EEE-6512: Image Processing and Computer Vision

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Lecture #11: Model-Fitting

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Outline

- Overview
- **Fitting Lines**
- Fitting Curves
- Addressing Noise
- Fitting Multiple Models

Line Fitting

Fitting: Methods

- Global optimization / Search for parameters
  - Ordinary Least Squares
  - Total Least Squares

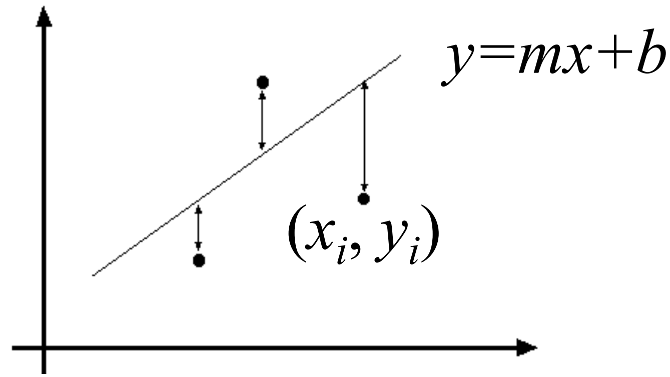
- Hypothesize and test
  - Generalized Hough transform
  - RANSAC

Ordinary Least Squares / Total Least Squares

Least squares line fitting

- Data: $(x_1, y_1), \ldots, (x_n, y_n)$
- Line equation: $y_i = m x_i + b$
- Find (m, b) to minimize

$$\boxed{E = \sum_{i=1}^{n} (y_i - m x_i - b)^2}$$



$$E = \sum_{i=1}^{n} \left(\begin{bmatrix} x_i & 1 \end{bmatrix} \begin{bmatrix} m \\ b \end{bmatrix} - y_i \right)^2 = \left\| \begin{bmatrix} x_1 & 1 \\ \vdots & \vdots \\ x_n & 1 \end{bmatrix} \begin{bmatrix} m \\ b \end{bmatrix} - \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \right\|^2 = \|\mathbf{Ap} - \mathbf{y}\|^2$$

$$= \mathbf{y}^T \mathbf{y} - 2(\mathbf{Ap})^T \mathbf{y} + (\mathbf{Ap})^T (\mathbf{Ap})$$

$$\frac{dE}{dp} = 2\mathbf{A}^T \mathbf{Ap} - 2\mathbf{A}^T \mathbf{y} = 0$$

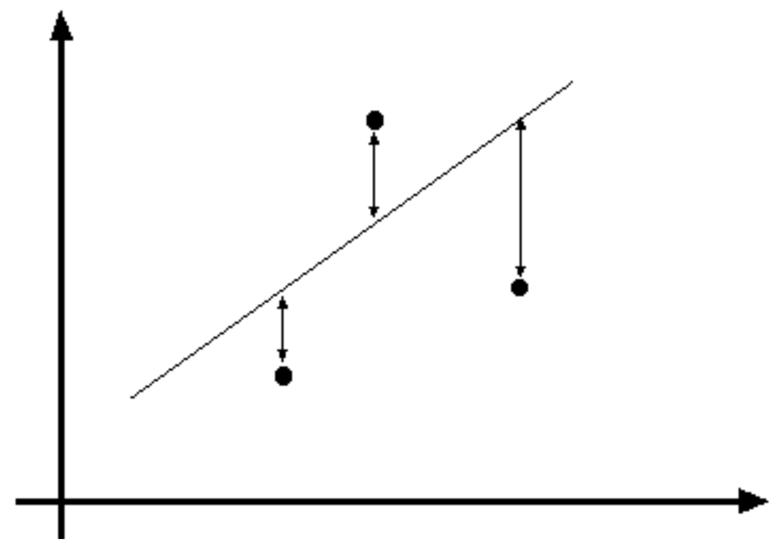
Matlab: `p = A \ y;`

$$\mathbf{A}^T \mathbf{Ap} = \mathbf{A}^T \mathbf{y} \Rightarrow \mathbf{p} = \left(\mathbf{A}^T \mathbf{A}\right)^{-1} \mathbf{A}^T \mathbf{y}$$

Modified from S. Lazebnik

Problem with "vertical" least squares

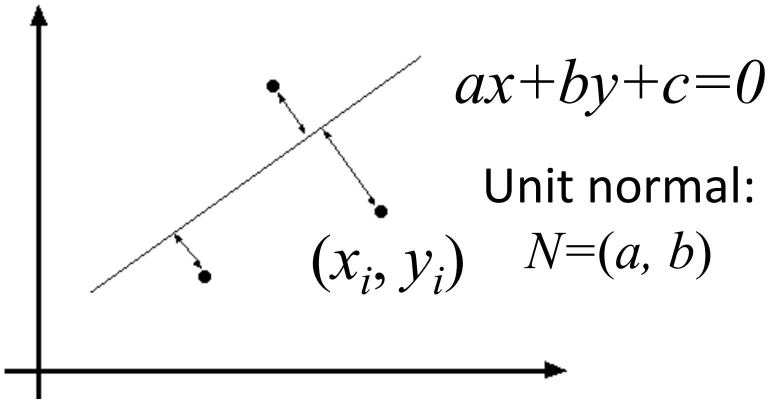
- Not rotation-invariant
- Fails completely for vertical lines



