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Process quality adjusted lot sizing and marketing interface in JIT environment

József Vörös*, Gábor Rappai

Faculty of Business and Economics, University of Pécs, Rákóczi 80, Pécs, Hungary

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ABSTRACT

Lots exist even in a JIT environment. However, there is a continuous effort to decrease set up time, and employees are empowered to signal quality problems. When signaled quality problems are not fixed inside the cycle time the assembly line stops and is resumed again after the correction. But good JIT systems are pragmatic, and sometimes products with known quality problems go through the assembly line. They are separated (in the clinic area), where they wait to be fixed. In this paper both the number of cars leaving the assembly line and the number of cars entering the clinic area are considered random variables. All possible outcomes of these two random variables are identified and the appropriate inventory positions are determined in order to calculate the expected values of the inventory costs and the cycle length respectively. Using these values we gain new lot- sizing formulas measuring the optimal lot size and the total costs. In a JIT environment orders are usually frozen before the time period when production is scheduled and production plans (demand) is smoothed as much as possible. Thus, demand practically is deterministic and constant throughout the month, while the production volume of a shift is stochastic. The paper analyzes the characteristics of the total cost as a function of demand, and suggests how to determine the optimum volume (the demand) in the contracts. It is shown that the minimum point of the total cost as a function of demand decreases in backlogging cost.

1. Introduction

Toyota Production System (TPS) has evolved as Toyota's response to the task of 'better cars for more people'. The concept of better cars means flawless quality, and for more people means affordable price with perfect timing. The CEO of Toyota, Mr. Watanabe, says the Toyota Way has two pillars: continuous improvement and respecting people (employees, suppliers and customers) [1]. Behind these pillars there are two important elements of TPS: the principles of Jidoka and Heijunka [2]. The task of Heijunka is multiple: to connect the total value chain from customers to suppliers, make what customers want and when they want, and smooth the system pulse. The production volume is streamlined as smooth as possible, but product mix is similarly spread out as evenly as possible. The result of this policy is that in each moment the sequence structure of different model types in the assembly line reflects the volume and the structure of the monthly and smoothed daily demand. For example, if the volume ratio of the monthly demand for models A, B, and C is 3:2:1, respectively, then the sequence of cars in the assembly line appears as AAABBCAAABBC, and so on. This way, the implementation of the Heijunka principle

* Corresponding author. Tel.: +36 707080014; fax: +36 72501553.
E-mail addresses: voros@ktk.pte.hu (J. Vörös), rappai@ktk.pte.hu (G. Rappai).

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determines the lot size of models, which is 3, 2 and 1 for models A, B and C, respectively. To make these lot sizes optimal or less expensive, concerted efforts are required to decrease setup cost to the desired level. Schniederjans and Cao [3], and later Cao and Schniederjans [4] published comparative models for inventory related costs under economic order quantity and JIT policy showing the efficacy of the more cost inclusive models. Jaber et al. [5] pointed out that implementing JIT might be accompanied with problems that could negatively influence continuous improvement efforts, and that these problems could be attributed to hidden costs inherent in many JIT systems.

But each production process faces deviations and variability. The Jidoka principle is intended to highlight the quality problem when it occurs and fix those issues. However, production problems influence lot sizes and we have to take into account the phenomenon when we estimate the necessary efforts to push down setup cost to make the appropriate lot size optimal. One of the purposes of this paper is to build a lot sizing model applicable to TPS, that takes into account the basic properties of the JIT manufacturing system and helps identify the ideal level of setup costs. But lot sizing is only one element in the sophisticated world of JIT; all processes are driven to be in control, repeatable, reliable and stable, i.e. the assembly line should exhibit a process with high quality.

There are numerous papers intending to analyze the Toyota Production System, and the comprehensive nature of TPS involves among others the problems of lot sizing, setup time reductions, and issues considering the improvement of the quality of the production process. Additionally, JIT systems work well when demand and production plans are smoothed, those are frozen in advance, but production problems frequently hamper the fulfillment of production plans.

The seminal papers of Haris [6] and Taft [7] on how much to order or produce at once have attracted many researchers and many new concepts have been created since. Probably Shih [8] is among the firsts who mentioned that there may be shortages in the delivery process as not every unit of the product in the accepted lot is of perfect quality. Two inventory models are extended to the case where the proportion of defective units in the accepted lot is a random variable with known probability distributions. As the pioneer works of Porteus [9,10] point it out, lot sizing and process quality are interrelated. He considers investment into setup cost reduction and process quality improvement possibilities in the EOQ model, and he found that the optimal lot size is strictly increasing with improvement in process quality. Also among the pioneers there is Chand [11], who discusses the results of worker learning that reduces setup cots and improves process quality, and finally we have small lot sizes, the base of the stockless production philosophy. As a result of the spreading popularity of JIT Mekler [12] considers the possibility of reducing setup costs in the dynamic lot sizing models and already in 1994 we had the first review of lot sizing with random yields by Yano and Lee [13], while Price et al. [14] give an overview of the Kanban based production systems. Jamal and Sarker [15] constructed a model to recommend a raw material ordering policy to deliver a fixed amount of finished product in JIT environment. Vörös [16] presents an economic productions quantity model where the assembly line is considered a binary machine in the Toyota Production System, as the assembly line must be stopped when quality problems are noticed. Grosfeld-Nir et al. [17] develop a model in which the number of defective units within a produced lot is not known in advance and it is possible that after examining the lot the number of conforming units is short of the demand, and further manufacturing and inspection are required – an expensive process. Analyzing the Toyota assembly line in Ihara, Givi et al. [18] suggest that firms have to pay attention to human factors when designing jobs that are labor intensive, or otherwise they risk negative effects on product and process quality and, subsequently, system performance.

The problem of the presence of defective items in a lot appears in economic order quantity models as well. Salameh and Jaber [19] open a new track with the model in which a portion of a lot is defective and this portion is a random variable. Papachristos and Konstantaras [20] argue that in case of defective items shortages will appear and there is no sufficient mathematical assumption for model parameters assuring the disappearance of shortages. Maddah and Jaber [21] suggest a new procedure to take the ratio of the expected values of inventory costs and the cycle length, respectively (the model presented in this paper applies this concept). Khan et al. [22] provide a comprehensive summary of these models. Vörös [23] shows that both the Salameh and Jaber [19] and the Maddah and Jaber [21] models give the correct cost estimates but in two different model settings. Jaber et al. [24] present an entropic EOQ model to provide the estimation of hidden inventory cost as result of disorder. Konstantaras et al. [25] studied the effect of learning in screening rate, which will result in increasing quality.

The stochastic nature appears in the dynamic lot sizing models as well, but mainly the periodic demand is considered random. Tarim and Kingsman [26] present a mixed integer programming formulation to determine the replenishment periods and the necessary adjustments as demand was realized. In the Hayya et al. [27] model both demand and lead time are random variable and they show how intractability may be resolved. Levi and Shi [28] develop a new algorithm to compute provably near-optimal policies for the multi-period stochastic lot-sizing inventory models with positive lead times, general demand distributions and dynamic forecast updates. In Sana's model [29] the demand depends on price and price is a random variable. It is worth noting that non planned shortages may occur in this case. Similarly to Porteus [9,10], in the model of Sana et al. [30] imperfect items increase with time as the probability of imperfect quality is described by exponential function. Yoo et al. [31] assume a two-way inspection policy where only a portion of the quantity produced is screened and defective items went to market are returned. Sana [32] develops a dynamic model in which product reliability may be increased by investments and production rates are determined for time-varying demand. At the same year Sana [33] published another model in which production may be in or out of control, and defective items are restored in original quality. Shih and Wang [34] considered a model in which system may similarly be in or out of control, and they minimized the total cost per product per cycle time.

In the model of Freimer et al. [35] the system produces time varying proportion of defective parts which can be repaired at some cost. They propose investments to improve process quality and reduce setup cost. Jaber [36] takes into account that workers are authorized to stop production if quality or production problems arise. Then production process is interrupted for quality maintenance to bring the process under control again. The lot sizing is characterized for setup reduction with rework and interruptions to restore process quality. Li et al. [37] develop a model where the return on investment is maximized under budget constraint and they find that as the investment budget increases, a fundamental shift of investment strategies – setup cost reduction vs quality improvement – may be necessary. Kulkarni's [38] study examines a joint lot sizing and process investment problem with random yield and backorders – using almost all the categories we are going to utilize. He allows for inspection and in the stochastic model the optimal inspection and lot sizing policy, and the optimal process investment to reduce the yield variance are determined. The backlogging concept also appears in the study of Skouri et al. [39], and in these studies backlogs are planned ones as opposed to our model where they may appear randomly. One of the latest contributions is made by Diaby et al. [40]. In a JIT environment they address the problem of investing into the reduction of setup time and defect rates where constraints prevail for budget limitations, manufacturing and warehouse capacity. Omar and Sarker [41] propose optimal policies in a JIT environment when both shipment intervals and lot sizes are varied.

