Solution of Moed C in Linear Algebra 1 2022

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1) Problem: Let $P_n(x)$ denote the vector space of all polynomials with coefficients in $\mathbf{C}$ whose degree is at most n. For $p(x) \in P_n(x)$, let $T : P_n(x) \to P_n(x)$ denote the linear map defined by $T(p(x)) = p(x) + p'(x)$. Prove that T is an isomorphism.

In addition, suppose that $n = 2$. Given $p(x) \in P_2(x)$, give an explicit formula for the linear map $T^{-1}(p(x))$.

Solution: Let $p(x) = a_0 + a_1x + \cdots + a_nx^n$. Then,

$$T(p(x)) = p(x) + p'(x) = (a_0 + a_1x + \cdots + a_nx^n) + (a_1 + 2a_2x + 3a_3x^2 + \cdots + na_nx^{n-1}) =$$

$$= (a_0 + a_1) + (a_1 + 2a_2)x + (a_2 + 3a_3)x^2 + \cdots + (a_{n-1} + na_n)x^{n-1} + a_nx^n$$

To prove that T is an isomorphism it is enough to prove that $\ker T = \{0\}$. Hence, if $T(p(x)) = 0$, we get from the above the following system of equations,

$$a_0 + a_1 = 0;\;\; a_1 + 2a_2 = 0;\;\; a_2 + 3a_3 = 0; a_{n-1} + na_n = 0;\;\; a_n = 0$$

In other words, the system of equations is given by $a_{i-1} + ia_i = 0;\;\; a_n = 0$. Here, $1 \le i \le n$. It is easy to prove by induction, that this system has only the trivial solution.

Assume that $n = 2$. To give an explicit formula for T^{-1}, it is enough to compute $T^{-1}(1);\; T^{-1}(x)$ and $T^{-1}(x^2)$. Assume that $T^{-1}(1) = b_0 + b_1x + b_2x^2$. Then, $1 = b_0T(1) + b_1T(x) + b_2T(x^2)$. This is the same as $1 = b_0 + b_1(x+1) + b_2(x^2+2x)$. Comparing coefficients we get $b_0 + b_1 = 1;\; b_1 + 2b_2 = 0$ and $b_2 = 0$. The solution is $b_0 = 1;\; b_1 = b_2 = 0$. Hence $T^{-1}(1) = 1$. To compute $T^{-1}(x)$, we need to solve the equation $x = b_0 + b_1(x+1) + b_2(x^2+2x)$. This gives us $b_0 = -1;\; b_1 = 1$, and $b_2 = 0$. Hence $T^{-1}(x) = -1 + x$. Similarly, the equation $x^2 = b_0 + b_1(x+1) + b_2(x^2+2x)$ implies $T^{-1}(x^2) = 2 - 2x + x^2$. Thus, if $p(x) = a + bx + cx^2$, then

$$T^{-1}(p(x)) = T^{-1}(a + bx + cx^2) = aT^{-1}(1) + bT^{-1}(x) + cT^{-1}(x^2) =$$

$$= a + b(-1 + x) + c(2 - 2x + x^2) = (a - b + 2c) + (b - 2c)x + cx^2$$

Remark: Its not easy to see it, but in general, for all n, we have

$$T^{-1}(p(x)) = p(x) - p^{(1)}(x) + p^{(2)}(x) - p^{(3)}(x) + \cdots + (-1)^n p^{(n)}(x)$$

Here $p^{(i)}(x)$ is the i-th derivative of $p(x)$.

2) Problem: Let V denote a vector space defined over the field $\mathbf{C}$. Let $T, S : V \to V$ denote two linear maps. Assume that

$$5T^4 - 2T^2 + 3TS + T - I = 0 \tag{1}$$

Prove that $TS = ST$.

Solution: Write the identity (1) as $5T^4 - 2T^2 + 3TS + T = I$. This implies that $T(5T^3 - 2T + 3S + I) = I$. Hence T is invertable and we have $(5T^3 - 2T + 3S + I)T = I$. This last identity follows from the fact that if A and B are two square matrices such that $AB = I$, then $BA = I$. It is also true for linear maps.

The equation $(5T^3 - 2T + 3S + I)T = I$ implies

$$5T^4 - 2T^2 + 3ST + T = I \tag{2}$$

Subtracting equation (1) from equation (2), we get $TS = ST$.

3) Problem: Let A denote the 5×5 such that every number between one and five appears exactly once in each row and each column. Prove that 75 divides the determinant of A.

Remark: An example of such a matrix is

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 1 & 2 & 3 & 4 \\ 4 & 5 & 1 & 2 & 3 \\ 3 & 4 & 5 & 1 & 2 \\ 2 & 3 & 4 & 5 & 1 \end{pmatrix}$$

Solution: For $2 \leq i \leq 5$ perform the row operation $R_1 \to R_1 + R_i$ on A. Then, we obtain a matrix B such that all entries in first row are $1 + 2 + 3 + 4 + 5 = 15$. For example, for the above matrix A we get

$$B = \begin{pmatrix} 15 & 15 & 15 & 15 & 15 \\ 5 & 1 & 2 & 3 & 4 \\ 4 & 5 & 1 & 2 & 3 \\ 3 & 4 & 5 & 1 & 2 \\ 2 & 3 & 4 & 5 & 1 \end{pmatrix}$$

