Approximation Algorithms

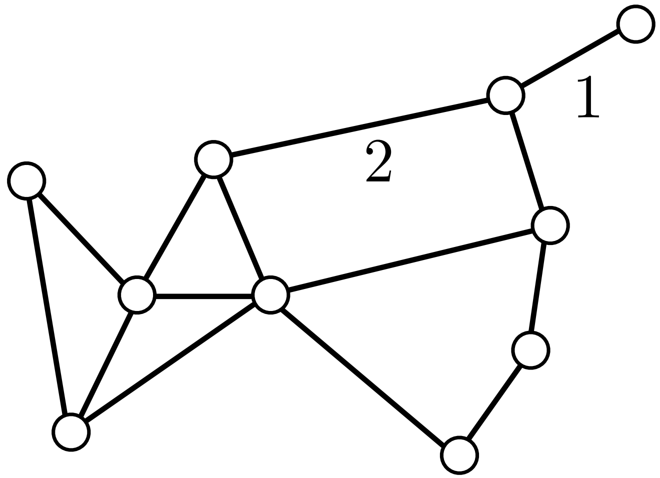
Lecture 4

11/02/10

k-Center

Problem: Metric k-Center.

Let $G=(V,E)$ be a complete graph with edge costs $c : E \to \mathbb{Q}^+$ satisfying the triangle inequality, k a positive integer. For any set $S \subseteq V$ and vertex $v \in V$ let $c(v,S)$ be the cost of the cheapest edge from v to a vertex in S. The k-Center Problem asks for a set $S \subseteq V$, $|S| = k$, minimizing $\max_{v \in V} c(v, S)$.



$k = 3$

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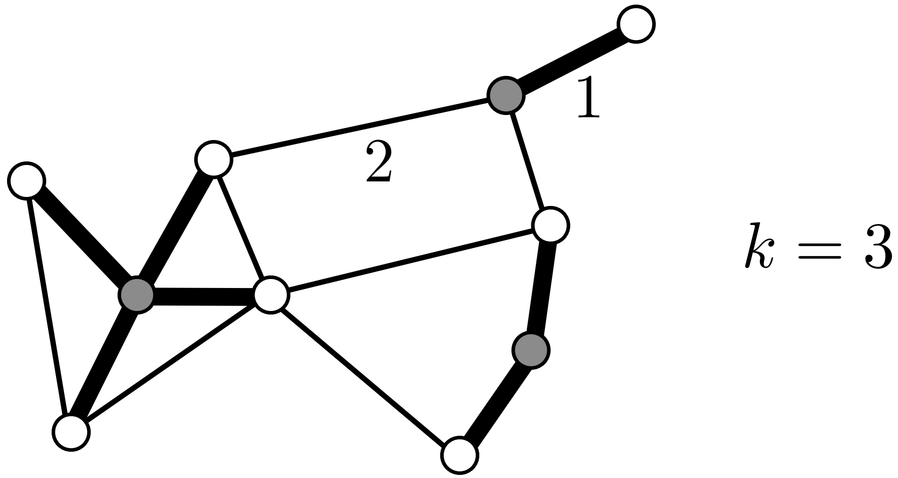


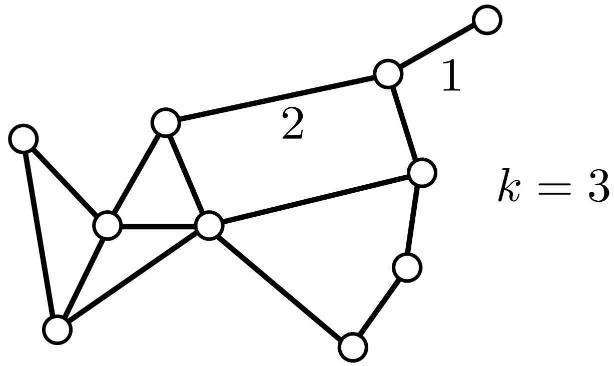
k-Center

Let $E = \{e_1, \ldots, e_m\}$, such that $c(e_1) \leq c(e_2) \leq \cdots \leq c(e_m)$.

