Solution for Final in Linear Algebra 1

Moad A 2009

1) For simplicity we shall assume that n is even. That is $n = 2m$. The case when n is odd is the same. It is given, that for $1 \le i \le 2m$ we have

$$\sum_{j=1}^{m} a_{i,2j-1} = \sum_{j=1}^{m} a_{i,2j}$$

Expressing $a_{i,1}$ in term of the others we obtain

$$a_{i,1} = \sum_{j=1}^{m} a_{i,2j} - \sum_{j=2}^{m} a_{i,2j-1}$$

Plugging this into the matrix A we obtain

$$|A| = \begin{vmatrix} \sum_{j=1}^{m} a_{1,2j} - \sum_{j=2}^{m} a_{1,2j-1} & a_{1,2} & \cdots & a_{1,2m} \\ \sum_{j=1}^{m} a_{2,2j} - \sum_{j=2}^{m} a_{2,2j-1} & a_{2,2} & \cdots & a_{2,2m} \\ \vdots & \vdots & & \vdots \\ \sum_{j=1}^{m} a_{2m,2j} - \sum_{j=2}^{m} a_{2m,2j-1} & a_{2m,2} & \cdots & a_{2m,2m} \end{vmatrix}$$

Using the additive in columns property of the determinant, we obtain

$$|A| = \sum_{j=1}^{m} \begin{vmatrix} a_{1,2j} & a_{1,2} & \cdots & a_{1,2m} \\ a_{2,2j} & a_{2,2} & \cdots & a_{2,2m} \\ \vdots & \vdots & & \vdots \\ a_{2m,2j} & a_{2m,2} & \cdots & a_{2m,2m} \end{vmatrix} - \sum_{j=2}^{m} \begin{vmatrix} a_{1,2j-1} & a_{1,2} & \cdots & a_{1,2m} \\ a_{2,2j-1} & a_{2,2} & \cdots & a_{2,2m} \\ \vdots & \vdots & & \vdots \\ a_{2m,2j-1} & a_{2m,2} & \cdots & a_{2m,2m} \end{vmatrix}$$

In each summand we have a determinant of a matrix which has two equal columns. Such a determinant is zero, and hence $|A| = 0$.

2) Multiplying both sides by $I - A$, we look for a number c which satisfies the identity $I = (I - A)(I - cA)$. This is the same as $I = I - (1 + c)A + cA^2$, or $(1 + c)A = cA^2$. From matrix multiplication it follows that the matrix A^2 is the matrix which has the value n at each of its entry. Hence, the equality $(1 + c)A = cA^2$ is equivalent to $(1 + c) = cn$. Hence, $c = \frac{1}{n-1}$.

3) Preforming the row operations $R_2 \to R_2 - (b+c)R_1$ and $R_3 \to R_3 - bcR_1$ we obtain

$$\begin{pmatrix} 1 & 1 & 1 \\ b+c & a+c & a+b \\ bc & ac & ab \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & 1 \\ 0 & a-b & a-c \\ 0 & c(a-b) & b(a-c) \end{pmatrix}$$

Next preform the row operation $R_3 \to R_3 - cR_2$. We obtain the matrix

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & a-b & a-c \\ 0 & 0 & (b-c)(a-c) \end{pmatrix}$$

Recall that row operation do not change the rank of the matrix. Therefore we have the following cases:

1) If $a \neq b$; $a \neq c$ and $b \neq c$ then the rank is three.

2) If $a = b = c$ then the rank is one.

3) In all other cases the rank is two.

4) It is easier to start with the image of T. In class we proved that $\mathrm{Im}T$ is spanned by an image of a basis. Let $B = \{1, x, x^2, \ldots, x^n\}$ denote the standard basis for V. We have

$$T(1) = 0, \qquad T(x) = x + 1 - x = 1, \qquad T(x^2) = (x+1)^2 - x^2 = 2x + 1$$

In general, for each $0 \leq i \leq n$ we have $T(x^i) = (x+1)^i - x^i$. Let $q_i(x) = (x+1)^i - x^i$. Since the factor x^i cancels, then $q_i(x)$ is a polynomial whose degree is $i-1$. In other words, we have $q_i(x) = a_{i-1}x^{i-1} + r_i(x)$, where $a_{i-1} \neq 0$, and $r_i(x)$ is a polynomial of degree at most $i-2$. From the above we have $\mathrm{Im}T = \mathrm{Sp}\{q_1(x), q_2(x), \ldots, q_n(x)\}$. Using induction we will prove that for all $0 \leq i \leq n$ we have $\mathrm{Sp}\{q_1(x), q_2(x), \ldots, q_i(x)\} = \mathrm{Sp}\{1, x, x^2, \ldots, x^{i-1}\}$. This clearly holds for $i = 0$, and assume it is true for $i-1$. Since each $q_i(x)$ is a polynomial of degree $i-1$, then clearly $\mathrm{Sp}\{q_1(x), q_2(x), \ldots, q_i(x)\} \subset \mathrm{Sp}\{1, x, x^2, \ldots, x^{i-1}\}$. By induction $\mathrm{Sp}\{q_1(x), q_2(x), \ldots, q_{i-1}(x)\} = \mathrm{Sp}\{1, x, x^2, \ldots, x^{i-2}\}$. Hence, $\mathrm{Sp}\{q_1(x), q_2(x), \ldots, q_i(x)\} \supset \mathrm{Sp}\{1, x, x^2, \ldots, x^{i-2}\}$. Also, from $q_i(x) = a_{i-1}x^{i-1} + r_i(x)$, since $a_{i-1} \neq 0$, then $x^{i-1} = \frac{1}{a_{i-1}}(q_i(x) + r_i(x)) \in \mathrm{Sp}\{q_1(x), q_2(x), \ldots, q_i(x)\}$. Hence we get $\mathrm{Sp}\{q_1(x), q_2(x), \ldots, q_i(x)\} = \mathrm{Sp}\{1, x, x^2, \ldots, x^{i-1}\}$. Plugging $i = n$ we get that the $\mathrm{Im}T = \mathrm{Sp}\{q_1(x), q_2(x), \ldots, q_n(x)\} = \mathrm{Sp}\{1, x, x^2, \ldots, x^{n-1}\}$. From this we deduce that $\dim\mathrm{Im}T = n$.

To compute the kernel, we first use the dimension theorem $\dim V = \dim\ker T + \dim\mathrm{Im}T$ to deduce that $\dim\ker T = 1$. Since we saw that $T(1) = 0$, it follows that $\ker T = \mathrm{Sp}\{1\}$.

5) Let $v \in \ker T$. Then $Tv = 0$. Hence, $0 = STv = Sv$ where the last equality follows from the identity $ST = S$. Hence $v \in \ker S$, and hence $\ker T \subset \ker S$. Similarly, we prove $\ker S \subset \ker T$, and hence $\ker T = \ker S$.

Next, $T^2 = TT = TST = TS = T$. Here, the second and the last equality follows from the identity $T = TS$, and the third equality follows from $ST = S$. The identity $S^2 = S$ is obtained in the same way.