Linear Algebra 1

Exercise Number 9

1) Find $a \in \mathbf{R}$ such that the three vectors $(1,-1,1),(2,0,3)$ and $(1,1,a)$ will be linearly dependent.

2) In $\mathbf{C}^2$, are the vectors $(1,i)$ and $(i,-1)$ linearly independent?

3) In each of the following cases, find a basis for the corresponding vector spaces:

a) The vector space which consists of all real solutions to the linear system

$$\begin{aligned} x+y \quad +t \quad &= 0 \\ 2x+4y+3z &= 0 \end{aligned}$$

b) All symmetric matrices of size $n \times n$.

c) The vector space $\mathbf{Q}(\sqrt{2})$ as a vector space over the field $\mathbf{Q}$.

d) The vector space $\mathbf{C}^2$ over the field $\mathbf{C}$.

e) The vector space $\mathbf{C}^2$ over the field $\mathbf{R}$.

f) Let $U = \{(a,b,c) : a+b+c=0\}$ and $W = \{(a,b,c) : a=b=c\}$ be two subspaces of F^3. Find a basis for $U \cap W$ and $U+W$.

g) In $V = P_3(x)$, let $W = \{p(x) \in P_3(x) : p(0) = p'(0) = p''(0) = 0\}$. Find a basis for W.

4) If A is a given matrix, prove that the dimension of the row space of A equals the rank of A. In other words, prove that $\dim R(A) = \mathrm{rank}(A)$.

5) In a given vector space V, prove that any subset of vectors which contains the zero vector, is linearly dependent.

6) Prove that if $K \subset V$ is a linearly independent set of vectors, then any subset of K is linearly independent.

7) If $T \subset K \subset V$, and if T is linearly dependent, prove that K is also linearly dependent.

8) Let $v_1 = (1,1,1,1)$ and $v_2 = (2,1,1,1)$. Complete the set $\{v_1, v_2\}$ to a basis of F^4.

9) In $\mathbf{R}^n$ let $W = \{(a_1, a_2, \ldots, a_n) : a_1 + a_2 + \ldots + a_n = 0\}$. What is $\dim W$? Find a subspace $U \subset \mathbf{R}^n$ such that $\mathbf{R}^n = U \oplus W$.

10) In $Mat_{n\times n}(F)$, let $W = \{A \in Mat_{n\times n}(F) : \mathrm{tr}(A) = 0\}$. Find a basis for W.

11) a) Let $A \in Mat_{n\times n}(\mathbf{R})$. Given $\lambda \in R$, define $V_\lambda = \{v \in Mat_{n\times 1}(\mathbf{R}) : Av = \lambda v\}$. Prove that V_λ is a subspace of $Mat_{n\times 1}(\mathbf{R})$.

b) Let $A = \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix}$, and let $\lambda = 2$. Find a basis and compute the dimension of V_2.