Linear Algebra 1

Exercise Number 6

In the following exercises, V will denote a vector field over a given field F.

1) a) Prove that in V there is a unique zero element 0.

b) Prove that for all $\alpha \in F$ we have $\alpha \cdot 0 = 0$.

c) Prove that for all $v \in V$ we have $0 \cdot v = 0$.

d) Prove that $\alpha \cdot v = 0$ if and only if $\alpha = 0$ or $v = 0$.

e) Prove that for all $v \in V$, we have $(-1)v = -v$.

2) Determine which of the following subsets are subspaces. Prove your claim!

a) $W = \{(\alpha_1, \ldots, \alpha_n) : \sum_{i=1}^{n} \alpha_i = 0\} \subset F^n$.

b) The collection of all solutions over $\mathbf{R}$ of the system of equation

$$\begin{aligned} 2x + 3y + 4z &= 1 \\ x - y \ + 2z &= 2 \end{aligned}$$

c) $W = \{(b_1, b_2) : b_1 b_2 = 0\} \subset \mathbf{R}^2$.

d) Denote by $\mathbf{R}^\infty$ the vector space of all sequences of real numbers. Let W denote the subset of $\mathbf{R}^\infty$ which consists of all sequences $(x_1, x_2, \ldots)$ such that there exists a number m such that $x_j = 0$ for all $j > m$. Is W a subspace of $\mathbf{R}^\infty$?

e) With the notations of part d) let W denote all sequences which are monotonic increasing. Is W a subspace of $\mathbf{R}^\infty$?

f) Same as part d) where now

$$W = \{(x_1, kx_1, k^2x_1, \ldots); x_1, k \in \mathbf{R}\}$$

g) Let $P_n(x)$ denote the vector space of all polynomials of degree up to n with coefficients in a given field F. Is $W = \{p(x) \in P_n(x) : p(1) = 0\}$ a subspace of $P_n(x)$?

3) For $1 \leq i \leq n$, let U_i be a subspace of V. Prove that $U_1 \bigcap U_2 \bigcap \ldots \bigcap U_n$ is a subspace of V.

4) Let $v_1, v_2 \in \mathbf{R}^3$. Prove that the set of all linear combinations of these two vectors, is a subspace of $\mathbf{R}^3$. In other words, let $W = \{\alpha_1 v_1 + \alpha_2 v_2 : \alpha_1, \alpha_2 \in \mathbf{R}\}$. Prove that W is a subspace of $\mathbf{R}^3$.

5) Let $V = F = \mathbf{R}$. Find all the subspace of V.

6) Let $W = \{A \in Mat_{n\times n}(F) : tr(A) = 0\} \subset Mat_{n\times n}(F)$. Is W a subspace?