Solution of Moed C in Linear Algebra 1 2022

David Ginzburg

1) Problem: Let $P_n(x)$ denote the vector space of all polynomials with coefficients in $\mathbf{C}$ whose degree is at most n. For $p(x) \in P_n(x)$, let $T : P_n(x) \to P_n(x)$ denote the linear map defined by $T(p(x)) = p(x) + p'(x)$. Prove that T is an isomorphism.

In addition, suppose that $n = 2$. Given $p(x) \in P_2(x)$, give an explicit formula for the linear map $T^{-1}(p(x))$.

Solution: Let $p(x) = a_0 + a_1x + \cdots + a_nx^n$. Then,

$$T(p(x)) = p(x) + p'(x) = (a_0 + a_1x + \cdots + a_nx^n) + (a_1 + 2a_2x + 3a_3x^2 + \cdots + na_nx^{n-1}) =$$

$$= (a_0 + a_1) + (a_1 + 2a_2)x + (a_2 + 3a_3)x^2 + \cdots + (a_{n-1} + na_n)x^{n-1} + a_nx^n$$

To prove that T is an isomorphism it is enough to prove that $\ker T = \{0\}$. Hence, if $T(p(x)) = 0$, we get from the above the following system of equations,

$$a_0 + a_1 = 0;\ \ a_1 + 2a_2 = 0;\ \ a_2 + 3a_3 = 0; a_{n-1} + na_n = 0;\ \ a_n = 0$$

In other words, the system of equations is given by $a_{i-1} + ia_i = 0;\ \ a_n = 0$. Here, $1 \le i \le n$. It is easy to prove by induction, that this system has only the trivial solution.

Assume that $n = 2$. To give an explicit formula for T^{-1}, it is enough to compute $T^{-1}(1)$; $T^{-1}(x)$ and $T^{-1}(x^2)$. Assume that $T^{-1}(1) = b_0 + b_1x + b_2x^2$. Then, $1 = b_0T(1) + b_1T(x) + b_2T(x^2)$. This is the same as $1 = b_0 + b_1(x+1) + b_2(x^2+2x)$. Comparing coefficients we get $b_0 + b_1 = 1$; $b_1 + 2b_2 = 0$ and $b_2 = 0$. The solution is $b_0 = 1$; $b_1 = b_2 = 0$. Hence $T^{-1}(1) = 1$. To compute $T^{-1}(x)$, we need to solve the equation $x = b_0 + b_1(x+1) + b_2(x^2+2x)$. This gives us $b_0 = -1$; $b_1 = 1$, and $b_2 = 0$. Hence $T^{-1}(x) = -1 + x$. Similarly, the equation $x^2 = b_0 + b_1(x+1) + b_2(x^2+2x)$ implies $T^{-1}(x^2) = 2 - 2x + x^2$. Thus, if $p(x) = a + bx + cx^2$, then

$$T^{-1}(p(x)) = T^{-1}(a + bx + cx^2) = aT^{-1}(1) + bT^{-1}(x) + cT^{-1}(x^2) =$$

$$= a + b(-1 + x) + c(2 - 2x + x^2) = (a - b + 2c) + (b - 2c)x + cx^2$$

Remark: Its not easy to see it, but in general, for all n, we have

$$T^{-1}(p(x)) = p(x) - p^{(1)}(x) + p^{(2)}(x) - p^{(3)}(x) + \cdots + (-1)^n p^{(n)}(x)$$

Here $p^{(i)}(x)$ is the i-th derivative of $p(x)$.

2) Problem: Let V denote a vector space defined over the field $\mathbf{C}$. Let $T, S : V \to V$ denote two linear maps. Assume that

$$5T^4 - 2T^2 + 3TS + T - I = 0 \tag{1}$$

Prove that $TS = ST$.

Solution: Write the identity (1) as $5T^4 - 2T^2 + 3TS + T = I$. This implies that $T(5T^3 - 2T + 3S + I) = I$. Hence T is invertable and we have $(5T^3 - 2T + 3S + I)T = I$. This last identity follows from the fact that if A and B are two square matrices such that $AB = I$, then $BA = I$. It is also true for linear maps.

The equation $(5T^3 - 2T + 3S + I)T = I$ implies

$$5T^4 - 2T^2 + 3ST + T = I \tag{2}$$

Subtracting equation (1) from equation (2), we get $TS = ST$.

3) Problem: Let A denote the 5×5 such that every number between one and five appears exactly once in each row and each column. Prove that 75 divides the determinant of A.

Remark: An example of such a matrix is

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 1 & 2 & 3 & 4 \\ 4 & 5 & 1 & 2 & 3 \\ 3 & 4 & 5 & 1 & 2 \\ 2 & 3 & 4 & 5 & 1 \end{pmatrix}$$

Solution: For $2 \leq i \leq 5$ perform the row operation $R_1 \to R_1 + R_i$ on A. Then, we obtain a matrix B such that all entries in first row are $1 + 2 + 3 + 4 + 5 = 15$. For example, for the above matrix A we get

$$B = \begin{pmatrix} 15 & 15 & 15 & 15 & 15 \\ 5 & 1 & 2 & 3 & 4 \\ 4 & 5 & 1 & 2 & 3 \\ 3 & 4 & 5 & 1 & 2 \\ 2 & 3 & 4 & 5 & 1 \end{pmatrix}$$

From the properties of determinants, we have $|A| = |B| = 15|C|$, where all entries of the first row of C are ones, and all other entries are as in the matrix B. In the above example, we have

$$C = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 5 & 1 & 2 & 3 & 4 \\ 4 & 5 & 1 & 2 & 3 \\ 3 & 4 & 5 & 1 & 2 \\ 2 & 3 & 4 & 5 & 1 \end{pmatrix}$$

Next, on C perform the column operation $C_1 \to C_1 + C_j$ for $2 \leq j \leq 5$. Then, we obtain a matrix D such that all entries of the first column are 15, except $d_{1,1} = 5$. In the above example, we have

$$D = \begin{pmatrix} 5 & 1 & 1 & 1 & 1 \\ 15 & 1 & 2 & 3 & 4 \\ 15 & 5 & 1 & 2 & 3 \\ 15 & 4 & 5 & 1 & 2 \\ 15 & 3 & 4 & 5 & 1 \end{pmatrix}$$

We have $|A| = 15|C| = 15|D|$. Clearly, five divides $|D|$. Hence, 75 divides $|A|$.

Remark: Many students wrote that it is "obvious" that using rows and columns permutations we can obtain any such matrix. Hence, if we start with such a matrix, one can change rows and columns and obtain the matrix A as described above. Since row and column changes, changes only the sign of the determinant, and hence it is enough to prove that the determinant of the above matrix A is divisible by 75.

Not only its not obvious that this argument is true, in fact in general it is not true. Consider the two 4×4 matrices

$$A_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 4 & 1 & 2 \\ 4 & 3 & 2 & 1 \end{pmatrix} \qquad A_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \\ 3 & 4 & 1 & 2 \\ 2 & 3 & 4 & 1 \end{pmatrix}$$

Then $|A_1| = 0$ and $|A_2| = -160$.

4) Problem: In this problem V is a finitely generated vector space over the field $\mathbf{C}$.
a) (**12 points**) Let $T, S : V \to V$ be two linear maps. Suppose that $V = \mathrm{Im}T + \mathrm{Im}S = \ker T + \ker S$. Prove that $V = \mathrm{Im}T \oplus \mathrm{Im}S = \ker T \oplus \ker S$.
b) (**13 points**) Let U_1, U_2 and U_3 denote three subspaces of V. Prove that

$$\dim(U_1 \cap U_2 \cap U_3) \geq \dim U_1 + \dim U_2 + \dim U_3 - 2\dim V$$

Solution: a) We need to prove that $\mathrm{Im}T \cap \mathrm{Im}S = \ker T \cap \ker S = \{0\}$. We have

$$2\dim V = \dim(\mathrm{Im}T + \mathrm{Im}S) + \dim(\ker T + \ker S) \tag{3}$$

The first dimension Theorem implies that

$$\dim(\mathrm{Im}T + \mathrm{Im}S) = \dim\mathrm{Im}T + \dim\mathrm{Im}S - \dim(\mathrm{Im}T \cap \mathrm{Im}S)$$

$$\dim(\ker T + \ker S) = \dim\ker T + \dim\ker S - \dim(\ker T \cap \ker S)$$

Plugging these two equations in equation (3), we obtain

$$2\dim V = \dim\mathrm{Im}T + \dim\mathrm{Im}S - \dim(\mathrm{Im}T \cap \mathrm{Im}S) + \dim\ker T + \dim\ker S - \dim(\ker T \cap \ker S) =$$

$$= (\dim\mathrm{Im}T + \dim\ker T) + (\dim\mathrm{Im}S + \dim\ker S) - \dim(\mathrm{Im}T \cap \mathrm{Im}S) - \dim(\ker T \cap \ker S) =$$

$$= 2\dim V - \dim(\mathrm{Im}T \cap \mathrm{Im}S) - \dim(\ker T \cap \ker S)$$

Here, we used the second dimension Theorem $\dim V = \dim\mathrm{Im}T + \dim\ker T$, and similarly for S.

From the above equation we deduce that $\dim(\mathrm{Im}T \cap \mathrm{Im}S) = \dim(\ker T \cap \ker S) = 0$.
b) Let L_1 and L_2 be two subspaces of V. Apply the first dimension Theorem to L_1 and L_2. We have $\dim(L_1 + L_2) = \dim L_1 + \dim L_2 - \dim(L_1 \cap L_2)$. This is the same as $\dim(L_1 \cap L_2) = \dim L_1 + \dim L_2 - \dim(L_1 + L_2)$. Since $\dim(L_1 + L_2) \leq \dim V$, then it follows from the above equation that $\dim(L_1 \cap L_2) \geq \dim L_1 + \dim L_2 - \dim V$. Use this inequality with $L_1 = U_1$ and $L_2 = U_2 \cap U_3$. Then we get

$$\dim(U_1 \cap U_2 \cap U_3) \geq \dim U_1 + \dim(U_2 \cap U_3) - \dim V$$

Use $\dim(L_1 \cap L_2) \geq \dim L_1 + \dim L_2 - \dim V$ once again, this time with $L_1 = U_2$ and $L_2 = U_3$. We get

$$\dim U_1 + \dim(U_2 \cap U_3) - \dim V \geq \dim U_1 + (\dim U_2 + \dim U_3 - \dim V) - \dim V$$

From this the result follows.

מועד א' 16.01.2019

בחינה באלגברה לינארית 1 א

דוד גינזבורג

- משך הבחינה 3 שעות.
- יש לענות על כל השאלות.
- אין להשתמש בכל חומר עזר לרבות מחשבון.
- יש לנמק היטב את דרך הפיתרון.

שאלה 1 *(25 נ')* לכל $\lambda \in \mathbb{R}$ מצאו את דרגת המטריצה

$$A = \begin{pmatrix} 2\lambda & -1 & 2 \\ -2 & 1+\lambda & 2-3\lambda \\ -3 & -1 & 5 \end{pmatrix}.$$

שאלה 2 *(25 נ')* חשבו את הדטרמיננטה של המטריצה $A=(a_{i,j})_{1\le i,j\le n}$. כאן $a_{i,j}=a_{j,i}$ לכל $1\le i,j\le n$, ובנוסף $a_{1,j}=1$ לכל $1\le j\le n$, ו- $a_{i,j}=(i-1)x$ לכל $2\le i\le j\le n$. במילים אחרות,

$$A = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 & 1 \\ 1 & x & x & \dots & x & x \\ 1 & x & 2x & \dots & 2x & 2x \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & x & 2x & \dots & (n-2)x & (n-2)x \\ 1 & x & 2x & \dots & (n-2)x & (n-1)x \end{pmatrix}.$$

שאלה 3 *(25 נ')* יהי V מרחב וקטורי מעל השדה F ויהי U תת מרחב לא טריביאלי של V. הוכיחו כי אם קיים תת מרחב <u>*יחיד*</u> W של V כך ש- $U\oplus W=V$, אז $U=V$.

שאלה 4 א. *(13 נ')* יהי V מרחב וקטורי ממימד n מעל השדה F, ותהי $T:V\to V$ העתקה לינארית המקיימת $T^2=0$ ו- $\dim(\mathrm{Ker}T)=r$. הוכיחו כי $n\le 2r$.

ב. *(12נ')* האם קיימות העתקות לינאריות $T:\mathbb{R}^{10}\to\mathbb{R}^{14}$ ו- $S:\mathbb{R}^{14}\to\mathbb{R}^{5}$ כך ש- T העתקה חד-חד-ערכית, S העתקת על ו- $S\circ T$ העתקת האפס ?

בהצלחה!

Solution 2: The computations are straightforward but the idea is to simplify them as possible. It is convenient to start by interchanging the first and second row, and then perform the row operations

$$\begin{pmatrix} -2 & 1+\lambda & 2-3\lambda \\ 2\lambda & -1 & 2 \\ -3 & -1 & 5 \end{pmatrix} \xrightarrow[R_3 \to 2R_3 - 3R_1]{R_2 \to R_2 + \lambda R_1} A_1 = \begin{pmatrix} -2 & 1+\lambda & 2-3\lambda \\ 0 & -1+\lambda(\lambda+1) & 2+\lambda(2-3\lambda) \\ 0 & -5-3\lambda & 4+9\lambda \end{pmatrix}$$

Since row operations do not change the rank of the matrices, then $\operatorname{rank}(A) = \operatorname{rank}(A_1)$. Let

$$B = \begin{pmatrix} -1+\lambda(\lambda+1) & 2+\lambda(2-3\lambda) \\ -5-3\lambda & 4+9\lambda \end{pmatrix}$$

Then we have $\operatorname{rank}(A) = \operatorname{rank}(A_1) = 1 + \operatorname{rank}(B)$. Indeed, this follows from the fact that $\operatorname{rank}(A)$ is the number of non-zero rows in the row echelon matrix which is obtained from A by row operations.

To determine $\operatorname{rank}(B)$, we notice first that $\operatorname{rank}(B) = 0$ if and only if $B = 0$. For $B = 0$ we must have $-5-3\lambda = 4+9\lambda = 0$. This cannot happen, and hence $\operatorname{rank}(B) = 1, 2$. Second, $\operatorname{rank}(B) = 2$ if and only if B is invertible, or if and only if $|B| \neq 0$. We have

$$|B| = [-1+\lambda(\lambda+1)][4+9\lambda] - [2+\lambda(2-3\lambda)][-5-3\lambda] = 4\lambda^2 + 11\lambda + 6 = (4\lambda+3)(\lambda+2)$$

Hence $|B| = 0$ if and only if $\lambda = -\frac{3}{4}, -2$.

From this we conclude that $\operatorname{rank}(A) = 2$ if $\lambda = -\frac{3}{4}, -2$, and in all other cases $\operatorname{rank}(A) = 3$.

Solution 2: Write the last row of the matrix A as

$$\begin{pmatrix}1 & x & 2x & \dots & (n-2)x & (n-1)x\end{pmatrix} =$$
$$\begin{pmatrix}1+0 & x+0 & 2x+0 & \dots & (n-2)x+0 & (n-2)x+x\end{pmatrix}$$

Then by the property of addition in rows we obtain

$$|A| = \begin{vmatrix} 1 & 1 & 1 & \dots & 1 & 1 \\ 1 & x & x & \dots & x & x \\ 1 & x & 2x & \dots & 2x & 2x \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & x & 2x & \dots & (n-2)x & (n-2)x \\ 1 & x & 2x & \dots & (n-2)x & (n-2)x \end{vmatrix} + \begin{vmatrix} 1 & 1 & 1 & \dots & 1 & 1 \\ 1 & x & x & \dots & x & x \\ 1 & x & 2x & \dots & 2x & 2x \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & x & 2x & \dots & (n-2)x & (n-2)x \\ 0 & 0 & 0 & \dots & 0 & x \end{vmatrix}$$

The first determinant on the right hand side is zero because it has two equal rows. Computing the second determinant on the right by expanding over the last row, we obtain

$$|A| = x \begin{vmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & x & x & \dots & x \\ 1 & x & 2x & \dots & 2x \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x & 2x & \dots & (n-2)x \end{vmatrix}$$

Thus, if we denote $A_n = A$, then the above equality is $|A_n| = x|A_{n-1}|$. Arguing by induction we obtain

$$|A| = |A_n| = x|A_{n-1}| = x^2|A_{n-2}| = x^{n-2}|A_2| = x^{n-2}\begin{vmatrix} 1 & 1 \\ 1 & x \end{vmatrix} = x^{n-2}(x-1)$$

Solution 3: We will assume that U is a proper subspace of V and derive a contradiction. Assume that $r = \dim U$ and $n = \dim V$. Thus, we assume that $r < n$. Let $\mathcal{B} = \{u_1, \ldots, u_r\}$ be a basis for U. Complete it to a basis $\mathcal{B}_1$ of V. Denote $\mathcal{B}_1 = \{u_1, \ldots, u_r, v_1, v_2, \ldots, v_{n-r}\}$. Let $W_1 = \mathrm{Sp}\{v_1, v_2, \ldots, v_{n-r}\}$. Then W_1 is a subspace of V which satisfies $U \oplus W_1 = V$.

Let $W_2 = \mathrm{Sp}\{u_r + v_1, v_2, \ldots, v_{n-r}\}$. We claim that $U \oplus W_2 = V$, and that $W_1 \neq W_2$. This will be a contradiction. Denote $\mathcal{B}_2 = \{u_1, \ldots, u_r, u_r + v_1, v_2, \ldots, v_{n-r}\}$. Since $\mathcal{B}_2$ is a set which contains n vectors, then to prove that it is a basis for V, it is enough to prove that $V = \mathrm{Sp}(\mathcal{B}_2)$. To do that it is enough to prove that $v_1 \in \mathrm{Sp}(\mathcal{B}_2)$. This follows from the trivial identity $v_1 = (-1)u_r + (u_r + v_1)$. Hence, $U + W_2 = V$. To prove that $U \cap W_2 = \{0\}$ let $w \in U \cap W_2$. Then

$$w = \alpha_1 u_1 + \cdots + \alpha_r u_r = \beta_1(u_r + v_1) + \beta_2 v_2 + \cdots + \beta_{n-r} v_{n-r}$$

It follows from the right hand side identity that

$$\alpha_1 u_1 + \cdots + (\alpha_r - \beta_1)u_r + (-\beta_1)v_1 + (-\beta_2)v_2 + \cdots + (-\beta_{n-r})v_{n-r} = 0$$

Since the set $\mathcal{B}_1$ is a basis for V, then it is an independent set, and hence all coefficients are zero. Thus $w = 0$. We proved that $U \oplus W_2 = V$.

To complete the proof we need to prove that $W_1 \neq W_2$. We claim that $v_1 \in W_1$ but $v_1 \notin W_2$. Clearly, $v_1 \in W_1$. Assume that $v_1 \in W_2$. Then by definition it is in $\mathrm{Sp}\{u_r + v_1, v_2, \ldots, v_{n-r}\}$. Hence there are scalars such that

$$v_1 = \alpha_1(u_r + v_1) + \alpha_2 v_2 + \cdots + \alpha_{n-r} v_{n-r}$$

Hence, moving the vector v_1 to the right hand side,

$$\alpha_1 u_r + (\alpha_1 - 1)v_1 + \alpha_2 v_2 + \cdots + \alpha_{n-r} v_{n-r} = 0$$

The set $\{u_r, v_1, v_2, \ldots, v_{n-r}\}$ is a subset of $\mathcal{B}_1$, and hence it is an independent set. This means that all coefficients in the above equality are zero. Thus we obtain $\alpha_1 = 0$ and $\alpha_1 - 1 = 0$ which is clearly a contradiction.

Solution 4: **a)** Let $\{v_1, \ldots, v_r\}$ denote a basis for $ker\ T$. Complete it to a basis

$$\{v_1, \ldots, v_r, w_1, \ldots, w_{n-r}\}$$

of V. Then $\{Tw_1, \ldots, Tw_{n-r}\}$ is a linearly independent set in V. The proof of that is similar to a corresponding proof in the dimension theorem. In some details, assume that $\alpha_1 Tw_1 + \cdots + \alpha_{n-r} Tw_{n-r} = 0$. Then $T(\alpha_1 w_1 + \cdots + \alpha_{n-r} w_{n-r}) = 0$. Hence $\alpha_1 w_1 + \cdots + \alpha_{n-r} w_{n-r} \in \ker T$. From this we obtain that there are scalars β_i such that

$$\alpha_1 w_1 + \cdots + \alpha_{n-r} w_{n-r} = \beta_1 v_1 + \cdots + \beta_r v_r$$

Since the set $\{v_1, \ldots, v_r, w_1, \ldots, w_{n-r}\}$ is a basis, then $\alpha_1 = \ldots = \alpha_{n-r} = 0$.

Because $T^2 = 0$, for all $1 \leq i \leq n-r$ we have $T(Tw_i) = T^2 w_i = 0$. Hence, $\{Tw_1, \ldots, Tw_{n-r}\}$ is a linearly independent set inside $ker\ T$. Since $\dim(kerT) = r$, we obtain $n - r \leq r$, or $n \leq 2r$.

b) Suppose that $S \circ T = 0$. Then $\mathrm{Im}T \subset \ker S$. Indeed, if $v \in \mathrm{Im}T$, then there is $u \in \mathbf{R}^{10}$ such that $v = Tu$. Hence $0 = (S \circ T)(u) = S(Tu) = Sv$. Hence $v \in \ker S$. From this we deduce that $\dim(\mathrm{Im}T) \leq \dim(\ker S)$.

Apply the dimension Theorem to the two linear maps. We have

$$10 = \dim \mathbf{R}^{10} = \dim(\ker T) + \dim(\mathrm{Im}T) = \dim(\mathrm{Im}T) \tag{2}$$

$$14 = \dim \mathbf{R}^{14} = \dim(\ker S) + \dim(\mathrm{Im}S) = \dim(\ker S) + 5 \tag{3}$$

The right most equality in equation (2) follows from the fact that T is one-to-one. The right most equality in equation (3) follows from the fact that S is onto. Hence, $\dim(\mathrm{Im}T) = 10$ and $\dim(\ker S) = 9$. This a contradiction to the above inequality $\dim(\mathrm{Im}T) \leq \dim(\ker S)$.

מועד ב' 17.02.2019

בחינה באלגברה לינארית 1 א

דוד גינזבורג

- משך הבחינה 3 שעות.
- יש לענות על כל השאלות.
- אין להשתמש בכל חומר עזר לרבות מחשבון.
- יש לנמק היטב את דרך הפיתרון.

שאלה 1 *(25 נ')* במרחב הוקטורי $\mathbb{R}^4$ נתונים שני תתי המרחב

$$W_1 = \mathrm{Sp}\{(1,2,3,4);\ (3,4,5,6);\ (7,8,9,10)\},$$
$$W_2 = \{(a,b,c,d)\ :\ a+b=0;\ a,b,c,d \in \mathbb{R}\}.$$

מצאו תת מרחב W_3 של W_2 כך ש- $W_1 \oplus W_3 = W_1 + W_2$.

שאלה 2 *(25 נ')* נתונה מערכת של n משוואות עם n נעלמים ופרמטר k,

$$\begin{cases} kx_1 + x_2 + \ldots + x_n = 1 \\ x_1 + kx_2 + \ldots + x_n = 1 \\ \ldots \quad \ldots \quad \ldots \quad \ldots \\ x_1 + x_2 + \ldots + kx_n = 1 \end{cases}.$$

לכל ערך של $k \in \mathbb{R}$ מצאו את כל הפתרונות של המערכת.

שאלה 3 *(25 נ')* יהי L תת המרחב של $Mat_{3\times3}(\mathbb{R})$ המוגדר על ידי

$L = \left\{ A \in Mat_{3\times3}(\mathbb{R})\ :\ A\begin{pmatrix}1\\2\\3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix} \right\}$. תנו דוגמה לבסיס עבור L.