Total least squares

If ($a^2+b^2=1$) then

Distance between point (x_i, y_i) and line $ax+by+c=0$ is $|ax_i + by_i + c|$



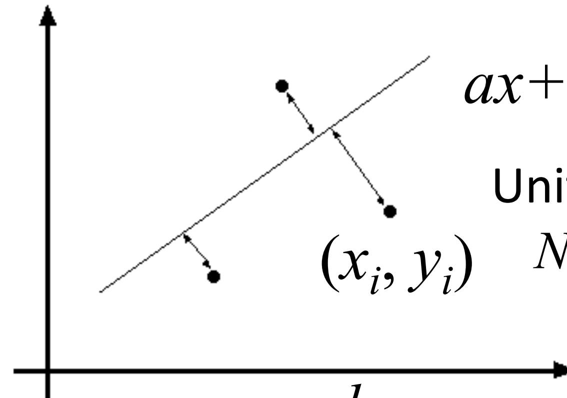
Find (a, b, c) to minimize the sum of squared perpendicular distances

$$\boxed{E = \sum_{i=1}^{n} (ax_i + by_i + c)^2}$$

Total least squares

Find (a, b, c) to minimize the sum of squared perpendicular distances

$$\boxed{E = \sum_{i=1}^{n}(ax_i + by_i + c)^2}$$



$$\frac{\partial E}{\partial c} = \sum_{i=1}^{n} 2(ax_i + by_i + c) = 0 \qquad c = -\frac{a}{n}\sum_{i=1}^{n} x_i - \frac{b}{n}\sum_{i=1}^{n} y_i = -a\bar{x} - b\bar{y}$$

$$E = \sum_{i=1}^{n}(a(x_i - \bar{x}) + b(y_i - \bar{y}))^2 = \left\| \begin{bmatrix} x_1 - \bar{x} & y_1 - \bar{y} \\ \vdots & \vdots \\ x_n - \bar{x} & y_n - \bar{y} \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \right\|^2 = \mathbf{p}^T \mathbf{A}^T \mathbf{A} \mathbf{p}$$

$$\text{minimize } \mathbf{p}^T \mathbf{A}^T \mathbf{A} \mathbf{p} \quad \text{s.t. } \mathbf{p}^T \mathbf{p} = 1 \quad \Rightarrow \quad \text{minimize } \frac{\mathbf{p}^T \mathbf{A}^T \mathbf{A} \mathbf{p}}{\mathbf{p}^T \mathbf{p}}$$

Solution is eigenvector corresponding to smallest eigenvalue of A^TA

Recap: Two Common Optimization Problems

Problem statement

$$\text{minimize } \|\mathbf{Ax}-\mathbf{b}\|^2$$

least squares solution to $\mathbf{Ax}=\mathbf{b}$

Solution

$$\mathbf{x}=\left(\mathbf{A}^T\mathbf{A}\right)^{-1}\mathbf{A}^T\mathbf{b}$$

$\mathbf{x} = \mathbf{A} \setminus \mathbf{b}$ (matlab)

Problem statement

$$\text{minimize } \mathbf{x}^T\mathbf{A}^T\mathbf{Ax} \quad \text{s.t. } \mathbf{x}^T\mathbf{x}=1$$

$$\text{minimize} \frac{\mathbf{x}^T\mathbf{A}^T\mathbf{Ax}}{\mathbf{x}^T\mathbf{x}}$$