Our approach differs in some ways from the models mentioned above. As mentioned before, in TPS employees are empowered and required to highlight problems when they appear. Many of these problems are fixed within the cycle time, but – depending on the process quality – this is not always the case, and the assembly line stops. Consequently the assembly output can be considered a random variable. Sometimes, although problems are highlighted, they are not fixed instantly and the assembly line is not stopped. This time cars are forwarded to the clinic area where they await attention. This is an obvious violation of the Jidoka principle and we consider the number of cars getting into the clinic area also a random variable. The next section identifies the five inventory positions as the outcomes of these random events, and calculates the appropriate inventory costs. As a result of this new model, Section 3 identifies the location of the probability fields where the different inventory positions appear, and calculates the expected values of inventory costs and cycle length as function of lot size, respectively. These formulas take into account the performance of the assembly line where the outcome of the assembly line depends on both the process yield and the achievement of the clinic area, giving this way a new insight into the behavior of lot sizing models when process quality extended with the clinic area contribution is considered. The new lot sizing rule is derived, and then we can determine the annual total cost function taking into account the quality of the process. Using the new lot sizing rule, the ideal level of setup cost is determined to economically adjust it into a JIT environment. In JIT environment demand is frozen quite before the production takes place. However, the outcome of the production line is a random yield, while the contracted volume (demand) is given for a day. Thus Section 4 considers the expected annual total costs as a function of the daily production plan (demand). Analyzing the nature of the total cost as function of demand we suggest an algorithm to determine the optimum level of the optimal volume to contract in order to have maximum profit under a given process quality. In this section it is proved that the minimum point of the total cost function is decreasing in demand, but this property remains valid only under assumption when the number of defective items with known quality problems exceeds the capacity of the clinic area. Section 5 gives the conclusions.

2. Model building

2.1. Problem description

The intention of the Jidoka principle in a JIT system is to make a production problem clear when and where it occurs. Colored tape marks out areas of the floor to specify where just about everything in sight belongs and any deviation from normal conditions must be signaled visually [2]. A green and red line marks the beginning and the end of a work station, respectively, and a yellow line in between marks a point by which 70% of the work had to be completed. If a team member is behind at this yellow line, or finds any other problem, the andon cord is pulled, and the team leader rushes to that work station and if the problem is correctable, the andon cord is pulled again. Otherwise the andon cord is not pulled and the assembly line stops. This stoppage directly reduces the number of cars produced in a shift. According to Mishina and Takeda [2] a team member, on average, pulled the andon cord nearly one dozen times per shift, and one of these andon pulls resulted in an actual line stoppage. Due to stoppages, sometimes the run ratio falls to 85% from 95%, and this 10-point drop meant a shortfall of 45 cars per shift (based on the 57 s line cycle time that is in Georgetown, Kentucky, in 1991. Let us note that the current cycle time is 40 s in Takaoka, Japan [1], 1 min at Suzuki plant, 2–3 min at the Mercedes plant, both in Hungary, and is 154 s at the Porsche Macan assembly line, Leipzig, (www.porsche-leipzig.com)). It is also interesting to note that when a particular seat problem occurred, the team leader could fix a familiar problem on-line in 30 s.

However, there are exemptions when Jidoka principles are not implemented perfectly. Three reasons are known: first, the final assembly people already knew of the problem; second, it was possible to finish building the car without a particular assembly; third, it was felt that stopping the line was too expensive [2]. In these cases, when a team member spots quality problems, she pulls the andon cord before installing the defective assembly. Then the team leader pulls the andon cord to signal okay and tags the car to alert quality inspectors to the problem. The car then goes through the rest of the assembly line, and upon line-off, the car moves to the clinic/overflow area and waits for the problem to be fixed.

Table 1
Notations used.

ξ	The number of units leaving the assembly line in a day (including those entering the clinic area), random variable with probability density function $f(x)$. ξ is the daily production rate.
η	The number of units arriving at the clinic/overflow area during a day with a particular known quality problem, random variable with probability density function $g(y)$
D	Daily demand in units, input parameter
m	Number of cars fixed in the clinic/overflow area during a day, input parameter
Q	The lot size in units, decision variable
s	The current setup cost, input parameter
h	Holding cost per unit per day, input parameter
b	Backlogging cost per unit per day, input parameter
z	$= (y/x - m/D)/(D - m)$, assuming $D > m$
K	Planned capacity of the assembly line per day, in units (the ratio of the duration of a day over the planned cycle time)
M	The lowest value of ξ with positive probability ($P(K \geq \xi \geq M > 0$, and $P(\xi < M)=0$)
N	Number of working days in a year
π	Unit net profit on the car (product)

2.2. Notations and assumptions

Table 1.

2.3. Problem formulation

Let ξ denote the number of cars leaving the assembly line during a day, including tagged cars with known defective problems. We consider ξ a random variable with the probability density function $f(x)$, and we suppose that there is no probability that the assembly line is completely in shutdown state during a day (shift). Let η denote the number of tagged cars arriving at the clinic/overflow area with a particular quality problem during a day. We consider η a random variable with probability density function $g(y)$. We suppose that m cars with the particular quality problem can be fixed during a day. m is a known input parameter.

For simplicity, we assume there is one shift per working day, thus shift or day may be used alternatively, and demand can be expressed as D units per model type per day, which is a known constant input parameter, as a result of applying the Heijunka principle. The planned capacity of the assembly line can be calculated as the ratio of the duration of a shift over the planned cycle time. Because of stoppages the real output of the assembly line is lower than the planned capacity.

We denote the daily holding and backlogging cost with h and b per unit, respectively, and the set up cost is denoted by s. We seek the optimum lot size, and the lot size is denoted by Q.

Let x denote a realization of ξ, while y a realization of η, i.e. x denotes the observed number of cars leaving the assembly line in a particular day, out of which y are observed as defective with known quality problem. In the followings we summarize all the possible outcomes of these observations in order to calculate the expected cost and cycle length. Case 1 considers events when in a day the number of cars arriving at the clinic area is not larger than the repair capacity of this cell, i.e. we may assume that $y \leq m$. Consequently, clinic area capacity will not contribute to the development of backlogs. So, if there are no serious quality problems with the assembly line, backlogs will not appear at all if the assembly line has the sufficient capacity on the observed day, i.e. $x \geq D$. We identify this situation as Case 1a, and the inventory build-up diagram is represented by Fig. 1a.

This is a well-known diagram and denoting the inventory related costs for Case 1a by H_{1a}, it can be written as:

$$H_{1a}(Q, x, y)/h = \left(Q^2/2\right)(1/D - 1/x). \tag{1a}$$

The explanation behind Fig. 1a is that producing Q units in one lot that leaves the assembly line requires Q/x time units and serves demand over Q/D time units. During run time inventory increases at the rate of $(x - D)$ cars per time unit, and when a lot is finished, inventory decreases at the rate of D cars per time unit.

On the other hand, even if $y \leq m$ (the clinic area is not lagging behind), and the assembly line suffers from serious quality problems, backlog may develop, i.e. $x < D$. We identify this situation as Case 1b, and its inventory build-up diagram is represented by Fig. 1b.

Fig. 1b indicates that there are no cars waiting for customers as in each time unit, the system produces less than the demand. The number of unsatisfied units is $(D - x)$per day, thus until the end of the production run $Q(D - x)/x$ units backlog develops as the length of the production run is Q/x time units. The volume of the accumulated backlog is produced under overtime and the unit cost of the backlog is denoted by b. The time required to produce the accumulated backlog is considered negligible, for simplicity. In fact, the backlog level is kept till the start of overtime, however the cost of this maybe inserted into b, thus the approach indicated by the figure may be accepted. Denoting the accumulated backlogging cost by H_{1b}, the following can be written:

$$H_{1b}(Q,\ x,\ y)/b = \left(Q^2/2\right)(D - x)/x^2. \tag{1b}$$

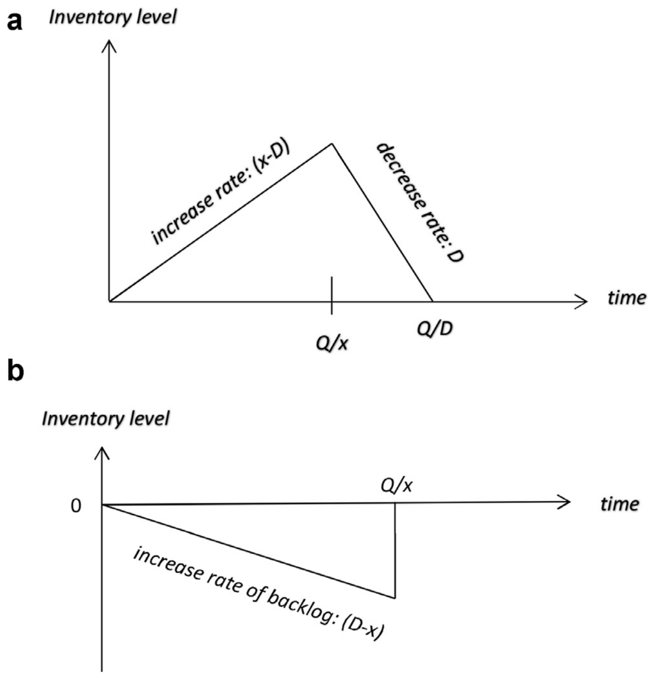

Fig. 1. (a) The inventory build-up diagram for Case 1a. (b) The inventory build-up diagram for Case 1b.

