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Discrete Optimization

Simultaneous optimization of classical objectives in JIT scheduling

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Abstract

Just-in-time production models have been developed in recent years in order to reduce costs of diversified small-lot productions. The methods aim at maintaining the production rate of each type of part as smooth as possible and therefore holding small inventory and shortage costs. Different ways of measuring the slack between a given schedule and the ideal no-inventory no-shortage production have been considered in the literature. This paper compares the three most studied objective functions and refutes several conjectures that have been formulated in the last few years.

Keywords: Multiple criteria analysis; Balanced schedules; Just-in-time

1. Introduction

Just-in-time (JIT) manufacturing environments have been developed in order to reduce costs of diversified small-lot production. This model aims at holding inventory and shortage costs as small as possible.

Monden [12] states that the most important goal of a JIT system is to keep the schedule as balanced as possible, *i.e.* to keep the production rate of each type of product per unit of time as smooth as possible. Different ways of measuring the deviation of the effective production from the ideal perfectly balanced schedule have been considered in the literature. The main objective of this paper is to compare the three most studied objective functions and to refute several conjectures that have been formulated in the last few years. It also describes tools and models characterizing instances for which some solutions optimize several objective functions simultaneously.

We use two different formulations of the JIT scheduling problem. The first one, a linear program, has been introduced by Miltenburg in [11]. The second one is a reformulation by Kubiak and Sethi [9] as an assignment problem.

Section 2 reviews previous research on JIT manufacturing. Section 3 details the constraints of the problem and the notations. In Section 4, a description of JIT problems as assignment problems [13,9] is presented and a certificate for the optimality of min-sum sequences is explained. In Section 5, we analyze the objective functions *sum of deviations*

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and *sum of deviations squared*. In particular, we obtain new results in the case where the maximum deviation is less or equal to 1. Using the cost structure of [9], we show that for these two objective functions the costs coincide. In Section 6, we include also the *maximum deviation* objective function and show that no simultaneous optimization is possible in general. The conclusion (Section 7) was specially designed to allow the reader (already aware of the definitions in JIT) to have an overview of the new results.

2. Literature review

In this section, we review previous researches on JIT manufacturing. The description by Monden [12] of Toyota's production system created the first interest in leveled scheduling. By proposing an optimization formulation of the problem, Miltenburg [11] has led to a considerable amount of research. This formulation aims at minimizing the *total deviation* or sum of all deviations of the real production from the ideal but rational production. When the deviations are convex, non-negative functions, Kubiak and Sethi [8,9] prove that this optimization model can be reformulated as an assignment problem. In this formulation, deviations from the ideal are represented by penalties for placing parts earlier or tardier than their location in the ideal sequence. Inman and Bulfin [5] give a pseudo-polynomial heuristic for total deviation problems with slightly different penalty functions. They consider that the due date of a part is its ideal instant of production and solve the problem with an Earliest Due Date Rule.

Another class of objective functions has been widely considered in the literature. The constraints remain the same and the goal is to minimize the *maximum deviation* of the real production from the ideal. Steiner and Yeomans [13] show that, when considered as a one-machine scheduling problem with release and due dates, the model could be reduced to a perfect matching problem in a bipartite graph. From this approach, they obtain a pseudo-polynomial time algorithm. Brauner and Crama [1] show that the maximum deviation JIT problem is in Co-NP. Note that it is even difficult to know whether or not total deviation or maximum deviation problems are in NP since the output is not polynomial in the size of the input. Similar issues arise for a larger class of high-multiplicity optimization problems (see for instance [2]).

3. Notations and constraints of the problem

3.1. Constraints

A diversified small-lot production consists of n part types with a demand $d_i \in \mathbb{N}$ for part type $i = 1,2,\ldots,n$. Each part is produced in one time period. Let $D = \sum_{i=1}^{n} d_i$ be the total demand. A schedule will be called *uniformly leveled* if, at each time period k, the line has assembled kd_i/D parts of type i. The proportion $r_i = d_i/D$ is called the *ideal production rate* and a JIT schedule tries to keep the effective production rate as close as possible to this ideal. Monden [12] states that it is the main goal of Toyota's JIT systems.

In order to formulate this problem as an optimization problem, we denote $x_{i,k}$, for $i = 1,2,\ldots,n$; $k = 1,2,\ldots,D$ the number of parts of type i produced in the time periods 1 to k.

Miltenburg [11] formulated the constraints of the problem as follows:

$$\sum_{i=1}^{n} x_{i,k} = k, \quad k = 1,2,\ldots,D, \tag{1a}$$

$$x_{i,D} = d_i, \quad i = 1,2,\ldots,n, \tag{1b}$$

$$0 \leqslant x_{i,k} - x_{i,k-1}, \quad i = 1,2,\ldots,n;\ k = 2,3,\ldots,D, \tag{1c}$$

$$x_{i,k} \in \mathbb{N}, \quad i = 1,2,\ldots,n;\ k = 1,2,\ldots,D. \tag{1d}$$

Equality (1a) indicates that k parts have to be produced in the first k time periods; equality (1b) means that all demands have to be satisfied at time period D; inequality (1c) states that, for a given type i, the number of parts produced cannot decrease with time. Hence, (1a) and (1c) together imply that exactly one part is produced per time period.

3.2. Objective functions

The objective function of the problem must describe the fact that we want to keep the effective production 'as close as possible' to the ideal and therefore minimize the distance between a feasible sequence and the ideal production. There is no consensus on which distance is the most adequate and many objective functions have been studied in the literature. In this paper, we consider *max-abs*, *sum-abs* and *sum-sqr* problems, which are the most widely studied.

