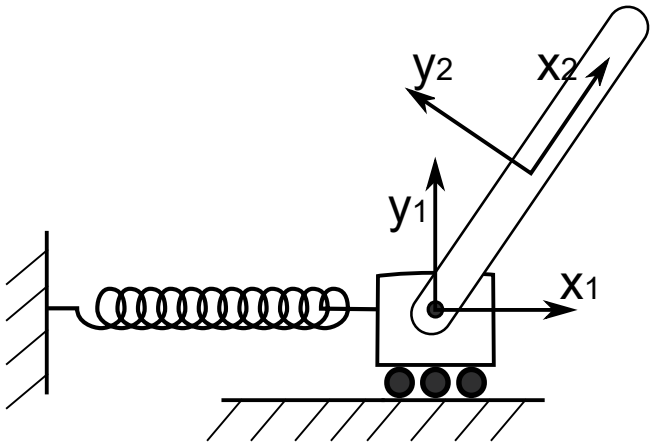
Homework #8

Problem 1



For the system shown above, block one slides along the x-axis and is attached via a pin to a link, body 2. The system has the following characteristics,

- $m^1 = 0.1\ kg$
- $k = 40\ N/m$
- $F_{spring} = -k(R_x^1 - 0.2)$
- $l^2 = 1\ m$
- $m^2 = 1\ kg$

Use the **embedding technique** to solve for the dynamic response for the system. Separate the generalized coordinates are separated into *dependent* and *independent* coordinates as such

$\mathbf{q} = [\mathbf{q}_d,\ \mathbf{q}_i]$.

- $\mathbf{q}_d = [R_y^1,\ \theta^1,\ R_x^2,\ R_y^2]$
- $\mathbf{q}_i = [R_x^1,\ \theta^2]$

This distinction creates two second order differential equations, where $n = 3 \times (\#\ bodies)$ and $n_c = number\ of\ constraints$. The equations of motion are as such

$$\mathbf{B^T M B} + \mathbf{B^T M} \gamma - \mathbf{B^T Q_e} = \mathbf{0}$$

where,

- $\mathbf{B} = \begin{bmatrix} -\mathbf{C}_{\mathbf{q_d}}^{-\mathbf{1}}\mathbf{C}_{\mathbf{q_i}} \\ \bar{I} \end{bmatrix}$
- $\gamma = \left[\begin{array}{c} ~\mathbf{-C_{q_d}^{-1}[(C_q\dot{q})\dot{q}+2C_{qt}\dot{q}+C_{tt}]} \\ \mathbf{0}\end{array} \right] = \left[\begin{array}{c} \mathbf{C}_{\mathbf{q_d}}^{-\mathbf{1}}[\mathbf{Q_d}] \\ \mathbf{0} \end{array} \right]$

- $\mathbf{C}_{\mathbf{q_d}}\ and\ \mathbf{C}_{\mathbf{q_i}}$ are the Jacobian of constraints for *dependent* and *independent* coordinates, respectively

You will define the following functions:

- `C_sys` the 4 constraint equations for the system
- `Cq_sys` the Jacobian of the system `d C_sys/dq`
- `Bi_link` the array $\mathbf{B}$ that transforms $\delta\mathbf{q_i}$ to $\delta\mathbf{q}$
- `eom_sys` the final equation of motion using the embedding technique

Define $\mathbf{A}$ and $\mathbf{A}_\theta$

Here, you create two functions to rotate from the body coordinate system at angle θ to the global coordinate system. The derivate, $\mathbf{A}_\theta$ will be used in the Jacobian and equations of motion.