Now we turn to Case 2, where we assume that the clinic/overflow area has not the sufficient capacity to fix each car arriving at the clinic area at the same day, i.e. $y > m$. We identify Case 2aa as when although $y > m$, the assembly line does not suffer from serious quality problems and has enough capacity to meet demand, i.e. $x - y + m \geq D$. This form indicates as well that the outcome (the fixed units) of the clinic area is utilized at once to satisfy demand, and defective cars are hold at the same warehouse wearing the same holding cost like non-defective cars, as they are fully assembled. Additionally, we assume that the length of the cycle (the time elapsing between two consecutive points when inventory level is zero – these points are called regeneration points in the dynamic lot sizing literature) is long enough to fix all cars parking at the clinic area. During the production run ($= Q/x$) the number of cars entering the clinic area is yQ/x, and it requires yQ/mx time units to fix them. As we have time to fix all the cars during the cycle, it must be valid that $yQ/mx \leq Q/D$. But in this case no backlog will develop throughout the cycle, thus we have the same inventory build-up diagram like in Case 1a. Denoting the inventory related cost in Case 2aa by $H_{2aa}(Q, x, y)$, i.e. when $y \geq m$, $x - y + m \geq D$ and $y/x \leq m/D$, the following can be written:

$$H_{2aa} = H_{1a}. \tag{2aa}$$

On the other hand, when although backlog does not accumulate during production time, but the length of the cycle is not enough to fix all the cars parking at the clinic/overflow area, backlogs may accumulate after the end of the production runtime.

Let us identify this situation as Case 2ab, i.e. the assumptions are: $y \geq m$, $x - y + m \geq D$ but $y/x > m/D$. Fig. 2b represents the inventory build-up diagram where after that production is terminated (at Q/x), backlog starts accumulating at a certain point. Again, the stock level realization diagram indicated by Fig. 2b includes both the flawless and defective items and the fixed unites at the clinic area are used to satisfy daily demand.

It is a natural consequence that $y \leq x$, i.e. the number of cars entering the clinic area may not be larger than the number of cars leaving the assembly line. Then from the assumption $y/x > m/D$ follows that $m/D < 1$ as well, so $m < D$ is valid, and Fig. 2a reflects this property. After production terminates non defective cars are sold at the rate of the demand, however from a certain point, when flawless inventory runs out, inventory may deplete at the rate of m instead of D because only the fixed cars can be sold, and non-fixed ones are still waiting at the clinic area. Let us note that, except the backlogging possibility, Fig. 2a is rather similar to that of the Moussawi-Haidar et al. [42] model, where repairing is scheduled after runtime.

The number of cars getting into the clinic area during the production run is yQ/x while during the cycle – which is Q/D time units long – only mQ/D units may be fixed. The volume of the accumulated backlog is $Q(y/x - m/D)$, and this volume accumulates during the time length of $Q(y/x - m/D)/(D - m)$ as the rate of backlog accumulation is $(D - m)$. Let us note that this backlog volume is staying at the clinic area in the form of defective cars as well, so positive and negative inventory exist simultaneously. We can say that backlog starts accumulating at time $[Q/D - Q(y/x - m/D)/(D - m)]$. The on hand inventory at this point is: $[Q(y/x - m/D) + mQ(y/x - m/D)/(D - m)]$. The inventory cost during the accumulation of the backlog this way

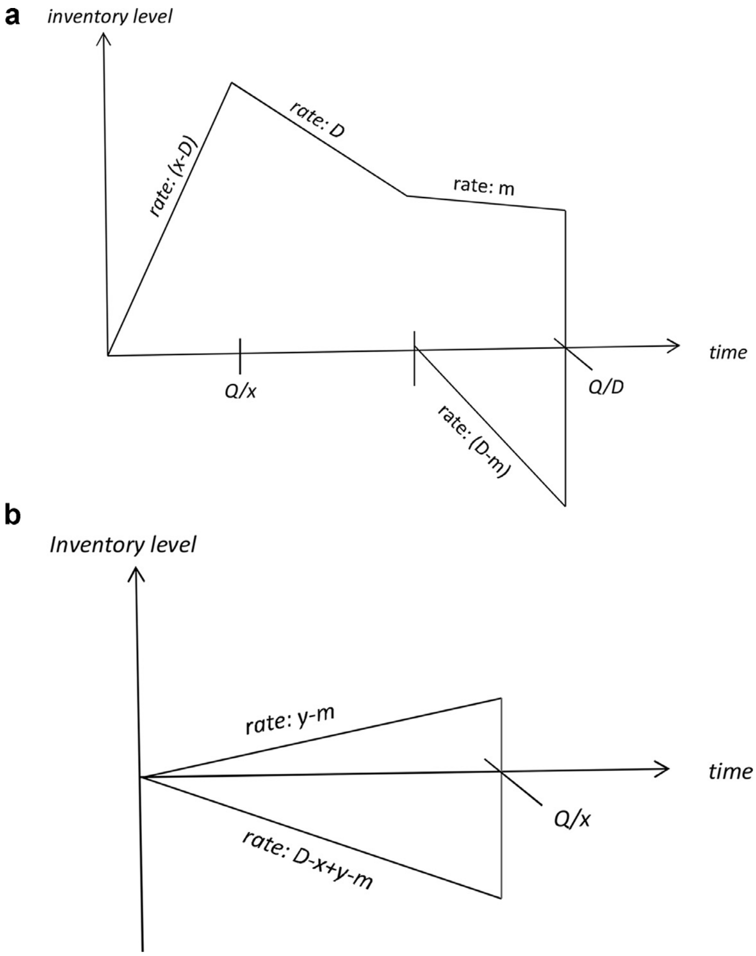

Fig. 2. (a) Inventory build-up diagram for Case 2ab. (b) The inventory build-up diagram for Case 2b.

is : $h[[Q(y/x - m/D)/(D - m)][2Q(y/x - m/D)+mQ(y/x - m/D)/(D - m)]]/2=h[Q(y/x - m/D)]^2[D - m/2]/(D - m)^2$. The occurring backlogging cost during the same interval is: $b[Q(y/x - m/D)]^2$.

By Fig. 2a, at point Q/x the inventory level is $(Q/x)(x - D)$, consequently the occurring holding cost from the beginning of the cycle till the point Q/x is: $hQ^2(x - D)/2x^2$ and from the point Q/x till the point when backlog starts developing the occurring holding cost is: $h[Q/D - Q/x - Q(y/x - m/D)/(D - m)][Q(y/x - m/D)(1+m/(D - m)) +Q (x - D)/x]/2=hQ^2(Dz +(x - D)/x)(1/D - 1/x - z)/2$, where we used the simplifying notation $z=(y/x - m/D)/(D - m)$. Denoting the all holding and backlogging costs in Case 2ab by H_{2ab}, adding the four types of costs it can be written that:

$$\begin{aligned} H_{2ab}(Q,x,y) &= \left(Q^2/2\right)\left(h\left[(x-D)/x^2+(Dz+(x-D)/x)(1/D-1/x-z)+z^2(2D-m)\right]+b(D-m)z^2\right) \\ &= \left(Q^2/2\right)\left[h(1/D-1/x)+(h+b)(D-m)z^2\right]. \end{aligned} \tag{2ab}$$

There is only one subcase waiting for identification inside Case 2: this is when demand cannot be satisfied from the very beginning of the cycle because of the frequent stoppages. Additionally the number of fixed cars is lower than the number of defective cars entering the clinic area, i.e. in Case 2b: $y > m$, and $x - y+m < D$. Fig. 2b gives the inventory build-up diagram, where we have simultaneously the defective items in the system not fixed yet in the clinic area as positive inventory, and the backlogs due to the not sufficient capacity, as negative inventory.

