EEE-6512: Image Processing and Computer Vision

October 18, 2017
Lecture #8: Frequency Domain Processing

Damon L. Woodard, Ph.D.
Dept. of Electrical and Computer Engineering
dwoodard@ece.ufl.edu

Lecture Outline

- Preliminaries
- Fourier Transform
- Discrete Fourier Transform (DFT)
- Two-Dimensional DFT
- Frequency-Domain Filtering
- Localizing Frequencies in Time
- Discrete Wavelet Transform

Preliminaries

How to Represent Signals?

- Option 1: Taylor series represents any function using polynomials.

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \ldots .$$

- Polynomials are not the best - unstable and not very physically meaningful.
- Easier to talk about “signals” in terms of its “frequencies”

  (how fast/often signals change, etc).

Jean Baptiste Joseph Fourier (1768-1830)

- Had crazy idea (1807):
- **Any** periodic function can be rewritten as a weighted sum of Sines and Cosines of different frequencies.
- Don’t believe it?
  - Neither did Lagrange, Laplace, Poisson and other big wigs
  - Not translated into English until 1878!
- But it’s true!
  - called Fourier Series
  - Possibly the greatest tool used in Engineering

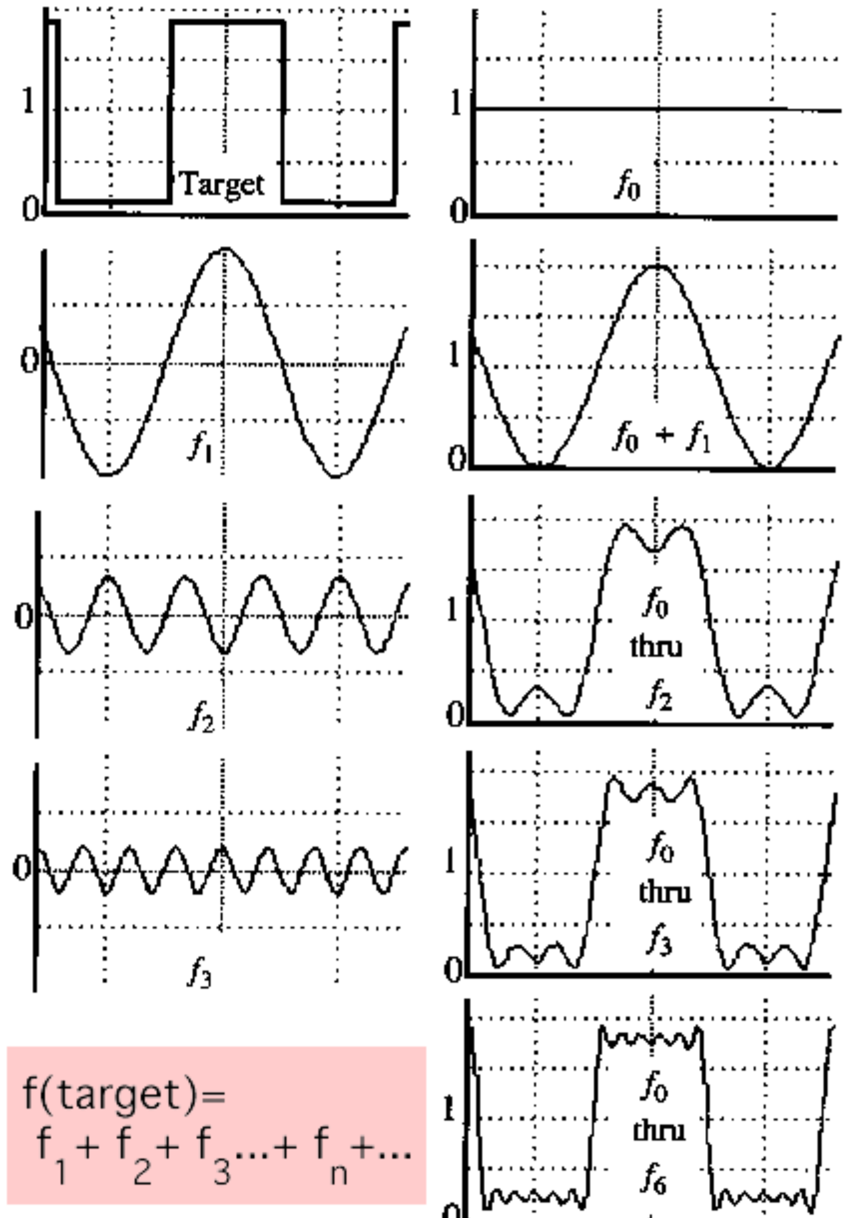
A Sum of Sinusoids

- Our building block:

$$A\sin(\omega x+\phi)$$

- Add enough of them to get any signal *f(x)* you want!

- How many degrees of freedom?



f(target)=
$f_1 + f_2 + f_3 ... + f_n + ...$

Why would we want to work in frequency domain?

