Solution of Moed A in Linear Algebra 1 2022

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1) Problem: Let $T : F^3 \to F^2$ given by $T(x, y, z) = (x + 2z, x + y + z)$. Find all bases $\mathcal{B} = \{(a, b, c); (1, 1, 1); (0, 0, 1)\}$ of F^3, and all bases $\mathcal{C} = \{(n_1, m_1); (n_2, m_2)\}$ of F^2 such that

$$[T]_{\mathcal{C}}^{\mathcal{B}} = \begin{pmatrix} 1 & 3 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

Solution: By definition we have $T((a, b, c)) = (a+2c, a+b+c) = 1(n_1, m_1)+1(n_2, m_2)$. Also, $T((1, 1, 1)) = (3, 3) = 3(n_1, m_1) + 0(n_2, m_2)$, and $T((0, 0, 1)) = (2, 1) = 0(n_1, m_1) + 1(n_2, m_2)$. From this we obtain $(n_1, m_1) = (1, 1)$ and $(n_2, m_2) = (2, 1)$. Hence $(a+2c, a+b+c) = (n_1, m_1) + (n_2, m_2) = (3, 2)$. Thus, we need to solve the system of equations $a + 2c = 3$ and $a+b+c = 2$. Subtracting the two equations, we obtain that the set of solutions is given by $\{(3 - 2c, c - 1, c) : c \in F\}$.

We need to check for what values of c, the set $\mathcal{B} = \{(3 - 2c, c - 1, c); (1, 1, 1); (0, 0, 1)\}$ is a base for F^3. Write $\alpha(3 - 2c, c - 1, c) + \beta(1, 1, 1) + \gamma(0, 0, 1) = 0$. This gives us the system of equations $(3 - 2c)\alpha + \beta = 0$, $(c - 1)\alpha + \beta = 0$ and $c\alpha + \beta + \gamma = 0$. Subtracting the first two equations, we obtain $(4-3c)\alpha = 0$. If $c \neq 4/3$ then $\alpha = 0$. This implies that $\beta = \gamma = 0$. Hence, if $c \neq 4/3$, we obtain that $\mathcal{B}$ is a base for F^3.

It is easy to check that $\mathcal{C} = \{(1, 1); (2, 1)\}$ is an independent set of vectors in F^2. Hence it is a base for F^2.

2) Problem: Let $a, b, c, d \in \mathbf{R}$ be four numbers such that one of them is zero and the other three are positive. Prove that the determinant of the matrix

$$A = \begin{pmatrix} a & -1 & -1 & -1 \\ -1 & b & -1 & -1 \\ -1 & -1 & c & -1 \\ -1 & -1 & -1 & d \end{pmatrix}$$

is a negative number.

Solution: Without loose of generality, we may assume that $a = 0$ and b, c, d are positive. Indeed, if $b = 0$, then the two operations $R_1 \leftrightarrow R_2$ and then $C_1 \leftrightarrow C_2$ will cause the $(1, 1)$ entry of the matrix to be zero. These two operations do not change the value of the determinant. Similarly if c or d are zero.

Assuming $a = 0$, for $2 \le i \le 4$, perform the operations $C_i \to C_i - C_1$. We obtain the matrix

$$B = \begin{pmatrix} 0 & -1 & -1 & -1 \\ -1 & b+1 & 0 & 0 \\ -1 & 0 & c+1 & 0 \\ -1 & 0 & 0 & d+1 \end{pmatrix}$$

Clearly, $|A| = |B|$. On the matrix B, perform $C_1 \to C_1 + \frac{1}{b+1}C_2$, then $C_1 \to C_1 + \frac{1}{c+1}C_3$, and $C_1 \to C_1 + \frac{1}{d+1}C_3$. Since b, c and d are all positive, these operations are well defined. We obtain

$$C = \begin{pmatrix} \alpha & -1 & -1 & -1 \\ 0 & b+1 & 0 & 0 \\ 0 & 0 & c+1 & 0 \\ 0 & 0 & 0 & d+1 \end{pmatrix}$$

where

$$\alpha = -\frac{1}{b+1} - \frac{1}{c+1} - \frac{1}{d+1}$$

We have $|B| = |C|$. Notice that $\alpha < 0$. Since $|A| = |C| = \alpha(b + 1)(c + 1)(d + 1)$, the result follows.

3) Problem: Let V and W denote two vector spaces over F. Let $S_1, S_2 : V \to W$ denote two linear maps such that $\ker S_1 = \ker S_2$. Prove that there is an invertible map $T : W \to W$ such that $S_1 = TS_2$.