Denoting the occurring holding and backlogging costs in Case 2b by H_{2b}, we can determine this cost like as:

$$H_{2b}(Q,x,y) = \left(Q^2/2\right)[h(y-m)+b(D-x+y-m)]/x^2. \tag{2b}$$

Let us note that the length of the cycles (elapsed time between two regenerations points) in Case 1a, Case 2aa, Case 2ab depends on a decision variable (Q), and in Case 1b, Case 2b depends on a decision and a random variable (Q and ξ).

Table 2 summarizes the identified situations, including the resulting inventory related costs. Let us note that these inventory costs summarized in Table 2, are random variables.

Table 2
All inventory cases with assumptions and resulting costs, assuming $D > m$.

Case 1: $y \le m$		Case 2: $y > m$		
Case 1a: $x \ge D$	Case 1b: $x < D$	Case 2a: $x - y + m \ge D$		Case 2b: $x - y + m < D$
		Case 2aa: $y/x - m/D \le 0$	Case 2ab: $y/x - m/D > 0$	
$h(Q^2/2)(1/D - 1/x)$	$b(Q^2/2)(D - x)/x^2$	$h(Q^2/2)(1/D - 1/x)$	$(Q^2/2)[h(1/D - 1/x)+(h+b)(D - m)z^2]$	$(Q^2/2)[h(y - m)+b(D - x+y - m)]/x^2$

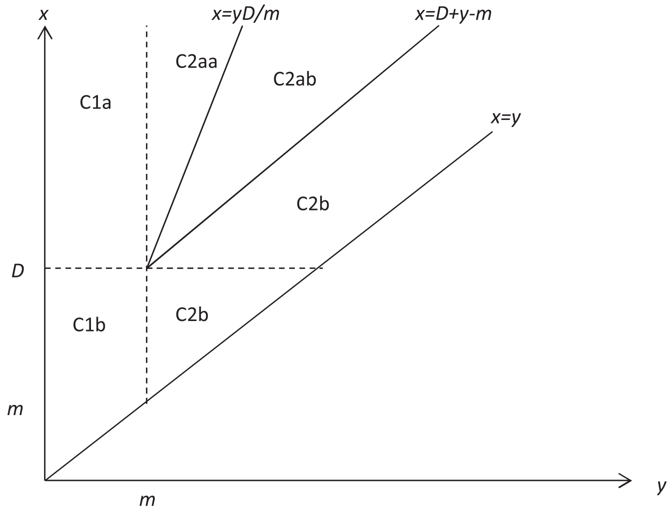

Fig. 3. Probability fields of inventory build-up diagrams with $D > m$ (C refers for case).

3. Determining the optimal lot size and minimum total cost

In order to ease the determination of the expected value of the inventory related costs, mainly based on the previous section, we develop a map that indicates the borders of the probability fields. In Fig. 3 we assume that $D > m$, i.e. the daily demand is larger than the repairing capacity of the clinic area. The complementary case will be discussed later. In Fig. 3 the vertical axis represents the observed number of cars leaving the assembly line during a day (denoted by x) and the horizontal one does the observed number of defective cars entering the clinic area during a day (denoted by y). It is a natural assumption that $x \ge y$, as cars entering the clinic area constitute a subset of cars leaving the assembly line. We have to note that there exists a theoretical limit on y (8 by the study of Mishina and Takeda [2]), however the limit is rarely kept. Anyway, the analysis is richer when we consider wider sets of possible outcomes, and we analyze the full upper half of the first quarter.

The vertical axis is divided into two segments by the horizontal line expressing the daily demand level, D. The horizontal axis is also divided into two sections, namely by the repairing capacity m represented by a vertical dashed line. The symbols among lines indicate the valid inventory models elaborated in the previous section.

Denoting the expected value of inventory related costs per cycle by EH_C, assuming that the two random variables are independent, this expected value is determined by the following expression:

$$\begin{aligned} EH_C(Q) = & \int_0^m \int_0^x H_{1b}(Q,\ x,\ y)g(y)dyf(x)dx + \int_m^D \int_0^m H_{1b}(Q,\ x,\ y)g(y)dyf(x)dx \\ & + \int_m^D \int_m^x H_{2b}(Q,\ x,\ y)g(y)dyf(x)dx \\ & + \int_D^K \int_0^m H_{1a}(Q,\ x,\ y)g(y)dyf(x)dx + \int_D^K \int_m^{xm/D} H_{2aa}(Q,\ x,\ y)g(y)dyf(x)dx \\ & + \int_D^K \int_{xm/D}^{x-D+m} H_{2ab}(Q,\ x,\ y)g(y)dyf(x)dx + \int_D^K \int_{x-D+m}^{x} H_{1b}(Q,\ x,\ y)g(y)dyf(x)dx. \end{aligned} \tag{3}$$

In (3) K denotes the planned capacity of the assembly line per day. Using the detailed formulas for the inventory costs, it can be rewritten that:

$$EH_C(Q) = \left(Q^2/2\right)H_C, \tag{4a}$$

where,

$$H_C = b\int_0^m \int_0^x \left((D-x)/x^2\right)g(y)dyf(x)dx + b\int_m^D \int_0^m \left((D-x)/x^2\right)g(y)dyf(x)dx$$
$$+ \int_m^D \int_m^x \left[[h(y-m) \ + \ b(D- \ x \ + \ y \ - \ m)]/x^2\right]g(y)dyf(x)dx$$
$$+ h\int_D^K \int_0^{xm/D} \left(\frac{1}{D} - \frac{1}{x}\right)g(y)dyf(x)dx$$
$$+ \int_D^K \int_{xm/D}^{x-D+m} \left[h(1/D - 1/x) \ + \ (h \ + \ b)(D-m)z^2\right]g(y)dyf(x)dx$$
$$+ \int_D^K \int_{x-D+m}^{x} \left[[h(y-m) \ + \ b(D-x+y-m)]/x^2\right]g(y)dyf(x)dx. \tag{4b}$$

Obviously, H_C is a positive constant, otherwise demand would not be satisfied. Next, we are going to determine the expected value of the length of the inventory cycles. As we noted, in Case 1a, Case 2aa and Case 2ab it depends on a decision variable (on Q/D), and in Case 1b, Case 2b depends on a decision and a random variable (on Q/x). Using the probability fields and the corresponding inventory build-up diagrams, denoting the expected value of the length of the inventory cycles by EL_c, it can be written that:

$$EL_c(Q) = \ L_C, \tag{5a}$$

where,

$$L_C = \int_0^m \int_0^x (1/x)g(y)dyf(x)dx + \int_m^D \int_0^m (1/x) \ g(y)dyf(x)dx$$
$$+ \int_m^D \int_m^x (1/x)g(y)dyf(x)dx + \int_D^K \int_0^m (1/D)g(y)dyf(x)dx + \int_D^K \int_m^{xm/D} (1/D)g(y)dyf(x)dx$$
$$+ \int_D^K \int_{xm/D}^{x-D+m} (1/D)g(y)dyf(x)dx + \int_D^K \int_{x-D+m}^{x} (1/x)g(y)dyf(x)dx, \tag{5b}$$

and L_C is positive constant as we suppose that there is no probability that the assembly line is completely in a shutdown state throughout a shift.

Denoting the expected cycle costs by EC_c, it can be written that:

$$EC_c(Q) = s + EH_c(Q) = s + \left(Q^2/2\right)H_C,$$

and we expect that $N/EL_c(Q)=N/(QL_c)$ cycles will appear in a year when N denotes the number of working days in a year. This way we can calculate the expected annual total cost (denoting it by ETC) as:

$$ETC(Q)/N = [1/(QL_c)]\left[s + \left(Q^2/2\right)H_C\right] = \ (s/L_c)/Q \ + (Q/2)(H_c/L_c). \tag{6}$$

Proposition 1. *Under Jidoka principle the optimal lot size (Q_{opt}) is determined by the following rule:*

$$Q_{opt} = \sqrt{2s}\sqrt{1/H_c} = \sqrt{2sD/h}\sqrt{h/DH_c}. \tag{7}$$

Now, let us suppose that from the implementation of the Heijunka principle it follows that the lot size must be Q_{heij}.

Corollary 1. *If the Heijunka principle determines the lot size in Q_{heij} units, the ideal setup cost (s_{ideal}) should be:*

$$s_{ideal} \ \le H_c Q_{heij}^2/2. \tag{8}$$

As in many cases $Q_{heij} = 1$, (8) reduces to: $s_{ideal} \le H_c/2$.

Example 1. Let the daily demand $D=100$, the repairing capacity of the clinic area $m=10$, and let both x and y have uniform distribution with the probability density functions:

$$f(x) = \begin{Bmatrix} \frac{1}{70} & \text{if } 430 \ \le x \ \le 500 \\ 0 & \text{otherwise} \end{Bmatrix} \text{ and } g(y) = \begin{Bmatrix} \frac{1}{50} & \text{if } 0 \ \le y \ \le 50 \\ 0 & \text{otherwise} \end{Bmatrix},$$

respectively. In this case Fig. 3 takes a shape which is represented by Fig. 4. As $D/m=10$, and $D - m=90$, the critical lines are: $x=10y$ and $x=y+90$.

