Mathematical Model

Sets

L: Set of working days, excluding holidays. For example, if August 2024 has only 22 valid working days after removing holidays, then $L = \{0, 1, 2, \ldots, 20, 21\}$.

K: Set of orders.

P: Set of printing shafts (bearing).

I: Set of order sides, where $I = \{0, 1\}$, with 0 representing the A-side and 1 representing the B-side.

Parameters

q_k: Number of products to be processed in order k, where $k \in K$.

v_k: Revenue obtained if order k is delivered on time, where $k \in K$.

u_k: Due date (delivery deadline) of order k, where $k \in K$.

r: Penalty rate for late delivery. A delay of one week incurs a penalty of 1% of the order's revenue.

a: Duration of Stage 1 processing. Typically, Stage 1 is assumed to require 8 days.

b: Daily processing capacity per shaft (bearing).

M: A sufficiently large constant used for model linearization.

z_l: The actual calendar date corresponding to the l-th day in the planning horizon. Since holidays are excluded, for August 2024, the planning day 0 corresponds to August 1, day 1 to August 2, and day 2 to August 5, and so on.

Decision Variables

x_{p0}^{lki}: Binary variable, equal to 1 if the first-stage processing of side i in order k starts on shaft (bearing) p on day l; 0 otherwise. $l \in L, k \in K, i \in I, p \in P$.

x_{p1}^{lki}: Binary variable, equal to 1 if the second-stage processing of side i in order k starts on shaft (bearing) p on day l; 0 otherwise. Since the second-stage small-batch production can begin simultaneously on multiple machines, its exact start time is not uniquely defined. $l \in L, k \in K, i \in I, p \in P$.

y_{p0}^{lki}: Binary variable, equal to 1 if the first-stage processing of side i in order k is being performed on shaft (bearing) p on day l; 0 otherwise. $l \in L, k \in K, i \in I, p \in P$.

y_{p1}^{lki}: Binary variable, equal to 1 if the second-stage processing of side i in order k is being performed on shaft (bearing) p on day l; 0 otherwise. $l \in L, k \in K, i \in I, p \in P$.

t_{ki}: Number of delayed days for side i of order k. $k \in K, i \in I$.

w_{ki}: Number of delayed weeks for side i of order k. $k \in K, i \in I$.

c_k: Number of delayed weeks for order k. $k \in K$

Model

$$\min \sum_{k \in K} v_k * r * c_k \tag{1}$$

Objective function (1): Minimize the total penalty cost incurred from late deliveries. The objective is to minimize the total penalty cost associated with delayed orders, which is proportional to the order revenue v_k, the penalty rate r, and the number of delayed weeks c_k.

s.t.

$$\sum_{k\in K}\sum_{i\in I} x_{p0}^{lki} \leq 1, \qquad \forall\, l \in L,\, p \in P \tag{2}$$

Constraints (2) ensure that at any given day, each shaft (bearing) on a machine can start at most one first-stage processing task.

$$\sum_{k\in K}\sum_{i\in I} \left(y_{p0}^{lki} + y_{p1}^{lki}\right) \leq 1, \qquad \forall\, l \in L,\, p \in P \tag{3}$$

Constraints (3) ensure that at any given day, each shaft can process at most one order, either in the first-stage or second-stage processing.

$$\sum_{l\in L}\sum_{p\in P} x_{p0}^{lki} = 1, \qquad \forall\, k \in K,\, i \in I \tag{4}$$

Constraints (4) ensure that each order side can start its first-stage processing on only one shaft (bearing) of one machine.

$$M(1 - x_{p0}^{lki}) + a \geq \sum_{l\leq l'\leq l+a} y_{p0}^{l'ki}, \qquad \forall\, l = 0, \ldots, |L| - a,\, k \in K,\, i \in I,\, p \in P \tag{5}$$

$$\sum_{l\leq l'\leq l+a} y_{p0}^{l'ki} \geq -M(1 - x_{p0}^{lki}) + a, \qquad \forall\, l = 0, \ldots, |L| - a,\, k \in K,\, i \in I,\, p \in P \tag{6}$$

Constraints (5) and (6) ensure that the first-stage processing duration for each order side must equal the predefined processing time a (e.g., 8 days). The following constraints ensure that once processing begins, it continues for a consecutive working days.

$$\sum_{l\in L} x_{p0}^{lki} * a \geq \sum_{l\in L} y_{p0}^{lki}, \qquad \forall\, k \in K,\, i \in I,\, p \in P \tag{7}$$

Constraints (7) ensure that, for each order side, grinding on a given shaft can occur only if a corresponding start time has been assigned.

$$M(1 - y_{p1}^{lki}) + l * y_{p1}^{lki} \geq \sum_{l\in L}\sum_{p\in P} x_{p0}^{lki} * (l + a), \qquad \forall\, k \in K,\, i \in I,\, l \in L,\, p \in P \tag{8}$$

Constraints (8) ensure that the second-stage process for each order side must begin only after the completion of its first-stage process.

$$\sum_{l\in L}\sum_{p\in P} y_{p1}^{lki} = \left\lceil \frac{q_k}{b} \right\rceil, \qquad \forall\, k \in K,\, i \in I \tag{9}$$

Constraints (9) ensure that the total processing time in the second stage must meet the required production quantity for each order side.

$$\sum_{k\in K}\sum_{i\in I}\sum_{p\in P} \left(y_{p0}^{lki} + y_{p1}^{lki}\right) \leq 50, \qquad \forall\, l \in L \tag{10}$$

Constraints (10) ensure that, on any given day, the total number of operating shafts (bearings) cannot exceed 100. Since each shaft group consists of two shafts, at most 50 shaft groups may be active simultaneously.

$$z_l * y_{p1}^{lki} - u_k \leq t_{ki}, \qquad \forall\, l \in L,\, k \in K,\, i \in I,\, p \in P \tag{11}$$

Constraints (11) ensure that the delivery delay (in days) for each order side is computed based on the actual completion date of the second-stage grinding and the order due date.

$$7 * (w_{ki} - 1) < t_{ki} \leq 7 * w_{ki}, \qquad \forall\, k \in K,\, i \in I \tag{12}$$

Constraints (12) ensure that the number of delayed weeks for each order side is determined from its delivery delay (in days).

$$w_{ki} \leq c_k, \qquad \forall\, k \in K,\, i \in I \tag{13}$$

Constraints (13) ensure that the total number of delayed weeks for an order must be no less than the delay of any individual side.