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ANALYTICAL SOLUTION OF LORENZ EQUATION USING HOMOTOPY ANALYSIS METHOD

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Abstract: In this study, a dynamical system of Lorenz equation is discussed. The main aim of this paper is to describe the nonlinear dynamics for the better understanding in biomedical field. Approximate analytical solution of Lorenz equation is obtained by using the Homotopy analysis method (HAM). Furthermore, in this work the numerical simulation of the problem is also reported using Scilab/Matlab program. An agreement between analytical and numerical results is noted.

Keywords: Lorenz equation, Homotopy analysis method, Mathematical modeling, Non-linear equation.

1. INTRODUCTION

The Lorenz equation has had a significant contribution to mathematics and many other disciplines. In particular, the Lorenz equation helped pioneer the study of chaos [1] and sensitive dependence on initial conditions. This equation behaves like any other family of equations, in that it has fixed and bifurcation points, which can be graphed accordingly. Unlike other families, however, this equation is chaotic for certain parameter values. When it is chaotic, it is known as the Lorenz attractor, which has certain chaotic properties. The Lorenz equation has made quantifying chaos possible which has inspired many mathematicians to research and study chaos [2]. Chaotic systems have many applications, in particular to nonlinear equations. Chaotic systems can be used to explain topics in engineering, geography and even the stock market [3].

The idea behind the Lorenz equation came in 1961, when Lorenz ran his program with data rounded off from a previous experiment. He recalls the moment he realised the 'chaos' present in weather systems:

“I typed in some of the intermediate conditions which the computer had printed out as new initial conditions to start another computation and then went out for a while. Afterwards, I found that the solution was not the same as the one I had before. I soon found that the reason was that the numbers I had typed in were not the same, but were rounded off numbers. The small difference between something retained to six decimal places and rounded off to three had amplified in the course of two months of simulated weather until the difference was as big as the signal itself. And to me this implied that if the real atmosphere behaved in this method then we simply couldn't make forecasts two months ahead. The small errors in observation would amplify until they became large.” [4]. This occurrence would become the basis for the Lorenz equation. The fact that two weather conditions, which

differ by less than 0.1%, can produce drastically different results underscores how chaos and sensitive dependence on initial conditions are integrated into meteorology.

In March 1963, Lorenz wrote that he wanted to introduce, “ordinary differential equations whose solutions afford the simplest example of deterministic non periodic flow and finite amplitude convection”. In his paper, he examines the work of meteorologist Barry Saltzman and physicist John Rayleigh while incorporating several physical phenomena [5-6]. Lorenz found that when applying the Fourier series to one of Rayleigh's convection equations that, “. . . all except three variables tended to zero, and that these three variables underwent irregular, apparently non periodic functions”. He then used these variables to construct a simple model based on the 2-dimensional representation of the earth's atmosphere.

Włodzimierz Klonowski et.al [7] describes the overview of nonlinear dynamics and its application in biomedicine like monitoring the depth of anaesthesia and of sedation, bright light therapy and seasonal affective disorder, analysis of posturographic signals, evoked EEG and photo-simulation and influence of electromagnetic fields generated by cellular phones using Lorenz equation. The purpose of this paper is to provide analytical expressions for the variables involved in Lorenz equation. We also investigate the influence of the constants over the state variable of the parameters of Lorenz equation.

2. MATHEMATICAL MODELLING

Let us consider the differential equations as follows [7]

$$\frac{dx}{dt} = -ax + ay \quad (1)$$

$$\frac{dy}{dt} = bx - y - xz \quad (2)$$

$$\frac{dz}{dt} = -cz + xy \quad (3)$$

where a, b, c are parameters. Initial condition for Eqn. (1) to (3)

$$\text{At } t = 0 \quad x(t) = 1, y(t) = 1, z(t) = 1 \quad (4)$$

3. ANALYTICAL SOLUTION OF LORENZ EQUATION USING HOMOTOPY ANALYSIS METHOD

Liao [8-10] proposed a powerful analytical method for solving the nonlinear problems, namely the Homotopy analysis method (see Appendix A). Different from all perturbation and non-perturbative techniques, the Homotopy analysis method [11-13] itself provides us with a convenient way to control and adjust the convergences region and rate of approximation series, when necessary. Briefly speaking, the Homotopy analysis method has the following advantages: It is valid even if a given nonlinear problem does not contain any small/large parameters at all; it can be employed to efficiently approximate a nonlinear problem by choosing different sets of base functions.