שאלה 4 א. *(13 נ')* יהי V מרחב וקטורי מעל השדה F. נתונה העתקה לינארית $T: V \to V$ המקיימת $T^2 = T$. הוכיחו כי $KerT \oplus \mathrm{Im}\,T = V$.

ב. *(12נ')* נתונה $A \in Mat_{n\times n}(\mathbb{R})$. יהי W תת המרחב של $Mat_{n\times n}(\mathbb{R})$ המוגדר על ידי $W = \{X \in V\ :\ AX = XA\}$. הוכיחו כי $\dim W \geq 2$.

בהצלחה!

Solutions

Solution 1: We start by computing $\dim(W_1 + W_2)$. For that we use the dimension Theorem

$$\dim(W_1 + W_2) = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2)$$

To compute $\dim W_1$, we find a basis for W_1. We write the three vectors as a matrix, and we perform the following row operations,

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 5 & 6 \\ 7 & 8 & 9 & 10 \end{pmatrix} \xrightarrow[R_3 \to R_3 - 7R_1]{R_2 \to R_2 - 3R_1} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -2 & -4 & -6 \\ 0 & -6 & -12 & -18 \end{pmatrix} \xrightarrow[R_2 \to -R_2]{R_3 \to R_3 - 3R_2} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Hence $\dim W_1 = 2$, and $\mathcal{B} = \{(1,2,3,4); (0,1,2,3)\}$ is a basis for W_1.

As for $\dim W_2$, we can write a vector in W_2 as

$$(a, -a, c, d) = a(1, -1, 0, 0) + c(0, 0, 1, 0) + d(0, 0, 0, 1)$$

from which we deduce that $\dim W_2 = 3$. To compute $\dim(W_1 \cap W_2)$ we take a vector in $w \in W_1 \cap W_2$ and write it as

$$w = \alpha(1, 2, 3, 4) + \beta(0, 1, 2, 3) = (a, -a, c, d)$$

Here α, β, a, c and d are all scalars in $\mathbf{R}$. Comparing the coordinates on the right hand side of the equation, we get the system of equations $\alpha = a$; $2\alpha + \beta = -a$; $3\alpha + 2\beta = c$, and $4\alpha + 3\beta = d$. It is easy to see that the solution is $\alpha = a$; $\beta = c = -3a$, and $d = -5a$. Hence $W_1 \cap W_2 = \{(a, -a, -3a, -5a) \ : \ a \in \mathbf{R}\} = \text{Sp}\{(1, -1, -3, -5)\}$. In particular we obtain $\dim(W_1 \cap W_2) = 1$. Hence, from the dimension Theorem stated above we obtain that $\dim(W_1 + W_2) = 2 + 3 - 1 = 4 = \dim \mathbf{R}^4$. This implies that $W_1 + W_2 = \mathbf{R}^4$. Thus, the problem is reduced to finding $W_3 \subset W_2$ such that $W_1 \oplus W_3 = \mathbf{R}^4$. To do that it is enough to find two vectors $u_1, u_2 \in W_3$ such that the set $\mathcal{B} \cup \{u_1, u_2\}$ is a basis for $\mathbf{R}^4$. It is easy to see that $u_1 = (0, 0, 1, 0)$ and $u_2 = (0, 0, 0, 1)$ is a such a choice.

Solution 2: Adding all the equations in the system we obtain

$$(k+n-1)(x_1+x_2+\cdots+x_n)=n$$

If $k=1-n$, then the left hand side of the above equation is zero, and the right hand side is equal to n. So in this case there are no solutions.

Assume that $k\neq 1-n$. Then, dividing by $k+n-1$ we obtain the equation

$$x_1+x_2+\cdots+x_n=\frac{n}{k+n-1} \tag{7}$$

Subtract this equation from the first equation of the system. We obtain the equation

$$(k-1)x_1=1-\frac{n}{k+n-1}$$

Repeating this subtraction for each equation of the system, we obtain

$$(k-1)x_i=1-\frac{n}{k+n-1} \qquad 1\leq i\leq n$$

If $k\neq 1$, we can divide by $k-1$, and obtain $x_i=\frac{1}{k+n-1}$. If $k=1$, the system of equations reduces to one equation which is $x_1+x_2+\cdots+x_n=1$.

Hence, the solutions of the system are as follows:

a) If $k=1$, the system has infinite number of solutions given by

$$\{(1-\alpha_2-\alpha_3-\ldots-\alpha_n,\alpha_2,\ldots,\alpha_n):\ \alpha_i\in\mathbf{R};\quad 2\leq i\leq n\}$$

b) If $k=1-n$, the system has no solutions.

c) If $k\neq 1,1-n$, the system has a unique solution given by

$$\left(\frac{1}{k+n-1},\frac{1}{k+n-1},\ldots,\frac{1}{k+n-1}\right).$$

Solution 3: Assume that $A = (a_{i,j})_{3\times 3} \in L$. Then, by matrix multiplication, the equation

$$A\begin{pmatrix}1\\2\\3\end{pmatrix} = 0$$

is equivalent to the system of equations

$$a_{1,1} + 2a_{1,2} + 3a_{1,3} = 0$$

$$a_{2,1} + 2a_{2,2} + 3a_{2,3} = 0$$

$$a_{3,1} + 2a_{3,2} + 3a_{3,3} = 0$$

This system has 9 variables and 3 equations. From this we easily deduce that it has 6 free variables. Hence, dim$L = 6$. An example for a basis for L is the set

$$\begin{pmatrix}-2 & 1 & 0\\0 & 0 & 0\\0 & 0 & 0\end{pmatrix};\begin{pmatrix}-3 & 0 & 1\\0 & 0 & 0\\0 & 0 & 0\end{pmatrix};\begin{pmatrix}0 & 0 & 0\\-2 & 1 & 0\\0 & 0 & 0\end{pmatrix};\begin{pmatrix}0 & 0 & 0\\-3 & 0 & 1\\0 & 0 & 0\end{pmatrix};\begin{pmatrix}0 & 0 & 0\\0 & 0 & 0\\-2 & 1 & 0\end{pmatrix};\begin{pmatrix}0 & 0 & 0\\0 & 0 & 0\\-3 & 0 & 1\end{pmatrix}$$

Solution 4: a) First we prove that $\ker T \cap \mathrm{Im} T = \{0\}$. Let $v \in \ker T \cap \mathrm{Im} T$. Then, $Tv = 0$ (since $v \in \ker T$), and there is $u \in V$ such that $v = Tu$ (since $v \in \mathrm{Im} T$). Hence, $0 = Tv = T(Tu) = T^2 u = Tu = v$. Here, we used the fact that $T^2 = T$. Using the two dimension Theorems we obtain

$$\dim(\ker T + \mathrm{Im} T) = \dim(\ker T) + \dim(\mathrm{Im} T) - \dim(\ker T \cap \mathrm{Im} T) =$$

$$= \dim(\ker T) + \dim(\mathrm{Im} T) = \dim V$$

Here, we used the fact that $\ker T \cap \mathrm{Im} T = \{0\}$. Hence, $V = \ker T + \mathrm{Im} T$, and from the fact that $\ker T \cap \mathrm{Im} T = \{0\}$ we deduce that $V = \ker T \oplus \mathrm{Im} T$.

Since the set $\{v_1, \ldots, v_r, w_1, \ldots, w_{n-r}\}$ is a basis, then $\alpha_1 = \ldots = \alpha_{n-r} = 0$.

Because $T^2 = 0$, for all $1 \leq i \leq n - r$ we have $T(Tw_i) = T^2 w_i = 0$. Hence, $\{Tw_1, \ldots, Tw_{n-r}\}$ is a linearly independent set inside $ker\ T$. Since $\dim(ker T) = r$, we obtain $n - r \leq r$, or $n \leq 2r$.

b) There are two cases to consider. First, assume that $A = aI$ for some $a \in \mathbf{R}$. Then it is easy to see that $W = V$. Hence, it is clear that $\dim W \geq 2$. In general it is clear that the matrices I and A are in W. Suppose that $A \neq aI$ for all $a \in \mathbf{R}$. Then the claim is that the set $\{I, A\}$ is a linearly independent set in V. Indeed, consider the equation $\alpha I + \beta A = 0$. If $\beta \neq 0$, then $A = -\frac{\alpha}{\beta} I$. This a contradiction to the assumption that $A \neq aI$. Hence $\beta = 0$, which implies that $\alpha = 0$. Hence, again we have $\dim W \geq 2$.

Solution of Moed A in Linear Algebra 1 2022

David Ginzburg and Evgeny Musicantov

1) Problem: Let $T : F^3 \to F^2$ given by $T(x, y, z) = (x + 2z, x + y + z)$. Find all bases $\mathcal{B} = \{(a, b, c); (1, 1, 1); (0, 0, 1)\}$ of F^3, and all bases $\mathcal{C} = \{(n_1, m_1); (n_2, m_2)\}$ of F^2 such that

$$[T]_{\mathcal{C}}^{\mathcal{B}} = \begin{pmatrix} 1 & 3 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

Solution: By definition we have $T((a, b, c)) = (a+2c, a+b+c) = 1(n_1, m_1)+1(n_2, m_2)$. Also, $T((1, 1, 1)) = (3, 3) = 3(n_1, m_1) + 0(n_2, m_2)$, and $T((0, 0, 1)) = (2, 1) = 0(n_1, m_1) + 1(n_2, m_2)$. From this we obtain $(n_1, m_1) = (1, 1)$ and $(n_2, m_2) = (2, 1)$. Hence $(a+2c, a+b+c) = (n_1, m_1) + (n_2, m_2) = (3, 2)$. Thus, we need to solve the system of equations $a + 2c = 3$ and $a+b+c = 2$. Subtracting the two equations, we obtain that the set of solutions is given by $\{(3 - 2c, c - 1, c) : c \in F\}$.

We need to check for what values of c, the set $\mathcal{B} = \{(3 - 2c, c - 1, c); (1, 1, 1); (0, 0, 1)\}$ is a base for F^3. Write $\alpha(3 - 2c, c - 1, c) + \beta(1, 1, 1) + \gamma(0, 0, 1) = 0$. This gives us the system of equations $(3 - 2c)\alpha + \beta = 0$, $(c - 1)\alpha + \beta = 0$ and $c\alpha + \beta + \gamma = 0$. Subtracting the first two equations, we obtain $(4-3c)\alpha = 0$. If $c \neq 4/3$ then $\alpha = 0$. This implies that $\beta = \gamma = 0$. Hence, if $c \neq 4/3$, we obtain that $\mathcal{B}$ is a base for F^3.

It is easy to check that $\mathcal{C} = \{(1, 1); (2, 1)\}$ is an independent set of vectors in F^2. Hence it is a base for F^2.

2) Problem: Let $a, b, c, d \in \mathbf{R}$ be four numbers such that one of them is zero and the other three are positive. Prove that the determinant of the matrix

$$A = \begin{pmatrix} a & -1 & -1 & -1 \\ -1 & b & -1 & -1 \\ -1 & -1 & c & -1 \\ -1 & -1 & -1 & d \end{pmatrix}$$

is a negative number.

Solution: Without loose of generality, we may assume that $a = 0$ and b, c, d are positive. Indeed, if $b = 0$, then the two operations $R_1 \leftrightarrow R_2$ and then $C_1 \leftrightarrow C_2$ will cause the $(1, 1)$ entry of the matrix to be zero. These two operations do not change the value of the determinant. Similarly if c or d are zero.

Assuming $a = 0$, for $2 \le i \le 4$, perform the operations $C_i \to C_i - C_1$. We obtain the matrix

$$B = \begin{pmatrix} 0 & -1 & -1 & -1 \\ -1 & b+1 & 0 & 0 \\ -1 & 0 & c+1 & 0 \\ -1 & 0 & 0 & d+1 \end{pmatrix}$$

Clearly, $|A| = |B|$. On the matrix B, perform $C_1 \to C_1 + \frac{1}{b+1}C_2$, then $C_1 \to C_1 + \frac{1}{c+1}C_3$, and $C_1 \to C_1 + \frac{1}{d+1}C_3$. Since b, c and d are all positive, these operations are well defined. We obtain

$$C = \begin{pmatrix} \alpha & -1 & -1 & -1 \\ 0 & b+1 & 0 & 0 \\ 0 & 0 & c+1 & 0 \\ 0 & 0 & 0 & d+1 \end{pmatrix}$$

where

$$\alpha = -\frac{1}{b+1} - \frac{1}{c+1} - \frac{1}{d+1}$$

We have $|B| = |C|$. Notice that $\alpha < 0$. Since $|A| = |C| = \alpha(b+1)(c+1)(d+1)$, the result follows.

3) Problem: Let V and W denote two vector spaces over F. Let $S_1, S_2 : V \to W$ denote two linear maps such that $\ker S_1 = \ker S_2$. Prove that there is an invertible map $T : W \to W$ such that $S_1 = TS_2$.

Solution: Let $K = \ker S_1 = \ker S_2$. Let U denote any subspace of V such that $V = K \oplus U$. Let $\{u_1, \dots, u_r\}$ denote a base for U. Then, $\mathrm{Im} S_1 = Sp\{S_1(u_1), \dots, S_1(u_r)\}$. Since $K \cap U = \{0\}$, then $\{S_1(u_1), \dots, S_1(u_r)\}$ is a linear independent set of vectors in W. (This was done in class: Indeed, if $\alpha_1 S_1(u_1) + \cdots + \alpha_r S_1(u_r) = 0$, then $S_1(\alpha_1 u_1 + \cdots + \alpha_r u_r) = 0$. Since $K \cap U = \{0\}$, then $\alpha_1 u_1 + \cdots + \alpha_r u_r = 0$, which implies $\alpha_i = 0$ for all $1 \le i \le r$.)

We deduce that $\{S_1(u_1), \dots, S_1(u_r)\}$ is a base for $\mathrm{Im} S_1$, and hence U is isomorphic to $\mathrm{Im} S_1$. In a similar way, U is isomorphic to $\mathrm{Im} S_2$. In particular $\mathrm{Im} S_1$ is isomorphic to $\mathrm{Im} S_2$. Write $W = W_1 \oplus \mathrm{Im} S_1$, and $W = W_2 \oplus \mathrm{Im} S_2$. Then W_1 is isomorphic to W_2. Let $T : W \to W$

denote a linear map which maps $S_2(u_i)$ to $S_1(u_i)$ for all $1 \leq i \leq r$, and which maps a base of W_2 to a base of W_1. This is possible since W_1 is isomorphic to W_2, and $\mathrm{Im}S_1$ is isomorphic to $\mathrm{Im}S_2$. Hence T is invertible, and $TS_2(u_i) = S_1(u_i)$ for all $1 \leq i \leq r$. Given $v \in V$, let $v = v_0 + (\alpha_1 u_1 + \cdots + \alpha_r u_r)$ where $v_0 \in \ker S_1 = \ker S_2$. Then

$$TS_2(v) = TS_2(v_0) + \alpha_1 TS_2(u_1) + \cdots + \alpha_r TS_2(u_r) = \alpha_1 TS_2(u_1) + \cdots + \alpha_r TS_2(u_r) =$$

$$= \alpha_1 S_1(u_1) + \cdots + \alpha_r S_1(u_r) = S_1(v_0) + S_1(\alpha_1 u_1 + \cdots + \alpha_r u_r) = S_1(v)$$

Thus, $S_1 = TS_2$.

4) Problem: a) (12 points) Let $T : V \to W$ be a surjective map with the following property. For every finite set of vectors K in V, if $\mathrm{Sp}\{T(K)\} = W$, then $\mathrm{Sp}\{K\} = V$. Prove that T is injective.
b) (13 points) Let A be an invertiable matrix of size $n \times n$. Define $T : Mat_{n\times n}(F) \to Mat_{n\times n}(F)$ by $T(B) = A^tB + B^tA$. Prove that the map T is not an injective map.

Solution: Assume that $v \in V$ is a nonzero vector in $\ker T$. We will derive a contradiction. Complete this vector to a base $\mathcal{B} = \{v, u_1, \ldots, u_r\}$ of V. We have $\mathrm{Im}T = \mathrm{Sp}\{Tv, Tu_1, \ldots, Tu_r\}$. Since T is surjective, then $\mathrm{Sp}\{Tv, Tu_1, \ldots, Tu_r\} = W$. Let $K = \{u_1, \ldots, u_r\} \subset V$. Since we assumed that $v \in \ker T$, then $\mathrm{Sp}\{T(K)\} = \mathrm{Sp}\{Tu_1, \ldots, Tu_r\} = \mathrm{Sp}\{Tv, Tu_1, \ldots, Tu_r\} = W$. From the property of T, we deduce that the set $K = \{u_1, \ldots, u_r\}$ spans V. But this is impossible since $\mathcal{B}$ is a base for V, and hence it is a minimal set which spans V. This is a contradiction, and hence $\ker T = \{0\}$, and T is injective.
b) Suppose that T is an injective map. Then T must be an isomorphism. This follows from a Theorem we proved in class that every injective linear map from a vector space to itself, must be an isomorphism. Hence T is a surjective map. However, the matrix $A^tB + B^tA$ is a symmetric matrix. In other words, we have $(A^tB+B^tA)^t = A^tB+B^tA$. Since in $Mat_{n\times n}(F)$ there are non-symmetric matrices, then T cannot be surjective, and hence cannot be injective.

Solution of Moad A in Linear Algebra 2022

David Ginzburg

1) Problem: For all $a, b \in \mathbf{R}$, let

$$A = \begin{pmatrix} 1 & 1 & a \\ 1 & 0 & 0 \\ 0 & 2 & b \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 1 & b \\ a & 0 & 0 \\ 0 & 3 & a \end{pmatrix}$$

Find conditions on a, b such not A is invertible, and rank$B = 3$.

Solution: For A to be invertible we need $|A| \neq 0$. Similarly, for rank$B = 3$ we also need $|B| \neq 0$. Since $|A| = b - 2a$ and $|B| = a(a - 3b)$, the conditions are $a \neq 0$, and $a \neq b/2$ and $a \neq b/3$.

2) Problem: Let A denote a 3×3 matrix with entries in $\mathbf{R}$. Assume that A is not invertible. Prove that there is a matrix B such that $AB = 0$.

Solution: Let $\{v_1, v_2, v_3\}$ denote the columns of the matrix A. View them as vectors in $\mathbf{R}^3$. Since A is not invertiable, then there are $\alpha_1, \alpha_2, \alpha_3 \in \mathbf{R}$ such that $\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = 0$. Let

$$B = \begin{pmatrix} \alpha_1 & \alpha_1 & \alpha_1 \\ \alpha_2 & \alpha_2 & \alpha_2 \\ \alpha_3 & \alpha_3 & \alpha_3 \end{pmatrix}$$

Then a matrix multiplication implies that $AB = 0$. Also, since not all α_i are zero, then $B \neq 0$.

3) Problem: In this problem F^n consists of column vectors.
Let $\{v_2, \ldots, v_n\}$ denote $n - 1$ vectors in F^n. For any $u \in F^n$, let $A(u, v_2, \ldots, v_n)$ denote the matrix of order n, whose columns are the vectors $\{u, v_2, \ldots, v_n\}$. Prove that there is a vector $v_0 \in F^n$, such that $\det A(u, v_2, \ldots, v_n) = v_0^t \cdot u$.

Remark: Here v_0^t denotes the transpose of v_0, and $v_0^t \cdot u$ denotes the matrix multiplication of v_0^t and u.

Solution: Denote $T(u) = \det(A(u, v_2, \dots, v_n))$. Then $T : F^n \to F$ is a linear map. Indeed, to prove it is linear we first notice that the addition by columns property of the determinant implies that for $u_1, u_2 \in F^n$ we have

$$\det(A(u_1 + u_2, v_2, \dots, v_n)) = \det(A(u_1, v_2, \dots, v_n)) + \det(A(u_2, v_2, \dots, v_n))$$

In terms of T, this implies that $T(u_1 + u_2) = T(u_1) + T(u_2)$. Similarly, for all $\alpha \in F$, we have $\det(A(\alpha u, v_2, \dots, v_n)) = \alpha \det(A(u, v_2, \dots, v_n))$. This implies $T(\alpha u) = \alpha T(u)$.

From the above it follows that $T : F^n \to F$ is a linear map. Similarly, if, for fixed $v_0 \in F^n$, we define $S(u) = v_0^t \cdot u$, then $S : F^n \to F$ is also a linear map. Because $\dim F = 1$, it follows from the dimension Theorem that two linear maps from F^n to F which have the same kernel, must be proportional.

In details, assume first that $\{v_2, \dots, v_n\}$ is a linear dependent set. Then, for all $u \in F^n$, $\det(A(u, v_2, \dots, v_n)) = 0$. In other words T is the zero map. In this case, choose $v_0 = 0$. Next, assume that $\{v_2, \dots, v_n\}$ is a linear independent set. Then, we can complete it to a base of F^n. Let $v_1 \in F^n$ be such that $\mathcal{B} = \{v_1, v_2, \dots, v_n\}$ is a base for F^n. Let $\alpha = \det(A(v_1, v_2, \dots, v_n))$. Then $\alpha \neq 0$. Form the system of linear equations given by $x^t A(v_1, v_2, \dots, v_n) = b$. Here $b = (\alpha, 0, \dots, 0)^t$. Since $A(v_1, v_2, \dots, v_n)$ is an invertible matrix, this equation has a unique solution. Denote this solution by v_0, and let $S(u) = v_0^t \cdot u$. Then, we have $T(v_1) = \alpha$. By comparing the first coordinate of the equation $b = v_0^t A(v_1, v_2, \dots, v_n)$ we get $S(v_1) = \alpha$. Also, we have $T(v_i) = S(v_i) = 0$ for all $2 \le i \le n$. Hence, T and S agree on $\mathcal{B}$. Thus, $T = S$.

4) Problem: In this problem V is an $n > 1$ dimensional vector space over a field F.
a) (12 points) Let U and W denote two subspaces of V such that $\dim U + \dim W \ge n$. Prove that there is a linear map $T : V \to V$ such that $\ker T \subset U$ and $\mathrm{Im}\, T \subset W$.
b) (13 points) Let $T : V \to V$ denote a linear map. Prove that there are two isomorphisms $T_1, T_2 : V \to V$ such that $T = T_1 + T_2$. In this part we assume that $F \neq \mathbf{Z}_2$.

Solution: a) Denote $\dim U = m$ and $\dim W = k$. Then $m + k \ge n$. Let $\{u_1, \dots, u_m\}$ denote a base for U, and let $\{w_1, \dots, w_k\}$ denote a base for W. Complete $\{u_1, \dots, u_m\}$ to a base $\{u_1, \dots, u_m, u_{m+1}, \dots, u_n\}$ of V, and similarly, complete $\{w_1, \dots, w_k\}$ to a base $\{w_1, \dots, w_k, w_{k+1}, \dots, w_n\}$ of V. Choose $m' \le m$ and $k' \le k$ such that $m' + k' = n$. To define

$T : V \to V$ it is enough to specify it on a base. Choose the base $\{u_1, \dots, u_m, u_{m+1}, \dots, u_n\}$. First, let $T(u_1) = T(u_2) = \dots = T(u_{m'}) = 0$. Then, set $T(u_{m'+1}) = w_1$; $T(u_{m'+2}) = w_2; \dots; T(u_{m'+n-m'}) = w_{n-m'}$. Since $m' \leq m$ then $\{u_1, \dots, u_{m'}\} \in U$. Hence ker $T \subset U$. Also, since $n - m' = k' \leq k$, then

$$\text{Im } T = \text{Sp}\{T(u_{m'+1}), T(u_{m'+2}) \dots, T(u_{m'+n-m'})\} = \text{Sp}\{w_1, \dots, w_{n-m'}\} \subset W$$

b) Let $\{u_1, \dots, u_m\}$ denote a base for ker T. Complete it to a base of V, say $\mathcal{B} = \{u_1, \dots, u_m, u_{m+1}, \dots, u_n\}$. Then Im T is isomorphic to $W = \text{Sp}\{u_{m+1}, \dots, u_n\}$. For this claim see problem **3)** in Moed A. Thus $V = \text{ker } T \oplus W$. From this we deduce that the set $\{T(u_{m+1}), \dots, T(u_n)\}$ is a linear independent set. Complete this set to a base of V, say $\mathcal{C} = \{e_1, \dots, e_m, T(u_{m+1}), \dots, T(u_n)\}$.