In [ ]:
```
import numpy as np
import matplotlib.pyplot as plt
import sympy
```

In [ ]:
```python
def rotA(theta):
    '''This function returns a 2x2 rotation matrix to convert the
    rotated coordinate to the global coordinate system
    input is angle in radians

    Parameters
    ----------
    theta : angle in radians

    Returns
    -------
    A : 2x2 array to rotate a coordinate system at angle theta to global x-y
    '''
    A=np.zeros((2,2))
    A=np.array([[np.cos(theta), -np.sin(theta)],
                [np.sin(theta), np.cos(theta)]])

    return A
```

In [ ]:
```python
def A_theta(theta):
    '''This function returns a 2x2 rotation matrix derivative
    input is angle in radians

    Parameters
    ----------
    theta : angle in radians

    Returns
    -------
    dAda : 2x2 array derivative of `rotA`
    '''
    dAda=np.array([[-np.sin(theta), -np.cos(theta)],
                   [np.cos(theta), -np.sin(theta)]])
    return dAda
```

Define Jacobian of pin constraint as `Cq_pin` .

In [ ]:
```
def Cq_pin(qi, qj, ui, uj):
    '''Jacobian of a pinned constraint for planar motion

    Parameters
    ----------
    qi : generalized coordinates of the first body, i [Rxi, Ryi, thetai]
    qj : generalized coordinates of the 2nd body, i [Rxj, Ryj, thetaj]
    ui : position of the pin the body-i coordinate system
    uj : position of the pin the body-j coordinate system

    Returns
    -------
    Cq_pin : 2 rows x 6 columns Jacobian of pin constraint Cpin
    '''

    Cq_1=np.block([np.eye(2), A_theta(qi[2])@ui[:,np.newaxis] ])
    Cq_2=np.block([-np.eye(2), -A_theta(qj[2])@uj[:,np.newaxis] ])
    Cq_pin=np.block([Cq_1, Cq_2])
    return Cq_pin
```

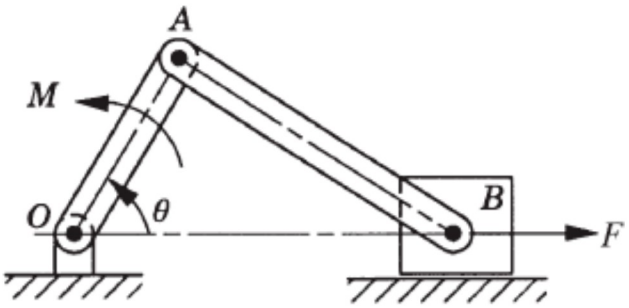
In [ ]:
```
m1 = 0.1
k = 40
l2 = 1
m2 = 1
```

In [ ]:
```
def C_sys(q,t):
    C= np.zeros(4)
    C[0] = q[0]- q[3] +l2/2*np.cos(q[5])
    C[1] = q[1] - q[4] + l2/2*np.sin(q[5])
    C[2] = q[1]
    C[3] = q[2]

    return C
```

In [ ]:
```
def Cq_sys(q,t):
    Cq = np.zeros([4,6])

    Cq[0:2,0:2] = np.eye(2)
    Cq[2:4,1:3] = np.eye(2)
    Cq[0:2,3:5] = -np.eye(2)
    Cq[0,-1] = -l2/2*np.sin(q[5])
    Cq[1,-1] = l2/2*np.cos(q[5])

    return Cq
```

In [ ]:
```
def Bi_link(Cq_sys):
    Cqd = Cq_sys[:,5]
    Cqi = Cq_sys[:,-1]
    B = np.zeros([6,2])
    B[:4,:] = np.linalg.solve(Cqd,Cqi)

    return B, Cqd,Cqi
```

In [ ]:
```
def eom_sys(q,dq,Q_e,t):
    C = C_sys(q,t)
    Cq = Cq_sys(q,t)
    B,Cqd,Cqi = Bi_link(Cq)

    Qd = np.zeros(4)
    Qd[0]= l2/2*dq[5]**2*np.cos(q[5])
    Qd[1]= l2/2*dq[5]**2*np.sin(q[5])

    gamma = np.array([np.linalg.inv(Cqd)@Qd,0])
    M = np.array([[m1,0],[0,m2]])
    eom = B.T@M@B - B.T@M@gamma - B.T@Q_e
    return eom
```



$m^2 = 0.5$ kg, $m^3 = 1$ kg, $m^4 = 0.2$ kg

$l^2 = 0.2$ m, $l^3 = 0.4$ m

Problem 2

Create the system of equations using the **augmented technique**. Solve these equations numerically using `solve_ivp` and plot the position and orientation of the links for one revolution of the crankshaft. Assume that M^2 = 10N-m, F^4 = 15N, θ^2 = 45 ^{o}, and $\dot{\theta}^2$=150 rad/s.

