Solution of Moed B in Linear Algebra 2011

David Ginzburg

2) a) Suppose that $A = (a_{i,j}) \in \ker T$. Then $T(A) = \sum_{i,j=1}^{3} a_{i,j} = 0$. Notice that T is onto, and hence, by the dimension theorem $\dim\ker T = 9 - 1 = 8$. Thus, a base is given by

$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \begin{pmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}; \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}; \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

b) The answer is yes. To prove it, for $1 \le k \le m$, let $\alpha_k \in \mathbf{C}$ be such that $\alpha_1 v_1 + \cdots + \alpha_m v_m = 0$. Write $\alpha_k = \beta_k + i\gamma_k$ where $\beta_k, \gamma_k \in \mathbf{R}$. Here i is the complex imaginary number. Thus, we have

$$(\beta_1 v_1 + \cdots \beta_m v_m) + i(\gamma_1 v_1 + \cdots + \gamma_m v_m) = 0$$

Since all entries of v_k are real, the above equality implies that $\beta_1 v_1 + \cdots \beta_m v_m = 0$ and $\gamma_1 v_1 + \cdots + \gamma_m v_m = 0$. Since $\{v_1, \ldots, v_m\}$ is linearly independent, it follows that all β_k and all γ_k, and hence all α_k are zeros.

3) a) A vector is in the subspace $\text{Sp}\{(1,2,3);\ (1,1,0);\ (1,1,1)\}$ if and only if it is of the form

$$x_1(1,2,3) + x_2(1,1,0) + x_3(1,1,1) = (x_1 + x_2 + x_3, 2x_1 + x_2 + x_3, 3x_1 + x_3)$$

It is not hard to check that the given three vectors are linearly independent. Hence, we are looking for a system $Ax = 0$ such that $Av = 0$ for all $v \in \mathbf{Z}_5^3$. The only such matrix is $A = 0$.

b) This is easily done by direct calculation. Expand along the first column. We obtain

$$\begin{vmatrix} a+b & c & c \\ a & b+c & a \\ b & b & a+c \end{vmatrix} = (a+b)[(b+c)(a+c) - ab] - a[c(a+c) - bc] + b[ca - c(b+c)]$$

Computing the right hand side, one gets that the determinant is equal to $4abc$. Hence, the answer is all triple numbers a, b and c such that at least one of them is even.

4) Let v be a vector in the intersection. Then $T^k v = 0$ and there exists $u \in V$ such that $v = T^k u$. Consider the set of vectors $\{u, Tu, \ldots, T^k u\}$. Since $k \geq n$, then this set is dependent. Hence, there exist $\alpha_i \in F$, not all zero such that

$$\alpha_0 u + \alpha_1 Tu + \cdots + \alpha_k T^k u = 0 \tag{1}$$

If $u = 0$, then $v = 0$ and we are done. Hence assume that $u \neq 0$. Then $T^{2k} u = T^k(T^k v) = 0$. Hence, there exist a number $m \geq 0$ such that $T^m u \neq 0$ but $T^{m+1} u = 0$. If $m + 1 \leq k$, then $v = T^k u = T^{k-m-1}(T^{m+1} u) = 0$ and we are done. Finally, assume that $k < m + 1$. We shall derive a contradiction. Apply T^m to (1). Since $T^{m+1} u = 0$, we obtain $\alpha_0 T^m u = 0$. Since $T^m u \neq 0$ we obtain $\alpha_0 = 0$. Then (1) is given by $\alpha_1 Tu + \cdots + \alpha_k T^k u = 0$. Apply T^{m-1} to this equation we obtain $\alpha_1 T^m u = 0$, and hence $\alpha_1 = 0$. Since $k < m + 1$ we can repeat this process, where the last step is by applying T^{m-k}. Thus we obtain that for all $0 \leq i \leq k$ we have $\alpha_i = 0$. This is a contradiction.