To define T_1 and T_2 it is enough to specify it on the base $\mathcal{B}$. Define T_1 as follows. First, for $1 \leq i \leq m$ set $T_1(u_i) = e_i$. Then, for $m \leq i \leq n$ set $T_1(u_i) = 2T(u_i)$. This is the point where we assume that $F \neq \mathbf{Z_2}$. For T_2 define $T_2(u_i) = -e_i$ when $1 \leq i \leq m$, and $T_2(u_i) = -T(u_i)$ when $m \leq i \leq n$. Then $T(u_i) = T_1(u_i) + T_2(u_i)$ and T_1 and T_2 are both isomorphisms.

SOLUTIONS MOED B LINEAR ALGEBRA 1 A 2020

SEMYON ALESKER AND DAVID GINZBURG

Problem 1: Let V denote the subspace of $Mat_{3\times 3}(\mathbf{R})$ consisting of all symmetric matrices with trace zero. Compute the dimension of V. Prove your claim.

Solution : A matrix A in $Mat_{3\times 3}(\mathbf{R})$ is V if and only if $A^t = A$ and the sum of all the diagonal entries of A is zero. Thus,

$$A = \begin{pmatrix} \alpha & a & b \\ a & \beta & c \\ b & c & -\alpha-\beta \end{pmatrix}$$

We have

$$\begin{pmatrix} \alpha & a & b \\ a & \beta & c \\ b & c & -\alpha-\beta \end{pmatrix} = \alpha \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} + \beta \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} + a \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + b \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} + c \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

From this we deduce that the five matrices written on the right hand side of the above equation span the vector space V. It is also easy to deduce from the above equation that these five matrices are linearly independent. From this we obtain that $\dim V = 5$.

Problem 2: Let $A = (a_{i,j})$ denote the matrix of size four defined by $a_{i,j} = x^{max\{i,j\}}$ for all $1 \le i, j \le 4$. Compute the determinant of A.

Solution: By definition we have

$$A = \begin{pmatrix} x & x^2 & x^3 & x^4 \\ x^2 & x^2 & x^3 & x^4 \\ x^3 & x^3 & x^3 & x^4 \\ x^4 & x^4 & x^4 & x^4 \end{pmatrix}$$

Performing the operation $R_4 \to R_4 - xR_3$, we obtain

$$|A| = \begin{vmatrix} x & x^2 & x^3 & x^4 \\ x^2 & x^2 & x^3 & x^4 \\ x^3 & x^3 & x^3 & x^4 \\ 0 & 0 & 0 & x^4 - x^5 \end{vmatrix}$$

Next perform the row operation $R_3 \to R_3 - xR_2$. We have

$$|A| = \begin{vmatrix} x & x^2 & x^3 & x^4 \\ x^2 & x^2 & x^3 & x^4 \\ 0 & 0 & x^3 - x^4 & x^4 - x^5 \\ 0 & 0 & 0 & x^4 - x^5 \end{vmatrix}$$

Finally perform $R_2 \to R_2 - xR_1$. We obtain

$$|A| = \begin{vmatrix} x & x^2 & x^3 & x^4 \\ 0 & x^2-x^3 & x^3-x^4 & x^4-x^5 \\ 0 & 0 & x^3-x^4 & x^4-x^5 \\ 0 & 0 & 0 & x^4-x^5 \end{vmatrix}$$

Hence, $|A| = x(x^2-x^3)(x^3-x^4)(x^4-x^5) = x^{10}(1-x)^3$.

Problem 3: Let V be a vector space defined over a field F. Let $T, S : V \to V$ be two linear maps. Prove that

$$T(ker(S \circ T)) = ImT \cap kerS$$

Solution : We will prove that each set is included in the other. Let $v \in T(ker(S \circ T))$. Then, there is $u \in ker(S \circ T)$ such that $Tu = v$. This implies that $v \in ImT$. Next, $Sv = S(Tu) = 0$ since by definition $u \in ker(S \circ T)$. Hence $v \in kerS$, and $v \in ImT \cap kerS$. This proves that $T(ker(S \circ T)) \subset ImT \cap kerS$.

The proof of the other inclusion is by reversing the argument. Let $v \in ImT \cap kerS$. Then, there is a $u \in V$ such that $Tu = v$ and also $Sv = 0$. Hence, $S(Tu) = Sv = 0$. Hence $u \in ker(S \circ T)$. Since $Tu = v$, we deduce that $v \in T(ker(S \circ T))$. Hence $ImT \cap kerS \subset T(ker(S \circ T))$.

Problem 4: Let V denote a vector space whose dimension is n. Let $T : V \to V$ be a linear map. Prove that there are bases $\mathcal{B}$ and $\mathcal{C}$ of V such that

$$[T]^{\mathcal{B}}_{\mathcal{C}} = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$$

Solution: Assume that $r = \dim ImT$. Then $n - r = \dim kerT$. Let $v_{r+1}, \ldots, v_n$ denote a base for $kerT$. Extend it to a base $\mathcal{B} = \{v_1, \ldots, v_r, v_{r+1}, \ldots, v_n\}$ of V. Denote $w_1 = T(v_1); \ldots; w_r = T(v_r)$. Then the set $\{w_1, \ldots, w_r\}$ is independent. Indeed, if $\alpha_1 w_1 + \cdots + \alpha_r w_r = 0$, then $T(\alpha_1 v_1 + \cdots + \alpha_r v_r) = \alpha_1 T(v_1) + \cdots + \alpha_r T(v_r) = \alpha_1 w_1 + \cdots + \alpha_r w_r = 0$. This implies that $\alpha_1 v_1 + \cdots + \alpha_r v_r \in kerT$. Since $v_{r+1}, \ldots, v_n$ is a base for $kerT$, we deduce that

$$\alpha_1 v_1 + \cdots + \alpha_r v_r = \beta_{r+1} v_{r+1} + \cdots + \beta_r v_r$$

or

$$\alpha_1 v_1 + \cdots + \alpha_r v_r - \beta_{r+1} v_{r+1} + \cdots - \beta_r v_r = 0$$

Since $\mathcal{B}$ is a base for V this implies that $\alpha_i = 0$ for all $1 \leq i \leq r$. Hence, $\{w_1, \ldots, w_r\}$ is independent. Extend it to a base $\mathcal{C} = \{w_1, \ldots, w_r, w_{r+1}, \ldots, w_n\}$ of V. Then the matrix $[T]^{\mathcal{B}}_{\mathcal{C}}$ is as above.

מועד א'
29/1/2020

בחינה באלגברה לינארית 1 א

סמיון אלסקר, דוד גינזבורג

יש לענות על כל השאלות. אין להשתמש בכל חומר עזר לרבות מחשבונים. לכל השאלות ניקוד שווה. בשאלה בה יש יותר מסעיף אחד, אם לא צויין אחרת, לכל סעיף ניקוד שווה. יש לנמק הייטב את דרך הפיתרון.

משך הבחינה: 3 שעות.

שאלה 1: נסמן ב $P_2(x)$ את המרחב הוקטורי של אוסף כל הפולינומים מדרגה עד וכולל שתיים עם מקדמים ממשיים. תהי $T : P_2(x) \to P_2(x)$ ההעתקה הלינארית המוגדרת על ידי $T(p(x)) = \alpha p''(x)+\beta p'(x)+\gamma p(x)$. מצאו את כל הערכים של $\alpha, \beta, \gamma \in \mathbf{R}$ כך שיתקיים $kerT = \{0\}$.

שאלה 2: תהי A מטריצה מסדר שלוש עם איברים בשדה $\mathbf{Q}$. נתונה המטריצה

$$B = \begin{pmatrix} 0 & 1 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

הוכיחו כי למערכת

$$A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} y \\ 2x \\ 0 \end{pmatrix}$$

יש פתרון לא טריביאלי אם ורק אם $|A-B| = 0$.

שאלה 3: יהי V מרחב וקטורי ויהיו W', W ו W'' שלושה תתי מרחב של V המקימים $V = W \oplus W' = W \oplus W''$. הוכיחו כי

$$\dim(W' \cap W'') \geq \dim V - 2\dim W$$

.

שאלה 4: יהי V מרחב וקטורי המוגדר מעל השדה $\mathbf{R}$. תהיינה $T, S : V \to \mathbf{R}$ שתי העתקות לינאריות שאינן העתקות האפס. נניח שמתקיים התנאי שלכל $v \in V$ אם $T(v) \geq 0$ אז $S(v) \geq 0$. הוכיחו כי קים סקלר ממשי $\alpha > 0$ המקיים $T = \alpha S$.

בהצלחה!

מועד ב'
25/2/2020

בחינה באלגברה לינארית 1 א

סמיון אלסקר, דוד גינזבורג

יש לענות על כל השאלות. אין להשתמש בכל חומר עזר לרבות מחשבונים. לכל השאלות ניקוד שווה. בשאלה בה יש יותר מסעיף אחד, אם לא צויין אחרת, לכל סעיף ניקוד שווה. יש לנמק הייטב את דרך הפיתרון.

משך הבחינה: 3 שעות.

שאלה 1: יהי V תת המרחב של $Mat_{3\times3}(\mathbf{R})$ המוגדר על ידי אוסף כל המטריצות הסימטריות עם עקבה אפס. חשבו את המימד של V. יש לנמק היטב את התשובה.

להזכירכם: אם $A=(a_{i,j})$ הינה מטריצה רבועית, אז העקבה של A מוגדרת להיות הסקלר $a_{1,1}+a_{2,2}+\cdots+a_{n,n}$.

שאלה 2: תהי $A=(a_{i,j})$ המטריצה מסדר ארבע המוגדרת על ידי $a_{i,j}=x^{max\{i,j\}}$ לכל $1\leq i,j\leq 4$. חשבו את $|A|$.

שאלה 3: יהי V מרחב וקטורי. תהיינה $T,S:V\to V$ שתי העתקות לינאריות. הוכיחו כי

$$T(ker(S\circ T))=ImT\cap kerS$$

.

שאלה 4: יהי V מרחב וקטורי ממימד n. תהי $T:V\to V$ העתקה לינארית. הוכיחו כי קיימים בסיסים $\mathcal{B}$ ו $\mathcal{C}$ של V וקיים מספר $0\leq r\leq n$ כך שמתקיים

$$[T]_{\mathcal{C}}^{\mathcal{B}}=\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$$

בהצלחה!

שאלה 1 *(33 נ')*

מצאו עבור אילו ערכים של הפרמטר $t\in\mathbb{R}$ למערכת

$$\begin{cases} -x_1 \qquad\quad + 2x_3 = 1 \\ 3x_1 + tx_2 - 6x_3 = -3 \\ -2x_1 - tx_2 + tx_3 = 3 \end{cases}$$

יש פתרון יחיד, אינסוף פתרונות, או אין פתרון. במקרים בהם יש פתרון רשמו את קבוצת הפתרונות.

שאלה 2 *(33 נ')*

תהי A מטריצה ממשית מסדר $n\times n$ המקיימת $(A+2I)^2=O$.

כאן O היא מטריצת האפס מסדר $n\times n$ ו- I מטריצת היחידה מסדר $n\times n$.

הוכיחו כי המטריצה $A+\lambda I$ הפיכה אם ורק אם $\lambda\neq 2$.

שאלה 3 *(34 נ')*

תהי A מטריצה מדורגת קנונית מסדר 3×3 עם איברים ב- $\mathbb{R}$. מצאו את כל הערכים

של $a,b\in\mathbb{R}$ כך שהמטריצה $B=\begin{pmatrix}1&2&3\\2&a&0\\3&0&b\end{pmatrix}$ תקיים $AB=O$.

כאן O היא מטריצת האפס מסדר 3×3.

בהצלחה!

<u>בחינה באלגברה לינארית 1</u>

דוד גינזבורג

משך הבחינה שלוש שעות.
אין להשתמש בכל חומר עזר לרבות מחשבונים.
יש לענות על כל השאלות.

<u>שאלה 1</u>

תהי A מטריצה מסדר n עם איברים בשדה F. נסתכל על עמודות A כעל וקטורים במרחב F^n. נניח כי סכום הוקטורים המתאימים לעמודות הזוגיות במטריצה, שווה לסכום הוקטורים המתאימים לעמודות האי זוגיות במטריצה. לחשב את הדטרמיננטה של A.

<u>שאלה 2</u>

תהי $A = (a_{i,j})$ המטריצה מסדר n המוגדרת על ידי $a_{i,j} = 1$ לכל $1 \le i, j \le n$. למצוא סקלר c כך שיתקיים $(I - A)^{-1} = I - cA$.

<u>הערה:</u> אין צורך להוכיח כי המטריצה $I - A$ הינה הפיכה.

<u>שאלה 3</u>

יהיו $a, b, c \in \mathbf{R}$. לחשב את דרגת המטריצה $\begin{pmatrix} 1 & 1 & 1 \\ b+c & a+c & a+b \\ bc & ac & ab \end{pmatrix}$

<u>שאלה 4</u>

יהי V המרחב הוקטורי המורכב מכל הפולינומים עם מקדמים בשדה F שדרגתם היא לכל היותר n. נגדיר העתקה לינארית $T : V \to V$ על ידי $T(p(x)) = p(x+1) - p(x)$. לחשב את הגרעין והתמונה של T.

<u>הערה:</u> אין צורך להוכיח כי ההעתקה היא לינארית.

<u>שאלה 5</u>

יהי V מרחב וקטורי. תהיינה $T, S : V \to V$ שתי העתקות לינאריות המקימות $TS = T$ וגם $ST = S$. להוכיח כי $\ker T = \ker S$, $T^2 = T$ וגם $S^2 = S$.

<u>בחינה באלגברה לינארית 1</u>

דוד גינזבורג

מועד א -2011

משך הבחינה שלוש שעות.
אין להשתמש בכל חומר עזר לרבות מחשבונים.
יש לענות על כל השאלות.

<u>שאלה 1</u>

להוכיח את המשפט הבא:
יהי V מרחב וקטורי מעל שדה F. תהי $K \subset V$ קבוצה סופית המכילה לפחות שני וקטורים. להוכיח כי K קבוצה תלויה אם ורק אם לפחות אחד מהוקטורים ב- K הינו צרוף לינארי של האחרים.

<u>שאלה 2</u>

יהי V מרחב וקטורי מעל שדה F. תהי $T : V \mapsto V$ העתקה לינארית. נגדיר:

$$U = \{S \in Hom_F(V,V) \ : \ S \circ T = 0\}$$

א. להוכיח כי U הינו תת מרחב של $Hom_F(V,V)$.
ב. אם $\dim V = 2$, מהו $\dim U$?

<u>שאלה 3</u>

יהיו V ו- W שני מרחבים וקטורים מעל שדה F. תהי $T : V \mapsto W$ העתקה לינארית.
לפניכם שלוש טענות. עבור כל אחת מהם יש לקבוע אם היא נכונה או לא. אם הטענה נכונה יש להוכיחה, ואם לא יש להביא דוגמא נגדית. יש לנמק היטב. אין קשר בין הסעיפים השונים.
א. אם $\dim V = 6$ וגם $\dim W = 4$ אז בהכרח $\ker T \neq \{0\}$.
ב. יהי q מספר רציונלי . נתונה המטריצה $A = \begin{pmatrix} 2 & 2 & 3 & 4 \\ 3q+4 & 4q+4 & 5q+6 & 2q+8 \\ 0 & -1 & 1 & 8 \\ 5 & 9 & 7 & a \end{pmatrix}$. אז קיים a רציונלי אחד ויחיד כך ש- $|A| = q$.
ג. תהי A מטריצה רבועית מסדר n ונניח כי $A = AA^t$. האם בהכרח $A^2 = A$?

<u>שאלה 4</u>

כמטריצה מעל הרציונלים, יש לקבוע את הדרגה של המטריצה הבאה:

$$\begin{pmatrix} a & -1 & 2 & 1 \\ -1 & a & 5 & 2 \\ 10 & -6 & 1 & 1 \end{pmatrix}$$

בהצלחה!

Linear Algebra Moed A 2015

David Ginzburg

1) Give an example for a linear map $T : F^4 \mapsto F^4$ such that

$$ImT = KerT = \text{Sp}\{(1,1,1,1);\ (1,1,1,0)\}$$

Solution : Complete the two vectors $(1,1,1,1)$ and $(1,1,1,0)$ to a basis in F^4. For example choose $(1,0,0,0)$ and $(0,1,0,0)$. Then we are looking for a map T such that

$$T((1,1,1,1)) = T((1,1,1,0)) = 0; \quad T((1,0,0,0)) = (1,1,1,1); \quad T((0,1,0,0) = (1,1,1,0))$$

To give an explicit formula for T, let $(x,y,z,w) \in F^4$. Then a simple computation implies

$$(x,y,z,w) = \alpha(1,1,1,1) + \beta(1,1,1,0) + (x-z)(1,0,0,0) + (y-z)(0,1,0,0)$$

Here α and β are some elements in F which we dont care about. Hence,

$$T((x,y,z,w)) = (x-z)T((1,0,0,0)) + (y-z)T((0,1,0,0)) =$$

$$(x+y-2z, x+y-2z, x+y-2z, x-z)$$

Clearly such a T is not unique.

2) Let U, V and W denote three vector spaces over a field F. Let $T : U \mapsto V$ and $S : V \mapsto W$ be two linear transformation such that $S \circ T$ is an isomorphism. Prove that $V = Im\ T \oplus KerS$.

Solution : To prove that the sum is direct we first prove that $Im\ T \bigcap KerS = \{0\}$. Let $v \in Im\ T \bigcap KerS$. Then $S(v) = 0$, and there is $u \in U$ such that $T(u) = v$. But then $(S(T(u)) = 0$. Since $S \circ T$ is an isomorphism, we get $u = 0$ and hence $v = 0$.

We give two proofs that $V = ImT + KerS$. The first is based mainly on dimension considerations. Notice that since $S \circ T$ is an isomorphism then T is one to one and S is onto. (**It is not true that T and S must be isomorphisms!!**). For example, to show that T

is one to one, let $Tu = 0$ for some $u \in U$. Then $0 = S(Tu) = (S \circ T)(u)$ which implies that $u = 0$ since $S \circ T$ is an isomorphism. Similarly, one proves that S is onto. Hence we have

$$\text{dim} KerT = 0; \quad \text{dim} ImS = \text{dim} W; \quad \text{dim} U = \text{dim} W \tag{1}$$

The first identity follows from the fact that T is one to one. The second follows from the fact that S is onto, and the third because $S \circ T$ is an isomorphism.

Applying the two dimension Theorems, and using identities (1) we get

$$\text{dim} U = \text{dim} ImT + \text{dim} KerT = \text{dim} ImT \tag{2}$$

$$\text{dim} V = \text{dim} ImS + \text{dim} KerS = \text{dim} W + \text{dim} KerS = \tag{3}$$

$$\text{dim} U + \text{dim} KerS + \text{dim} ImT + \text{dim} KerS$$

where the last equality is obtained by plugging in identity (2). We obtain,

$$\text{dim} V = \text{dim} ImT + \text{dim} KerS$$

We also have the dimension theorem

$$\text{dim}(ImT + KerS) = \text{dim} ImT + \text{dim} KerS - \text{dim}(ImT \cap KerS) =$$

$$= \text{dim} ImT + \text{dim} KerS = \text{dim} V$$

From this we deduce that $V = ImT + KerS$, and we are done.

In the second proof the idea is to define a certain map from V to itself. To do that, let $K : W \mapsto U$ denote the inverse of the map $S \circ T$. Then $L = T \circ K \circ S$ is a linear map from V to itself.

Let $v \in V$. Then $L(v) = T((K \circ S)(v))$. Hence $L(v) \in Im\ T$. Also, since $S \circ T \circ K = I_V$ then

$$S(v - L(v)) = S(v) - (S \circ L)(v) = S(v) - (S \circ T \circ K \circ S)(v) = S(v) - S(v) = 0$$

Hence $v - L(v) \in KerS$. The identity $v = L(v) + (v - L(v))$ implies that $V = Im\ T + KerS$.

3) Write down all the matrices A of size three such that the vector space of all the solutions to the homogeneous system $Ax = 0$ will be generated by the vector $(1, 2, 3)^t$.

Solution : We first determine all the row echelon matrices with this property. Since $V = \mathrm{Sp}\{(1,2,3)^t\}$ is one dimensional, then we must have $\mathrm{rank}(A) = 2$. So we have two possible cases for the corresponding row echelon matrix. They are

$$P = \begin{pmatrix} 1 & & a \\ & 1 & b \\ & & \end{pmatrix} \qquad Q = \begin{pmatrix} 1 & a & b \\ & & 1 \\ & & \end{pmatrix}$$

Since $Q(1,2,3)^t \neq 0$, then Q is not good. On the other hand $P(1,2,3)^t = 0$ has a unique solution which is $a = -1/3$ and $b = -2/3$. In other words we get

$$P = \begin{pmatrix} 1 & & -1/3 \\ & 1 & -2/3 \\ & & \end{pmatrix}$$

The conclusion is that all matrices A which satisfies the requirements are EP where E is any invertible matrix of size three.

4) Let u, v and w be three vectors in an inner product vector space over the Real numbers. Assume that $u + v + w = 0$ and that $||u|| = ||v|| = ||w|| = 1$. Prove that $(u, v) = -\frac{1}{2}$.

Solution : We have $w = -(u + v)$. Hence

$$1 = ||w||^2 = || - (u+v)||^2 = ||(u+v)||^2 = (u+v, u+v) =$$

$$= (u,u) + (u,v) + (v,u) + (v,v) = 1 + 2(u,v) + 1$$

Here we used the fact that $||u|| = ||v|| = 1$ and that $(u,v) = (v,u)$ because it is a Real inner product space. Comparing both sides of the above equation we get $(u,v) = -\frac{1}{2}$.

5) Let A be a matrix of size four whose entries are all ± 1. Prove that $8|\det A$.

Solution : If $\det A = 0$ the result follows. Multiplying each column by ± 1, the value of $|A|$ is changed by ± 1. Hence we may assume that all the entries of the first row of A are all one. Multiplying the last three rows by ± 1, it is enough to prove the statement for

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & \pm 1 & \pm 1 & \pm 1 \\ -1 & \pm 1 & \pm 1 & \pm 1 \\ -1 & \pm 1 & \pm 1 & \pm 1 \end{pmatrix}$$

Add the first row to each of the three last rows. Then

$$|A| = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 2a_1 & 2a_2 & 2a_3 \\ 0 & 2a_4 & 2a_5 & 2a_6 \\ 0 & 2a_7 & 2a_8 & 2a_9 \end{vmatrix} = \begin{vmatrix} 2a_1 & 2a_2 & 2a_3 \\ 2a_4 & 2a_5 & 2a_6 \\ 2a_7 & 2a_8 & 2a_9 \end{vmatrix}$$

From which the claim easily follows.

בחינה באלגברה לינארית 1

דוד גינזבורג

משך הבחינה שלוש שעות.
אין להשתמש בכל חומר עזר לרבות מחשבונים.
יש לענות על כל השאלות.

שאלה 1

לכל $a,b,c,d,e,f,g \in \mathbf{R}$ לחשב

$$\begin{vmatrix} a & b & b \\ c & d & e \\ f & g & g \end{vmatrix} + \begin{vmatrix} a & b & b \\ e & c & d \\ f & g & g \end{vmatrix} + \begin{vmatrix} a & b & b \\ d & e & c \\ f & g & g \end{vmatrix}$$

שאלה 2

יהי V מרחב וקטורי ממימד סופי. יהיו U,W תתי מרחב של V. נניח כי קימת פונקציה $f : V \to \mathbf{R}$ המקימת $f(u) < f(w)$ לכל $0 \neq u \in U$ ולכל $0 \neq w \in W$. להוכיח כי

$$\text{dim}W + \text{dim}U \leq \text{dim}V$$

שאלה 3

יהי $B = \{v_1, v_2, v_3\}$ בסיס למרחב F^3. תהי $T : F^3 \to F^3$ העתקה לינארית המקימת

$$[T]_B = \begin{pmatrix} -1 & -1 & -3 \\ -5 & -2 & -6 \\ 2 & 1 & 3 \end{pmatrix}$$

למצוא את הגרעין של T.

