

Workshop 2: calibrating a stochastic model

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Chapter 1

Objectives

In this workshop, you will learn to perform history matching with emulation on models with stochasticity, using the [hmer](#) package. This workshop should be addressed after working on [Workshop 1](#), which shows how to perform history matching with emulation on deterministic models.

Show: Code to load relevant libraries and helper functions on P54

Chapter 2

An overview of history matching with emulation and hmer

In this section we briefly recap the history matching with emulation workflow and we explain how various steps of the process can be performed using hmer functions.

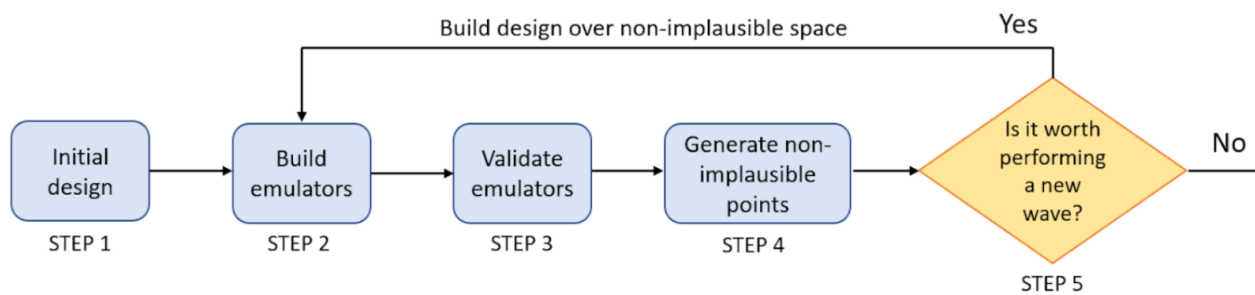


Figure 2.1: History matching with emulation workflow

As shown in the above image, history matching with emulation proceeds in the following way:

1. The model is run on the initial design points, a manageable number of parameter sets that are spread out uniformly across the input space.
2. Emulators are built using the training data provided by step 1.
3. Emulators are tested, to check whether they are good approximations of the model outputs.
4. Emulators are evaluated at a large number of parameter sets. The implausibility of each of these is then assessed. Parameter sets classified as non-implausible will be used in the next wave of the process.
5. The process is stopped or a new wave of the process is performed, going back to step 2.

The table below associates each step of the process with the corresponding hmer function. Note that step 1 and 5 are not in the table, since they are specific to each model and calibration task. The last column of the table indicates what section of this workshop deals with each step of the process.

Step of the HME process	Hmer function			Relevant section of the workshop
	Name	Input	Output	
Build stochastic emulators (step 2)	<i>variance_emulator_from_data</i>	<ul style="list-style-type: none"> The training data The outputs to emulate The ranges of parameters 	An emulator of the variance and an emulator of the mean for each output of interest.	4 Emulators
Validate emulators (step 3)	<i>validation_diagnostics</i>	<ul style="list-style-type: none"> The trained emulators The targets to match to The validation data 	Three diagnostics for each emulator of interest.	6 Emulator diagnostics
Generate non-implausible points (step 4)	<i>generate_new_runs</i>	<ul style="list-style-type: none"> The trained emulators The number of points to generate The targets to match to 	A dataframe of points deemed non-implausible by all emulators	7 Proposing new points

Figure 2.2: The three main hmer functions

Chapter 3

Introduction to the model

In this section we introduce the model that we will work with throughout our workshop. To facilitate the comparison between the deterministic and the stochastic setting, we will work with the SEIRS model which we used in [Workshop 1](#), but this time we introduce stochasticity. The deterministic SEIRS model used in [Workshop 1](#) was described by the following differential equations:

$$\frac{dS}{dt} = bN - \frac{\beta(t)IS}{N} + \omega R - \mu S \quad (3.1)$$

$$\frac{dE}{dt} = \frac{\beta(t)IS}{N} - \epsilon E - \mu E \quad (3.2)$$

$$\frac{dI}{dt} = \epsilon E - \gamma I - (\mu + \alpha)I \quad (3.3)$$

$$\frac{dR}{dt} = \gamma I - \omega R - \mu R \quad (3.4)$$

where N is the total population, varying over time, and the parameters are as follows:

- b is the birth rate,
- μ is the rate of death from other causes,
- $\beta(t)$ is the infection rate between each infectious and susceptible individual,
- ϵ is the rate of becoming infectious after infection,
- α is the rate of death from the disease,
- γ is the recovery rate and
- ω is the rate at which immunity is lost following recovery.

The rate of infection $\beta(t)$ is set to be a simple linear function interpolating between points, where the points in question are $\beta(0) = \beta_1$, $\beta(100) = \beta(180) = \beta_2$, $\beta(270) = \beta_3$ and where $\beta_2 < \beta_1 < \beta_3$. This choice was made to represent an infection rate that initially drops due to external (social) measures and then raises when a more infectious variant appears. Here t is taken to measure days. Below we show a graph of the infection rate over time when $\beta_1 = 0.3$, $\beta_2 = 0.1$ and $\beta_3 = 0.4$:

In order to obtain the solutions of the stochastic SEIR model for a given set of parameters, we will use a helper function, `get_results` (which is defined in the R-script). This function assumes an initial population of 900 susceptible individuals, 100 exposed individuals, and no infectious or recovered individuals, and

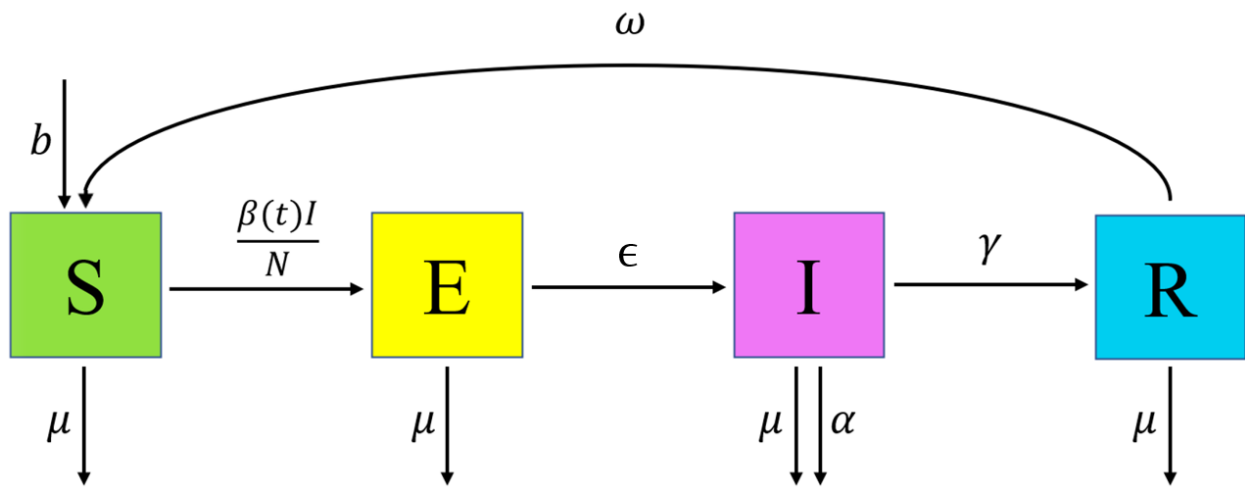


Figure 3.1: SEIRS Diagram

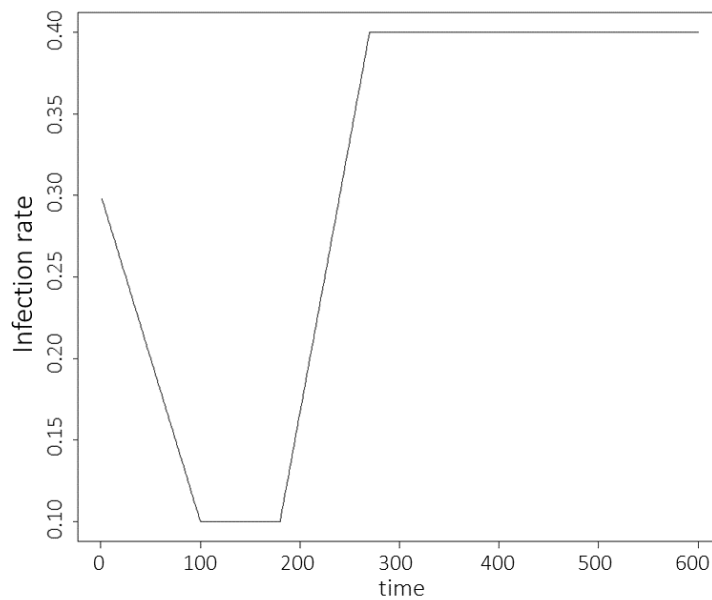


Figure 3.2: Infection rate graph

uses the Gillespie algorithm to generate statistically correct trajectories of the model. The minimum specifications for this function are: the parameter set(s) to run the model on, a set of outputs (e.g. `c("S", "R")`), and a set of times (e.g. `c(10, 100, 150)`) that we are interested in. The default behaviour of `get_results` is to run the model 100 times on the parameter set(s) provided, but more or less repetitions can be obtained, using the argument `nreps`. The function `get_results` returns a dataframe containing a row for each repetition at each parameter set provided, with the first nine columns showing the values of the parameters and the subsequent columns showing the requested outputs at the requested times. If `raw` is set to `TRUE`, all outputs and all times are instead returned: this is useful if we want to plot “continuous” trajectories.

As done in [Workshop 1](#), the outputs obtained with the parameter set

```
chosen_params <- c(
  b = 1/(76*365),
  mu = 1/(76*365),
  beta1 = 0.214, beta2 = 0.107, beta3 = 0.428,
  epsilon = 1/7,
  alpha = 1/50,
  gamma = 1/14,
  omega = 1/365
)
```

will be used to define the target bounds for our calibration task.

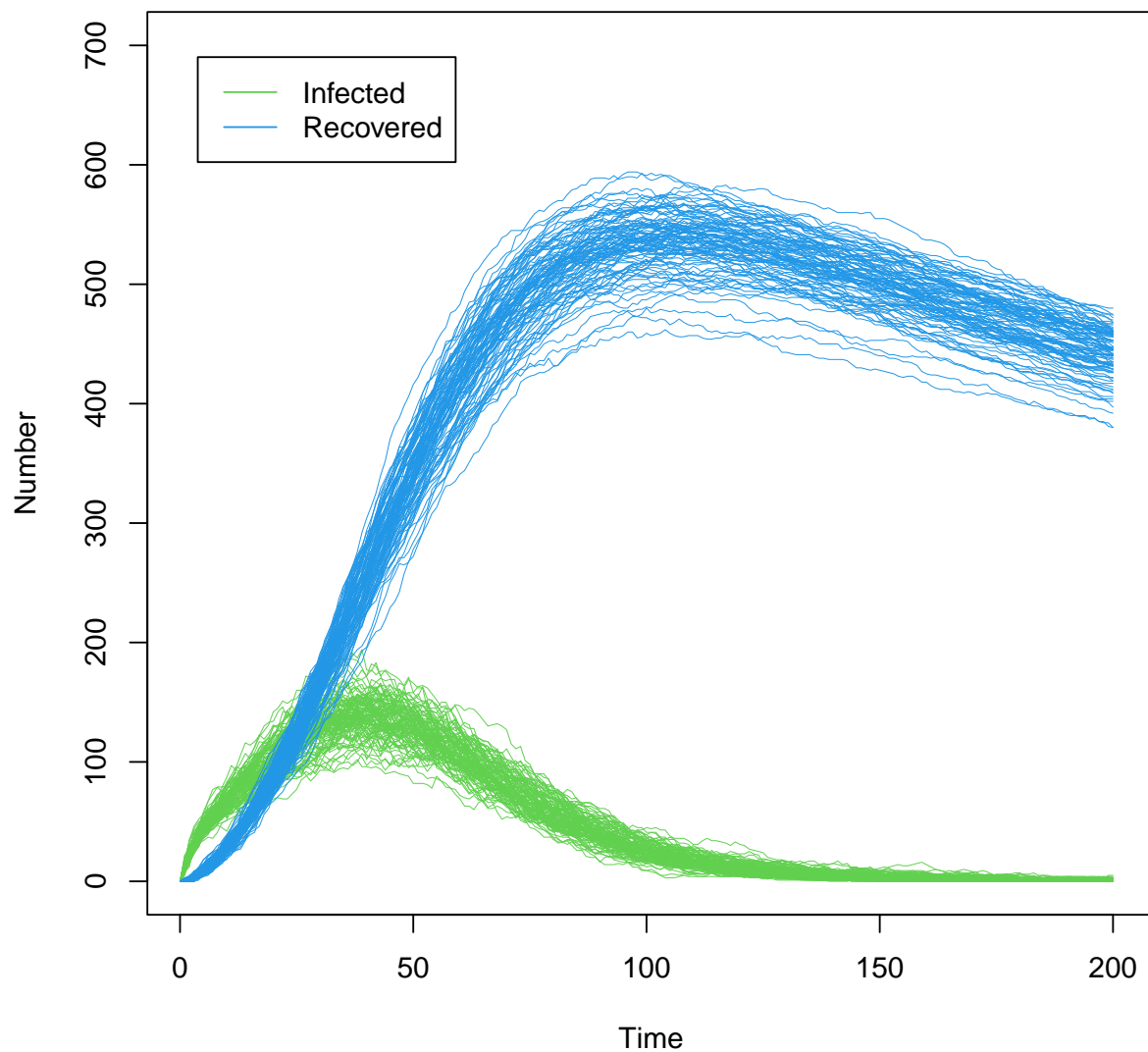
Using `get_results` on `chosen_params`

```
solution <- get_results(chosen_params, outs = c("I", "R"),
  times = c(25, 40, 100, 200), raw = TRUE)
```

we get a 3-dimensional array `solution` such that `solution[t,j,i]` contains the number of individuals in the j -th compartment at time t for the i -th run of the model at `chosen_params`. In particular, t can take values 1, 2, ..., 201, j can take values 1, 2, 3, 4, 5 corresponding to S, E, I, R, D (where D stands for the number of deaths occurred), and i can be 1, 2, 3, ..., 100.

