Linear Algebra 1

Exercise Number 12

1) Let $S, T : \mathbf{R}^2 \to \mathbf{R}^2$ be two linear transformations defined as $S(x, y) = (x + y, 0)$ and $T(x, y) = (-y, x)$. Compute the transformations $5S - 3T, ST, TS, S^2$ and T^2.

2) Let $T : \mathbf{R}^3 \to \mathbf{R}^3$ be the linear transformation defined by $T(x, y, z) = (x+z, x-z, y)$. Prove that T is invertible and find T^{-1}.

3) Let $D : P_4(x) \to P_4(x)$ denote the linear transformation defined by $D(p) = p'$. For all $n \in \mathbf{N}$ compute D^n.

4) Let V be a vector space. Let $T : V \to V$ denote a linear transformation. Suppose that $\dim \mathrm{Im} T = \dim \mathrm{Im} T^2$, prove that $\mathrm{Ker} T \cap \mathrm{Im} T = \{0\}$.

5) Let $T : U \to V$ and $R : V \to W$ be two linear transformations. Prove that

a) $\mathrm{Im} RT \subset \mathrm{Im} R$.

b) $\mathrm{Ker} T \subset \mathrm{Ker} RT$.

c) $\mathrm{Im} RT = R(\mathrm{Im} T)$.

6) Let $V = P_2(x)$. Let

$$B = \{2x, 3x + x^2, -1\} \qquad B' = \{1, 1 + x, 1 + x + x^2\}$$

be two bases for V. Find the transformation matrix M from basis B to basis B'. Verify that for all $v \in V$, $[v]_B = M[v]_{B'}$.

7) Repeat the previous exercise with $V = F^3$ and

$$B = \{(1, 1, 1), (0, 1, 1), (0, 0, 1)\} \qquad B' = \{(1, 1, 0), (1, 2, 0), (1, 2, 1)\}$$

8) Let $V = F^4$. Let $B = \{(1, 0, 1, 0), (1, 1, 0, 0), (1, 0, 0, 0), (1, 1, 1, 1)\}$. Given the matrix

$$M = \begin{pmatrix} 2 & 3 & 1 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 5 \end{pmatrix}$$

find a basis B' of V such that the matrix M is the transformation matrix from basis B to basis B'.

9) Let $T : F^2 \to F^3$ denote the linear transformation $T(x, y) = (ax, bx + cy, dy)$. Here $a, b, c, d \in F$. Given the two bases

$$B = \{(1, 0), (1, 1)\} \qquad C = \{(1, 0, 0), (1, 1, 0), (1, 1, 1)\}$$

of F^2 and F^3 respectively, compute the matrix $[T]_C^B$. Verify that $[Tv]_C = [T]_C^B[v]_B$ for all $v \in F^2$.

10) Let $V = \mathbf{C}$ be a vector space over the field $F = \mathbf{R}$. Define the linear transformation $T : V \to V$ by $Tz = \bar{z}$. Compute the matrix $[T]_B$ where $B = \{1 + i, 2 + i\}$.

11) Let $V = F^n$. Define the linear transformation $T : V \to V$ by $T((a_1, \ldots, a_n)) = (0, a_1, \ldots, a_{n-1})$. Let B denote the standard basis of V. For $k \in \mathbf{N}$ compute the matrix $[T^k]_B$.

12) Let V be a vector space and let $T : V \to V$ be a linear transformation. A subspace W of V will be said to be invariant under T, if $T(W) \subset W$. Let W and U be two invariant subspaces under T and assume that $V = U \oplus W$. Let $\dim U = n$ and $\dim W = m$. Prove that there is a basis B of V such that

$$[T]_B = \begin{pmatrix} A_1 & \\ & A_2 \end{pmatrix} \quad A_1 \in Mat_{n\times n}, \quad A_2 \in Mat_{m\times m}$$

13) Let $T : P_3(x) \to P_3(x)$ be a linear transformation defined by $T(p(x)) = xp'(x)$. Let

$$B = \{1 + x, 1 - x, x^2, x^3\} \qquad C = \{1, 1 + x, (1 + x)^2, (1 + x)^3\}$$

be two bases for $P_3(x)$. Compute $[T]_B$ and $[T]_C$. Find the transformation matrix M from basis B to basis C and verify the relation $[T]_C = M^{-1}[T]_C M$.

14) a) If A is similar to B, prove that A^t is similar to B^t.

b) Let A be an invertible matrix. Prove that AB is similar to BA.

c) Prove that two similar matrices have the same trace.

d) Prove that there are no square matrices A and B such that $AB - BA = I$.