For the maximum deviation problem, the objective is of the form $F_{\max} = \max_{1 \leqslant i \leqslant n, 1 \leqslant k \leqslant D} F_i(x_{i,k} - kr_i)$. In [13,7,1] for instance, $F_i(x) = |x|$, for $i = 1,2,\ldots,n$ and hence the objective function is

$F_{\max} = \max_{i,k}|x_{i,k} - kr_i|$ which leads to the *max-abs* problem. A result concerning other choices for the functions F_i is presented in Section 6. In Section 4.1, the max-abs problem is formulated as a perfect matching problem in a bipartite graph as in [13].

For the total deviation problem, the objective function is of the form $F_{\text{sum}} = \sum_{k=1}^{D}\sum_{i=1}^{n}F_i(x_{i,k} - kr_i)$. The problem is then denoted *total deviation* problem or *min-sum* problem. In [9], the deviations F_i, for $i = 1,2,\ldots,n$, are convex functions verifying

$$F_i(0) = 0, \quad i = 1,2,\ldots,n \quad \text{and}$$
$$F_i(y) > 0 \quad \text{for } y \neq 0,\ i = 1,2,\ldots,n. \tag{2}$$

For the min-sum problems, the most studied values for F_i are $F_i(x) = |x|$ and $F_i(x) = x^2$ [11,9,7]. In the first case, $F_{\text{sum}} = \sum_{k=1}^{D}\sum_{i=1}^{n}|x_{i,k} - kr_i|$, and we obtain the *sum-abs* problem. In the second case, $F_{\text{sum}} = \sum_{k=1}^{D}\sum_{i=1}^{n}(x_{i,k} - kr_i)^2$ and we refer to *sum-sqr* problems. In Section 4.2, JIT problems with total deviation are formulated as assignment problems [9].

3.3. Notations

We use the following notations.

- The j-th part of type i is denoted (i,j).
- The set of all integers $i \in [a,b]$ is denoted $[a..b]$.
- The ceiling $\lceil a \rceil$ of a real number a is the smallest integer greater than or equal to a.
- The flooring $\lfloor a \rfloor$ of a real number a is the greatest integer smaller than or equal to a.

4. Just-in-time scheduling formulations

In this section, we describe different formulations for max-abs (Section 4.1) and sum-abs (Section 4.2) problems and we present an optimality certificate for sum-abs problems (Section 4.3).

4.1. Maximum deviation and bipartite graphs

The max-abs problem, also denoted MDJIT (maximum deviation JIT problem) in [1], has been analyzed by Steiner and Yeomans [13] and Brauner and Crama [1]. All results of this section are derived from those two papers. Consider the decision version of problem (1) with a max-abs objective function:

max-abs decision problem:
Input:

- $n \in \mathbb{N}$: number of part types
- $d_i \in \mathbb{N}$: demand for part type i, $i = 1,2,\ldots,n$
- $B \in \mathbb{Q}$: a bound

Question: Does there exist an $n \times D$ matrix $x = (x_{i,k})$ such that:

$$\begin{aligned}
&\max_{1\leqslant i\leqslant n, 1\leqslant k\leqslant D} |x_{i,k} - kr_i| \leqslant B\\
&\sum_{i=1}^{n} x_{i,k} = k, \quad k = 1,2,\ldots,D,\\
&x_{i,D} = d_i, \quad i = 1,2,\ldots,n,\\
&0 \leqslant x_{i,k} - x_{i,k-1}, \quad i = 1,2,\ldots,n;\ k = 2,3,\ldots,D,\\
&x_{i,k} \in \mathbb{N}, \quad i = 1,2,\ldots,n;\ k = 1,2,\ldots,D.
\end{aligned} \tag{3}$$

Consider an instance $(n, d_1, d_2, \ldots, d_n, B)$ of the max-abs decision problem. Denote by (i,j) the j-th part of type i. The earliest and latest feasible production periods of part (i,j) are defined by

$$E(i,j) = \left\lceil \frac{j-B}{r_i} \right\rceil \quad \text{and}$$
$$L(i,j) = \left\lfloor \frac{j-1+B}{r_i} + 1 \right\rfloor. \tag{4}$$

In any feasible schedule, part (i,j) lies in the interval $[E(i,j)..L(i,j)]$.

The max-abs decision problem can be formulated as a perfect matching problem in the bipartite graph $G = (V_1 \cup V_2, E)$ as follows: the vertex set $V_1 = [1..D]$ represents the production time periods and V_2 is the set of all parts (i,j). An edge $(k,(i,j))$ is in E if and only if part (i,j) can be produced in time period k, *i.e.* if and only if $k \in [E(i,j)..L(i,j)]$ (see Fig. 1).

The following statement describes a necessary and sufficient condition for the feasibility of a solution:

Proposition 1 [1]. *The max-abs decision problem has a feasible solution if and only if the graph G has a perfect matching.*

From the previous proposition, we can trivially deduce:

Proposition 2. *The optimal objective value of the max-abs problem is the smallest B such that the corresponding graph G has a perfect matching.*

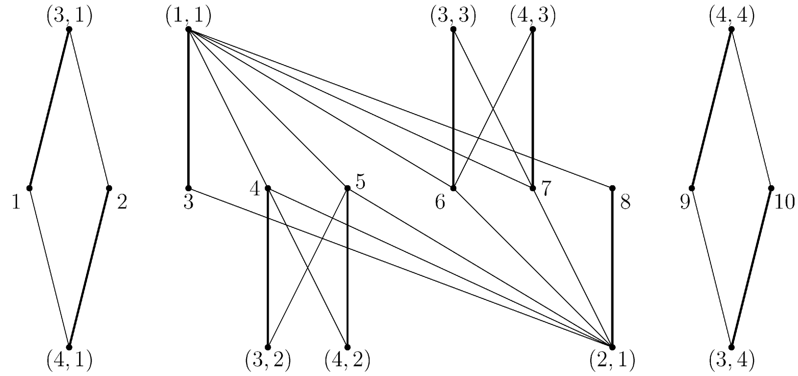

Fig. 1. Bipartite graph for $d_1 = d_2 = 1$, $d_3 = d_4 = 4$ and $B = \frac{7}{10}$.

