Answers for Problem 0

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1 Exercise 1

(a) A(B+C) = AB + AC
The (i, j)-th element of the left hand side is:

$$\begin{aligned} [A(B+C)]_{ij} &= \sum_k [A]_{ik}[B+C]_{kj} \\ &= \sum_k [A]_{ik}[B]_{kj} + [A]_{ik}[C]_{kj} \\ &= [AB]_{ij} + [AC]_{ij} \\ &= [AB+AC]_{ij} \end{aligned} \tag{1}$$

Hence, A(B+C) = AB + AC

(b) (AB)C = A(BC) AB $\neq$ BA
The (i, j)-th element of the left hand side is:

$$\begin{aligned} [(AB)C]_{ij} &= \sum_k [AB]_{ik}[C]_{kj} \\ &= \sum_k \sum_m [A]_{im}[B]_{mk}[C]_{kj} \\ &= \sum_m [A]_{im} \sum_k [B]_{mk}[C]_{kj} \\ &= \sum_m [A]_{im}[BC]_{mj} = [A(BC)]_{ij} \end{aligned} \tag{2}$$

Hence, (AB)C = A(BC)

(c) (A+B)^T = A^T + B^T
The (i, j)-th element of the left hand side is:

$$\begin{aligned} [(A+B)^T]_{ij} &= [A+B]_{ji} \\ &= [A]_{ji} + [B]_{ji} \\ &= [A^T]_{ij} + [B^T]_{ij} \\ &= [A^T + B^T]_{ij} \end{aligned} \tag{3}$$

Hence, (A+B)^T = A^T + B^T

(d) (AB)^T = B^{T}A^T
The (i, j)-th element of the left hand side is:

$$
\begin{aligned}
[(AB)^T]_{ij} &= [AB]_{ji} \\
&= \sum_k [A]_{jk}[B]_{ki} \\
&= \sum_k [B^T]_{ik}[A^T]_{kj} \\
&= [B^T A^T]_{ij}
\end{aligned} \tag{4}
$$

Hence, $(\mathrm{AB})^T = \mathrm{B}^T\mathrm{A}^T$

2 Exercise 2

(a) show that the sum of the diagonal elements of the outer product xx^T is equal to $\|x\|^2$
The sum of the diagonal elements of the outer product is:

$$
\begin{aligned}
\sum_{i=1}^{m} [xx^T]_{ii} &= \sum_{i=1}^{m} [x]_{i1}[x]_{i1} \\
&= \sum_{i=1}^{m} [x]_i^2 \\
&= \|x\|^2
\end{aligned} \tag{5}
$$

(b) $\mathrm{AA}^T = \mathrm{nR}$
The (i, j)-th element of AA^T is:

$$
\begin{aligned}
[AA^T]_{ij} &= \sum_{k=1}^{n} [A]_{ik}[A^T]_{kj} \\
&= \sum_{k=1}^{n} [A]_{ik}[A]_{jk} \\
&= \sum_{k=1}^{n} [A]_{ik}[A]_{jk}
\end{aligned} \tag{6}
$$

The (i, j)-th element of R is:

$$
\begin{aligned}
[R]_{ij} &= \frac{1}{n} \sum_{k=1}^{n} [x_k]_i [x_k]_j \\
&= \frac{1}{n} \sum_{k=1}^{n} [A]_{ik}[A]_{jk}
\end{aligned} \tag{7}
$$

Since,

$$[AA^T]_{ij} = n[R]_{ij}$$

Hence, $\mathrm{AA}^T = \mathrm{nR}$

(c) $\mathrm{BB}^T = \mathrm{nC}$
The (i, j)-th element of BB^T is:

$$
\begin{aligned}
[BB^T]_{ij} &= \sum_{k=1}^{n}[B]_{ik}[B^T]_{kj} \\
&= \sum_{k=1}^{n}[B]_{ik}[B]_{jk} \\
&= \sum_{k=1}^{n}[B]_{ik}[B]_{jk}
\end{aligned} \tag{8}
$$

The (i, j)-th element of C is:

$$
\begin{aligned}
[C]_{ij} &= \frac{1}{n}\sum_{k=1}^{n}[x_k - \mu]_i[x_k - \mu]_j \\
&= \frac{1}{n}\sum_{k=1}^{n}[B]_{ik}[B]_{jk}
\end{aligned} \tag{9}
$$

Hence, BB^T = nC

(d) C = R - $\mu\mu^T$
The (i, j)-th element of C is:

$$
\begin{aligned}
[C]_{ij} &= \frac{1}{n}\sum_{k=1}^{n}[x_k - \mu]_i[x_k - \mu]_j \\
&= \frac{1}{n}\sum_{k=1}^{n}([x_k]_i[x_k]_j + [\mu]_i[\mu]_j - [x_k]_i[\mu]_j - [x_k]_j[\mu]_i) \\
&= \frac{1}{n}\sum_{k=1}^{n}[x_k]_i[x_k]_j + [\mu]_i[\mu]_j - \frac{1}{n}\sum_{k=1}^{n}([x_k]_i[\mu]_j + [x_k]_j[\mu]_i) \\
&= \frac{1}{n}\sum_{k=1}^{n}[x_k]_i[x_k]_j + [\mu]_i[\mu]_j - 2[\mu]_i[\mu]_j \\
&= \frac{1}{n}\sum_{k=1}^{n}[x_k]_i[x_k]_j - [\mu]_i[\mu]_j
\end{aligned} \tag{10}
$$

The (i, j)-th element of R is:

$$
[R]_{ij} = \frac{1}{n}\sum_{k=1}^{n}[x_k]_i[x_k]_j \tag{11}
$$

The (i, j)-th element of R - $\mu\mu^T$ is:

$$
[R - \mu\mu^T]_{ij} = \frac{1}{n}\sum_{k=1}^{n}[x_k]_i[x_k]_j - [\mu]_i[\mu]_j \tag{12}
$$

Hence, C = R - $\mu\mu^T$

(e) x^{T}Wx = $\sum_{j=1}^{m} w_j {x_j}^2$
The k-th element of Wx is:

$$
\begin{aligned}
[Wx]_k &= \sum_i [W]_{ki}[x]_i \\
&\text{when i = k}, [W]_{ik} = w_k \\
&= w_k x_k
\end{aligned}
\tag{13}
$$

$$
\begin{aligned}
x^T W x &= \sum_k [x^T]_{1k}[Wx]_{k1} \\
&= \sum_k [x]_k [Wx]_k \\
&= \sum_k [x]_k \cdot [w]_k [x]_k \\
&= \sum_{k=1}^{m} [w]_k {[x]_k}^2
\end{aligned}
\tag{14}
$$

Hence, $x^T W x = \left\|W^{1/2}x\right\|^2 = \sum_{j=1}^{m} w_j {x_j}^2$