Linear Algebra 1

Exercise Number 15

1) If $u = (x_1, x_2)$ and $v = (y_1, y_2)$, prove that $(u, v) = x_1y_1 - x_1y_2 - x_2y_1 + 3x_2y_2$ defines an inner product on $\mathbf{R}^2$.

2) Consider $\mathbf{R}^3$ with the standard inner product. For the vector $u = (2, 1, -1)$, find a vector in the direction of u which have norm 1.

3) Let V be an inner product space. Prove:
a) $||v|| \geq 0$, and $||v|| = 0$ if and only if $v = 0$.
b) $||u + v|| \leq ||u|| + ||v||$. (this is the triangular inequality)
c) $||u + v||^2 + ||u - v||^2 = 2(||u||^2 + ||v||^2)$.

4) Consider $\mathbf{R}^5$ with the standard inner product. Let W denote the subspace of $\mathbf{R}^5$ which is spanned by the vectors $(1, 2, 3, -1, 2)$ and $(2, 4, 7, 2, -1)$. Find a basis for $W^{\perp}$.

5) Let $u = (z_1, z_2)$ and $v = (w_1, w_2)$ be two vector in the vector space $\mathbf{C}^2$ over the field $\mathbf{C}$. Prove that

$$(u, v) = z_1\bar{w}_1 + (1 + i)z_1\bar{w}_2 + (1 - i)z_2\bar{w}_1 + 3z_2\bar{w}_2$$

defines an inner product. Compute the norm of the vector $(1 - 2i, 2 + 3i)$ with respect to this inner product.

6) Find an orthonormal basis for the following vector spaces with respect to the indicated inner product:
a) $V = M_{2\times 2}(\mathbf{R})$ with respect to $(A, B) = tr(B^tA)$.
b) $V = \{(a, b, c, d) : a + b + c + d = 0\}$ with respect to the standard inner product in $\mathbf{R}^4$.
c) $W = \mathrm{Sp}\{(1, i, 1), (1 + i, 0, 2)\} \subset \mathbf{C}^3$ with respect to the standard inner product in $\mathbf{C}^3$.

7) (Bessel inequality) Let $\{w_1, \ldots, w_n\}$ be an orthonormal set of vectors in V. Prove that for all $v \in V$

$$\sum_{i=1}^{n} |(w_i, v)|^2 \leq ||v||^2$$

Hint: Write $v = w + \widetilde{w}$ where $w = \mathrm{Sp}\{w_1, \ldots, w_n\}$ and $\widetilde{w} = \mathrm{Sp}\{w_1, \ldots, w_n\}^{\perp}$.

8) Prove that if $\{w_1, \ldots, w_n\}$ is an orthonormal set of vectors in V, such that $\sum_{i=1}^{n} |(w_i, v)|^2 = ||v||^2$ for all $v \in V$, then $\{w_1, \ldots, w_n\}$ is a basis for V.

9) Let $W \subset V$ be a subspace of V. Let $v \in V$ be a vector which satisfy $(v, w) + (w, v) \leq (w, w)$ for all $w \in W$. Prove that $(v, w) = 0$ for al $w \in W$.

10) Let V be an inner product vector space . Define on V the distance function $d(u, v)$ between the two vectors $u, v \in V$ by $d(u, v) = ||u - v||$.

a) Let $V = \mathbf{R}^2$ with the standard inner product. If $u = (x_1, y_1)$ and $v = (x_2, y_2)$, write the distance $d(u, v)$ in terms of the coordinates x_i, y_i.

b) Prove that $d(u, v) \geq 0$, and $d(u, v) = 0$ if and only if $u = v$.

c) For any two vectors $d(u, v) = d(v, u)$.

d) $d(u, v) \leq d(u, w) + d(w, v)$ this is the triangular inequality.

11) Let V be an inner product vector space over the field F. Let $T : V \to F$ be a linear transformation. Prove that there is a vector $u_0 \in V$ such that $Tv = (v, u_0)$ for all $v \in V$.