- Convolution in spatial domain is computationally more costly in the spatial domain. (Convolution = Multiplication in Frequency Domain)
- Frequency Domain filtering is much more efficient in convolving images with large kernels.
- Frequency filtering ignores the presence of object regions in the image and processes the entire image at once. This is useful in terms of noise reduction, smoothing, compression, etc.

Fourier Transform

Forward Transform

- **Fourier transform *G*(*f*):** the integration of the signal after first multiplying by a certain complex exponential:

$$G(f) \equiv \mathcal{F}\{g\} \equiv \int_{-\infty}^{\infty} g(t)e^{-j2\pi ft}\,dt$$

- If *t* is measured in seconds, then *f* is measured in inverse seconds, also known as hertz.

Forward Transform (cont'd)

- By applying **Euler's formula:**

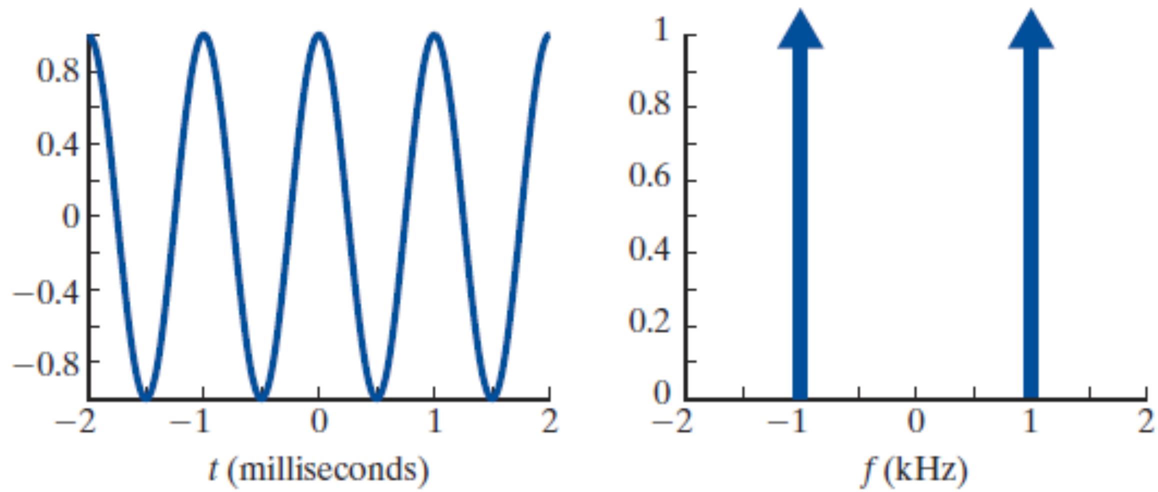
$$G(f) = \underbrace{\int_{-\infty}^{\infty} g(t)\cos 2\pi ft\,dt}_{G_{even}} + j\underbrace{\int_{-\infty}^{\infty} -g(t)\sin 2\pi ft\,dt}_{G_{odd}}$$

- Or

$$G(f) = G_{even}(f) + jG_{odd}(f)$$

Forward Transform (cont'd)

Figure 6.1 A continuous time-domain signal (left) and its Fourier transform (right). The latter reveals that the signal is a pure sinusoid with frequency 1000 Hz, since it contains two infinite spikes (Dirac deltas) at $f = 1$ kHz and $f = -1$ kHz. Note that the multiplicative factor $\frac{1}{2}$ has no effect on the display. See the text for an explanation of the negative frequency.