Plotting the results for “I” and “R,” we have

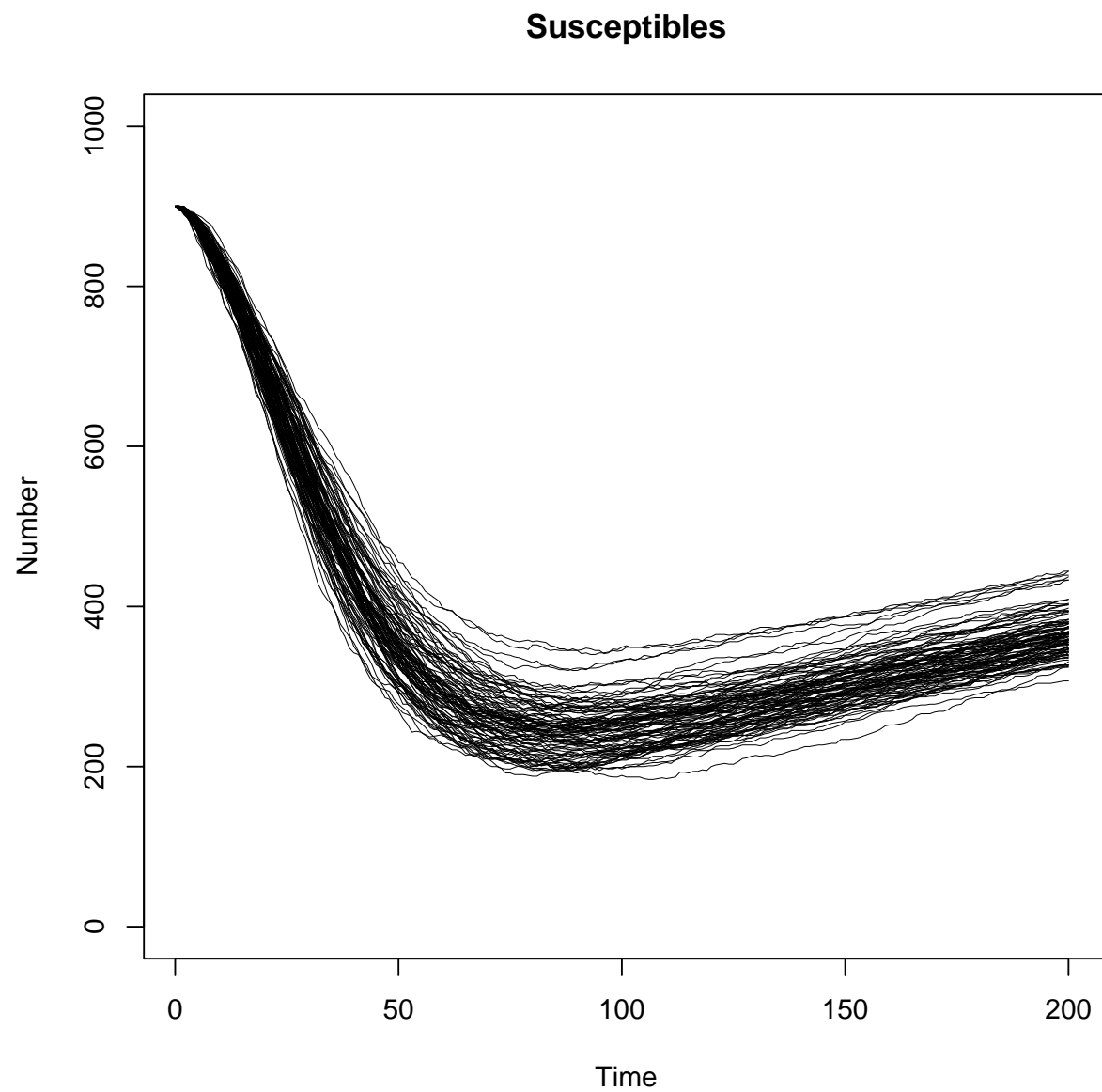
```
plot(0:200, ylim=c(0,700), ty="n", xlab = "Time", ylab = "Number")
for(j in 3:4) for(i in 1:100) lines(0:200, solution[,j,i], col=(3:4)[j-2], lwd=0.3)
legend('topleft', legend = c('Infected', "Recovered"), lty = 1,
  col = c(3,4), inset = c(0.05, 0.05))
```



The plot above clearly shows the stochasticity of our model. Furthermore we can also appreciate how the spread of the runs varies across the time: for example, the variance in the number of infected individuals is largest around $t = 40$ and quite small at later times.

Plotting the results for “S” also shows stochasticity:

```
plot(0:200, ylim=c(0,1000), ty="n", xlab = "Time", ylab = "Number", main = "Susceptibles")
for(i in 1:100) lines(0:200, solution[,1,i], col='black', lwd=0.3,
                     xlab = "Time", ylab = "Number", main = "Susceptibles")
```



Task 1

If you would like, familiarise yourself with the model. Investigate how the plots change as you change the values of the parameters.

Show: R tip on P55

Show: Solution on P45

Chapter 4

‘waveo’ - parameter ranges, targets and design points

In this section we set up the emulation task, defining the input parameter ranges, the calibration targets and all the data necessary to build the first wave of emulators. All these data are the same as the ones set in [Workshop 1](#). Note that, for the sake of clarity, in this workshop we will adopt the word ‘data’ only when referring to the set of runs of the model that are used to train emulators. Real-world observations, that inform our choice of targets, will instead be referred to as ‘observations.’

First of all, let us set the parameter ranges:

```
ranges = list(  
  b = c(1e-5, 1e-4), # birth rate  
  mu = c(1e-5, 1e-4), # rate of death from other causes  
  beta1 = c(0.2, 0.3), # infection rate at time t=0  
  beta2 = c(0.1, 0.2), # infection rates at time t=100  
  beta3 = c(0.3, 0.5), # infection rates at time t=270  
  epsilon = c(0.07, 0.21), # rate of becoming infectious after infection  
  alpha = c(0.01, 0.025), # rate of death from the disease  
  gamma = c(0.05, 0.08), # recovery rate  
  omega = c(0.002, 0.004) # rate at which immunity is lost following recovery  
)
```

We then turn to the targets we will match: the number of infectious individuals I and the number of recovered individuals R at times $t = 25, 40, 100, 200$. For each of these outputs, we define a pair (val, sigma), where ‘val’ represents the measured value of the output and ‘sigma’ represents its standard deviation:

```
targets <- list(  
  I25 = list(val = 115.88, sigma = 5.79),  
  I40 = list(val = 137.84, sigma = 6.89),  
  I100 = list(val = 26.34, sigma = 1.317),  
  I200 = list(val = 0.68, sigma = 0.034),  
  R25 = list(val = 125.12, sigma = 6.26),  
  R40 = list(val = 256.80, sigma = 12.84),  
  R100 = list(val = 538.99, sigma = 26.95),  
  R200 = list(val = 444.23, sigma = 22.21)  
)
```

The 'sigmas' in our `targets` list represent the uncertainty we have about the observations. Note that in general we can also choose to include model uncertainty in the 'sigmas' to reflect how accurate we think our model is. Also note that we can also define targets using lower and upper bounds, instead of value and standard deviation. For example, if we want to add a target `I60` with lower bound 100 and upper bound 120, then we would add `I60 = c(100,120)` in the list above.

For this workshop, we generate parameter sets using a [Latin Hypercube](#) design. Through the function `maximinLHS` in the package `lhs`, we create two hypercube designs with 100 parameter sets and 50 parameter sets, one for the training set and one for the validation set.

```
initial_lhs <- maximinLHS(100, 9)
validation_lhs <- lhs::randomLHS(50, 9)
```

Note that in `initial_lhs` and in `validation_lhs` each parameter is distributed on $[0, 1]$. This is not exactly what we need, since each parameter has a different range. We therefore re-scale each component in `initial_lhs` and in `validation_lhs` multiplying it by the difference between the upper and lower bounds of the range of the corresponding parameter and then we add the lower bound for that parameter. In this way we obtain `initial_points` and `validation_points`, which contain parameter values in the correct ranges.

```
initial_points <- setNames(data.frame(t(apply(initial_lhs, 1,
                                             function(x) x * purrr::map_dbl(ranges, diff) +
                                             purrr::map_dbl(ranges, ~.[[1]])))), names(ranges))

validation_points <- setNames(data.frame(t(apply(validation_lhs, 1,
                                                  function(x) x * purrr::map_dbl(ranges, diff) +
                                                  purrr::map_dbl(ranges, ~.[[1]])))), names(ranges))
```

We then run the model for the parameter sets in `initial_points` and in `validation_points` through the `get_results` function, specifying that we are interested in the outputs for I and R at times $t = 25, 40, 100, 200$.

```
output_list <- list()
for (i in 1:nrow(initial_points)) {
  model_out <- get_results(unlist(initial_points[i,]), nreps = 50, outs = c("I", "R"),
                          times = c(25, 40, 100, 200))
  output_list[[i]] <- model_out
}
validation_list <- list()
for (i in 1:nrow(validation_points)) {
  model_out <- get_results(unlist(validation_points[i,]), nreps = 50, outs = c("I", "R"),
                          times = c(25, 40, 100, 200))
  validation_list[[i]] <- model_out
}
```

Note that `output_list` is a list of length 5000, since it has a row for each of the 50 repetitions of the 100 parameter sets in `initial_points`. Similarly, `validation_list` is a list of length 2500, since it has a row for each of the 50 repetitions of the 50 parameter sets in `validation_points`. Finally, we bind all elements in `output_list` and all elements in `validation_list` to obtain two data frames `all_output` and `all_valid`.

```
all_output <- data.frame(do.call('rbind', output_list))
all_valid <- data.frame(do.call('rbind', validation_list))
output_names <- c("I25", "I40", "I100", "I200",
                 "R25", "R40", "R100", "R200")
```


Chapter 5

Emulators

In this section we will train stochastic emulators, which take into account the fact that running the model twice on a parameter set will produce different outputs.

5.1 Brief recap on the structure of an emulator

We very briefly recap the general structure of a univariate emulator (see [Workshop 1](#) for more details):

$$f(x) = g(x)^T \xi + u(x),$$

where $g(x)^T \xi$ is a regression term and $u(x)$ is a [weakly stationary process](#) with mean zero.

The regression term, which mimics the global behaviour of the model output, is specified by a vector of functions of the parameters $g(x)$ which determine the shape and complexity of the regression hypersurface we fit to the training data, and a vector of regression coefficients ξ .

The [weakly stationary process](#) $u(x)$ (similar to a [Gaussian process](#)), accounts for the local deviations (or residuals) of the output from the regression hypersurface.

In this workshop, $u(x)$ is assumed to be a Gaussian process, with covariance structure given by

$$\text{Cov}(u(x), u(x')) = \sigma^2 c(x, x')$$

where c is the square-exponential correlation function

$$c(x, x') := \exp \left(\frac{-\sum_i (x_i - x'_i)^2}{\theta^2} \right)$$

where x_i is the i th-component of the parameter set x . The term σ^2 is the **emulator variance**, i.e. the variance of $u(x)$, and reflects how far we expect the output to be from the regression hypersurface, while θ is the **correlation length** of the process, and determines how close two parameter sets must be in order for the corresponding residuals to be non-negligibly correlated. In the deterministic workshop, we said that in order to train an emulator, the `emulator_from_data` function first estimated the value for σ , thanks to the provided training data. It is important to note that **this estimated value of σ was constant**, i.e., it was the same for all parameter sets.

When working with stochastic models, different areas of the parameter space often give rise to quite different levels of stochasticity in the outputs. Taking this phenomenon into account when dealing with stochastic models will allow us to train more accurate and efficient emulators.

5.2 Stochastic emulators

To train stochastic emulators we use the function `variance_emulator_from_data`, which requires the training data, the names of the outputs to emulate, and the ranges of the parameters:

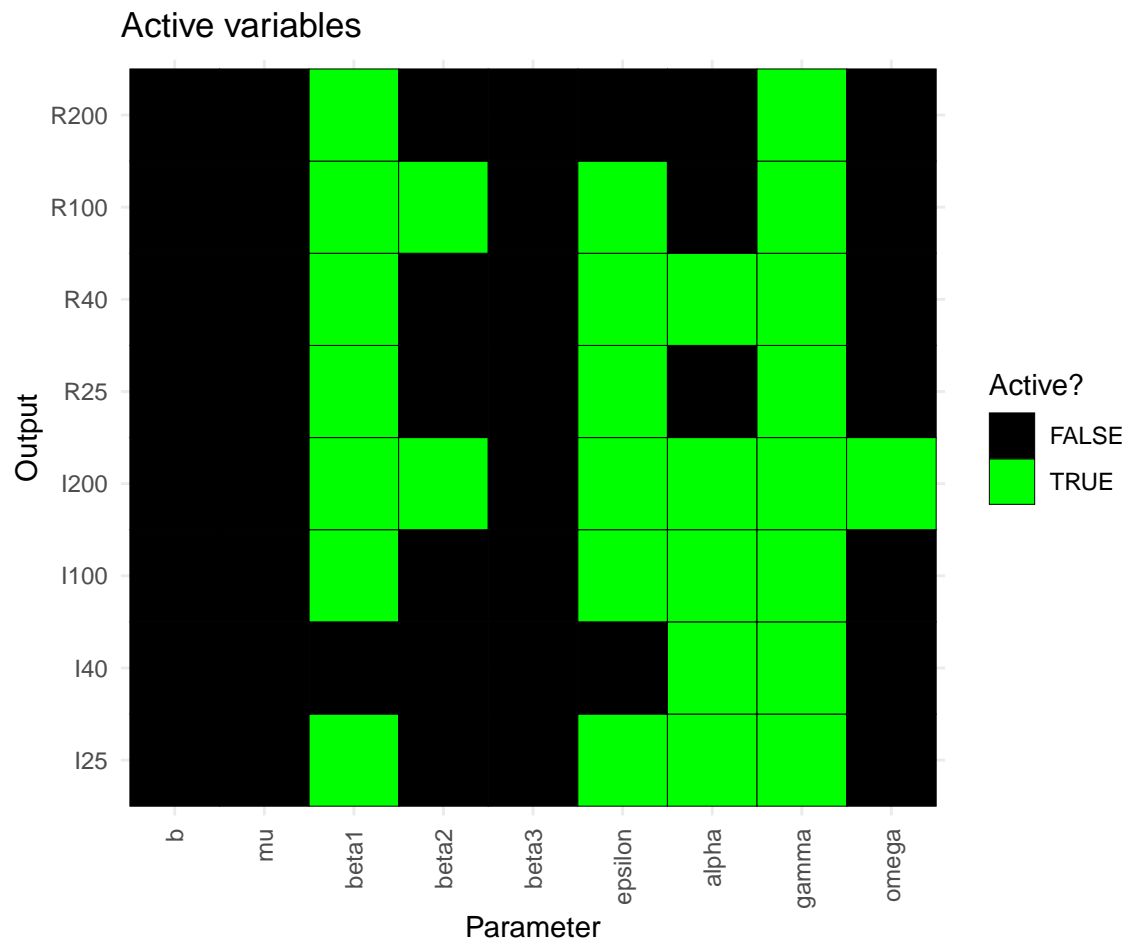
```
stoch_emulators <- variance_emulator_from_data(all_output, output_names, ranges)
```

The function `variance_emulator_from_data` returns two sets of emulators: one for the variance and one for the expectation of each model output. Behind the scenes, `variance_emulator_from_data` does the following:

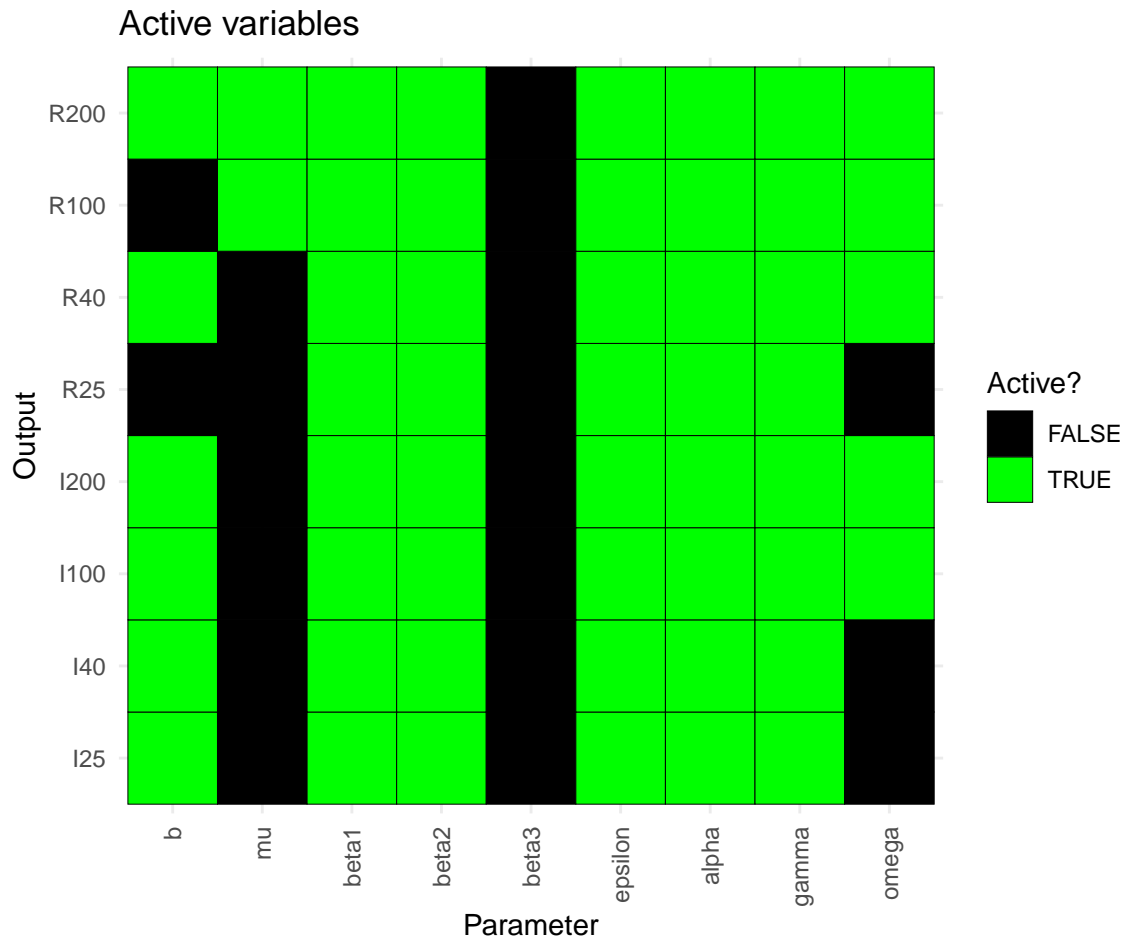
- First, it calculates the variance of each model output of interest at each provided parameter set: this is possible since `all_output` contains several model runs at each parameter set.
- Using the obtained variance data, it then trains an emulator for the variance of each of the outputs of interest. These emulators will be referred to as **variance emulators** and can be accessed typing `stoch_emulators$variance`.
- Finally, emulators for the mean of each output are built. These emulators will be referred to as **mean emulators** and can be accessed typing `stoch_emulators$expectation`. The main difference in the training of emulators between the stochastic and the deterministic case is that here, for each output, we use the variance emulator to inform our choice of σ : for each parameter set x , the variance emulator gives us an estimate of the variance of the model output at x . This means that the prior for σ varies from parameter set to parameter set, and is therefore estimated more adequately across the input space, compared to the deterministic case, where σ was constant.

Let's take a look at the two sets of emulators stored in `stoch_emulators`, starting with plots of active variables:

```
plot_actives(stoch_emulators$variance)
```



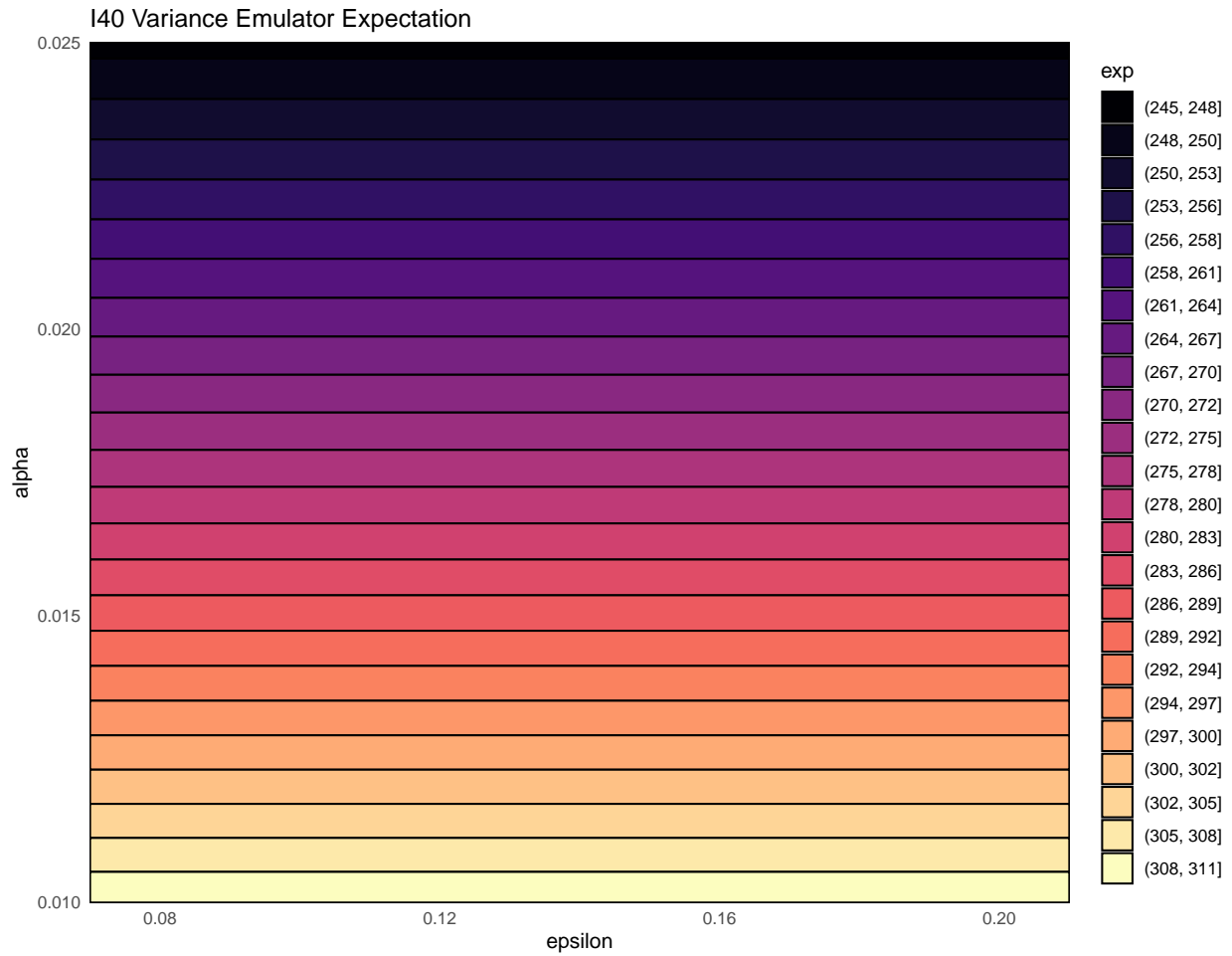
```
plot_actives(stoch_emulators$expectation)
```



We see that b and μ are inactive for almost all outputs in the variance emulators, even though they are active for some outputs in the mean emulators. Parameters β_1 , β_2 , ϵ , α are active for most or all outputs in the mean emulators.

As in deterministic case, the `emulator_plot` can be used to visualise how emulators represent the output space. Of course, since we have two sets of emulators, we have several options in terms of plots. Since the default behaviour of `emulator_plot` is to plot the expectation, if we want to plot the expectation of the variance emulator for $I40$, we just need the following command:

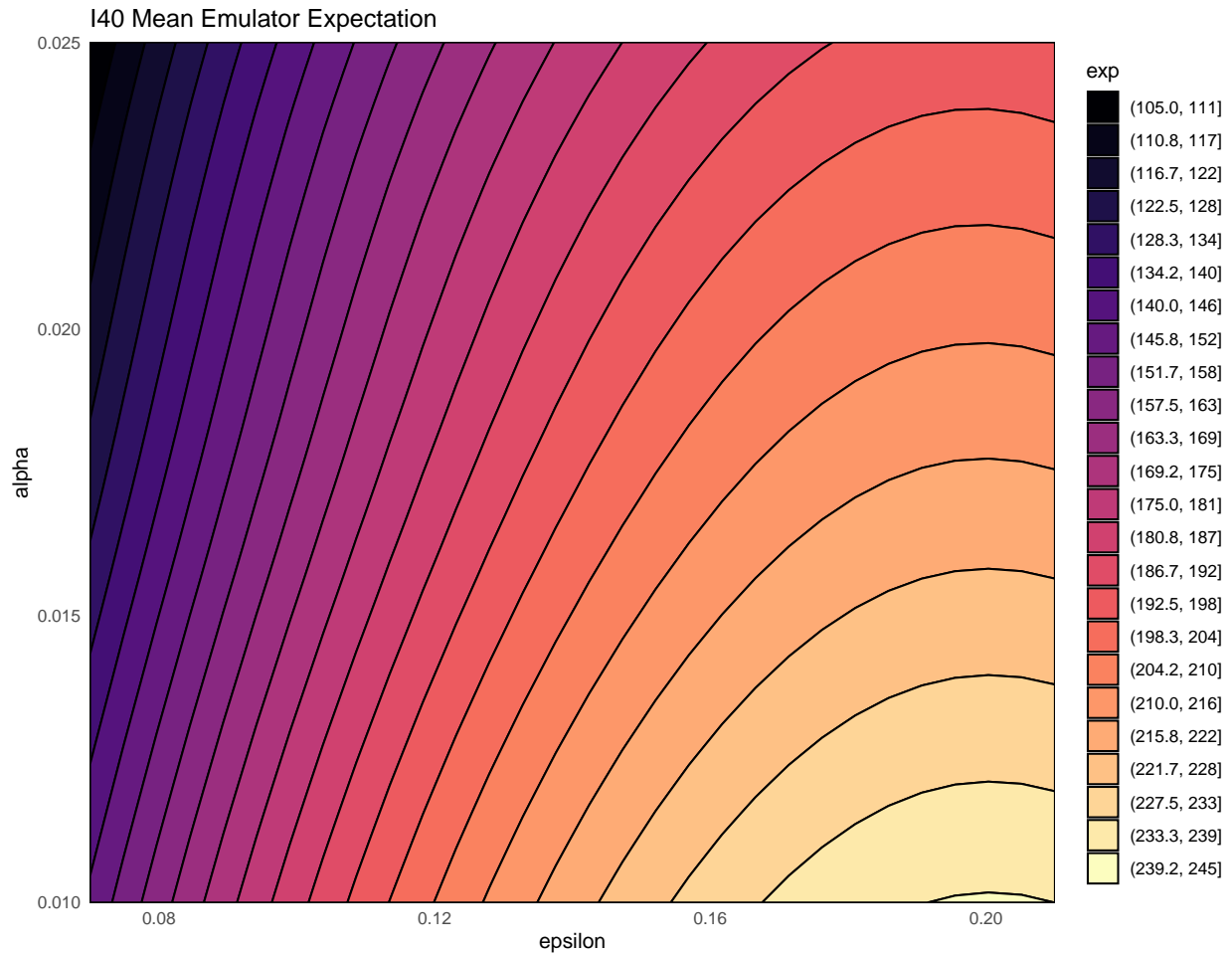
```
emulator_plot(stoch_emulators$variance$I40, params = c('epsilon', 'alpha'))
```



where we chose to show the two parameters ϵ and α .

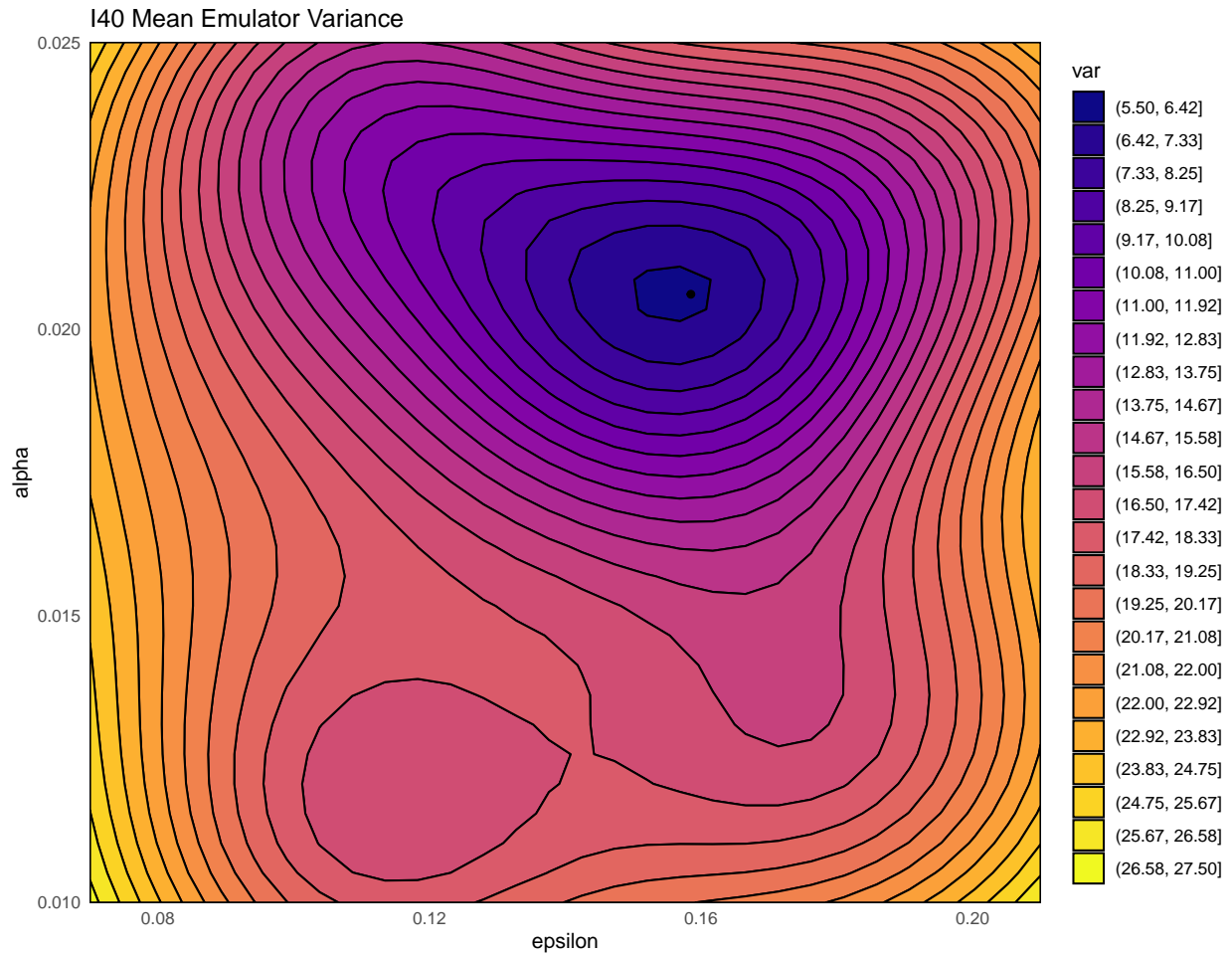
To produce a similar plot for the expectation of the mean emulator for I_{40} , we type:

```
emulator_plot(stoch_emulators$expectation$I40, params = c('epsilon', 'alpha'))
```



It is important to stress the following point: unlike in the deterministic case, here mean emulators do not have zero variance at ‘known’ points, i.e., at points in the training set. Let’s verify this, by plotting the variance of the mean emulator for *I40* in the (ϵ, α) -plane, with all unshown parameters to be as in the first row of `all_output`:

```
emulator_plot(stoch_emulators$expectation$I40, params = c('epsilon', 'alpha'),
  fixed_vals = all_output[1, names(ranges)[-c(6,7)]], plot_type = 'var') +
  geom_point(data = all_output[1,], aes(x = epsilon, y = alpha))
```



We see that the black point in the dark blue area, corresponding to the first point in `a11_output`, does not have variance zero. This is because the mean emulator is trying to predict the ‘true’ mean at the point, but it has only been given a sample mean, which it cannot assume is equal to the ‘true’ mean. The emulator internally compensates for this incomplete information, resulting in the variance not being zero, even at points in the training set. This compensation also implies that we can work with different numbers of replicates at different parameter sets. This can be very useful: for example, if we know that a model is expensive to run in some part of the input space, we can do fewer replicates there and the emulators will take the disparity in the number of model runs into account.