non - trivial lsq solution to $\mathbf{Ax}=0$

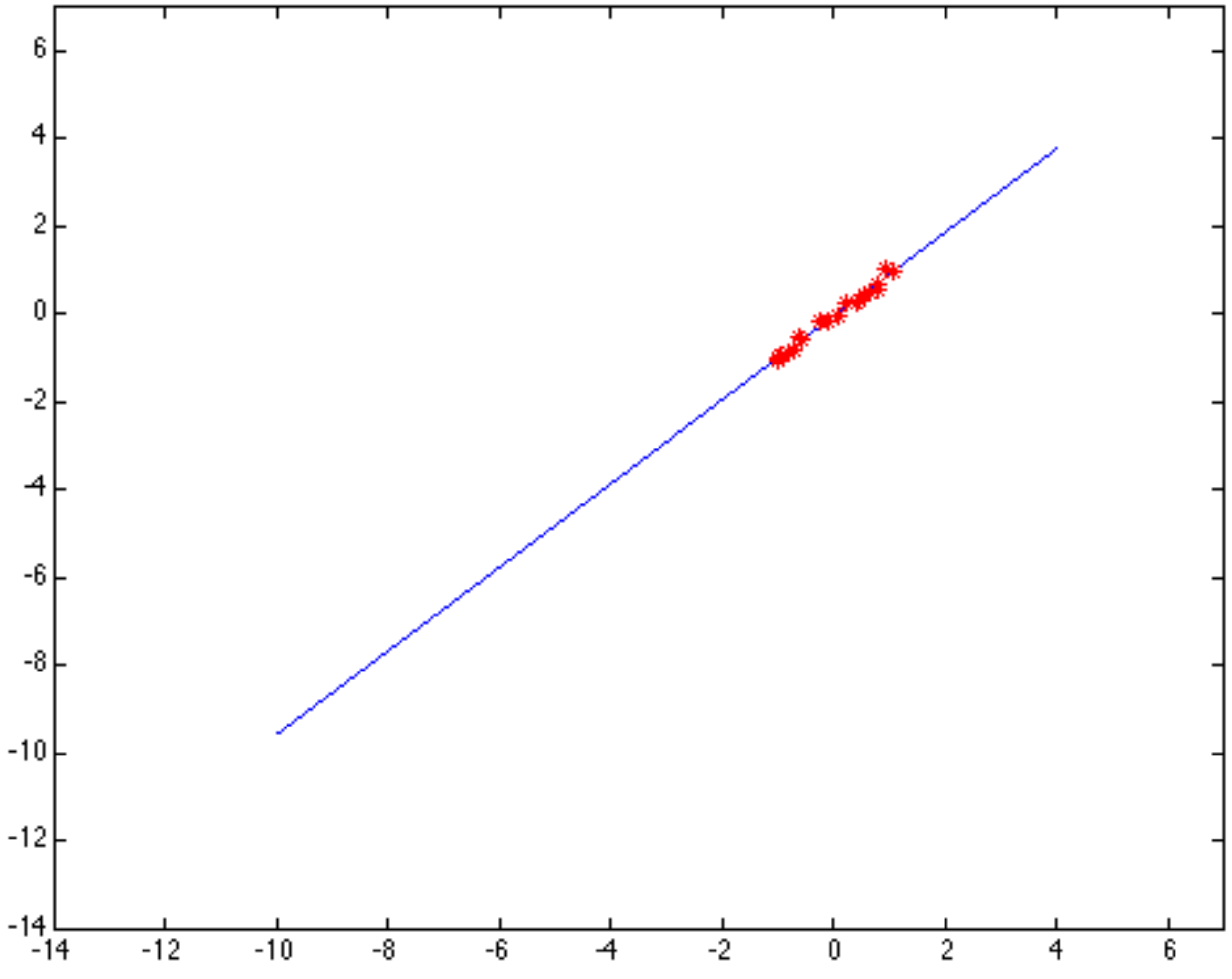
Solution

$$[\mathbf{v},\lambda]=\mathrm{eig}(\mathbf{A}^T\mathbf{A})$$