$\cos 2000\pi t$

$\frac{1}{2}(\delta(f-1000)+\delta(f+1000))$

Inverse Transform

- An **inverse Fourier transform** is defined in exactly the same way as the forward Fourier transform except for the sign in the exponent, and the fact that the integral is computed over frequency rather than over time:

$$g(t) = \mathcal{F}^{-1}\{G\} \equiv \int_{-\infty}^{\infty} G(f) e^{j2\pi ft}\, df$$

Sampling and Aliasing

- Some information is lost in the process of sampling, thereby making it impossible to reconstruct the original signal from its samples.

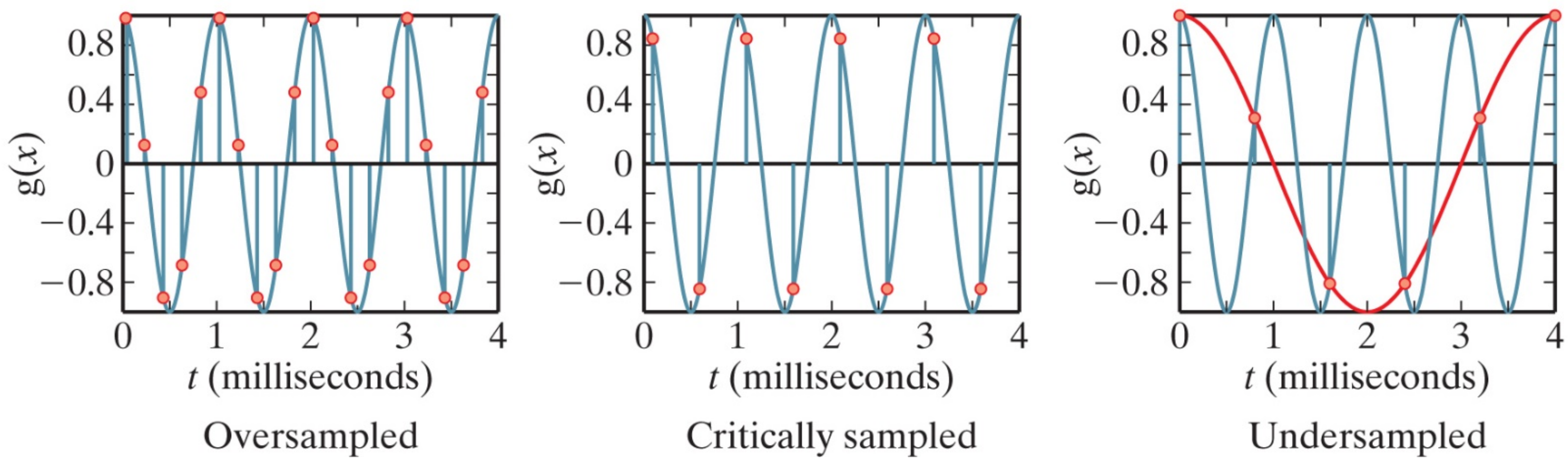
- **Nyquist-Shannon sampling theorem**: if a certain condition holds true, then the discrete samples contain just as much information as the original signal, so that the original signal can be reconstructed *exactly* from the discrete samples.
  - The sampling rate must be greater than the **Nyquist rate**, which is twice the highest frequency in the signal.

Sampling and Aliasing (cont'd)

- **Oversampled:** when the sampling frequency is greater than the Nyquist rate.
  - Perfect reconstruction is possible.
- **Undersampled:** when the sampling frequency is lower than the Nyquist rate.
  - Important information about the signal is irrecoverably lost.
  - When a signal is undersampled, **aliasing** occurs.
- **Critically sampled:** when the sampling frequency is exactly the Nyquist rate.
  - The original signal is also unrecoverable.