Now, more and more researchers have been successfully applying this method to various nonlinear problems in science and engineering. In this paper we employ HAM to solve the nonlinear differential equations (Eqns. (3) - (5)). The basic concept of Homotopy analysis method is given in Appendix A. Using Homotopy analysis method (see Appendix -B) we can obtain the following new approximate solution by solving the equations (1), (2) and (3).

$$x(t) = e^{-at} + \frac{ah}{(a-1)}(e^{-at} - e^{-t}) \quad (5)$$

$$y(t) = e^{-t} + \frac{bh}{(1-a)}(e^{-t} - e^{-at}) + \frac{h}{(1-a-c)}(e^{-t(a+c)} - e^{-t}) \quad (6)$$

$$z(t) = e^{-ct} + \frac{h}{(c-a-1)}(e^{-ct} - e^{-t(a+1)}) \quad (7)$$

When t is small, the above equation becomes,

$$x(t) = 1 - at(1+h) \quad (8)$$

$$y(t) = 1 - t[1 + h(b-1)] \quad (9)$$

$$z(t) = 1 - t(c+h) \quad (10)$$

4. NUMERICAL SIMULATION

In order to investigate the accuracy of the HAM solution with a finite number of terms, the system of differential equations were solved numerically. To show the efficiency of the present method for our problem in comparison with the numerical solution (Matlab program), we report our results graphically. The function ode45 (Range-Kutta method) in Matlab software [17] which is a function of solving the initial value problems is used to solve Eqns. (1) to (3). The Matlab program [18] is also given in Appendix C. The numerical results are also compared with the analytical solution obtained by using HAM method.

5. RESULT AND DISCUSSION

Equations (5), (6) and (7) are the new and simple analytical expressions of state variable for all values of parameters a , b and c . Figure (1), represents the state variable x versus time t for small values of constants a , b and c . From this figure, it is inferred that the state variable x decreases when the constant a increases for the fixed values of $b = 1.2$ and $c = 0.2$. In this figure, it is also evident that the value of state variable x gradually decreases as t increases. Figure (2) represents the state variable y versus time t for fixed value of b and some different values of the constants a and c . It is clear that the state variable y decreases when the constant a increases at the same time another constant c decreases for the fixed value of $b = 1.5$. Figure (3) represents the state variable z versus time t for small values of constants a , b and c . It is clear that the state variable z decreases when the constant c increases for the fixed values of $a = 2$ and $b = 0.01$. But when time t increases the state variables decreases gradually and reaches the steady state.

6. CONCLUSIONS

Approximate analytical solutions of the Lorenz equations are presented using Homotopy analysis method. A simple, straight forward and a new method of estimating the state variables are derived. This solution procedure can be easily extended to all kinds of system of non-linear differential equations with various complex boundary conditions in enzyme-substrate reaction diffusion

processes. Furthermore, in this work the numerical simulation of the problem is also reported using Scilab/Matlab program. An agreement between analytical and numerical results is noted.

