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The performance evaluation of a multi-stage JIT production system with stochastic demand and production capacities

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ABSTRACT

This paper discusses a single-item, multi-stage, serial Just-in-Time (JIT) production system with stochastic demand and production capacities. The JIT production system is modeled as a discrete-time, M/G/1-type Markov chain. A necessary and sufficient condition, or a stability condition, under which the system has a steady-state distribution is derived. A performance evaluation algorithm is then developed using the matrix analytic methods. In numerical examples, the optimal numbers of kanbans are determined by the proposed algorithm. The optimal numbers of kanbans are robust for the variations in production capacity distribution and demand distribution.

1. Introduction

Just-in-Time (JIT) production systems have been created based on the main objective of reducing cost by eliminating waste during the production process. The fundamental property of the system is using the "pull" control mechanism. In JIT production systems, each subsequent process withdraws the necessary parts from a preceding process at the necessary point in time, and the preceding process produces those parts withdrawn by the subsequent process. Kanbans are simple tools used to implement this pull-production control mechanism throughout the system. Among the many studies on the subject, Monden (1993) explains the JIT production systems in detail. According to him, JIT production systems require both input stock in the form of parts and output stock in the form of products at each stage. To maintain these systems, two types of kanbans, or production-ordering kanban and withdrawal kanban, are used as tools to control the production and withdrawal quantities at each stage, respectively.

Much work has been devoted to evaluating the performance of JIT production systems controlled only by production-ordering kanbans. Many authors use simulations to make evaluations (e.g., Ardalan, 1997; Chu and Shin, 1992); however, some authors have developed analytical methods. JIT production systems using only production-ordering kanbans have been evaluated using analytical methods in the following papers. Deleersnyder et al. (1989) analyzed a JIT production system using a discrete-time Markov process: numerical computations were used to study the effects of the number of kanbans, machine reliability, demand variability and safety stock requirements on the performance of the system. Mitra and Mitrani (1990, 1991) studied a multi-stage, serial JIT production system: the subsystem corresponding to each stage was analyzed precisely and an approximation algorithm was devised using a decomposition technique. Wang and Wang (1990) studied multi-item JIT production systems using Markovian queues, and determined the optimal numbers of kanbans for serial, merge- or split-type JIT production systems. Mascolo et al. (1996) used synchronization mechanisms and broke down the original system into a set of subsystems, each of which was analyzed using product-form approximation: an iterative procedure was developed to determine the performance measures of the system. Matta et al. (2005) considered two different kanban release policies (i.e., independent and simultaneous systems), and compared them by approximate analytical methods. Kreig and Kuhn (2004) proposed a decomposition-based analytical evaluation method for a single-stage, multi-product JIT production system with state-dependent setups and lost sales: performance measures of the original system were obtained using an approximation technique.

The JIT production systems using two types of kanbans have been evaluated using analytical methods in the following papers.

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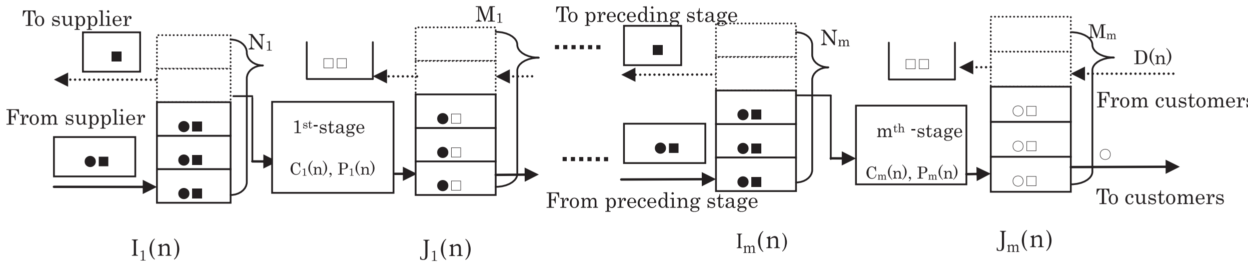

Fig. 1. Multi-stage JIT production system.

Kamarkar and Kerme (1989) constructed Markov models for the JIT production systems and studied the effect of a batch-sizing policy and the number of kanbans on the expected inventory and back-order costs. Ohno et al. (1995) studied a single-stage JIT production system, derived a stability condition for the system and proposed an algorithm for determining the optimal numbers of kanbans using the probability-generating function of the total backlogged demand. Gurgur and Altiok (2004) broke down a JIT production into subsystems using two-node decomposition and obtained an analytical solution for each subsystem: a fixed-point iteration algorithm was proposed for evaluating the performance of the system.

Yang (2000) made a comparison of pull-production control policies using production ordering kanbans and two types of kanbans. As an extension of JIT production systems, Buzacott and Shanthikumar (1993) proposed a wide class of inventory-production systems using four kinds of tags, which they call "PAC systems." PAC systems with specialized parameters can express the base stock, MRP, CONWIP, kanban and other systems. Recently, Kotani (2007) studied a new kanban system with the *e*-Kanban, which utilizes computers and communications networks to transfer information to suppliers, instead of the withdrawal kanban.

This paper discusses a single-item, multi-stage, serial Just-in-Time (JIT) production system using two types of kanbans in which customer demands and production capacities are stochastic. The system is modeled as a discrete-time, M/G/1-type Markov chain with the unit of time being one withdrawal cycle. A necessary and sufficient condition, or a stability condition, under which the JIT production system has a steady-state distribution is derived. A performance evaluation algorithm is then developed using the matrix analytic methods. This paper shows how the matrix analytic methods can be applied for the performance evaluation of the JIT production system under the stability condition. It is therefore essentially an extension of Ohno et al. (1995) to a multi-stage case.