Define $E_i = \{e_1 \ldots, e_i\}$ and *pruned* graphs $G_i = (V, E_i)$.

A *dominating set* in an undirected graph $G=(V,E)$ is a subset of vertices $S \subseteq V$, such that every vertex in $V \backslash S$ is adjacent to a vertex in S. Let $\mathrm{dom}(G)$ denote the size of the smallest dominating set in G.

The k-Center Problem is equivalent to finding the smallest index i^*, such that G_{i^*} has a dominating set of size k.



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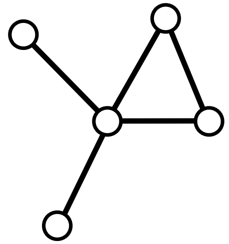

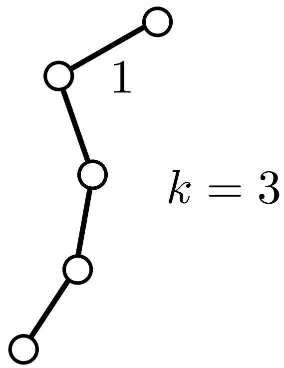

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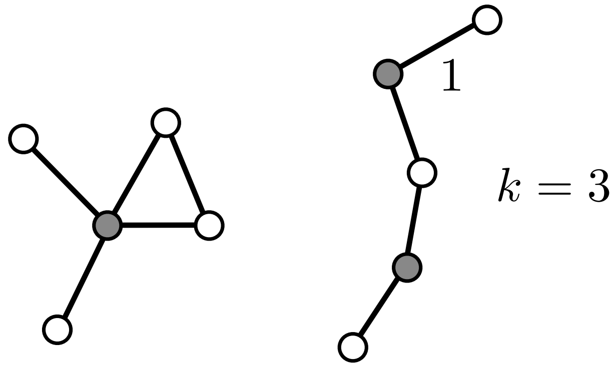


k-Center

For a graph $G=(V,E)$ define $G^2 = (V, E^2)$, where

$$E^2 = \Big\{ \{u, v\} \,\Big|\, \exists \text{ a path of length} \le 2 \text{ between } u \ne v \text{ in } G \Big\}.$$

The cost of an edge in G^2 is equal to the length of the path in G.

Lemma 4

Given a graph G, let I be an independent set in G^2. Then

$$|I| \le \mathrm{dom}(G).$$

Algorithm 7: Metric k-Center.

1. Construct $G_1^2, G_2^2, \ldots, G_m^2$.
2. Compute a maximal independent set, M_i, in each G_i^2.
3. Find the smallest index i, such that $|M_i| \le k$, say j.
4. Return M_j .

Minimum k-Cut

Lemma 5

For j as defined in the k-Center algorithm, $c(e_j) \leq \mathrm{opt}.$

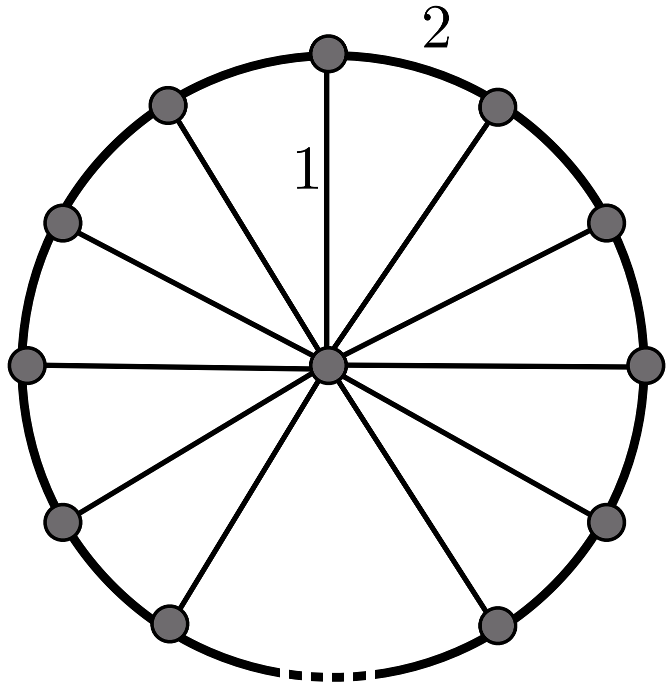
Theorem 7

Algorithm 7 is a factor 2 approximation algorithm for the metric k-Center Problem.