שאלה 4

תהי $A = (a_{i,j})$ מטריצה מסדר n המוגדרת באופן הבא. יהי c מספר ממשי. אם $i+j$ הינו מספר זוגי אז $a_{i,j} = c$ ואם $i+j$ איזוגי אז $a_{i,j} = 0$. למצוא k מינימלי כך שהקבוצה $\{A, A^2, \ldots, A^k\}$ תהיה תלויה לינארית.

שאלה 5

תהי $T : Mat_{2\times2}(\mathbf{Q}) \to Mat_{2\times2}(\mathbf{Q})$ העתקה לינארית המקימת $T(AB) = T(A)T(B)$ לכל $A, B \in Mat_{2\times2}(\mathbf{Q})$. להוכיח כי $T\left(\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right) \neq I$.

<u>בחינה באלגברה לינארית 1</u>

דוד גינזבורג

מועד ב-2011

משך הבחינה שלוש שעות. הניקוד על כל שאלה שווה. אין להשתמש בכל חומר עזר לרבות מחשבונים. יש לענות על כל השאלות. אם לא נאמר אחרת, אין קשר בין הסעיפים השונים.

<u>שאלה 1</u>

להוכיח את המשפט הבא:
תהי $Ax = b$ מערכת משוואות לינארית. להוכיח כי למערכת יש פתרון אם ורק אם דרגת המטריצה המצומצמת של המערכת שווה לדרגת המטריצה המורחבת של המערכת.

<u>שאלה 2</u>

א. תהי $T : \mathrm{Mat}_{3\times3}(\mathbf{R}) \mapsto \mathbf{R}$ ההעתקה הלינארית המוגדרת באופן הבא. אם $A = (a_{i,j}) \in \mathrm{Mat}_{3\times3}(\mathbf{R})$ אז $T(A) = \sum_{i,j=1}^{3} a_{i,j}$. למצוא בסיס עבור $\ker T$.
ב. תהי $\{v_1, \ldots, v_m\}$ קבוצה בלתי תלויה של וקטורים ב- $\mathbf{R}^n$. נסתכל על קבוצה זו כעל קבוצת וקטורים במרחב $\mathbf{C}^n$. האם במרחב זה , קבוצת וקטורים זו הינה בהכרח בלתי תלויה?

<u>שאלה 3</u>

א.(15 נקודות) יהי $F = \mathbf{Z}_5$ השדה הסופי עם חמישה איברים. לרשום מערכת משוואות הומוגנית שמרחב הפתרונות שלה נתון על ידי

$$Sp\{(1,2,3);\ (1,1,0);\ (1,1,1)\}$$

ב.(10 נקודות) למצוא את כל המספרים השלמים b, a ו- c כך שהדטרמיננטה הבאה

$$\begin{vmatrix} a+b & c & c \\ a & b+c & a \\ b & b & a+c \end{vmatrix}$$

תתחלק ללא שארית בשמונה.

<u>שאלה 4</u>

יהי V מרחב וקטורי ממימד n. תהי $T : V \to V$ העתקה לינארית. להוכיח כי לכל $k \geq n$ החתוך של $kerT^k$ עם ImT^k הינו אפס.

בהצלחה!

Solution of Moed A in Linear Algebra 2011

David Ginzburg

2) a) Let $S_1, S_2 \in U \subset Hom_F(V,V)$ and let $\alpha_1, \alpha_2 \in F$. Then, by definition of U, we have $S_1 \circ T = S_2 \circ T = 0$. Hence $(\alpha_1 S_1 + \alpha_2 S_2) \circ T = \alpha_1 S_1 \circ T + \alpha_2 S_2 \circ T = 0 + 0 = 0$. Hence $\alpha_1 S_1 + \alpha_2 S_2 \in U$.
b) Let $\mathcal{B}$ be a basis of V. We know that the map $S \mapsto [S]_{\mathcal{B}}$ defines an isomorphism between $Hom_F(V,V)$ and $Mat_{2\times 2}(F)$. We also know that $[S \circ T]_{\mathcal{B}} = [S]_{\mathcal{B}}[T]_{\mathcal{B}}$. Let $A = [T]_{\mathcal{B}}$. Define

$$U' = \{B \in Mat_{2\times 2}(F) \ : \ BA = 0\}$$

Then from the above we deduce that U is isomorphic to U' and hence dim $U' =$ dim U. To compute dim U' we consider several cases according to the rank of the matrix A:
1) Assume rank$A = 2$. Then A is invertible, and hence $BA = 0$ implies $BAA^{-1} = 0$ and $B = 0$. Hence dim $U' = 0$ in this case.
2) Assume rank$A = 0$. Then $A = 0$ and therefore $U' = Mat_{2\times 2}(F)$. Hence dim $U' = 4$.
3) Assume rank$A = 1$. There are two cases. First assume that $A = \begin{pmatrix} 0 & 0 \\ a & b \end{pmatrix}$ and a and b are not both zero. Let $B = \begin{pmatrix} x & y \\ z & r \end{pmatrix}$. Then

$$BA = \begin{pmatrix} x & y \\ z & r \end{pmatrix} \begin{pmatrix} 0 & 0 \\ a & b \end{pmatrix} = \begin{pmatrix} ya & yb \\ ra & rb \end{pmatrix} = 0$$

Since either a or b are not zero we deduce that $y = r = 0$. Hence

$$U' = \{ \begin{pmatrix} x & 0 \\ z & 0 \end{pmatrix} \ : x, z \in F\}$$

Thus dim $U' = 2$. The second case is when $A = \begin{pmatrix} a & b \\ \alpha a & \alpha b \end{pmatrix}$ where $\alpha \in F$ and a and b are not both zero. Thus

$$BA = \begin{pmatrix} x & y \\ z & r \end{pmatrix} \begin{pmatrix} a & b \\ \alpha a & \alpha b \end{pmatrix} = \begin{pmatrix} a(x+\alpha y) & b(x+\alpha y) \\ a(z+\alpha r) & b(z+\alpha r) \end{pmatrix} = 0$$

Since either a or b are not zero we deduce that $x + \alpha y = z + \alpha r = 0$. Hence

$$U' = \{\begin{pmatrix} -\alpha y & y \\ -\alpha r & r \end{pmatrix} : y, r \in F\}$$

Thus $\dim U' = 2$.

3) a) The statement is true. Apply the dimension theorem $\dim V = \dim \ker T + \dim \operatorname{Im} T$. Clearly, $\dim \operatorname{Im} T \leq \dim W = 4$. Hence $6 = \dim V \leq \dim \ker T + 4$ which implies that $\dim \ker T \geq 2$. Thus $\ker T$ is not zero.

b) Here the answer depends on q. Using the additive in rows property of the determinant, we obtain

$$\begin{vmatrix} 2 & 2 & 3 & 4 \\ 3q+4 & 4q+4 & 5q+6 & 2q+8 \\ 0 & -1 & 1 & 8 \\ 5 & 9 & 7 & a \end{vmatrix} = \begin{vmatrix} 2 & 2 & 3 & 4 \\ 3q & 4q & 5q & 2q \\ 0 & -1 & 1 & 8 \\ 5 & 9 & 7 & a \end{vmatrix} + \begin{vmatrix} 2 & 2 & 3 & 4 \\ 4 & 4 & 6 & 8 \\ 0 & -1 & 1 & 8 \\ 5 & 9 & 7 & a \end{vmatrix} = q \begin{vmatrix} 2 & 2 & 3 & 4 \\ 3 & 4 & 5 & 2 \\ 0 & -1 & 1 & 8 \\ 5 & 9 & 7 & a \end{vmatrix}$$

We also used the fact that if two rows in a matrix are proportional, then the determinant of this matrix is zero. Consider two cases. If $q = 0$ then for all a we obtain $|A| = 0$ and the answer is no.

Assume that $q \neq 0$. Thus the question we need to answer is if there is a unique rational number a such that

$$\begin{vmatrix} 2 & 2 & 3 & 4 \\ 3 & 4 & 5 & 2 \\ 0 & -1 & 1 & 8 \\ 5 & 9 & 7 & a \end{vmatrix} = 1$$

Expand this determinant according to the last row. Then we obtain

$$a \begin{vmatrix} 2 & 2 & 3 \\ 3 & 4 & 5 \\ 0 & -1 & 1 \end{vmatrix} = 1 - \alpha$$

Here α is a rational number, in fact an integer, which is obtained from all the other terms in the development of the determinant. Thus, there will be a unique rational a if the determinant

$$\begin{vmatrix} 2 & 2 & 3 \\ 3 & 4 & 5 \\ 0 & -1 & 1 \end{vmatrix} = \begin{vmatrix} 2 & 3 \\ 3 & 5 \end{vmatrix} + \begin{vmatrix} 2 & 2 \\ 3 & 4 \end{vmatrix}$$

is not zero. This is clearly true, and hence there is a unique a such that $|A| = q$.

c) The statement is true. Apply transpose to $A = AA^t$ to obtain $A^t = (AA^t)^t = AA^t = A$. Hence $A = AA^t = AA = A^2$.

4) Since the rank of a matrix is equal to the rank of its transpose, and since we can rearrange the rows and columns as we want, its enough to compute the rank of the matrix

$$\begin{pmatrix} 1 & 2 & 1 \\ 2 & 5 & 1 \\ -1 & a & -6 \\ a & -1 & 10 \end{pmatrix}$$

We apply the following row operations: $R_2 \to R_2 - 2R_1$; $R_3 \to R_3 + R_1$ and $R_4 \to R_4 - aR_1$. Then

$$\begin{pmatrix} 1 & 2 & 1 \\ 2 & 5 & 1 \\ -1 & a & -6 \\ a & -1 & 10 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 0 & a+2 & -5 \\ 0 & -2a-1 & 10-a \end{pmatrix}$$

Next we preform $R_3 \to R_3 - (a+2)R_2$ and $R_4 \to R_4 + (1+2a)R_2$. We obtain

$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 0 & a+2 & -5 \\ 0 & -2a-1 & 10-a \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & a-3 \\ 0 & 0 & -3(a-3) \end{pmatrix}$$

Hence, if $a = 3$ the rank is 2 and if $a \neq 3$ the rank is 3.

Solution of Moad A in Linear Algebra

David Ginzburg

1) We know that the systems $Ax = 0$ and $Bx = 0$ will have exactly the same set of solutions if and only if the row echelon forms of these two matrices is the same. We start with the matrix B. Preforming the row operations $R_1 \to R_2 - R_1$, then $R_3 \to R_3 - 2R_1$, and then $R_3 \to R_3 - R_2$ we obtain the matrix

$$B_1 = \begin{pmatrix} 1 & \gamma & 2 & \delta & 1 \\ 0 & \gamma & \gamma & -\gamma & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

On A we operate by $R_3 \to R_3 - R_1$ and then by $R_3 \to R_3 - R_2$ and we obtain the matrix

$$A_1 = \begin{pmatrix} 1 & \alpha & 2 & \beta - 1 & 1 \\ 0 & \alpha & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Hence, to get from these two matrices the same echelon form matrix, we must have $\alpha \neq 0$ and $\gamma \neq 0$. Assuming that we operate on B_1 by $R_1 \to R_1 - R_2$ and then $R_2 \to \gamma^{-1} R_2$. On A_1 we operate by $R_1 \to R_1 - R_2$. We obtain the matrices

$$A_2 = \begin{pmatrix} 1 & 0 & 1 & \beta & 1 \\ 0 & \alpha & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad B_2 = \begin{pmatrix} 1 & 0 & 2-\gamma & \gamma+\delta & 1 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Therefore, $\alpha = 1$, $2 - \gamma = 1$ and $\beta = \gamma + \delta$. Thus, the values which will give the same set of solutions are $\alpha = \gamma = 1$ and $\beta = \delta + 1$.

2) a) Let $v \in W \cap U$. We need to prove that $Tv \in W \cap U$. Since $v \in W$, and it is given that W is T invariant, then $Tv \in W$. Similarly, since $v \in U$, and it is given that U is T invariant, then $Tv \in U$. Hence $Tv \in W \cap U$.

b) Suppose that $\dim V = n$. Let W denote an arbitrary subspace of V whose dimension is $n-2$. Pick a base $\{w_1, \ldots, w_{n-2}\}$ of W. By a theorem we proved in class, we can complete it to a base of V. So assume that $\{w_1, \ldots, w_{n-2}, u_1, u_2\}$ is a base for V. Let $U_1 = Sp\{w_1, \ldots, w_{n-2}, u_1\}$ and $U_2 = Sp\{w_1, \ldots, w_{n-2}, u_2\}$. Then $dim U_i = n-1$ for $i = 1, 2$. Since it is given that every subspace of V whose dimension is $n-1$ is T invariant, it follows that U_1 and U_2 are T invariant. By part **a**) we have that $W = U_1 \cap U_2$ is T invariant. Thus we proved that every subspace of V whose dimension is $\dim V - 2$, is T invariant. Continuing by induction we deduce that every subspace of dimension one is T invariant. Choose a base $B = \{v_1, \ldots, v_n\}$ of V. Since $Sp\{v_i\}$ is T invariant, it follows that $Tv_i = \alpha_i v_i$. Thus with respect to the base B we have

$$[T]_B = \begin{pmatrix} \alpha_1 & 0 & 0 & \cdots & 0 \\ 0 & \alpha_2 & 0 & \cdots & 0 \\ 0 & 0 & \alpha_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \alpha_n \end{pmatrix}$$

3) First, for all $1 \le i \le n-1$, we apply the row operations $R_i \to R_i - R_n$. This does not change the value of the determinant. Hence, we obtain that $|A| = |B|$, where

$$B = \begin{pmatrix} b & 0 & 0 & \ldots & 0 & -b \\ 0 & b & 0 & \ldots & 0 & -b \\ 0 & 0 & b & \ldots & 0 & -b \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a & a & a & \ldots & a & a+b \end{pmatrix}$$

Next, for each $1 \le i \le n-1$, preform the column operation $C_i \to C_i + C_n$. Once again, this does not change the value of the determinant, and we obtain that $|A| = |C|$ where C is now lower diagonal, whose diagonal is $(b, b, b, \ldots, b, na + b)$. Therefore, the determinant of C is the product of all diagonal elements, and we obtain that $|A| = (na + b)b^{n-1}$.

Another way is to argue by induction on n. Denote $\Delta_n = |A|$. Let B denote the matrix obtained from A by the row operation $R_1 \to R_1 - R_2$. Then $|A| = |B|$. The first row of B is $\begin{pmatrix} b & -b & 0 & \cdots & 0 \end{pmatrix}$. All other rows of B are as the rows of A. Develop the determinant of B using the first row. We obtain $\Delta_n = b|C_{n-1}| + b|D_{n-1}|$. Here C_{n-1} is the matrix of size $n-1$ whose diagonal elements are $a+b$ and all other entries are a. Therefore, by induction, we have $|C_{n-1}| = \Delta_{n-1}$. The matrix D_{n-1} is defined as follows

$$D_{n-1} = \begin{pmatrix} a & a & a & \ldots & a \\ a & a+b & a & \ldots & a \\ a & a & a+b & \ldots & a \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a & a & a & \ldots & a+b \end{pmatrix}$$

In other words, all the diagonal elements of D_{n-1} except the $(1,1)$ entry, are $a+b$. All other entries of D_{n-1} are a. For $2 \le i \le n-1$, preform on this matrix the row operations $R_i \to R_i - R_1$. Each of these operations does not change the value of the determinant, and therefore $|D_{n-1}| = |E_{n-1}|$, where

$$E_{n-1} = \begin{pmatrix} a & a & a & \dots & a \\ 0 & b & 0 & \dots & 0 \\ 0 & 0 & b & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & b \end{pmatrix}$$

Since E_{n-1} is upper diagonal its determinant is the product of all diagonal elements.

Overall we obtain $\Delta_n = b\Delta_{n-1} + ab^{n-1}$. To compute this explicitly, we use induction to obtain $\Delta_{n-1} = b\Delta_{n-2} + ab^{n-2}$. Plugging this above, we obtain $\Delta_n = b^2\Delta_{n-2} + 2ab^{n-1}$. Continuing this process by induction we obtain $\Delta_n = b\Delta_1 + (n-1)ab^{n-1}$. Since $\Delta_1 = a + b$, we obtain $\Delta_n = b^n + nab^{n-1}$.

4) We have $T(1) = 1 + x^2$. Hence if we choose the vector $1 + x^2$ to be the first vector in the base C, then by definition we have

$$[T]_B^C = \begin{pmatrix} 1 & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix}$$

Here the stars indicates entries to be determined. Next we have $T(x + 1) = 2 + x + 2x^2$. Write $2 + x + 2x^2 = \alpha(1 + x^2) + p(x)$. Clearly $p(x) = x$ and $\alpha = 2$. Therefore, if we choose the second vector in C to be x, then by definition

$$[T]_B^C = \begin{pmatrix} 1 & 2 & * \\ 0 & 1 & * \\ 0 & 0 & * \end{pmatrix}$$

Notice that the set $\{1+x^2, x\}$ is linearly independent. Finally, we have $T(x^2+1) = 1-x+2x^2$. Write $1 - x + 2x^2 = \alpha(1 + x^2) + \beta x + q(x)$. Then $\alpha = 2$, $\beta = -1$, and $q(x) = -1$. Thus, if we choose as the third element of C the vector -1, we get

$$[T]_B^C = \begin{pmatrix} 1 & 2 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

It is clear that $C = \{x^2 + 1, x, -1\}$ is a base for $P_2(x)$.

5) a) To prove the statement we need to prove that if $\alpha_1 v_1 + \alpha_2 v_2 = 0$, then $\alpha_1 = \alpha_2 = 0$. Since $|a_{1,1}| > |a_{2,1}|$ then $a_{1,1} \neq 0$. Therefore $v_1 \neq 0$. Similarly, $v_2 \neq 0$. Hence, if $\alpha_1 = 0$ then $\alpha_1 v_1 + \alpha_2 v_2 = 0$ implies that $\alpha_2 = 0$. Similarly, if $\alpha_2 = 0$ then $\alpha_1 = 0$. Therefore we may assume that both α_1 and α_2 are nonzero. We shall derive a contradiction. From $\alpha_1 v_1 + \alpha_2 v_2 = 0$ we deduce that $\alpha_1 a_{1,1} + \alpha_2 a_{2,1} = 0$. Hence, we obtain $|\alpha_1||a_{1,1}| = |\alpha_2||a_{2,1}|$.

From the fact that $|a_{1,1}| > |a_{2,1}|$ and that $\alpha_2 \neq 0$ it follows that $|\alpha_1||a_{1,1}| = |\alpha_2||a_{2,1}| < |\alpha_2||a_{1,1}|$. Since $a_{1,1} \neq 0$, it follows that $|\alpha_1| < |\alpha_2|$. Similarly, using the fact that $\alpha_1 \neq 0$, the equation $\alpha_1 a_{1,2} + \alpha_2 a_{2,2} = 0$ implies that $|\alpha_2| < |\alpha_1|$. Thus we derived a contradiction, and hence the set $\{v_1, v_2\}$ is independent.

Another way to prove it is to observe that the linear system $\alpha_1 a_{1,1} + \alpha_2 a_{2,1} = 0$ and $\alpha_1 a_{1,2} + \alpha_2 a_{2,2} = 0$ can be written as

$$\begin{pmatrix} a_{1,1} & a_{2,1} \\ a_{1,2} & a_{2,2} \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = 0$$

Let A denote the left most matrix in the above equality. Therefore, the set $\{v_1, v_2\}$ is linearly independent if and only if the above system has only the trivial solution if and only if $\det A = 0$. By the triangular inequality we have $|\det A| = |a_{1,1}a_{2,2} - a_{1,2}a_{2,1}| \geq ||a_{1,1}a_{2,2}| - |a_{1,2}a_{2,1}|| > 0$ where the last inequality follows from the fact that $|a_{1,1}| > |a_{2,1}|$ and $|a_{2,2}| > |a_{1,2}|$. Thus $\det A \neq 0$.

b) Since $\ker S$ is not contained in $\ker T$ it follows that $T \neq 0$. Therefore $\dim \mathrm{Im} T \geq 1$, and since it is given that $\dim W = 1$, it follows that $\mathrm{Im} T = W$. It follows from the dimension theorem ($\dim V = \dim\ker T + \dim \mathrm{Im} T$) that $\dim\ker T = \dim V - 1$. Using again the fact that $\ker S$ is not contained in $\ker T$, we deduce that $\ker S + \ker T = V$. Indeed, since $\dim V = n$, we can choose a base $\{v_1, \ldots, v_{n-1}\}$ for $\ker T$. Let u be a vector in $\ker S$ not in $\ker T$. Then $\{v_1, \ldots, v_{n-1}, u\}$ is a base for V. Hence, from the first dimension theorem,

$$\dim V = \dim(\ker S + \ker T) = \dim\ker S + \dim\ker T - \dim(\ker S \cap \ker T) =$$

$$= \dim\ker S + \dim V - 1 - \dim(\ker S \cap \ker T)$$

from which the result follows.

Solution of Moed B in Linear Algebra 2011

David Ginzburg

2) a) Suppose that $A = (a_{i,j}) \in \ker T$. Then $T(A) = \sum_{i,j=1}^{3} a_{i,j} = 0$. Notice that T is onto, and hence, by the dimension theorem $\dim\ker T = 9 - 1 = 8$. Thus, a base is given by

$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \begin{pmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}; \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}; \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

b) The answer is yes. To prove it, for $1 \le k \le m$, let $\alpha_k \in \mathbf{C}$ be such that $\alpha_1 v_1 + \cdots + \alpha_m v_m = 0$. Write $\alpha_k = \beta_k + i\gamma_k$ where $\beta_k, \gamma_k \in \mathbf{R}$. Here i is the complex imaginary number. Thus, we have

$$(\beta_1 v_1 + \cdots \beta_m v_m) + i(\gamma_1 v_1 + \cdots + \gamma_m v_m) = 0$$

Since all entries of v_k are real, the above equality implies that $\beta_1 v_1 + \cdots \beta_m v_m = 0$ and $\gamma_1 v_1 + \cdots + \gamma_m v_m = 0$. Since $\{v_1, \ldots, v_m\}$ is linearly independent, it follows that all β_k and all γ_k, and hence all α_k are zeros.

3) a) A vector is in the subspace $\mathrm{Sp}\{(1,2,3);\ (1,1,0);\ (1,1,1)\}$ if and only if it is of the form

$$x_1(1,2,3) + x_2(1,1,0) + x_3(1,1,1) = (x_1 + x_2 + x_3, 2x_1 + x_2 + x_3, 3x_1 + x_3)$$

It is not hard to check that the given three vectors are linearly independent. Hence, we are looking for a system $Ax = 0$ such that $Av = 0$ for all $v \in \mathbf{Z}_5^3$. The only such matrix is $A = 0$.

b) This is easily done by direct calculation. Expand along the first column. We obtain

$$\begin{vmatrix} a+b & c & c \\ a & b+c & a \\ b & b & a+c \end{vmatrix} = (a+b)[(b+c)(a+c) - ab] - a[c(a+c) - bc] + b[ca - c(b+c)]$$

Computing the right hand side, one gets that the determinant is equal to $4abc$. Hence, the answer is all triple numbers a, b and c such that at least one of them is even.

