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בחינה באלגברה לינארית 1 א

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- משך הבחינה 3 שעות.
- יש לענות על כל השאלות.
- אין להשתמש בכל חומר עזר לרבות מחשבון.
- יש לנמק היטב את דרך הפיתרון.

שאלה 1 *(25 נ')* לכל $\lambda \in \mathbb{R}$ מצאו את דרגת המטריצה

$$A=\begin{pmatrix} 2\lambda & -1 & 2 \\ -2 & 1+\lambda & 2-3\lambda \\ -3 & -1 & 5 \end{pmatrix}.$$

שאלה 2 *(25 נ')* חשבו את הדטרמיננטה של המטריצה $A=(a_{i,j})_{1\le i,j\le n}$. כאן $a_{i,j}=a_{j,i}$ לכל $1\le i,j\le n$, ובנוסף $a_{1,j}=1$ לכל $1\le j\le n$, ו- $a_{i,j}=(i-1)x$ לכל $2\le i\le j\le n$. במילים אחרות,

$$A=\begin{pmatrix} 1 & 1 & 1 & \dots & 1 & 1 \\ 1 & x & x & \dots & x & x \\ 1 & x & 2x & \dots & 2x & 2x \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & x & 2x & \dots & (n-2)x & (n-2)x \\ 1 & x & 2x & \dots & (n-2)x & (n-1)x \end{pmatrix}.$$

שאלה 3 *(25 נ')* יהי V מרחב וקטורי מעל השדה F ויהי U תת מרחב לא טריביאלי של V. הוכיחו כי אם קיים תת מרחב <u>יחיד</u> W של V כך ש- $U\oplus W=V$, אז $U=V$.

שאלה 4 א. *(13 נ')* יהי V מרחב וקטורי ממימד n מעל השדה F, ותהי $T:V\to V$ העתקה לינארית המקיימת $T^2=0$ ו- $\dim(\mathrm{Ker}T)=r$. הוכיחו כי $n\le 2r$.

ב. *(12נ')* האם קיימות העתקות לינאריות $T:\mathbb{R}^{10}\to\mathbb{R}^{14}$ ו- $S:\mathbb{R}^{14}\to\mathbb{R}^{5}$ כך ש- T העתקה חד-חד-ערכית, S העתקת על ו- $S\circ T$ העתקת האפס ?

בהצלחה!

Solution 2: The computations are straightforward but the idea is to simplify them as possible. It is convenient to start by interchanging the first and second row, and then perform the row operations

$$\begin{pmatrix} -2 & 1+\lambda & 2-3\lambda \\ 2\lambda & -1 & 2 \\ -3 & -1 & 5 \end{pmatrix} \xrightarrow[R_3 \to 2R_3 - 3R_1]{R_2 \to R_2 + \lambda R_1} A_1 = \begin{pmatrix} -2 & 1+\lambda & 2-3\lambda \\ 0 & -1+\lambda(\lambda+1) & 2+\lambda(2-3\lambda) \\ 0 & -5-3\lambda & 4+9\lambda \end{pmatrix}$$

Since row operations do not change the rank of the matrices, then $\operatorname{rank}(A) = \operatorname{rank}(A_1)$. Let

$$B = \begin{pmatrix} -1+\lambda(\lambda+1) & 2+\lambda(2-3\lambda) \\ -5-3\lambda & 4+9\lambda \end{pmatrix}$$

Then we have $\operatorname{rank}(A) = \operatorname{rank}(A_1) = 1 + \operatorname{rank}(B)$. Indeed, this follows from the fact that $\operatorname{rank}(A)$ is the number of non-zero rows in the row echelon matrix which is obtained from A by row operations.

To determine $\operatorname{rank}(B)$, we notice first that $\operatorname{rank}(B) = 0$ if and only if $B = 0$. For $B = 0$ we must have $-5-3\lambda = 4+9\lambda = 0$. This cannot happen, and hence $\operatorname{rank}(B) = 1, 2$. Second, $\operatorname{rank}(B) = 2$ if and only if B is invertible, or if and only if $|B| \neq 0$. We have

$$|B| = [-1+\lambda(\lambda+1)][4+9\lambda] - [2+\lambda(2-3\lambda)][-5-3\lambda] = 4\lambda^2 + 11\lambda + 6 = (4\lambda+3)(\lambda+2)$$

Hence $|B| = 0$ if and only if $\lambda = -\frac{3}{4}, -2$.

