Linear Algebra 1

Exercise Number 14

1) Let A be a matrix of size $n \times n$. Prove that $|A| \neq 0$, if and only if the rows of A form an independent set of vectors in F^n.

2) Let A denote an $n \times n$ matrix with the property that in every column, the sum of all entries is zero. Compute $|A|$.

3 Prove the identity

$$\begin{vmatrix} a_{1,2}+a_{1,3} & a_{1,3}+a_{1,1} & a_{1,1}+a_{1,2} \\ a_{2,2}+a_{2,3} & a_{2,3}+a_{2,1} & a_{2,1}+a_{2,2} \\ a_{3,2}+a_{3,3} & a_{3,3}+a_{3,1} & a_{3,1}+a_{3,2} \end{vmatrix} = 2 \begin{vmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{vmatrix}$$

4) Given the matrix

$$A = \begin{pmatrix} 0 & a_{1,2} & a_{1,3} & \dots & a_{1,n} \\ -a_{1,2} & 0 & a_{2,3} & \dots & a_{2,n} \\ -a_{1,3} & -a_{2,3} & 0 & \dots & a_{3,n} \\ \vdots & \vdots & \vdots & \dots & \vdots \\ -a_{1,n} & -a_{2,n} & -a_{3,n} & \dots & 0 \end{pmatrix}$$

Prove that if n is odd then $|A| = 0$.

5) Given the $n \times n$ matrix

$$A = \begin{pmatrix} 1-n & 1 & 1 & \dots & 1 \\ 1 & 1-n & 1 & \dots & 1 \\ 1 & 1 & 1-n & \dots & 1 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 1 & 1 & 1 & \dots & 1-n \end{pmatrix}$$

prove that $|A| = 0$.

6) Consider the system of equations of the following two quadratic equations

$$a_0x^2 + a_1x + a_2 = 0$$
$$b_0x^2 + b_1x + b_2 = 0$$

where $a_0 \neq 0$ and $b_0 \neq 0$. It can be shown (you dont need to do it), that this system has a solution if and only if

$$\begin{vmatrix} a_0 & a_1 & a_2 & 0 \\ 0 & a_0 & a_1 & a_2 \\ b_0 & b_1 & b_2 & 0 \\ 0 & b_0 & b_1 & b_2 \end{vmatrix} = 0$$

Use this to prove that for all $\alpha, \beta \in \mathbf{R}$ the system

$$\begin{aligned} \alpha x^2 + x + &(1-\alpha) = 0 \\ (1-\beta)x^2 + x \quad & +\beta = 0 \end{aligned}$$

has a solution.

7) Use determinants to compute the area of the triangular ABC where $A = (3,3), B = (7,6)$ and $C = (3,9)$.

8) An equation of the type $ax + by + c = 0$, where a and b are not both zero, represents a line in the plane. Prove that the equation of the line which passes through two distinct points $A = (\alpha_1, \alpha_2)$ and $B = (\beta_1, \beta_2)$ is given by the equation

$$\begin{vmatrix} x & y & 1 \\ \alpha_1 & \alpha_2 & 1 \\ \beta_1 & \beta_2 & 1 \end{vmatrix} = 0$$

9) Find the equation of the plane which passes through the three points $(1,2,-1), (0,1,0)$ and $(1,1,1)$.

10) Do the four points $(1,0,2), (2,1,3), (4,0,1)$ and $(5,1,1)$ are all on the same plane in the space?

11) Compute the area of the parallelogram which is enclosed by the four lines $y = x-1, y = x+3, y = -2x+2$ and $y = -2x+5$.