Solution of Moed A in Linear Algebra 2011

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2) a) Let $S_1, S_2 \in U \subset Hom_F(V,V)$ and let $\alpha_1, \alpha_2 \in F$. Then, by definition of U, we have $S_1 \circ T = S_2 \circ T = 0$. Hence $(\alpha_1 S_1 + \alpha_2 S_2) \circ T = \alpha_1 S_1 \circ T + \alpha_2 S_2 \circ T = 0 + 0 = 0$. Hence $\alpha_1 S_1 + \alpha_2 S_2 \in U$.

b) Let $\mathcal{B}$ be a basis of V. We know that the map $S \mapsto [S]_{\mathcal{B}}$ defines an isomorphism between $Hom_F(V,V)$ and $Mat_{2\times 2}(F)$. We also know that $[S \circ T]_{\mathcal{B}} = [S]_{\mathcal{B}}[T]_{\mathcal{B}}$. Let $A = [T]_{\mathcal{B}}$. Define

$$U' = \{B \in Mat_{2\times 2}(F) \ : \ BA = 0\}$$

Then from the above we deduce that U is isomorphic to U' and hence dim $U' =$ dim U. To compute dim U' we consider several cases according to the rank of the matrix A:

1) Assume rank$A = 2$. Then A is invertible, and hence $BA = 0$ implies $BAA^{-1} = 0$ and $B = 0$. Hence dim $U' = 0$ in this case.

2) Assume rank$A = 0$. Then $A = 0$ and therefore $U' = Mat_{2\times 2}(F)$. Hence dim $U' = 4$.

3) Assume rank$A = 1$. There are two cases. First assume that $A = \begin{pmatrix} 0 & 0 \\ a & b \end{pmatrix}$ and a and b are not both zero. Let $B = \begin{pmatrix} x & y \\ z & r \end{pmatrix}$. Then

$$BA = \begin{pmatrix} x & y \\ z & r \end{pmatrix} \begin{pmatrix} 0 & 0 \\ a & b \end{pmatrix} = \begin{pmatrix} ya & yb \\ ra & rb \end{pmatrix} = 0$$

Since either a or b are not zero we deduce that $y = r = 0$. Hence

$$U' = \{ \begin{pmatrix} x & 0 \\ z & 0 \end{pmatrix} \ : x, z \in F\}$$

Thus dim $U' = 2$. The second case is when $A = \begin{pmatrix} a & b \\ \alpha a & \alpha b \end{pmatrix}$ where $\alpha \in F$ and a and b are not both zero. Thus

$$BA = \begin{pmatrix} x & y \\ z & r \end{pmatrix} \begin{pmatrix} a & b \\ \alpha a & \alpha b \end{pmatrix} = \begin{pmatrix} a(x+\alpha y) & b(x+\alpha y) \\ a(z+\alpha r) & b(z+\alpha r) \end{pmatrix} = 0$$

Since either a or b are not zero we deduce that $x+\alpha y = z+\alpha r = 0$. Hence

$$U' = \{\begin{pmatrix} -\alpha y & y \\ -\alpha r & r \end{pmatrix} \ : y, r \in F\}$$

Thus $\dim U' = 2$.

3) a) The statement is true. Apply the dimension theorem $\dim V = \dim \ker T + \dim \operatorname{Im} T$. Clearly, $\dim \operatorname{Im} T \le \dim W = 4$. Hence $6 = \dim V \le \dim \ker T + 4$ which implies that $\dim \ker T \ge 2$. Thus $\ker T$ is not zero.

b) Here the answer depends on q. Using the additive in rows property of the determinant, we obtain

$$\begin{vmatrix} 2 & 2 & 3 & 4 \\ 3q+4 & 4q+4 & 5q+6 & 2q+8 \\ 0 & -1 & 1 & 8 \\ 5 & 9 & 7 & a \end{vmatrix} = \begin{vmatrix} 2 & 2 & 3 & 4 \\ 3q & 4q & 5q & 2q \\ 0 & -1 & 1 & 8 \\ 5 & 9 & 7 & a \end{vmatrix} + \begin{vmatrix} 2 & 2 & 3 & 4 \\ 4 & 4 & 6 & 8 \\ 0 & -1 & 1 & 8 \\ 5 & 9 & 7 & a \end{vmatrix} = q\begin{vmatrix} 2 & 2 & 3 & 4 \\ 3 & 4 & 5 & 2 \\ 0 & -1 & 1 & 8 \\ 5 & 9 & 7 & a \end{vmatrix}$$

We also used the fact that if two rows in a matrix are proportional, then the determinant of this matrix is zero. Consider two cases. If $q=0$ then for all a we obtain $|A|=0$ and the answer is no.

Assume that $q \ne 0$. Thus the question we need to answer is if there is a unique rational number a such that

$$\begin{vmatrix} 2 & 2 & 3 & 4 \\ 3 & 4 & 5 & 2 \\ 0 & -1 & 1 & 8 \\ 5 & 9 & 7 & a \end{vmatrix} = 1$$

Expand this determinant according to the last row. Then we obtain

$$a\begin{vmatrix} 2 & 2 & 3 \\ 3 & 4 & 5 \\ 0 & -1 & 1 \end{vmatrix} = 1-\alpha$$

Here α is a rational number, in fact an integer, which is obtained from all the other terms in the development of the determinant. Thus, there will be a unique rational a if the determinant

$$\begin{vmatrix} 2 & 2 & 3 \\ 3 & 4 & 5 \\ 0 & -1 & 1 \end{vmatrix} = \begin{vmatrix} 2 & 3 \\ 3 & 5 \end{vmatrix} + \begin{vmatrix} 2 & 2 \\ 3 & 4 \end{vmatrix}$$

is not zero. This is clearly true, and hence there is a unique a such that $|A| = q$.

c) The statement is true. Apply transpose to $A = AA^t$ to obtain $A^t = (AA^t)^t = AA^t = A$. Hence $A = AA^t = AA = A^2$.

4) Since the rank of a matrix is equal to the rank of its transpose, and since we can rearrange the rows and columns as we want, its enough to compute the rank of the matrix

$$\begin{pmatrix} 1 & 2 & 1 \\ 2 & 5 & 1 \\ -1 & a & -6 \\ a & -1 & 10 \end{pmatrix}$$

We apply the following row operations: $R_2 \to R_2 - 2R_1$; $R_3 \to R_3 + R_1$ and $R_4 \to R_4 - aR_1$. Then

$$\begin{pmatrix} 1 & 2 & 1 \\ 2 & 5 & 1 \\ -1 & a & -6 \\ a & -1 & 10 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 0 & a+2 & -5 \\ 0 & -2a-1 & 10-a \end{pmatrix}$$

Next we preform $R_3 \to R_3 - (a+2)R_2$ and $R_4 \to R_4 + (1+2a)R_2$. We obtain

$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 0 & a+2 & -5 \\ 0 & -2a-1 & 10-a \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & a-3 \\ 0 & 0 & -3(a-3) \end{pmatrix}$$

Hence, if $a = 3$ the rank is 2 and if $a \neq 3$ the rank is 3.