$$\lambda_1 < \lambda_{2..n} : \mathbf{x}=\mathbf{v}_1$$

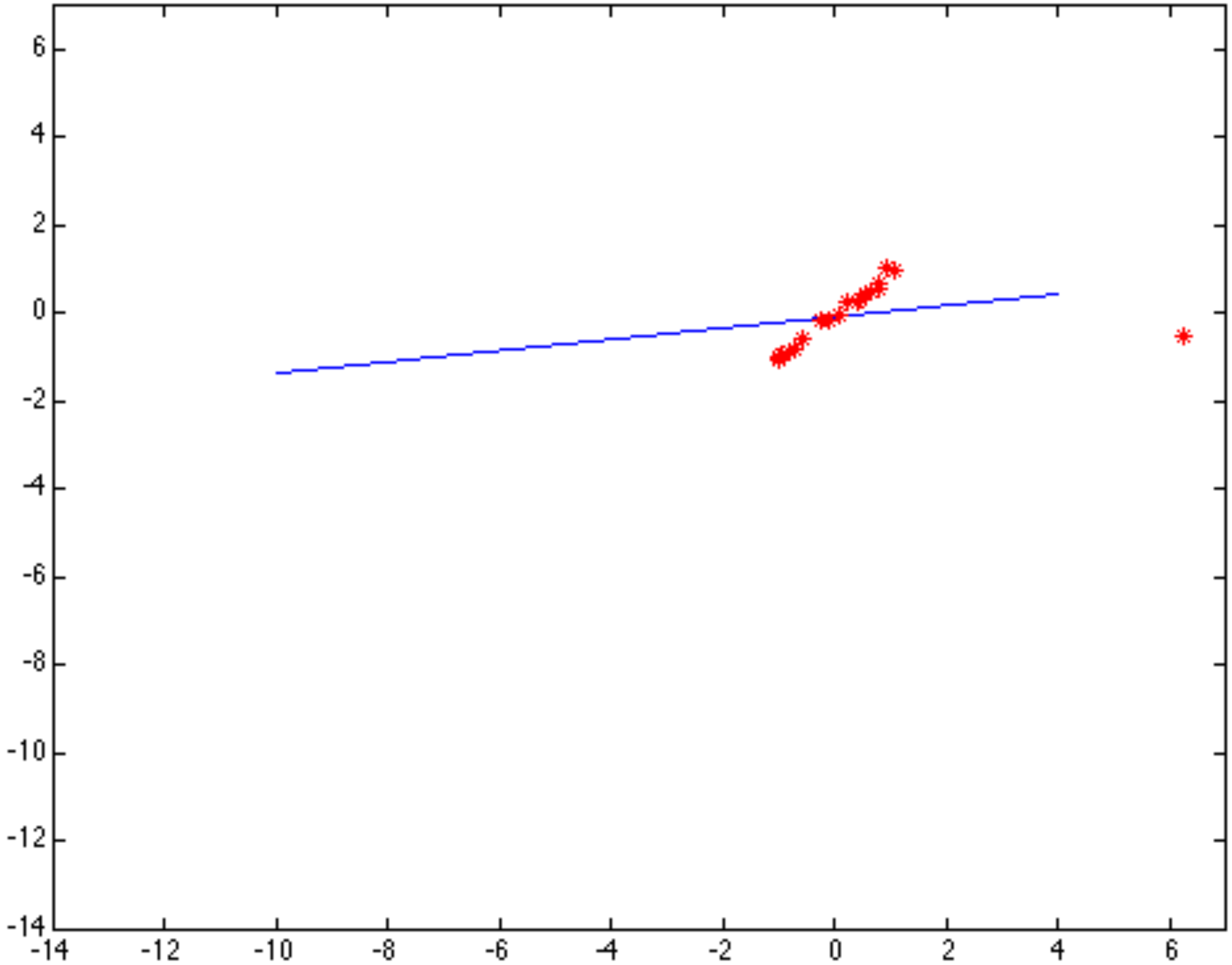
Least squares: Robustness to noise

Least squares fit to the red points:



Least squares: Robustness to noise

Least squares fit with an outlier:



Problem: squared error heavily penalizes outliers

Least squares (global) optimization

Good

- Clearly specified objective
- Optimization is easy

Bad

- Sensitive to outliers
  - Bad matches, extra points
- Doesn't allow you to get multiple good fits
  - Detecting multiple objects, lines, etc.

Other ways to search for parameters
(for when no closed form solution exists)

- **Line search**
  1. For each parameter, step through values and choose value that gives best fit
  2. Repeat (1) until no parameter changes
- **Grid search**
  1. Propose several sets of parameters, evenly sampled in the joint set
  2. Choose best (or top few) and sample joint parameters around the current best; repeat
- **Gradient descent**
  1. Provide initial position (e.g., random)
  2. Locally search for better parameters by following gradient

Hypothesize and Test

Hypothesize and Test

1. **Propose parameters**
   - Try all possible
   - Each point votes for all consistent parameters
   - Repeatedly sample enough points to solve for parameters

2. **Score the given parameters**

   Number of consistent points, possibly weighted by distance

3. **Choose from among the set of parameters**

   Global or local maximum of scores

4. **Possibly refine parameters using inliers**

Hough Transform: Outline

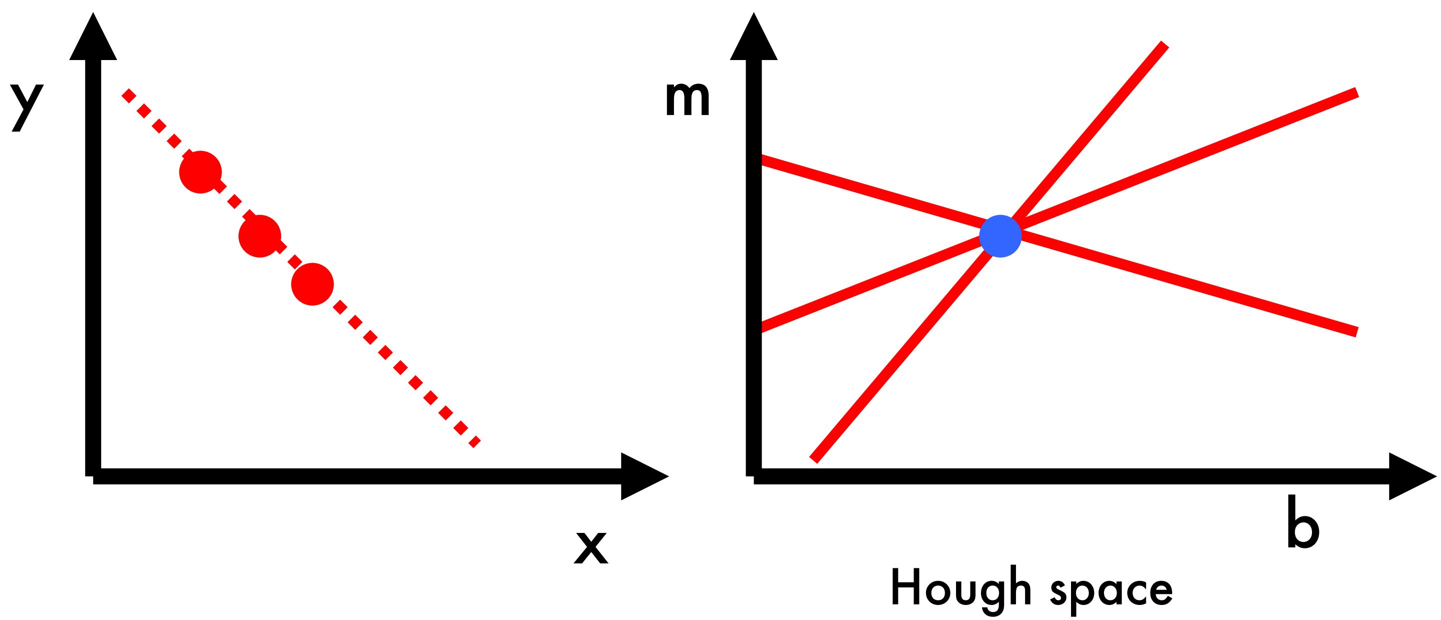
1. Create a grid of parameter values

2. Each point votes for a set of parameters, incrementing those values in grid

3. Find maximum or local maxima in grid

Hough transform

Given a set of points, find the curve or line that explains the data points best

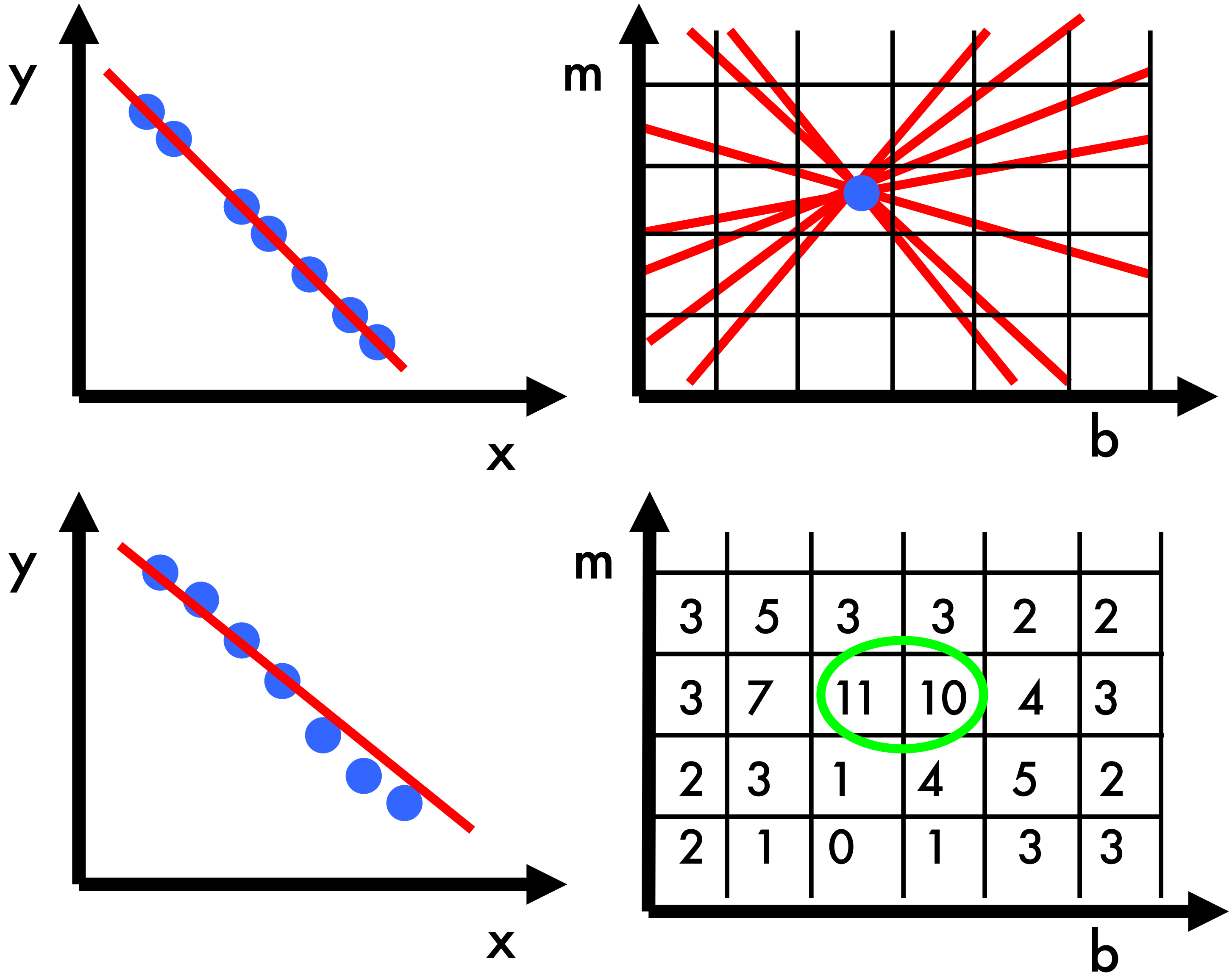


$$y = m\,x + b$$

Slide from S. Savarese

Connection Between Image and Hough Spaces

Hough transform

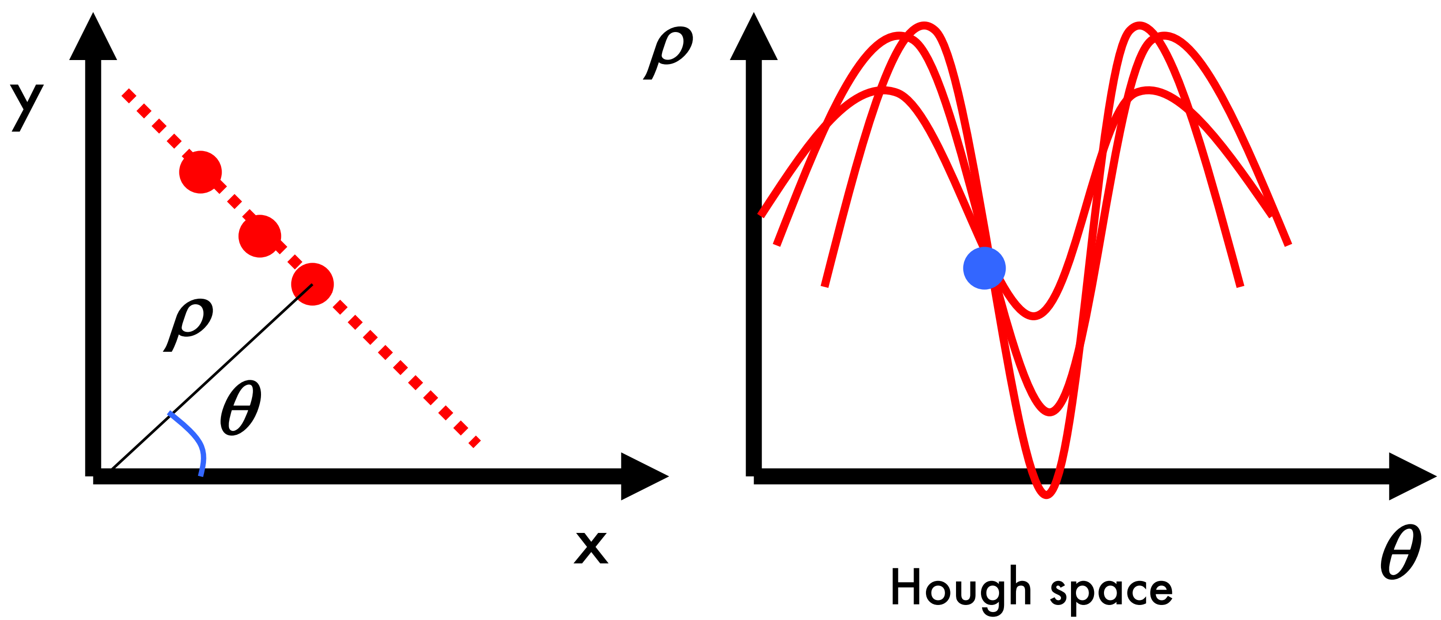


Slide from S. Savarese

Hough transform

Issue : parameter space [m,b] is unbounded…

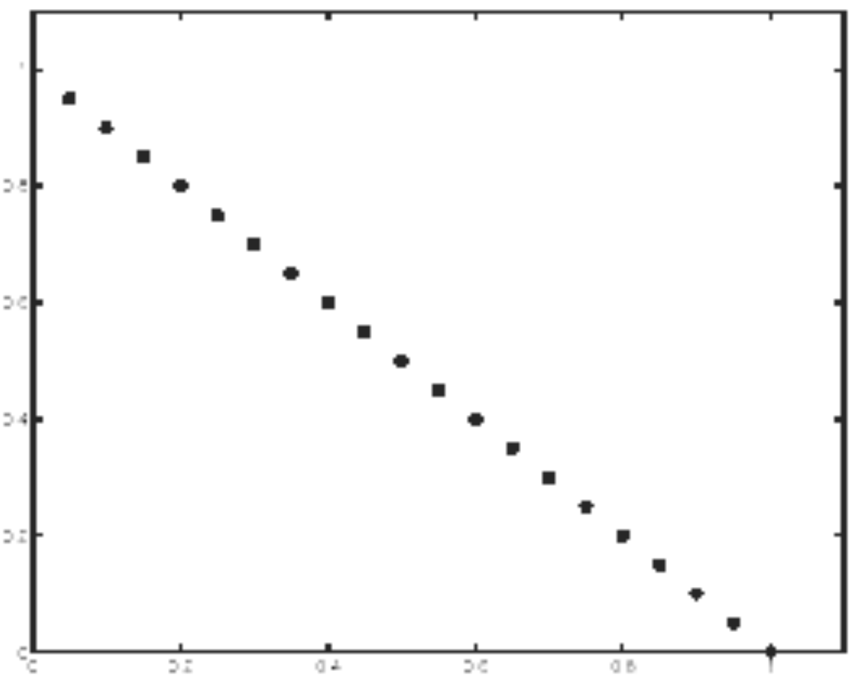
Use a polar representation for the parameter space



