School of Computing
National University of Singapore
CS5240 Theoretical Foundations in Multimedia

Exercise 2

Vector Products

Objectives

- This exercise illustrates the relationship and difference between matrix inner product and outer product.
- You should learn to work out the answers **yourself** without referring to Google, Wikipedia, etc., or consulting others.
- This exercise uses the knowledge that you have gained in Exercise 1.
- Work out the answers using the simplest, cleanest and most concise method.

Exercise Questions

A vector $\mathbf{x}$ is regarded as a column matrix $[\ x_1 \ \cdots \ x_m\ ]^\top$. It has only one column. So, the $(i,1)$-th element of $\mathbf{x}$ is $[\mathbf{x}]_{i1}$. The **dot product** $\mathbf{x} \cdot \mathbf{x}$ and the **norm** $\|\mathbf{x}\|$ of $\mathbf{x}$ are related by

$$\mathbf{x} \cdot \mathbf{x} = \|\mathbf{x}\|^2 = \sum_{j=1}^{m} x_j^2.$$

The **inner product** of $\mathbf{x}$ with itself is a 1×1 matrix whose element is its dot product:

$$\mathbf{x}^\top \mathbf{x} = [\ \mathbf{x} \cdot \mathbf{x}\ ] = [\ \|\mathbf{x}\|^2\ ].$$

For notational convenience, we usually write $\mathbf{x}^\top \mathbf{x} = \|\mathbf{x}\|^2$, dropping the matrix symbols. The **outer product** of $\mathbf{x}$ with itself is an $m \times m$ matrix $\mathbf{x}\mathbf{x}^\top$. The (i,j)-th element of $\mathbf{x}\mathbf{x}^\top$ is

$$[\mathbf{x}\mathbf{x}^\top]_{ij} = \sum_{k=1}^{1} [\mathbf{x}]_{ik} [\mathbf{x}^\top]_{kj} = [\mathbf{x}]_{i1} [\mathbf{x}]_{j1}.$$

(a) Show that the sum of the diagonal elements of the outer product $\mathbf{x}\mathbf{x}^\top$ is equal to $\|\mathbf{x}\|^2$.

(b) Consider n vectors $\mathbf{x}_i = [x_{i1} \ \cdots \ x_{im}]^\top$, $i = 1, \ldots, n$. The auto-correlation matrix $\mathbf{R}$ of $\mathbf{x}_i$ is defined as:

$$\mathbf{R} = \frac{1}{n} \sum_{i=1}^{n} \mathbf{x}_i \mathbf{x}_i^\top.$$

Form the matrix $\mathbf{A} = [\mathbf{x}_1 \ \cdots \ \mathbf{x}_n]$. What are the (i,j)-th elements of $\mathbf{A}\mathbf{A}^\top$ and $\mathbf{R}$, respectively? Show that $\mathbf{A}\mathbf{A}^\top = n\mathbf{R}$.

(c) The mean vector $\boldsymbol{\mu}$ and covariance matrix $\mathbf{C}$ of $\mathbf{x}_i$ are defined as follows:

$$\begin{aligned}\boldsymbol{\mu} &= \frac{1}{n}\sum_{i=1}^{n}\mathbf{x}_i \\ \mathbf{C} &= \frac{1}{n}\sum_{i=1}^{n}(\mathbf{x}_i-\boldsymbol{\mu})(\mathbf{x}_i-\boldsymbol{\mu})^\top.\end{aligned}$$

Form the matrix $\mathbf{B} = [(\mathbf{x}_1-\boldsymbol{\mu}) \ \cdots \ (\mathbf{x}_n-\boldsymbol{\mu})]$. Show that $\mathbf{B}\mathbf{B}^\top = n\mathbf{C}$.

(d) Show that $\mathbf{C} = \mathbf{R} - \boldsymbol{\mu}\boldsymbol{\mu}^\top$.

(e) Form diagonal matrix $\mathbf{W}$ with scalar weights w_j as follows:

$$\mathbf{W} = \operatorname{diag}(w_1, \cdots, w_m) = \begin{bmatrix} w_1 & 0 & \cdots & 0 \\ 0 & w_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & w_m \end{bmatrix}.$$

Then, $\mathbf{W}^{1/2} = \operatorname{diag}\left(\sqrt{w_1}, \cdots, \sqrt{w_m}\right)$. Show that, for any vector $\mathbf{x} = [x_1 \ \cdots \ x_m]^\top$, its weighted inner product is equal to its weighted norm:

$$\mathbf{x}^\top\mathbf{W}\mathbf{x} = \|\mathbf{W}^{1/2}\mathbf{x}\|^2 = \sum_{j=1}^{m} w_j x_j^2.$$