By Lemma 5, a dominating set of size k in G_j would correspond to an optimal k-center in G. What's left to prove is that an independent set in G_j^2 gives "almost" as good a k-center in G.

Multiway Cut

A tight example:



Theorem 8

There is no poly-time factor $2 - \varepsilon$ approximation algorithm for the metric k-Center Problem for any $\varepsilon > 0$, unless $\mathrm{P{=}NP}$.

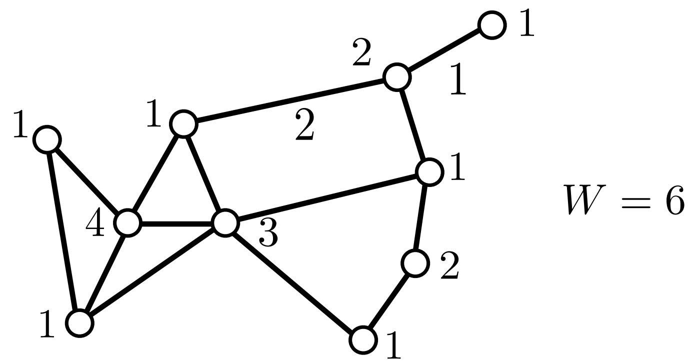
Weighted k-Center

Problem: Metric Weighted k-Center.

Let $G=(V,E)$ be a complete graph with edge costs $c : E \to \mathbb{Q}^+$ satisfying the triangle inequality, $w : V \to \mathbb{Q}^+$ a weight function on vertices and $W \in \mathbb{Q}^+$ a weight bound. The Weighted k-Center Problem asks for a set $S \subseteq V$,

$$\sum_{v \in S} w(v) \leq W,$$

minimizing $\max_{v \in V} c(v, S)$.



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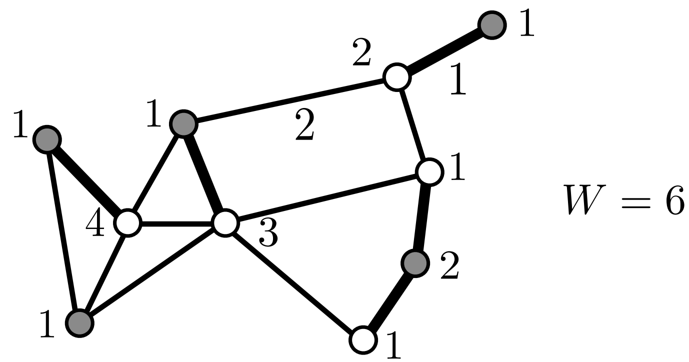


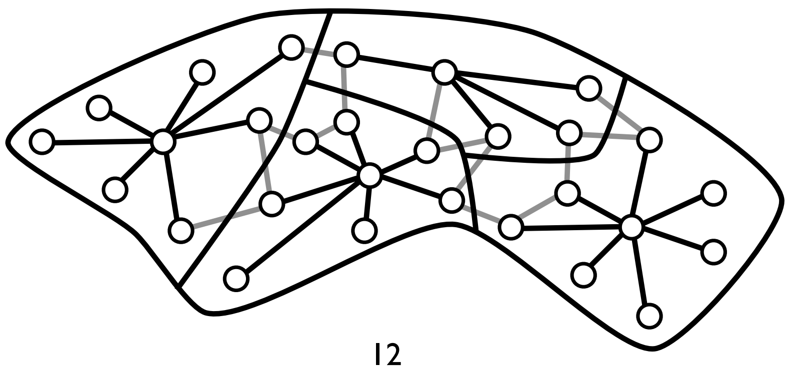
Weighted k-Center

For a vertex weighted graph $G=(V,E)$ we let $\mathrm{wdom}(G)$ denote the weight of a minimum weight dominating set in G.