Sampling and Aliasing (cont'd)

Figure 6.2 A continuous 1 kHz time-domain sinusoid sampled with 3 different sampling frequencies: 5000 Hz (left), 2000 Hz (middle), and 1250 Hz (right). The Nyquist rate, which is twice the frequency of the signal, is 2000 Hz. Sampling at higher than the Nyquist rate preserves the information in the signal, while sampling at lower than the Nyquist rate leads to aliasing. In this case the frequency of the aliased signal (red curve) is 250 Hz.



Four Versions of the Fourier Transform

	continuous	discrete
infinite duration	**Fourier transform** $G(f) = \int_{-\infty}^{\infty} g(t) e^{-j2\pi ft}\, dt$ $g(t) = \int_{-\infty}^{\infty} G(f) e^{j2\pi ft}\, df$ $t \in \mathbb{R} \quad f \in \mathbb{R}$	**Discrete-time Fourier transform (DTFT)** $G(f) = \sum_{x=-\infty}^{\infty} g(x) e^{-j2\pi fx}$ $g(x) = \int_{-\frac{1}{2}}^{\frac{1}{2}} G(f) e^{j2\pi fx}\, df$ $x \in \mathbb{Z} \quad f \in [-\frac{1}{2}, \frac{1}{2}]$
finite duration (periodic)	**Fourier series** $G(k) = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} g(t) e^{-j2\pi kt/T}\, dt$ $g(t) = \sum_{k=-\infty}^{\infty} G(k) e^{j2\pi kt/T}$ $t \in [-\frac{T}{2}, \frac{T}{2}] \quad k \in \mathbb{Z}$	**Discrete Fourier transform (DFT)** $G(k) = \sum_{x=0}^{w-1} g(x) e^{-j2\pi kx/w}$ $g(x) = \frac{1}{w} \sum_{k=0}^{w-1} G(k) e^{j2\pi kx/w}$ $x \in \mathbb{Z}_{0:w-1} \quad k \in \mathbb{Z}_{0:w-1}$

TABLE 6.1 The four versions of the Fourier transform.

Forward Transform

- Let *g*(*x*) be a 1D discrete signal with *w* samples. The DFT of *g* is defined as the summation of the signal after multiplying by a certain complex exponential:

$$G(k) = \mathcal{F}\{g(x)\} = \sum_{x=0}^{w-1} g(x) e^{-j2\pi kx/w}$$

where *x* and *k* are integers.

- Input is real-values and output is complex.
- All modern implementations of the DFT use some variation of the FFT algorithm.

Inverse Transform

- Given the DFT of a signal, the original signal can be recovered by applying the **inverse DFT**:

$$g(x) = \mathcal{F}^{-1}\{G(k)\} = \frac{1}{w}\sum_{k=0}^{w-1} G(k)e^{j2\pi kx/w}$$

Properties

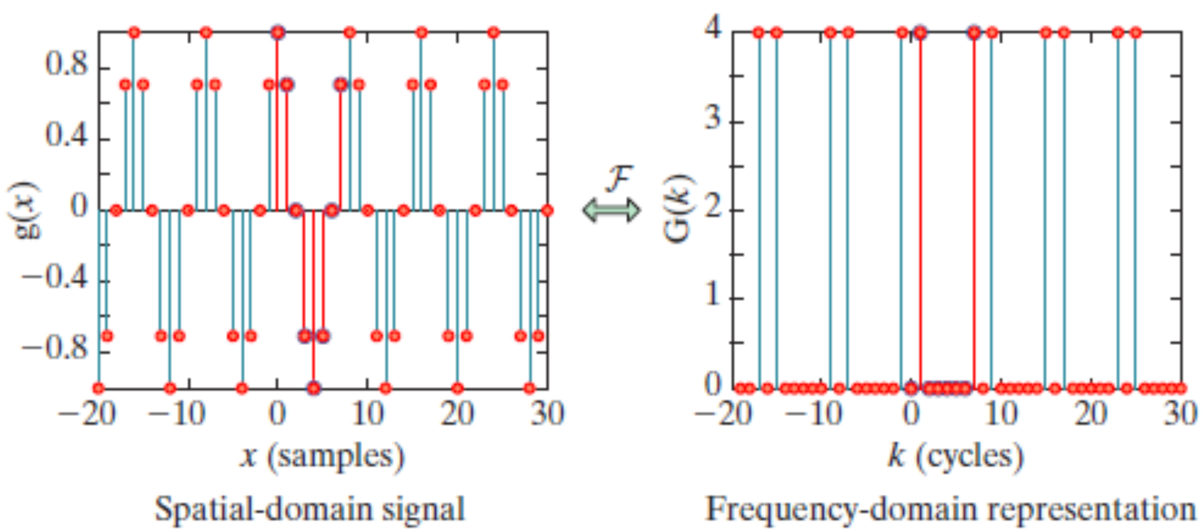
- The DFT is **linear**.

