Linear Algebra 1

Exercise Number 13

1) Let $\mathbf{C}$ be a vector space over $\mathbf{R}$. Let $B = \{1+i, 1-i\}$ and $C = \{1, i\}$ be two bases.
a) For the linear transformation $Tz = \bar{z}$, write down the matrices $[T]_B$ and $[T]_C$.
b) Find the transformation matrix M between the bases B and C, and write down the relation between M, $[T]_B$ and $[T]_C$.
c) Repeat parts a) and b) for the linear transformation $Sz = iz$.

2) a) Let $A \in Mat_{n\times n}(F)$ a matrix which satisfies $A^n = 0$ and $A^{n-1} \neq 0$ for some positive integer n. (Such a matrix is called a nilpotent matrix of order n). Let $v \in F^n$ be such that $A^{n-1}v \neq 0$. Prove that $\{v, Av, A^2v, \ldots, A^{n-1}v\}$ is a basis for F^n.
b) Let B be another nilpotent matrix of order n. Prove that A and B are similar. (*Hint:* Find the matrix $[T_A]$ with respect to the basis in part a). Here $T_A : F^n \to F^n$ is the linear transformation defined by $T_A u = Au$ for all $u \in F^n$).

3) Compute the determinant of the following matrices:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 0 & 0 & 2 \\ 2 & 1 & 3 & 5 \\ -1 & 1 & 1 & 1 \end{pmatrix}$$

$$C = \begin{pmatrix} 1 & 0 & 1 & 1 & 3 \\ 2 & 1 & -1 & 1 & 2 \\ 5 & 4 & 0 & -2 & 1 \\ 1 & -1 & 0 & 0 & 2 \\ 0 & 1 & 2 & -1 & -1 \end{pmatrix} \qquad D = \begin{pmatrix} 0 & i & -i & 1+i \\ i & 0 & i & -1 \\ -i & i & 0 & i \\ 1+i & -1 & i & 0 \end{pmatrix}$$

4) If $AA^t = I$, prove that $|A| = \pm 1$.

5) a) Let $A \in Mat_{r\times r}(F)$ and $B \in Mat_{(n-r)\times(n-r)}(F)$. Prove that $\det \begin{pmatrix} A & \\ & B \end{pmatrix} = \det A \det B$.
b) In the notations of part a) prove that $\det \begin{pmatrix} I_r & C \\ & B \end{pmatrix} = \det B$.
c) From a) and b) prove that $\det \begin{pmatrix} A & C \\ & B \end{pmatrix} = \det A \det B$.

6) If A is invertible, prove that $|A^{-1}| = |A|^{-1}$.

7) Give an example that $\det(A+B) \neq \det A + \det B$.

8) Using Cramer's rule, solve the following two linear system of equations:

$$\begin{aligned} x + y + z &= 1 \\ 2x - y &= 2 \\ x + 2y - z &= 0 \end{aligned} \qquad\qquad \begin{aligned} 2x + y &= 1 \\ x + 2y - 3z &= 0 \\ x + y + 2z + t &= 0 \\ -x - y + z + 2t &= 1 \end{aligned}$$

9) Compute the determinant of the following matrices

$$A = \begin{pmatrix} x & y & x+y \\ y & x+y & x \\ x+y & x & y \end{pmatrix} \qquad B = \begin{pmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{pmatrix}$$

$$C = \begin{pmatrix} 1 & a_1 & a_2 & \dots & a_n \\ 1 & a_1 + b_1 & a_2 & \dots & a_n \\ 1 & a_1 & a_2 + b_2 & \dots & a_n \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 1 & a_1 & a_2 & \dots & a_n + b_n \end{pmatrix}$$

10) Let A, B, C and D be matrices of size $n \times n$. Assume that $\det A \neq 0$. Prove that

$$\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(D - CA^{-1}B)\det A$$

(*Hint:* Multiply on the left by a suitable matrix whose determinant you can compute.)

11) Compute the adjoint matrix of the following two matrices. Use it to find the inverse matrix of each one of them.

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 2 & 4 \\ -1 & -1 & -1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 0 & 2 & 3 \\ 4 & 1 & 0 & 1 \\ -1 & -1 & 1 & 1 \\ 0 & 0 & -3 & -2 \end{pmatrix}$$