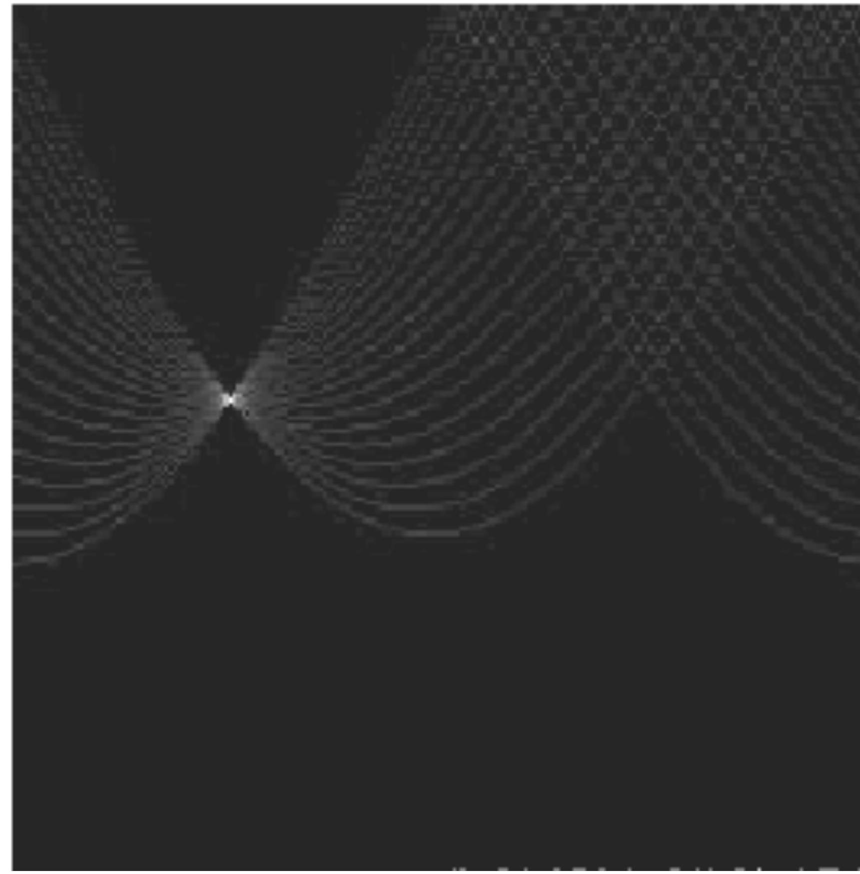
$$x\cos\theta + y\sin\theta = \rho$$

Slide from S. Savarese

Hough transform - experiments



features



votes

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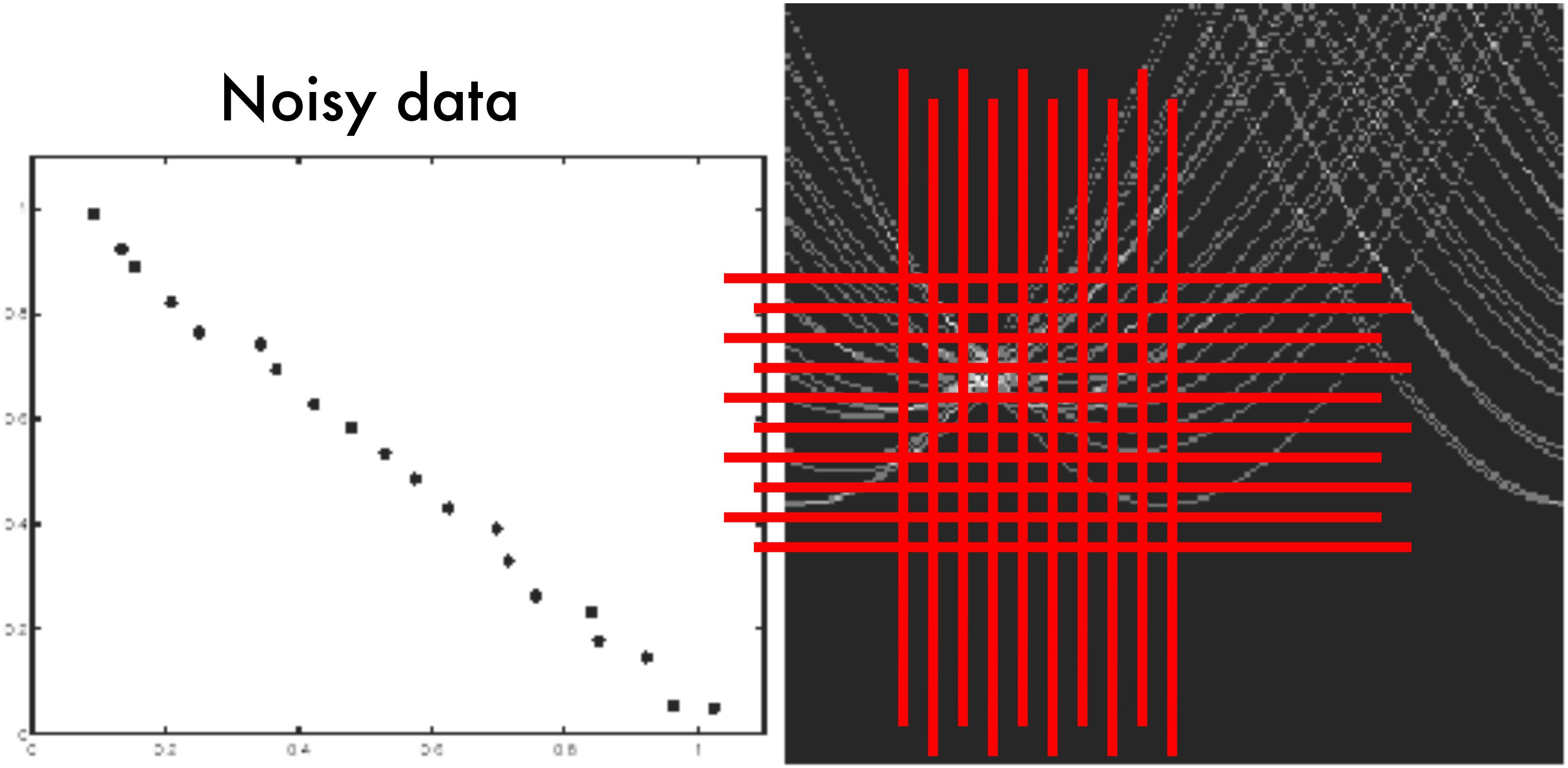
Hough transform - experiments

