$$\frac{d^2x}{dt^2} + b\frac{dx}{dt} + \sin x = a\cos wt$$

$$|x| \ll 1 \Rightarrow \sin x \simeq x \Rightarrow x_{tt} + bx_t + x = a\cos wt \qquad x(t) = x_h(t) + x_p(t)$$

Homogeneous solution: $x_{tt} + bx_t + x = 0$ Let $x = e^{rt}$

$$r^2e^{rt} + bre^{rt} + e^{rt} = 0$$

$$r^2 + br + 1 = 0$$

$$r = \frac{-b \pm \sqrt{b^2-4}}{2}$$

Overdamped: Assume $b^2 > 4 \Rightarrow x_h(t) = C_1e^{r_1t} + C_2e^{r_2t}$

Critically Damped: Assume $b^2 = 4 \Rightarrow r = -\frac{b}{2} = \pm 1 \qquad x_h(t) = (C_1 + C_2t)e^{-\frac{b}{2}t}$

Underdamped: Assume $b^2 < 4 \Rightarrow r = \underset{\alpha}{\underset{\shortparallel}{-\frac{b}{2}}} \pm \underset{\beta}{\underset{\shortparallel}{\frac{i\sqrt{4-b^2}}{2}}} \qquad x_h(t) = C_1\cos\beta t e^{\alpha t} + C_2\sin\beta t e^{\alpha t}$

Particular Solution: Assume form $x_p(t) = C_3\cos wt + C_4\sin wt$

$$x_p'(t) = -C_3w\sin wt + C_4w\cos wt \qquad x_p''(t) = -C_3w^2\cos wt - C_4w^2\sin wt$$

Sub into original equation:

$$-(C_3w^2\cos wt + C_4w^2\sin wt) - b(C_3w\sin wt - C_4w\cos wt) + (C_3\cos wt + C_4\sin wt) = a\cos wt$$

$$\cos wt(-C_3w^2 + C_4bw + C_3) + \sin wt(-C_4w^2 - C_3bw + C_4) = a\cos wt$$

$\cos wt = 0$: $-C_4w^2 - C_3bw + C_4 = 0$

$$C_4(1-w^2) = C_3bw$$

$$C_4 = \frac{C_3bw}{1-w^2} = \frac{abw}{1-w^2+\frac{b^2w^2}{1-w^2}} \times \frac{\frac{1-w^2}{1}}{}$$

$$C_4 = \frac{abw}{1+\frac{b^2w^2}{(1+w^2)^2}}$$

$\sin wt = 0$: $-C_3w^2 + C_4bw + C_3 = a$

$$C_3(1-w^2) + \frac{C_3b^2w^2}{1-w^2} = a$$

$$C_3\left(1-w^2+\frac{b^2w^2}{1-w^2}\right) = a$$

$$C_3 = \frac{a}{1-w^2+\frac{b^2w^2}{1-w^2}}$$

$$\Rightarrow x_p(t) = \frac{a\cos wt}{1-w^2+\frac{b^2w^2}{1-w^2}} + \frac{abw\sin wt}{1+\frac{b^2w^2}{(1+w^2)^2}}$$

$$\therefore x(t) = x_h(t) + \frac{a\cos wt}{1-w^2+\frac{b^2w^2}{1-w^2}} + \frac{abw\sin wt}{1+\frac{b^2w^2}{(1+w^2)^2}}$$

Assume initial conditions $x(t=0)=x_0$, $x'(t=0)=0$

$$x_h(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}$$

$$r = \frac{-b}{2} = \pm 1 \qquad x_h(t) = (C_1 + C_2 t) e^{-\frac{b}{2}t}$$

$$r = \underset{\alpha}{\underset{\shortparallel}{\frac{-b}{2}}} \pm \underset{\beta}{\underset{\shortparallel}{\frac{i\sqrt{4-b^2}}{2}}} \qquad x_h(t) = C_1 \cos\beta t\, e^{\alpha t} + C_2 \sin\beta t\, e^{\alpha t}$$

$$C_1 = x_0 \qquad x_h'(0) = C_2 - \frac{b}{2} C_1 = 0$$

$$C_2 = \frac{b}{2} x_0$$