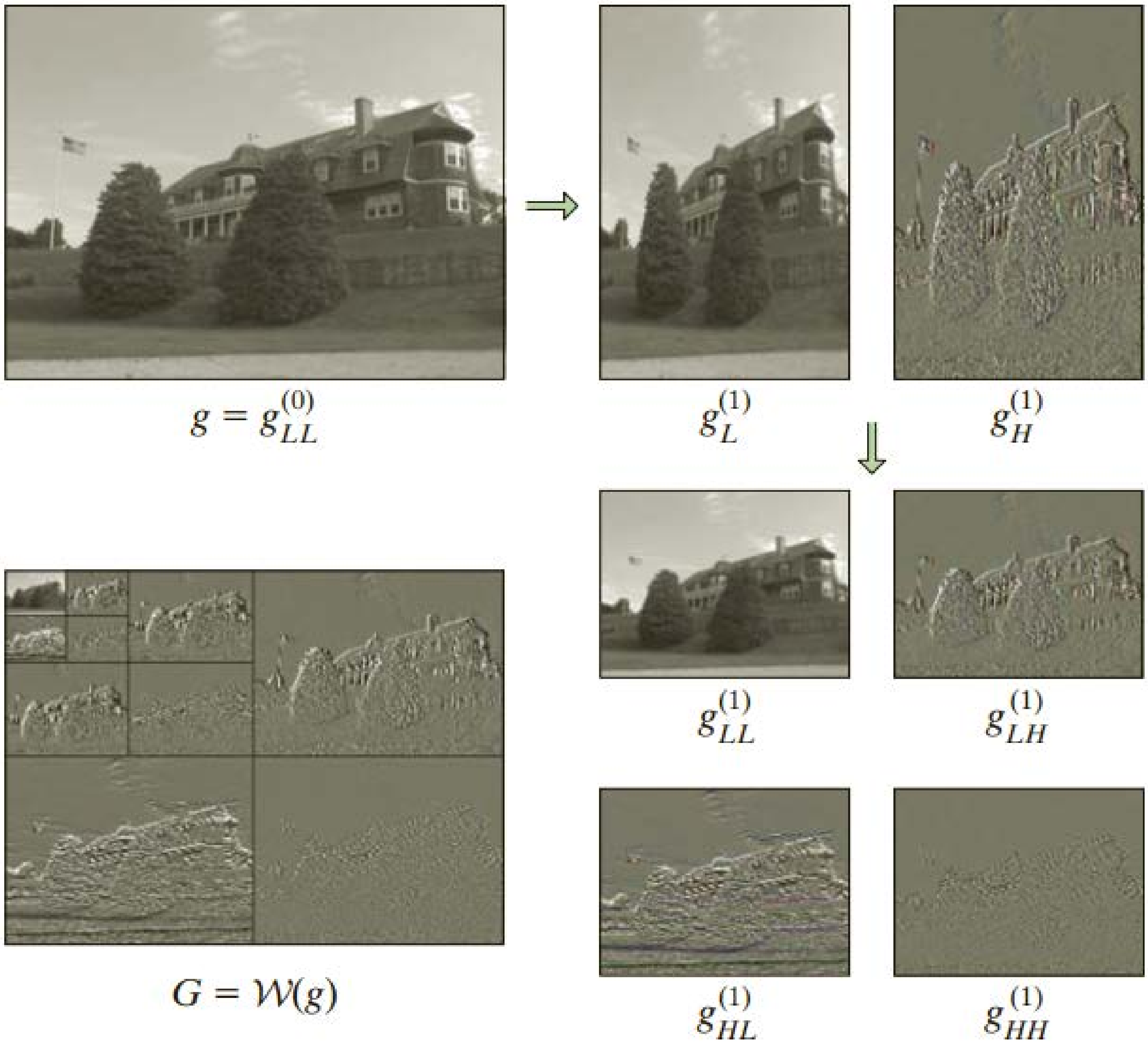
- The 2D wavelets are separable so that they can be expressed as the multiplication of two 1D wavelets

$$g_{LL}^{(i+1)}(x, y) = \sum_{k_x} \sum_{k_y} g_{LL}^{(i)}(2x + k_x, 2y + k_y) \phi(k_x) \phi(k_y)$$

$$g_{HL}^{(i+1)}(x, y) = \sum_{k_x} \sum_{k_y} g_{LL}^{(i)}(2x + k_x, 2y + k_y) \psi(k_x) \phi(k_y)$$

$$g_{LH}^{(i+1)}(x, y) = \sum_{k_x} \sum_{k_y} g_{LL}^{(i)}(2x + k_x, 2y + k_y) \phi(k_x) \psi(k_y)$$

$$g_{HH}^{(i+1)}(x, y) = \sum_{k_x} \sum_{k_y} g_{LL}^{(i)}(2x + k_x, 2y + k_y) \psi(k_x) \psi(k_y)$$


$g = g_{LL}^{(0)}$
$g_L^{(1)}$
$g_H^{(1)}$
$g_{LL}^{(1)}$
$g_{LH}^{(1)}$
$G = \mathcal{W}(g)$
$g_{HL}^{(1)}$
$g_{HH}^{(1)}$

Gabor Wavelets

- **Gabor wavelet**: a complex sinusoid multiplied by a Gaussian window.

- In 1D, the wavelet is given by: $\psi(x) = \underbrace{e^{-\alpha x^2}}_{\text{Gaussian}} \cdot \underbrace{e^{j\omega x}}_{\text{sinusoid}}$

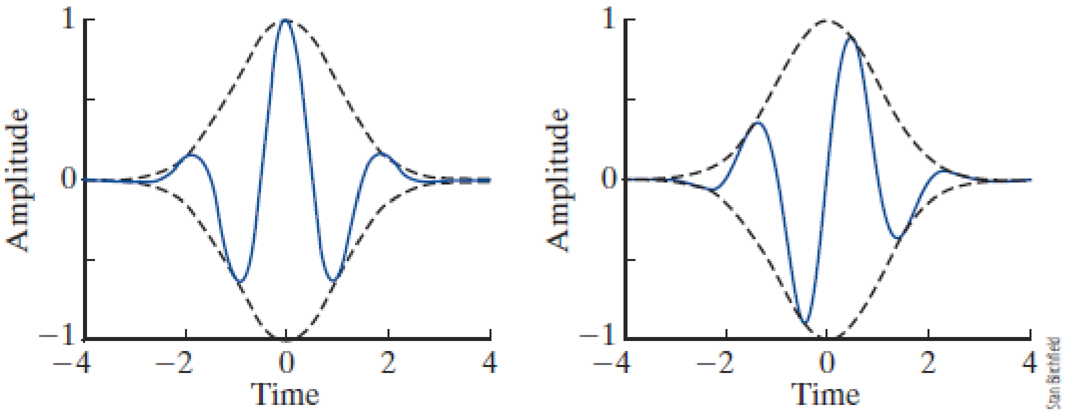
- The Gabor wavelet consists of even and odd components:

$$\psi(x) = \psi_{even}(x) + j\psi_{odd}(x)$$

$$\psi_{even}(x) = e^{-\alpha x^2}\cos(\omega x) \qquad \psi_{odd}(x) = e^{-\alpha x^2}\sin(\omega x)$$

1D Gabor Wavelet

Figure 6.30 1D Gabor wavelet, showing even (left) and odd (right) components, using $\sigma = 1$ and $\tau = 2$.



2D Gabor Wavelets

Figure 6.31 Gabor 2D wavelets are achieved by multiplying a plane wave sinusoid with a Gaussian window function aligned with the direction of the wave propagation. Shown are the even (top) and odd (bottom) components, both as a 3D plot and as an image, using $\sigma = 1, \tau = 2, \theta = 30°$, and $\beta = \alpha/4$.