Solution of Moad A in Linear Algebra

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1) We know that the systems $Ax = 0$ and $Bx = 0$ will have exactly the same set of solutions if and only if the row echelon forms of these two matrices is the same. We start with the matrix B. Preforming the row operations $R_1 \to R_2 - R_1$, then $R_3 \to R_3 - 2R_1$, and then $R_3 \to R_3 - R_2$ we obtain the matrix

$$B_1 = \begin{pmatrix} 1 & \gamma & 2 & \delta & 1 \\ 0 & \gamma & \gamma & -\gamma & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

On A we operate by $R_3 \to R_3 - R_1$ and then by $R_3 \to R_3 - R_2$ and we obtain the matrix

$$A_1 = \begin{pmatrix} 1 & \alpha & 2 & \beta - 1 & 1 \\ 0 & \alpha & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Hence, to get from these two matrices the same echelon form matrix, we must have $\alpha \neq 0$ and $\gamma \neq 0$. Assuming that we operate on B_1 by $R_1 \to R_1 - R_2$ and then $R_2 \to \gamma^{-1}R_2$. On A_1 we operate by $R_1 \to R_1 - R_2$. We obtain the matrices

$$A_2 = \begin{pmatrix} 1 & 0 & 1 & \beta & 1 \\ 0 & \alpha & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad B_2 = \begin{pmatrix} 1 & 0 & 2 - \gamma & \gamma + \delta & 1 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Therefore, $\alpha = 1$, $2 - \gamma = 1$ and $\beta = \gamma + \delta$. Thus, the values which will give the same set of solutions are $\alpha = \gamma = 1$ and $\beta = \delta + 1$.

2) a) Let $v \in W \cap U$. We need to prove that $Tv \in W \cap U$. Since $v \in W$, and it is given that W is T invariant, then $Tv \in W$. Similarly, since $v \in U$, and it is given that U is T invariant, then $Tv \in U$. Hence $Tv \in W \cap U$.

b) Suppose that $\dim V = n$. Let W denote an arbitrary subspace of V whose dimension is $n - 2$. Pick a base $\{w_1, \ldots, w_{n-2}\}$ of W. By a theorem we proved in class, we can complete it to a base of V. So assume that $\{w_1, \ldots, w_{n-2}, u_1, u_2\}$ is a base for V. Let $U_1 = Sp\{w_1, \ldots, w_{n-2}, u_1\}$ and $U_2 = Sp\{w_1, \ldots, w_{n-2}, u_2\}$. Then $dim U_i = n - 1$ for $i = 1, 2$. Since it is given that every subspace of V whose dimension is $n - 1$ is T invariant, it follows that U_1 and U_2 are T invariant. By part **a**) we have that $W = U_1 \cap U_2$ is T invariant. Thus we proved that every subspace of V whose dimension is $\dim V - 2$, is T invariant. Continuing by induction we deduce that every subspace of dimension one is T invariant. Choose a base $B = \{v_1, \ldots, v_n\}$ of V. Since $Sp\{v_i\}$ is T invariant, it follows that $Tv_i = \alpha_i v_i$. Thus with respect to the base B we have

$$[T]_B = \begin{pmatrix} \alpha_1 & 0 & 0 & \cdots & 0 \\ 0 & \alpha_2 & 0 & \cdots & 0 \\ 0 & 0 & \alpha_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \alpha_n \end{pmatrix}$$

3) First, for all $1 \leq i \leq n-1$, we apply the row operations $R_i \to R_i - R_n$. This does not change the value of the determinant. Hence, we obtain that $|A| = |B|$, where

$$B = \begin{pmatrix} b & 0 & 0 & \ldots & 0 & -b \\ 0 & b & 0 & \ldots & 0 & -b \\ 0 & 0 & b & \ldots & 0 & -b \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a & a & a & \ldots & a & a+b \end{pmatrix}$$

Next, for each $1 \leq i \leq n-1$, preform the column operation $C_i \to C_i + C_n$. Once again, this does not change the value of the determinant, and we obtain that $|A| = |C|$ where C is now lower diagonal, whose diagonal is $(b, b, b, \ldots, b, na+b)$. Therefore, the determinant of C is the product of all diagonal elements, and we obtain that $|A| = (na+b)b^{n-1}$.

Another way is to argue by induction on n. Denote $\Delta_n = |A|$. Let B denote the matrix obtained from A by the row operation $R_1 \to R_1 - R_2$. Then $|A| = |B|$. The first row of B is $\begin{pmatrix} b & -b & 0 & \cdots & 0 \end{pmatrix}$. All other rows of B are as the rows of A. Develop the determinant of B using the first row. We obtain $\Delta_n = b|C_{n-1}| + b|D_{n-1}|$. Here C_{n-1} is the matrix of size $n-1$ whose diagonal elements are $a+b$ and all other entries are a. Therefore, by induction, we have $|C_{n-1}| = \Delta_{n-1}$. The matrix D_{n-1} is defined as follows

$$D_{n-1} = \begin{pmatrix} a & a & a & \ldots & a \\ a & a+b & a & \ldots & a \\ a & a & a+b & \ldots & a \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a & a & a & \ldots & a+b \end{pmatrix}$$

In other words, all the diagonal elements of D_{n-1} except the $(1,1)$ entry, are $a+b$. All other entries of D_{n-1} are a. For $2 \leq i \leq n-1$, preform on this matrix the row operations $R_i \to R_i - R_1$. Each of these operations does not change the value of the determinant, and therefore $|D_{n-1}| = |E_{n-1}|$, where

$$E_{n-1} = \begin{pmatrix} a & a & a & \dots & a \\ 0 & b & 0 & \dots & 0 \\ 0 & 0 & b & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & b \end{pmatrix}$$

Since E_{n-1} is upper diagonal its determinant is the product of all diagonal elements.