$$\mathcal{F}\{ag(x) + bh(x)\} = a\mathcal{F}\{g(x)\} + b\mathcal{F}\{h(x)\}$$

- The DFT is **periodic**.

$$g(x + nw) = g(x) \overset{DFT}{\Longleftrightarrow} G(k) = G(k + nw), \quad x, k, n, w \in \mathbb{Z}$$

Figure 6.3 Periodicity of the DFT. The discrete signal consisting of eight samples $x = 0, \ldots, 7$ (red, left) gives rise to the DFT consisting of eight samples $k = 0, \ldots, 7$ (red, right). If the DFT is evaluated for other values of k, or if the inverse DFT of the DFT is evaluated for other values of x, the signal repeats with period $w = 8$.



Properties (cont'd)

- **Shift theorem**: computing the DFT of a shifted signal is the same as multiplying the DFT of the original, unshifted signal by an appropriate complex exponential.

$$g(x) \overset{DFT}{\Longleftrightarrow} G(k)$$

$$g(x - x_0) \overset{DFT}{\Longleftrightarrow} G(k)e^{-j2\pi k x_0/w}$$

- **Modulation**: states that multiplying a signal by a complex exponential causes a shift in the frequency domain:

$$g(x) \overset{DFT}{\Longleftrightarrow} G(k)$$

$$g(x)e^{j2\pi k_0 x/w} \overset{DFT}{\Longleftrightarrow} G(k - k_0)$$

$$g(x)(-1)^x \overset{DFT}{\Longleftrightarrow} G\left(k - \frac{w}{2}\right)$$

Properties (cont'd)

- The **scaling property** says that if the signal is stretched in the spatial domain, then the Fourier transform is compressed in the frequency domain, and vice versa:

$$g(x) \overset{\mathcal{F}}{\Longleftrightarrow} G(k)$$
$$g(ax) \overset{\mathcal{F}}{\Longleftrightarrow} \frac{1}{a}G\left(\frac{k}{a}\right)$$

- The DFT of a real-valued signal exhibits **Hermitian** symmetry.
  - Its real component is even-symmetric, and its imaginary component is odd-symmetric.

Properties (cont'd)

- The DFT of a real-valued, even-symmetric signal is also real-valued and even-symmetric.
- **Parseval's theorem:** the energy is preserved in the frequency domain, where the energy is defined as the sum of the squares of the magnitudes of the elements:

$$\sum_{x=0}^{w-1} |g(x)|^2 = \sum_{k=0}^{w-1} |G(k)|^2$$

Properties (cont'd)

- The **DC component** of the signal is captured by $G(0)$, which is the sum of the values in *g(x).*

- Convolution in the time (or spatial) domain is equivalent to multiplication in the frequency domain, and vice versa:

$$g_1(x) \circledast g_2(x) \quad \stackrel{DFT}{\Longleftrightarrow} \quad G_1(k)G_2(k)$$

$$g_1(x)g_2(x) \quad \stackrel{DFT}{\Longleftrightarrow} \quad \frac{1}{w}G_1(k) \circledast G_2(k)$$

Zero Padding

- Multiplication in the frequency domain is equivalent to *circular* convolution in the spatial domain.
- If regular convolution is desired, we must first zero pad one of the signals.
- If the two signals are of different lengths, then their Fourier transforms will have different lengths, thus precluding their multiplication; zero-padding is the answer!