This statement is used to show optimality results for the max-abs problems appearing in Section 6.

Remark. When considering possible optimal values for the max-abs objective function, one can consider only the values $B = \frac{q}{D}$ with $q \in [1..D]$ since deviations are multiples of $\frac{1}{D}$ and since the maximum deviation of an optimal schedule is always lower than 1.

4.2. Total deviation and assignment problem

This section describes the formulation of problems with total deviation (like sum-abs and sum-sqr) as assignment problems [9]. This formulation allows to find optimal solutions for those problems.

In [9], the authors show that, for any convex functions F_i, $i = 1, 2, \dots, n$ verifying condition (2), problem (1) with the objective of minimizing the total deviation F_{sum} can be reformulated as an assignment problem.

The ideal location $Z^*_{i,j} = \lceil k_{i,j} \rceil$ of (i,j) is the ceiling of the unique instant $k_{i,j}$ satisfying

$$F_i(j - k_{i,j} r_i) = F_i(j - 1 - k_{i,j} r_i). \tag{5}$$

Let $C_{i,j,k}$ be the costs induced by placing (i,j) in the k-th position:

$$C_{i,j,k} = \begin{cases} \sum_{p=k}^{Z^*_{i,j}-1} \psi_{i,j,p} & k < Z^*_{i,j}, \\ 0 & k = Z^*_{i,j}, \\ \sum_{p=Z^*_{i,j}}^{k-1} \psi_{i,j,p} & k > Z^*_{i,j}, \end{cases} \tag{6}$$

where $\psi_{i,j,p}$ is defined by

$$\psi_{i,j,p} = |F_i(j - pr_i) - F_i(j - 1 - pr_i)|. \tag{7}$$

If (i,j) is produced before the ideal instant $Z^*_{i,j}$, then $\psi_{i,j,p}$ represent *inventory costs* and when (i,j) is produced after the ideal instant $Z^*_{i,j}$, they represent *shortage costs*.

The assignment variables are defined as

$$y_{i,j,k} = \begin{cases} 1 & \text{if } (i,j) \text{ is produced in time period } k, \\ 0 & \text{otherwise.} \end{cases}$$

We can now describe the assignment problem as:

$$\begin{aligned} \min \quad & \sum_{k=1}^{D} \sum_{i=1}^{n} \sum_{j=1}^{d_i} C_{i,j,k} y_{i,j,k} \\ \text{s.t.} \quad & \sum_{i=1}^{n} \sum_{j=1}^{d_i} y_{i,j,k} = 1, \quad \forall k = 1, 2, \dots, D, \\ & \sum_{k=1}^{D} y_{i,j,k} = 1, \quad \forall i = 1, 2, \dots, n, \ \forall j = 1, 2, \dots, d_i. \end{aligned} \tag{8}$$

The constraints indicate that one part is produced at time period k and that part (i,j) is produced exactly once.

4.3. Optimality certificate for min-sum problems

In order to prove the optimality of solutions for problems with total deviation objective, we will consider the assignment problem in terms of graphs. The objective is to find a smallest perfect matching in the weighted-bipartite graph $G_w = (V_1 \cup V_2, E)$ where, as in Section 4.1, the vertex set $V_1 = [1..D]$ represents the production time periods and V_2 is the set of all parts (i,j).

Note that in a total deviation problem, a part (i,j) can be produced at any instant k. Therefore G_w is a complete bipartite graph. An edge $e = (k, (i,j)) \in E$

has the weight $w_e = C_{i,j,k}$. Denote by A the vertex-edge incidence matrix of the graph G_w (*i.e.* the $2D \times D^2$ matrix such that $a_{l,m} = 1$ if the l-th vertex is incident to the m-th edge and 0 otherwise).

We can rewrite the problem as

$$\begin{aligned} \min \quad & wy \\ \text{s.t.} \quad & Ay = 1, \\ & y \in \mathbb{N}. \end{aligned} \tag{9}$$

It is well known that the vertex-edge incidence matrix of a matching is totally unimodular. Therefore, the polytope associated with problem (9) is integer. Hence, one can solve the linear relaxation of the integer program (9) to obtain the optimal objective value.

The dual of the linear relaxation of (9) is

$$\begin{aligned} \max \quad & 1z \\ \text{s.t.} \quad & zA \leqslant w, \\ & z \text{ unconstrained}. \end{aligned} \tag{10}$$

This dual problem (10) can be interpreted as finding vertex weights z such that the sum of z_l for all $l \in V_1 \cup V_2$ is maximal and such that for all edges $(k,(i,j))$, one has $z_k + z_{i,j} \leqslant C_{i,j,k}$. Therefore, one has the following statement:

Proposition 3. *The objective value s of a total deviation problem is optimal if and only if there is a perfect matching M and a weight function $z : V_1 \cup V_2 \to \mathbb{R}$ such that*

$$\begin{cases} z_k + z_{i,j} \leqslant C_{i,j,k}, & \forall (k,(i,j)) \in E, \\ \sum_{k \in V_1} z_k + \sum_{(i,j) \in V_2} z_{i,j} = \sum_{(k,(i,j)) \in M} C_{i,j,k} = s. \end{cases} \tag{11}$$

This statement allows to prove in a compact way that a solution is optimal for a given min-sum problem by giving the corresponding perfect matching and the vertex weight values in the graph G_w.