M/G/1-type Markov chains, pioneered by Neuts (1989), provide a framework that enables the exact analysis of frequently encountered classes of queueing models. The matrix analytic methods for M/G/1-type Markov chains are widely used to evaluate communications network systems. One feature of the matrix analytic methods is that accurate numerical solutions can be obtained for problems in which exact solutions are difficult to obtain otherwise. Iwase and Ohno (2008) derived a performance evaluation method for a one-stage, make-to-order production inventory system using the matrix analytic methods for M/G/1-type Markov chains. However, to our knowledge, with the exception of that paper, the matrix analytic methods for M/G/1-type Markov chains have not been applied to evaluate production inventory systems.

This paper is organized as follows: the JIT production system is described in Section 2, the associated M/G/1-type Markov chain is presented in Section 3, the stochastic properties and the stability condition for the Markov chain are derived in Section 4, the cost structure of the system and an algorithm for evaluating system performance are presented in Section 5, and the optimal numbers of kanbans are determined by the proposed algorithm in Section 6.

2. The multi-stage JIT production system

2.1. The kanban system

In the kanban system, production-ordering kanbans are attached to the containers which are periodically withdrawn according to the withdrawal kanbans in the withdrawal kanban post at the subsequent process. When a container is withdrawn, the production-ordering kanban is detached from the container and placed in the production-ordering kanban post. Next, a withdrawal kanban is attached to the container and it is taken to the subsequent process. At the preceding process, the parts are processed according to the ordinal sequence of the production-ordering kanbans in the production-ordering kanban post. Once the first part in a container has been used, the withdrawal kanban is detached from the container and placed into the withdrawal kanban post.

2.2. System description

The JIT production system discussed in this paper is shown in Fig. 1. The system has "m" stages in series, produces a single type of product and there is an infinite supply of raw materials at the input point of the first stage. From upstream-to-downstream, the stages are numbered 1st-stage to mth-stage, and in each stage, two types of kanbans (i.e., withdrawal and production-ordering kanbans) are used. The lead-time for the delivery of the parts, number of production-ordering kanbans and number of withdrawal kanbans in the ith-stage are represented as L_i, M_i and N_i, respectively. It is assumed that the each container's capacity is equal to one.

The discrete-time formulation is adapted: the constant withdrawal cycle is set as one period, and period "n" is set as the interval from time n to immediately before the time $n+1$. For each customer demand arriving in period n, one unit of product is supplied at the beginning of period $n+1$ if there is an inventory of products. The customer demand is backlogged (i.e., backlogged demand) if there is no inventory of products.

The detached production kanbans associated with the parts consumed in the 1st-stage in period n allow parts from an outside supplier to be delivered to the 1st-stage at the beginning of period $n+L_1+1$. For $2 \leqslant i$, the quantity of parts consumed in the ith-stage in period n is delivered to the ith-stage from the $(i-1)$-stage at the beginning of period $n+L_i+1$ or later depending on the upstream state at the beginning of period n. It is assumed that the production capacities and customer demand in each period are independent random variables. The total backlogged demand is set as the sum of backlogged demand and the number of production-ordering kanbans in the mth-stage production-ordering post. The following notations are used:

$B'(n)$	backlogged demand at the beginning of period n,
$B(n)$	total backlogged demand at the beginning of period n,
$C_i(n)$	production capacity in the ith-stage and period n,
$C_{i,\max}$	maximum production capacity in the ith-stage per period (nominal production capacity),
$D(n)$	customer demand in period n,
D	average customer demand per period,
$d_{\min}$	minimum customer demand per period,
$I_i(n)$	inventory of parts in the ith-stage at the beginning of period n,
$J_i(n)$	inventory of products in the ith-stage at the beginning of period n,
$P_i(n)$	production quantity in the ith-stage and period n,
$T_1(n)$	quantity of parts dispatched from an outside supplier to the 1st-stage at the beginning of period n,
$T_i(n)$ $(i = 2, K, m)$	quantity of parts dispatched from $(i\text{th} - 1)$-stage to the ith-stage at the beginning of period n, and
$T_{m+1}(n)$	quantity of products supplied from the mth-stage to customers at the beginning of period n.

Let $F_i(z)$ be the probability mass function of the ith-stage production capacity per period and $G(z)$ be the probability mass function of customer demand per period. It therefore follows that:

$$F_i(z) = \Pr\{C_i(n) = z\}, \quad z = 0, \ldots, C_{i,\max} \tag{1}$$

and

$$G(z) = \Pr\{D(n) = z\}, \quad z = 0, 1, \ldots \tag{2}$$

$M_i - J_i(n)$ represents the number of production-ordering kanbans in the ith-stage production-ordering kanban post at the beginning of period n. Hence, the total backlogged demand $B'(n)$, backlogged demand $B'(n)$ and number of products in the mth-stage $J_m(n)$ satisfy the following relations.

$$B(n) = B'(n) + (M_m - J_m(n)), \tag{3}$$

$$B'(n) = (B(n) - M_m)^+, \tag{4}$$

$$J_m(n) = (M_m - B(n))^+. \tag{5}$$

The production quantity in each period is the minimum inventory of parts, the number of production-ordering kanbans in the production-ordering kanban post and the production capacity. Hence, it follows that:

$$P_i(n) = \min\{I_i(n), M_i - J_i(n), C_i(n)\} = 1, \ldots, m. \tag{6}$$

From Eqs. (5) and (6), it follows that:

$$P_m(n) = \min\{I_m(n), M_m, B(n), C_m(n)\}. \tag{7}$$

The quantity of parts dispatched at the beginning of period $n+1$ is determined as follows.

$$T_1(n+1) = N_1 - \sum_{j=0}^{L_1-1} T_1(n-j) - I_1(n) + P_1(n), \tag{8}$$

$$T_i(n+1) = \min\left\{N_i - \sum_{j=0}^{L_i-1} T_i(n-j) - I_i(n) + P_i(n), J_{i-1}(n) + P_{i-1}(n)\right\}, \quad i = 2, \ldots, m. \tag{9}$$

The quantity of products supplied to customers at the beginning of period $n+1$ is determined as follows.

$$T_{m+1}(n+1) = \min\{B'(n) + D(n), J_m(n) + P_m(n)\}. \tag{10}$$

The inventory of the parts and the inventory of products in the ith-stage at the beginning of period $n+1$ are determined as follows.

$$I_i(n+1) = I_i(n) + T_i(n - L_i + 1) - P_i(n), \quad i = 1, \ldots, m, \tag{11}$$

$$J_i(n+1) = J_i(n) + P_i(n) - T_{i+1}(n+1), \quad i = 1, \ldots, m. \tag{12}$$

Since the customer demand is backlogged, the total backlogged demand at the beginning of period $n+1$ is determined as follows.

$$B(n+1) = B(n) - P_m(n) + D(n). \tag{13}$$

Let $\boldsymbol{x}_i^n$ be as follows.

$$\boldsymbol{x}_i^n = (I_i(n), J_i(n)), \quad L_i = 0, \quad i = 1, \ldots, m-1, \tag{14}$$

$$\boldsymbol{x}_i^n = (T_i(n), \ldots, T_i(n - L_i + 1), I_i(n), J_i(n)), \quad L_i \geqslant 1, \quad i = 1, \ldots, m-1, \tag{15}$$

$$\boldsymbol{x}_m^n = (I_m(n), B(n)), \quad L_m = 0, \tag{16}$$

$$\boldsymbol{x}_m^n = (T_m(n), \ldots, T_m(n - L_m + 1), I_m(n), B(n)), \quad L_m \geqslant 1. \tag{17}$$

For $i = 1, \ldots, m-1$, $\boldsymbol{x}_i^n$ is a vector that represents the state of the ith-stage immediately after dispatching parts at the beginning of period n. $\boldsymbol{x}_m^n$ is a vector that represents the state of the mth-stage immediately after supplying the products to customers at the beginning of period n. Set $\boldsymbol{X}_n = (\boldsymbol{x}_1^n, \ldots, \boldsymbol{x}_m^n)$. $\boldsymbol{X}_n$ is given as a vector that represents the system state at the beginning of period n. Set $\boldsymbol{T}_i$, $i = 1, \ldots, m$ and $\boldsymbol{T}$ as follows:

$$\boldsymbol{T}_1 = \left\{(s_{1,1}, \ldots, s_{1,L_1+2}) : s_{1,j} \in Z_+ (j = 1, \ldots, L_1 + 2), \times \sum_{j=1}^{L_1+1} s_{1,j} = N_1, s_{1,L_1+2} \leqslant M_1\right\}, \tag{18}$$

$$\boldsymbol{T}_i = \left\{(s_{i,1}, \ldots, s_{i,L_i+2}) : s_{i,j} \in Z_+ (j = 1, \ldots, L_i + 2), \times \sum_{j=1}^{L_i+1} s_{i,j} \leqslant N_i, s_{i,L_i+2} \leqslant M_i\right\}, \quad i = 2, \ldots, m-1, \tag{19}$$

$$\boldsymbol{T}_m = \left\{(s_{m,1}, \ldots, s_{m,L_m+2}) : s_{m,j} \in Z_+ (j = 1, \ldots, L_m + 2), \sum_{j=1}^{L_m+1} s_{m,j} \leqslant N_m\right\}, \tag{20}$$

$$\boldsymbol{T} = \left\{(\boldsymbol{s}_1, \ldots, \boldsymbol{s}_m) : \boldsymbol{s}_i = (s_{i,1}, \ldots, s_{i,L_i+2}) \in T_i, \wedge_{i=1}^{m-1} \left(s_{i,L_i+2} = 0 \vee \sum_{j=1}^{L_{i+1}+1} s_{i+1\,j} = N_{i+1}\right)\right\}, \tag{21}$$

where, Z_+ is a set of non-negative integers, $\bigwedge$ is a logical product and $\bigvee$ is a logical sum.

Set $\boldsymbol{X}_0$ as any fixed state in $\boldsymbol{T}$. Then $\{\boldsymbol{X}_n\}$ is a Markov chain which represents the state transition with the state space $\boldsymbol{T}$.

Hereafter, to avoid inessential complexities, it is assumed that $L_i \geqslant 1$ for $i = 1, \ldots, m$. The cases where $L_i = 0$ for some $i \in \{1, \ldots, m\}$ can be dealt with in a similar manner. For $\boldsymbol{s} \in \boldsymbol{T}$, $\boldsymbol{c} = (c_1, \ldots, c_m)$ and $d \geqslant 0$, let $\psi(\boldsymbol{s}, \boldsymbol{c}, d) \in \boldsymbol{T}$ be the state of $\boldsymbol{X}_{n+1}$ when $\boldsymbol{X}_n = \boldsymbol{s}$, $C_i(n) = c_i$ for $i = 1, \ldots, m$ and $D(n) = d$. Set $\boldsymbol{s}' = \psi(\boldsymbol{s}, \boldsymbol{c}, d)$. Let $\boldsymbol{s} = (\boldsymbol{s}_1, \ldots, \boldsymbol{s}_m)$, $\boldsymbol{s}_i = (s_{i,1}, \ldots, s_{i,L_i+2}) \in T_i$, $\boldsymbol{s}' = (\boldsymbol{s}'_1, \ldots, \boldsymbol{s}'_m)$ and $\boldsymbol{s}'_i = (s'_{i,1}, \ldots, s'_{i,L_i+2}) \in \boldsymbol{T}_i$. Let $p_i = P_i(n)$ for $i = 1, \ldots, m$. From Eqs. (6)–(17), the following relations hold:

$$p_i = \min\{s_{i,L_i+1}, M_i - s_{i,L_i+2}, c_i\}, \quad i = 1, \ldots, m-1, \tag{22}$$

$$p_m = \min\{s_{m,L_m+1}, M_m, s_{m,L_m+2}, c_m\}, \quad (23)$$

$$s'_{1,1} = N_1 - \sum_{j=1}^{L_1+1} s_{1,j+p_1}, \quad (24)$$

$$s'_{i,1} = \min\left\{N_i - \sum_{j=1}^{L_i+1} s_{i,j} + p_i, s_{i-1,L_{i-1}+2} + p_{i-1}\right\}, \quad i = 2,\ldots,m \quad (25)$$

$$s'_{i,j} = s_{i,j-1}, \quad j = 2,\ldots,L_i, \quad i = 1,\ldots,m, \quad (26)$$

$$s'_{i,L_i+1} = s_{i,L_i+1} + s_{i,L_i} - p_i, \quad i = 1,\ldots,m, \quad (27)$$

$$s'_{i,L_i+2} = s_{i,L_i+2} + p_i - \min\left\{N_{i+1} - \sum_{j=1}^{L_{i+1}+1} s_{i+1,j} + p_{i+1}, s_{i,L_i+2} + p_i\right\}, \quad i = 1,\ldots,m-1, \quad (28)$$

$$s'_{m,L_m+2} = s_{m,L_m+2} + d - p_m. \quad (29)$$

Then, the transition probability of $\{\boldsymbol{X}_n\}$ is given as follows:

$$\Pr(\boldsymbol{X}_{n+1} = \boldsymbol{s}'|\boldsymbol{X}_n = \boldsymbol{s}) = \sum_{(\boldsymbol{c},d)\in K(\boldsymbol{s},\boldsymbol{s}')} F(\boldsymbol{c}) \times G(d), \quad (30)$$

where, $F(\boldsymbol{c}) = \Pi_{i=1}^{m} F_i(c_i)$ and $K(\boldsymbol{s},\boldsymbol{s}') = \{(\boldsymbol{c},d)|\boldsymbol{s}' = \psi(\boldsymbol{s},\boldsymbol{c},d)\}$.

3. Transition probability matrix of $\{X_n\}$

Let $M_{C_m} = \min\{M_m, C_{m,\max}\}$. For $0 \leqslant i$, let **level *i*** be a subset of $\boldsymbol{T}$ defined as follows.

$$\textbf{level } \boldsymbol{i} = \{\boldsymbol{s} = (\boldsymbol{s}_1,\ldots,\boldsymbol{s}_m) \in T : \boldsymbol{s}_i \in T_i \ (i = 1,\ldots,m), \boldsymbol{s}_m = (s_{m,1},\ldots,s_{m,L_m+2}), iM_{C_m} \leqslant s_{m,L_m+2} < (i+1)M_{C_m}\}. \quad (31)$$

The state space $\boldsymbol{T}$ is written as follows.

$$\boldsymbol{T} = \bigcup_{i=0}^{+\infty} \textbf{level } \boldsymbol{i}. \quad (32)$$

For $0 \leqslant i$, **level *i*** has an equal number of states. Let r be this number. Arrange the states in **level 0** to $\boldsymbol{a}_{0,1},\ldots,\boldsymbol{a}_{0,r}$, and set:

$$\boldsymbol{a}_{0,j} = (\boldsymbol{b}_{j,1},\ldots,\boldsymbol{b}_{j,m}), \quad (33)$$

where $\boldsymbol{b}_{j,i} = (b_{j,i,1},\ldots,b_{j,i,L_i+2}) \in T_i$, $i = 1,\ldots,m$.

For $1 \leqslant i$, let $\boldsymbol{a}_{i,j}$ be a state in **level *i*** defined as follows:

$$\boldsymbol{a}_{i,j} = (\boldsymbol{b}_{j,1},\ldots,\boldsymbol{b}_{j,m-1},\boldsymbol{b}^i_{j,m}), \quad (34)$$

where $\boldsymbol{b}^i_{j,m} = (b_{j,m,1},\cdots,b_{j,m,L_m+1}, b_{j,m,L_m+2} + iM_{C_m}) \in T_m$.

Since the maximum decrease in the total backlogged demand in one transition is not greater than M_{C_m}, $\Pr(\boldsymbol{X}_{n+1} = \boldsymbol{a}_{i+k,t}|\boldsymbol{X}_n = \boldsymbol{a}_{k,j}) > 0$ implies $i \geqslant -1$. For $1 \leqslant k$, k' and $j = 1,\ldots,r$, let $\boldsymbol{a}_{k,j} = (\boldsymbol{b}_{j,1},\ldots,\boldsymbol{b}_{j,m-1},\boldsymbol{b}^k_{j,m}) \in$ **level *k*** and $\boldsymbol{a}_{k',j} = (\boldsymbol{b}_{j,1},\ldots,\boldsymbol{b}_{j,m-1},\boldsymbol{b}^{k'}_{j,m}) \in$ **level *k*′**, respectively. Since $b^k_{j,m\cdot L_m+2} \geqslant M_{C_m}$ an $db^{k'}_{j,m\cdot L_m+2} \geqslant M_{C_m}$, from Eqs. (22)–(29), it follows that the equation $\Pr(\boldsymbol{X}_{n+1} = \boldsymbol{a}_{i+k\prime,t} \mid \boldsymbol{X}_n = \boldsymbol{a}_{k,j}) = \Pr(\boldsymbol{X}_{n+1} = \boldsymbol{a}_{i+k\prime,t} \mid \boldsymbol{X}_n = \boldsymbol{a}_{k',j})$ holds for $i = -1,0,\ldots$, $t = 1,\ldots,r$. Hence, $\Pr(\boldsymbol{X}_{n+1} = \boldsymbol{a}_{i+k,t}|\boldsymbol{X}_n = \boldsymbol{a}_{k,j})$ is determined by (t,j,i) for $k \geqslant 1$. Therefore, arranging all of the states in the ascending order of levels and arranging the states in **level *i*** as $(\boldsymbol{a}_{i,1},\ldots,\boldsymbol{a}_{i,r})$, the transition probability matrix $\boldsymbol{P}$ of $\{\boldsymbol{X}_n\}$ is given as follows:

$$\boldsymbol{P} = \begin{pmatrix} \boldsymbol{B}_0 & \boldsymbol{B}_1 & \boldsymbol{B}_2 & \boldsymbol{B}_3 & \cdots \\ \boldsymbol{A}_0 & \boldsymbol{A}_1 & \boldsymbol{A}_2 & \boldsymbol{A}_3 & \cdots \\ \boldsymbol{0} & \boldsymbol{A}_0 & \boldsymbol{A}_1 & \boldsymbol{A}_2 & \cdots \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{A}_0 & \boldsymbol{A}_1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad (35)$$

where, $\boldsymbol{B}_i$ and $\boldsymbol{A}_i$ are $r \times r$ matrices. $\boldsymbol{B}_i$ represents the transition probability from **level 0** to **level *i*** $(i \geqslant 0)$ and $\boldsymbol{A}_i$ represents the transition probability from **level *k*** to **level *k* + *i*** -1 $(k \geqslant 1, i \geqslant 0)$.

4. Properties of the Markov chain $\{X_n\}$

Let $\boldsymbol{A} = \sum_{i=0}^{+\infty} \boldsymbol{A}_i$, and let **phase *j*** $= \{\boldsymbol{a}_{i,j} : 0 \leqslant i\}$ for $j = 1,\ldots,r$. By relabeling the phases if necessary, the stochastic matrix $\boldsymbol{A}$ and non-negative matrices $\boldsymbol{A}_i$ for $i = 0,1,\ldots$ have the following canonical forms.

$$\boldsymbol{A} = \begin{pmatrix} \boldsymbol{H}(1) & \boldsymbol{0} & \cdots & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \ddots & & \vdots & \vdots \\ \vdots & & \ddots & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \cdots & \boldsymbol{0} & \boldsymbol{H}(c) & \boldsymbol{0} \\ \boldsymbol{T}(1) & \cdots & \cdots & \boldsymbol{T}(c) & \boldsymbol{T}(0) \end{pmatrix} \quad (36)$$

$$\boldsymbol{A}_i = \begin{pmatrix} \boldsymbol{H}_i(1) & \boldsymbol{0} & \cdots & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \ddots & & \vdots & \vdots \\ \vdots & & \ddots & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \cdots & \boldsymbol{0} & \boldsymbol{H}_i(c) & \boldsymbol{0} \\ \boldsymbol{T}_i(1) & \cdots & \cdots & \boldsymbol{T}_i(c) & \boldsymbol{T}_i(0) \end{pmatrix}, \quad i = 0,1,\ldots \quad (37)$$

In Eqs. (36) and (37), $\boldsymbol{H}(i)$ and $\boldsymbol{H}_j(i)$ are $t_i \times t_i$ matrices, and $\boldsymbol{H}(i)$ is an irreducible stochastic matrix. $\boldsymbol{T}(i)$ and $\boldsymbol{T}_j(i)$ are $t_0 \times t_i$ matrices, and $\boldsymbol{I}_{t_0} - \boldsymbol{T}(0)$ is an invertible matrix, where $\boldsymbol{I}_{t_0}$ is a $t_0 \times t_0$ identity matrix. The positive integer c is the number of irreducible classes of $\boldsymbol{A}$. In Appendix A, an example of the state space $\boldsymbol{T}$ and associated matrices for a two-stage JIT production system are shown. Since $\boldsymbol{H}(i)$ is an irreducible stochastic matrix, a steady-state probability vector $\boldsymbol{p}_i$ exists which satisfies

$$\boldsymbol{p}_i\boldsymbol{H}(i) = \boldsymbol{p}_i, \quad \boldsymbol{p}_i\boldsymbol{e}^T_{t_i} = 1, \quad i = 1,\ldots,c. \quad (38)$$

Next, let

$$\rho_i = \boldsymbol{p}_i \sum_{j=1}^{+\infty} j\boldsymbol{H}_j(i)\boldsymbol{e}^T_{t_i}, \quad i = 1,\ldots,c. \quad (39)$$

The following theorems hold.

Theorem 1. *For i = 1,…,c, ρ_i in Eq. (39) is an equal value.*

From Theorem 1, we set ρ_i as ρ.

Theorem 2. *$\rho$is expressed as $\rho = \frac{D}{M_{C_m}} + b$, where, the constant b is determined by the production capacity distribution F_i, numbers of kanbans M_i and N_i, and the lead time for the delivery of the parts L_i for i = 1,…,m.*

Let $\boldsymbol{\alpha}_i \in T_i$ $(i = 1,\ldots,m)$, and $\alpha \in T$ be defined as follows:

$$\boldsymbol{\alpha}_i = (\overbrace{d_{\min},\ldots,d_{\min}}^{L_i}, N_i - L_i d_{\min}, M_i - d_{\min}), \quad i = 1,\ldots,m-1, \quad (40)$$

$$\alpha_m = (\overbrace{d_{\min},\ldots,d_{\min}}^{L_m}, N_m - L_m d_{\min}, d_{\min}) \quad (41)$$

and

$$\alpha = (\alpha_1,\ldots,\alpha_m). \quad (42)$$

Let $\boldsymbol{U} \subseteq \boldsymbol{T}$ be defined as $\boldsymbol{U} = \{\boldsymbol{s} \in \boldsymbol{T} : \boldsymbol{s}$ is accessible from $\alpha\}$. Then the following theorems hold.

Theorem 3. *If the conditions $(L_i+1)d_{min} < N_i$, $d_{min} < M_i$, $d_{min} < C_{i,max}$ for $i = 1,\ldots,m$ are satisfied, α is accessible from any state in T, and U is a unique aperiodic irreducible class of $\{X_n\}$.*

Theorem 4. *If the conditions in Theorem 3 are not satisfied, the total backlogged demand B(n) diverges to $+\infty$ as $n \to +\infty$ with proability 1.*

Theorem 5. *Any state $s \in U$ is positive recurrent if, and only if, $\rho < 1$.*

For the given system parameters, the value ρ can be calculated from Theorems 1 and 2, using the state classification algorithm (Fox and Landi, 1968). From Theorem 5, it follows that $\rho < 1$ is the stability condition for the system. From Theorems 3 and 4, if $\rho < 1$ the conditions of Theorem 3 are satisfied and α is accessible from any state in $\boldsymbol{T}$.

Let Dsup be defined as

$$\text{Dsup} = M_{C_m}(1-b), \tag{43}$$

where, b is the constant determined in Theorem 2. Then the following theorem holds.

Theorem 6. *For given F_i, M_i, N_i, L_i $i = 1,\ldots,m$, the following conditions are equivlent:*

(i) $\rho < 1$.
(ii) The average customer demand per period D satisfies $0 < D <$ Dsup.

When the stability condition is satisfied, performance measures of the system can be obtained as stated in Section 5. All the proofs of the above theorems are given in Appendix B.

5. Evaluation of system performance

5.1. Steady-state analysis

Let $\boldsymbol{\pi} = (\boldsymbol{\pi}_0, \boldsymbol{\pi}_1, \ldots)$ be the steady-state distribution of $\{\boldsymbol{X}_n\}$ that satisfies the following equation, where $\boldsymbol{\pi}_i = (\pi_{i,1},\ldots,\pi_{i,r})$ is the steady-state probability vector of **level *i***:

$$\boldsymbol{\pi P} = \boldsymbol{\pi}, \quad \boldsymbol{\pi e}_\infty^T = 1, \tag{44}$$

where, $\boldsymbol{e}_\infty = (1,1,\ldots)$.

Let B_∞ be the total backlogged demand with the steady-state distribution. Let $I_{i\infty}$ and $J_{i\infty}$ be the inventory of parts and inventory of products, respectively, in the ith-stage at the beginning of a period with the steady-state distribution.

Let $\pi(z)$ be the probability-generating function vector defined as follows:

$$\boldsymbol{\pi}(z) = \sum_{i=0}^{\infty} z^i \pi_i, |z| \leqslant 1. \tag{45}$$

Let $\boldsymbol{y}_i = (b_{1,i,L_i+2},\ldots,b_{r,i,L_i+2})$ and $\boldsymbol{z}_i = (b_{1,i,L_i+1},\ldots,b_{r,i,L_i+1})$. The average of these variables can then be written as follows:

$$E(B_\infty) = \boldsymbol{\pi}(1)\boldsymbol{y}_m^T + M_{C_m}\pi'(1)\boldsymbol{e}_r^T, \tag{46}$$

$$E(J_{i\infty}) = \boldsymbol{\pi}(1)\boldsymbol{y}_i^T, \quad i = 1,\ldots,m-1, \tag{47}$$

$$E(I_{i\infty}) = \boldsymbol{\pi}(1)\boldsymbol{z}_i^T, \quad i = 1,\ldots,m, \tag{48}$$

$$E(J_{m\infty}) = \sum_{iM_{C_m}+b_{j,m,L_m+2} \leqslant M_m} \boldsymbol{\pi}_{i,j}(M_m - (iM_{C_m} + b_{j,m,L_m+2})). \tag{49}$$

Let B'_∞ be the backlogged demand at the beginning of a period with the steady-state distribution. From Eq. (3), the average of B'_∞ is written as:

$$E(B'_\infty) = E(B_\infty) + E(J_{m\infty}) - M_m. \tag{50}$$

The probability of $B'_\infty > 0$ is written as:

$$\Pr\{B'_\infty > 0\} = 1 - \sum_{iM_{C_m}+b_{j,m,L_m+2} \leqslant M_m} \pi_{i,j} \tag{51}$$

Neuts (1989) derived an algorithm for obtaining the steady-state performance parameters for a M/G/1-type Markov chain. But the description is not clear if the associated stochastic matrix $\boldsymbol{A}$ has multiple irreducible classes. On the other hand, the M/G/1-type Markov chain considered in this paper can have this property (Appendix A). Hence, we explain a method that can be used to obtain the performance parameters for the general stochastic matrix $\boldsymbol{A}$ in Appendix C. $\pi(1)$ and $\pi'(1)\boldsymbol{e}_r^T$ are derived in Appendix C.1–C.3. $E(B_\infty)$, $E(I_{k\infty})$, $k = 1,\ldots,m$, $E(J_{k\infty})$, $k = 1,\ldots,m-1$ are obtained from Eqs. (46)–(48) using $\pi(1)$, $\pi\prime(1)\boldsymbol{e}_r^T$. For $i = 0,1,\ldots,\pi_i$ is derived inductively in Appendix C.4. $E(J_{m\infty})$, $E(B'_\infty)$ and $\Pr\{B'_\infty > 0\}$ are obtained from Eqs. (49)–(51) using π_i, $i = 0,1,\ldots$

5.2. System cost structure

For given withdrawal kanbans $\boldsymbol{N} = (N_1,\ldots,N_m)$ and production-ordering kanbans $\boldsymbol{M} = (M_1,\ldots,M_m)$, let $C_1(\boldsymbol{N},\boldsymbol{M})$ and $C_2(\boldsymbol{N},\boldsymbol{M})$ represent the average inventory/order cost and average backlogged cost, respectively. Let C $(\boldsymbol{N},\boldsymbol{M})$ represent the total cost. The following equations are thus derived:

$$C_1(\boldsymbol{N},\boldsymbol{M}) = \sum_{i=1}^{m} A_{I_i}(E(I_{i\infty}) - \frac{1}{2}D) + \sum_{i=1}^{m} B_{I_i}E(J_{i\infty}) + (A_o + A_w)D, \tag{52}$$

$$C_2(\boldsymbol{N},\boldsymbol{M}) = A_B E(B'_\infty) + C_B \Pr\{B'_\infty > 0\} \text{ and} \tag{53}$$

$$C(\boldsymbol{N},\boldsymbol{M}) = C_1(\boldsymbol{N},\boldsymbol{M}) + C_2(\boldsymbol{N},\boldsymbol{M}), \tag{54}$$

where

A_{I_i} = inventory cost of one part in ith-stage per period,
B_{I_i} = inventory cost of one product in ith-stage per period,
A_O = ordering cost of one part,
A_w = withdrawal cost of one part,
A_B = backlogged cost of one product per period,
C_B = backlogged cost per once

5.3. Performance evaluation algorithm

Let $\nu(i)$ be the first passage time to **level 0** of $\{\boldsymbol{X}_n\}$ when $\boldsymbol{X}_0 = \boldsymbol{a}_{1,i} \in$ **level 1**. Let $G_{i,j}$ be defined by $G_{i,j} = \sum_{n=1}^{\infty} \Pr(\boldsymbol{X}_{\nu(i)} = \boldsymbol{a}_{0,j}, \nu(i) = n | \boldsymbol{X}_0 = \boldsymbol{a}_{1,i})$. Set a $r \times r$ stochastic matrix $\boldsymbol{G} = (G_{i,j})$ $(\boldsymbol{i},\boldsymbol{j} = 1,\ldots,r)$. The matrix $\boldsymbol{G}$ is a minimal nonnegative solution of $\boldsymbol{X} = \sum_{k=0}^{+\infty} \boldsymbol{A_k X^k}$, $\boldsymbol{Xe}_r^T = \boldsymbol{e}_r^T$ and obtained by an iteration method (Neuts, 1989).

For the m-stage JIT system with production-ordering kanbans $\boldsymbol{M} = (M_1,\ldots,M_m)$ and withdrawal kanbans $\boldsymbol{N} = (N_1,\ldots,N_m)$, the performance evaluation algorithm is as follows:

Step 1. Select two sufficiently small, positive real numbers ε_1 and ε_2.
Step 2. Set a positive integer k_0 that satisfies $(\boldsymbol{I}_r - \sum_{i=0}^{k_0} \boldsymbol{A}_i)\boldsymbol{e}_r^T < \varepsilon_1 \boldsymbol{e}_r^T$ and $(\boldsymbol{I}_r - \sum_{i=0}^{k_0} \boldsymbol{B}_i)\boldsymbol{e}_r^T < \varepsilon_1 \boldsymbol{e}_r^T$.
Step 3. Let $\boldsymbol{A} = \sum_{i=0}^{k_0} \boldsymbol{A}_i$. Relabel the phases, if necessary, to set the matrix $\boldsymbol{A}$ as a standardized stochastic matrix, Eq. (36). To obtain a standardized stochastic matrix, use the state classification algorithm (Fox and Landi, 1968).
Step 4. Calculate ρ using Eq. (39). If $1 \leqslant \rho$, then stop. (In this case, the system does not satisfy the stability condition.)
Step 5. $n \leftarrow 1$; $\boldsymbol{G}_n \leftarrow \boldsymbol{I}_r$