Applying the inventory costs determined in (1a) or (2aa) and (2ab), it can be written that:

$$H_c = \frac{1}{70}\int_{430}^{500} \left[\frac{h}{50}\int_0^{x/10} \left(\frac{1}{100} - \frac{1}{x}\right)dy + \frac{1}{50}\int_{x/10}^{50} \left(h\left(\frac{1}{100} - \frac{1}{x}\right) + \frac{h+b}{90}\left(\frac{y}{x} - 0.1\right)^2\right)dy\right]dx \ ,$$

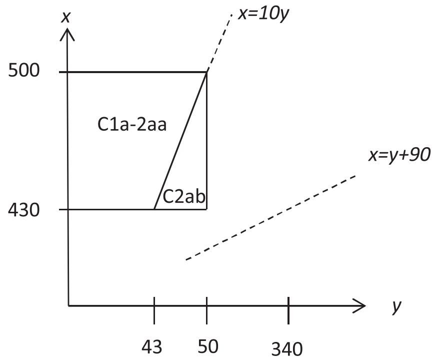

Fig. 4. The probability fields of Example 1.

which can be rewritten as:

$$H_c = \frac{1}{70}\int_{430}^{500}\left[\frac{h}{50}\int_0^{50}\left(\frac{1}{100}-\frac{1}{x}\right)dy + \frac{h+b}{4500}\int_{x/10}^{50}\left(\frac{y}{x}-0.1\right)^2 dy\right]dx$$
$$= \frac{1}{70}\int_{430}^{500}\left[h\left(\frac{1}{100}-\frac{1}{x}\right) + \frac{h+b}{4500}\int_{x/10}^{50}\left(\frac{y}{x}-0.1\right)^2 dy\right]dx. \quad (9)$$

We have that,

$$\int_{x/10}^{50}\left(\frac{y}{x}-0.1\right)^2 dy = \left(\frac{50}{x}-0.1\right)^3 x\frac{1}{3}\ .$$

It can be shown that:

$$\int_{430}^{500}\left(\frac{50}{x}-0.1\right)^3 x\frac{1}{3}dx = \frac{1}{3}\int_{430}^{500}\left(50^3/x^2 - 0.3\frac{2500}{x} + 1.5 - 0.001x\right)dx = 0.01.$$

Moreover,

$$\int_{430}^{500}\left(\frac{1}{100}-\frac{1}{x}\right)dx = 0.55, \quad (10)$$

thus (9) can be calculated as:

$$H_c = \frac{1}{70}(0.55h + 0.01(h+b)/4500).$$

This expression indicates that low repairing capacity plays a role when the cost of backlogging is very high under the distribution functions defined in our Example 1.

It is worth noting that in the traditional economic production quantity model the term $(Q^2/2)$ is multiplied by the expression $(1/D - 1/x)$, where x is deterministic. Taking the expected value of ξ, we have $(1/D - 1/x) = (1/100 - 1/465) = 0.0078$. The first term of (9) exactly gives this value as using the result in (10), $0.55/70 = 0.0078$. So, the second term of (9) gives the clinic area contribution to the lot sizing problem.

Now, we turn to the case when the daily demand is not larger than the repairing capacity of the clinic area, i.e. $D \le m$. Fig. 5 summarizes the possible outcomes.

When $D \le m$, backlogging appears only in one situation, namely when the actual capacity of the assembly line drops below the demand due to frequent stoppages, i.e. it happens that $x < D$. Because the repairing capacity is higher than the number of cars entering the clinic area, i.e. $y < m$, there are no cars waiting at the clinic area. So, the inventory build-up diagram behaves like in Fig. 1b, and we have Case 1b.

Backlogs may not accumulate when $x \ge D$ and $y \le m$, because the actual capacity of the assembly line exceeds the demand. Although x may include defective cars getting into the clinic area, but the repairing capacity is high enough and the cars are fixed during the same shift. Thus we have Case 1a, and the inventory diagram is represented by Fig. 1a.

Backlogs similarly will not appear in cases when $y > m$. The inventory level in this case is $x - y + m - D$, and this expression may not be negative as $x - y \ge 0$, and $m - D \ge 0$. Thus inventory build-up diagram C2aa applies as Case 2aa. But the inventory diagrams for Case 2aa and Case 1a are the same, thus the expected value of the inventory costs for a cycle can be calculated as follows:

$$EH_C(Q) = \left(Q^2/2\right)H_C, \quad (11a)$$

where,

$$H_C = b\int_0^D\int_0^x \left((D-x)/x^2\right)g(y)dyf(x)dx + h\int_D^K\int_0^x\left(\frac{1}{D}-\frac{1}{x}\right)g(y)dyf(x)dx. \quad (11b)$$

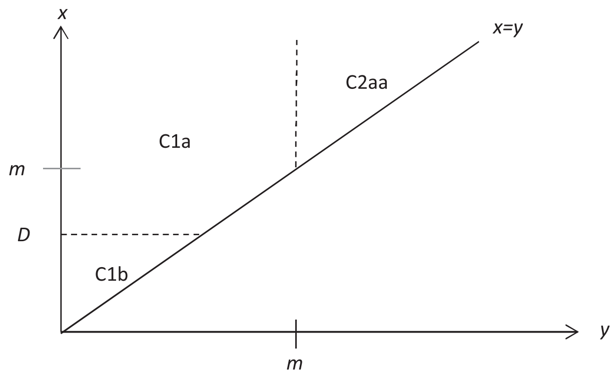

Fig. 5. Probability fields of inventory build-up diagrams with $D \le m$ (C refers for case).

The expected length of the cycle can be calculated as:

$$EL_c(Q) = QL_C, \tag{12a}$$

where,

$$L_C = \int_0^D \int_0^x (1/x)g(y)dyf(x)dx + \int_D^K \int_0^x \left(\frac{1}{D}\right)g(y)dyf(x)dx, \tag{12b}$$

and L_c is a positive number as we suppose that there is probability that the assembly line is completely shut down during a shift.

This way formally the optimal lot sizing problem behaves like in the previous situation, and formulas determined in (6) and (7) can be used for cases when $D \le m$.

Corollary 2. *Increasing the repairing capacity of the clinic area helps avoid accumulating backlogs, and lifting it above the daily demand, backlogs stemming from scarce capacity of the clinic area will not occur.*

Probably, one of the most frequently appearing critical situations can be identified as Case 2ab, as assembly line systems are designed to be sufficiently robust to avoid critical cases like Case 1b and Case 2b which indicate very serious process quality problems. In Case 1b and Case 2b the capacity of the system may be far below the demand. Thus focusing better on Case 2ab, in (2ab) we can see that the expression $z=(y/x - m/D)/(D - m)$ plays a key role in shaping backlogging related costs when $D > m$. This term does not exist when $D \le m$, and Fig. 4 indicates that a case of backlogging may occur only in one situation, namely when the bottleneck is the assembly capacity. Stoppages happen so frequently that demand cannot be met.

The formal proof of Corollary 2 will be utilized in the following section: compared to cases C1a or C2aa, the second term of (2ab) gives the clinic area 'contribution' to the costs, which is $(h+b)(D - m)z^2$, where $z=(y/x - m/D) / (D - m)$. Rewriting these expression, we have $(h+b)(D - m)z^2=(h+b)(y/x - m/D)^2 / (D - m)$. We have to see that when $m \to D$, then not only the denominator tends to zero but the numerator as well. In this case $m/D \to 1$ in the numerator, but $y/x \to 1$, too. The explanation comes from Fig. 3 as when $m \to D$, lines $x=Dy/m$ and $x=D+y - m$ tends to overlap, i.e. $x \to y$. Thus by the L'Hopital rule, we can write the following:

$$\lim_{m\to D} \frac{\left(\frac{y}{x}-\frac{m}{D}\right)^2}{D-m} = \frac{\lim_{m\to D} \partial\left(\frac{y}{x}-\frac{m}{D}\right)^2/\partial m}{\lim_{m\to D} \partial(D-m)/\partial m} = \frac{\lim_{m\to D} -2\left(\frac{y}{x}-\frac{m}{D}\right)}{-1} = \frac{0}{-1} = 0.$$

4. The marketing-operations interface

Interestingly, in a JIT environment demand (daily production requirement) may not vary during a short time horizon. There are important pre-assumptions to use lean production system, and level loading plays primary role among the pre-requisites [43]. Ones established, productions schedules are relatively fixed over time and this fact provides certainty to the system. According to Mishina and Takeda [2] a monthly demand (of May) is frozen a couple of months before (in January), and the monthly production plan is smoothed as much as possible and it is desirable to produce in small lots and to spread the production of the different products throughout the day to achieve smooth production [43]. But even in the most efficient lean production system production problems and stoppages exist, which are random events in every system. Thus the question offers itself: under a given process quality sales and marketing department should increase or decrease the number of sold units (cars) for a month, or a particular time period? The question raised looks like the newsboy problem [44], but there are significant differences. First of all in our model not the demand is random, but the assembly line yield, and even more importantly, there may exist unsatisfied demand and positive inventory (cars at the clinic area) simultaneously.