The **augmented technique** is a generalized method to solve for the dynamic response in a system of moving parts. The system of *differential algebraic equations* (DAE) are solve for all generalized coordinates and constraint forces in a system of equations,

$$\begin{bmatrix} \mathbf{M} & \mathbf{C_q}^T \\ \mathbf{C_q} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{q}} \\ \lambda \end{bmatrix} = \begin{bmatrix} \mathbf{Q_e} \\ \mathbf{Q_d} \end{bmatrix}$$

- Plot the angles of the two connecting links and the position of the sliding block at point B.
- Plot the reaction forces at O, A, and B.

In [ ]:
```
from scipy.integrate import solve_ivp
```

In [ ]:
```
def Qd_slidercrank(q, dq, t):
    '''return slidercrank Qd = Cq@ddq

    Parameters
    ----------
    q : numpy array for 9 generalized coordinates for bodies 1-3 in the slider cran
        q = [q1, q2, q3]
    t : current time
    Returns
    -------
    Qd : 1D array with length 9
    '''
    l1, l2 = 0.2,0.4
    q1 = q[0:3]
    q2 = q[3:6]
    q3 = q[6:9]
    dq1 = dq[0:3]
    dq2 = dq[3:6]
    dq3 = dq[6:9]

    Qd=np.zeros(8)
    Qd[0:2] = dq1[2]**2*rotA(q1[2])@np.array([-l1/2, 0])
    Qd[2:4] = dq1[2]**2*rotA(q1[2])@np.array([l1/2, 0]) -\
              dq2[2]**2*rotA(q2[2])@np.array([-l2/2, 0])
    Qd[4:6] = dq2[2]**2*rotA(q2[2])@np.array([l2/2, 0])
    Qd[6:8] = 0
    return Qd
```