4) Let v be a vector in the intersection. Then $T^k v = 0$ and there exists $u \in V$ such that $v = T^k u$. Consider the set of vectors $\{u, Tu, \ldots, T^k u\}$. Since $k \geq n$, then this set is dependent. Hence, there exist $\alpha_i \in F$, not all zero such that

$$\alpha_0 u + \alpha_1 Tu + \cdots + \alpha_k T^k u = 0 \tag{1}$$

If $u = 0$, then $v = 0$ and we are done. Hence assume that $u \neq 0$. Then $T^{2k} u = T^k(T^k v) = 0$. Hence, there exist a number $m \geq 0$ such that $T^m u \neq 0$ but $T^{m+1} u = 0$. If $m + 1 \leq k$, then $v = T^k u = T^{k-m-1}(T^{m+1} u) = 0$ and we are done. Finally, assume that $k < m + 1$. We shall derive a contradiction. Apply T^m to (1). Since $T^{m+1} u = 0$, we obtain $\alpha_0 T^m u = 0$. Since $T^m u \neq 0$ we obtain $\alpha_0 = 0$. Then (1) is given by $\alpha_1 Tu + \cdots + \alpha_k T^k u = 0$. Apply T^{m-1} to this equation we obtain $\alpha_1 T^m u = 0$, and hence $\alpha_1 = 0$. Since $k < m + 1$ we can repeat this process, where the last step is by applying T^{m-k}. Thus we obtain that for all $0 \leq i \leq k$ we have $\alpha_i = 0$. This is a contradiction.

Solution for Final in Linear Algebra 1

Moad A 2009

1) For simplicity we shall assume that n is even. That is $n = 2m$. The case when n is odd is the same. It is given, that for $1 \le i \le 2m$ we have

$$\sum_{j=1}^{m} a_{i,2j-1} = \sum_{j=1}^{m} a_{i,2j}$$

Expressing $a_{i,1}$ in term of the others we obtain

$$a_{i,1} = \sum_{j=1}^{m} a_{i,2j} - \sum_{j=2}^{m} a_{i,2j-1}$$

Plugging this into the matrix A we obtain

$$|A| = \begin{vmatrix} \sum_{j=1}^{m} a_{1,2j} - \sum_{j=2}^{m} a_{1,2j-1} & a_{1,2} & \cdots & a_{1,2m} \\ \sum_{j=1}^{m} a_{2,2j} - \sum_{j=2}^{m} a_{2,2j-1} & a_{2,2} & \cdots & a_{2,2m} \\ \vdots & \vdots & & \vdots \\ \sum_{j=1}^{m} a_{2m,2j} - \sum_{j=2}^{m} a_{2m,2j-1} & a_{2m,2} & \cdots & a_{2m,2m} \end{vmatrix}$$

Using the additive in columns property of the determinant, we obtain

$$|A| = \sum_{j=1}^{m} \begin{vmatrix} a_{1,2j} & a_{1,2} & \cdots & a_{1,2m} \\ a_{2,2j} & a_{2,2} & \cdots & a_{2,2m} \\ \vdots & \vdots & & \vdots \\ a_{2m,2j} & a_{2m,2} & \cdots & a_{2m,2m} \end{vmatrix} - \sum_{j=2}^{m} \begin{vmatrix} a_{1,2j-1} & a_{1,2} & \cdots & a_{1,2m} \\ a_{2,2j-1} & a_{2,2} & \cdots & a_{2,2m} \\ \vdots & \vdots & & \vdots \\ a_{2m,2j-1} & a_{2m,2} & \cdots & a_{2m,2m} \end{vmatrix}$$

In each summand we have a determinant of a matrix which has two equal columns. Such a determinant is zero, and hence $|A| = 0$.

2) Multiplying both sides by $I - A$, we look for a number c which satisfies the identity $I = (I - A)(I - cA)$. This is the same as $I = I - (1 + c)A + cA^2$, or $(1 + c)A = cA^2$. From matrix multiplication it follows that the matrix A^2 is the matrix which has the value n at each of its entry. Hence, the equality $(1 + c)A = cA^2$ is equivalent to $(1 + c) = cn$. Hence, $c = \frac{1}{n-1}$.

3) Preforming the row operations $R_2 \to R_2 - (b+c)R_1$ and $R_3 \to R_3 - bcR_1$ we obtain

$$\begin{pmatrix} 1 & 1 & 1 \\ b+c & a+c & a+b \\ bc & ac & ab \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & 1 \\ 0 & a-b & a-c \\ 0 & c(a-b) & b(a-c) \end{pmatrix}$$

Next preform the row operation $R_3 \to R_3 - cR_2$. We obtain the matrix

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & a-b & a-c \\ 0 & 0 & (b-c)(a-c) \end{pmatrix}$$

Recall that row operation do not change the rank of the matrix. Therefore we have the following cases:
1) If $a \neq b$; $a \neq c$ and $b \neq c$ then the rank is three.
2) If $a = b = c$ then the rank is one.
3) In all other cases the rank is two.

4) It is easier to start with the image of T. In class we proved that $\mathrm{Im}T$ is spanned by an image of a basis. Let $B = \{1, x, x^2, \ldots, x^n\}$ denote the standard basis for V. We have

$$T(1) = 0, \qquad T(x) = x + 1 - x = 1, \qquad T(x^2) = (x+1)^2 - x^2 = 2x + 1$$

In general, for each $0 \leq i \leq n$ we have $T(x^i) = (x+1)^i - x^i$. Let $q_i(x) = (x+1)^i - x^i$. Since the factor x^i cancels, then $q_i(x)$ is a polynomial whose degree is $i-1$. In other words, we have $q_i(x) = a_{i-1}x^{i-1} + r_i(x)$, where $a_{i-1} \neq 0$, and $r_i(x)$ is a polynomial of degree at most $i-2$. From the above we have $\mathrm{Im}T = \mathrm{Sp}\{q_1(x), q_2(x), \ldots, q_n(x)\}$. Using induction we will prove that for all $0 \leq i \leq n$ we have $\mathrm{Sp}\{q_1(x), q_2(x), \ldots, q_i(x)\} = \mathrm{Sp}\{1, x, x^2, \ldots, x^{i-1}\}$. This clearly holds for $i = 0$, and assume it is true for $i-1$. Since each $q_i(x)$ is a polynomial of degree $i-1$, then clearly $\mathrm{Sp}\{q_1(x), q_2(x), \ldots, q_i(x)\} \subset \mathrm{Sp}\{1, x, x^2, \ldots, x^{i-1}\}$. By induction $\mathrm{Sp}\{q_1(x), q_2(x), \ldots, q_{i-1}(x)\} = \mathrm{Sp}\{1, x, x^2, \ldots, x^{i-2}\}$. Hence, $\mathrm{Sp}\{q_1(x), q_2(x), \ldots, q_i(x)\} \supset \mathrm{Sp}\{1, x, x^2, \ldots, x^{i-2}\}$. Also, from $q_i(x) = a_{i-1}x^{i-1} + r_i(x)$, since $a_{i-1} \neq 0$, then $x^{i-1} = \frac{1}{a_{i-1}}(q_i(x) + r_i(x)) \in \mathrm{Sp}\{q_1(x), q_2(x), \ldots, q_i(x)\}$. Hence we get $\mathrm{Sp}\{q_1(x), q_2(x), \ldots, q_i(x)\} = \mathrm{Sp}\{1, x, x^2, \ldots, x^{i-1}\}$. Plugging $i = n$ we get that the $\mathrm{Im}T = \mathrm{Sp}\{q_1(x), q_2(x), \ldots, q_n(x)\} = \mathrm{Sp}\{1, x, x^2, \ldots, x^{n-1}\}$. From this we deduce that $\dim\mathrm{Im}T = n$.

To compute the kernel, we first use the dimension theorem $\dim V = \dim\ker T + \dim\mathrm{Im}T$ to deduce that $\dim\ker T = 1$. Since we saw that $T(1) = 0$, it follows that $\ker T = \mathrm{Sp}\{1\}$.

5) Let $v \in \ker T$. Then $Tv = 0$. Hence, $0 = STv = Sv$ where the last equality follows from the identity $ST = S$. Hence $v \in \ker S$, and hence $\ker T \subset \ker S$. Similarly, we prove $\ker S \subset \ker T$, and hence $\ker T = \ker S$.

Next, $T^2 = TT = TST = TS = T$. Here, the second and the last equality follows from the identity $T = TS$, and the third equality follows from $ST = S$. The identity $S^2 = S$ is obtained in the same way.

Solution for Final in Linear Algebra 1

Moad B 2009

1) Using the property of addition by rows the sum of the three determinants is equal to

$$\begin{vmatrix} a & b & b \\ c+e+d & c+e+d & c+e+d \\ f & g & g \end{vmatrix}$$

Since the last two columns are equal we get zero.

2) We claim that $U \cap W = \{0\}$. Indeed, if $v \in U \cap W$ and $v \neq 0$, then it follows that $f(v) < f(v)$, which is impossible. Hence $U \cap W = \{0\}$. Thus, if follows from the dimension theorem

$$\dim V \geq \dim(U + W) = \dim U + \dim W - \dim(U \cap W) = \dim U + \dim W$$

Here, the first inequality follows from the fact that $U + W$ is a subspace of V, and the last equality follows from the fact that $U \cap W = \{0\}$.

3) A vector (α, β, γ) is in the kernel of T if and only if

$$\begin{pmatrix} -1 & -1 & -3 \\ -5 & -2 & -6 \\ 2 & 1 & 3 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = 0 \tag{1}$$

To see this, apply the definition of a matrix representing a linear transformation to obtain

$$\begin{aligned} Tv_1 &= -v_1 - 5v_2 + 2v_1 \\ Tv_2 &= -v_1 - 2v_2 + v_3 \\ Tv_3 &= -3v_1 - 6v_2 + 3v_3 \end{aligned}$$

If $v = \alpha v_1 + \beta v_2 + \gamma v_3$ is in the kernel of T then $Tv = 0$. Hence

$$0 = Tv = T(\alpha v_1 + \beta v_2 + \gamma v_3) = \alpha Tv_1 + \beta Tv_2 + \gamma Tv_3 =$$

$$= \alpha(-v_1 - 5v_2 + 2v_1) + \beta(-v_1 - 2v_2 + v_3) + \gamma(-3v_1 - 6v_2 + 3v_3)$$

From this it follows that we need to solve the system (2). Applying row operations, we obtain the system

$$\begin{pmatrix} -1 & -1 & -3 \\ & 1 & 3 \\ & & \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = 0$$

From this we deduce that $\ker T = \mathrm{Sp}\{(0,-3,1)\}$.

4) (**To make things clear I wrote a detailed answer, much more then needed.**)

To see what is going on, we first write the matrix A for $n = 4$ and $n = 5$. We have

$$A = \begin{pmatrix} c & & c & \\ & c & & c \\ c & & c & \\ & c & & c \end{pmatrix} \qquad A = \begin{pmatrix} c & & c & & c \\ & c & & c & \\ c & & c & & c \\ & c & & c & \\ c & & c & & c \end{pmatrix}$$

If $c = 0$, then $A = 0$ and so $k = 1$ is the minimal number. Assume that $c \neq 0$. We are looking for the smallest number k, such that the equation

$$\alpha_1 A + \alpha_2 A^2 + \cdots + \alpha_k A^k = 0 \tag{2}$$

has a nontrivial solution. Let $A^r = (a_{i,j}(r))$. In other words, $a_{i,j}(r)$ denotes the (i,j)-th entry of the matrix A^r. The following is proved using matrix multiplication:

1) For all r, if $i + j$ is odd, then $a_{i,j}(r) = 0$.

2) If $n = 2m$, and $i + j$ is even then $a_{i,j}(r) = a(r,c)$, where $a(r,c)$ is a fixed *nonzero* number depending only on r and c and not on i or j. For example $a(1,c) = c$; $a(2,c) = mc^2$; $a(3,c) = m^3c^4$ etc.

3) If $n = 2m + 1$, and $i + j$ is even, then there are two cases for the values of $a_{i,j}(r)$. The first, which we denote by $a(r,c)$ corresponds to the nonzero entries in the odd rows of A^r, and the other we denote by $b(r,c)$ which corresponds to the nonzero entry in the even rows of A^r.

For example,

$$A^2 = \begin{pmatrix} 2c^2 & & 2c^2 & \\ & 2c^2 & & 2c^2 \\ 2c^2 & & 2c^2 & \\ & 2c^2 & & 2c^2 \end{pmatrix} \qquad A^2 = \begin{pmatrix} 3c^2 & & 3c^2 & & 3c^2 \\ & 2c^2 & & 2c^2 & \\ 3c^2 & & 3c^2 & & 3c^2 \\ & 2c^2 & & 2c^2 & \\ 3c^2 & & 3c^2 & & 3c^2 \end{pmatrix}$$

Thus, from the definition of scalar multiplication and from the definition of addition of matrices, it follows that the matrix on the left hand side of (2) has the same structure. More precisely, in the matrix $\alpha_1 A + \alpha_2 A^2 + \cdots + \alpha_k A^k$ we have:

1) If $i+j$ is odd then the (i,j)-th entry of this matrix is zero.
2) If n is even, and $i+j$ is even then the (i,j)-th entry of this matrix is $\alpha_1 a(1,c)+\alpha_2 a(2,c)+\cdots+\alpha_k a(k,c)$.
3) If n is odd, and $i+j$ is even then the (i,j)-th entry of this matrix is either $\alpha_1 a(1,c)+\alpha_2 a(2,c)+\cdots+\alpha_k a(k,c)$ or $\alpha_1 b(1,c)+\alpha_2 b(2,c)+\cdots+\alpha_k b(k,c)$.

Assume that n is even, from the above discussion it follows that the set of solutions to equation (2) is the same as the set of solutions of the equation

$$\alpha_1 a(1,c)+\alpha_2 a(2,c)+\cdots+\alpha_k a(k,c)=0$$

We are looking for a minimal number k such that the above equation has a nontrivial solution. Since all $a(r,c)\neq 0$, then clearly $k=2$ is the minimal number. In other words, the equation $\alpha_1 a(1,c)+\alpha_2 a(2,c)=0$ has a nontrivial solution. Thus, when n is even, and $c\neq 0$, then the set $\{A,A^2\}$ is linearly dependent.

Assume that n is odd. Then the set of solutions to equation (2) is the same as the set of solutions of the system of equations

$$\begin{gathered}\alpha_1 a(1,c)+\alpha_2 a(2,c)+\cdots+\alpha_k a(k,c)=0\\ \alpha_1 b(1,c)+\alpha_2 b(2,c)+\cdots+\alpha_k b(k,c)=0\end{gathered}$$

In this case $k=2$ will not work. Indeed, since $a(1,c)=b(1,c)=c$, and since $a(2,c)\neq b(2,c)$ then the system $\alpha_1 a(1,c)+\alpha_2 a(2,c)=0$ and $\alpha_1 b(1,c)+\alpha_2 b(2,c)=0$ has only the trivial solution. When $k=3$ then we obtain a homogenous system with two equations and three unknowns. This has a nontrivial solution. Hence, when n is odd, and $c\neq 0$, then the set $\{A,A^2,A^3\}$ is linearly dependent, and this is the minimal set.

5) Since T is a linear transformation then $T(0)=0$. Here 0 is the zero matrix. Matrix multiplication shows that if $X=\begin{pmatrix}0&0\\1&0\end{pmatrix}$ then $X^2=0$. Plugging $A=B=X$ in the identity $T(AB)=T(A)T(B)$, we obtain $0=T(0)=T(X^2)=T(X)T(X)$. Thus if $T(X)=I$ we derive a contradiction.

Linear Algebra 1

Exercise Number 4

1) For each of the following matrices find its inverse if it exists.

$$A = \begin{pmatrix} 6 & 1 \\ 2 & 4 \end{pmatrix} \qquad B = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 3 & 1 \\ -1 & 4 & 0 \end{pmatrix} \qquad C = \begin{pmatrix} 5 & 2 & 6 \\ 1 & 2 & 2 \\ -1 & 2 & 0 \end{pmatrix}$$

$$D = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \end{pmatrix} \qquad E = \begin{pmatrix} i & 1 \\ 2 & -i \end{pmatrix}$$

2) Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. If $ad - bc \neq 0$, show that $A^{-1} = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

3) Let A be a $m \times n$ matrix, and let B be a $n \times m$ matrix. Suppose that $m > n$. Prove that AB is not invertible. (*Hint:* Consider the linear system $Bx = 0$).

4) a) Let A and B be two square matrices of the same size. Prove that if AB is invertible and A is invertible, then B is invertible.

b) Prove that if AB is invertible, then both A and B are invertible.

c) From part b) conclude that if $AB = I$ then $BA = I$.

d) Let A denote a square matrix which satisfies the identity $2A^3 + 3A^2 - 4A - 6I = 0$. Prove that A is invertible, and express the matrix A^{-1} in terms of the matrix A.

5) Let A, B and P be square matrices of the same size which satisfy the relation $B = P^{-1}AP$. Let $f(x)$ be a polynomial in x.

a) Prove that for any natural number k we have $B^k = P^{-1}A^kP$.

b) Prove that if $f(A) = 0$ then $f(B) = 0$.

6) Write down all elementary matrices of size 2×2.

7) Let J be an invertible matrix, and let A be a matrix which satisfies $A^tJA = J$. Prove that A is invertible, and that A^{-1} satisfies $(A^{-1})^tJA^{-1} = J$.

8) If A is a matrix such that $A^2 = 0$, prove that $I + A$ is invertible and find $(I + A)^{-1}$.

9) Find at lease three different matrices A of size 2×2 which are different from I or $-I$ and which satisfy $A^2 = I$.

Linear Algebra 1

Exercise Number 5

1) Let F be a field. Prove that the set,

$$\{\begin{pmatrix} a & -b \\ b & a \end{pmatrix} : a, b \in F\}$$

with addition and multiplication defined as in matrices, is itself a field.

2) Define in the set $\mathbf{Q}(\sqrt{2}) = \{a + \sqrt{2}b : a, b \in \mathbf{Q}\}$ the following operations. Define addition and multiplication by

$$(a + \sqrt{2}b) + (c + \sqrt{2}d) = (a + c) + \sqrt{2}(b + d)$$

$$(a + \sqrt{2}b)(c + \sqrt{2}d) = (ac + 2bd) + \sqrt{2}(ad + bc)$$

Prove that with two operations, the set $\mathbf{Q}(\sqrt{2})$ is a field.

3) Prove that $\mathbf{Z}_{20}$ is not a field.

4) Solve the following equations over the fields $\mathbf{Z}_2, \mathbf{Z}_3, \mathbf{Z}_5, \mathbf{Q}$ and $\mathbf{R}$.

a) $x^2 - 2 = 0$ b) $x^2 + 1 = 0$

c) $2x^2 - 2x + 1 = 0$ d) $x^4 - 1 = 0$.

5) Solve the following system of linear equations over $\mathbf{Z}_2$ and over $\mathbf{R}$.

$$x_1 + x_2 - x_3 = 1$$

$$x_1 - x_2 + x_3 = 1$$

$$x_1 - x_2 - x_3 = 1$$

6) Solve the following equation over the complex numbers $\mathbf{C}$.

$$ix_1 + x_2 - ix_3 = i$$

$$-x_1 - ix_2 + x_3 = -i$$

$$x_1 + x_2 + x_3 = 1$$

7) a) Prove that in a field F there is a unique identity element. That is, there is a unique element $e \in F$, such that $ae = ea = a$ for all $a \in F$.

b) Prove that if in a field F we have $ab = ac$, and $a \neq 0$, then $b = c$. Here $a, b, c \in F$.

c) Prove that in a field F, if $a \neq 0$, then the equation $ax = b$ has a unique solution. Here $a, b \in F$.

d) Prove that in a field F, the identity $ab = 0$ implies that $a = 0$ or $b = 0$. Here $a, b \in F$.

SOLUTIONS MOED A LINEAR ALGEBRA 1 A 2020

SEMYON ALESKER AND DAVID GINZBURG

1. Moed A

Problem 1: Let $P_2(x)$ denote the vector space consisting of all polynomials of degree at most two, with coefficients in a field F. Let T denote the linear map $T : P_2(x) \to P_2(x)$ defined by $T(p(x)) = \alpha p''(x) + \beta p'(x) + \gamma p(x)$. Find all $\alpha, \beta, \gamma \in F$, such that $kerT = \{0\}$.

Solution: Let $p(x) = a + bx + cx^2$. Then $p'(x) = b + 2cx$ and $p''(x) = 2c$. Hence,

$$T(p(x)) = \alpha p''(x) + \beta p'(x) + \gamma p(x) = 2c\alpha + \beta(b + 2cx) + \gamma(a + bx + cx^2) =$$
$$= 2\alpha c + \beta b + \gamma a + (2\beta c + \gamma b)x + \gamma cx^2$$

Hence $T(p(x)) = 0$ if and only if

$$\gamma c = 0; \quad 2\beta c + \gamma b = 0; \quad 2\alpha c + \beta b + \gamma a = 0$$

Suppose first that $\gamma \neq 0$. Then we must have $c = 0$. The second equation is then $\gamma b = 0$ which implies that $b = 0$. From this the third equation is $\gamma a = 0$ which implies that $a = 0$. Thus, if $\gamma \neq 0$ then $kerT = \{0\}$ for all $\alpha, \beta \in F$.

If $\gamma = 0$, then the polynomial $p(x) = a$ is in $kerT$. Thus if $\gamma = 0$ then $kerT \neq \{0\}$.

Problem 2: Let A be a matrix of size three defined over $\mathbf{Q}$. Let

$$B = \begin{pmatrix} 0 & 1 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Consider the system of equations given by

$$A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} y \\ 2x \\ 0 \end{pmatrix} \tag{1}$$

Prove that this system has a nontrivial solution if and only if $|A - B| = 0$.

Solution: Notice that

$$B \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} y \\ 2x \\ 0 \end{pmatrix} \tag{2}$$

Subtracting equation (2) from equation (1), we obtain

$$(A - B) \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \tag{3}$$

From this we deduce that $|A-B|=0$ if and only if there is a nonzero vector (a,b,c) which is a solution to equation (3). This last equation holds if and only if

$$0=(A-B)\begin{pmatrix}a\\b\\c\end{pmatrix}=A\begin{pmatrix}a\\b\\c\end{pmatrix}-B\begin{pmatrix}a\\b\\c\end{pmatrix}=A\begin{pmatrix}a\\b\\c\end{pmatrix}-\begin{pmatrix}b\\2a\\0\end{pmatrix}$$

This holds if and only if

$$A\begin{pmatrix}a\\b\\c\end{pmatrix}=\begin{pmatrix}b\\2a\\0\end{pmatrix}$$

Problem 3: Let V be a vector space and let W, W' and W'' be three subspaces of V such that $V=W\oplus W'=W\oplus W''$. Prove that

$$\dim(W'\cap W'')\geq \dim V-2\dim W \tag{4}$$

Solution: Applying the first dimension Theorem we deduce that

$$\dim V=\dim W+\dim W'$$

and

$$\dim V=\dim W+\dim W''$$

Adding the two equations we get

$$2\dim V=2\dim W+\dim W'+\dim W'' \tag{5}$$

Applying the same Theorem again we have

$$\dim(W'+W'')=\dim W'+\dim W''-\dim(W'\cap W'')$$

or

$$\dim(W'+W'')+\dim(W'\cap W'')=\dim W'+\dim W''$$

Plugging this into equation (5) we obtain

$$2\dim V=2\dim W+\dim(W'+W'')+\dim(W'\cap W'') \tag{6}$$

Since $W'+W''$ is a subspace of V, we have $\dim(W'+W'')\leq \dim V$. Hence, it follows from equation (6) that

$$2\dim V=2\dim W+\dim(W'+W'')+\dim(W'\cap W'')\leq 2\dim W+\dim V+\dim(W'\cap W'')$$

Or

$$\dim V\leq 2\dim W+\dim(W'\cap W'')$$

This implies equation (4).

Problem 4: Let V denote a vector space over $\mathbf{R}$. Let $T,\ S:V\to\mathbf{R}$ denote two nonzero linear maps. Suppose that for all $v\in V$, if $T(v)\geq 0$, then $S(v)\geq 0$. Prove that $T=\alpha S$ where $\alpha\in\mathbf{R}$ and $\alpha>0$.

