Solution of Moad A in Linear Algebra 2022

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1) Problem: For all $a, b \in \mathbf{R}$, let

$$A = \begin{pmatrix} 1 & 1 & a \\ 1 & 0 & 0 \\ 0 & 2 & b \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 1 & b \\ a & 0 & 0 \\ 0 & 3 & a \end{pmatrix}$$

Find conditions on a, b such not A is invertible, and $\text{rank}B = 3$.

Solution: For A to be invertible we need $|A| \neq 0$. Similarly, for $\text{rank}B = 3$ we also need $|B| \neq 0$. Since $|A| = b - 2a$ and $|B| = a(a - 3b)$, the conditions are $a \neq 0$, and $a \neq b/2$ and $a \neq b/3$.

2) Problem: Let A denote a 3×3 matrix with entries in $\mathbf{R}$. Assume that A is not invertible. Prove that there is a matrix B such that $AB = 0$.

Solution: Let $\{v_1, v_2, v_3\}$ denote the columns of the matrix A. View them as vectors in $\mathbf{R}^3$. Since A is not invertiable, then there are $\alpha_1, \alpha_2, \alpha_3 \in \mathbf{R}$ such that $\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = 0$. Let

$$B = \begin{pmatrix} \alpha_1 & \alpha_1 & \alpha_1 \\ \alpha_2 & \alpha_2 & \alpha_2 \\ \alpha_3 & \alpha_3 & \alpha_3 \end{pmatrix}$$

Then a matrix multiplication implies that $AB = 0$. Also, since not all α_i are zero, then $B \neq 0$.

3) Problem: In this problem F^n consists of column vectors.
Let $\{v_2, \ldots, v_n\}$ denote $n-1$ vectors in F^n. For any $u \in F^n$, let $A(u, v_2, \ldots, v_n)$ denote the matrix of order n, whose columns are the vectors $\{u, v_2, \ldots, v_n\}$. Prove that there is a vector $v_0 \in F^n$, such that $\det A(u, v_2, \ldots, v_n) = v_0^t \cdot u$.

Remark: Here v_0^t denotes the transpose of v_0, and $v_0^t \cdot u$ denotes the matrix multiplication of v_0^t and u.

Solution: Denote $T(u) = \det(A(u, v_2, \ldots, v_n))$. Then $T : F^n \to F$ is a linear map. Indeed, to prove it is linear we first notice that the addition by columns property of the determinant implies that for $u_1, u_2 \in F^n$ we have

$$\det(A(u_1 + u_2, v_2, \ldots, v_n)) = \det(A(u_1, v_2, \ldots, v_n)) + \det(A(u_2, v_2, \ldots, v_n))$$

In terms of T, this implies that $T(u_1 + u_2) = T(u_1) + T(u_2)$. Similarly, for all $\alpha \in F$, we have $\det(A(\alpha u, v_2, \ldots, v_n)) = \alpha \det(A(u, v_2, \ldots, v_n))$. This implies $T(\alpha u) = \alpha T(u)$.

From the above it follows that $T : F^n \to F$ is a linear map. Similarly, if, for fixed $v_0 \in F^n$, we define $S(u) = v_0^t \cdot u$, then $S : F^n \to F$ is also a linear map. Because $\dim F = 1$, it follows from the dimension Theorem that two linear maps from F^n to F which have the same kernel, must be proportional.

In details, assume first that $\{v_2, \ldots, v_n\}$ is a linear dependent set. Then, for all $u \in F^n$, $\det(A(u, v_2, \ldots, v_n)) = 0$. In other words T is the zero map. In this case, choose $v_0 = 0$. Next, assume that $\{v_2, \ldots, v_n\}$ is a linear independent set. Then, we can complete it to a base of F^n. Let $v_1 \in F^n$ be such that $\mathcal{B} = \{v_1, v_2, \ldots, v_n\}$ is a base for F^n. Let $\alpha = \det(A(v_1, v_2, \ldots, v_n))$. Then $\alpha \neq 0$. Form the system of linear equations given by $x^t A(v_1, v_2, \ldots, v_n) = b$. Here $b = (\alpha, 0, \ldots, 0)^t$. Since $A(v_1, v_2, \ldots, v_n)$ is an invertible matrix, this equation has a unique solution. Denote this solution by v_0, and let $S(u) = v_0^t \cdot u$. Then, we have $T(v_1) = \alpha$. By comparing the first coordinate of the equation $b = v_0^t A(v_1, v_2, \ldots, v_n)$ we get $S(v_1) = \alpha$. Also, we have $T(v_i) = S(v_i) = 0$ for all $2 \leq i \leq n$. Hence, T and S agree on $\mathcal{B}$. Thus, $T = S$.