Magnitude and Phase

- It is convenient to convert the real and imaginary components of the Fourier transform into **polar coordinates**:

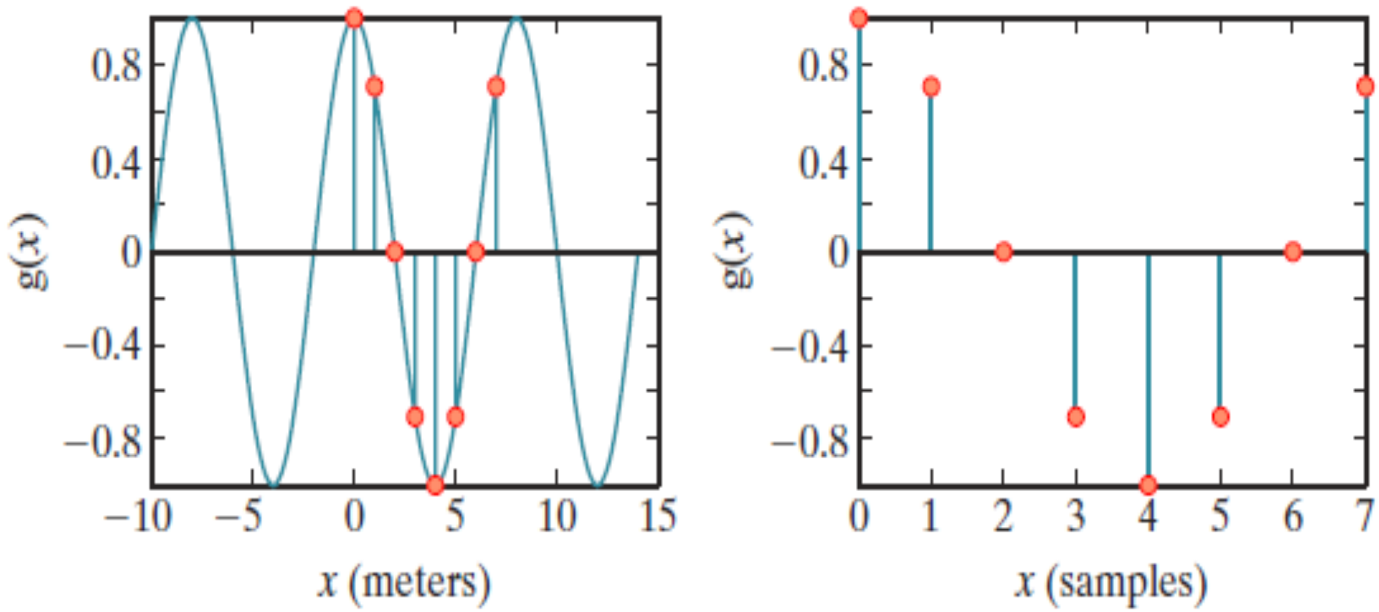
$$G(k) = G_{even}(k) + jG_{odd}(k) = |G(k)|e^{j\angle G(k)}$$