Solution: We first prove that $kerT \subset kerS$. Indeed, suppose that $v \in kerT$. Then $T(v) = 0$, and also $T(-v) = 0$. Hence, we deduce that $S(v) \geq 0$ and also $S(-v) \geq 0$. The last condition implies $-S(v) \geq 0$, or $S(v) \leq 0$. Hence $S(v) = 0$, which implies $v \in kerS$. Thus $kerT \subset kerS$.

Since $T, S : V \to \mathbf{R}$ are nonzero, then $ImT = ImS = \mathbf{R}$. Hence $\dim(ImT) = \dim(ImS) = 1$. It follows from the dimension identity $\dim V = \dim(kerT) + \dim(ImT)$ that

$$\dim V = \dim(kerT) + 1$$

and similarly,

$$\dim V = \dim(kerS) + 1$$

Hence, $\dim(kerT) = \dim(kerS)$, and since $kerT \subset kerS$, we deduce that $kerT = kerS$.

Suppose that $r = \dim V$. Then, from the above equalties we deduce that $\dim kerT = \dim kerS = r - 1$. Let $\{v_1, \ldots, v_{r-1}\}$ denote a base for $kerT = kerS$. Choose v_r such that $T(v_r) \neq 0$. If needed, by replacing v_r by $-v_r$, we may assume that $T(v_r) > 0$. Hence $S(v_r) \geq 0$. We cant have $S(v_r) = 0$ since $kerT = kerS$. Hence $S(v_r) > 0$. Denote $\alpha = T(v_r)/S(v_r)$. Then $\alpha > 0$, and

$$\alpha S(v_r) = \frac{T(v_r)}{S(v_r)} S(v_r) = T(v_r)$$

Since $T(v_i) = \alpha S(v_i) = 0$ for all $1 \leq i \leq r-1$ we deduce that T and αS are equal on a base of V, and hence they are equal for all $v \in V$.

מועד א'
28/1/2015

בחינה באלגברה לינארית 1 א

דוד גינזבורג

יש לענות על כל השאלות. אין להשתמש בכל חומר עזר לרבות מחשבונים. לכל השאלות ניקוד שווה. בשאלה בה יש יותר מסעיף אחד, אם לא צויין אחרת, לכל סעיף ניקוד שווה. יש לנמק הייטב את דרך הפיתרון.

שאלה 1: לתת דוגמא להעתקה לינארית $T: F^4 \to F^4$ המקימת

$$ImT = KerT = \mathrm{Sp}\{(1,1,1,1);\ (1,1,1,0)\}$$

שאלה 2: יהיו U, V ו־ W שלושה מרחבים וקטורים מעל שדה F. תהיינה $T: U \to V$ ו־ $S: V \to W$ שתי העתקות לינאריות כך שההעתקה $S \circ T$ הינה איזומורפיזם. להוכיח כי $V = ImT \oplus KerS$.

שאלה 3: לרשום את כל המטריצות הרבועיות A מסדר שלוש כך שמרחב הפתרונות של המערכת ההומוגנית $Ax = 0$ הינו $\mathrm{Sp}\{\begin{pmatrix}1\\2\\3\end{pmatrix}\}$.

שאלה 4: יהיו $u,\ v$ ו־ w שלושה וקטורי יחידה במרחב עם מכפלה פנימית ממשית. נניח כי $u+v+w=0$. להוכיח כי מתקים $(u,v) = -\frac{1}{2}$.

שאלה 5: תהי A מטריצה רבועית מסדר ארבע שכל איבריה הם המספרים ± 1. להוכיח כי המספר שמונה מחלק את $\det A$.

בהצלחה!

מועד ב'
1/3/2015

בחינה באלגברה לינארית 1 א

דוד גינזבורג

יש לענות על כל השאלות. אין להשתמש בכל חומר עזר לרבות מחשבונים. לכל השאלות ניקוד שווה. בשאלה בה יש יותר מסעיף אחד, אם לא צויין אחרת, לכל סעיף ניקוד שווה. יש לנמק הייטב את דרך הפיתרון.

שאלה 1: עבור אלו מספרים ממשיים a ו־ b, למערכת הבאה

$$ax + y + bz = 1$$
$$-x + 3y - z = 1$$
$$x - y + 3z = 1$$

יש אין סוף פתרונות.

שאלה 2: יהי $f(x)$ פולינום מעל הממשיים שדרגתו היא n. להוכיח כי לכל פולינום ממשי $g(x)$ שדרגתו היא לכל היותר n, קיימים מספרים ממשיים $\alpha_0, \alpha_1, \ldots, \alpha_n$ כך שמתקיים

$$g(x) = \alpha_0 f(x) + \alpha_1 f^{(1)}(x) + \cdots + \alpha_{n-1} f^{(n-1)}(x) + \alpha_n f^{(n)}(x)$$

כאן $f^{(i)}(x)$ הינה הנגזרת ה i־ית של $f(x)$.

שאלה 3: תהי A מטריצה ממשית מסדר שתים, ונניח כי קים מספר ממשי $a > 0$ כך שמתקיים $|A^2 + aI| = 0$.
א) להוכיח כי קים וקטור $v \in \mathbf{R}^2$ השונה מאפס כך שמתקיים $(A^2 + aI)v = 0$.
ב) להוכיח כי הקבוצה $\{v, Av\}$ הינה קבוצה בלתי תלויה לינארית.
ג) לרשום את $|A|$ בצורה מפורשת באמצעות a.

שאלה 4: תהי $T : Mat_{n\times n}(F) \to Mat_{n\times n}(F)$ העתקה המוגדרת על ידי $T(A) = A + aA^t$. כאן F הינו שדה ו־ $a \in F$.
א) להוכיח כי אם $a \neq \pm 1$ אז T הינו איזומורפיזם.
ב) לתת נוסחה מפורשת עבור T^{-1}.

שאלה 5: תהי $v \cdot u$ המכפלה הפנימית הסטנדרטית המוגדרת על $\mathbf{R}^n$ באופן הבא. אם $v = (x_1, \ldots, x_n)$ ו־ $u = (y_1, \ldots, y_n)$ אז $v \cdot u = x_1y_1 + \cdots + x_ny_n$. תהי A מטריצה ממשית מסדר n. להוכיח כי $(v, u) = (Av) \cdot (Au)$ מגדירה מכפלה פנימית מעל $\mathbf{R}^n$ אם ורק אם $\mathrm{rank}A = n$.

בהצלחה!

13/12/2019

בוחן ביניים באלגברה לינארית 1 א

דוד גינזבורג

יש לענות על כל השאלות. אין להשתמש בכל חומר עזר לרבות מחשבונים. לכל השאלות ניקוד שווה. בשאלה בה יש יותר מסעיף אחד, אם לא צויין אחרת, לכל סעיף ניקוד שווה. יש לנמק הייטב את דרך הפיתרון.

משך הבחינה : שעה וחצי.

שאלה 1: רשמו את כל המטריצות A מסדר $m \times n$ כך שגם A וגם A^t שתיהן תהיינה מטריצות בצורה קנונית.

שאלה 2: נתונה המערכת הבאה מעל הממשיים:

$$\text{x+2y+z=0}$$

$$\text{-x+}(\beta\text{-1})\text{y-2z=1}$$

$$\text{x+}(2\beta\text{+4})\text{y+}(\alpha^2\text{-2})\text{z=1}$$

עבור אילו ערכים של α ושל β למערכת יש אין סוף פתרונות? אין צורך לרשום את הפתרונות.

שאלה 3: יהי $p \geq 3$ מספר ראשוני. כמה מן המטריצות

$$\begin{pmatrix} 1 & a \\ b & 2 \end{pmatrix}$$

הן הפיכות מעל השדה $\mathbf{Z}_p$?

Linear Algebra 1 A

Midterm 2018

1) Find for what values of $t \in \mathbf{R}$ the system of equations

$$\begin{array}{llll} -x_1 & & +2x_3 & = 1 \\ 3x_1 & +tx_2 & -6x_3 & = -3 \\ -2x_1 & -tx_2 & +tx_3 & = 3 \end{array}$$

has a unique solution, has infinite number of solutions, or no solutions at all. In those cases where there are solutions write the solution set for each case.

Solution: Perform the two row operations: $R_2 \to R_2 + 3R_1$ and $R_3 \to R_3 - 2R_1$. Then we obtain the system

$$\begin{array}{llll} -x_1 & & +2x_3 & = 1 \\ & tx_2 & & = 0 \\ & -tx_2 & +(t-4)x_3 & = 1 \end{array}$$

Perform the operation $R_3 \to R_3 + R_2$. We obtain

$$\begin{array}{llll} -x_1 & & +2x_3 & = 1 \\ & tx_2 & & = 0 \\ & & (t-4)x_3 & = 1 \end{array}$$

Suppose $t = 4$. Then the last equation becomes $0 = 1$, and hence there are no solutions. If $t = 0$, then we obtain the system

$$\begin{array}{lll} -x_1 & +2x_3 & = 1 \\ & -4x_3 & = 1 \end{array}$$

From this we deduce that the system has an infinite number of solutions which are given by the set $\{(-\frac{3}{2}, \alpha, -\frac{1}{4}) \ : \ \alpha \in \mathbf{R}\}$.

Finally, if $t \neq 0, 4$, then we obtain the system

$$\begin{array}{llll} -x_1 & & +2x_3 & = 1 \\ & x_2 & & = 0 \\ & & x_3 & = \frac{1}{t-4} \end{array}$$

From this we deduce that there is a unique solution which is given by $(\frac{2}{t-4} - 1, 0, \frac{1}{t-4})$.

2) Let A denote a square matrix with entries in $\mathbf{R}$, which satisfies $(A+2I)^2 = 0$. Prove that $A + \lambda I$ is invertible if and only if $\lambda \neq 2$.

Solution: Denote $B = A + \lambda I$, Then

$$(A+2I)^2 = (A+\lambda I - \lambda I + 2I)^2 = (B + (2-\lambda)I)^2 = B^2 + 2(2-\lambda)B + (2-\lambda)^2 I$$

Hence $(A+2I)^2 = 0$ if and only if $B^2+2(2-\lambda)B+(2-\lambda)^2I = 0$, or $B^2+2(2-\lambda)B = -(2-\lambda)^2I$. This is equivalent to $B(B+2(2-\lambda)I) = -(2-\lambda)^2I$. If $\lambda = 2$, then we get $B^2 = 0$. Then clearly B is not invertible. If $\lambda \neq 2$, we obtain the equation

$$B\left[\frac{1}{-(2-\lambda)^2}(B+2(2-\lambda)I)\right] = I$$

This also implies

$$\left[\frac{1}{-(2-\lambda)^2}(B+2(2-\lambda)I)\right]B = I$$

From this we deduce that B is invertible, and we obtain

$$B^{-1} = \frac{1}{-(2-\lambda)^2}(B+2(2-\lambda)I)$$

3) Let A be a 3×3 matrix defined over $\mathbf{R}$. Assume that A is a reduced row echelon form (a canonical form) matrix. Let

$$B = \begin{pmatrix} 1 & 2 & 3 \\ 2 & a & 0 \\ 3 & 0 & b \end{pmatrix}$$

Find all values of $a, b \in \mathbf{R}$ such that $AB = 0$.

Solution: This depends on A. So we need to check all possibilities. Assume first that A has no zero rows ($\text{rank}(A) = 3$). Then $A = I$, the identity matrix. Hence $AB = IB = B$. Thus, the condition $AB = 0$ is equivalent in this case to $B = 0$. This can never happen.

Next assume that A has exactly one row of zeros ($\text{rank}(A) = 2$). Then there are three options. They are

$$A_1 = \begin{pmatrix} 1 & 0 & \gamma \\ 0 & 1 & \delta \\ 0 & 0 & 0 \end{pmatrix} \qquad A_2 = \begin{pmatrix} 1 & \gamma & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \qquad A_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

If $A = A_1$ then the $(1,2)$ entry of AB is equal to 2. Hence $AB \neq 0$ for all a, b. If $A = A_2$ or $A = A_3$ then the $(2,1)$ entry of AB is 3, and hence there are no a, b such that $AB = 0$.

Next we consider the case when there are 2 zero rows ($\text{rank}(A) = 1$). There are 3 options

$$A_4 = \begin{pmatrix} 1 & \gamma & \delta \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad A_5 = \begin{pmatrix} 0 & 1 & \gamma \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad A_6 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Assume first that $A = A_4$. Then

$$AB = \begin{pmatrix} 1+2\gamma+3\delta & 2+\gamma a & 3+\delta b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Hence the equation $AB = 0$ can hold only if γ and δ are such that $2\gamma + 3\delta = -1$. If that is the case, then to have a solution we must have $2 + \gamma a = 0$ and $3 + \delta b = 0$. If γ and δ are both nonzero and satisfy $2\gamma + 3\delta = -1$ then there is a unique solution for a and b given by $a = -2/\gamma$ and $b = -3/\delta$. If $\gamma = 0$, then $2+\gamma a = 0$ has no solution. If $\delta = 0$, then $3+\delta b = 0$ has no solution.

When $A = A_5$, we have

$$AB = \begin{pmatrix} 2+3\gamma & a & \gamma b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

In this case we have a solution if $\gamma = -2/3$. The solution is $a = b = 0$. In the case of $A = A_6$, we obtain that the $(1, 1)$ entry of AB is 3, and hence there are no solutions.

Finally, we also need to consider the case when $A = 0$. In this case we clearly have $AB = 0$ for all a and b.

Solution of Midterm Linear Algebra 1 2019

David Ginzburg

1) Problem: Write down all matrices A such that both A and A^t are canonical matrices.

Solution: If $A = 0$, then $A^t = 0$ and both are canonical matrices. Assume that $A \neq 0$. Write $A = (a_{i,j})$. Consider the first row of A. For $A \neq 0$ to be in canonical form, its first row should be nonzero. If $a_{1,1} = 0$, then the first column of A is zero, and then the first row of $A^t \neq 0$ must be the zero row. This means that A^t is not a canonical matrix. Hence, we must have $a_{1,1} = 1$. Hence, $a_{i,1} = 0$ for all $i \geq 2$. If $a_{1,j} \neq 0$ for some $j \geq 2$, then A^t is not a canonical matrix. From this we conclude that

$$A = \begin{pmatrix} 1 & 0_{1\times(n-1)} \\ 0_{(n-1)\times 1} & A_1 \end{pmatrix}$$

In general, we denote by $0_{k,l}$ the zero matrix of order $k \times l$. It is clear that A and A^t are both in canonical form if and only if A_1 and A_1^t are. Continuing by induction we deduce that

$$A = \begin{pmatrix} I_r & 0_{r\times(n-r)} \\ 0_{(m-r)\times r} & 0_{(m-r)\times(n-r)} \end{pmatrix} \qquad 0 \leq r \leq \min(m, n)$$

2) Problem: For values of real numbers α and β, the following system

$$x + 2y + z = 0$$

$$-x + (\beta - 1)y - 2z = 1$$

$$x + (2\beta + 4)y + (\alpha^2 - 2)z = 1$$

has infinite number of solutions? There is no need to write the solutions.

Solution The extended matrix of the system is

$$\begin{pmatrix} 1 & 2 & 1 & 0 \\ -1 & \beta - 1 & -2 & 1 \\ 1 & 2\beta + 4 & \alpha^2 - 2 & 1 \end{pmatrix}$$

Perform the row operations $R_2 \to R_2 + R_1$ and $R_3 \to R_3 - R_1$. We get

$$\begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & \beta+1 & -1 & 1 \\ 0 & 2\beta+2 & \alpha^2-3 & 1 \end{pmatrix}$$

Next, perform the row operation $R_3 \to R_3 - 2R_2$. We obtain

$$\begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & \beta+1 & -1 & 1 \\ 0 & 0 & \alpha^2-1 & -1 \end{pmatrix}$$

Suppose that $\alpha^2 - 1 = 0$. Then the system has no solution. Thus we have $\alpha^2 - 1 \neq 0$. If $\beta + 1 \neq 0$, then the reduced matrix of the system will be row equivalent to the identity which means that the system has a unique solution. Thus we have $\beta = -1$. From this we obtain the two equations $-z = 1$ and $(\alpha^2 - 1)z = -1$. Hence $z = -1$ and $\alpha = \pm\sqrt{2}$.

To summarize, the system has infinite number of solutions if and onlt if $\beta = -1$ and $\alpha = \pm\sqrt{2}$.

3) Problem: Let $p \geq 3$ denote a prime number. How many matrices of the form

$$A = \begin{pmatrix} 1 & a \\ b & 2 \end{pmatrix}$$

are invertible over the field $\mathbf{Z}_p$?

Solution: A matrix is invertible if and only if it is row equivalent to the identity matrix. Applying the row operation $R_2 \to R_2 - bR_1$, we obtain that A is row equivalent to the matrix

$$B = \begin{pmatrix} 1 & a \\ 0 & 2-ab \end{pmatrix}$$

Clearly, the matrix B is row equivalent to the identity matrix if and only if $ab \neq 2$. To count how many possibilities there are for $ab \neq 2$, it is easier to count the number of solutions of $ab = 2$. Clearly $a, b \neq 0$. Given $b \neq 0$ it determines a uniquely, that is, we have $a = 2b^{-1}$. Hence, the equation $ab = 2$ has exactly $p - 1$ solutions. Therefore , the number of choices for $ab \neq 2$ is $p^2 - (p-1) = p^2 - p + 1$.

Linear Algebra Moed b 2015

David Ginzburg

1) Consider the following system of equations over the Real numbers.

$$ax + y + bz = 1$$

$$-x + 3y - z = 1$$

$$x - y + 3z = 1$$

For what values of a and b, the system has infinite number of solutions?

Solution : Write the extended matrix of the system. We have

$$\begin{pmatrix} a & 1 & b & 1 \\ -1 & 3 & -1 & 1 \\ 1 & -1 & 3 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 3 & 1 \\ -1 & 3 & -1 & 1 \\ a & 1 & b & 1 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 3 & 1 \\ 0 & 2 & 2 & 2 \\ 0 & 1+a & b-3a & 1-a \end{pmatrix} \to$$

$$\begin{pmatrix} 1 & -1 & 3 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1+a & b-3a & 1-a \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 3 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & b-4a-1 & -2a \end{pmatrix}$$

If $b - 4a - 1 \neq 0$, then the system has a unique solution. Hence $b - 4a - 1 = 0$. If $-2a \neq 0$, the system has no solution. Hence $a = 0$ and then $b = 1$. In this case the system has infinite number of solutions.

2) Let $f(x)$ be a polynomial of degree n over the Real numbers. Prove that for all polynomial $g(x)$ whose degree is at most n, there are Real numbers α_i for $0 \leq i \leq n$ such that

$$g(x) = \alpha_0 f(x) + \alpha_1 f^{(1)}(x) + \cdots + \alpha_{n-1} f^{(n-1)}(x) + \alpha_n f^{(n)}(x)$$

Here $f^{(i)}(x)$ is the i-th derivative of $f(x)$.

Solution : This can be proved by induction on n. Its easy to check for $n = 1$. Assume its true for $n - 1$, and prove for n. Assume that

$$f(x) = a_n x^n + \cdots + a_1 x + a_0 \qquad g(x) = b_n x^n + \cdots + b_1 x + b_0$$

where we know that $a_n \neq 0$. Define the polynomial $h(x) = g(x) - \frac{b_n}{a_n} f(x)$. Then the degree of $h(x)$ is at most $n-1$. By the induction hypothesis, applied to $f^{(1)}(x)$, which is a polynomial of degree $n-1$, we have constants β_i such that

$$h(x) = \beta_1 f^{(1)}(x) + \beta_2 f^{(2)}(x) + \cdots + \beta_{n-1} f^{(n-1)}(x)$$

Plugging the definition of $h(x)$ into the last equation, the result follows for $g(x)$.

3) Let A be a matrix of size 2 over the Real numbers, and assume that there is a Real number $a > 0$ such that $|A^2 + aI| = 0$.
a) Prove that there is a nonzero vector $v \in \mathbf{R}^2$ such that $(A^2 + aI)v = 0$.
b) Prove that the set $\{v, Av\}$ is a linearly independent set.
c) Express $|A|$ in terms of a only.

Solution : a) This follows immediately from the fact that $|A^2 + aI| = 0$. Indeed, if the homogeneous system $(A^2 + aI)x = 0$ has only the trivial solution then $|A^2 + aI| \neq 0$.
b) Suppose that the set $\{v, Av\}$ is a linearly dependent set. Then there is a nonzero Real number λ such that $Av = \lambda v$. Hence

$$0 = (A^2 + aI)v = (A^2 v + av) = A(Av) + av = A(\lambda v) + av = \lambda^2 v + av = (\lambda^2 + a)v$$

By assumption $v \neq 0$, and since $a > 0$ is a Real number, then $\lambda^2 + a \neq 0$. Hence we obtain a contradiction, and hence the set $\{v, Av\}$ is a linearly independent set.
c) It follows from the first two parts that $\{v, Av\}$ is a base for $\mathbf{R}^2$. Also, we have

$$(A^2 + aI)Av = A(A^2 + aI)v = 0$$

Hence v and Av is a basic for the solution space of the system $(A^2 + aI)x = 0$. This implies that $\text{rank}(A^2 + aI) = 0$ or that $A^2 + aI = 0$. Hence $A^2 = -aI$ and $|A|^2 = a^2$. Since $a > 0$, then $|A| = a$.

4) Let $T : Mat_{n\times n}(F) \to Mat_{n\times n}(F)$ denote the map defined by $T(A) = A + aA^t$. Here F is a field and $a \in F$.
a) If $a \neq \pm 1$, prove that T is an isomorphism.
b) Give an explicit formula for T^{-1}.

Solution : a) To prove that T is linear we have

$$T(\alpha A + \beta B) = (\alpha A + \beta B) + a(\alpha A + \beta B)^t = \alpha(A + aA^t) + \beta(B + aB^t) = \alpha T(A) + \beta T(B)$$

To prove that it is one to one we consider its kernel. Assume that $T(A) = 0$. Then $A + aA^t = 0$ or $A = -aA^t$. Taking transpose on this last equation we obtain $A^t = -aA$. Plugging this into the first equation we obtain $A = a^2 A$. Since $a \neq \pm 1$ we obtain that $A = 0$. Hence $\ker T = 0$. Since every one to one linear map from a vector space to itself is an isomorphism, the first part follows.

b) Assume that $T^{-1}(B) = A$. Then $B = T(A) = A + aA^t$. Taking transpose we obtain $B^t = A^t + aA$. Hence $A^t = B^t - aA$. Plugging into the first equation we obtain $B = A + a(B^t - aA)$, or $A = \frac{1}{1-a^2}(B - aB^t)$.

5) Let $v \cdot u$ denote the standard inner product on $\mathbf{R}^n$. In other words, if $v = (x_1, \ldots, x_n)$ and $u = (y_1, \ldots, y_n)$, then $v \cdot u = x_1 y_1 + \cdots + x_n y_n$. Let A denote a matrix of order n with entries in $\mathbf{R}$. Prove that $< v, u >= (Av) \cdot (Au)$ defines an inner product on $\mathbf{R}^n$ if and only if $\text{rank} A = n$.

Solution : We apply the definition. First, linearity. We have

$$< \alpha v + \beta w, u >= (A(\alpha v + \beta w)) \cdot (Au) = (\alpha Av + \beta Aw) \cdot (Au) =$$

$$= \alpha (Av) \cdot (Au) + \beta (Aw) \cdot (Au) = \alpha < v, u > + \beta < v, w >$$

where the third equality follows from the fact that $v \cdot u$ is a linear map. Next we prove that $< v, u >=< u, v >$. Indeed,

$$< v, u >= (Av) \cdot (Au) = (Au) \cdot (Av) =< v, u >$$

Finally, we need to prove that if $< v, v >= 0$, then $v = 0$. By definition $< v, v >= 0$ is equivalent to $(Av) \cdot (Av) = 0$. Since $v \cdot u$ is an inner product, then $(Av) \cdot (Av) = 0$ is equivalent to $Av = 0$. If $\text{rank} A = n$, then the only solution to the system $Ax = 0$ is $x = 0$. This follows from the theorem we proved that the dimension of the space of all solutions to $Ax = 0$ is equal to $n - \text{rank} A$.