4.4. B-bounded min-sum problems

In problem (8), there is no constraint concerning the maximum deviation. However we will later consider min-sum problems with bounded maximum deviation, in which case the edge set E can be described as before: an edge $(k,(i,j))$ is in E if and only if part (i,j) can be produced in time period k, *i.e.* if and only if $k \in [E(i,j)..L(i,j)]$.

Such a total deviation problem with bounded maximum deviation will be denoted *B-bounded min-sum* problem.

5. 1-Bounded-sum-abs and sum-sqr problems

The set of all possible solutions of a 1-bounded problem is considerably smaller than the set of its unbounded version. Considering 1-bounded problems instead of unbounded ones therefore allows computational improvements on exact min-sum methods. Namely, computing the smallest perfect matching can be done in $O(nD^2 \log D)$ for a 1-bounded min-sum problem instead of $O(D^3 \log D)$ for its unbounded version with $D \gg n$ (see [14]). For all instances, it is possible to find a sequence with maximum deviation lower than 1 [13]. Hence, it is interesting to know if there always exists a 1-bounded sequence that minimizes total deviation problems, *i.e.* if the 1-bounded min-sum problems have the same optimal objective value as their unbounded versions. In [7], the authors test the two following questions over 100,000 randomly chosen instances:

- Is there an optimal sum-abs sequence such that the maximum deviation is lower than 1?
- Is there an optimal sum-abs sequence that optimizes sum-sqr?

For all the tested instances, the answers were both "yes". Furthermore, in [7], the authors noticed that the optimal 1-bounded sum-abs sequence was always optimal for the sum-sqr problem. Exhaustive testing shows that such a 1-bounded optimal sequence does not exist for all instances (see Section 6).

The following lemma compares the costs of all edges $(k,(i,j))$ of the graph corresponding to the 1-bounded problems for the sum-abs and sum-sqr problems.

Lemma 4. *We consider the bound $B = 1$ for the deviations. For all $i = 1,2,\ldots,n$, $j = 1,2,\ldots,d_i$, and $k = 1,2,\ldots,D$, one has*

$$k \in [E(i,j)..L(i,j)] \Rightarrow C^a_{i,j,k} = C^s_{i,j,k},$$

where C^a is the cost matrix calculated for the sum-abs problem and C^s is the cost matrix calculated for the sum-sqr problem.

Proof. Denote by $\lceil k^a_{i,j} \rceil$ and $\lceil k^s_{i,j} \rceil$ the ideal locations of (i,j) in the production sequence for the sum-abs problem and the sum-sqr problem respectively. Since $|x| = |x-1|$ if and only if $x = \frac{1}{2}$ and $x^2 = (x-1)^2$ if and only if $x = \frac{1}{2}$, one has $k^a_{i,j} = k^s_{i,j}$.

Therefore, we will denote $Z^*_{i,j} = \lceil k^a_{i,j} \rceil = \lceil k^s_{i,j} \rceil$ the ideal location of (i,j) in the production sequence for both problems.

Let $C^a_{i,j,k}$ and $C^s_{i,j,k}$ be the costs induced by placing (i,j) in the k-th position and $\psi^a_{i,j,p}$ and $\psi^s_{i,j,p}$ the inventory and shortage costs for each problem. The function $f(x) = |x^2 - (x-1)^2| - ||x| - |x-1||$ is equal to zero if and only if $x \in [0,1]$. Therefore we have the following statement:

$$\psi^a_{i,j,p} = \psi^s_{i,j,p} \quad \text{if and only if } j - pr_i \in [0,1].$$

Consider the part (i,j) and an instant k and suppose that $k \in [E(i,j)..L(i,j)]$, with $E(i,j)$ and $L(i,j)$ corresponding to the bound $B = 1$.

Since $E(i,j) \leqslant k$, one has $\lceil \frac{j-1}{r_i} \rceil \leqslant k$. Therefore $r_i \lceil \frac{j-1}{r_i} \rceil \leqslant kr_i$.

Hence, we deduce that $j - 1 \leqslant kr_i$ and then

$$j - kr_i \leqslant 1.$$

We shall consider three cases depending on the respective positions of k and $Z^*_{i,j}$.

Case 1: $k < Z^*_{i,j}$

For all $p = k,\ k+1, \ldots, Z^*_{i,j} - 1$, one has

$$j - pr_i > \frac{1}{2}$$

since the function $g(x) = j - xr_i$ is decreasing and $g(k_{i,j}) = \frac{1}{2}$.

Therefore, one has $0 \leqslant j - pr_i \leqslant 1$ and hence

$$\forall p = k, k+1, \ldots, Z^*_{i,j} - 1, \quad \psi^a_{i,j,p} = \psi^s_{i,j,p}.$$

Case 2: $k > Z^*_{i,j}$

Since $k \leqslant L(i,j)$, one has $k \leqslant \lfloor \frac{j}{r_i} \rfloor + 1$. Therefore, $(k-1)r_i \leqslant \lfloor \frac{j}{r_i} \rfloor r_i$.

Hence, we have $(k-1)r_i\ j$. We finally obtain

$$0 \leqslant j - (k-1)r_i.$$

As the function g is decreasing, for all $p = Z^*_{i,j}, Z^*_{i,j} + 1, \ldots, k-1$, one has $j - p\ r_i \in [0,1]$. Hence

$$\forall p = Z^*_{i,j}, Z^*_{i,j} + 1, \ldots, k-1, \quad \psi^a_{i,j,p} = \psi^s_{i,j,p}.$$

Case 3: $k = Z^*_{i,j}$

In this case, both costs are zero.

Therefore, according to the definition of the costs, one has

$$C^a_{i,j,k} = C^s_{i,j,k}$$

in all the three cases. Hence, the lemma holds. □

Theorem 5 below was independently proved in [4]. We give a short alternative proof based on Lemma 4.