Solution: Let $K = \ker S_1 = \ker S_2$. Let U denote any subspace of V such that $V = K \oplus U$. Let $\{u_1, \ldots, u_r\}$ denote a base for U. Then, $\mathrm{Im}S_1 = Sp\{S_1(u_1), \ldots, S_1(u_r)\}$. Since $K \cap U = \{0\}$, then $\{S_1(u_1), \ldots, S_1(u_r)\}$ is a linear independent set of vectors in W. (This was done in class: Indeed, if $\alpha_1 S_1(u_1) + \cdots + \alpha_r S_1(u_r) = 0$, then $S_1(\alpha_1 u_1 + \cdots + \alpha_r u_r) = 0$. Since $K \cap U = \{0\}$, then $\alpha_1 u_1 + \cdots + \alpha_r u_r = 0$, which implies $\alpha_i = 0$ for all $1 \le i \le r$.)

We deduce that $\{S_1(u_1), \ldots, S_1(u_r)\}$ is a base for $\mathrm{Im}S_1$, and hence U is isomorphic to $\mathrm{Im}S_1$. In a similar way, U is isomorphic to $\mathrm{Im}S_2$. In particular $\mathrm{Im}S_1$ is isomorphic to $\mathrm{Im}S_2$. Write $W = W_1 \oplus \mathrm{Im}S_1$, and $W = W_2 \oplus \mathrm{Im}S_2$. Then W_1 is isomorphic to W_2. Let $T : W \to W$

denote a linear map which maps $S_2(u_i)$ to $S_1(u_i)$ for all $1 \leq i \leq r$, and which maps a base of W_2 to a base of W_1. This is possible since W_1 is isomorphic to W_2, and $\mathrm{Im}S_1$ is isomorphic to $\mathrm{Im}S_2$. Hence T is invertible, and $TS_2(u_i) = S_1(u_i)$ for all $1 \leq i \leq r$. Given $v \in V$, let $v = v_0 + (\alpha_1 u_1 + \cdots + \alpha_r u_r)$ where $v_0 \in \ker S_1 = \ker S_2$. Then

$$TS_2(v) = TS_2(v_0) + \alpha_1 TS_2(u_1) + \cdots + \alpha_r TS_2(u_r) = \alpha_1 TS_2(u_1) + \cdots + \alpha_r TS_2(u_r) =$$

$$= \alpha_1 S_1(u_1) + \cdots + \alpha_r S_1(u_r) = S_1(v_0) + S_1(\alpha_1 u_1 + \cdots + \alpha_r u_r) = S_1(v)$$

Thus, $S_1 = TS_2$.

4) Problem: a) (12 points) Let $T : V \to W$ be a surjective map with the following property. For every finite set of vectors K in V, if $\mathrm{Sp}\{T(K)\} = W$, then $\mathrm{Sp}\{K\} = V$. Prove that T is injective.

b) (13 points) Let A be an invertiable matrix of size $n \times n$. Define $T : Mat_{n\times n}(F) \to Mat_{n\times n}(F)$ by $T(B) = A^t B + B^t A$. Prove that the map T is not an injective map.

Solution: Assume that $v \in V$ is a nonzero vector in $\ker T$. We will derive a contradiction. Complete this vector to a base $\mathcal{B} = \{v, u_1, \ldots, u_r\}$ of V. We have $\mathrm{Im}T = \mathrm{Sp}\{Tv, Tu_1, \ldots, Tu_r\}$. Since T is surjective, then $\mathrm{Sp}\{Tv, Tu_1, \ldots, Tu_r\} = W$. Let $K = \{u_1, \ldots, u_r\} \subset V$. Since we assumed that $v \in \ker T$, then $\mathrm{Sp}\{T(K)\} = \mathrm{Sp}\{Tu_1, \ldots, Tu_r\} = \mathrm{Sp}\{Tv, Tu_1, \ldots, Tu_r\} = W$. From the property of T, we deduce that the set $K = \{u_1, \ldots, u_r\}$ spans V. But this is impossible since $\mathcal{B}$ is a base for V, and hence it is a minimal set which spans V. This is a contradiction, and hence $\ker T = \{0\}$, and T is injective.

b) Suppose that T is an injective map. Then T must be an isomorphism. This follows from a Theorem we proved in class that every injective linear map from a vector space to itself, must be an isomorphism. Hence T is a surjective map. However, the matrix $A^t B + B^t A$ is a symmetric matrix. In other words, we have $(A^t B + B^t A)^t = A^t B + B^t A$. Since in $Mat_{n\times n}(F)$ there are non-symmetric matrices, then T cannot be surjective, and hence cannot be injective.