From the properties of determinants, we have $|A| = |B| = 15|C|$, where all entries of the first row of C are ones, and all other entries are as in the matrix B. In the above example, we have

$$C = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 5 & 1 & 2 & 3 & 4 \\ 4 & 5 & 1 & 2 & 3 \\ 3 & 4 & 5 & 1 & 2 \\ 2 & 3 & 4 & 5 & 1 \end{pmatrix}$$

Next, on C perform the column operation $C_1 \to C_1 + C_j$ for $2 \le j \le 5$. Then, we obtain a matrix D such that all entries of the first column are 15, except $d_{1,1} = 5$. In the above example, we have

$$D = \begin{pmatrix} 5 & 1 & 1 & 1 & 1 \\ 15 & 1 & 2 & 3 & 4 \\ 15 & 5 & 1 & 2 & 3 \\ 15 & 4 & 5 & 1 & 2 \\ 15 & 3 & 4 & 5 & 1 \end{pmatrix}$$

We have $|A| = 15|C| = 15|D|$. Clearly, five divides $|D|$. Hence, 75 divides $|A|$.

Remark: Many students wrote that it is "obvious" that using rows and columns permutations we can obtain any such matrix. Hence, if we start with such a matrix, one can change rows and columns and obtain the matrix A as described above. Since row and column changes, changes only the sign of the determinant, and hence it is enough to prove that the determinant of the above matrix A is divisible by 75.

Not only its not obvious that this argument is true, in fact in general it is not true. Consider the two 4×4 matrices

$$A_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 4 & 1 & 2 \\ 4 & 3 & 2 & 1 \end{pmatrix} \qquad A_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \\ 3 & 4 & 1 & 2 \\ 2 & 3 & 4 & 1 \end{pmatrix}$$

Then $|A_1| = 0$ and $|A_2| = -160$.

4) Problem: In this problem V is a finitely generated vector space over the field $\mathbf{C}$.
a) (**12 points**) Let $T, S : V \to V$ be two linear maps. Suppose that $V = \mathrm{Im}T + \mathrm{Im}S = \ker T + \ker S$. Prove that $V = \mathrm{Im}T \oplus \mathrm{Im}S = \ker T \oplus \ker S$.
b) (**13 points**) Let U_1, U_2 and U_3 denote three subspaces of V. Prove that

$$\dim(U_1 \cap U_2 \cap U_3) \geq \dim U_1 + \dim U_2 + \dim U_3 - 2\dim V$$

Solution: a) We need to prove that $\mathrm{Im}T \cap \mathrm{Im}S = \ker T \cap \ker S = \{0\}$. We have

$$2\dim V = \dim(\mathrm{Im}T + \mathrm{Im}S) + \dim(\ker T + \ker S) \tag{3}$$

The first dimension Theorem implies that

$$\dim(\mathrm{Im}T + \mathrm{Im}S) = \dim\mathrm{Im}T + \dim\mathrm{Im}S - \dim(\mathrm{Im}T \cap \mathrm{Im}S)$$

$$\dim(\ker T + \ker S) = \dim\ker T + \dim\ker S - \dim(\ker T \cap \ker S)$$

Plugging these two equations in equation (3), we obtain

$$2\dim V = \dim\mathrm{Im}T + \dim\mathrm{Im}S - \dim(\mathrm{Im}T \cap \mathrm{Im}S) + \dim\ker T + \dim\ker S - \dim(\ker T \cap \ker S) =$$

$$= (\dim\mathrm{Im}T + \dim\ker T) + (\dim\mathrm{Im}S + \dim\ker S) - \dim(\mathrm{Im}T \cap \mathrm{Im}S) - \dim(\ker T \cap \ker S) =$$

$$= 2\dim V - \dim(\mathrm{Im}T \cap \mathrm{Im}S) - \dim(\ker T \cap \ker S)$$

Here, we used the second dimension Theorem $\dim V = \dim\mathrm{Im}T + \dim\ker T$, and similarly for S.

From the above equation we deduce that $\dim(\mathrm{Im}T \cap \mathrm{Im}S) = \dim(\ker T \cap \ker S) = 0$.
b) Let L_1 and L_2 be two subspaces of V. Apply the first dimension Theorem to L_1 and L_2. We have $\dim(L_1 + L_2) = \dim L_1 + \dim L_2 - \dim(L_1 \cap L_2)$. This is the same as $\dim(L_1 \cap L_2) = \dim L_1 + \dim L_2 - \dim(L_1 + L_2)$. Since $\dim(L_1 + L_2) \leq \dim V$, then it follows from the above equation that $\dim(L_1 \cap L_2) \geq \dim L_1 + \dim L_2 - \dim V$. Use this inequality with $L_1 = U_1$ and $L_2 = U_2 \cap U_3$. Then we get

$$\dim(U_1 \cap U_2 \cap U_3) \geq \dim U_1 + \dim(U_2 \cap U_3) - \dim V$$

Use $\dim(L_1 \cap L_2) \geq \dim L_1 + \dim L_2 - \dim V$ once again, this time with $L_1 = U_2$ and $L_2 = U_3$. We get

$$\dim U_1 + \dim(U_2 \cap U_3) - \dim V \geq \dim U_1 + (\dim U_2 + \dim U_3 - \dim V) - \dim V$$

From this the result follows.