Linear Algebra 1

Exercise Number 8

1) Let $P(x)$ denote the vector space of all polynomials in x with coefficients in a field F. Prove that $P(x)$ cannot be spanned by a finite set of polynomials.

2) Compute the rank of the following two matrices

$$A = \begin{pmatrix} 2 & 1 & -1 \\ 0 & 2 & 1 \\ -1 & 0 & 1 \\ 3 & 1 & 2 \end{pmatrix} \qquad B = \begin{pmatrix} 7 & 2 & 1 & 3 & 5 \\ 2 & 2 & 0 & 1 & 2 \\ 11 & 6 & 1 & 5 & 9 \end{pmatrix}$$

3) Let A be a square matrix of size n. Prove that A is invertible if and only if $\text{rank}A = n$.

4) Let A be a matrix of size $n \times n$, and let B be a matrix of size $n \times m$. If A is invertible, prove that $\text{rank}AB = \text{rank}B$. (*Hint:* Reduce it to the case when A is an elementary matrix.)

5) Given the matrices

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 3 & 4 & 1 \\ 5 & 3 & 9 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & -2 & 7 \\ 2 & -3 & 12 \\ 3 & -4 & 17 \end{pmatrix}$$

Is it true that $R(A) = R(B)$?

6) In F^3 let $W = \{(0, b, c) : b, c \in F\}$ and $U = \{(a, a, a) : a \in F\}$. Prove that $F^3 = U \oplus W$.

7) Let $V = Mat_{n\times n}(F)$. Let $U \subset V$ denote the subspace of all symmetric matrices, and let $W \subset V$ denote the subspace of all antisymmetric matrices. Prove that $V = U \oplus W$. (A matrix A is symmetric if $A^t = A$ and antisymmetric if $A^t = -A$.)

8) Let U, V and W be three subspaces of a certain vector space. Prove that

$$(U \cap V) + (U \cap W) \subset U \cap (V + W)$$

Give an example in $\mathbf{R}^2$ that the inclusion can be proper.

9) Let V be a vector space, and let $U, W \subset V$ be two subspaces. Prove that $V = U \oplus W$ if and only if the following two statements hold. 1) $V = U + W$, 2) The only way to represent the zero vector in V as a sum of a vector from U with a vector in W is $0 = 0 + 0$.

10) In F^3, let $U = \{(a,b,c) : a+b+c=0\}$, $V = \{(a,b,c) : a=c\}$ and $W = \{(0,0,c) : c \in F\}$. Is it true that $F^3 = U + V$? $F^3 = U + W$? $F^3 = V + W$? When are these sums direct sums?