APPENDIX A:

Basic concepts of Liao's Homotopy Analysis Method

Consider the following differential equation [15]:

$$N[u(x)] = 0 \quad (A1)$$

Where N is a nonlinear operator, x denotes an independent variable; $u(x)$ is an unknown function. For simplicity, we ignore all boundary or initial conditions, which can be treated in the similar way. By means of generalizing the conventional Homotopy method, Liao [14] constructed the so-called zero-order deformation equation as:

$$(1-p)L[\varphi(x;p) - u_0(x)] = phH(x)N[\varphi(x;p)] \quad (A2)$$

where $p \in [0,1]$ is the embedding parameter, $h \neq 0$ is a nonzero auxiliary parameter, $H(x) \neq 0$ is an auxiliary function, L is an auxiliary linear operator, $u_0(x)$ is an initial guess of $u(x)$, $\varphi(x;p)$ is an unknown function. It is important, that one has great freedom to choose auxiliary unknowns in HAM. Obviously, when $p = 0$ and $p = 1$, it holds:

$$\varphi(x;0) = u_0(x) \text{ and } \varphi(x;1) = u(x) \quad (A3) \text{ respectively. Thus, as } p \text{ increases}$$

from 0 to 1, the solution $\varphi(x;p)$ varies from the initial guess $u_0(x)$ to the solution $u(x)$. Expanding $\varphi(x;p)$ in Taylor series with respect to p , we have:

$$\varphi(x;p) = u_0(x) + \sum_{m=1}^{+\infty} u_m(x)p^m \quad (A4)$$

where

$$u_m(x) = \frac{1}{m!} \frac{\partial^m \varphi(x;p)}{\partial p^m} \Big|_{p=0} \quad (A5)$$

If the auxiliary linear operator, the initial guess, the auxiliary parameter h , and the auxiliary function are so properly chosen, the series (A4) converges at $p=1$ then we have:

$$u(x) = u_0(x) + \sum_{m=1}^{+\infty} u_m(x) \quad (A6)$$

$$\text{Define the vector } \vec{u} = \{u_0, u_1, \dots, u_n\} \quad (A7)$$

Differentiating Eqn. (A2) for m times with respect to the embedding parameter p , and then setting $p = 0$ and finally dividing them by $m!$, we will have the so-called m^{th} -order deformation equation as:

$$L[u_m - \chi_m u_{m-1}] = hH(x)\Re_m(\vec{u}_{m-1}) \quad (A8)$$

where

$$\Re_m(\vec{u}_{m-1}) = \frac{1}{(m-1)!} \frac{\partial^{m-1} N[\varphi(x;p)]}{\partial p^{m-1}} \quad (A9)$$

and

$$\chi_m = \begin{cases} 0, & m \leq 1, \\ 1, & m > 1. \end{cases} \quad (\text{A10})$$

Applying L^{-1} on both side of equation (A8), we get

$$u_m(x) = \chi_m u_{m-1}(x) + hL^{-1}[H(x)\mathfrak{R}_m(u_{m-1})] \quad (\text{A11})$$

In this way, it is easily to obtain u_m for $m \geq 1$, at M^{th} order, we have

$$u(x) = \sum_{m=0}^M u_m(x)$$

When $M \rightarrow +\infty$, we get an accurate approximation of the original equation (A1). For the convergence of the above method we refer the reader to Liao [16]. If Eqn. (A1) admits unique solution, then this method will produce the unique solution. If Eqn. (A1) does not possess unique solution, the HAM will give a solution among many other (possible) solutions.

APPENDIX B:

APPROXIMATE ANALYTICAL SOLUTIONS FOR EQN. (1), (2) AND (3) USING HAM:

In order to solve Eqns. (1), (2) and (3) by means of the HAM, we first construct the Zeroth order deformation equation by taking $H(t) = 1$.

$$(1-p)\left[\frac{dx}{dt} + ax\right] = ph\left[\frac{dx}{dt} + ax - ay\right] \quad (\text{B1})$$

$$(1-p)\left[\frac{dy}{dt} + y\right] = ph\left[\frac{dy}{dt} - bx + y + xz\right] \quad (\text{B2})$$

$$(1-p)\left[\frac{dz}{dt} + cz\right] = ph\left[\frac{dz}{dt} + cz - xy\right] \quad (\text{B3})$$

