SOLUTIONS MOED B LINEAR ALGEBRA 1 A 2020

SEMYON ALESKER AND DAVID GINZBURG

Problem 1: Let V denote the subspace of $Mat_{3\times3}(\mathbf{R})$ consisting of all symmetric matrices with trace zero. Compute the dimension of V. Prove your claim.

Solution : A matrix A in $Mat_{3\times3}(\mathbf{R})$ is V if and only if $A^t = A$ and the sum of all the diagonal entries of A is zero. Thus,

$$A = \begin{pmatrix} \alpha & a & b \\ a & \beta & c \\ b & c & -\alpha-\beta \end{pmatrix}$$

We have

$$\begin{pmatrix} \alpha & a & b \\ a & \beta & c \\ b & c & -\alpha-\beta \end{pmatrix} = \alpha \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} + \beta \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} + a \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + b \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} + c \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

From this we deduce that the five matrices written on the right hand side of the above equation span the vector space V. It is also easy to deduce from the above equation that these five matrices are linearly independent. From this we obtain that $\dim V = 5$.

Problem 2: Let $A = (a_{i,j})$ denote the matrix of size four defined by $a_{i,j} = x^{max\{i,j\}}$ for all $1 \le i, j \le 4$. Compute the determinant of A.

Solution: By definition we have

$$A = \begin{pmatrix} x & x^2 & x^3 & x^4 \\ x^2 & x^2 & x^3 & x^4 \\ x^3 & x^3 & x^3 & x^4 \\ x^4 & x^4 & x^4 & x^4 \end{pmatrix}$$

Performing the operation $R_4 \to R_4 - xR_3$, we obtain

$$|A| = \begin{vmatrix} x & x^2 & x^3 & x^4 \\ x^2 & x^2 & x^3 & x^4 \\ x^3 & x^3 & x^3 & x^4 \\ 0 & 0 & 0 & x^4 - x^5 \end{vmatrix}$$

Next perform the row operation $R_3 \to R_3 - xR_2$. We have

$$|A| = \begin{vmatrix} x & x^2 & x^3 & x^4 \\ x^2 & x^2 & x^3 & x^4 \\ 0 & 0 & x^3 - x^4 & x^4 - x^5 \\ 0 & 0 & 0 & x^4 - x^5 \end{vmatrix}$$

Finally perform $R_2 \to R_2 - xR_1$. We obtain

$$|A| = \begin{vmatrix} x & x^2 & x^3 & x^4 \\ 0 & x^2 - x^3 & x^3 - x^4 & x^4 - x^5 \\ 0 & 0 & x^3 - x^4 & x^4 - x^5 \\ 0 & 0 & 0 & x^4 - x^5 \end{vmatrix}$$

Hence, $|A| = x(x^2 - x^3)(x^3 - x^4)(x^4 - x^5) = x^{10}(1-x)^3$.

Problem 3: Let V be a vector space defined over a field F. Let $T, S : V \to V$ be two linear maps. Prove that

$$T(ker(S \circ T)) = ImT \cap kerS$$

Solution : We will prove that each set is included in the other. Let $v \in T(ker(S \circ T))$. Then, there is $u \in ker(S \circ T)$ such that $Tu = v$. This implies that $v \in ImT$. Next, $Sv = S(Tu) = 0$ since by definition $u \in ker(S \circ T)$. Hence $v \in kerS$, and $v \in ImT \cap kerS$. This proves that $T(ker(S \circ T)) \subset ImT \cap kerS$.

The proof of the other inclusion is by reversing the argument. Let $v \in ImT \cap kerS$. Then, there is a $u \in V$ such that $Tu = v$ and also $Sv = 0$. Hence, $S(Tu) = Sv = 0$. Hence $u \in ker(S \circ T)$. Since $Tu = v$, we deduce that $v \in T(ker(S \circ T))$. Hence $ImT \cap kerS \subset T(ker(S \circ T))$.

Problem 4: Let V denote a vector space whose dimension is n. Let $T : V \to V$ be a linear map. Prove that there are bases $\mathcal{B}$ and $\mathcal{C}$ of V such that

$$[T]_{\mathcal{C}}^{\mathcal{B}} = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$$

Solution: Assume that $r = \dim ImT$. Then $n - r = \dim kerT$. Let $v_{r+1}, \ldots, v_n$ denote a base for $kerT$. Extend it to a base $\mathcal{B} = \{v_1, \ldots, v_r, v_{r+1}, \ldots, v_n\}$ of V. Denote $w_1 = T(v_1); \ldots; w_r = T(v_r)$. Then the set $\{w_1, \ldots, w_r\}$ is independent. Indeed, if $\alpha_1 w_1 + \cdots + \alpha_r w_r = 0$, then $T(\alpha_1 v_1 + \cdots + \alpha_r v_r) = \alpha_1 T(v_1) + \cdots + \alpha_r T(v_r) = \alpha_1 w_1 + \cdots + \alpha_r w_r = 0$. This implies that $\alpha_1 v_1 + \cdots + \alpha_r v_r \in kerT$. Since $v_{r+1}, \ldots, v_n$ is a base for $kerT$, we deduce that

$$\alpha_1 v_1 + \cdots + \alpha_r v_r = \beta_{r+1} v_{r+1} + \cdots + \beta_r v_r$$

or

$$\alpha_1 v_1 + \cdots + \alpha_r v_r - \beta_{r+1} v_{r+1} + \cdots - \beta_r v_r = 0$$

Since $\mathcal{B}$ is a base for V this implies that $\alpha_i = 0$ for all $1 \leq i \leq r$. Hence, $\{w_1, \ldots, w_r\}$ is independent. Extend it to a base $\mathcal{C} = \{w_1, \ldots, w_r, w_{r+1}, \ldots, w_n\}$ of V. Then the matrix $[T]_{\mathcal{C}}^{\mathcal{B}}$ is as above.