$$|G(k)| = \sqrt{G_{even}^2(k) + G_{odd}^2(k)}$$

$$\angle G(k) = \tan^{-1}\left(\frac{G_{odd}(k)}{G_{even}(k)}\right)$$

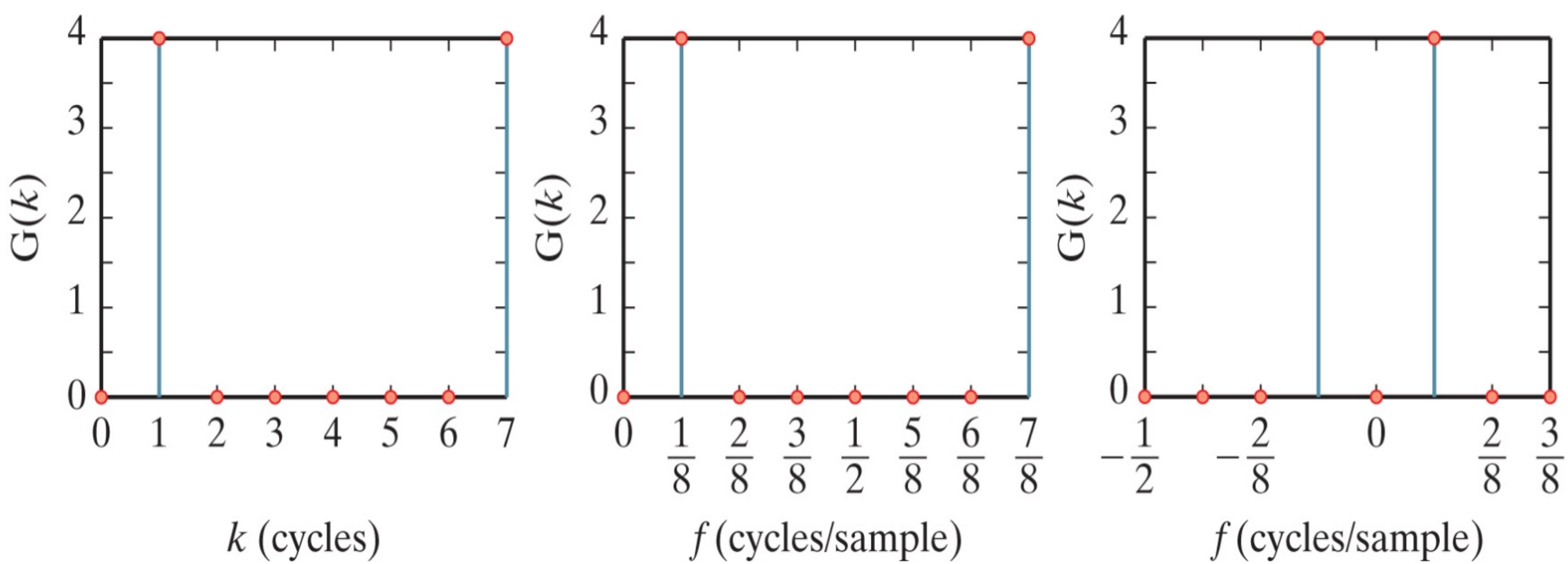
Interpreting Discrete Frequencies

Figure 6.4 LEFT: A continuous spatial-domain signal $\cos \frac{2\pi}{8} x$ is sampled at locations $x = 0, 1, \ldots, 7$. RIGHT: The discrete signal resulting from the sampling. Note that the units for the domain have changed from meters to samples.



Interpreting Discrete Frequencies (cont.)

Figure 6.5 LEFT: The DFT of the discrete signal shown in the right side of Figure 6.4, shown as a function of the discrete index k. MIDDLE: The DFT shown as a function of $f = k/w$, where $w = 8$ is the number of samples in the original discrete signal. RIGHT: The DFT of the discrete signal shifted to show positive and negative frequencies.



DFT Basis Functions

- DFT illustrates a fundamental concept in signal analysis
- A **basis function** is a scalar function defined over the same domain as the original signal that, when linearly combined with other basis functions, yields the signal:

$$g(x) = \sum_{k} \alpha_k \psi_k(x)$$

- The basis functions define a transform such that:

$$\alpha_k = \sum_{x=0}^{w-1} g(x) e_k(x)$$

$$g(x) = \sum_{k=0}^{w-1} \alpha_k e_k(x)$$

Basis Function (cont.)

- The forward DFT performs an analysis of the signal by determining the contributions of the various frequencies in the signal.

- The inverse DFT performs a synthesis of the signal as a weighted sum of sines and cosines. The sines and cosines at different frequencies are the basis for the functions of the DFT.

Questions?