Noisy data



features

votes

Need to adjust grid size or smooth

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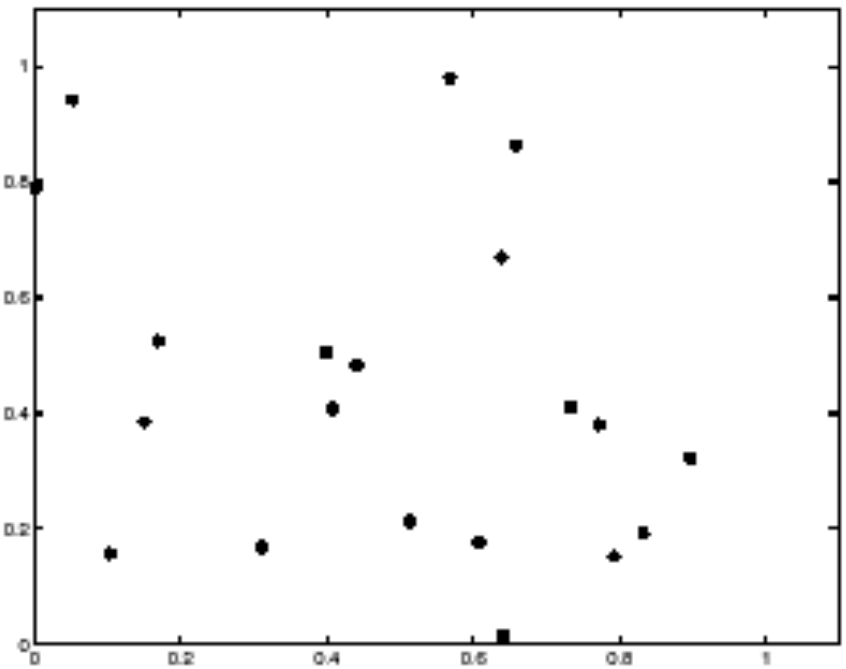
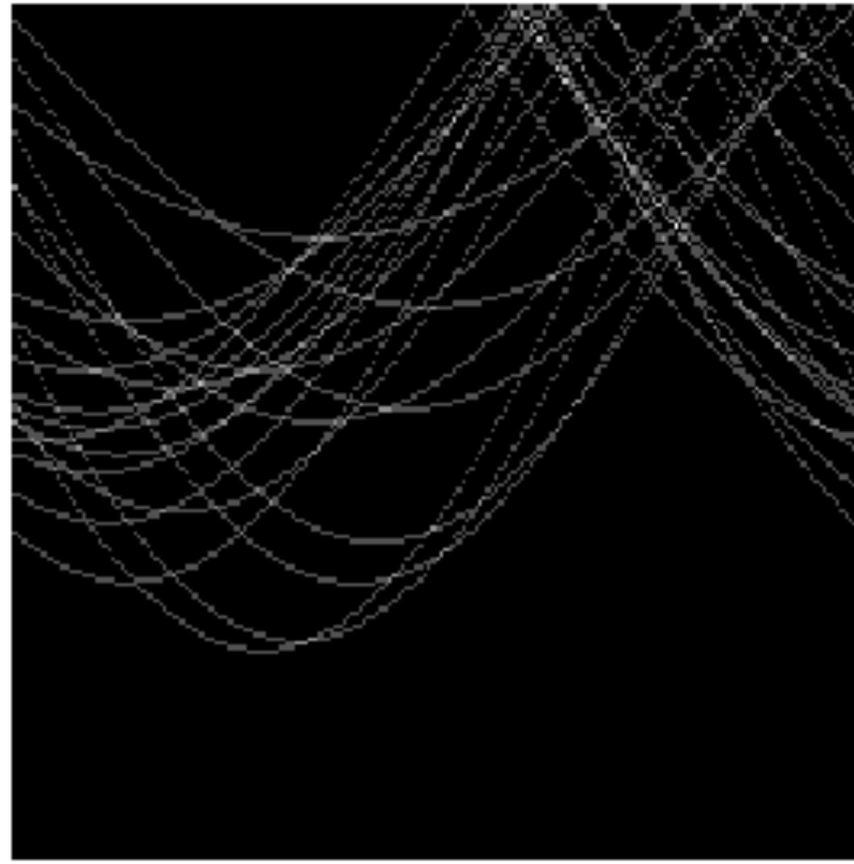
Hough transform - experiments



features



votes

Issue: spurious peaks due to uniform noise

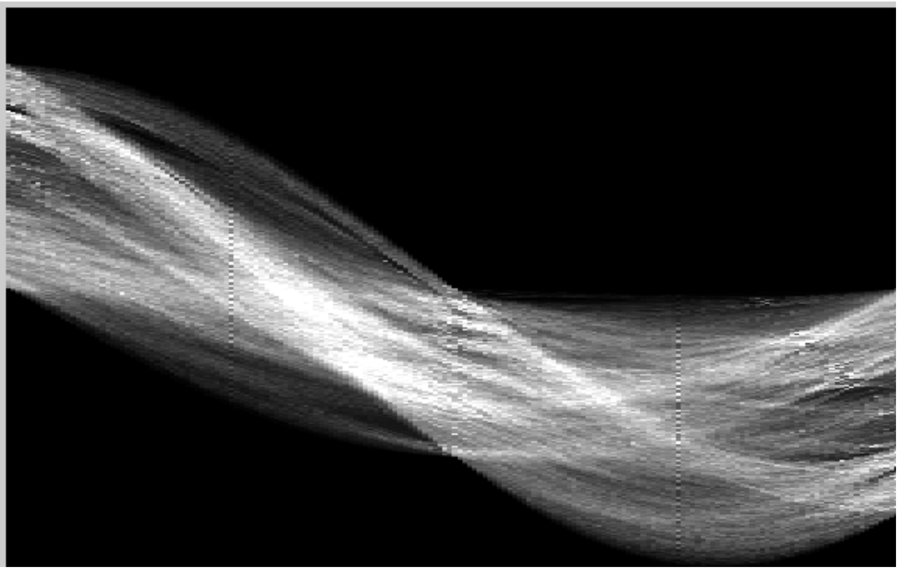
Slide from S. Savarese

1. Image → Canny

2. Canny → Hough votes

3. Hough votes → Edges

Find peaks and post-process



Next Time

- Fitting Curves (Generalized Hough)
- Dealing with Noise (RANSAC)

Questions?

Slide Credits

Some slides from Dr. Stanley Birchfield, Dr. Kristen Grauman, Dr. Svetlana Lazebnik, Dr. Jia-Bin Huang, Dr. Silvio Savarese.