From this we conclude that $\operatorname{rank}(A) = 2$ if $\lambda = -\frac{3}{4}, -2$, and in all other cases $\operatorname{rank}(A) = 3$.

Solution 2: Write the last row of the matrix A as

$$\begin{pmatrix}1 & x & 2x & \dots & (n-2)x & (n-1)x\end{pmatrix} =$$
$$\begin{pmatrix}1+0 & x+0 & 2x+0 & \dots & (n-2)x+0 & (n-2)x+x\end{pmatrix}$$

Then by the property of addition in rows we obtain

$$|A| = \begin{vmatrix} 1 & 1 & 1 & \dots & 1 & 1 \\ 1 & x & x & \dots & x & x \\ 1 & x & 2x & \dots & 2x & 2x \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & x & 2x & \dots & (n-2)x & (n-2)x \\ 1 & x & 2x & \dots & (n-2)x & (n-2)x \end{vmatrix} + \begin{vmatrix} 1 & 1 & 1 & \dots & 1 & 1 \\ 1 & x & x & \dots & x & x \\ 1 & x & 2x & \dots & 2x & 2x \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & x & 2x & \dots & (n-2)x & (n-2)x \\ 0 & 0 & 0 & \dots & 0 & x \end{vmatrix}$$

The first determinant on the right hand side is zero because it has two equal rows. Computing the second determinant on the right by expanding over the last row, we obtain

$$|A| = x \begin{vmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & x & x & \dots & x \\ 1 & x & 2x & \dots & 2x \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x & 2x & \dots & (n-2)x \end{vmatrix}$$

Thus, if we denote $A_n = A$, then the above equality is $|A_n| = x|A_{n-1}|$. Arguing by induction we obtain

$$|A| = |A_n| = x|A_{n-1}| = x^2|A_{n-2}| = x^{n-2}|A_2| = x^{n-2}\begin{vmatrix} 1 & 1 \\ 1 & x \end{vmatrix} = x^{n-2}(x-1)$$

Solution 3: We will assume that U is a proper subspace of V and derive a contradiction. Assume that $r = \dim U$ and $n = \dim V$. Thus, we assume that $r < n$. Let $\mathcal{B} = \{u_1, \ldots, u_r\}$ be a basis for U. Complete it to a basis $\mathcal{B}_1$ of V. Denote $\mathcal{B}_1 = \{u_1, \ldots, u_r, v_1, v_2, \ldots, v_{n-r}\}$. Let $W_1 = \mathrm{Sp}\{v_1, v_2, \ldots, v_{n-r}\}$. Then W_1 is a subspace of V which satisfies $U \oplus W_1 = V$.

Let $W_2 = \mathrm{Sp}\{u_r + v_1, v_2, \ldots, v_{n-r}\}$. We claim that $U \oplus W_2 = V$, and that $W_1 \neq W_2$. This will be a contradiction. Denote $\mathcal{B}_2 = \{u_1, \ldots, u_r, u_r + v_1, v_2, \ldots, v_{n-r}\}$. Since $\mathcal{B}_2$ is a set which contains n vectors, then to prove that it is a basis for V, it is enough to prove that $V = \mathrm{Sp}(\mathcal{B}_2)$. To do that it is enough to prove that $v_1 \in \mathrm{Sp}(\mathcal{B}_2)$. This follows from the trivial identity $v_1 = (-1)u_r + (u_r + v_1)$. Hence, $U + W_2 = V$. To prove that $U \cap W_2 = \{0\}$ let $w \in U \cap W_2$. Then

$$w = \alpha_1 u_1 + \cdots + \alpha_r u_r = \beta_1(u_r + v_1) + \beta_2 v_2 + \cdots + \beta_{n-r} v_{n-r}$$

It follows from the right hand side identity that

$$\alpha_1 u_1 + \cdots + (\alpha_r - \beta_1)u_r + (-\beta_1)v_1 + (-\beta_2)v_2 + \cdots + (-\beta_{n-r})v_{n-r} = 0$$

Since the set $\mathcal{B}_1$ is a basis for V, then it is an independent set, and hence all coefficients are zero. Thus $w = 0$. We proved that $U \oplus W_2 = V$.