Overall we obtain $\Delta_n = b\Delta_{n-1} + ab^{n-1}$. To compute this explicitly, we use induction to obtain $\Delta_{n-1} = b\Delta_{n-2} + ab^{n-2}$. Plugging this above, we obtain $\Delta_n = b^2\Delta_{n-2} + 2ab^{n-1}$. Continuing this process by induction we obtain $\Delta_n = b\Delta_1 + (n-1)ab^{n-1}$. Since $\Delta_1 = a+b$, we obtain $\Delta_n = b^n + nab^{n-1}$.

4) We have $T(1) = 1 + x^2$. Hence if we choose the vector $1 + x^2$ to be the first vector in the base C, then by definition we have

$$[T]_B^C = \begin{pmatrix} 1 & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix}$$

Here the stars indicates entries to be determined. Next we have $T(x+1) = 2 + x + 2x^2$. Write $2 + x + 2x^2 = \alpha(1 + x^2) + p(x)$. Clearly $p(x) = x$ and $\alpha = 2$. Therefore, if we choose the second vector in C to be x, then by definition

$$[T]_B^C = \begin{pmatrix} 1 & 2 & * \\ 0 & 1 & * \\ 0 & 0 & * \end{pmatrix}$$

Notice that the set $\{1+x^2, x\}$ is linearly independent. Finally, we have $T(x^2+1) = 1-x+2x^2$. Write $1 - x + 2x^2 = \alpha(1 + x^2) + \beta x + q(x)$. Then $\alpha = 2$, $\beta = -1$, and $q(x) = -1$. Thus, if we choose as the third element of C the vector -1, we get

$$[T]_B^C = \begin{pmatrix} 1 & 2 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

It is clear that $C = \{x^2 + 1, x, -1\}$ is a base for $P_2(x)$.

5) a) To prove the statement we need to prove that if $\alpha_1 v_1 + \alpha_2 v_2 = 0$, then $\alpha_1 = \alpha_2 = 0$. Since $|a_{1,1}| > |a_{2,1}|$ then $a_{1,1} \neq 0$. Therefore $v_1 \neq 0$. Similarly, $v_2 \neq 0$. Hence, if $\alpha_1 = 0$ then $\alpha_1 v_1 + \alpha_2 v_2 = 0$ implies that $\alpha_2 = 0$. Similarly, if $\alpha_2 = 0$ then $\alpha_1 = 0$. Therefore we may assume that both α_1 and α_2 are nonzero. We shall derive a contradiction. From $\alpha_1 v_1 + \alpha_2 v_2 = 0$ we deduce that $\alpha_1 a_{1,1} + \alpha_2 a_{2,1} = 0$. Hence, we obtain $|\alpha_1||a_{1,1}| = |\alpha_2||a_{2,1}|$.

From the fact that $|a_{1,1}| > |a_{2,1}|$ and that $\alpha_2 \neq 0$ it follows that $|\alpha_1||a_{1,1}| = |\alpha_2||a_{2,1}| < |\alpha_2||a_{1,1}|$. Since $a_{1,1} \neq 0$, it follows that $|\alpha_1| < |\alpha_2|$. Similarly, using the fact that $\alpha_1 \neq 0$, the equation $\alpha_1 a_{1,2} + \alpha_2 a_{2,2} = 0$ implies that $|\alpha_2| < |\alpha_1|$. Thus we derived a contradiction, and hence the set $\{v_1, v_2\}$ is independent.

Another way to prove it is to observe that the linear system $\alpha_1 a_{1,1} + \alpha_2 a_{2,1} = 0$ and $\alpha_1 a_{1,2} + \alpha_2 a_{2,2} = 0$ can be written as

$$\begin{pmatrix} a_{1,1} & a_{2,1} \\ a_{1,2} & a_{2,2} \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = 0$$

Let A denote the left most matrix in the above equality. Therefore, the set $\{v_1, v_2\}$ is linearly independent if and only if the above system has only the trivial solution if and only if $\det A = 0$. By the triangular inequality we have $|\det A| = |a_{1,1}a_{2,2} - a_{1,2}a_{2,1}| \geq ||a_{1,1}a_{2,2}| - |a_{1,2}a_{2,1}|| > 0$ where the last inequality follows from the fact that $|a_{1,1}| > |a_{2,1}|$ and $|a_{2,2}| > |a_{1,2}|$. Thus $\det A \neq 0$.

b) Since $\ker S$ is not contained in $\ker T$ it follows that $T \neq 0$. Therefore $\dim \mathrm{Im} T \geq 1$, and since it is given that $\dim W = 1$, it follows that $\mathrm{Im} T = W$. It follows from the dimension theorem ($\dim V = \dim\ker T + \dim \mathrm{Im} T$) that $\dim\ker T = \dim V - 1$. Using again the fact that $\ker S$ is not contained in $\ker T$, we deduce that $\ker S + \ker T = V$. Indeed, since $\dim V = n$, we can choose a base $\{v_1, \ldots, v_{n-1}\}$ for $\ker T$. Let u be a vector in $\ker S$ not in $\ker T$. Then $\{v_1, \ldots, v_{n-1}, u\}$ is a base for V. Hence, from the first dimension theorem,

$$\dim V = \dim(\ker S + \ker T) = \dim\ker S + \dim\ker T - \dim(\ker S \cap \ker T) =$$

$$= \dim\ker S + \dim V - 1 - \dim(\ker S \cap \ker T)$$

from which the result follows.