Linear Algebra 1

Exercise Number 1

1) Use induction to prove:

a)

$$\frac{1^2}{1\cdot 3}+\frac{2^2}{3\cdot 5}+\ldots+\frac{n^2}{(2n-1)(2n+1)}=\frac{n(n+1)}{2(2n+1)}$$

b)

$$2+\frac{3}{(1\cdot 2)^2}+\frac{5}{(2\cdot 3)^2}+\ldots+\frac{2n+1}{n^2(n+1)^2}=3-\frac{1}{(n+1)^2}$$

2) In the set of complex numbers $\mathbf{C}$ define the following numbers $z_1=2-i$, $z_2=3i$, $z_3=-1+3i$ and $z_4=5$. Compute the following numbers: $4z_1-5z_2+2\bar{z}_3-z_4$, $z_1\bar{z}_2$, $\frac{z_3}{z_1}$, z_1^2, $\frac{1}{z_3}$, $\mathrm{Im}(z_2-3z_1)$, $\mathrm{Re}(z_2-3z_4)$, $|z_1+z_2|$, $|z_1-z_2|$.

3) Prove the following:

a) $|\frac{z_1}{z_2}|=\frac{|z_1|}{|z_2|}$ b) $|z_1z_2|=|z_1||z_2|$ c) $z\bar{z}=|z|^2$

d) $\overline{z_1\pm z_2}=\bar{z_1}\pm\bar{z}_2$ e) $\bar{\bar{z}}=z$ f) $z+\bar{z}=2\mathrm{Re}(z)$

4) Use induction to prove that $|z^n|=|z|^n$ for all $z\in\mathbf{C}$ and all $n\in\mathbf{N}$.

5) a) Prove that $|z|^2=(\mathrm{Re}z)^2+(\mathrm{Im}z)^2$.

b) Use part a) to prove that $|\mathrm{Re}z|\le|z|$ and $|\mathrm{Im}z|\le|z|$.

Linear Algebra 1

Exercise Number 2

1) Solve the following linear systems over the real numbers:

$$\begin{aligned} x_1 - x_2 + x_3 &= 2 \\ 3x_1 - x_2 + 2x_3 &= -6 \\ 3x_1 + x_2 + x_3 &= -18 \end{aligned}$$

$$\begin{aligned} x_1 - x_2 + x_3 + x_4 &= -2 \\ x_1 + x_2 - x_3 + 2x_4 &= 1 \\ 3x_1 - x_2 + x_3 + 2x_4 &= 2 \end{aligned}$$

$$\begin{aligned} x_1+x_2+x_3+x_4 &= 1 \\ x_1+x_2+x_3-x_4 &= 2 \\ x_1+x_2-x_3-x_4 &= 3 \\ x_1-x_2-x_3-x_4 &= 4 \end{aligned}$$

2) Find for what values of t the following linear system has a unique solution, no solution, infinite number of solutions. Work over the real numbers. You dont need to find the explicit sets of solutions.

$$\begin{aligned} tx + 3y - z &= 1 \\ x + 2y - z &= 2 \\ -tx + y + 2z &= -1 \end{aligned}$$

3) Let A be the matrix that represents a linear system with n unknowns. Let B be the reduced echelon matrix obtained from A using elementary row operations. Show that the number of free variables is n minus the number of leading variables in B.

4) Solve the following linear system over the complex numbers

$$\begin{aligned} ix+(1+i)y+2z &= 0 \\ 2x-(1-i)y+iz &= 0 \end{aligned}$$

5) Suppose that linear system

$$ax + by = 0$$
$$cx + dy = 0$$

has a unique solution. Prove that for any choice of numbers c_1 and c_2, the linear system

$$ax + by = c_1$$
$$cx + dy = c_2$$

has a unique solution.

6) Given a linear system, denote by A the corresponding reduced echelon matrix.

a) If the system is homogenous, and if A has five leading terms, what is the minimal number of variables that the system can possibly have?

b) If the system is homogenous and has 5 variables and a unique solution, what is the minimal and maximal number of leading terms in the matrix A

c) Is it possible that the leading term which appears in the fifth row of A, be located in the sixth column of A? in its fourth column? Explain!

d) If A has five columns, what is the minimal number of leading terms the matrix A can have? Explain!

7) From all the citizens of a certain country, which start the year in this country, 80 percent stay and 20 percent leave the country. From all the citizens which live outside the country, 90 percent stay outside and 10 percent return to the country. If at the end of the year there were 200 million people in the country, and 30 millions live outside the country, write down a system of linear equations to find the number of citizens leaving in the country and outside it, at the beginning of the year. No need to solve the system.

Linear Algebra 1

Exercise Number 3

1) Denote

$$A = \begin{pmatrix} 2 & -1 & 0 \\ 1 & -1 & 3 \end{pmatrix} \qquad B = \begin{pmatrix} -1 & 1 & 1 \\ -1 & -2 & 3 \end{pmatrix}$$

$$C = \begin{pmatrix} 2 & 0 & 1 & 1 \\ 5 & -1 & 2 & 3 \\ 0 & 0 & 1 & 2 \end{pmatrix} \qquad \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$$

Compute the matrices $3A - 4B,\ CD,\ B^tA,\ DD^t,\ AC,\ BD.$

2) a) Let $\begin{pmatrix} -1 & 0 \\ 2 & -1 \end{pmatrix}$. Compute A^n. If $p(x) = 2x^3 - x + 1$ compute $p(A)$.

b) Let $A = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$. Compute A^4.

3) Let A and B be two matrices of order n. Determine which of the following statements is true and which not. Explain!

a) For all $p, q \in \mathbf{N}$ we have $A^pA^q = A^{p+q}$.

b) $(A+B)^2 = A^2 + 2AB + B^2$.

c) $(A+B)(A-B) = A^2 - B^2$.

4) Let A, B and C be three matrices of order n which satisfy the relations $A = B + C, C^2 = 0$ and $BC = CB$. Using induction, prove that for all natural number k we have

$$A^{k+1} = B^k(B + (k+1)C)$$

5) The sum of all diagonal entries of a square matrix A is called the trace of A, and is denoted by $tr(A)$. In other words, if $A = (a_{i,j})$ is a square matrix of order n, then $tr(A) = a_{1,1} + a_{2,2} + \ldots + a_{n,n}$. Prove that:

a) For any two square matrices A and B of the same size, we have $tr(A+B) = tr(A) + tr(B)$.

b) If C is a matrix of size $m \times n$ and D is a matrix of size $n \times m$, then $tr(CD) = tr(DC)$.

6) Let $S = (s_{i,j})$ denote the $n \times n$ matrix such that $s_{i,i+1} = 1$ and zero for all other entries. For all $p \in \mathbf{N}$ compute S^p.

7) An $n \times n$ matrix is a permutation matrix if its entries are zeros and ones, such that every row and every column has exactly one entry which is the number one, and all other entries are zeros. Let W_n denote the collection of all permutation matrices of order n. For example, we have

$$W_2 = \{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} , \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \}$$

a) Write down all matrices in W_3.
b) How many matrices are there in the set W_n?
c) Prove that if $w_1, w_2 \in W_n$, then $w_1 w_2 \in W_n$. Also, prove that $w_1^t \in W_n$. Compute the product $w_1 w_1^t$.

8) A square matrix A is symmetric if $A = A^t$, and it is anti-symmetric if $A^t = -A$.
a) Write down all symmetric matrices of size 3×3.
b) Prove that for any matrix A, the matrix AA^t is a symmetric matrix.
c) Prove that if A is symmetric or anti-symmetric, then $AA^t = A^t A$.
d) Two square matrices A and B are said to commute, if $AB = BA$. Prove that if A and B commute, then AB is symmetric.
e) Show by an example that if A and B symmetric then AB need not be symmetric.
f) Prove that any matrix A can be written as $A = B + C$ where B is symmetric and C is anti-symmetric.

9) Let J denote a 4×4 matrix. Let A and B be two 4×4 matrices which satisfy $A^t JA = J$ and $B^t JB = J$. Prove that $(AB)^t J(AB) = J$.

10) A square matrix A is called hermitian if $A = (\bar{A})^t$. Here $\bar{A}$ denotes the complex conjugate of A. For example, if

$$A = \begin{pmatrix} 1-i & 2 \\ i & 1+i \end{pmatrix} \quad \text{then} \quad \bar{A} = \begin{pmatrix} 1+i & 2 \\ -i & 1-i \end{pmatrix}$$

a) Give an example of a 3×3 hermitian matrix.
b) Denote $A^* = (\bar{A})^t$. Then A hermitian if $A = A^*$. Prove that for any two matrices A and B, we have $(A^*)^* = A$ and that $(A + B)^* = A^* + B^*$.
c) For any matrix A prove that AA^* and A^*A are both hermitian.

Linear Algebra 1

Exercise Number 6

In the following exercises, V will denote a vector field over a given field F.

1) a) Prove that in V there is a unique zero element 0.

b) Prove that for all $\alpha \in F$ we have $\alpha \cdot 0 = 0$.

c) Prove that for all $v \in V$ we have $0 \cdot v = 0$.

d) Prove that $\alpha \cdot v = 0$ if and only if $\alpha = 0$ or $v = 0$.

e) Prove that for all $v \in V$, we have $(-1)v = -v$.

2) Determine which of the following subsets are subspaces. Prove your claim!

a) $W = \{(\alpha_1, \ldots, \alpha_n) : \sum_{i=1}^{n} \alpha_i = 0\} \subset F^n$.

b) The collection of all solutions over $\mathbf{R}$ of the system of equation

$$\begin{aligned} 2x + 3y + 4z &= 1 \\ x - y \ + 2z &= 2 \end{aligned}$$

c) $W = \{(b_1, b_2) : b_1 b_2 = 0\} \subset \mathbf{R}^2$.

d) Denote by $\mathbf{R}^\infty$ the vector space of all sequences of real numbers. Let W denote the subset of $\mathbf{R}^\infty$ which consists of all sequences $(x_1, x_2, \ldots)$ such that there exists a number m such that $x_j = 0$ for all $j > m$. Is W a subspace of $\mathbf{R}^\infty$?

e) With the notations of part d) let W denote all sequences which are monotonic increasing. Is W a subspace of $\mathbf{R}^\infty$?

f) Same as part d) where now

$$W = \{(x_1, kx_1, k^2 x_1, \ldots); x_1, k \in \mathbf{R}\}$$

g) Let $P_n(x)$ denote the vector space of all polynomials of degree up to n with coefficients in a given field F. Is $W = \{p(x) \in P_n(x) : p(1) = 0\}$ a subspace of $P_n(x)$?

3) For $1 \leq i \leq n$, let U_i be a subspace of V. Prove that $U_1 \bigcap U_2 \bigcap \ldots \bigcap U_n$ is a subspace of V.

4) Let $v_1, v_2 \in \mathbf{R}^3$. Prove that the set of all linear combinations of these two vectors, is a subspace of $\mathbf{R}^3$. In other words, let $W = \{\alpha_1 v_1 + \alpha_2 v_2 : \alpha_1, \alpha_2 \in \mathbf{R}\}$. Prove that W is a subspace of $\mathbf{R}^3$.

5) Let $V = F = \mathbf{R}$. Find all the subspace of V.

6) Let $W = \{A \in Mat_{n\times n}(F) : tr(A) = 0\} \subset Mat_{n\times n}(F)$. Is W a subspace?

Linear Algebra 1

Exercise Number 7

1) a) If possible, express the vector $(7,3,6,14)$ as a linear combination of the three vectors $(0,2,0,0); (1,0,0,0); (2,1,3,7)$.
b) Express the polynomial $q(x) = 2x^4+3x^2+3$ as a linear combination of the two polynomials $p_1(x) = x^4 + 3x$ and $p_2(x) = x^2 - 2x + 1$.

2) Find a spanning set for the space of solutions of the linear system

$$\begin{aligned} 2x + 3y + 4z &= 0 \\ x - y \ \ + z \ &= 0 \end{aligned}$$

3) Given the matrix $A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 4 & 8 \\ 3 & 9 & 27 \end{pmatrix}$, find a spanning set for the space of solutions for the linear system $Ax = 0$.

4) Can the space $Sp\{(1,2,0,-1),(2,4,0,2),(-3,-6,0,3)\}$ be spanned with less than three vectors?

5) a) Assume that $U = Sp\{u_1, u_2, \ldots, u_r\}$ and $W = Sp\{u_2, \ldots, u_r\}$. Prove that $U = W$ if and only if the vector u_1 is a linear combination of $u_2, \ldots, u_r$.
b) Let A be a reduced echelon matrix of size $m \times n$. Consider the nonzero rows of A as vectors $v_1, \ldots, v_r$ in the vector space F^n. Let $v_{i_1}, \ldots, v_{i_l}$ be a proper subset of the set of vectors $v_1, \ldots v_r$. Prove that it is impossible that

$$Sp\{v_{i_1}, \ldots, v_{i_l}\} = Sp\{v_1, \ldots, v_r\}$$

6) Let

$$Sp\{(1,-1,2,3),(-1,0,1,1),(2,0,-1,1)\} \subset \mathbf{R}^4$$

What are the conditions on the numbers a, b, c, d in $\mathbf{R}$ so that $(a,b,c,d) \in W$?

Linear Algebra 1

Exercise Number 8

1) Let $P(x)$ denote the vector space of all polynomials in x with coefficients in a field F. Prove that $P(x)$ cannot be spanned by a finite set of polynomials.

2) Compute the rank of the following two matrices

$$A = \begin{pmatrix} 2 & 1 & -1 \\ 0 & 2 & 1 \\ -1 & 0 & 1 \\ 3 & 1 & 2 \end{pmatrix} \qquad B = \begin{pmatrix} 7 & 2 & 1 & 3 & 5 \\ 2 & 2 & 0 & 1 & 2 \\ 11 & 6 & 1 & 5 & 9 \end{pmatrix}$$

3) Let A be a square matrix of size n. Prove that A is invertible if and only if $\text{rank} A = n$.

4) Let A be a matrix of size $n \times n$, and let B be a matrix of size $n \times m$. If A is invertible, prove that $\text{rank} AB = \text{rank} B$. (*Hint:* Reduce it to the case when A is an elementary matrix.)

5) Given the matrices

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 3 & 4 & 1 \\ 5 & 3 & 9 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & -2 & 7 \\ 2 & -3 & 12 \\ 3 & -4 & 17 \end{pmatrix}$$

Is it true that $R(A) = R(B)$?

6) In F^3 let $W = \{(0, b, c) : b, c \in F\}$ and $U = \{(a, a, a) : a \in F\}$. Prove that $F^3 = U \oplus W$.

7) Let $V = Mat_{n\times n}(F)$. Let $U \subset V$ denote the subspace of all symmetric matrices, and let $W \subset V$ denote the subspace of all antisymmetric matrices. Prove that $V = U \oplus W$. (A matrix A is symmetric if $A^t = A$ and antisymmetric if $A^t = -A$.)

8) Let U, V and W be three subspaces of a certain vector space. Prove that

$$(U \cap V) + (U \cap W) \subset U \cap (V + W)$$

Give an example in $\mathbf{R}^2$ that the inclusion can be proper.

9) Let V be a vector space, and let $U, W \subset V$ be two subspaces. Prove that $V = U \oplus W$ if and only if the following two statements hold. 1) $V = U + W$, 2) The only way to represent the zero vector in V as a sum of a vector from U with a vector in W is $0 = 0 + 0$.

10) In F^3, let $U = \{(a, b, c) : a + b + c = 0\}$, $V = \{(a, b, c) : a = c\}$ and $W = \{(0, 0, c) : c \in F\}$. Is it true that $F^3 = U + V$? $F^3 = U + W$? $F^3 = V + W$? When are these sums direct sums?

Linear Algebra 1

Exercise Number 9

1) Find $a \in \mathbf{R}$ such that the three vectors $(1,-1,1),(2,0,3)$ and $(1,1,a)$ will be linearly dependent.

2) In $\mathbf{C}^2$, are the vectors $(1,i)$ and $(i,-1)$ linearly independent?

3) In each of the following cases, find a basis for the corresponding vector spaces:

a) The vector space which consists of all real solutions to the linear system

$$\begin{aligned} x+y \quad +t &= 0 \\ 2x+4y+3z &= 0 \end{aligned}$$

b) All symmetric matrices of size $n \times n$.

c) The vector space $\mathbf{Q}(\sqrt{2})$ as a vector space over the field $\mathbf{Q}$.

d) The vector space $\mathbf{C}^2$ over the field $\mathbf{C}$.

e) The vector space $\mathbf{C}^2$ over the field $\mathbf{R}$.

f) Let $U=\{(a,b,c): a+b+c=0\}$ and $W=\{(a,b,c): a=b=c\}$ be two subspaces of F^3. Find a basis for $U \cap W$ and $U+W$.

g) In $V=P_3(x)$, let $W=\{p(x)\in P_3(x): p(0)=p'(0)=p''(0)=0\}$. Find a basis for W.

4) If A is a given matrix, prove that the dimension of the row space of A equals the rank of A. In other words, prove that $\dim R(A)=\mathrm{rank}(A)$.

5) In a given vector space V, prove that any subset of vectors which contains the zero vector, is linearly dependent.

6) Prove that if $K \subset V$ is a linearly independent set of vectors, then any subset of K is linearly independent.

7) If $T \subset K \subset V$, and if T is linearly dependent, prove that K is also linearly dependent.

8) Let $v_1=(1,1,1,1)$ and $v_2=(2,1,1,1)$. Complete the set $\{v_1,v_2\}$ to a basis of F^4.

9) In $\mathbf{R}^n$ let $W=\{(a_1,a_2,\ldots,a_n): a_1+a_2+\ldots+a_n=0\}$. What is $\dim W$? Find a subspace $U \subset \mathbf{R}^n$ such that $\mathbf{R}^n=U\oplus W$.

10) In $Mat_{n\times n}(F)$, let $W=\{A\in Mat_{n\times n}(F): \mathrm{tr}(A)=0\}$. Find a basis for W.

11) a) Let $A \in Mat_{n\times n}(\mathbf{R})$. Given $\lambda \in R$, define $V_\lambda = \{v \in Mat_{n\times 1}(\mathbf{R}) : Av = \lambda v\}$. Prove that V_λ is a subspace of $Mat_{n\times 1}(\mathbf{R})$.

b) Let $A = \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix}$, and let $\lambda = 2$. Find a basis and compute the dimension of V_2.

Linear Algebra 1

Exercise Number 10

1) Let V be a vector space over a field F. Assume that V has a basis with n vectors. Prove:

a) Every set of vectors in V which contains more than n vectors is linearly dependent.

b) Every set of vectors in V which contains less than n vectors does not span V.

c) Every linearly independent set which contains n vectors is a basis.

d) Every set which spans V and contains n vectors is a basis.

e) In every basis for V there are exactly n vectors.

2) Let V a vector space and let $W \subset V$ be a subspace of V. Prove that there is a subspace U of V such that $V = W \oplus U$.

3) Let V be a vector space which is finitely generated. Let U be a subspace of V. Prove that $\dim U = \dim V$ if and only if $U = V$.

4) In each of the following cases W is a subspace of a given vector space V. In each case find a basis for W, and complete it to basis of V.

a) $V = P_3(x)$ $\quad W = \{p(x) \in P_3(x) : p'(0) = 0\}$

b) $V = F^n$ $\quad W = \{(a_1, \ldots, a_n) : \sum_{i=1}^n a_i = 0\}$

c) $V = F^5$ $\quad W = Sp\{(-1, 1, 0, 1, 1), (0, 0, 1, 1, 1)\}$

d) $V = F^4$ $\quad W = Sp\{(1, 2, 3, 1), (0, 1, 0, 0), (2, 5, 6, 2)\}$

e) $V = F^2$ $\quad W = \{v \in F^2 : Av = 0\}$ where

$$A = \begin{pmatrix} 2 & 1 & 0 & 6 & 1 & 1 & 2 \\ 4 & 1 & 0 & 6 & 2 & 2 & 1 \end{pmatrix}$$

f) $V = \mathbf{C}^2$ over the field $\mathbf{C}$. $W = Sp\{(i, 1)\}$.

5) Prove that every two planes in $\mathbf{R}^3$ which pass through the origin, intersect in more than one point. Is it true in $\mathbf{R}^4$?

6) In $V = P_5(x)$, let $W = \{p(x) \in P_5(x) : p''(0) = 0\}$ and $R = \{p(x) \in P_5(x) : p(0) + p'(0) = 0\}$. Compute $\dim(W + R)$.

7) A square matrix is upper triangular, if all entries below the diagonal are zeros. Let V

denote the vector space of all upper triangular matrices of size 4×4 with entries in $\mathbf{R}$. Let

$$J_1 = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & -1 & & \\ -1 & & & \end{pmatrix} \qquad J_2 = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{pmatrix}$$

For $i = 1, 2$ let

$$W_i = \{v \in V : v^t J_i + J_i v = 0\}$$

Compute $\dim(W_1 + W_2)$ and find a basis for $W_1 + W_2$.

8) Prove that for all matrix A we have $\text{rank}A = \text{rank}A^t$.

9) For any two matrices A and B such that AB is defined, prove that $\text{rank}(AB) \leq \min(\text{rank}A, \text{rank}B)$.

10) Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Prove that $\text{rank}A = 2$ if and only if $ad - bc \neq 0$.

11) Prove or disprove the following statement: For any two matrices of the same size we have $\text{rank}(A + B) = \text{rank}A + \text{rank}B$.

Linear Algebra 1

Exercise Number 11

1) Prove that if $T : V \to W$ is a linear transformation, then $T(0) = 0$.

2) Determine which of the following maps define a linear transformation.

a) $T : F^2 \to F^2$ defined by $T(x, y) = (2x - y, ax + by)$ where $a, b \in F$.

b) $T : F^2 \to F^2$ defined by $T(x, y) = (|x|, y)$.

c) $T : P(x) \to P(x)$ defined by $T(p) = 2p + 3p' - p''$.

d) $T : \mathbf{C}^2 \to \mathbf{C}^2$ defined by $T(x, y) = (ix + (2 + i)y, -3ix)$.

e) $T : \mathbf{R}^3 \to \mathbf{R}$ defined by $T(x, y, z) = ax + by + cz + d$ where $a, b, c, d \in \mathbf{R}$.

f) $T : \mathbf{R}^\infty \to \mathbf{R}^\infty$ defined by $T((x_1, x_2, x_3, \ldots)) = (0, x_1, x_2, x_3, \ldots)$.

g) Let $A \in Mat_{m\times n}(F)$. Define $T_A : Mat_{n\times p}(F) \to Mat_{m\times p}(F)$ by $T_A(B) = AB$. Is T_A linear?

3) Write down all linear transformations $T : F^2 \to F^2$ such that $T(1, 1) = (3, -2)$ and $T(2, 1) = (1, 2)$.

4) Give an example of a linear transformation $T : F^3 \to F^2$ such that $\text{Ker}T = \text{Sp}\{(1, 0, 1), (0, 0, 1)\}$.

5) Give an example of a linear transformation $T : P_2(x) \to P_3(x)$ such that $\text{Im}T = \text{Sp}\{x^3 + 1, x^2, 2x^3 + 2x^2 + 3\}$.

6) Give an example of a linear transformation $T : F^3 \to F^3$ such that $T(1, -1, 1) = (2, 0, 0)$.

7) Let V denote an n dimensional vector space over the field F. Let $B = \{v_1, \ldots, v_n\}$ be a basis for V. Define the map $T_B : V \to F^n$ as follows. If $v = \alpha_1 v_1 + \ldots + \alpha_n v_n$ then $T_B(v) = (\alpha_1, \ldots, \alpha_n)$. Check that the map T_B is well defined and prove that it is a linear transformation.

8) For each of the linear transformations in exercise **2)**, determine which is one to one, and which is onto. Do the same for the map defined in exercise **7)**.

9) Give an example for two different isomorphisms between $\mathbf{R}^4$ and $Mat_{2\times 2}(\mathbf{R})$.