Chapter 6

Implausibility

We quickly review the definition of implausibility measure. For a given model output and a given target, the implausibility measures the difference between the emulator output and the target, taking into account all sources of uncertainty. For a parameter set x , the general form for the implausibility $\text{Imp}(x)$ is

$$\text{Imp}(x) = \frac{|f(x) - z|}{\sqrt{V_0 + V_c(x) + V_s + V_m}},$$

where $f(x)$ is the emulator output, z the target, and the terms in the denominator refer to various forms of uncertainty. In particular

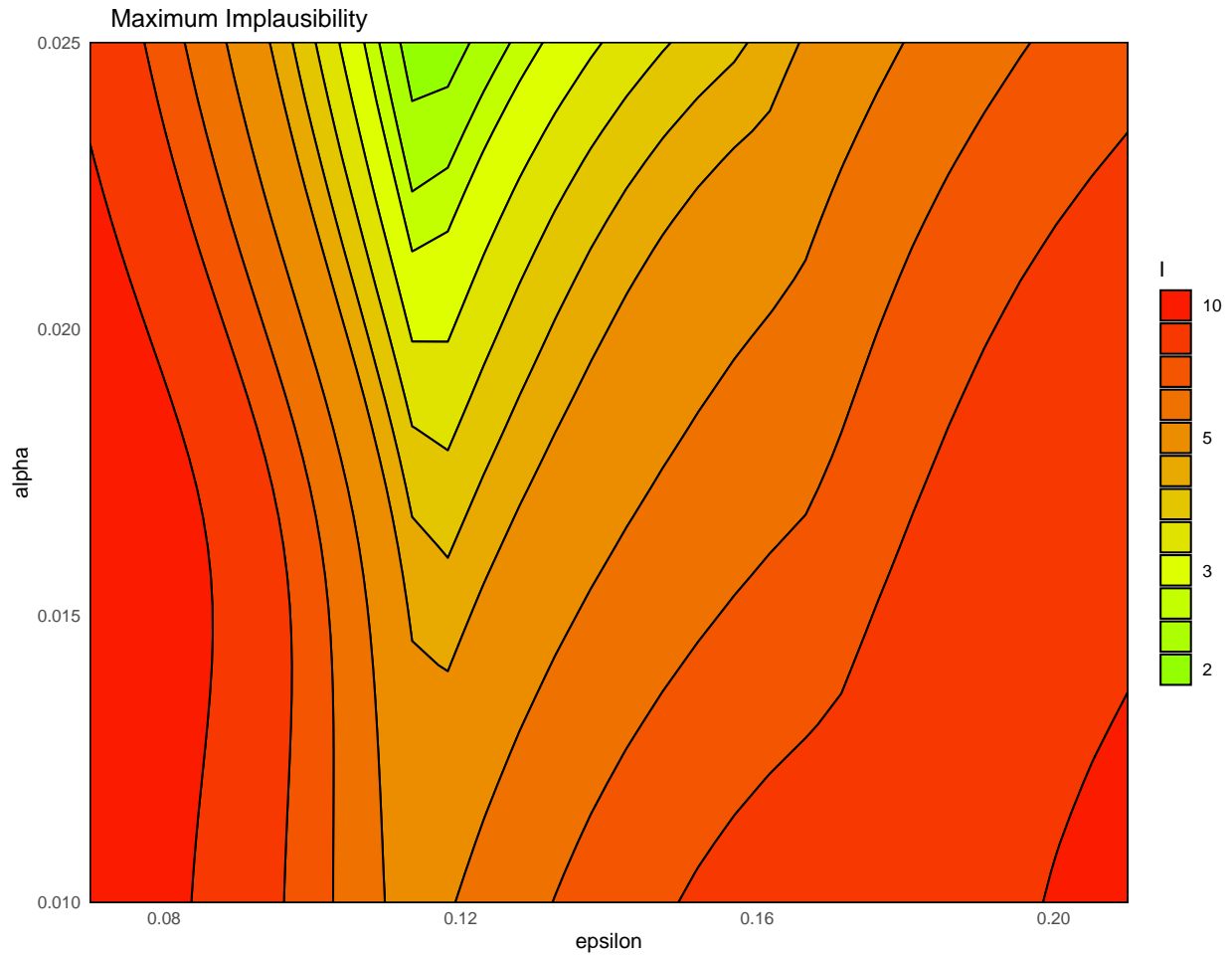
- V_0 is the variance associated with the observation uncertainty;
- $V_c(x)$ refers to the uncertainty one introduces when using the emulator output instead of the model output itself;
- V_s is the ensemble variability and represents the stochastic nature of the model;
- V_m is the model discrepancy, accounting for possible mismatches between the model and reality.

Since in this case study we want to emulate our model, without reference to a real-life analogue, the model represents the reality perfectly. For this reason we have $V_m = 0$. The term V_s , which was zero when we worked on the deterministic SEIRS model ([Workshop 1](#)), is now different from zero, in order to account for the stochasticity of the model.

A very large value of $\text{Imp}(x)$ means that we can be confident that the parameter set x does not provide a good match to the observed data, even factoring in the additional uncertainty that comes with the use of emulators.

To plot the maximum implausibility (across all emulators), we set `plot_type` to `nimp` in the call to `emulator_plot`:

```
emulator_plot(stoch_emulators, plot_type = 'nimp',
              targets = targets, params = c('epsilon', 'alpha'))
```

**Task 2**

Use the argument `fixed_vals` to set the parameters that are not shown in the plot to be as in `chosen_params`. Verify that the implausibility at `chosen_params` is below 3.

Show: Solution on P50

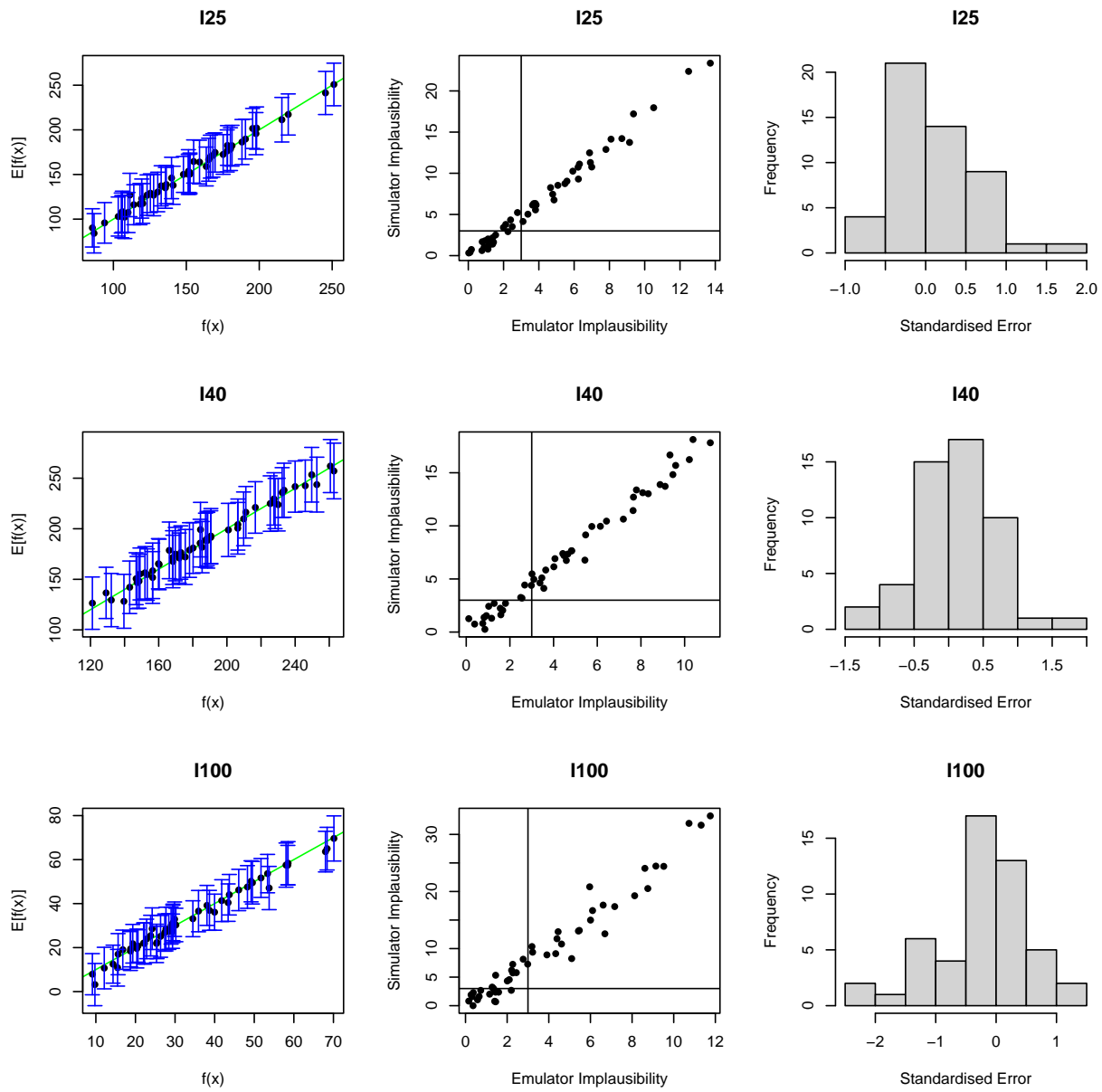
Chapter 7

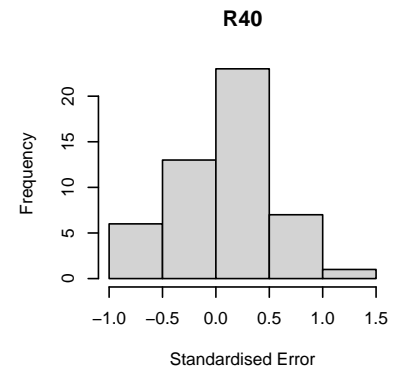
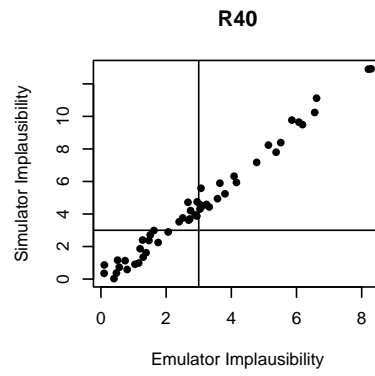
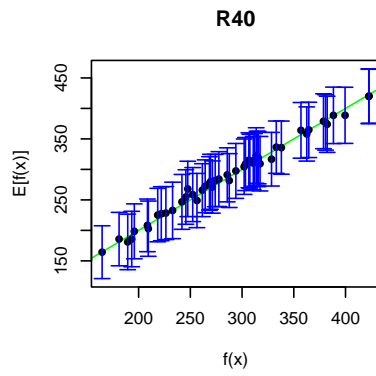
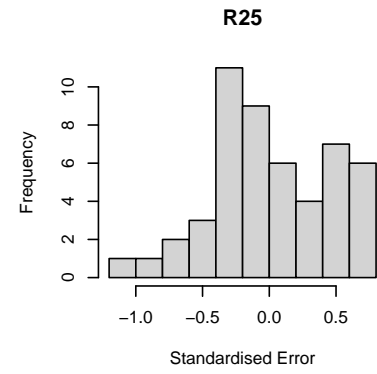
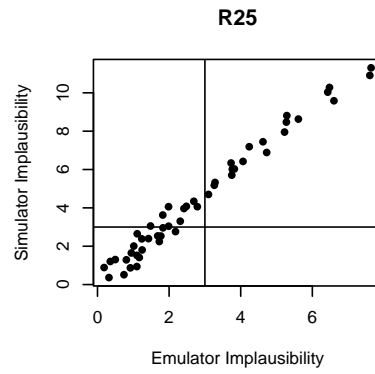
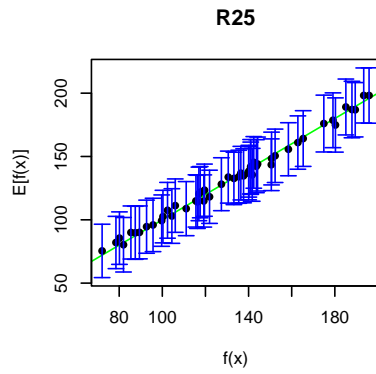
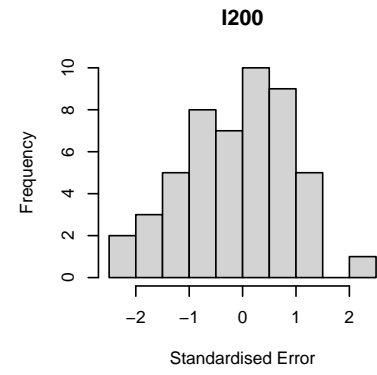
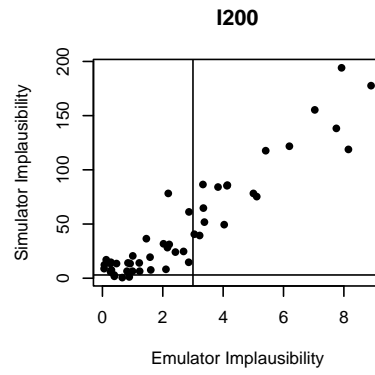
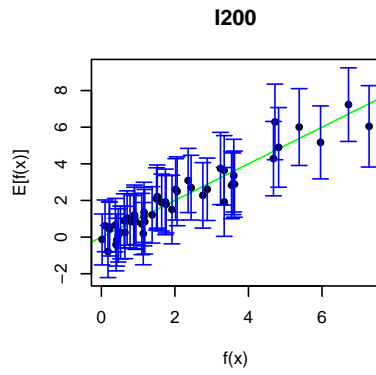
Emulator diagnostics

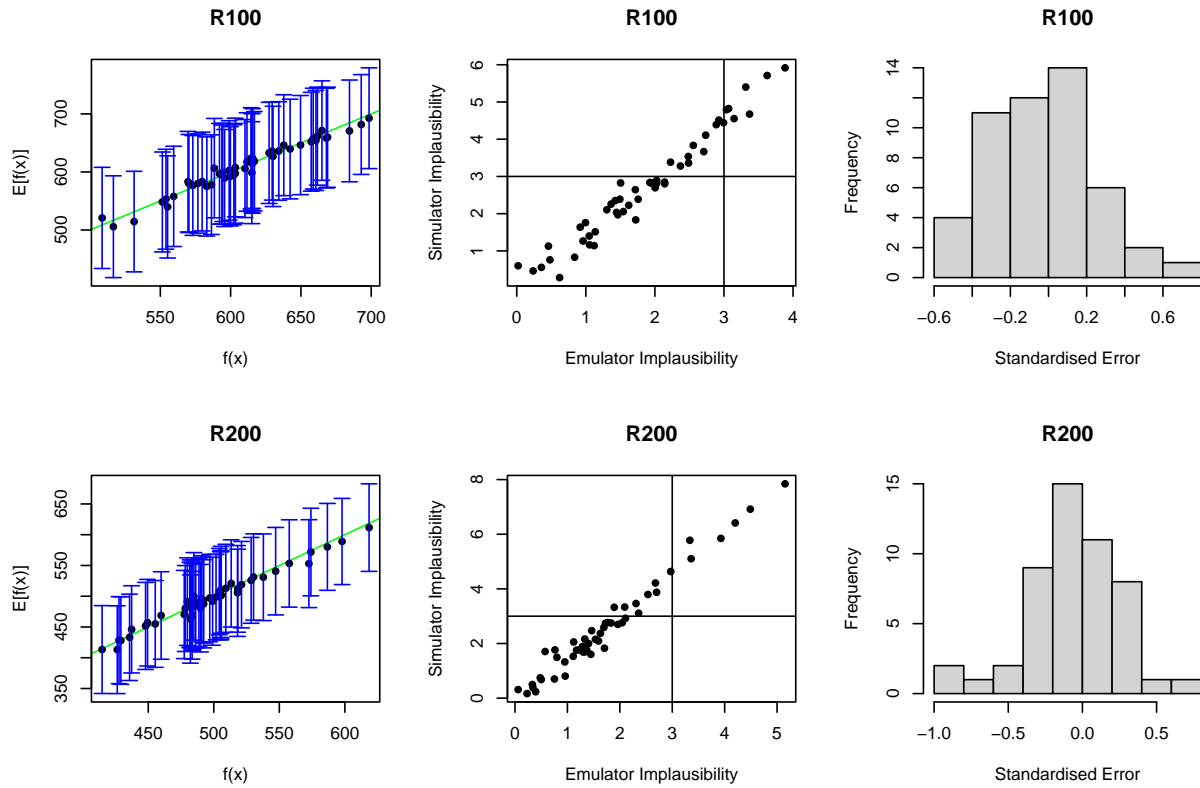
The function `validation_diagnostics` can be used as in the deterministic case, to get three diagnostics for each emulated output.