The approximate solutions of Eqn. (B1) - (B3) are as follows

$$x = x_0 + px_1 + p^2x_2 + \dots \quad (\text{B4})$$

$$y = y_0 + py_1 + p^2y_2 + \dots \quad (\text{B5})$$

$$z = z_0 + pz_1 + p^2z_2 + \dots \quad (\text{B6})$$

Substituting (B4) in Eqn. (B1) and equating the like powers of p we get

$$p^0 : \frac{dx_0}{dt} + ax_0 = 0 \quad (\text{B7})$$

$$p^1 : \frac{dx_1}{dt} + ax_1 = (h+1)\left[\frac{dx_0}{dt} + ax_0\right] + h(-ay_0) \quad (\text{B8})$$

Substituting (B5) in Eqn. (B2) and equating the like powers of p we get

$$p^0 : \frac{dy_0}{dt} + y_0 = 0 \quad (\text{B9})$$

$$p^1 : \frac{dy_1}{dt} + y_1 = (h+1)\left[\frac{dy_0}{dt} + y_0\right] + h[-bx_0 + x_0z_0] \quad (\text{B10})$$

Substituting (B6) in Eqn. (B2) and equating the like powers of p we get

$$p^0 : \frac{dz_0}{dt} + cz_0 = 0 \quad (B11)$$

$$p^1 : \frac{dz_1}{dt} + cz_1 = (h+1) \left[\frac{dz_0}{dt} + cz_0 \right] + h(-x_0z_0) \quad (B12)$$

The boundary conditions in Eqn. (4) becomes

$$x_0 = 1, y_0 = 1 \text{ and } z_0 = 1 \text{ when } t = 0 \quad (B13)$$

$$x_1 = 0, y_1 = 0 \text{ and } z_1 = 0 \text{ when } t = 0 \quad (B14)$$

Now applying the boundary conditions (B13) in Eqns. (B7), (B9) and (B11) we get

$$x_0 = e^{-at} \quad (B15)$$

$$y_0 = e^{-t} \quad (B16)$$

$$z_0 = e^{-ct} \quad (B17)$$

Substituting the values of x_0, y_0 and z_0 in Eqn. (B8) and (B10) and solving the equations using the boundary conditions (B14) we obtain the following results:

$$x_1 = \frac{ah}{(a-1)} [e^{-at} - e^{-t}] \quad (B18)$$

$$y_1 = \frac{bh}{(1-a)} [e^{-t} - e^{-at}] + \frac{h}{(1-a-c)} [e^{-t(a+c)} - e^{-t}] \quad (B19)$$

$$z_1 = \frac{h}{(c-a-1)} [e^{-ct} - e^{-t(a+1)}] \quad (B20)$$

Adding Eqns. (B15) and (B18), we get Eqn. (5) in the text. Similarly we get Eqns. (6) and (7) in the text.

APPENDIX C:

Matlap program for the numerical solution of the system of non-linear equations (1) – (3)

functions p4num

options =odeset('RelTel', 1e-6, 'stats', 'on');

%initial condions

Xo = [1; 1; 1;];

tspan = [0,1];

tic

[t, x] = ode45(@TestFuctions, tspan, Xo, options);

toc

figure

hold on

%plot(t, x(:, 1), '-')

```
%plot(t, x(:, 2), '-')
%plot(t, x(:,3), '-')
legend('x1', 'x2', 'x3')
ylabel('x')
xlabel('t')
return
function [dx_dt] = TestFuction (t, x)
a=1; b=1; c=1;
dx_dt(1) = -a*x(1)+a*x(2);
dx_dt(2) = b*x(1)-x(2)-x(1)*x(3);
dx_dt(3) = -c*x(3)+x(1)*x(2);
dx_dt = dx_dt';
return
```