4) Problem: In this problem V is an $n > 1$ dimensional vector space over a field F.
a) (12 points) Let U and W denote two subspaces of V such that $\dim U + \dim W \geq n$. Prove that there is a linear map $T : V \to V$ such that $\ker T \subset U$ and $\mathrm{Im}\, T \subset W$.
b) (13 points) Let $T : V \to V$ denote a linear map. Prove that there are two isomorphisms $T_1, T_2 : V \to V$ such that $T = T_1 + T_2$. In this part we assume that $F \neq \mathbf{Z}_2$.

Solution: a) Denote $\dim U = m$ and $\dim W = k$. Then $m + k \geq n$. Let $\{u_1, \ldots, u_m\}$ denote a base for U, and let $\{w_1, \ldots, w_k\}$ denote a base for W. Complete $\{u_1, \ldots, u_m\}$ to a base $\{u_1, \ldots, u_m, u_{m+1}, \ldots, u_n\}$ of V, and similarly, complete $\{w_1, \ldots, w_k\}$ to a base $\{w_1, \ldots, w_k, w_{k+1}, \ldots, w_n\}$ of V. Choose $m' \leq m$ and $k' \leq k$ such that $m' + k' = n$. To define

$T : V \to V$ it is enough to specify it on a base. Choose the base $\{u_1, \ldots, u_m, u_{m+1}, \ldots, u_n\}$. First, let $T(u_1) = T(u_2) = \ldots = T(u_{m'}) = 0$. Then, set $T(u_{m'+1}) = w_1$; $T(u_{m'+2}) = w_2; \ldots; T(u_{m'+n-m'}) = w_{n-m'}$. Since $m' \leq m$ then $\{u_1, \ldots, u_{m'}\} \in U$. Hence $\ker T \subset U$. Also, since $n - m' = k' \leq k$, then

$$\text{Im } T = \text{Sp}\{T(u_{m'+1}), T(u_{m'+2}) \ldots, T(u_{m'+n-m'})\} = \text{Sp}\{w_1, \ldots, w_{n-m'}\} \subset W$$

b) Let $\{u_1, \ldots, u_m\}$ denote a base for $\ker T$. Complete it to a base of V, say $\mathcal{B} = \{u_1, \ldots, u_m, u_{m+1}, \ldots, u_n\}$. Then Im T is isomorphic to $W = \text{Sp}\{u_{m+1}, \ldots, u_n\}$. For this claim see problem **3)** in Moed A. Thus $V = \ker T \oplus W$. From this we deduce that the set $\{T(u_{m+1}), \ldots, T(u_n)\}$ is a linear independent set. Complete this set to a base of V, say $\mathcal{C} = \{e_1, \ldots, e_m, T(u_{m+1}), \ldots, T(u_n)\}$.

To define T_1 and T_2 it is enough to specify it on the base $\mathcal{B}$. Define T_1 as follows. First, for $1 \leq i \leq m$ set $T_1(u_i) = e_i$. Then, for $m \leq i \leq n$ set $T_1(u_i) = 2T(u_i)$. This is the point where we assume that $F \neq \mathbf{Z_2}$. For T_2 define $T_2(u_i) = -e_i$ when $1 \leq i \leq m$, and $T_2(u_i) = -T(u_i)$ when $m \leq i \leq n$. Then $T(u_i) = T_1(u_i) + T_2(u_i)$ and T_1 and T_2 are both isomorphisms.