Linear Algebra 1

Exercise Number 10

1) Let V be a vector space over a field F. Assume that V has a basis with n vectors. Prove:

a) Every set of vectors in V which contains more than n vectors is linearly dependent.

b) Every set of vectors in V which contains less than n vectors does not span V.

c) Every linearly independent set which contains n vectors is a basis.

d) Every set which spans V and contains n vectors is a basis.

e) In every basis for V there are exactly n vectors.

2) Let V a vector space and let $W \subset V$ be a subspace of V. Prove that there is a subspace U of V such that $V = W \oplus U$.

3) Let V be a vector space which is finitely generated. Let U be a subspace of V. Prove that $\dim U = \dim V$ if and only if $U = V$.

4) In each of the following cases W is a subspace of a given vector space V. In each case find a basis for W, and complete it to basis of V.

a) $V = P_3(x)$ $\quad W = \{p(x) \in P_3(x) : p'(0) = 0\}$

b) $V = F^n$ $\quad W = \{(a_1, \ldots, a_n) : \sum_{i=1}^n a_i = 0\}$

c) $V = F^5$ $\quad W = Sp\{(-1, 1, 0, 1, 1), (0, 0, 1, 1, 1)\}$

d) $V = F^4$ $\quad W = Sp\{(1, 2, 3, 1), (0, 1, 0, 0), (2, 5, 6, 2)\}$

e) $V = F^2$ $\quad W = \{v \in F^2 : Av = 0\}$ where

$$A = \begin{pmatrix} 2 & 1 & 0 & 6 & 1 & 1 & 2 \\ 4 & 1 & 0 & 6 & 2 & 2 & 1 \end{pmatrix}$$

f) $V = \mathbf{C}^2$ over the field $\mathbf{C}$. $W = Sp\{(i, 1)\}$.

5) Prove that every two planes in $\mathbf{R}^3$ which pass through the origin, intersect in more than one point. Is it true in $\mathbf{R}^4$?

6) In $V = P_5(x)$, let $W = \{p(x) \in P_5(x) : p''(0) = 0\}$ and $R = \{p(x) \in P_5(x) : p(0) + p'(0) = 0\}$. Compute $\dim(W + R)$.

7) A square matrix is upper triangular, if all entries below the diagonal are zeros. Let V

denote the vector space of all upper triangular matrices of size 4×4 with entries in $\mathbf{R}$. Let

$$J_1 = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & -1 & & \\ -1 & & & \end{pmatrix} \qquad J_2 = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{pmatrix}$$

For $i = 1, 2$ let

$$W_i = \{v \in V : v^t J_i + J_i v = 0\}$$

Compute $\dim(W_1 + W_2)$ and find a basis for $W_1 + W_2$.

8) Prove that for all matrix A we have $\text{rank}A = \text{rank}A^t$.

9) For any two matrices A and B such that AB is defined, prove that $\text{rank}(AB) \leq \min(\text{rank}A, \text{rank}B)$.

10) Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Prove that $\text{rank}A = 2$ if and only if $ad - bc \neq 0$.

11) Prove or disprove the following statement: For any two matrices of the same size we have $\text{rank}(A + B) = \text{rank}A + \text{rank}B$.