REFERENCES

- [1] Gleick, J. Chaos: Making a New Science. Penguin Books, New York, NY, 1987.
- [2] S.H. Strogatz. Nonlinear Dynamics and Chaos. Westview, 1994.
- [3] STEWART, I. Does God Play Dice? The Mathematics of Chaos. Blackwell, Cambridge, MA, 1989.
- [4] Peitgen, H. O., H. Jürgens, D. Saupe, & C. Zahlten. Fractals: An Animated Discussion. Videocassette. New York: Freeman, 1990.
- [5] Bradley, Larry. Chaos and Fractals. 2010. Web <<http://www.stsci.edu/~lbradley/seminar/attractors.html>>
- [6] Viswanath, Divakar. The Fractal Property of the Lorenz Attractor. Physica D: Nonlinear Phenomena, Volume 190, Issues 1_2, March 2004.
- [7] Wlodzimierz Klonowski, From conformons to human brains: an informal overview of nonlinear dynamics and its applications in biomedicine, Nonlinear Biomedical Physics 2007, 1:5 doi:10.1186/1753-4631-1-5
- [8] Liao, S.J.; Ph. D., thesis, Shanghai Jiao Tong University, 1992.
- [9] G. Domairry.; M. Fazeli, Comm. Nonlinear Sci.Numer.Sim.2009,14,489- 499.
- [10] Liao, S.J.; Tan, Y., A general approach to obtain series solutions of nonlinear differential equations, Studies in Applied Mathematics, 2007, 119, 297-354.
- [11] Liao, S. J., Commun. Nonlinear Sci. Numer. Simul. 2010, 15, 2003-2016.
- [12] Sohoul, A. R.; Famouri, M.; Kimiaefar, A.; Domairry, G., Comm. Nonlinear. Sci. Numer.Simulat. 2010, 15, 1691-1699.
- [13] Mastroberardino, A.,Comm. Nonlinear Sci. Numer. Simulat. 2011, 16(7), 2730- 2736
- [14] S.J. Liao, An optimal Homotopy-analysis approach for strongly nonlinear differential equations, Commun. Nonlinear Sci. Numer. Simulat. 15 (2010), 2003–2016
- [15] G. Domairry, H. Bararnia, An Approximation of the analytic solution of some nonlinear heat transfer equations: a survey by using Homotopy analysis method, Adv. Studies Theor. Phys. 2 (2008) 507–518.
- [16] S.J. Liao, Beyond Perturbation Introduction to the Homotopy Analysis Method, 1st edn., Chapman and Hall, CRC, Press, Boca Raton, 2003, p. 336.
- [17] MATLAB 6.1, The Math Works Inc., Natick, MA, 2000.

- [18] R.D. Skeel, M. Berzins, A Method for the Spatial Discretization of Parabolic Equations in One Space Variable. SIAM Journal on Scientific and Statistical Computing 11 (1990), 1-32.