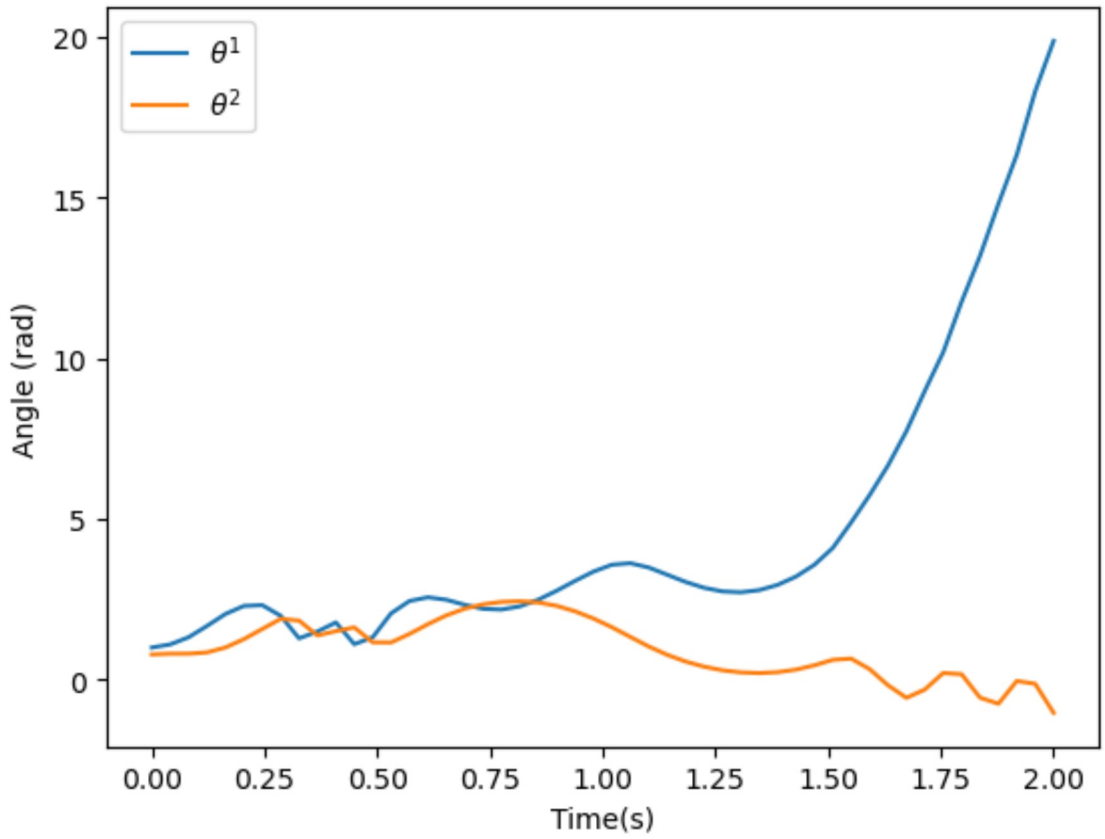
In [ ]:
```
def Cq_slidercrank(q,t):
    '''return Jacobian of C_slidercrank(q,t) = dC/dq_i
       |dC1/dR1x dC1/dR1y ... dC9/da3 |
       |dC2/dR1x dC2/dR1y ... dC9/da3 |
       |... ..     .           ...    |
       |            .                 |
       |             .                |
       |dC9/dR1x ...         dC9/da3  |
    Parameters
    ----------
    q : numpy array for 9 generalized coordinates for bodies 1-3 in the slider cran
        q = [q1, q2, q3]
    t : current time
    Returns
    -------
    Cq : 9 rows x 9 columns Jacobian of constraints `C_slidercrank`
    '''
    l1, l2 = 0.2,0.4
    q1 = q[0:3]
    q2 = q[3:6]
    q3 = q[6:9]

    Cq=np.zeros((8,9))
    Cq[0:2, 0:3] = Cq_pin(q1, np.array([0, 0, 0]),np.array([-l1/2, 0]),np.array([0,
    Cq[2:4, 0:6] = Cq_pin(q1, q2, np.array([l1/2, 0]), np.array([-l2/2, 0]))
    Cq[4:6, 3:10] = Cq_pin(q2, q3, np.array([l2/2, 0]), np.array([0, 0]))
    Cq[6:8, 7:10] = np.eye(2)
    return Cq
```

In [ ]:
```
m1 = 0.5
m2= 1
m3=0.2
torque = 10
F4 = 15
g= 9.81
theta2 = np.deg2rad(45)
theta_dot2 = 150
l1=0.2
l2=0.4
def eom_slider(t,y):
    q = y[0:9]
    dq = y[9:]
    Cq = Cq_slidercrank(q,t)
    Qd = Qd_slidercrank(q, dq,t)
    Qe = np.array([0,
                   -m1*g,
                   torque,
                   0,
                   -m2*g,
                   0,
                   F4,
                   -m3*g,
                   0])

    M = np.diag([m1,m1,m1*l1**2/l2,
                 m2,m2,m2*l2**2/l2,
                 m3,m3,m3])
    Aug = np.block([[M, Cq.T],
                    [Cq, np.zeros([8,8])]])
    rhs = np.hstack([Qe,Qd])

    ddq_lambda = np.linalg.solve(Aug,rhs)
    dy = np.zeros(18)
    dy[0:9] = dq
    dy[9:] = ddq_lambda[0:9]
    forces = ddq_lambda[9:]
    return dy,forces
```

In [ ]:
```
y = np.ones(18)
y[5] = theta2
solution = solve_ivp(lambda t,y: eom_slider(0,y)[0],[0,2],y,t_eval=np.linspace(0,2)
```

