Solution for Final in Linear Algebra 1

Moad B 2009

1) Using the property of addition by rows the sum of the three determinants is equal to

$$\begin{vmatrix} a & b & b \\ c+e+d & c+e+d & c+e+d \\ f & g & g \end{vmatrix}$$

Since the last two columns are equal we get zero.

2) We claim that $U \cap W = \{0\}$. Indeed, if $v \in U \cap W$ and $v \neq 0$, then it follows that $f(v) < f(v)$, which is impossible. Hence $U \cap W = \{0\}$. Thus, if follows from the dimension theorem

$$\dim V \geq \dim(U+W) = \dim U + \dim W - \dim(U \cap W) = \dim U + \dim W$$

Here, the first inequality follows from the fact that $U+W$ is a subspace of V, and the last equality follows from the fact that $U \cap W = \{0\}$.

3) A vector (α, β, γ) is in the kernel of T if and only if

$$\begin{pmatrix} -1 & -1 & -3 \\ -5 & -2 & -6 \\ 2 & 1 & 3 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = 0 \tag{1}$$

To see this, apply the definition of a matrix representing a linear transformation to obtain

$$\begin{aligned} Tv_1 &= -v_1 - 5v_2 + 2v_1 \\ Tv_2 &= -v_1 - 2v_2 + v_3 \\ Tv_3 &= -3v_1 - 6v_2 + 3v_3 \end{aligned}$$

If $v = \alpha v_1 + \beta v_2 + \gamma v_3$ is in the kernel of T then $Tv = 0$. Hence

$$0 = Tv = T(\alpha v_1 + \beta v_2 + \gamma v_3) = \alpha T v_1 + \beta T v_2 + \gamma T v_3 =$$

$$= \alpha(-v_1 - 5v_2 + 2v_1) + \beta(-v_1 - 2v_2 + v_3) + \gamma(-3v_1 - 6v_2 + 3v_3)$$

From this it follows that we need to solve the system (2). Applying row operations, we obtain the system

$$\begin{pmatrix} -1 & -1 & -3 \\ & 1 & 3 \\ & & \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = 0$$

From this we deduce that $\ker T = \mathrm{Sp}\{(0, -3, 1)\}$.

4) (**To make things clear I wrote a detailed answer, much more then needed.**)

To see what is going on, we first write the matrix A for $n = 4$ and $n = 5$. We have

$$A = \begin{pmatrix} c & & c & \\ & c & & c \\ c & & c & \\ & c & & c \end{pmatrix} \qquad A = \begin{pmatrix} c & & c & & c \\ & c & & c & \\ c & & c & & c \\ & c & & c & \\ c & & c & & c \end{pmatrix}$$

If $c = 0$, then $A = 0$ and so $k = 1$ is the minimal number. Assume that $c \neq 0$. We are looking for the smallest number k, such that the equation

$$\alpha_1 A + \alpha_2 A^2 + \cdots + \alpha_k A^k = 0 \tag{2}$$

has a nontrivial solution. Let $A^r = (a_{i,j}(r))$. In other words, $a_{i,j}(r)$ denotes the (i, j)-th entry of the matrix A^r. The following is proved using matrix multiplication:

1) For all r, if $i + j$ is odd, then $a_{i,j}(r) = 0$.

2) If $n = 2m$, and $i + j$ is even then $a_{i,j}(r) = a(r, c)$, where $a(r, c)$ is a fixed *nonzero* number depending only on r and c and not on i or j. For example $a(1, c) = c$; $a(2, c) = mc^2$; $a(3, c) = m^3c^4$ etc.

3) If $n = 2m + 1$, and $i + j$ is even, then there are two cases for the values of $a_{i,j}(r)$. The first, which we denote by $a(r, c)$ corresponds to the nonzero entries in the odd rows of A^r, and the other we denote by $b(r, c)$ which corresponds to the nonzero entry in the even rows of A^r.

For example,

$$A^2 = \begin{pmatrix} 2c^2 & & 2c^2 & \\ & 2c^2 & & 2c^2 \\ 2c^2 & & 2c^2 & \\ & 2c^2 & & 2c^2 \end{pmatrix} \qquad A^2 = \begin{pmatrix} 3c^2 & & 3c^2 & & 3c^2 \\ & 2c^2 & & 2c^2 & \\ 3c^2 & & 3c^2 & & 3c^2 \\ & 2c^2 & & 2c^2 & \\ 3c^2 & & 3c^2 & & 3c^2 \end{pmatrix}$$

Thus, from the definition of scalar multiplication and from the definition of addition of matrices, it follows that the matrix on the left hand side of (2) has the same structure. More precisely, in the matrix $\alpha_1 A + \alpha_2 A^2 + \cdots + \alpha_k A^k$ we have:

1) If $i+j$ is odd then the (i,j)-th entry of this matrix is zero.
2) If n is even, and $i+j$ is even then the (i,j)-th entry of this matrix is $\alpha_1 a(1,c)+\alpha_2 a(2,c)+\cdots+\alpha_k a(k,c)$.
3) If n is odd, and $i+j$ is even then the (i,j)-th entry of this matrix is either $\alpha_1 a(1,c)+\alpha_2 a(2,c)+\cdots+\alpha_k a(k,c)$ or $\alpha_1 b(1,c)+\alpha_2 b(2,c)+\cdots+\alpha_k b(k,c)$.

Assume that n is even, from the above discussion it follows that the set of solutions to equation (2) is the same as the set of solutions of the equation

$$\alpha_1 a(1,c)+\alpha_2 a(2,c)+\cdots+\alpha_k a(k,c)=0$$

We are looking for a minimal number k such that the above equation has a nontrivial solution. Since all $a(r,c)\neq 0$, then clearly $k=2$ is the minimal number. In other words, the equation $\alpha_1 a(1,c)+\alpha_2 a(2,c)=0$ has a nontrivial solution. Thus, when n is even, and $c\neq 0$, then the set $\{A, A^2\}$ is linearly dependent.

Assume that n is odd. Then the set of solutions to equation (2) is the same as the set of solutions of the system of equations

$$\begin{gathered}\alpha_1 a(1,c)+\alpha_2 a(2,c)+\cdots+\alpha_k a(k,c)=0\\ \alpha_1 b(1,c)+\alpha_2 b(2,c)+\cdots+\alpha_k b(k,c)=0\end{gathered}$$

In this case $k=2$ will not work. Indeed, since $a(1,c)=b(1,c)=c$, and since $a(2,c)\neq b(2,c)$ then the system $\alpha_1 a(1,c)+\alpha_2 a(2,c)=0$ and $\alpha_1 b(1,c)+\alpha_2 b(2,c)=0$ has only the trivial solution. When $k=3$ then we obtain a homogenous system with two equations and three unknowns. This has a nontrivial solution. Hence, when n is odd, and $c\neq 0$, then the set $\{A, A^2, A^3\}$ is linearly dependent, and this is the minimal set.

5) Since T is a linear transformation then $T(0)=0$. Here 0 is the zero matrix. Matrix multiplication shows that if $X=\begin{pmatrix}0&0\\1&0\end{pmatrix}$ then $X^2=0$. Plugging $A=B=X$ in the identity $T(AB)=T(A)T(B)$, we obtain $0=T(0)=T(X^2)=T(X)T(X)$. Thus if $T(X)=I$ we derive a contradiction.