Theorem 5 [4]. *Any optimal sum-abs sequence y with maximal deviation smaller than 1 is an optimal sequence for the sum-sqr problem.*

Our Proof of Theorem 5. The function $f(x) = |x^2 - (x-1)^2| - ||x| - |x-1||$ is non-negative. We therefore obtain

$$\forall i, j, k, \quad C^a_{i,j,k} \leqslant C^s_{i,j,k}.$$

Hence, for any feasible solution y of the JIT problem

$$\sum_{i=1}^{n} \sum_{j=1}^{d_i} \sum_{k=1}^{D} C^a_{i,j,k} y_{i,j,k} \leqslant \sum_{i=1}^{n} \sum_{j=1}^{d_i} \sum_{k=1}^{D} C^s_{i,j,k} y_{i,j,k}. \tag{12}$$

Let y be a solution with maximum deviation smaller than 1. Suppose that $y_{i,j,k} = 1$. Since the maximum deviation for y is lower than 1, we have $k \in [E(i,j)..L(i,j)]$ with $E(i,j)$ and $L(i,j)$ corresponding to the bound $B = 1$. According to Lemma 4, it implies that $C^s_{i,j,k} = C^a_{i,j,k}$. Therefore, for any $y \in P$,

$$y_{i,j,k} = 1 \Rightarrow C^s_{i,j,k} = C^a_{i,j,k}.$$

Let $y^* \in P$ be an optimal solution of the sum-abs problem. Hence for any feasible schedule y we have

$$\sum_{i=1}^{n} \sum_{j=1}^{d_i} \sum_{k=1}^{D} C^s_{i,j,k} y^*_{i,j,k} = \sum_{i=1}^{n} \sum_{j=1}^{d_i} \sum_{k=1}^{D} C^a_{i,j,k} y^*_{i,j,k}$$
$$\leqslant \sum_{i=1}^{n} \sum_{j=1}^{d_i} \sum_{k=1}^{D} C^a_{i,j,k} y_{i,j,k}. \tag{13}$$

Combining (12) and (13), we obtain that, for any feasible solution y,

$$\sum_{i=1}^{n} \sum_{j=1}^{d_i} \sum_{k=1}^{D} C^s_{i,j,k} y^*_{i,j,k} \leqslant \sum_{i=1}^{n} \sum_{j=1}^{d_i} \sum_{k=1}^{D} C^s_{i,j,k} y_{i,j,k}$$

and hence y^* is an optimal solution of the sum-sqr problem. □

Corollary 6. *If for an instance, there exists a pair of optimal solutions (y^*, Y^*) of the sum-abs and sum-sqr problems respectively verifying*

$$\sum_{i=1}^{n}\sum_{j=1}^{d_i}\sum_{k=1}^{D} C^s_{i,j,k} Y^*_{i,j,k} > \sum_{i=1}^{n}\sum_{j=1}^{d_i}\sum_{k=1}^{D} C^a_{i,j,k} y^*_{i,j,k},$$

then the sum-abs problem has no optimal solution with maximum deviation lower or equal to 1.

Proof. Directly from the proof of Theorem 5. □

Theorem 5 raises some natural questions:

- A sequence optimizing sum-abs with maximum deviation lower than 1 is optimal for sum-sqr, but what about a sequence optimizing sum-sqr with maximum deviation lower than 1? *i.e.* can sum-abs and sum-sqr be exchanged in Theorem 5?
- To apply Theorem 5, we need the existence of an optimal sequence for sum-abs with maximum deviation lower than 1. Can we always find such a sequence? *i.e.* is there for all instances a sequence that optimizes sum-abs with maximum deviation lower than 1?
- If not, can we nonetheless optimize sum-abs and sum-sqr simultaneously for all instances?
- Can we find for all instances a 1-bounded optimal sum-sqr sequence?

Section 6 answers those questions by proposing instances for which there exists or not sequences optimizing the different criteria simultaneously.

6. Comparison of the objective functions

6.1. Max-type objective functions

One may wonder why maximum deviation criteria is studied only with $F_i(x) = |x|$ when considering $\max_{i,k} F_i(x_{i,k} - kr_i)$.

The following result explains that one can take any function instead, provided that all F_i are identical and equal to a function F that penalizes lateness and earliness equally.

Proposition 7. *Let B be a bound and $F : \mathbb{R} \to \mathbb{R}^+$ a function strictly increasing on $\mathbb{R}^+$ and such that $F(x) = F(-x)$. Suppose that all deviations F_i are equal to F. Then, a feasible solution $X = (x_{i,k})$ (with respect to (1)) has maximum deviation lower than B if and only if*

$$\max_{i,k} F(x_{i,k}) \leqslant F(B).$$

In particular, X is optimal for $\max_{i,k}|x_{i,k} - kr_i|$ if and only if X is optimal for $\max_{i,k} F(x_{i,k} - kr_i)$.

Hint of the proof [6]. Let $F : \mathbb{R}^+ \to \mathbb{R}^+$ be non-decreasing, and let A be a finite subset of $\mathbb{R}^+$. Then

$$\max_{a\in A}\{F(a)\} = F\left(\max_{a\in A}\{a\}\right).$$

Moreover, if F is strictly increasing the left-hand side is attained by and only by the maximum element $a \in A$. □

6.2. Optimizing maximum and sum of the deviations simultaneously

In order to find the examples of this section, we used the software CPLEX to solve the linear programs describing the different problems and we adopted for them efficient formulations. A complete description of those formulations can be found in [10].