Show: Remind me of what each diagnostics means on P55

```
vd <- validation_diagnostics(stoch_emulators, targets, all_valid, plt=TRUE)
```







As in deterministic case, we can enlarge the σ values of the mean emulators to obtain more conservative emulators, if needed. Based on the diagnostics above, all emulators perform quite well, so we won't modify any σ values here.

Note that the output `vd` of the `validation_diagnostics` function is a dataframe containing all parameter sets in the validation dataset that fail at least one of the three diagnostics. This dataframe can be used to automate the validation step of the history matching process. For example, one can consider the diagnostic check to be successful if `vd` contains at most 5% of all points in the validation dataset. In this way, the user is not required to manually inspect each validation diagnostic for each emulator at each wave of the process.

Chapter 8

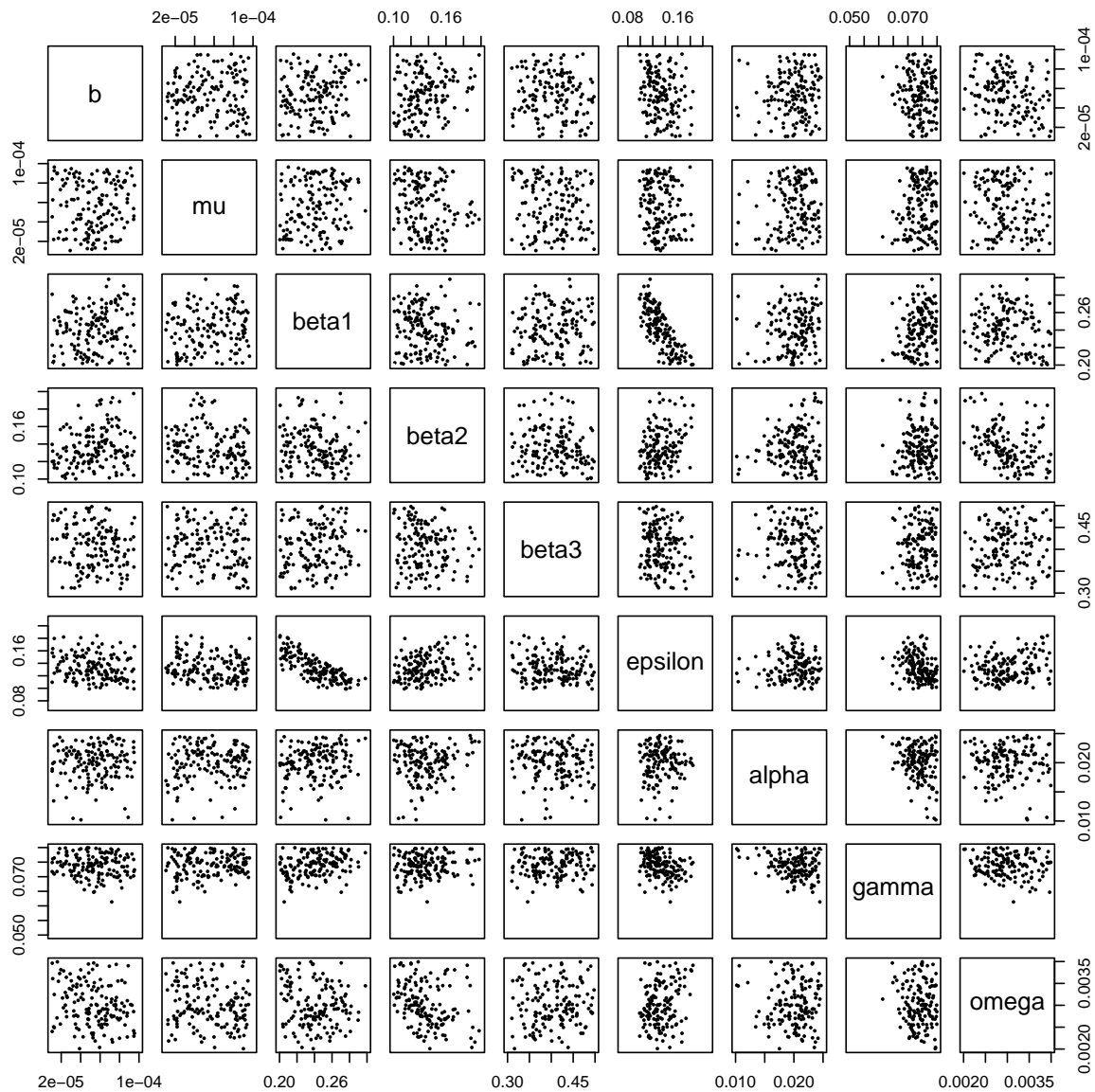
Proposing new points

To generate a set of non-implausible points, based on the trained emulators, we use the function `generate_new_runs`, exactly as in the deterministic case:

```
new_points <- generate_new_runs(stoch_emulators, 150, targets)
```

Here we generated 150 points, since we are going to use 100 of them to train new emulators and 50 of them to validate new emulators (as done in the first wave). We can visualise the non-implausible space at the end of this first wave using `plot_wrap`:

```
plot_wrap(new_points, ranges)
```



Here we see which parameters are more or less constrained at the end of the first wave. For example, it seems clear that low values of γ cannot produce a match (cf. penultimate column). We can also deduce relationships between parameters: β_1 and ϵ are an example of negatively-correlated parameters. If β_1 is large then ϵ needs to be small, and vice versa. Other parameters, such as ω or μ , are instead still spread out across their initial range.

Chapter 9

Second wave

To perform a second wave of history matching and emulation we follow the same procedure as in the previous sections, with two caveats. First, since the parameter sets in `new_points` tend to lie in a small region inside the original input space, we will train the new emulators only on the non-implausible region found in wave one. To do this we define new ranges for the parameters:

```
min_val <- list()
max_val <- list()
new_ranges <- list()
for (i in 1:length(ranges)) {
  par <- names(ranges)[[i]]
  min_val[[par]] <- max(min(new_points[,par])-0.05*diff(range(new_points[,par])),
    ranges[[par]][1])
  max_val[[par]] <- min(max(new_points[,par])+0.05*diff(range(new_points[,par])),
    ranges[[par]][2])
  new_ranges[[par]] <- c(min_val[[par]], max_val[[par]])
}
```

Show: Remind me of why the new lower and upper bounds are defined as above on P56

We now split the `new_points` in two subsets, to train and validate the new emulators. We then run the model on these two subsets as done in the first wave, producing the dataset `new_all_output` with the training data, and the dataset `new_all_valid` with the validation data.

```
t_sample <- sample(1:nrow(new_points), 100)
new_training <- new_points[t_sample,]
new_validation <- new_points[-t_sample,]

new_output_list <- list()
for (i in 1:nrow(new_training)) {
  new_model_out <- get_results(unlist(new_training[i,]), nreps = 50, outs = c("I", "R"),
    times = c(25, 40, 100, 200))
  new_output_list[[i]] <- new_model_out
}
new_validation_list <- list()
for (i in 1:nrow(new_validation)) {
  new_model_out <- get_results(unlist(new_validation[i,]), nreps = 50, outs = c("I", "R"),
    times = c(25, 40, 100, 200))
```

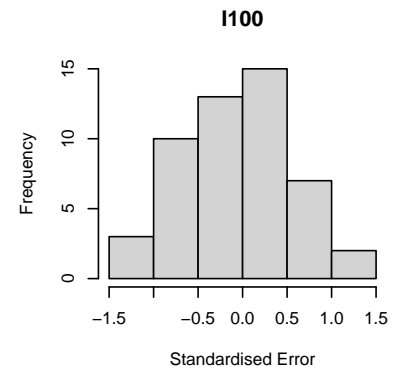
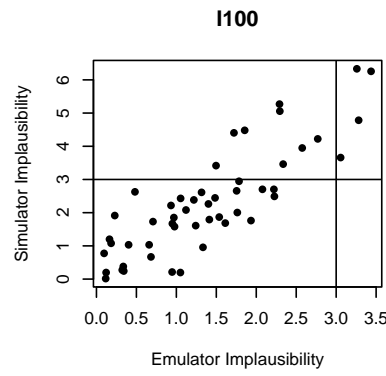
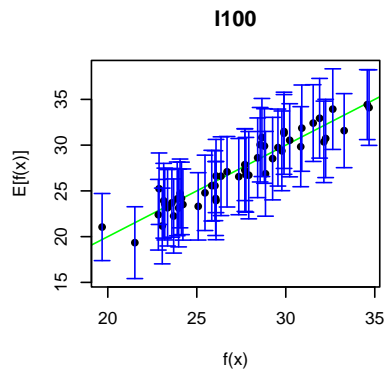
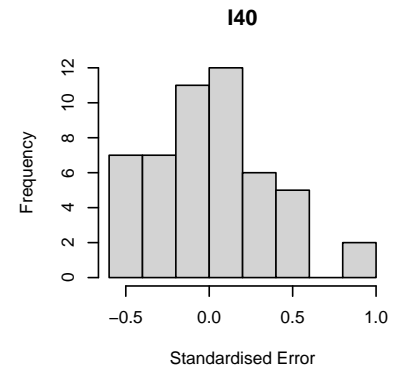
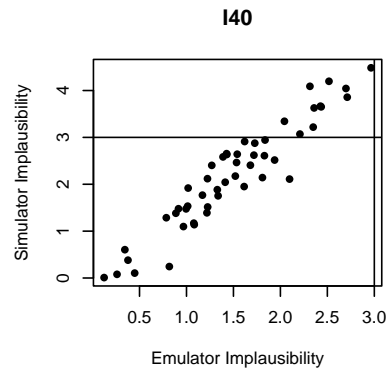
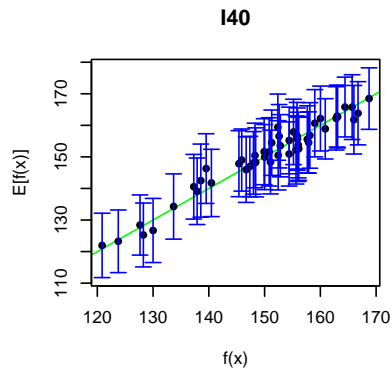
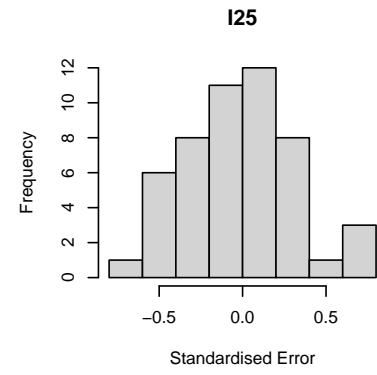
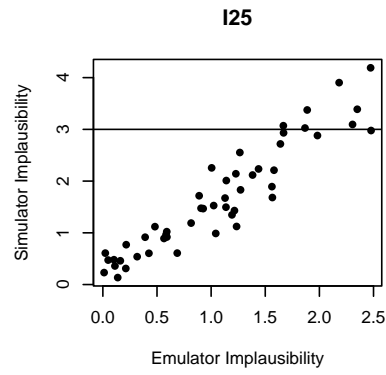
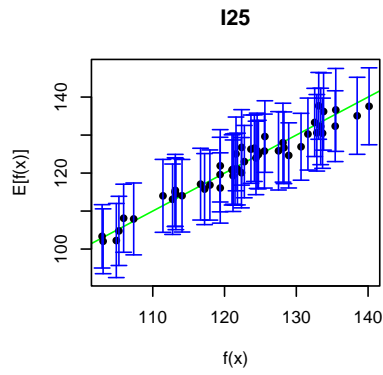
```
    new_validation_list[[i]] <- new_model_out
  }
new_all_output <- data.frame(do.call('rbind',new_output_list))
new_all_valid <- data.frame(do.call('rbind',new_validation_list))
```

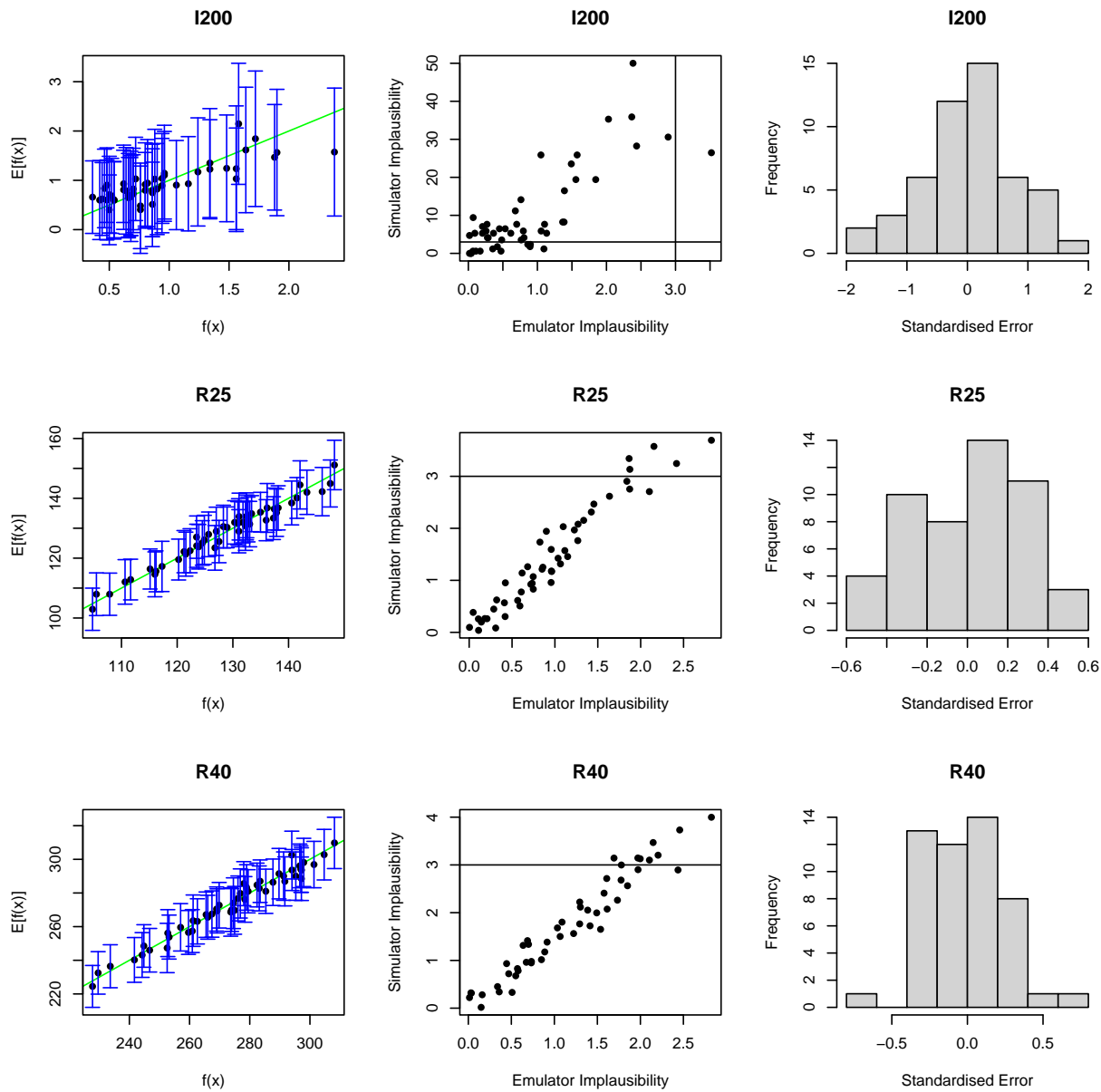
To train new emulators we use `variance_emulator_from_data`, passing the new training data, the outputs names and the new ranges.