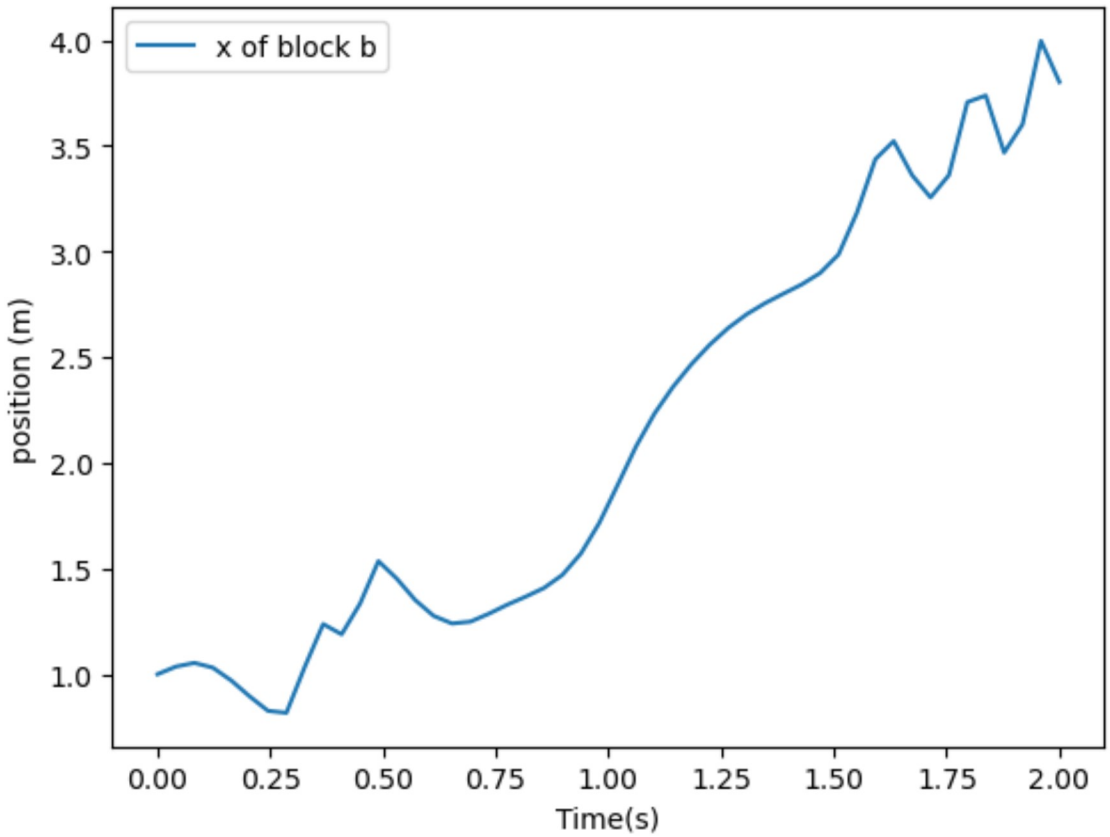
In [ ]:
```
plt.plot(solution.t,solution.y[2],label=r'$\theta^1$')
plt.plot(solution.t,solution.y[5],label=r'$\theta^2$')
plt.legend()
plt.xlabel('Time(s)')
plt.ylabel('Angle (rad)')
```

Out[ ]:
```
Text(0, 0.5, 'Angle (rad)')
```



In [ ]:
```
plt.plot(solution.t,solution.y[6],label='x of block b')
plt.xlabel('Time(s)')
plt.ylabel('position (m)')
plt.legend()
```

Out[ ]:
```
<matplotlib.legend.Legend at 0x2d2f6f164c0>
```


x of block b
position (m)
Time(s)
4.0
3.5
3.0
2.5
2.0
1.5
1.0
0.00
0.25
0.50
0.75
1.00
1.25
1.50
1.75
2.00

In [ ]: