SOLUTIONS MOED A LINEAR ALGEBRA 1 A 2020

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1. Moed A

Problem 1: Let $P_2(x)$ denote the vector space consisting of all polynomials of degree at most two, with coefficients in a field F. Let T denote the linear map $T : P_2(x) \to P_2(x)$ defined by $T(p(x)) = \alpha p''(x) + \beta p'(x) + \gamma p(x)$. Find all $\alpha, \beta, \gamma \in F$, such that $kerT = \{0\}$.

Solution: Let $p(x) = a + bx + cx^2$. Then $p'(x) = b + 2cx$ and $p''(x) = 2c$. Hence,

$$T(p(x)) = \alpha p''(x) + \beta p'(x) + \gamma p(x) = 2c\alpha + \beta(b + 2cx) + \gamma(a + bx + cx^2) =$$
$$= 2\alpha c + \beta b + \gamma a + (2\beta c + \gamma b)x + \gamma cx^2$$

Hence $T(p(x)) = 0$ if and only if

$$\gamma c = 0; \qquad 2\beta c + \gamma b = 0; \qquad 2\alpha c + \beta b + \gamma a = 0$$

Suppose first that $\gamma \neq 0$. Then we must have $c = 0$. The second equation is then $\gamma b = 0$ which implies that $b = 0$. From this the third equation is $\gamma a = 0$ which implies that $a = 0$. Thus, if $\gamma \neq 0$ then $kerT = \{0\}$ for all $\alpha, \beta \in F$.

If $\gamma = 0$, then the polynomial $p(x) = a$ is in $kerT$. Thus if $\gamma = 0$ then $kerT \neq \{0\}$.

Problem 2: Let A be a matrix of size three defined over $\mathbf{Q}$. Let

$$B = \begin{pmatrix} 0 & 1 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Consider the system of equations given by

$$A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} y \\ 2x \\ 0 \end{pmatrix} \tag{1}$$

Prove that this system has a nontrivial solution if and only if $|A - B| = 0$.

Solution: Notice that

$$B \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} y \\ 2x \\ 0 \end{pmatrix} \tag{2}$$

Subtracting equation (2) from equation (1), we obtain

$$(A - B) \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \tag{3}$$

From this we deduce that $|A - B| = 0$ if and only if there is a nonzero vector (a, b, c) which is a solution to equation (3). This last equation holds if and only if

$$0 = (A - B) \begin{pmatrix} a \\ b \\ c \end{pmatrix} = A \begin{pmatrix} a \\ b \\ c \end{pmatrix} - B \begin{pmatrix} a \\ b \\ c \end{pmatrix} = A \begin{pmatrix} a \\ b \\ c \end{pmatrix} - \begin{pmatrix} b \\ 2a \\ 0 \end{pmatrix}$$

This holds if and only if

$$A \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} b \\ 2a \\ 0 \end{pmatrix}$$

Problem 3: Let V be a vector space and let W, W' and W'' be three subspaces of V such that $V = W \oplus W' = W \oplus W''$. Prove that

$$\dim(W' \cap W'') \geq \dim V - 2\dim W \tag{4}$$

Solution: Applying the first dimension Theorem we deduce that

$$\dim V = \dim W + \dim W'$$

and

$$\dim V = \dim W + \dim W''$$

Adding the two equations we get

$$2\dim V = 2\dim W + \dim W' + \dim W'' \tag{5}$$

Applying the same Theorem again we have

$$\dim(W' + W'') = \dim W' + \dim W'' - \dim(W' \cap W'')$$

or

$$\dim(W' + W'') + \dim(W' \cap W'') = \dim W' + \dim W''$$

Plugging this into equation (5) we obtain

$$2\dim V = 2\dim W + \dim(W' + W'') + \dim(W' \cap W'') \tag{6}$$

Since $W' + W''$ is a subspace of V, we have $\dim(W' + W'') \leq \dim V$. Hence, it follows from equation (6) that

$$2\dim V = 2\dim W + \dim(W' + W'') + \dim(W' \cap W'') \leq 2\dim W + \dim V + \dim(W' \cap W'')$$

Or

$$\dim V \leq 2\dim W + \dim(W' \cap W'')$$

This implies equation (4).

Problem 4: Let V denote a vector space over $\mathbf{R}$. Let $T,\ S : V \to \mathbf{R}$ denote two nonzero linear maps. Suppose that for all $v \in V$, if $T(v) \geq 0$, then $S(v) \geq 0$. Prove that $T = \alpha S$ where $\alpha \in \mathbf{R}$ and $\alpha > 0$.

Solution: We first prove that $kerT \subset kerS$. Indeed, suppose that $v \in kerT$. Then $T(v) = 0$, and also $T(-v) = 0$. Hence, we deduce that $S(v) \geq 0$ and also $S(-v) \geq 0$. The last condition implies $-S(v) \geq 0$, or $S(v) \leq 0$. Hence $S(v) = 0$, which implies $v \in kerS$. Thus $kerT \subset kerS$.

Since $T, S : V \to \mathbf{R}$ are nonzero, then $ImT = ImS = \mathbf{R}$. Hence $\dim(ImT) = \dim(ImS) = 1$. It follows from the dimension identity $\dim V = \dim(kerT) + \dim(ImT)$ that

$$\dim V = \dim(kerT) + 1$$

and similarly,

$$\dim V = \dim(kerS) + 1$$

Hence, $\dim(kerT) = \dim(kerS)$, and since $kerT \subset kerS$, we deduce that $kerT = kerS$.

Suppose that $r = \dim V$. Then, from the above equalties we deduce that $\dim kerT = \dim kerS = r - 1$. Let $\{v_1, \ldots, v_{r-1}\}$ denote a base for $kerT = kerS$. Choose v_r such that $T(v_r) \neq 0$. If needed, by replacing v_r by $-v_r$, we may assume that $T(v_r) > 0$. Hence $S(v_r) \geq 0$. We cant have $S(v_r) = 0$ since $kerT = kerS$. Hence $S(v_r) > 0$. Denote $\alpha = T(v_r)/S(v_r)$. Then $\alpha > 0$, and

$$\alpha S(v_r) = \frac{T(v_r)}{S(v_r)} S(v_r) = T(v_r)$$

Since $T(v_i) = \alpha S(v_i) = 0$ for all $1 \leq i \leq r-1$ we deduce that T and αS are equal on a base of V, and hence they are equal for all $v \in V$.