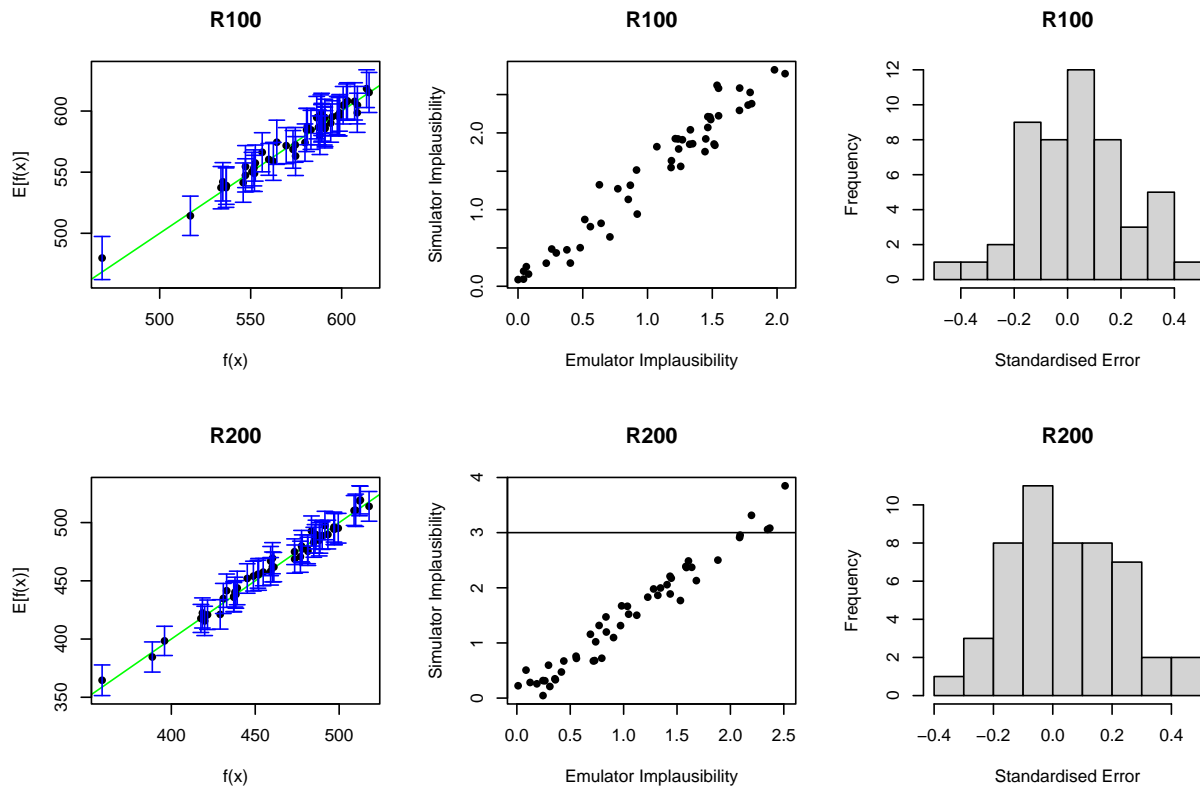
```
new_stoch_emulators <- variance_emulator_from_data(new_all_output, output_names, new_ranges)
```

As usual, before using the obtained emulators, we need to check their diagnostics. We use the function `validation_diagnostics` which takes the new emulators, the targets and the new validation data:

```
vd <- validation_diagnostics(new_stoch_emulators, targets, new_all_valid, plt=TRUE)
```







Since these diagnostics look good, we can generate new non-implausible points. Here is the second caveat: we now need to pass both `new_stoch_emulators` and `stoch_emulators` to the `generate_new_runs` function, since a point needs to be non-implausible for all emulators trained so far, and not just for emulators trained in the current wave:

```
new_new_points <- generate_new_runs(c(new_stoch_emulators, stoch_emulators), 150, targets)
```

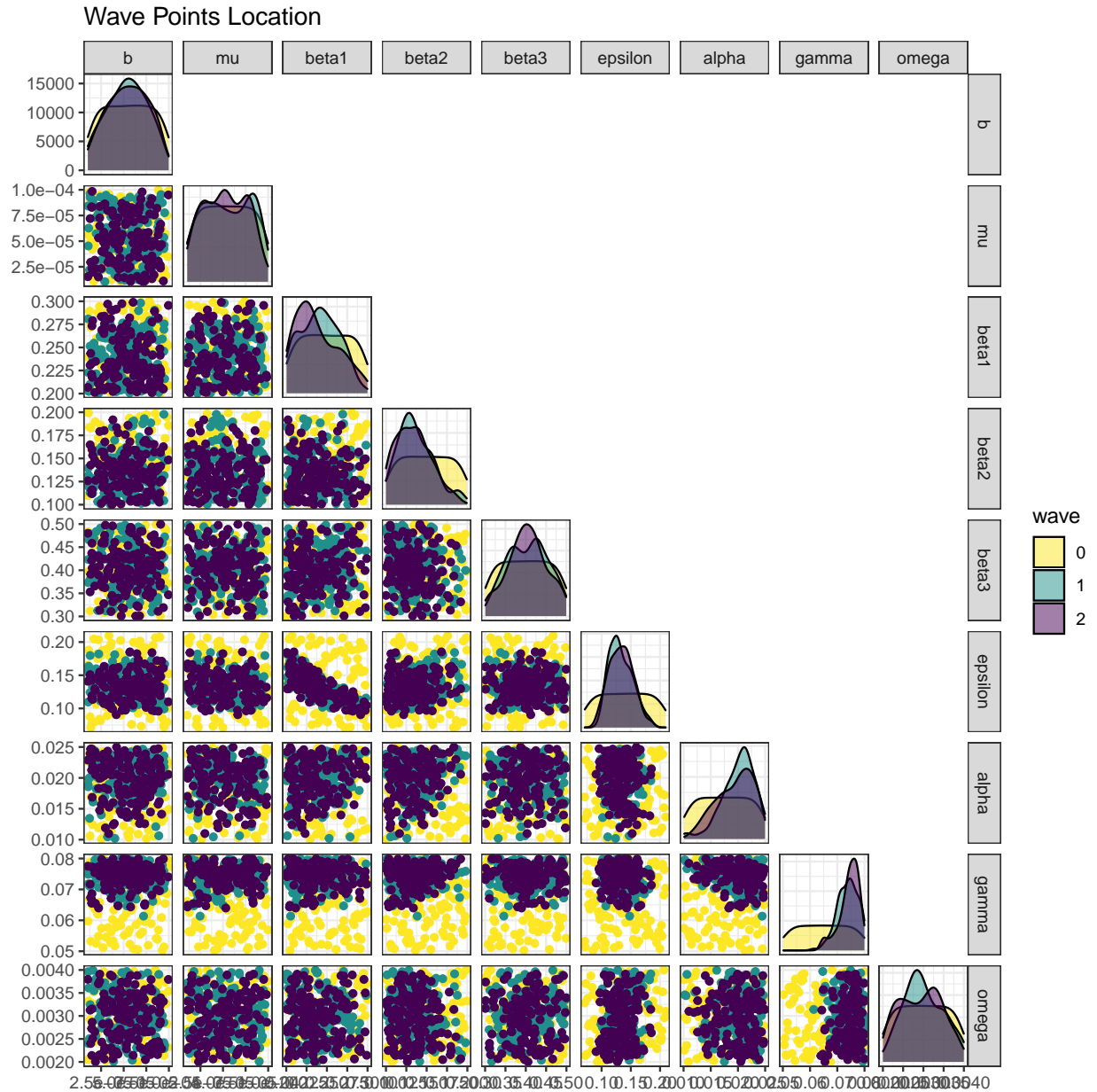

Chapter 10

Visualisation of the non-implausible space by wave

In this last section we present three visualisations that can be used to compare the non-implausible space identified at different waves of the process.

The first visualisation, obtained through the function `wave_points`, shows the distribution of the non-implausible space for the waves of interest. For example, let us plot the distribution of parameter sets at the beginning, at the end of wave one and at the end of wave two:

```
wave_points(list(initial_points, new_points, new_new_points), input_names = names(ranges))
```



Here we can easily see that the distributions of the second and third wave points are more narrow than those of the first wave.

The second visualisation allows us to assess how much better parameter sets at later waves perform compared to the original `initial_points`. Let us start by evaluating the model at parameter sets in `new_new_points`:

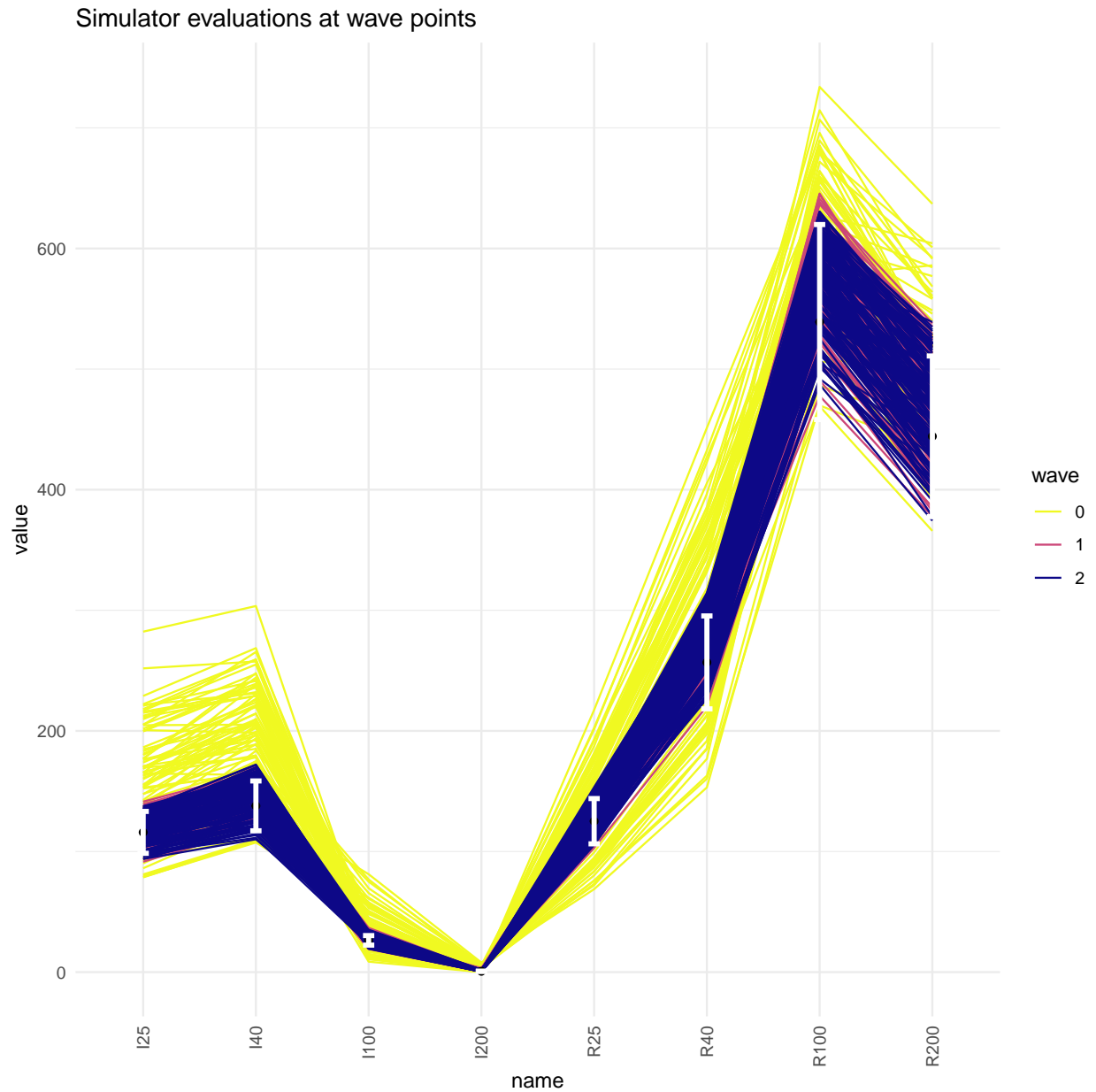
```
new_new_output_list <- list()
for (i in 1:nrow(new_new_points)) {
  new_new_model_out <- get_results(unlist(new_new_points[i,]), nreps = 50, outs = c("I", "R"),
    times = c(25, 40, 100, 200, 300, 350, 450))
  new_new_output_list[[i]] <- new_new_model_out
}
new_new_all_output <- data.frame(do.call('rbind', new_new_output_list))
```


Since our model is stochastic, for a given parameter set x , we are interested in the mean of each model output at x across different realisations, rather than in the output of each realisation at x . For this reason, we will use the helper function `aggregate_points`, which calculates the mean of each output across different realisations.

```
all_output_aggregated <- aggregate_points(all_output, names(ranges))
new_all_output_aggregated <- aggregate_points(new_all_output, names(ranges))
new_new_all_output_aggregated <- aggregate_points(new_new_all_output, names(ranges))
all_aggregated <- list(all_output_aggregated, new_all_output_aggregated,
                      new_new_all_output_aggregated)
```

We are now ready to compare the performance of parameter sets at the end of each wave, passing the list `all_aggregated` and the list `targets` to the function `simulator_plot`:

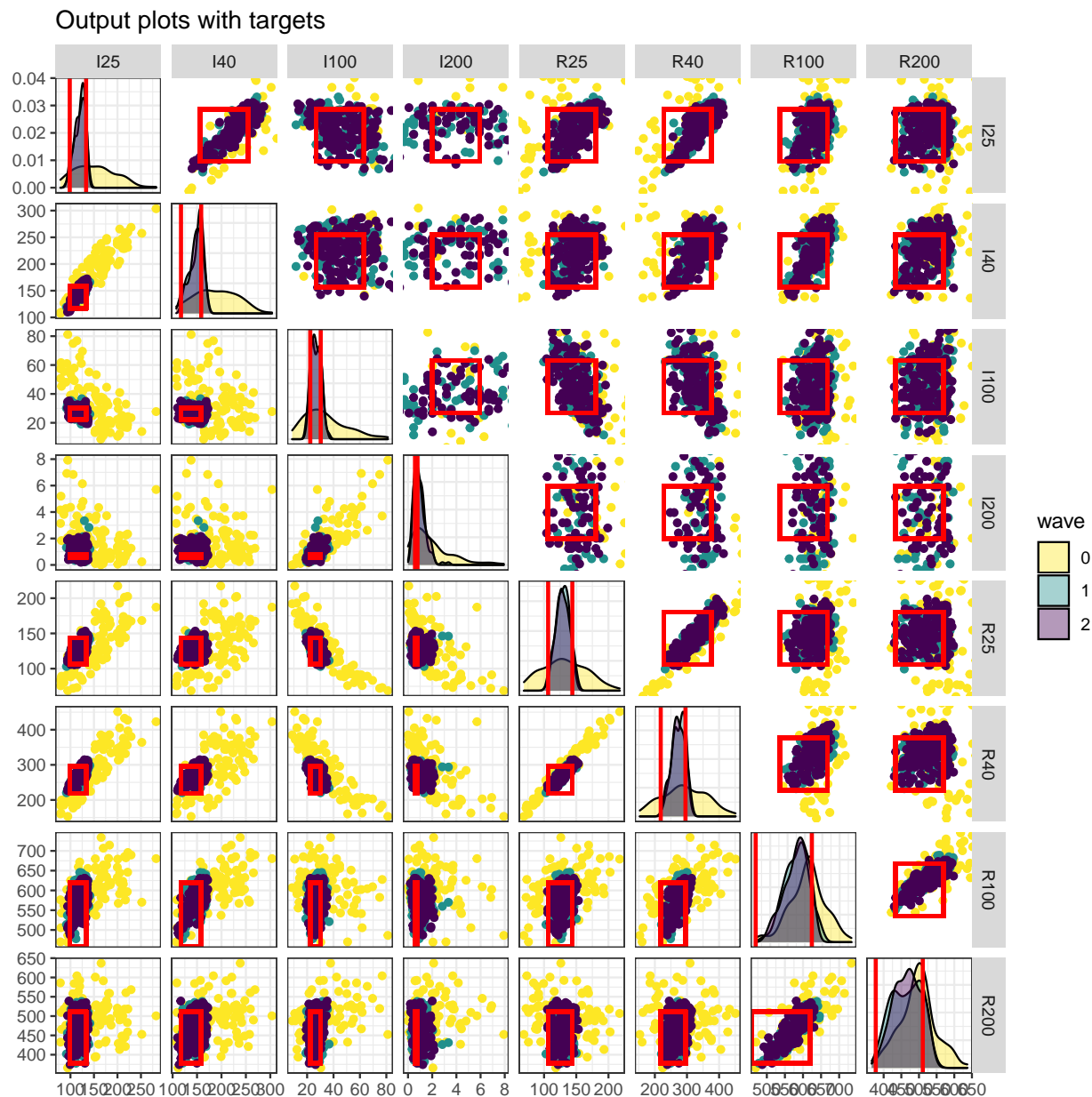
```
simulator_plot(all_aggregated, targets, barcol = "white")
```



Note that in this call we set `barcol="white"` in order to have the target intervals in white, instead of black. This plot clearly shows that parameter sets at the end of the first and second wave perform better than the initial set of points.

In the third visualisation, output values for non-implausible parameter sets at each wave are shown for each combination of two outputs:

```
wave_values(all_aggregated, targets, l_wid=1)
```



The argument `l_wid` is optional and helps customise the width of the red lines that create the target boxes. The main diagonal shows the distribution of each output at the end of each wave, with the vertical red lines indicating the lower and upper bounds of the target. Above and below the main diagonal are plots for each pair of targets, with rectangles indicating the target area where full fitting points should lie (the ranges are normalised in the figures above the diagonals). These graphs can provide additional information on output distributions, such as correlations between them. For example, here we see positive correlations between *I25* and *R25* and *R40*.

In this workshop we have shown how to perform the first two waves of the history matching process on a stochastic model. Of course, more waves are required, in order to complete the calibration task.

Show: Remind me of possible stopping criteria on P56

We conclude by noting that in this workshop we dealt with stochasticity, but not with bimodality. The calibration of stochastic models that present a bimodal behaviour is addressed in Workshop 3.

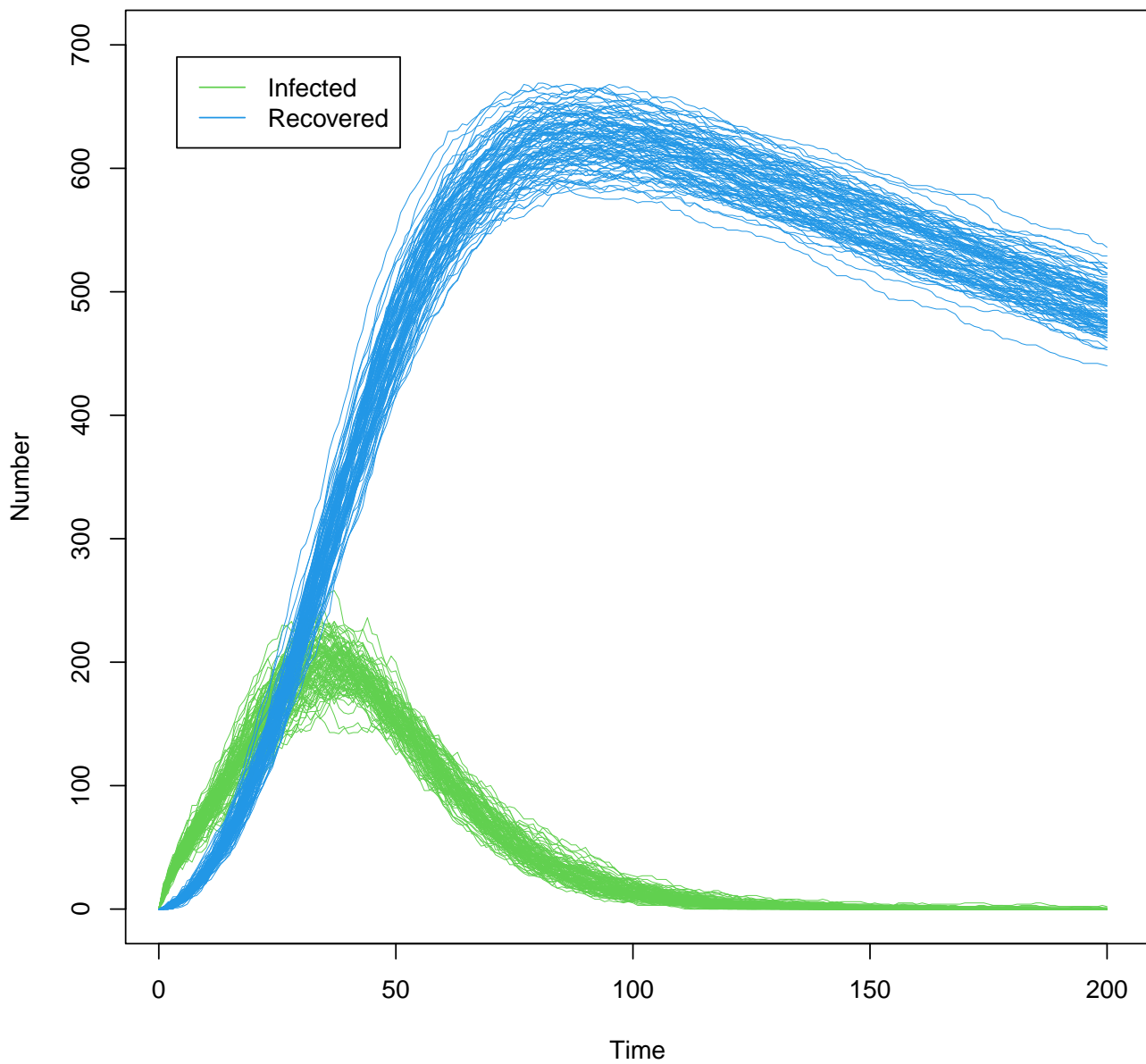
Appendix A

Answers

Solution 1

Let us see what happens when a higher force of infection is considered:

```
higher_foi_params <- c(
  b = 1/(76*365),
  mu = 1/(76*365),
  beta1 = 0.3, beta2 = 0.1, beta3 = 0.5,
  epsilon = 1/7,
  alpha = 1/50,
  gamma = 1/14,
  omega = 1/365
)
higher_foi_solution <- get_results(higher_foi_params, outs = c("I", "R"),
                                   times = c(25, 40, 100, 200), raw = TRUE)
plot(0:200, ylim=c(0,700), ty="n", xlab = "Time", ylab = "Number")
for(j in 3:4) for(i in 1:100) lines(0:200, higher_foi_solution[,j,i], col=(3:4)[j-2], lwd=0.3)
legend('topleft', legend = c('Infected', "Recovered"), lty = 1,
      col = c(3,4), inset = c(0.05, 0.05))
```



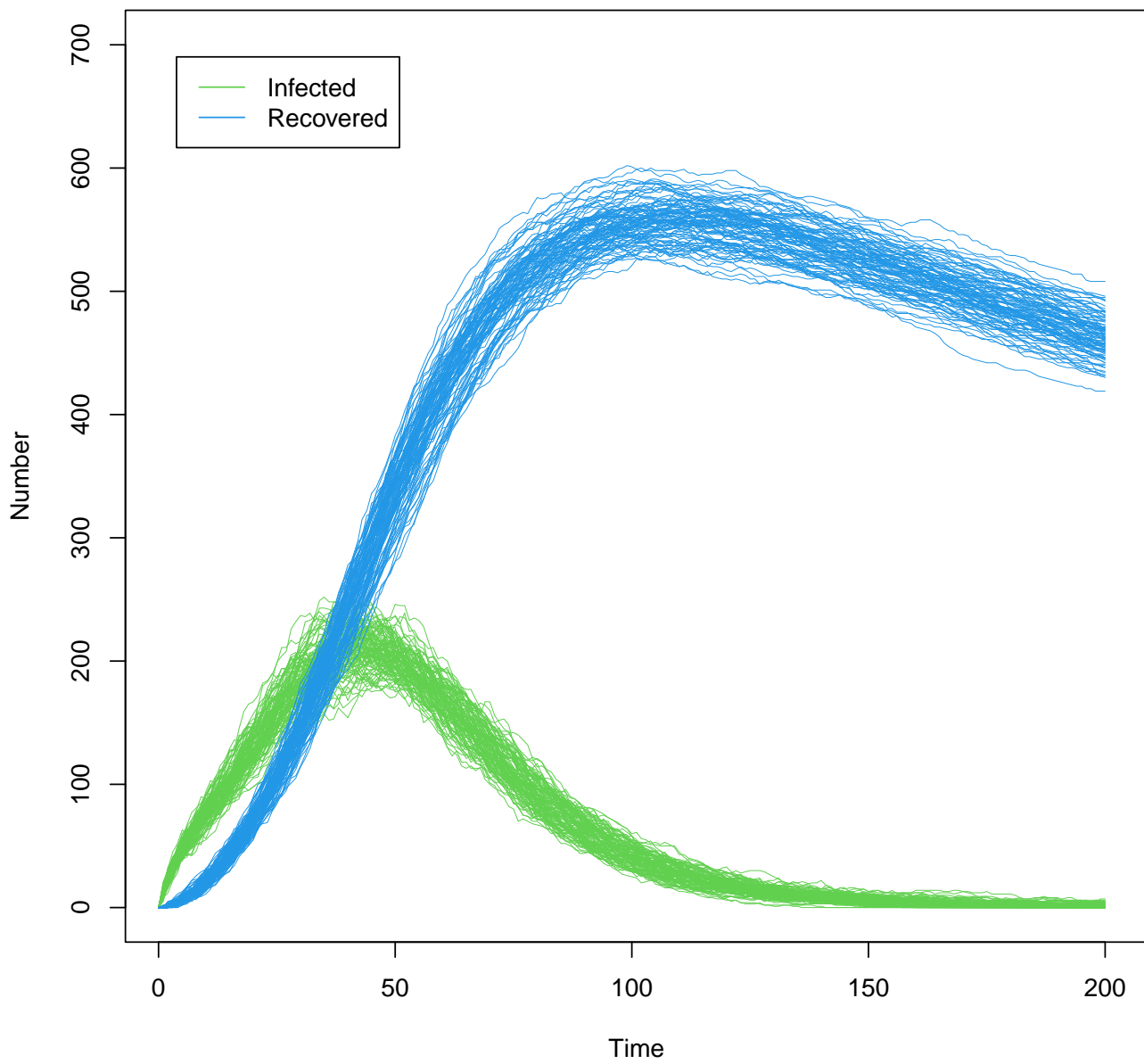
Comparing this with the plot produced before the task, we see that the peaks of recovered and infectious individuals have now increased, as expected.

