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בחינה באלגברה לינארית 1 א

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- משך הבחינה 3 שעות.
- יש לענות על כל השאלות.
- אין להשתמש בכל חומר עזר לרבות מחשבון.
- יש לנמק היטב את דרך הפיתרון.

שאלה 1 *(25 נ')* במרחב הוקטורי $\mathbb{R}^4$ נתונים שני תתי המרחב

$$W_1 = \mathrm{Sp}\{(1,2,3,4);\ (3,4,5,6);\ (7,8,9,10)\},$$
$$W_2 = \{(a,b,c,d)\ :\ a+b=0;\ a,b,c,d \in \mathbb{R}\}.$$

מצאו תת מרחב W_3 של W_2 כך ש- $W_1 \oplus W_3 = W_1 + W_2$.

שאלה 2 *(25 נ')* נתונה מערכת של n משוואות עם n נעלמים ופרמטר k,

$$\begin{cases} kx_1 + x_2 + \ldots + x_n = 1 \\ x_1 + kx_2 + \ldots + x_n = 1 \\ \ldots \quad \ldots \quad \ldots \quad \ldots \\ x_1 + x_2 + \ldots + kx_n = 1 \end{cases}.$$

לכל ערך של $k \in \mathbb{R}$ מצאו את כל הפתרונות של המערכת.

שאלה 3 *(25 נ')* יהי L תת המרחב של $Mat_{3\times3}(\mathbb{R})$ המוגדר על ידי

$L = \left\{ A \in Mat_{3\times3}(\mathbb{R})\ :\ A\begin{pmatrix}1\\2\\3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix} \right\}$. תנו דוגמה לבסיס עבור L.

שאלה 4 א. *(13 נ')* יהי V מרחב וקטורי מעל השדה F. נתונה העתקה לינארית $T: V \to V$ המקיימת $T^2 = T$. הוכיחו כי $KerT \oplus \mathrm{Im}\, T = V$.

ב. *(12נ')* נתונה $A \in Mat_{n\times n}(\mathbb{R})$. יהי W תת המרחב של $Mat_{n\times n}(\mathbb{R})$ המוגדר על ידי $W = \{X \in V\ :\ AX = XA\}$. הוכיחו כי $\dim W \geq 2$.

בהצלחה!

Solutions

Solution 1: We start by computing $\dim(W_1 + W_2)$. For that we use the dimension Theorem

$$\dim(W_1 + W_2) = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2)$$

To compute $\dim W_1$, we find a basis for W_1. We write the three vectors as a matrix, and we perform the following row operations,

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 5 & 6 \\ 7 & 8 & 9 & 10 \end{pmatrix} \xrightarrow[R_3 \to R_3 - 7R_1]{R_2 \to R_2 - 3R_1} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -2 & -4 & -6 \\ 0 & -6 & -12 & -18 \end{pmatrix} \xrightarrow[R_2 \to -R_2]{R_3 \to R_3 - 3R_2} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Hence $\dim W_1 = 2$, and $\mathcal{B} = \{(1, 2, 3, 4); (0, 1, 2, 3)\}$ is a basis for W_1.

As for $\dim W_2$, we can write a vector in W_2 as

$$(a, -a, c, d) = a(1, -1, 0, 0) + c(0, 0, 1, 0) + d(0, 0, 0, 1)$$

from which we deduce that $\dim W_2 = 3$. To compute $\dim(W_1 \cap W_2)$ we take a vector in $w \in W_1 \cap W_2$ and write it as

$$w = \alpha(1, 2, 3, 4) + \beta(0, 1, 2, 3) = (a, -a, c, d)$$

Here α, β, a, c and d are all scalars in $\mathbf{R}$. Comparing the coordinates on the right hand side of the equation, we get the system of equations $\alpha = a$; $2\alpha + \beta = -a$; $3\alpha + 2\beta = c$, and $4\alpha + 3\beta = d$. It is easy to see that the solution is $\alpha = a$; $\beta = c = -3a$, and $d = -5a$. Hence $W_1 \cap W_2 = \{(a, -a, -3a, -5a) \ : \ a \in \mathbf{R}\} = \mathrm{Sp}\{(1, -1, -3, -5)\}$. In particular we obtain $\dim(W_1 \cap W_2) = 1$. Hence, from the dimension Theorem stated above we obtain that $\dim(W_1 + W_2) = 2 + 3 - 1 = 4 = \dim \mathbf{R}^4$. This implies that $W_1 + W_2 = \mathbf{R}^4$. Thus, the problem is reduced to finding $W_3 \subset W_2$ such that $W_1 \oplus W_3 = \mathbf{R}^4$. To do that it is enough to find two vectors $u_1, u_2 \in W_3$ such that the set $\mathcal{B} \cup \{u_1, u_2\}$ is a basis for $\mathbf{R}^4$. It is easy to see that $u_1 = (0, 0, 1, 0)$ and $u_2 = (0, 0, 0, 1)$ is a such a choice.

