Linear Algebra 1

Exercise Number 11

1) Prove that if $T : V \to W$ is a linear transformation, then $T(0) = 0$.

2) Determine which of the following maps define a linear transformation.

a) $T : F^2 \to F^2$ defined by $T(x, y) = (2x - y, ax + by)$ where $a, b \in F$.

b) $T : F^2 \to F^2$ defined by $T(x, y) = (|x|, y)$.

c) $T : P(x) \to P(x)$ defined by $T(p) = 2p + 3p' - p''$.

d) $T : \mathbf{C}^2 \to \mathbf{C}^2$ defined by $T(x, y) = (ix + (2 + i)y, -3ix)$.

e) $T : \mathbf{R}^3 \to \mathbf{R}$ defined by $T(x, y, z) = ax + by + cz + d$ where $a, b, c, d \in \mathbf{R}$.

f) $T : \mathbf{R}^\infty \to \mathbf{R}^\infty$ defined by $T((x_1, x_2, x_3, \ldots)) = (0, x_1, x_2, x_3, \ldots)$.

g) Let $A \in Mat_{m\times n}(F)$. Define $T_A : Mat_{n\times p}(F) \to Mat_{m\times p}(F)$ by $T_A(B) = AB$. Is T_A linear?

3) Write down all linear transformations $T : F^2 \to F^2$ such that $T(1, 1) = (3, -2)$ and $T(2, 1) = (1, 2)$.

4) Give an example of a linear transformation $T : F^3 \to F^2$ such that $\text{Ker}T = \text{Sp}\{(1, 0, 1), (0, 0, 1)\}$.

5) Give an example of a linear transformation $T : P_2(x) \to P_3(x)$ such that $\text{Im}T = \text{Sp}\{x^3 + 1, x^2, 2x^3 + 2x^2 + 3\}$.

6) Give an example of a linear transformation $T : F^3 \to F^3$ such that $T(1, -1, 1) = (2, 0, 0)$.

7) Let V denote an n dimensional vector space over the field F. Let $B = \{v_1, \ldots, v_n\}$ be a basis for V. Define the map $T_B : V \to F^n$ as follows. If $v = \alpha_1 v_1 + \ldots + \alpha_n v_n$ then $T_B(v) = (\alpha_1, \ldots, \alpha_n)$. Check that the map T_B is well defined and prove that it is a linear transformation.

8) For each of the linear transformations in exercise **2)**, determine which is one to one, and which is onto. Do the same for the map defined in exercise **7)**.

9) Give an example for two different isomorphisms between $\mathbf{R}^4$ and $Mat_{2\times 2}(\mathbf{R})$.

10) For each of the following linear transformations, find $\text{Ker}T$ and $\text{Im}T$.

a) $T : \mathbf{Q}^3 \to \mathbf{Q}^3$ defined by $T(x, y, z) = (x + y, x + z, y + z)$.

b) $T : \mathbf{Z}_2^3 \to \mathbf{Z}_2^3$ defined by $T(x, y, z) = (x + y, x + z, y + z)$.

c) $T : P(x) \to P(x)$ defined by $T(p(x)) = xp(x)$.

11) Give an example of a linear transformation $T : \mathbf{R}^2 \to \mathbf{R}^2$ such that $\mathrm{Ker}T = \mathrm{Im}T$.

12) a) Give an example of a linear transformation $T : \mathbf{R}^2 \to \mathbf{R}^2$ such that $\mathrm{Ker}T \subsetneq \mathrm{Im}T$.

b) Give an example of a linear transformation $T : \mathbf{R}^2 \to \mathbf{R}^2$ such that $\mathrm{Im}T \subsetneq \mathrm{Ker}T$.

13) a) Let $T : \mathbf{C} \to \mathbf{C}$ be defined by $Tz = \bar{z}$. Prove that T is linear if $\mathbf{C}$ is viewed as a vector space over the field $\mathbf{R}$, and it is not linear if $\mathbf{C}$ is viewed as a vector space over the field $\mathbf{C}$

b) Does the same hold for the map $Sz = iz$.

c) In each case where T or S are linear determine their kernel and their image.

14) Let $T : V \to V$ be a linear transformation. Let $W \subset V$ be a subspace of V. Define a map $S : W \to V$ by $S(w) = T(w)$ for all $w \in W$.

a) Prove that S is linear.

b) Prove that $\mathrm{Ker}S = \mathrm{Ker}T \cap W$.

c) Prove that $\dim T(W) \leq \dim W \leq \dim T(W) + \dim \mathrm{Ker}T$. Here, $T(W) = \{v \in V : \text{there is a } w \in W \text{ such that } T(w) = v\}$.