Our testing aimed at obtaining instances with sequences optimizing several criteria simultaneously. For a given instance, we introduce five boolean variables denoted AM, SM, $AM1$, $SM1$ and AS to describe which criteria are simultaneously optimized. In those notations, A stands for sum-abs, S stands for sum-sqr, M stands for max-abs, $M1$ stands for maximum deviation lower than 1. For a given instance, AM is true if and only if there is a sequence optimizing sum-abs and max-abs simultaneously, SM is true if and only if a sequence optimizes sum-sqr and max-abs simultaneously, $AM1$ is true if and only if an optimal sum-abs sequence has maximum deviation lower than 1, $SM1$ is true if and only if an optimal sum-sqr sequence has maximum deviation lower than 1 and AS is true if and only if a sequence optimizes sum-abs and sum-sqr simultaneously.

We denote T the boolean value *True* and F the boolean value *False*. Not all values are possible for the quintuplet $(AM, SM, AM1, SM1, AS)$. Note first that, for an instance, if AM is true, then $AM1$ is true as the optimal value of max-abs is always lower than 1. Likewise, SM true implies $SM1$ true. Implications $AM \Rightarrow SM \wedge AS$ and $AM1 \Rightarrow SM1 \wedge AS$ are direct consequences of Theorem 5. In addition, we have the following statement:

Proposition 8. *$SM \wedge AM1 \Rightarrow AM$.*

Proof. Consider an instance of the JIT problem and suppose that SM and $AM1$ are true for this instance. Denote Y^* a sequence optimizing sum-sqr and max-abs simultaneously and y^* a minimal sequence of sum-abs with maximum deviation lower than 1.

The solution Y^* is optimal for sum-sqr. Therefore,

$$\sum_{i,j,k} C^s_{i,j,k} y^*_{i,j,k} \geqslant \sum_{i,j,k} C^s_{i,j,k} Y^*_{i,j,k}.$$

Maximum deviation for Y^* is equal to the max-abs optimum and is hence lower than 1. Therefore, from the proof of Theorem 5, we can deduce

$$\sum_{i,j,k} C^s_{i,j,k} Y^*_{i,j,k} = \sum_{i,j,k} C^a_{i,j,k} Y^*_{i,j,k}.$$

Likewise, since the solution y^* has maximum deviation lower than 1, we have

$$\sum_{i,j,k} C^s_{i,j,k} y^*_{i,j,k} = \sum_{i,j,k} C^a_{i,j,k} y^*_{i,j,k}.$$

Therefore, we obtain the inequality

$$\sum_{i,j,k} C^a_{i,j,k} y^*_{i,j,k} \geqslant \sum_{i,j,k} C^a_{i,j,k} Y^*_{i,j,k}.$$

Since the sequence y^* is minimal for sum-abs, Y^* is also minimal for sum-abs. Since Y^* is optimal for max-abs and sum-abs, AM is true. □

Fig. 2 gives an illustration of the possible values of the quintuplet $(AM, SM, AM1, SM1, AS)$. For $X \in \{AM, SM, AM1, SM1, AS\}$, let IX represent the set of all instances such that X is true. Since $AM \iff SM \wedge AM1$ (by Theorem 5 and Proposition 8), the set IAM is equal to the intersection of the sets $IAM1$ and ISM. In order to make Fig. 2 easier to read, the set IAM is therefore not drawn. The numbers in the sets represent the seven examples of Table 1.

Table 1 presents examples for all possible values of quintuplet $(AM, SM, AM1, SM1, AS)$ except the quintuplet (F, F, F, F, F). Corresponding instances are given using a compact notation of the vector of demands. We denote $d = (d_a^p, d_b^q)$ the vector of demands of the instance with $n = p + q$ part types whose demands of the p first part types are equal to d_a and of the q following are equal to d_b. For all instances, if one of the boolean variables AM to AS is true, we give a sequence proving it. This sequence is also represented in a compact form. If part types i and i' are such that $d_i = d_{i'}$, one can exchange the parts (i,j) and (i',j) without modifying the sum or the maximum of deviations. Since for each example, the part types have at most two different values d_a and d_b, we denote a_j, $j = 1, 2, \ldots, d_a$ all parts (i,j) such that $d_i = d_a$ and b_j, $j = 1, 2, \ldots, d_b$ all parts (i,j) such that $d_i = d_b$. For example, if $d = (1, 1, 4, 4) = (1^2, 4^2)$, the sequence of parts $((3,1),(4,1),(3,2),(4,2),(1,1),(2,1),(3,3),(4,3),(3,4),(4,4))$ is denoted by $(b_1, b_1, b_2, b_2, a_1, a_1, b_3, b_3, b_4, b_4)$ or $(b_1^2, b_2^2, a_1^2, b_3^2, b_4^2)$. In Table 1, we denote *s-a* the sum-abs objective, *s-s* the sum-sqr objective and *m-a* the max-abs objective.

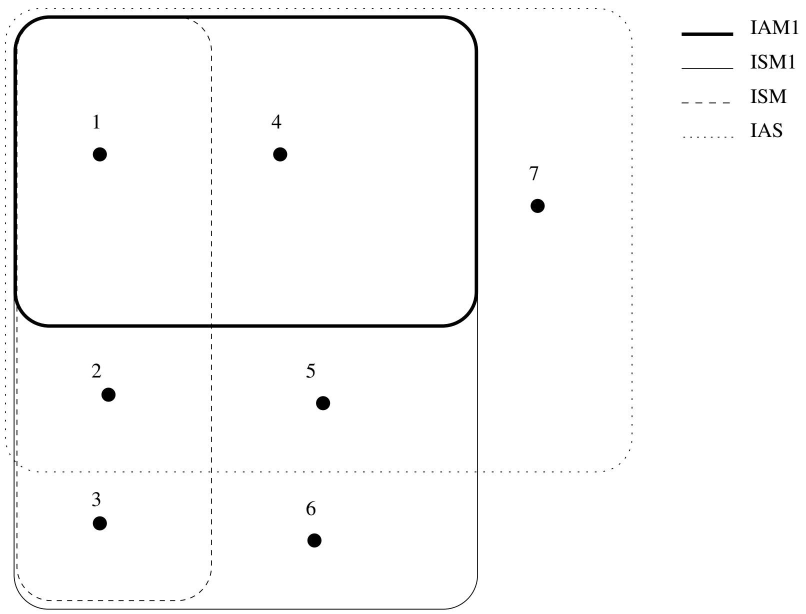

Fig. 2. Representation of the sets of instances corresponding to the values of the quintuplet $(AM, SM, AM1, SM1, AS)$.