10) For each of the following linear transformations, find $\text{Ker}T$ and $\text{Im}T$.

a) $T : \mathbf{Q}^3 \to \mathbf{Q}^3$ defined by $T(x, y, z) = (x + y, x + z, y + z)$.

b) $T : \mathbf{Z}_2^3 \to \mathbf{Z}_2^3$ defined by $T(x, y, z) = (x + y, x + z, y + z)$.

c) $T : P(x) \to P(x)$ defined by $T(p(x)) = xp(x)$.

11) Give an example of a linear transformation $T : \mathbf{R}^2 \to \mathbf{R}^2$ such that $\mathrm{Ker}T = \mathrm{Im}T$.

12) a) Give an example of a linear transformation $T : \mathbf{R}^2 \to \mathbf{R}^2$ such that $\mathrm{Ker}T \subsetneq \mathrm{Im}T$.
b) Give an example of a linear transformation $T : \mathbf{R}^2 \to \mathbf{R}^2$ such that $\mathrm{Im}T \subsetneq \mathrm{Ker}T$.

13) a) Let $T : \mathbf{C} \to \mathbf{C}$ be defined by $Tz = \bar{z}$. Prove that T is linear if $\mathbf{C}$ is viewed as a vector space over the field $\mathbf{R}$, and it is not linear if $\mathbf{C}$ is viewed as a vector space over the field $\mathbf{C}$
b) Does the same hold for the map $Sz = iz$.
c) In each case where T or S are linear determine their kernel and their image.

14) Let $T : V \to V$ be a linear transformation. Let $W \subset V$ be a subspace of V. Define a map $S : W \to V$ by $S(w) = T(w)$ for all $w \in W$.
a) Prove that S is linear.
b) Prove that $\mathrm{Ker}S = \mathrm{Ker}T \cap W$.
c) Prove that $\dim T(W) \leq \dim W \leq \dim T(W) + \dim \mathrm{Ker}T$. Here, $T(W) = \{v \in V : \text{there is a } w \in W \text{ such that } T(w) = v\}$.

Linear Algebra 1

Exercise Number 12

1) Let $S, T : \mathbf{R}^2 \to \mathbf{R}^2$ be two linear transformations defined as $S(x, y) = (x + y, 0)$ and $T(x, y) = (-y, x)$. Compute the transformations $5S - 3T, ST, TS, S^2$ and T^2.

2) Let $T : \mathbf{R}^3 \to \mathbf{R}^3$ be the linear transformation defined by $T(x, y, z) = (x + z, x - z, y)$. Prove that T is invertible and find T^{-1}.

3) Let $D : P_4(x) \to P_4(x)$ denote the linear transformation defined by $D(p) = p'$. For all $n \in \mathbf{N}$ compute D^n.

4) Let V be a vector space. Let $T : V \to V$ denote a linear transformation. Suppose that $\dim \mathrm{Im} T = \dim \mathrm{Im} T^2$, prove that $\mathrm{Ker} T \cap \mathrm{Im} T = \{0\}$.

5) Let $T : U \to V$ and $R : V \to W$ be two linear transformations. Prove that

a) $\mathrm{Im} RT \subset \mathrm{Im} R$.

b) $\mathrm{Ker} T \subset \mathrm{Ker} RT$.

c) $\mathrm{Im} RT = R(\mathrm{Im} T)$.

6) Let $V = P_2(x)$. Let

$$B = \{2x, 3x + x^2, -1\} \qquad B' = \{1, 1 + x, 1 + x + x^2\}$$

be two bases for V. Find the transformation matrix M from basis B to basis B'. Verify that for all $v \in V$, $[v]_B = M[v]_{B'}$.

7) Repeat the previous exercise with $V = F^3$ and

$$B = \{(1, 1, 1), (0, 1, 1), (0, 0, 1)\} \qquad B' = \{(1, 1, 0), (1, 2, 0), (1, 2, 1)\}$$

8) Let $V = F^4$. Let $B = \{(1, 0, 1, 0), (1, 1, 0, 0), (1, 0, 0, 0), (1, 1, 1, 1)\}$. Given the matrix

$$M = \begin{pmatrix} 2 & 3 & 1 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 5 \end{pmatrix}$$

find a basis B' of V such that the matrix M is the transformation matrix from basis B to basis B'.

9) Let $T : F^2 \to F^3$ denote the linear transformation $T(x, y) = (ax, bx + cy, dy)$. Here $a, b, c, d \in F$. Given the two bases

$$B = \{(1, 0), (1, 1)\} \qquad C = \{(1, 0, 0), (1, 1, 0), (1, 1, 1)\}$$

of F^2 and F^3 respectively, compute the matrix $[T]_C^B$. Verify that $[Tv]_C = [T]_C^B[v]_B$ for all $v \in F^2$.

10) Let $V = \mathbf{C}$ be a vector space over the field $F = \mathbf{R}$. Define the linear transformation $T : V \to V$ by $Tz = \bar{z}$. Compute the matrix $[T]_B$ where $B = \{1 + i, 2 + i\}$.

11) Let $V = F^n$. Define the linear transformation $T : V \to V$ by $T((a_1, \ldots, a_n)) = (0, a_1, \ldots, a_{n-1})$. Let B denote the standard basis of V. For $k \in \mathbf{N}$ compute the matrix $[T^k]_B$.

12) Let V be a vector space and let $T : V \to V$ be a linear transformation. A subspace W of V will be said to be invariant under T, if $T(W) \subset W$. Let W and U be two invariant subspaces under T and assume that $V = U \oplus W$. Let $\dim U = n$ and $\dim W = m$. Prove that there is a basis B of V such that

$$[T]_B = \begin{pmatrix} A_1 & \\ & A_2 \end{pmatrix} \quad A_1 \in Mat_{n\times n}, \quad A_2 \in Mat_{m\times m}$$

13) Let $T : P_3(x) \to P_3(x)$ be a linear transformation defined by $T(p(x)) = xp'(x)$. Let

$$B = \{1 + x, 1 - x, x^2, x^3\} \qquad C = \{1, 1 + x, (1 + x)^2, (1 + x)^3\}$$

be two bases for $P_3(x)$. Compute $[T]_B$ and $[T]_C$. Find the transformation matrix M from basis B to basis C and verify the relation $[T]_C = M^{-1}[T]_C M$.

14) a) If A is similar to B, prove that A^t is similar to B^t.
b) Let A be an invertible matrix. Prove that AB is similar to BA.
c) Prove that two similar matrices have the same trace.
d) Prove that there are no square matrices A and B such that $AB - BA = I$.

Linear Algebra 1

Exercise Number 13

1) Let $\mathbf{C}$ be a vector space over $\mathbf{R}$. Let $B = \{1+i, 1-i\}$ and $C = \{1, i\}$ be two bases.
a) For the linear transformation $Tz = \bar{z}$, write down the matrices $[T]_B$ and $[T]_C$.
b) Find the transformation matrix M between the bases B and C, and write down the relation between M, $[T]_B$ and $[T]_C$.
c) Repeat parts a) and b) for the linear transformation $Sz = iz$.

2) a) Let $A \in Mat_{n\times n}(F)$ a matrix which satisfies $A^n = 0$ and $A^{n-1} \neq 0$ for some positive integer n. (Such a matrix is called a nilpotent matrix of order n). Let $v \in F^n$ be such that $A^{n-1}v \neq 0$. Prove that $\{v, Av, A^2v, \ldots, A^{n-1}v\}$ is a basis for F^n.
b) Let B be another nilpotent matrix of order n. Prove that A and B are similar. (*Hint:* Find the matrix $[T_A]$ with respect to the basis in part a). Here $T_A : F^n \to F^n$ is the linear transformation defined by $T_A u = Au$ for all $u \in F^n$).

3) Compute the determinant of the following matrices:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 0 & 0 & 2 \\ 2 & 1 & 3 & 5 \\ -1 & 1 & 1 & 1 \end{pmatrix}$$

$$C = \begin{pmatrix} 1 & 0 & 1 & 1 & 3 \\ 2 & 1 & -1 & 1 & 2 \\ 5 & 4 & 0 & -2 & 1 \\ 1 & -1 & 0 & 0 & 2 \\ 0 & 1 & 2 & -1 & -1 \end{pmatrix} \qquad D = \begin{pmatrix} 0 & i & -i & 1+i \\ i & 0 & i & -1 \\ -i & i & 0 & i \\ 1+i & -1 & i & 0 \end{pmatrix}$$

4) If $AA^t = I$, prove that $|A| = \pm 1$.

5) a) Let $A \in Mat_{r\times r}(F)$ and $B \in Mat_{(n-r)\times(n-r)}(F)$. Prove that $\det \begin{pmatrix} A & \\ & B \end{pmatrix} = \det A \det B$.
b) In the notations of part a) prove that $\det \begin{pmatrix} I_r & C \\ & B \end{pmatrix} = \det B$.
c) From a) and b) prove that $\det \begin{pmatrix} A & C \\ & B \end{pmatrix} = \det A \det B$.

6) If A is invertible, prove that $|A^{-1}| = |A|^{-1}$.

7) Give an example that $\det(A+B) \neq \det A + \det B$.

8) Using Cramer's rule, solve the following two linear system of equations:

$$\begin{aligned} x + y + z &= 1 \\ 2x - y &= 2 \\ x + 2y - z &= 0 \end{aligned} \qquad\qquad \begin{aligned} 2x + y &= 1 \\ x + 2y - 3z &= 0 \\ x + y + 2z + t &= 0 \\ -x - y + z + 2t &= 1 \end{aligned}$$

9) Compute the determinant of the following matrices

$$A = \begin{pmatrix} x & y & x+y \\ y & x+y & x \\ x+y & x & y \end{pmatrix} \qquad B = \begin{pmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{pmatrix}$$

$$C = \begin{pmatrix} 1 & a_1 & a_2 & \dots & a_n \\ 1 & a_1+b_1 & a_2 & \dots & a_n \\ 1 & a_1 & a_2+b_2 & \dots & a_n \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 1 & a_1 & a_2 & \dots & a_n+b_n \end{pmatrix}$$

10) Let A, B, C and D be matrices of size $n \times n$. Assume that $\det A \neq 0$. Prove that

$$\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(D - CA^{-1}B)\det A$$

(*Hint:* Multiply on the left by a suitable matrix whose determinant you can compute.)

11) Compute the adjoint matrix of the following two matrices. Use it to find the inverse matrix of each one of them.

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 2 & 4 \\ -1 & -1 & -1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 0 & 2 & 3 \\ 4 & 1 & 0 & 1 \\ -1 & -1 & 1 & 1 \\ 0 & 0 & -3 & -2 \end{pmatrix}$$

Linear Algebra 1

Exercise Number 14

1) Let A be a matrix of size $n \times n$. Prove that $|A| \neq 0$, if and only if the rows of A form an independent set of vectors in F^n.

2) Let A denote an $n \times n$ matrix with the property that in every column, the sum of all entries is zero. Compute $|A|$.

3 Prove the identity

$$\begin{vmatrix} a_{1,2}+a_{1,3} & a_{1,3}+a_{1,1} & a_{1,1}+a_{1,2} \\ a_{2,2}+a_{2,3} & a_{2,3}+a_{2,1} & a_{2,1}+a_{2,2} \\ a_{3,2}+a_{3,3} & a_{3,3}+a_{3,1} & a_{3,1}+a_{3,2} \end{vmatrix} = 2 \begin{vmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{vmatrix}$$

4) Given the matrix

$$A = \begin{pmatrix} 0 & a_{1,2} & a_{1,3} & \dots & a_{1,n} \\ -a_{1,2} & 0 & a_{2,3} & \dots & a_{2,n} \\ -a_{1,3} & -a_{2,3} & 0 & \dots & a_{3,n} \\ \vdots & \vdots & \vdots & \dots & \vdots \\ -a_{1,n} & -a_{2,n} & -a_{3,n} & \dots & 0 \end{pmatrix}$$

Prove that if n is odd then $|A| = 0$.

5) Given the $n \times n$ matrix

$$A = \begin{pmatrix} 1-n & 1 & 1 & \dots & 1 \\ 1 & 1-n & 1 & \dots & 1 \\ 1 & 1 & 1-n & \dots & 1 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 1 & 1 & 1 & \dots & 1-n \end{pmatrix}$$

prove that $|A| = 0$.

6) Consider the system of equations of the following two quadratic equations

$$a_0x^2 + a_1x + a_2 = 0$$
$$b_0x^2 + b_1x + b_2 = 0$$

where $a_0 \neq 0$ and $b_0 \neq 0$. It can be shown (you dont need to do it), that this system has a solution if and only if

$$\begin{vmatrix} a_0 & a_1 & a_2 & 0 \\ 0 & a_0 & a_1 & a_2 \\ b_0 & b_1 & b_2 & 0 \\ 0 & b_0 & b_1 & b_2 \end{vmatrix} = 0$$

Use this to prove that for all $\alpha, \beta \in \mathbf{R}$ the system

$$\begin{aligned} \alpha x^2 + x + & (1-\alpha) = 0 \\ (1-\beta)x^2 + x \quad & +\beta = 0 \end{aligned}$$

has a solution.

7) Use determinants to compute the area of the triangular ABC where $A = (3, 3), B = (7, 6)$ and $C = (3, 9)$.

8) An equation of the type $ax + by + c = 0$, where a and b are not both zero, represents a line in the plane. Prove that the equation of the line which passes through two distinct points $A = (\alpha_1, \alpha_2)$ and $B = (\beta_1, \beta_2)$ is given by the equation

$$\begin{vmatrix} x & y & 1 \\ \alpha_1 & \alpha_2 & 1 \\ \beta_1 & \beta_2 & 1 \end{vmatrix} = 0$$

9) Find the equation of the plane which passes through the three points $(1, 2, -1), (0, 1, 0)$ and $(1, 1, 1)$.

10) Do the four points $(1, 0, 2), (2, 1, 3), (4, 0, 1)$ and $(5, 1, 1)$ are all on the same plane in the space?

11) Compute the area of the parallelogram which is enclosed by the four lines $y = x - 1, y = x + 3, y = -2x + 2$ and $y = -2x + 5$.

Linear Algebra 1

Exercise Number 15

1) If $u = (x_1, x_2)$ and $v = (y_1, y_2)$, prove that $(u, v) = x_1y_1 - x_1y_2 - x_2y_1 + 3x_2y_2$ defines an inner product on $\mathbf{R}^2$.

2) Consider $\mathbf{R}^3$ with the standard inner product. For the vector $u = (2, 1, -1)$, find a vector in the direction of u which have norm 1.

3) Let V be an inner product space. Prove:

a) $||v|| \geq 0$, and $||v|| = 0$ if and only if $v = 0$.

b) $||u + v|| \leq ||u|| + ||v||$. (this is the triangular inequality)

c) $||u + v||^2 + ||u - v||^2 = 2(||u||^2 + ||v||^2)$.

4) Consider $\mathbf{R}^5$ with the standard inner product. Let W denote the subspace of $\mathbf{R}^5$ which is spanned by the vectors $(1, 2, 3, -1, 2)$ and $(2, 4, 7, 2, -1)$. Find a basis for $W^\perp$.

5) Let $u = (z_1, z_2)$ and $v = (w_1, w_2)$ be two vector in the vector space $\mathbf{C}^2$ over the field $\mathbf{C}$. Prove that

$$(u, v) = z_1\bar{w}_1 + (1 + i)z_1\bar{w}_2 + (1 - i)z_2\bar{w}_1 + 3z_2\bar{w}_2$$

defines an inner product. Compute the norm of the vector $(1 - 2i, 2 + 3i)$ with respect to this inner product.

6) Find an orthonormal basis for the following vector spaces with respect to the indicated inner product:

a) $V = M_{2\times2}(\mathbf{R})$ with respect to $(A, B) = tr(B^tA)$.

b) $V = \{(a, b, c, d) : a + b + c + d = 0\}$ with respect to the standard inner product in $\mathbf{R}^4$.

c) $W = \mathrm{Sp}\{(1, i, 1), (1 + i, 0, 2)\} \subset \mathbf{C}^3$ with respect to the standard inner product in $\mathbf{C}^3$.

7) (Bessel inequality) Let $\{w_1, \ldots, w_n\}$ be an orthonormal set of vectors in V. Prove that for all $v \in V$

$$\sum_{i=1}^{n} |(w_i, v)|^2 \leq ||v||^2$$

Hint: Write $v = w + \widetilde{w}$ where $w = \mathrm{Sp}\{w_1, \ldots, w_n\}$ and $\widetilde{w} = \mathrm{Sp}\{w_1, \ldots, w_n\}^\perp$.

8) Prove that if $\{w_1, \ldots, w_n\}$ is an orthonormal set of vectors in V, such that $\sum_{i=1}^{n} |(w_i, v)|^2 = ||v||^2$ for all $v \in V$, then $\{w_1, \ldots, w_n\}$ is a basis for V.

9) Let $W \subset V$ be a subspace of V. Let $v \in V$ be a vector which satisfy $(v, w) + (w, v) \leq (w, w)$ for all $w \in W$. Prove that $(v, w) = 0$ for al $w \in W$.

10) Let V be an inner product vector space . Define on V the distance function $d(u, v)$ between the two vectors $u, v \in V$ by $d(u, v) = ||u - v||$.
a) Let $V = \mathbf{R}^2$ with the standard inner product. If $u = (x_1, y_1)$ and $v = (x_2, y_2)$, write the distance $d(u, v)$ in terms of the coordinates x_i, y_i.
b) Prove that $d(u, v) \geq 0$, and $d(u, v) = 0$ if and only if $u = v$.
c) For any two vectors $d(u, v) = d(v, u)$.
d) $d(u, v) \leq d(u, w) + d(w, v)$ this is the triangular inequality.

11) Let V be an inner product vector space over the field F. Let $T : V \to F$ be a linear transformation. Prove that there is a vector $u_0 \in V$ such that $Tv = (v, u_0)$ for all $v \in V$.

Solutions Linear Algebra 1 Moed A 2014

D. Ginzburg

Problem 1: Let A and B of size n be two matrices such that $AB = 0$. Prove that $\text{rank}A + \text{rank}B \leq n$.

Solution: Let $V = \{v \in F^n : Av = 0\}$. From a Theorem we proved in class we have $\text{rank}A = n - \text{dim}V$. Plugging this into $\text{rank}A + \text{rank}B \leq n$, it is equivalent to $\text{rank}B \leq \text{dim}V$. Thus it is enough to prove the last inequality. Let $v_1, \ldots, v_n \in F^n$ denote the columns of the matrix B. Then, from matrix multiplication we deduce that $AB = 0$ is equivalent to $Av_i = 0$ for all $1 \leq i \leq n$. Hence $v_i \in V$ for all $1 \leq i \leq n$. From this we obtain $C(B) = Sp\{v_1, \ldots, v_n\} \subset V$. Hence $\text{rank}B = \text{dim}C(B) \leq \text{dim}V$.

Problem 2: Let A be a matrix of size n such that rank $A = 1$ and that $n - 2$ rows of A are the zero rows. Is it true that $det(A + I) = tr(A) + 1$?

Solution: The above identity is true. To prove it, let i and k be the non zero rows of A. Since rank $A = 1$, these two rows are proportional. Assume that the $k - th$ row is β times the $i - th$ row. Thus

$$A + I = \begin{pmatrix} 1 & & & & & & & & \\ & \ddots & & & & & & & \\ \alpha_1 & \ldots & \alpha_i + 1 & & \ldots & & \alpha_k & & \\ & & & 1 & & & & & \\ & & & & \ddots & & & & \\ & & & & & 1 & & & \\ \beta\alpha_1 & \ldots & \beta\alpha_i & & \ldots & & \beta\alpha_k + 1 & & \\ & & & & & & & 1 & \\ & & & & & & & & \ddots \end{pmatrix}$$

In other words, all diagonal elements of $A + I$ are ones except the $i - th$ and $k - th$ rows. Expending the determinant $|A + I|$ by the first row, then by the second, then by the $i - 1$, then by the $i + 1$ and so on, we obtain that

$$|A + I| = \begin{vmatrix} \alpha_i + 1 & \alpha_k \\ \beta\alpha_i & \beta\alpha_k + 1 \end{vmatrix} = (\alpha_i + 1)(\beta\alpha_k + 1) - \alpha_k\beta\alpha_i = \alpha_i + \beta\alpha_k + 1$$

On the other hand, the diagonal entries of A are all zeros except at the $i-th$ and $k-th$ rows. In these rows the diagonal elements are α_i and $\beta\alpha_k$. Hence $\mathrm{tr}A + 1 = \alpha_i + \beta\alpha_k + 1 = |A+I|$.

Problem 3: Let V be a vector space over F. Let $T : V \mapsto V$ be a linear map. Suppose that v_1, v_2, v_3 are nonzero vectors in V. Assume that there are scalers a_1, a_2, a_3, all distinct, such that $Tv_i = a_i v_i$. Prove that $\{v_1, v_2, v_3\}$ is linearly independent.

Solution: By rearranging the order, we may assume that if one of the a_i is zero then $i = 3$. Thus, we may assume that $a_1, a_2 \neq 0$. Write

$$\beta_1 v_1 + \beta_2 v_2 + \beta_3 v_3 = 0. \tag{1}$$

Apply T to this equation, and use the fact that $Tv_i = a_i v_i$ to obtain

$$a_1\beta_1 v_1 + a_2\beta_2 v_2 + a_3\beta_3 v_3 = 0. \tag{2}$$

Multiply (1) by a_1 and subtract equation (2). We obtain

$$(a_1 - a_2)\beta_2 v_2 + (a_1 - a_3)\beta_3 v_3 = 0. \tag{3}$$

Apply T to this equation,

$$a_2(a_1 - a_2)\beta_2 v_2 + a_3(a_1 - a_3)\beta_3 v_3 = 0. \tag{4}$$

Multiply (3) by a_2 and subtract equation (4). We obtain

$$(a_2 - a_3)(a_1 - a_3)\beta_3 v_3 = 0$$

Since all the a_i are distinct, we deduce that $\beta_3 = 0$. Going back to equation (3) we obtain $\beta_2 = 0$, and then, from equation (1) we obtain $\beta_1 = 0$.

Problem 4: For what values of t, the vector $v = \begin{pmatrix} 1 \\ t \\ t^2 \end{pmatrix}$ is in the column space of the matrix

$$A = \begin{pmatrix} 1+t & 1 & 1 \\ 1 & 1+t & 1 \\ 1 & 1 & 1+t \end{pmatrix}$$

All values are assumed to be in a fixed field F.

Solution: For those values of t such that $|A| \neq 0$, we have $C(A) = F^3$ and then $v \in C(A)$. Performing the two row operations $R_1 \to R_1 - (1+t)R_3$ and $R_2 \to R_2 - R_3$, the value of the determinant does not change and we obtain

$$|A| = \begin{vmatrix} 0 & -t & 1-(1+t)^2 \\ 0 & t & -t \\ 1 & 1 & 1+t \end{vmatrix} = \begin{vmatrix} -t & 1-(1+t)^2 \\ t & -t \end{vmatrix} = t^2(t+3)$$

Hence, if $t \neq 0, -3$ we have $v \in C(A)$.

When $t = 0$, then $v = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $C(A) = Sp\{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\}$. So $v \notin C(A)$.

When $t = -2$, we see that the columns of A all have the property that the sum of the coordinates is zero. On the other hand, in this case $v = \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix}$, and this vector does not have this property. Hence $v \notin C(A)$.

Problem 5: Let A, B and C be three square matrices of size n. Assume that $C(I+AB) = I$, compute the matrix $(I - BCA)(I + BA)$. Write it in the simplest form possible.

Solution: We have

$$(I - BCA)(I + BA) = I + BA - BCA - BCABA = I + BA - BCA - BC[(I + AB) - I]A =$$

$$= I + BA - BCA - BC(I + AB)A - BCA = I + BA - BCA - BIA + BCA$$

In the last equality we used the fact that $C(I + AB) = I$. Thus

$$(I - BCA)(I + BA) = I + BA - BCA - BA + BCA = I$$