This section answers the operations-marketing interface question, whether under a given process quality the volume of contract for a time period should be increased or decreased. To facilitate it, we characterize the possible forms of the optimal expected annual total cost function as the function of demand, D. Turning back to the optimal lot size we gave in (7): $Q_{opt} = \sqrt{2s}\sqrt{1/H_c}$. Substituting this in the expected total cost function given in (6), we have the expected annual minimum total cost, which can be determined as:

$$ETC(Q_{opt}) = \frac{N}{L_c}\,\sqrt{2sH_c} = \frac{\sqrt{H_c}}{L_c}(\text{constant}). \tag{13}$$

This form indicates that we have to focus on the features of $\frac{\sqrt{H_c}}{L_c}$, but first of all on the nature of H_c and L_c. It is enough to restrict our analysis in the rest of the paper for cases when $D > m$.

Property 1. L_c is decreasing and H_c has local minimum point as a function of D.

Based on (5b) we can write that:

$$L_C = \int_m^D \int_0^x (1/x)\ g(y)dyf(x)dx + \int_D^K \int_0^{x-D+m} (1/D)g(y)dyf(x)dx + \int_D^K \int_{x-D+m}^x (1/x)g(y)dyf(x)dx.$$

We recall (the generalized form of the Leibniz integral rule) that:

$$\frac{d}{dz}\int_{k(z)}^{l(z)} m(z,p)dp = l_z \cdot m(z,l(z)) - k_z \cdot m(z,k(z)) + \int_{k(z)}^{l(z)} m_z(z,p)dp,$$

where $l_z = \frac{dl}{dz}$.

Taking the derivative of L_c with respect to D, we have:

$$\begin{aligned}\frac{dL_c}{dD} &= f(D)\int_0^D (1/D)\ g(y)dy + \int_D^K \left\{\int_0^{x-D+m}\left(-1/D^2\right)g(y)dy - g(x-D+m)/D\right\}f(x)dx - f(D)\int_0^m (1/D)g(y)dy \\ &\quad + \int_D^K (1/x)g(x-D+m)f(x)dx - f(D)\int_m^D (1/D)g(y)dy \\ &= -\int_D^K \int_0^{x-D+m}\left(\frac{1}{D^2}\right)g(y)dyf(x)dx - \int_D^K \left(\frac{1}{D}-\frac{1}{x}\right)g(x-D+m)f(x)dx, \end{aligned} \tag{14}$$

which is a negative term, thus L_c as function of D is decreasing.

Turning to H_c, from (4b) we can write that:

$$\begin{aligned} H_C &= b\int_m^D \int_0^m \left((D-x)/x^2\right)g(y)dyf(x)dx \\ &\quad + \int_m^D \int_m^x \left[[h(y-m)\ +\ b(D-x+y-m)]/x^2\right]g(y)dyf(x)dx \\ &\quad + h\int_D^K \int_0^{x-D+m}\left(\frac{1}{D}-\frac{1}{x}\right)g(y)dyf(x)dx \\ &\quad + \int_D^K \int_{xm/D}^{x-D+m}\left[(h\ +\ b)\left(\frac{y}{x}-\frac{m}{D}\right)^2\ /\ (D-m)\right]g(y)dyf(x)dx \\ &\quad + \int_D^K \int_{x-D+m}^x \left[[h(y-m)\ +\ b(D-x+y-m)]/x^2\right]g(y)dyf(x)dx. \end{aligned}$$

Then,

$$\begin{aligned}\frac{dH_c}{dD} &= b\int_m^D \int_0^x \left(1/x^2\right)g(y)dyf(x)dx + f(D)\int_m^D \frac{(h+b)(y-m)}{D^2}g(y)dy \\ &\quad + -h\int_D^K \left\{\int_0^{x-D+m}\left(\frac{1}{D^2}\right)g(y)dy + \left(\frac{1}{D}-\frac{1}{x}\right)g(x-D+m)\right\}f(x)dx \\ &\quad + \int_D^K \left\{\int_{xm/D}^{x-D+m}\frac{h\ +\ b}{(D-m)^2}\left[2\left(\frac{y}{x}-\frac{m}{D}\right)\left(\frac{m}{D^2}\right)(D-m) - \left(\frac{y}{x}-\frac{m}{D}\right)^2\right]g(y)dy \right. \\ &\quad \left. -\frac{h+b}{D-m}g(x-D+m)\left(\frac{x-D+m}{x}-\frac{m}{D}\right)^2\right\}f(x)dx \\ &\quad + \int_D^K \left\{\int_{x-D+m}^x \frac{b}{x^2}g(y)dy + \frac{h(x-D)g(x-D+m)}{x^2}\right\}f(x)dx - f(D)\int_m^D \frac{(h+b)(y-m)}{D^2}g(y)dy. \end{aligned}$$

This simplifies to:

$$\frac{dH_c}{dD} = b\int_m^D \int_0^x \left(1/x^2\right)g(y)dy f(x)dx - h\int_D^K \left\{\int_0^{x-D+m}\left(\frac{1}{D^2}\right)g(y)dy + \left(\frac{1}{D}-\frac{1}{x}\right)g(x-D+m)\right\}f(x)dx$$
$$+ \int_D^K \left\{\int_{xm/D}^{x-D+m} \frac{h\ +\ b}{(D-m)^2}\left[2\left(\frac{y}{x}-\frac{m}{D}\right)\left(\frac{m}{D^2}\right)(D-m) - \left(\frac{y}{x}-\frac{m}{D}\right)^2\right]g(y)dy\right.$$
$$\left. -\frac{h+b}{D-m}g(x-D+m)\left(\frac{x-D+m}{x}-\frac{m}{D}\right)^2\right\}f(x)dx$$
$$+ \int_D^K \left\{\int_{x-D+m}^{x} \frac{b}{x^2}g(y)dy + \frac{h(x-D)g(x-D+m)}{x^2}\right\}f(x)dx. \tag{15}$$

For $D=K$ we have,

$$\frac{dH_c}{dD} = b\int_m^D \int_0^x \left(1/x^2\right)g(y)dy f(x)dx,$$

which is a positive expression. Now, let us consider the situation when $D\to m$. Then, utilizing the mathematics of Corollary 2 as well:

$$\lim_{D\to m}\frac{dH_c}{dD} = -h\int_m^K \left\{\int_0^x \left(\frac{1}{m^2}\right)g(y)dy + \left(\frac{1}{m}-\frac{1}{x}\right)g(x)\right\}f(x)dx,$$

which is a negative term. Thus the derivative of H_c changes sign at the interval *(m, K]*. Consequently, H_c has local minimum.

Property 2. If there is no probability that $\eta > m$, then both H_c and L_c are convex.

In this case:

$$\frac{dH_c}{dD} = b\int_m^D \left(1/x^2\right)f(x)dx - h\int_D^K \left(\frac{1}{D^2}\right)f(x)dx, \tag{16a}$$

and

$$\frac{dL_c}{dD} = -\int_D^K \left(\frac{1}{D^2}\right)f(x)dx. \tag{16b}$$

From these we have:

$$\frac{d^2H_c}{dD^2} = \left(b/D^2\right)f(D) + 2h\int_D^K \left(\frac{1}{D^3}\right)f(x)dx + hf(D)/D^2, \tag{17a}$$

and

$$\frac{d^2L_c}{dD^2} = 2\int_D^K \left(\frac{1}{D^3}\right)f(x)dx + f(D)/D^2, \tag{17b}$$

where both expressions are positive, thus H_c and L_c are convex, respectively.

Corollary 3. *The optimal expected annual total cost as a function of D, ETC(Q_{opt}), may have both local minimum and maximum points and is increasing at K.*

Based on (13), the shape of the expected annual total cost function basically depends on the expression $\frac{\sqrt{H_c}}{L_c}$. Taking its derivative with respect to D, we have:

$$\frac{d\sqrt{H_c}/L_c}{dD} = \frac{\dfrac{L_c H'_c}{2\sqrt{H_c}} - L'_c\sqrt{H_c}}{L_c^2}. \tag{18}$$

As we pointed out, L'_c (the derivative of L_c with respect to D) is negative, thus the derivative of the total cost function in (18) will not change its sign if H'_c does not take negative values. Property 1 proves that H'_c does have negative values. At K L'_c is negative, H'_c is positive, thus the numerator of (18) is positive. Consequently, the annual cost function increases at K. The following example indicates that the total cost as function of D may have both local maximum and minimum points as well.

