Linear Algebra 1

Exercise Number 5

1) Let F be a field. Prove that the set,

$$\{\begin{pmatrix} a & -b \\ b & a \end{pmatrix} : a, b \in F\}$$

with addition and multiplication defined as in matrices, is itself a field.

2) Define in the set $\mathbf{Q}(\sqrt{2}) = \{a + \sqrt{2}b : a, b \in \mathbf{Q}\}$ the following operations. Define addition and multiplication by

$$(a + \sqrt{2}b) + (c + \sqrt{2}d) = (a + c) + \sqrt{2}(b + d)$$

$$(a + \sqrt{2}b)(c + \sqrt{2}d) = (ac + 2bd) + \sqrt{2}(ad + bc)$$

Prove that with two operations, the set $\mathbf{Q}(\sqrt{2})$ is a field.

3) Prove that $\mathbf{Z}_{20}$ is not a field.

4) Solve the following equations over the fields $\mathbf{Z}_2, \mathbf{Z}_3, \mathbf{Z}_5, \mathbf{Q}$ and $\mathbf{R}$.

a) $x^2 - 2 = 0$ b) $x^2 + 1 = 0$

c) $2x^2 - 2x + 1 = 0$ d) $x^4 - 1 = 0$.

5) Solve the following system of linear equations over $\mathbf{Z}_2$ and over $\mathbf{R}$.

$$x_1 + x_2 - x_3 = 1$$

$$x_1 - x_2 + x_3 = 1$$

$$x_1 - x_2 - x_3 = 1$$

6) Solve the following equation over the complex numbers $\mathbf{C}$.

$$\begin{aligned} ix_1 + x_2 \ - ix_3 &= i \\ -x_1 - ix_2 + x_3 &= -i \\ x_1 + x_2 \ + x_3 &= 1 \end{aligned}$$

7) a) Prove that in a field F there is a unique identity element. That is, there is a unique element $e \in F$, such that $ae = ea = a$ for all $a \in F$.

b) Prove that if in a field F we have $ab = ac$, and $a \neq 0$, then $b = c$. Here $a, b, c \in F$.

c) Prove that in a field F, if $a \neq 0$, then the equation $ax = b$ has a unique solution. Here $a, b \in F$.

d) Prove that in a field F, the identity $ab = 0$ implies that $a = 0$ or $b = 0$. Here $a, b \in F$.