Table 1
Example of instances with sequences optimizing several criteria

	Existence of a sequence optimizing					Instance	Proof
	AM *s-a* and *m-a*	SM *s-s* and *m-a*	AM1 *s-a* with *m-a* $\leqslant 1$	SM1 *s-s* with *m-a* $\leqslant 1$	AS *s-a* and *s-s*		
1	T	T	T	T	T	$d=(1)$	$S=(a_1)$ is optimal for all problems
2	F	T	F	T	T	$d=(1^9,7^3)$	$S=(b_1^3,b_2^3,a_1^3,b_3^3,a_1,b_4^3,a_1^2,b_5^3,a_1^3,b_6^3,b_7^3)$ is optimal for sum-abs and sum-sqr. $S'=(b_1^3,a_1,b_2^3,a_1^2,b_3^3,a_1,b_4^3,a_1^2,b_5^3,a_1^2,b_6^3,a_1,b_7^3)$ is optimal for sum-sqr and max-abs
3	F	T	F	T	F	$d=(1^{12},10^2)$	$S=(b_1^2,a_1,b_2^2,a_1,b_3^2,a_1^2,b_4^2,a_1,b_5^2,a_1^2,b_6^2,a_1,b_7^2,a_1^2,b_8^2,a_1,b_9^2,a_1,b_{10}^2)$ is optimal for sum-sqr and max-abs
4	F	F	T	T	T	$d=(1^2,4^2)$	$S=(b_1^2,b_2^2,a_1^2,b_3^2,b_4^2)$ is optimal for sum-abs with maximum deviation lower than 1
5	F	F	F	T	T	$d=(1^{10},6^5)$	$S=(b_1^5,a_1,b_2^5,a_1^3,b_3^5,a_1^2,b_4^5,a_1^3,b_5^5,a_1,b_6^5)$ is optimal for sum-sqr with maximum deviation lower than 1 and $S'=(b_1^5,b_2^5,a_1^3,b_3^5,a_1^4,b_4^5,a_1^3,b_5^5,b_6^5)$ is optimal for sum-abs and sum-sqr
6	F	F	F	T	F	$d=(1^9,6^4)$	$S=(b_1^4,a_1,b_2^4,a_1^2,b_3^4,a_1^3,b_4^4,a_1^2,b_5^4,a_1,b_6^4)$ is optimal for sum-sqr and is 1-bounded
7	F	F	F	F	T	$d=(1^7,6^4)$	$S=(b_1^4,b_2^4,a_1^2,b_3^4,a_1^3,b_4^4,a_1^2,b_5^4,b_6^4)$ is optimal for sum-abs and sum-sqr

In order to prove that the boolean variables AM to AS are true for a given instance, one can propose a sequence with the desired characteristics. The optimality of this sequence for the sum-abs or the sum-sqr objective can be shown using Proposition 3: one constructs the complete bipartite graph G_w (see Section 4.3). The sequence is represented by a perfect matching. Giving the vertex weights z_k and $z_{i,j}$ associated with this matching is sufficient to show the optimality of the sequence.

Therefore, when AS is not already implied by AM or $AM1$, one can show that AS is true by giving the perfect matching representing the common optimal solution to the sum-abs and sum-sqr problems and by proposing associated vertex weights for each complete bipartite graph.

To prove that an optimal min-sum sequence is B-bounded (*i.e.* that the corresponding variable AM to $SM1$ is true), one can either compute the maximum deviation of the sequence and verify that it is indeed lower than B, or construct the graph corresponding to the B-bounded problem (see Section 4.1) and verify that all the edges of the sequence (*i.e.* of the min-sum optimal perfect matching) belong to this restricted graph. Finding the optimal max-abs value can be done as explained in Section 4.1.

To prove that the boolean variables AM to $SM1$ are false, one has to compare the optimal min-sum objective values of the bounded and unbounded problems. It means that one has to compute optimal perfect matching and vertex weights for the unbounded and bounded problems. If the values of the two matchings are different, then the corresponding variable is false. For instance we propose here a proof that AM is false for the instance $d=(1^2,4^2)$.

The graph of Fig. 3 represents all the edges and edge costs (for the sum-abs problem) of the graph G_w associated with the instance $d=(1^2,4^2)$ and the bound $B=0.7$, which is the optimal value of the max-abs objective. The thick edges represent a minimum perfect matching denoted M^*. The dual vertex weights are not represented on the graph, but one can take $z_1=z_{10}=z_5=z_6=0.2$, $z_3=z_8=0.8$, $z_{a_1}=-0.2$ and any other vertex weight equal to zero, and obtain a dual optimal solution that proves that the matching is optimal.

The graph of Fig. 4 represents a matching of the bipartite complete graph associated with G_w. This matching is strictly better than M^*. Therefore, one can conclude that M^* is not optimal for the unbounded sum-abs problem and AM is false.