Example 2. Let us consider the distribution functions for ξ and η as given in Example 1, respectively, and let $m=10$. Let us identify the expected annual total cost function as function of D.

The problem entails seven segments for D. Table 3 summarizes the intervals and gives reasons why the distinctions are required.

Table 3
The segmentation of D.

Segments for D	*Reasons*
$10 < D \le 86$	Backlogs never occur, see Fig. 6a.
$86 < D \le 100$	Line $x = Dy/10$ enter the rectangle at point $(y, x) = (50, 430)$. The Fig. 4 of Example 1 applies here.
$100 < D \le 390$	Line $x = Dy/10$ does not cross the right border of the rectangle.
$390 < D \le 430$	Line $x = y + D - 10$ enter the rectangle.
$430 < D \le 460$	Demand is larger than the lower border of the rectangle, C1b and C2b may occure.
$460 < D \le 500$	Line $x = y + D - 10$ does not cross the right border of the rectangle.
$500 < D$	D surpasses the upper border of the rectangle, there are always backlogs and overtime.

As it takes serious time and attention to follow the identification of H_c function in the segments, we concentrate for the most interesting ones, namely for the first, third and fifth segments (of course we have calculated all). Finding the functions of the first segment is relatively easy (for seven as well):

$$H_c = \frac{h}{3500}\int_{430}^{500}\int_0^{50}\left(\frac{1}{D} - \frac{1}{x}\right)dydx = h/D - (h/70)ln(500/430).$$

On the other hand:

$$L_c = \frac{h}{3500}\int_{430}^{500}\int_0^{50}\left(\frac{1}{D}\right)dydx = 1/D.$$

Thus,

$$ETC(Q_{opt}) = (constant)\,\frac{\sqrt{H_c}}{L_c} = (constant)\,D\sqrt{\frac{h}{D} - \frac{h}{70}ln(500/430)},$$

which is increasing in D (despite of the fact that H_c decreases).

Fig. 6a shows the appropriate situation for $10 < D \le 86$.

Then let us turn to the third interval, i.e. when $100 < D \le 390$. Fig. 6b shows the relevant probability fields.

For this interval the functions can be revealed as:

$$\begin{aligned}3500H_c &= h\int_{430}^{500}\int_0^{50}\left(\frac{1}{D} - \frac{1}{x}\right)dydx + \frac{h+b}{D-10}\int_{430}^{500}\int_{10x/D}^{50}\left(\frac{y}{x} - \frac{m}{D}\right)^2 dydx\\ &= 50h(70/D - ln(500/430))\\ &\quad + \frac{1}{3}\frac{h+b}{D-10}\left(-\frac{50^3}{500} + \frac{50^3}{430} - \frac{500}{D^3}\left(500^2 - 430^2\right) - \frac{75000}{D}\ln\frac{500}{430} + 70\frac{15000}{D^2}\right),\end{aligned} \tag{19}$$

and $L_c = 1/D$ again.

Fig. 6c depicts the $\frac{\sqrt{H_c}}{L_c}$ function for $h = b = 100$, and the expected value of the minimum annual total cost has maximum point at this interval.

The calculation of H_c and L_c functions for the interval $430 < D \le 460$ requires the largest efforts, but provides interesting results. Fig. 6d gives the details, and according to this, the functions are as follows:

$$\begin{aligned}3500\,H_c &= b\int_{430}^{D}\int_0^{10}\frac{D-x}{x^2}dydx + \int_{430}^{D}\int_{10}^{50}\frac{[h(y-10) + b(D-x+y-10)]}{x^2}dydx\\ &\quad + h\int_D^{500}\int_0^{10x/D}\left(\frac{1}{D} - \frac{1}{x}\right)dydx + \int_D^{D+40}\int_{10x/D}^{x-D+10}\left[\frac{(h+b)\left(\frac{y}{x} - \frac{10}{D}\right)^2}{D-10} + h\left(\frac{1}{D} - \frac{1}{x}\right)\right]dydx\\ &\quad + \int_{D+40}^{500}\int_{10x/D}^{50}\left[\frac{(h+b)\left(\frac{y}{x} - \frac{10}{D}\right)^2}{D-10} + h\left(\frac{1}{D} - \frac{1}{x}\right)\right]dydx\\ &\quad + \int_D^{D+40}\int_{x-D+10}^{50}\frac{[h(y-10) + b(D-x+y-10)]}{x^2}dydx,\end{aligned}$$

and,

$$\begin{aligned}3500L_c &= \int_{430}^{D}\int_0^{50}\frac{1}{x}dydx + \int_D^{D+40}\int_0^{x-D+10}\frac{1}{D}\,dydx + \int_{D+40}^{500}\int_0^{50}\frac{1}{D}\,dydx\\ &\quad + \int_D^{D+40}\int_{x-D+10}^{50}\frac{1}{x}dydx = 50ln(D/430) + 1200/D + 50(460-D)/D + (40+D)ln((D+40)/D) - 40.\end{aligned}$$

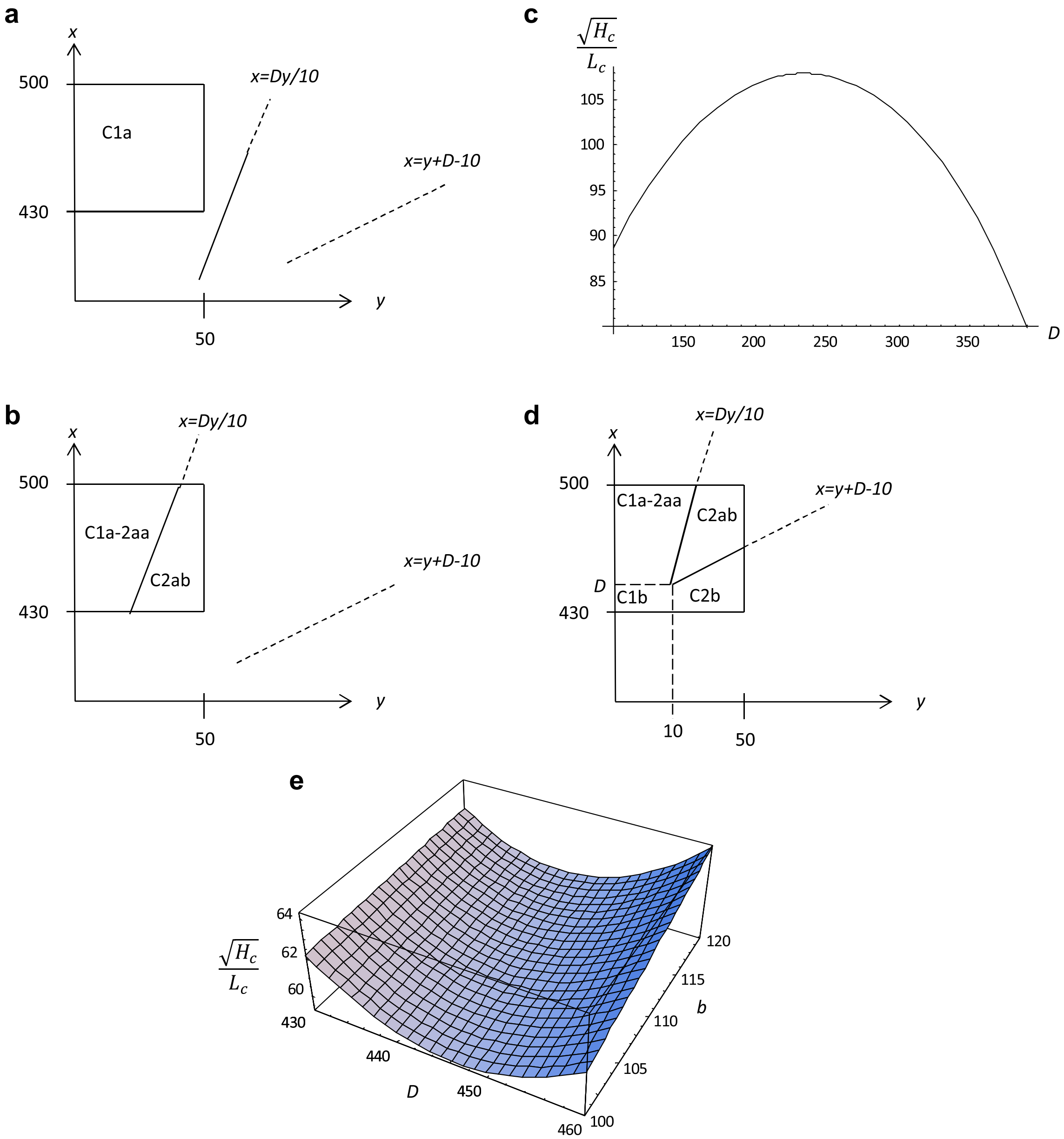

Fig. 6. (a) Probability fields for the case when $10 < D \leq 86$. (b) Probability fields for the case when $100 < D \leq 390$. (c) $\frac{\sqrt{H_c}}{L_c}$ values with $h = b = 100$. (d) Probability fields for the case when $430 < D \leq 460$. (e) $\frac{\sqrt{H_c}}{L_c}$ values for $h = 100$ and $100 \leq b \leq 120$, at the interval of $430 < D \leq 460$.