What happens when a lower value of the recovery rate γ is used?

```

smaller_gamma_params <- c(
  b = 1/(60*365),
  mu = 1/(76*365),
  beta1 = 0.214, beta2 = 0.107, beta3 = 0.428,
  epsilon = 1/7,
  alpha = 1/50,
  gamma = 1/20,
  omega = 1/365
)
smaller_gamma_solution <- get_results(smaller_gamma_params, outs = c("I", "R"),
                                     times = c(25, 40, 100, 200), raw = TRUE)
plot(0:200, ylim=c(0,700), ty="n", xlab = "Time", ylab = "Number")
for(j in 3:4) for(i in 1:100) lines(0:200, smaller_gamma_solution[,j,i],
                                   col=(3:4)[j-2], lwd=0.3)
legend('topleft', legend = c('Infected', "Recovered"), lty = 1,
      col = c(3,4), inset = c(0.05, 0.05))

```



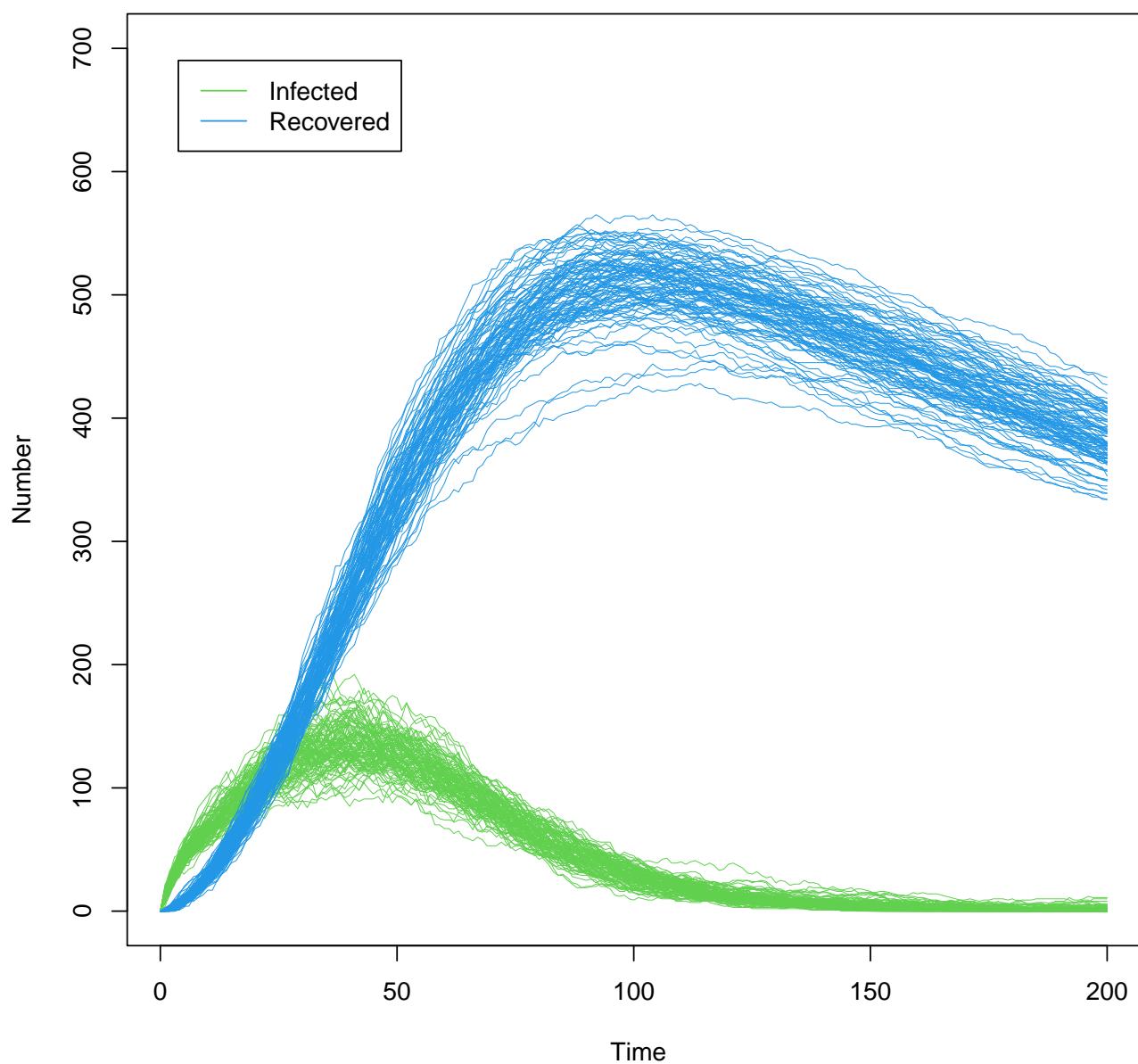
As one expects, this causes the peak of infectious individuals to increase.

Finally, what happens when we increase the rate at which immunity is lost ω ?


```

higher_omega_params <- c(
  b = 1/(60*365),
  mu = 1/(76*365),
  beta1 = 0.214, beta2 = 0.107, beta3 = 0.428,
  epsilon = 1/7,
  alpha = 1/50,
  gamma = 1/14,
  omega = 0.004
)
higher_omega_solution <- get_results(higher_omega_params, outs = c("I", "R"),
                                     times = c(25, 40, 100, 200), raw = TRUE)
plot(0:200, ylim=c(0,700), ty="n", xlab = "Time", ylab = "Number")
for(j in 3:4) for(i in 1:100) lines(0:200, higher_omega_solution[,j,i],
                                   col=(3:4)[j-2], lwd=0.3)
legend('topleft', legend = c('Infected', "Recovered"), lty = 1,
      col = c(3,4), inset = c(0.05, 0.05))

```



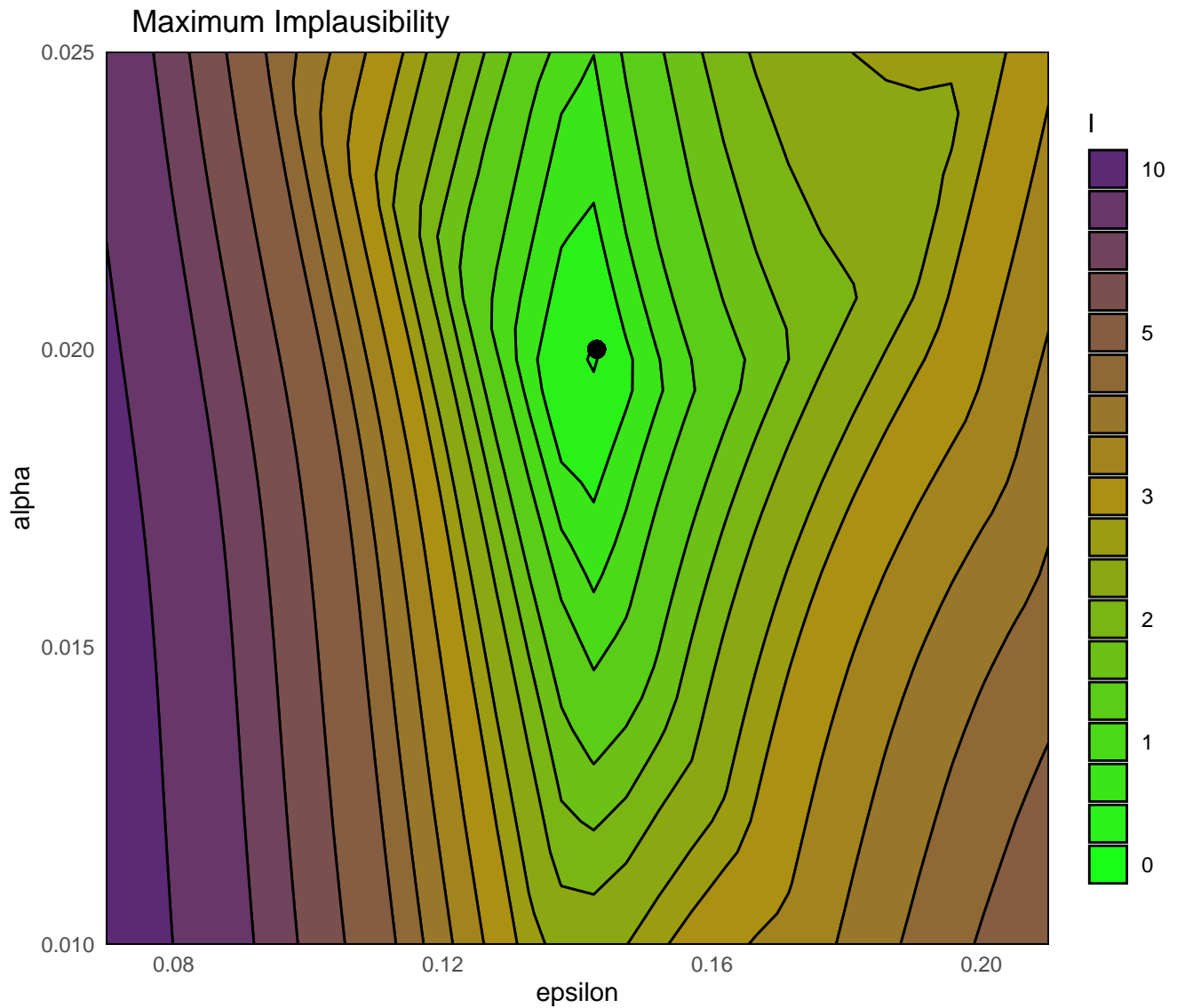
As one expects, this causes the peak of recovered individuals to decrease.

[Return to task on P13](#)

Solution 2

We set `fixed_vals` to `chosen_params[!names(chosen_params) %in% c('epsilon', 'alpha')]`, plot the maximum implausibility and add a point corresponding to the values of σ and α in `chosen_params`:

```
emulator_plot(stoch_emulators, plot_type = 'nimp', targets = targets,
              params = c('epsilon', 'alpha'),
              fixed_vals = chosen_params[!names(chosen_params) %in% c('epsilon', 'alpha')],
              cb=T) +geom_point(aes(x=1/7, y=1/50), size=3)
```



The plot shows what we expected: when ϵ and α are equal to their values in `chosen_params`, the implausibility measure is below the threshold 3 (cf. black point in the box).

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Appendix B

Additional information

Code to load relevant libraries and helper functions

```

library(hmer)
library(lhs)
library(deSolve)
library(ggplot2)
library(reshape2)
library(purrr)
library(tidyverse)
##### HELPER FUNCTIONS #####
# Function that calculates the mean of the model outputs from several runs
# at the same parameter set
aggregate_points <- function(data, input_names, func = 'mean') {
  unique_points <- unique(data[,input_names])
  uids <- apply(unique_points, 1, rlang::hash)
  data_by_point <- purrr::map(uids, function(h) {
    data[apply(data[,input_names], 1, rlang::hash) == h,]
  })
  aggregate_func <- get(func)
  aggregate_values <- do.call('rbind.data.frame', purrr::map(data_by_point, function(x) {
    output_data <- x[,!(names(x) %in% input_names)]
    apply(output_data, 2, aggregate_func)
  })))
  return(setNames(cbind.data.frame(unique_points, aggregate_values), names(data)))
}
# Function that implements the SEIRS stochastic model using the Gillespie algorithm
gillespie=function(N, T=400, dt=1, ...)
{
  tt=0
  n=T/%dt
  x=N$M
  S=t(N$Post-N$Pre)
  u=nrow(S)
  v=ncol(S)
  xmat=matrix(0,ncol=u,nrow=n)
  i=1
  target=0
  repeat {
    h=N$h(x, tt, ...)
    h0=sum(h)
    if (h0<1e-10)
      tt=1e99
    else
      tt=tt+rexp(1,h0)
    while (tt>=target) {
      xmat[i,]=x
      i=i+1
      target=target+dt
      if (i>n)
        return(xmat)
    }
    j=sample(v,1,prob=h)
    x=x+S[,j]
  }
}
Num <- 1000
N=list()
N$M=c(900,100,0,0,0)
N$Pre = matrix(c(0,0,0,0,0,
                  1,0,0,0,0,
                  1,0,0,0,0,
                  0,1,0,0,0,
                  0,1,0,0,0,

```

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R tip

Copy the code below, modify the value of (some) parameters and run it.

```
example_params <- c(
  b = 1/(76*365),
  mu = 1/(76*365),
  beta1 = 0.214, beta2 = 0.107, beta3 = 0.428,
  epsilon = 1/7,
  alpha = 1/50,
  gamma = 1/14,
  omega = 1/365
)
solution <- get_results(example_params, outs = c("I", "R"),
  times = c(25, 40, 100, 200), raw = TRUE)
plot(0:200, ylim=c(0,700), ty="n", xlab = "Time", ylab = "Number")
for(j in 3:4) for(i in 1:100) lines(0:200, solution[,j,i], col=(3:4)[j-2], lwd=0.3)
legend('topleft', legend = c('Infected', "Recovered"), lty = 1,
  col = c(3,4), inset = c(0.05, 0.05))
```

[Return to P13](#)

Remind me of what each diagnostics means

The first plot shows the emulator outputs plotted against the model outputs. In particular, the emulator expectation is plotted against the model output for each validation point, providing the dots in the graph. The emulator uncertainty at each validation point is shown in the form of a vertical interval that goes from 3σ below to 3σ above the emulator expectation, where σ is the emulator variance at the considered point. An 'ideal' emulator would exactly reproduce the model results: this behaviour is represented by the green line $f(x) = E[f(x)]$ (this is a diagonal line, visible here only in the bottom left and top right corners). Any parameter set whose emulated prediction lies more than 3σ away from the model output is highlighted in red. Note that we do not need to have no red points for the test to be passed: since we are plotting 3σ bounds, statistically speaking it is ok to have up to 5% of validation points in red (see [Pukelsheim's 3 \$\sigma\$ rule](#)).

The second column compares the emulator implausibility to the equivalent model implausibility (i.e. the implausibility calculated replacing the emulator output with the model output). There are three cases to consider:

- The emulator and model both classify a set as implausible or non-implausible (bottom-left and top-right quadrants). This is fine. Both are giving the same classification for the parameter set.
- The emulator classifies a set as non-implausible, while the model rules it out (top-left quadrant): this is also fine. The emulator should not be expected to shrink the parameter space as much as the model does, at least not on a single wave. Parameter sets classified in this way will survive this wave, but may be removed on subsequent waves as the emulators grow more accurate on a reduced parameter space.
- The emulator rules out a set, but the model does not (bottom-right quadrant): these are the problem sets, suggesting that the emulator is ruling out parts of the parameter space that it should not be ruling out.

As for the first test, we should be alarmed only if we spot a systematic problem, with 5% or more of the points in the bottom-right quadrant.

Finally, the third column gives the standardised errors of the emulator outputs in light of the model output: for each validation point, the difference between the emulator output and the model output is calculated, and then divided by the standard deviation σ of the emulator at the point. The general rule is that we want our standardised errors

to be somewhat normally distributed around 0, with 95% of the probability mass between -2 and 2 . When looking at the standard errors plot, we should ask ourselves at least the following questions:

- Is more than 5% of the probability mass outside the interval $[-2, 2]$? If the answer is yes, this means that, even factoring in all the uncertainties in the emulator and in the observed data, the emulator output is too often far from the model output.
- Is 95% of the probability mass concentrated in a considerably smaller interval than $[-2, 2]$ (say, for example, $[-0.5, 0.5]$)? For this to happen, the emulator uncertainty must be quite large. In such case the emulator, being extremely cautious, will cut out a small part of the parameter space and we will end up needing many more waves of history matching than are necessary, or, even worse, we just won't be able to reduce the non-implausible parameter space.
- Is the histogram skewing significantly in one direction or the other? If this is the case, the emulator tends to either overestimate or underestimate the model output.

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Remind me of why the new lower and upper bounds are defined as above

The list `new_ranges` contains lower and upper bounds for each parameter. The upper bound for a given parameter is determined in the following way: - Among all points in `new_points`, the maximum value of the parameter is identified. - 5% of the original range of the parameter is added to the maximum value found in the previous point. This step enlarges slightly the new range, to make sure that we are including all non-implausible points. - The minimum between the value found above and the upper bound in the original ranges list is selected: this ensure that we do not end up with a new upper bound which is larger than the original one.

A similar calculation was adopted to determine the new lower bounds of parameters.

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Remind me of possible stopping criteria

Since this is an iterative process, at the end of each wave we need to decide whether to perform a new wave or to stop. One possible stopping criterion consists of comparing the emulator uncertainty and the target uncertainty. If the former is larger, another wave can be performed, since new, more confident emulators can potentially help further reduce the non-implausible space. If the uncertainty of emulators is smaller than the uncertainty in the targets, improving the performance of emulators would not make a substantial difference, and additional waves would not be beneficial. We may also choose to stop the iterations when we get emulators that provide us with full fitting points at a sufficiently high rate. In such a case, rather than spending time training new emulators, we can simply use the current emulators and generate points until we find enough full fitting ones. Finally, we might end up with all the input space deemed implausible at the end of a wave. In this situation, we would deduce that there are no parameter sets that give an acceptable match with the data: in particular, this would raise doubts about the adequacy of the chosen model, or input and/or output ranges.

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