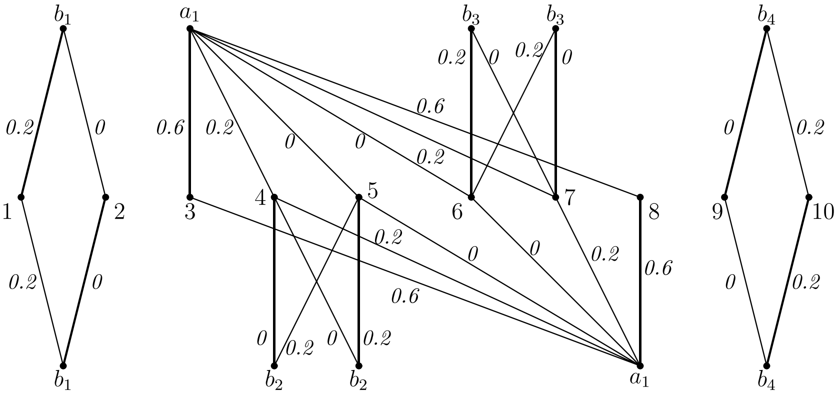

Fig. 3. Bipartite weighted graph for $d_1 = d_2 = 1$, $d_3 = d_4 = 4$ and $B = \frac{7}{10}$.

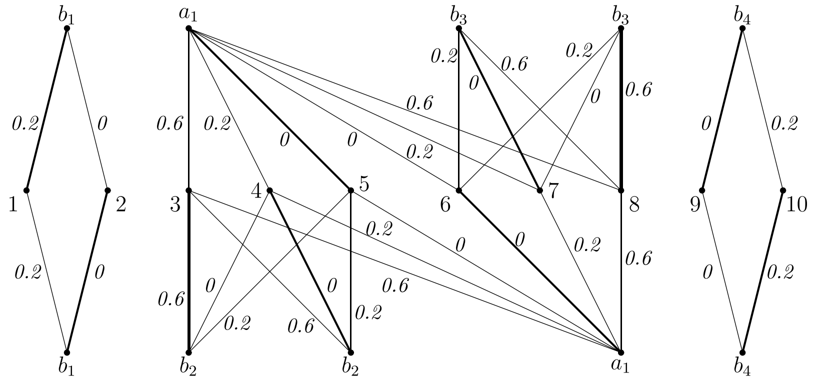

Fig. 4. A subgraph of the bipartite weighted graph for $d_1 = d_2 = 1$, $d_3 = d_4 = 4$.

The boolean variable AS is false when the sum-abs and sum-sqr problems have no common optimal solution, *i.e.* when the sets of all the optimal solutions of sum-abs and sum-sqr problems are disjoint. To show this result, one can, for each problem, generate the weighted graph G_w and all the minimum perfect matchings in this graph. It can be done using the algorithm proposed in [3].

7. Conclusion and possible extensions

In this paper, we studied the possibility of optimizing simultaneously two criteria for the JIT scheduling problem. Three objective functions have been examined: the maximum deviation (max-abs), the total deviation (sum-abs) and the total squared deviation (sum-sqr). We also studied the optimization of sum-abs (resp. sum-sqr) restricting the maximum deviation to be less than 1. In most cases, we provided instances for which there is no solution achieving simultaneous optimization. However, we tried to give a detailed picture of the interaction between these criteria.

A starting and inspiring conjecture was the following [8]:

Conjecture 1 [8]. *For any instance of the JIT problem, there is a solution optimizing simultaneously the maximum deviation (max-abs) and the total deviation (sum-abs).*

This conjecture was already claimed not to hold [7], but counter-examples were of total demand D at least 100 and none of them was explicitly provided. Here we gave the counter-example with the smallest total demand ($D = 10$): the instance $d = (1, 1, 4, 4)$. We provided a method to check that it is a counter-example.

Since for any instance, there exists a solution with maximum deviation less than 1 [13] the following conjecture is weaker.

Conjecture 2 [14]. *For any instance, there is a sequence that is optimal for sum-abs with maximum deviation less than* 1.

This conjecture is also studied in [7]. The authors concluded then that this conjecture holds, but further testing had provided a counter-example with $D = 100$ (stated in [4]). In Fig. 2 and Table 1 we provided several smaller counter-examples (labeled $2, 3, 5, 6, 7$) which we used for sharper statements.

Related to Conjecture 2 is Theorem 5, already proved in [4], stating that *if* a solution with max-abs less than 1 is optimal for sum-abs, *then* it is also optimal for sum-sqr.

We noticed that each of the examples labeled $2, 3, 5, 6$ shows that the role of sum-abs and sum-sqr cannot be made symmetric in Theorem 5.

The following is also related to Conjecture 2 and remains open although we tested all instances with $D \leqslant 100$.

Conjecture 3. *For $n = 3$ there are instances such that the sum-abs problem has no optimal solution with maximum deviation lower than* 1.

Section 6 does not present any example with $(AM, SM, AM1, SM1, AS) = (F, F, F, F, F)$. We have indeed found none and wonder whether or not at least one of $SM1$ and AS is always true:

Conjecture 4. *For any instance, there is a solution S^* satisfying at least one of the following:*

- *S^* is optimum for both sum-sqr and sum-abs.*
- *S^* is optimum for sum-sqr and has maximum deviation less than* 1.

In Proposition 7 we noticed that minimizing the maximum deviation is equivalent to minimize the maximum of more general functions, for instance the squared deviation.

Let us now turn to questions which our work suggests but for which it does not provide insight directly.

It is likely that if an optimal solution of the sum-abs problem has maximum deviation lower than 1, it will not imply that all sum-abs optimal solutions have maximum deviation lower or equal to 1.

Since sum-abs and sum-sqr problems can have no 1-bounded solutions, computational improvement proposed by Steiner and Yeomans in [14] cannot be applied to the general case, but studying 1-bounded solutions may reveal that they can approximate the solutions of the unbounded problems with a low ratio.

Another interesting problem could be to bound the maximum deviation with the constraint that the total deviation is minimal: we saw that we cannot impose a maximum deviation lower than 1 but is it possible to replace 1 by another constant?

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