Linear Algebra 1

Exercise Number 4

1) For each of the following matrices find its inverse if it exists.

$$A = \begin{pmatrix} 6 & 1 \\ 2 & 4 \end{pmatrix} \qquad B = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 3 & 1 \\ -1 & 4 & 0 \end{pmatrix} \qquad C = \begin{pmatrix} 5 & 2 & 6 \\ 1 & 2 & 2 \\ -1 & 2 & 0 \end{pmatrix}$$

$$D = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \end{pmatrix} \qquad E = \begin{pmatrix} i & 1 \\ 2 & -i \end{pmatrix}$$

2) Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. If $ad - bc \neq 0$, show that $A^{-1} = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

3) Let A be a $m \times n$ matrix, and let B be a $n \times m$ matrix. Suppose that $m > n$. Prove that AB is not invertible. (*Hint:* Consider the linear system $Bx = 0$).

4) a) Let A and B be two square matrices of the same size. Prove that if AB is invertible and A is invertible, then B is invertible.

b) Prove that if AB is invertible, then both A and B are invertible.

c) From part b) conclude that if $AB = I$ then $BA = I$.

d) Let A denote a square matrix which satisfies the identity $2A^3 + 3A^2 - 4A - 6I = 0$. Prove that A is invertible, and express the matrix A^{-1} in terms of the matrix A.

5) Let A, B and P be square matrices of the same size which satisfy the relation $B = P^{-1}AP$. Let $f(x)$ be a polynomial in x.

a) Prove that for any natural number k we have $B^k = P^{-1}A^kP$.

b) Prove that if $f(A) = 0$ then $f(B) = 0$.

6) Write down all elementary matrices of size 2×2.

7) Let J be an invertible matrix, and let A be a matrix which satisfies $A^tJA = J$. Prove that A is invertible, and that A^{-1} satisfies $(A^{-1})^tJA^{-1} = J$.

8) If A is a matrix such that $A^2 = 0$, prove that $I + A$ is invertible and find $(I + A)^{-1}$.

9) Find at lease three different matrices A of size 2×2 which are different from I or $-I$ and which satisfy $A^2 = I$.