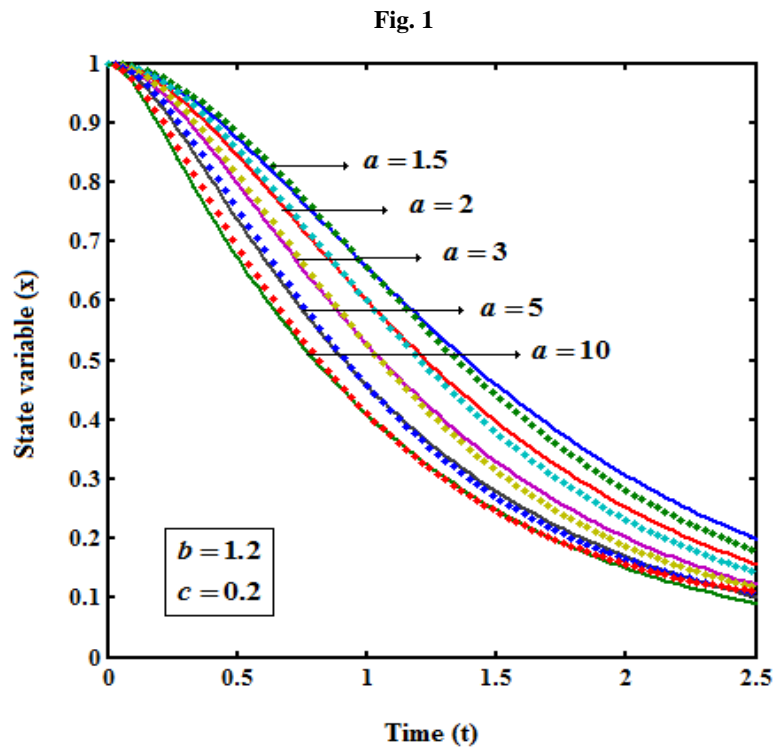


Fig. 1 The state variable x versus time t are plotted using Eqn. (5) for the fixed values of some parameters and various values of a . The key to the graph: solid line represents Eqn. (7) and dotted line represents the numerical simulation.

Fig. 2

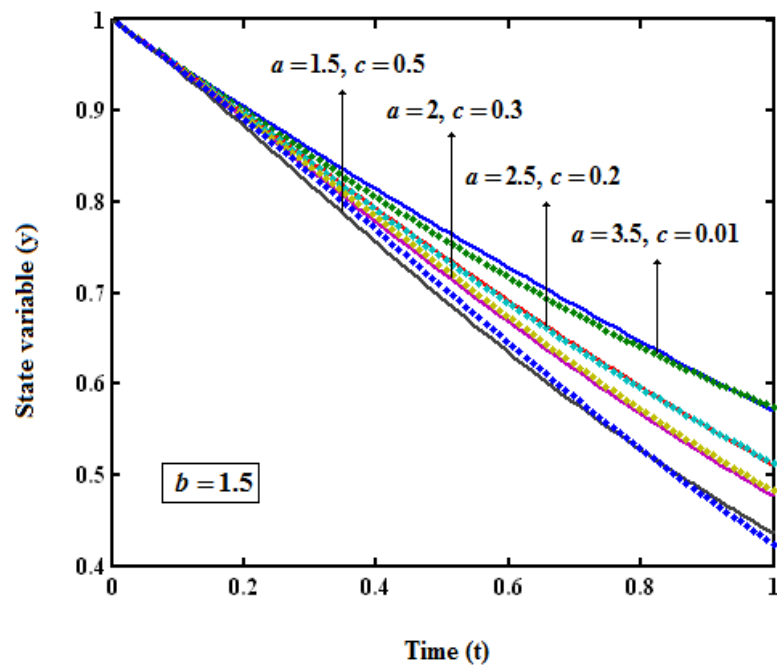


Fig. 2 The state variable y versus time t are plotted using Eqn. (6) for the fixed values of b and some various values of a and c . The key to the graph: solid line represents Eqn. (8) and dotted line represents the numerical simulation.