To complete the proof we need to prove that $W_1 \neq W_2$. We claim that $v_1 \in W_1$ but $v_1 \notin W_2$. Clearly, $v_1 \in W_1$. Assume that $v_1 \in W_2$. Then by definition it is in $\mathrm{Sp}\{u_r + v_1, v_2, \ldots, v_{n-r}\}$. Hence there are scalars such that

$$v_1 = \alpha_1(u_r + v_1) + \alpha_2 v_2 + \cdots + \alpha_{n-r} v_{n-r}$$

Hence, moving the vector v_1 to the right hand side,

$$\alpha_1 u_r + (\alpha_1 - 1)v_1 + \alpha_2 v_2 + \cdots + \alpha_{n-r} v_{n-r} = 0$$

The set $\{u_r, v_1, v_2, \ldots, v_{n-r}\}$ is a subset of $\mathcal{B}_1$, and hence it is an independent set. This means that all coefficients in the above equality are zero. Thus we obtain $\alpha_1 = 0$ and $\alpha_1 - 1 = 0$ which is clearly a contradiction.

Solution 4: a) Let $\{v_1, \ldots, v_r\}$ denote a basis for $ker\ T$. Complete it to a basis

$$\{v_1, \ldots, v_r, w_1, \ldots, w_{n-r}\}$$

of V. Then $\{Tw_1, \ldots, Tw_{n-r}\}$ is a linearly independent set in V. The proof of that is similar to a corresponding proof in the dimension theorem. In some details, assume that $\alpha_1 Tw_1 + \cdots + \alpha_{n-r} Tw_{n-r} = 0$. Then $T(\alpha_1 w_1 + \cdots + \alpha_{n-r} w_{n-r}) = 0$. Hence $\alpha_1 w_1 + \cdots + \alpha_{n-r} w_{n-r} \in \ker T$. From this we obtain that there are scalars β_i such that

$$\alpha_1 w_1 + \cdots + \alpha_{n-r} w_{n-r} = \beta_1 v_1 + \cdots + \beta_r v_r$$

Since the set $\{v_1, \ldots, v_r, w_1, \ldots, w_{n-r}\}$ is a basis, then $\alpha_1 = \ldots = \alpha_{n-r} = 0$.

Because $T^2 = 0$, for all $1 \leq i \leq n-r$ we have $T(Tw_i) = T^2 w_i = 0$. Hence, $\{Tw_1, \ldots, Tw_{n-r}\}$ is a linearly independent set inside $ker\ T$. Since $\dim(kerT) = r$, we obtain $n - r \leq r$, or $n \leq 2r$.

b) Suppose that $S \circ T = 0$. Then $\mathrm{Im}T \subset \ker S$. Indeed, if $v \in \mathrm{Im}T$, then there is $u \in \mathbf{R}^{10}$ such that $v = Tu$. Hence $0 = (S \circ T)(u) = S(Tu) = Sv$. Hence $v \in \ker S$. From this we deduce that $\dim(\mathrm{Im}T) \leq \dim(\ker S)$.

Apply the dimension Theorem to the two linear maps. We have

$$10 = \dim \mathbf{R}^{10} = \dim(\ker T) + \dim(\mathrm{Im}T) = \dim(\mathrm{Im}T) \tag{2}$$

$$14 = \dim \mathbf{R}^{14} = \dim(\ker S) + \dim(\mathrm{Im}S) = \dim(\ker S) + 5 \tag{3}$$

The right most equality in equation (2) follows from the fact that T is one-to-one. The right most equality in equation (3) follows from the fact that S is onto. Hence, $\dim(\mathrm{Im}T) = 10$ and $\dim(\ker S) = 9$. This a contradiction to the above inequality $\dim(\mathrm{Im}T) \leq \dim(\ker S)$.