Linear Algebra 1

Exercise Number 3

1) Denote

$$A = \begin{pmatrix} 2 & -1 & 0 \\ 1 & -1 & 3 \end{pmatrix} \qquad B = \begin{pmatrix} -1 & 1 & 1 \\ -1 & -2 & 3 \end{pmatrix}$$

$$C = \begin{pmatrix} 2 & 0 & 1 & 1 \\ 5 & -1 & 2 & 3 \\ 0 & 0 & 1 & 2 \end{pmatrix} \qquad \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$$

Compute the matrices $3A - 4B,\ CD,\ B^tA,\ DD^t,\ AC,\ BD$.

2) a) Let $\begin{pmatrix} -1 & 0 \\ 2 & -1 \end{pmatrix}$. Compute A^n. If $p(x) = 2x^3 - x + 1$ compute $p(A)$.
b) Let $A = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$. Compute A^4.

3) Let A and B be two matrices of order n. Determine which of the following statements is true and which not. Explain!
a) For all $p, q \in \mathbf{N}$ we have $A^pA^q = A^{p+q}$.
b) $(A+B)^2 = A^2 + 2AB + B^2$.
c) $(A+B)(A-B) = A^2 - B^2$.

4) Let A, B and C be three matrices of order n which satisfy the relations $A = B+C, C^2 = 0$ and $BC = CB$. Using induction, prove that for all natural number k we have

$$A^{k+1} = B^k(B + (k+1)C)$$

5) The sum of all diagonal entries of a square matrix A is called the trace of A, and is denoted by $tr(A)$. In other words, if $A = (a_{i,j})$ is a square matrix of order n, then $tr(A) = a_{1,1} + a_{2,2} + \ldots + a_{n,n}$. Prove that:
a) For any two square matrices A and B of the same size, we have $tr(A+B) = tr(A)+tr(B)$.
b) If C is a matrix of size $m \times n$ and D is a matrix of size $n \times m$, then $tr(CD) = tr(DC)$.

6) Let $S = (s_{i,j})$ denote the $n \times n$ matrix such that $s_{i,i+1} = 1$ and zero for all other entries. For all $p \in \mathbf{N}$ compute S^p.

7) An $n \times n$ matrix is a permutation matrix if its entries are zeros and ones, such that every row and every column has exactly one entry which is the number one, and all other entries are zeros. Let W_n denote the collection of all permutation matrices of order n. For example, we have

$$W_2 = \{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\}$$

a) Write down all matrices in W_3.

b) How many matrices are there in the set W_n?

c) Prove that if $w_1, w_2 \in W_n$, then $w_1 w_2 \in W_n$. Also, prove that $w_1^t \in W_n$. Compute the product $w_1 w_1^t$.

8) A square matrix A is symmetric if $A = A^t$, and it is anti-symmetric if $A^t = -A$.

a) Write down all symmetric matrices of size 3×3.

b) Prove that for any matrix A, the matrix AA^t is a symmetric matrix.

c) Prove that if A is symmetric or anti-symmetric, then $AA^t = A^t A$.

d) Two square matrices A and B are said to commute, if $AB = BA$. Prove that if A and B commute, then AB is symmetric.

e) Show by an example that if A and B symmetric then AB need not be symmetric.

f) Prove that any matrix A can be written as $A = B + C$ where B is symmetric and C is anti-symmetric.

9) Let J denote a 4×4 matrix. Let A and B be two 4×4 matrices which satisfy $A^t J A = J$ and $B^t J B = J$. Prove that $(AB)^t J (AB) = J$.

10) A square matrix A is called hermitian if $A = (\bar{A})^t$. Here $\bar{A}$ denotes the complex conjugate of A. For example, if

$$A = \begin{pmatrix} 1-i & 2 \\ i & 1+i \end{pmatrix} \quad \text{then} \quad \bar{A} = \begin{pmatrix} 1+i & 2 \\ -i & 1-i \end{pmatrix}$$

a) Give an example of a 3×3 hermitian matrix.

b) Denote $A^* = (\bar{A})^t$. Then A hermitian if $A = A^*$. Prove that for any two matrices A and B, we have $(A^*)^* = A$ and that $(A + B)^* = A^* + B^*$.

c) For any matrix A prove that AA^* and A^*A are both hermitian.