Linear Algebra 1 A

Midterm 2018

1) Find for what values of $t \in \mathbf{R}$ the system of equations

$$\begin{array}{llll} -x_1 & & +2x_3 & = 1 \\ 3x_1 & +tx_2 & -6x_3 & = -3 \\ -2x_1 & -tx_2 & +tx_3 & = 3 \end{array}$$

has a unique solution, has infinite number of solutions, or no solutions at all. In those cases where there are solutions write the solution set for each case.

Solution: Perform the two row operations: $R_2 \to R_2 + 3R_1$ and $R_3 \to R_3 - 2R_1$. Then we obtain the system

$$\begin{array}{llll} -x_1 & & +2x_3 & = 1 \\ & tx_2 & & = 0 \\ & -tx_2 & +(t-4)x_3 & = 1 \end{array}$$

Perform the operation $R_3 \to R_3 + R_2$. We obtain

$$\begin{array}{llll} -x_1 & & +2x_3 & = 1 \\ & tx_2 & & = 0 \\ & & (t-4)x_3 & = 1 \end{array}$$

Suppose $t = 4$. Then the last equation becomes $0 = 1$, and hence there are no solutions. If $t = 0$, then we obtain the system

$$\begin{array}{lll} -x_1 & +2x_3 & = 1 \\ & -4x_3 & = 1 \end{array}$$

From this we deduce that the system has an infinite number of solutions which are given by the set $\{(-\frac{3}{2}, \alpha, -\frac{1}{4}) \ : \ \alpha \in \mathbf{R}\}$.

Finally, if $t \neq 0, 4$, then we obtain the system

$$\begin{array}{llll} -x_1 & & +2x_3 & = 1 \\ & x_2 & & = 0 \\ & & x_3 & = \frac{1}{t-4} \end{array}$$

From this we deduce that there is a unique solution which is given by $(\frac{2}{t-4} - 1, 0, \frac{1}{t-4})$.

2) Let A denote a square matrix with entries in $\mathbf{R}$, which satisfies $(A+2I)^2 = 0$. Prove that $A+\lambda I$ is invertible if and only if $\lambda \neq 2$.

Solution: Denote $B = A + \lambda I$, Then

$$(A+2I)^2 = (A+\lambda I - \lambda I + 2I)^2 = (B + (2-\lambda)I)^2 = B^2 + 2(2-\lambda)B + (2-\lambda)^2 I$$

Hence $(A+2I)^2 = 0$ if and only if $B^2+2(2-\lambda)B+(2-\lambda)^2I = 0$, or $B^2+2(2-\lambda)B = -(2-\lambda)^2I$. This is equivalent to $B(B+2(2-\lambda)I) = -(2-\lambda)^2I$. If $\lambda = 2$, then we get $B^2 = 0$. Then clearly B is not invertible. If $\lambda \neq 2$, we obtain the equation

$$B\left[\frac{1}{-(2-\lambda)^2}(B+2(2-\lambda)I)\right] = I$$

This also implies

$$\left[\frac{1}{-(2-\lambda)^2}(B+2(2-\lambda)I)\right]B = I$$

From this we deduce that B is invertible, and we obtain

$$B^{-1} = \frac{1}{-(2-\lambda)^2}(B+2(2-\lambda)I)$$

3) Let A be a 3×3 matrix defined over $\mathbf{R}$. Assume that A is a reduced row echelon form (a canonical form) matrix. Let

$$B = \begin{pmatrix} 1 & 2 & 3 \\ 2 & a & 0 \\ 3 & 0 & b \end{pmatrix}$$

Find all values of $a, b \in \mathbf{R}$ such that $AB = 0$.

Solution: This depends on A. So we need to check all possibilities. Assume first that A has no zero rows ($\text{rank}(A) = 3$). Then $A = I$, the identity matrix. Hence $AB = IB = B$. Thus, the condition $AB = 0$ is equivalent in this case to $B = 0$. This can never happen.

Next assume that A has exactly one row of zeros ($\text{rank}(A) = 2$). Then there are three options. They are

$$A_1 = \begin{pmatrix} 1 & 0 & \gamma \\ 0 & 1 & \delta \\ 0 & 0 & 0 \end{pmatrix} \qquad A_2 = \begin{pmatrix} 1 & \gamma & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \qquad A_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

If $A = A_1$ then the $(1,2)$ entry of AB is equal to 2. Hence $AB \neq 0$ for all a, b. If $A = A_2$ or $A = A_3$ then the $(2,1)$ entry of AB is 3, and hence there are no a, b such that $AB = 0$.

Next we consider the case when there are 2 zero rows ($\text{rank}(A) = 1$). There are 3 options

$$A_4 = \begin{pmatrix} 1 & \gamma & \delta \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad A_5 = \begin{pmatrix} 0 & 1 & \gamma \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad A_6 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Assume first that $A = A_4$. Then

$$AB = \begin{pmatrix} 1+2\gamma+3\delta & 2+\gamma a & 3+\delta b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Hence the equation $AB = 0$ can hold only if γ and δ are such that $2\gamma + 3\delta = -1$. If that is the case, then to have a solution we must have $2 + \gamma a = 0$ and $3 + \delta b = 0$. If γ and δ are both nonzero and satisfy $2\gamma + 3\delta = -1$ then there is a unique solution for a and b given by $a = -2/\gamma$ and $b = -3/\delta$. If $\gamma = 0$, then $2 + \gamma a = 0$ has no solution. If $\delta = 0$, then $3 + \delta b = 0$ has no solution.

When $A = A_5$, we have

$$AB = \begin{pmatrix} 2+3\gamma & a & \gamma b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

In this case we have a solution if $\gamma = -2/3$. The solution is $a = b = 0$. In the case of $A = A_6$, we obtain that the $(1,1)$ entry of AB is 3, and hence there are no solutions.

Finally, we also need to consider the case when $A = 0$. In this case we clearly have $AB = 0$ for all a and b.