Lemma 6

Given a vertex weighted graph G, let I be an independent set in G^2. For each $v \in I$, let $s(v)$ denote a lightest neighbor of v in G (including v), and define $S = \{s(v) \mid v \in I\}$. Then

$$w(S) \leq \mathrm{wdom}(G).$$



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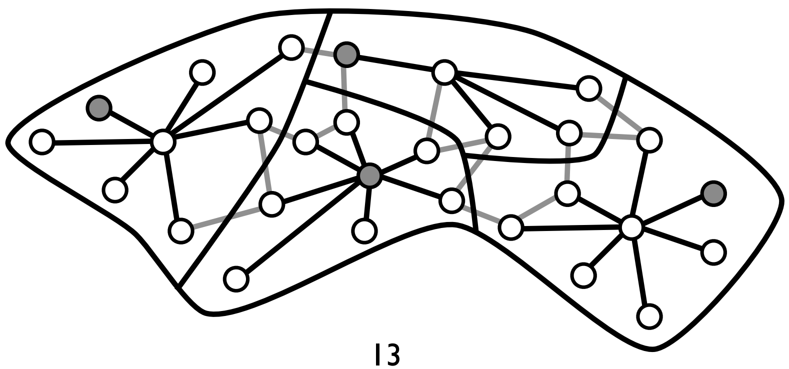


Weighted k-Center

Let $E = \{e_1, \ldots, e_m\}$, such that $c(e_1) \leq c(e_2) \leq \cdots \leq c(e_m)$. Define $G_i = (V, E_i)$ as in the unweighted case and let $s_i(v)$ denote a lightest neighbor of $v \in V$ (including itself) in G_i.

Algorithm 8: Metric Weighted k-Center.

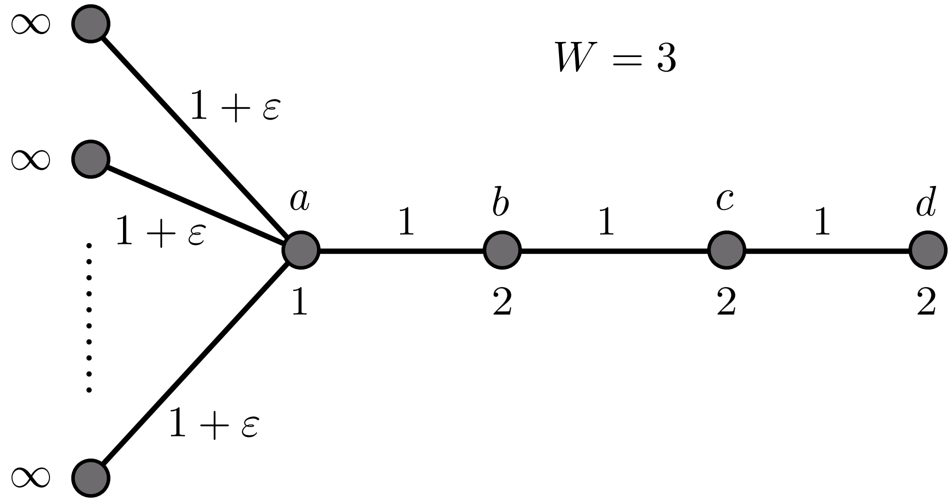
1. Construct $G_1^2, G_2^2, \ldots, G_m^2$.
2. Compute a maximal independent set, M_i, in each G_i^2.
3. Compute $S_i = \{s_i(v) \mid v \in M_i\}$.
4. Find the smallest index i, such that $w(S_i) \leq W$, say j.
5. Return S_j.

Theorem 9

Algorithm 8 is a factor *3* approximation algorithm for the Metric Weighted k-Center Problem.

Multiway Cut

A tight example:



The k-center $\{a,c\}$ has cost $1+\varepsilon$.

We have $w(S_i) = \infty$ for $i < n+3$.

In G^2_{n+3}, Algorithm 8 might pick $M_{n+3} = \{b\}$, thus $S_{n+3} = \{a\}$.