Fig. 3

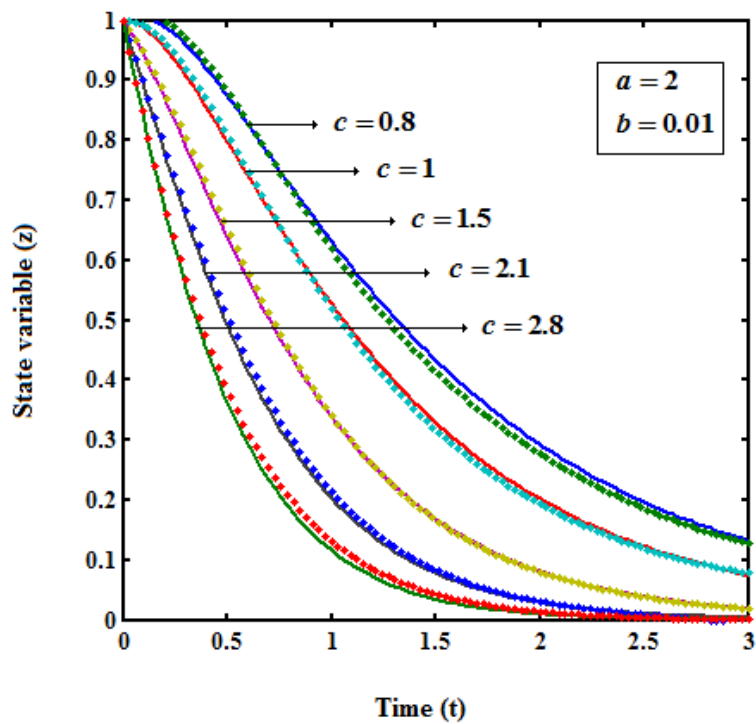


Fig. 3 The state variable z versus time t are plotted using Eqn. (7) for the fixed values of some parameters and various values of c . The key to the graph: solid line represents Eqn. (9) and dotted line represents the numerical simulation.

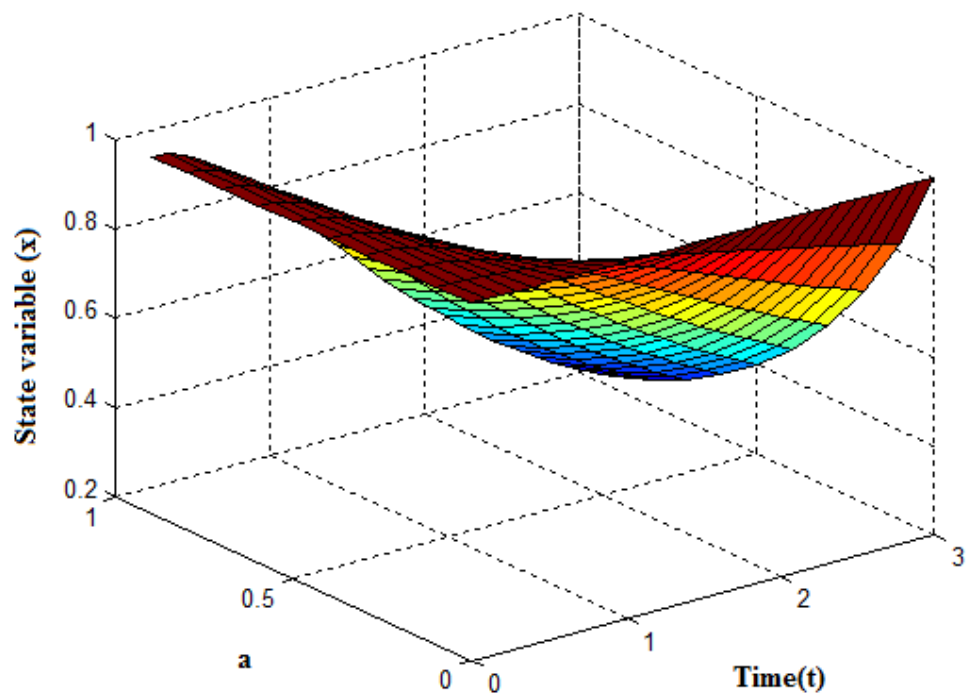


Fig.4 Plot of State variable x calculated using Eqn. (5) for some fixed values of parameters $b = 1.2$, $c = 0.2$ and $h = -1$.