Step 6. $n \leftarrow n + 1$.
Step 7. $\boldsymbol{G}_n \leftarrow \sum_{i=0}^{k_0} \boldsymbol{A}_i \boldsymbol{G}_{n-1}^i$.
Step 8. If the condition $\max_{1 \leqslant i \leqslant r, 1 \leqslant j \leqslant r} |(\boldsymbol{G}_n)_{i,j} - (\boldsymbol{G}_n - 1)_{i,j}| \geqslant \varepsilon_2$ is satisfied, then go to Step 6; otherwise go to Step 9.
Step 9. $\boldsymbol{G} \leftarrow \boldsymbol{G}_n$
Step 10. $\boldsymbol{Q} \leftarrow \sum_{i=0}^{k_0} \boldsymbol{B}_i \boldsymbol{G}^i$
Step 11. Solve the linear equation $\boldsymbol{kQ} = \boldsymbol{k},\ \boldsymbol{ke}_r^T = 1,\ \boldsymbol{k} \geqslant \boldsymbol{0}$.
Step 12. Applying 5.1, 5.2 and Appendix C, calculate $E(I_{i\infty}),\ E(J_{i\infty}),\ E(B_\infty),\ E(B'_\infty)$ and $\Pr\{B_\infty > 0\}$.
Step 13. Applying 5.3, calculate $C_1(\boldsymbol{N},\boldsymbol{M})$, $C_2(\boldsymbol{N},\boldsymbol{M})$ and $C(\boldsymbol{N},\boldsymbol{M})$.

6. Numerical results

The performance evaluation algorithm is applied to the following system: number of stages $m = 2$ and $L_i = 1$, $C_{i,max} = 3$ for $i = 1,2$. The following probability mass functions are used for production capacity distribution.

1. $F(3) = 1.0$ (no breakdown, average production capacity = 3.0, variance of production capacity = 0.0)
2. $F(3) = 0.8$, $F(2) = 0.1$, $F(0) = 0.1$ (average production capacity = 2.6, variance of production capacity = 0.84)
3. $F(3) = 0.7$, $F(1) = 0.2$, $F(0) = 0.1$ (average production capacity = 2.3, variance of production capacity = 1.21)

The following two types of demand distribution are considered.

(i) Poisson distribution with parameter 1.2.
(ii) Binomial distribution $B(6, 0.2)$.

It holds that $D = 1.2$ for these demand distributions. The parameters ε_1 and ε_2 in 5.3 are both set as 10^{-7}. The cost parameters are set as $A_{I_1} = 2,\ A_{I_2} = 10,\ B_{I_1} = 7,\ B_{I_2} = 20,\ C_B = 200$ and $A_B = 0$. The cost functions are caluculated for $3 \leqslant N_i \leqslant 6$ and $2 \leqslant M_i \leqslant 5$, $i = 1,2$. Table 1 shows the minimum total cost C^* and the optimal numbers of kanbans $N_1^*,\ N_2^*,\ M_1^*,\ M_2^*$ for nine production capacity distributions when the demand distribution is (i). Table 2 shows results similar to those of Table 1 when the demand distribution is (ii). From Tables 1 and 2, it is observed that the optimal numbers of kanbans are relatively robust for the variations in production capacity distribution and demand distribution. For these nine production capacity distributions and $D = 1.2$, the values of $\rho < 1$, if and only if, the conditions $3 \leqslant N_i$ and $2 \leqslant M_i$ for $i = 1,2$ are satisfied.

The computations were performed on a personal computer equipped with the Windows XP operating system, a 3.0 GHz CPU and 3.2 GB of memory. The computation time required to obtain the costs for one set of parameters ranged from approximately 20–22,000 seconds. When the size of the stochastic matrix $\boldsymbol{A}$ is large or the value of ρ is approximately 1.0, the time for converging the stochastic matrix $\boldsymbol{G}$ increases, which results in an increase in computation time.

7. Conclusion

In this paper, it is shown that the matrix analytic methods can be applied for the performance evaluation of a serial multi-stage JIT production system with two types of kanbans. The system is modeled as a discrete-time, M/G/1-type Markov chain. The stochastic properties and the stability condition of the system are derived, and a performance evaluation algorithm using the matrix analytic methods is developed. Numerical results are obtained using the proposed algorithm. They show that the optimal numbers of kanbans are relatively robust for variations in production capacity distribution and demand distribution. This paper demonstrates the feasibility of using the matrix analytic methods for M/G/1 type Markov chain to evaluate a multi-stage JIT production system.

Table 1
Optimal numbers of kanbans and minimal total cost for demand distribution (i).

F_1	F_2	N_1^*	N_2^*	M_1^*	M_2^*	ρ	C^*
(a)	(a)	4	4	2	3	0.7333	79.935
(a)	(b)	4	4	2	3	0.7973	89.163
(a)	(c)	4	4	2	4	0.8482	97.574
(b)	(a)	4	4	2	3	0.8000	82.159
(b)	(b)	4	5	2	3	0.8249	92.795
(b)	(c)	4	5	2	3	0.8424	100.179
(c)	(a)	4	4	3	3	0.8255	86.378
(c)	(b)	4	4	3	3	0.8473	95.930
(c)	(c)	4	4	3	4	0.8656	103.177

Table 2
Optimal numbers of kanbans and minimal total cost for demand distribution (ii).

F_1	F_2	N_1^*	N_2^*	M_1^*	M_2^*	ρ	C^*
(a)	(a)	4	4	2	3	0.7333	70.332
(a)	(b)	4	4	2	3	0.7000	77.526
(a)	(c)	4	4	2	3	0.8482	85.327
(b)	(a)	4	4	2	3	0.8000	70.966
(b)	(b)	4	4	2	3	0.8443	79.885
(b)	(c)	4	5	2	3	0.8424	88.398
(c)	(a)	4	4	2	3	0.8667	74.462
(c)	(b)	4	4	3	3	0.8473	83.590
(c)	(c)	4	4	3	3	0.8656	90.954

Appendix A. Supplementary data

Supplementary data associated with this article can be found, in the online version, at doi:10.1016/j.ejor.2011.04.014.

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