Fig. 6e shows the outcome for $h = 100$ and for an interval of b values. For $b = 100$, $\frac{\sqrt{H_c}}{L_c}$ has local minimum at $D_0 = 446.2$. We can say, that under the process quality given by the distribution functions, the expected annual total cost function attains its minimum value when the number of cars produced is 446 per day. (Let us note that the total cost function strictly increases for $460 < D$.) Fig. 6e represents well one of the key questions raised by this paper, namely whether the decision concerning the sales volume contracted by sales and marketing department is optimal. Like above, denoting the minimum location of the total cost function by D_0, then, certainly, if there is profit on the product produced, the contracted volume should be beyond D_0. Let π denote the unit net profit on the product. Then the next algorithm provides an optimal solution to determine the optimal contract volume:

Algorithm.

Step 1: Determine functions H_c, L_c, L_c' and H_c'. Use approximation methods when it is required.

Step 2: Determine the lowest value of $N\sqrt{2s}\frac{d\sqrt{H_c/L_c}}{dD}$.

Step 3: If this lowest value is not lower thank π, then Stop: capacities are overutilized, to satisfy backlogs is very expensive. Otherwise, next step.

Step 4: Determine the value of $N\sqrt{2s}\frac{d\sqrt{H_c}/L_c}{dD}$ at $D = K$.

Step 5: If this value is lower than π, then Stop: sales volume can be increased till K. Otherwise next step.

Step 6: Take the largest value of D that is the solution of the equation:

$$\pi = N\sqrt{2s}\frac{d\sqrt{H_c}/L_c}{dD}. \tag{20}$$

The solution of Eq. (20) then assures the largest profit volume.

Proposition 2. *When the expected annual total cost function $ETC(Q_{opt})$ has local minimum at D_0, this minimum point is decreasing in backlogging cost (b) if $\frac{\partial dH_c/dD}{\partial b}$ is positive, i.e. if the derivative of the derivative of H_c with respect to demand is increasing in b.*

The total cost function has local minimum if the expression $L_cH_c' - 2L_c' H_c$ (based on (18)) changes its sign at D_0, i.e. there is D_0 for which $L_cH_c' - 2L_c'H_c = 0$. Let $R(D, b) = L_cH_c' - 2L_c'H_c$. We recall that (the implicit function rule):

$$\frac{\partial R}{\partial D}\frac{dD}{db} + \frac{\partial R}{\partial b} = 0, \tag{21}$$

and implementing this rule, we have:

$$\frac{\partial R}{\partial b} = \frac{\partial dH_c/dD}{\partial b}L_c + \frac{\partial L_c}{\partial b}\frac{dH_c}{dD} - 2\frac{\partial dL_c/dD}{\partial b}H_c - 2\frac{\partial H_c}{\partial b}\frac{dL_c}{dD}.$$

Function L_c has no element containing b, thus $\frac{\partial L_c}{\partial b} = \frac{\partial dL_c/dD}{\partial b} = 0$, and based on (4b):

$$\begin{aligned}\frac{\partial H_c}{\partial b} = &\int_0^m \int_0^x \left((D-x)/x^2\right)g(y)dyf(x)dx + \int_m^D \int_0^m \left((D-x)/x^2\right)g(y)dyf(x)dx\\ &+ \int_m^D \int_m^x \left[(D-x\ +\ y-m)/x^2\right]g(y)dyf(x)dx\\ &+ \int_D^K \int_{xm/D}^{x-D+m} \left[(D-m)z^2\right]g(y)dyf(x)dx\\ &+ \int_D^K \int_{x-D+m}^{x} \left[(D-x+y-m)/x^2\right]g(y)dyf(x)dx,\end{aligned}$$

which is a positive expression. We know that $\frac{dL_c}{dD}$ is negative and if $\frac{\partial dH_c/dD}{\partial b}$ is positive, then $\frac{\partial R}{\partial b}$ is positive as well. We also know that R changes its sign at D_0 from negative to positive, being minimum point. Thus R is increasing, consequently $\frac{\partial R}{\partial D}$ is positive. Finally, (21) is valid if $\frac{dD}{db}$ is negative, which means that the minimum point D_0 decreases in b if $\frac{\partial dH_c/dD}{\partial b} > 0$.

The reason we require the $\frac{\partial dH_c/dD}{\partial b} > 0$ assumption is explained by the following analysis. Let us suppose $m = 0$, i.e. because of some reasons the clinic area capacity has dropped to zero. Using (15), it can be written that:

$$\begin{aligned}\frac{\partial dH_c/dD}{\partial b} = &\int_0^D \int_0^x \frac{1}{x^2}g(y)dyf(x)dx\\ &+ \int_D^K \left\{\int_0^{x-D} \frac{1}{D^2}\left[-\left(\frac{y}{x}\right)^2\right]g(y)dy - \frac{1}{D}g(x-D)\left(\frac{x-D}{x}\right)^2\right\}f(x)dx\\ &+ \int_D^K \int_{x-D}^x \frac{1}{x^2}g(y)dyf(x)dx.\end{aligned} \tag{22}$$

Then, $\lim\limits_{D\to 0} \frac{\partial dH_c/dD}{\partial b} = \lim\limits_{D\to 0} \frac{1}{D}\int_0^K\{\int_0^x \frac{1}{D}[-(\frac{y}{x})^2]g(y)dy - g(x)\}f(x)dx \to -\infty$.

Proposition 3. *When the expected annual total cost function $ETC(Q_{opt})$ has local minimum at D_0, this minimum point is decreasing in backlogging cost (b) if there is no probability that $\eta > m$.*

When there is no probability that $\eta > m$, L_c' and H_c' are given by (16a) and (b), and based on (16a):

$$\frac{\partial dH_c/dD}{\partial b} = \int_m^D \left(1/x^2\right)f(x)dx,$$

which is a positive term. Considering the proof of Proposition 2, this is the only additional thing we have to see.

Propositions 2 and 3 prove an important managerial point: implementing practices that decrease backlogging cots, for example making the overtime shifts more efficient, encourages sales and marketing departments to contract for larger volumes. The extant is given by the algorithm.

5. Conclusions

One of the key components of Just-in-Time manufacturing is the capability of developing short setup time technics that must be low enough to facilitate the lowest possible lot size, which is sometimes one. When we want to see the optimal lot size levels, we have to take into account other tools of the Toyota Production System as well, probably the two most immediate ones, the principles of Heijunka and Jidoka. Heijunka is coupled to complete smoothing and connecting supplies to customer demand, while Jidoka is the way of signaling process quality problems and to solve them instantaneously. As the occurrence of problems (deviations and variability) is natural, we consider both the output of the assembly line and the frequency of violating the Jidoka principle (cars with quality problem are tagged and go through the assembly line arriving at the clinic area and waiting for repairing) random variables.

These two random factors result in new lot sizing rules and modify the traditional formulas. As an outcome of the random factors, four different inventory build-up diagrams develop and the contribution of the different inventory positions to the total cost is influenced by the types of the distribution function. It is reasonable that the daily production plans are reconciled with the daily output, however when the daily output randomly varies management is frequently in trouble when production plans are not met. Thus we have focused on the factors that shape the shortages and overstocking. The typical managerial question occurs: whether sales plan is optimal or not. To answer the question we have analyzed the nature of the expected annual total cost as function of demand. To facilitate smooth production, in JIT environment production plans are frozen quite before the actual production starts, and under relatively long periods, the planned daily production volumes are constant while assembly yield may vary day by day. We revealed that the total cost function may have minimum point in demand (the daily production plan originated form the volume contracted) and we gave a procedure to determine the optimal volume to contract.

We found that the minimum point of the total cost function decreases when backlogging costs increase, but not trivially. The analysis of the case emphasizes the importance of the inclusion of the clinic area into the discussion not only because in JIT systems clinic area exists, but the last two propositions indicate that core problems may behave differently.

We hope this interpretation of the JIT operations system may attract other researchers and unveil many new unsolved questions. The identification of the distribution functions describing process quality is also a challenging area and would contribute significantly to our knowledge, but probably most manufacturers will consider it as business sensitive case – quite understandably.

The present scientific contribution is dedicated to the 650th anniversary of the foundation of the University of Pécs, Hungary.

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