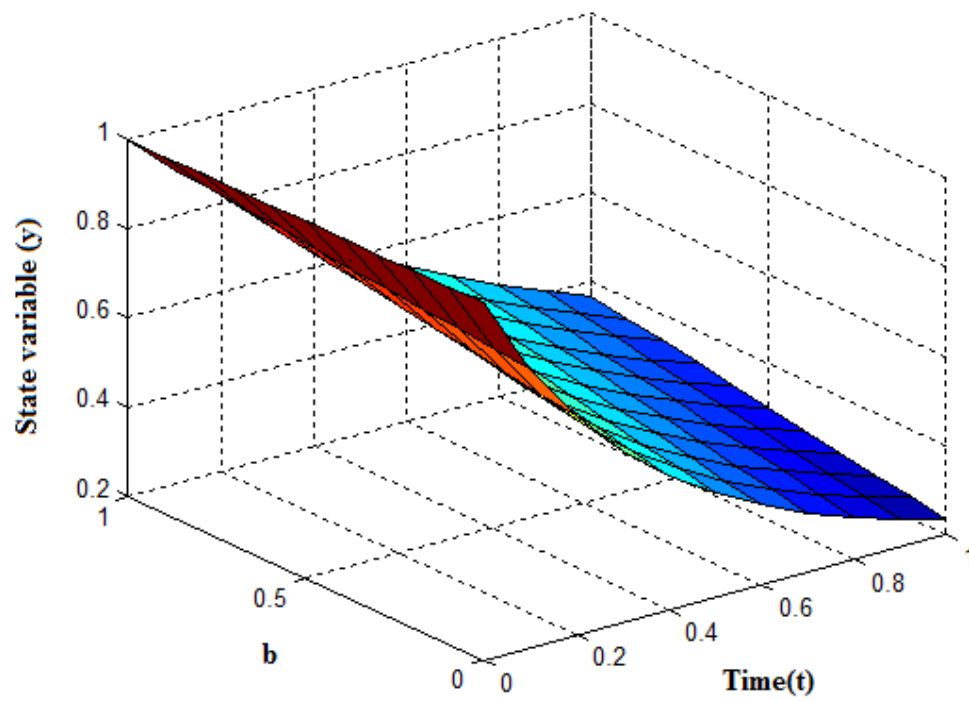


Fig.5 Plot of State variable y calculated using Eqn. (6) for some fixed values of parameters $b = 1.2$, $c = 0.2$ and $h = -1$.

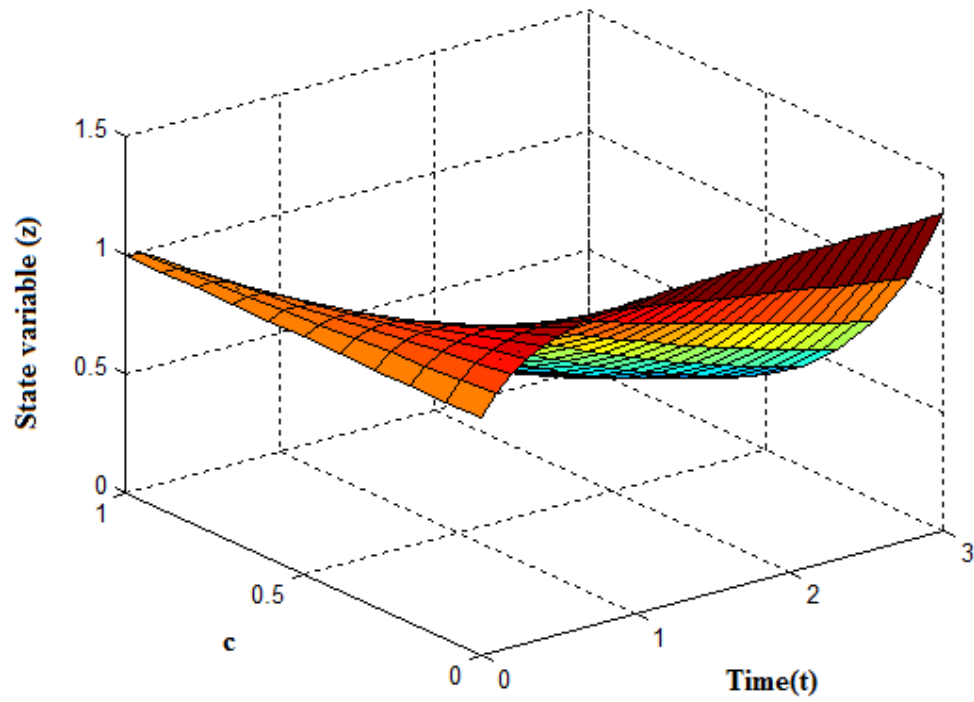


Fig.6 Plot of State variable z calculated using Eqn. (7) for some fixed values of parameters $a = 2$, $b = 0.01$ and $h = -1$.