Solution 2: Adding all the equations in the system we obtain

$$(k+n-1)(x_1+x_2+\cdots+x_n)=n$$

If $k=1-n$, then the left hand side of the above equation is zero, and the right hand side is equal to n. So in this case there are no solutions.

Assume that $k\neq 1-n$. Then, dividing by $k+n-1$ we obtain the equation

$$x_1+x_2+\cdots+x_n=\frac{n}{k+n-1} \tag{7}$$

Subtract this equation from the first equation of the system. We obtain the equation

$$(k-1)x_1=1-\frac{n}{k+n-1}$$

Repeating this subtraction for each equation of the system, we obtain

$$(k-1)x_i=1-\frac{n}{k+n-1} \qquad 1\leq i\leq n$$

If $k\neq 1$, we can divide by $k-1$, and obtain $x_i=\frac{1}{k+n-1}$. If $k=1$, the system of equations reduces to one equation which is $x_1+x_2+\cdots+x_n=1$.

Hence, the solutions of the system are as follows:

a) If $k=1$, the system has infinite number of solutions given by

$$\{(1-\alpha_2-\alpha_3-\ldots-\alpha_n,\alpha_2,\ldots,\alpha_n):\ \alpha_i\in\mathbf{R};\quad 2\leq i\leq n\}$$

b) If $k=1-n$, the system has no solutions.

c) If $k\neq 1,1-n$, the system has a unique solution given by

$$\left(\frac{1}{k+n-1},\frac{1}{k+n-1},\ldots,\frac{1}{k+n-1}\right).$$

Solution 3: Assume that $A = (a_{i,j})_{3\times 3} \in L$. Then, by matrix multiplication, the equation

$$A\begin{pmatrix}1\\2\\3\end{pmatrix} = 0$$

is equivalent to the system of equations

$$a_{1,1} + 2a_{1,2} + 3a_{1,3} = 0$$

$$a_{2,1} + 2a_{2,2} + 3a_{2,3} = 0$$

$$a_{3,1} + 2a_{3,2} + 3a_{3,3} = 0$$

This system has 9 variables and 3 equations. From this we easily deduce that it has 6 free variables. Hence, $\dim L = 6$. An example for a basis for L is the set

$$\begin{pmatrix}-2&1&0\\0&0&0\\0&0&0\end{pmatrix};\begin{pmatrix}-3&0&1\\0&0&0\\0&0&0\end{pmatrix};\begin{pmatrix}0&0&0\\-2&1&0\\0&0&0\end{pmatrix};\begin{pmatrix}0&0&0\\-3&0&1\\0&0&0\end{pmatrix};\begin{pmatrix}0&0&0\\0&0&0\\-2&1&0\end{pmatrix};\begin{pmatrix}0&0&0\\0&0&0\\-3&0&1\end{pmatrix}$$

Solution 4: a) First we prove that $\ker T \cap \operatorname{Im} T = \{0\}$. Let $v \in \ker T \cap \operatorname{Im} T$. Then, $Tv = 0$ (since $v \in \ker T$), and there is $u \in V$ such that $v = Tu$ (since $v \in \operatorname{Im} T$). Hence, $0 = Tv = T(Tu) = T^2 u = Tu = v$. Here, we used the fact that $T^2 = T$. Using the two dimension Theorems we obtain

$$\dim(\ker T + \operatorname{Im} T) = \dim(\ker T) + \dim(\operatorname{Im} T) - \dim(\ker T \cap \operatorname{Im} T) =$$

$$= \dim(\ker T) + \dim(\operatorname{Im} T) = \dim V$$

Here, we used the fact that $\ker T \cap \operatorname{Im} T = \{0\}$. Hence, $V = \ker T + \operatorname{Im} T$, and from the fact that $\ker T \cap \operatorname{Im} T = \{0\}$ we deduce that $V = \ker T \oplus \operatorname{Im} T$.

Since the set $\{v_1, \ldots, v_r, w_1, \ldots, w_{n-r}\}$ is a basis, then $\alpha_1 = \ldots = \alpha_{n-r} = 0$.

Because $T^2 = 0$, for all $1 \leq i \leq n-r$ we have $T(Tw_i) = T^2 w_i = 0$. Hence, $\{Tw_1, \ldots, Tw_{n-r}\}$ is a linearly independent set inside $ker\ T$. Since $\dim(ker T) = r$, we obtain $n - r \leq r$, or $n \leq 2r$.

b) There are two cases to consider. First, assume that $A = aI$ for some $a \in \mathbf{R}$. Then it is easy to see that $W = V$. Hence, it is clear that $\dim W \geq 2$. In general it is clear that the matrices I and A are in W. Suppose that $A \neq aI$ for all $a \in \mathbf{R}$. Then the claim is that the set $\{I, A\}$ is a linearly independent set in V. Indeed, consider the equation $\alpha I + \beta A = 0$. If $\beta \neq 0$, then $A = -\frac{\alpha}{\beta} I$. This a contradiction to the assumption that $A \neq aI$. Hence $\beta = 0$, which implies that $\alpha = 0$. Hence, again we have $\dim W \geq 2$.