Solution of Midterm Linear Algebra 1 2019

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1) Problem: Write down all matrices A such that both A and A^t are canonical matrices.

Solution: If $A = 0$, then $A^t = 0$ and both are canonical matrices. Assume that $A \neq 0$. Write $A = (a_{i,j})$. Consider the first row of A. For $A \neq 0$ to be in canonical form, its first row should be nonzero. If $a_{1,1} = 0$, then the first column of A is zero, and then the first row of $A^t \neq 0$ must be the zero row. This means that A^t is not a canonical matrix. Hence, we must have $a_{1,1} = 1$. Hence, $a_{i,1} = 0$ for all $i \geq 2$. If $a_{1,j} \neq 0$ for some $j \geq 2$, then A^t is not a canonical matrix. From this we conclude that

$$A = \begin{pmatrix} 1 & 0_{1\times(n-1)} \\ 0_{(n-1)\times 1} & A_1 \end{pmatrix}$$

In general, we denote by $0_{k,l}$ the zero matrix of order $k \times l$. It is clear that A and A^t are both in canonical form if and only if A_1 and A_1^t are. Continuing by induction we deduce that

$$A = \begin{pmatrix} I_r & 0_{r\times(n-r)} \\ 0_{(m-r)\times r} & 0_{(m-r)\times(n-r)} \end{pmatrix} \qquad 0 \leq r \leq \min(m, n)$$

2) Problem: For values of real numbers α and β, the following system

$$x + 2y + z = 0$$

$$-x + (\beta - 1)y - 2z = 1$$

$$x + (2\beta + 4)y + (\alpha^2 - 2)z = 1$$

has infinite number of solutions? There is no need to write the solutions.

Solution The extended matrix of the system is

$$\begin{pmatrix} 1 & 2 & 1 & 0 \\ -1 & \beta - 1 & -2 & 1 \\ 1 & 2\beta + 4 & \alpha^2 - 2 & 1 \end{pmatrix}$$

Perform the row operations $R_2 \to R_2 + R_1$ and $R_3 \to R_3 - R_1$. We get

$$\begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & \beta+1 & -1 & 1 \\ 0 & 2\beta+2 & \alpha^2-3 & 1 \end{pmatrix}$$

Next, perform the row operation $R_3 \to R_3 - 2R_2$. We obtain

$$\begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & \beta+1 & -1 & 1 \\ 0 & 0 & \alpha^2-1 & -1 \end{pmatrix}$$

Suppose that $\alpha^2 - 1 = 0$. Then the system has no solution. Thus we have $\alpha^2 - 1 \neq 0$. If $\beta+1 \neq 0$, then the reduced matrix of the system will be row equivalent to the identity which means that the system has a unique solution. Thus we have $\beta = -1$. From this we obtain the two equations $-z = 1$ and $(\alpha^2 - 1)z = -1$. Hence $z = -1$ and $\alpha = \pm\sqrt{2}$.

To summarize, the system has infinite number of solutions if and onlt if $\beta = -1$ and $\alpha = \pm\sqrt{2}$.

3) Problem: Let $p \geq 3$ denote a prime number. How many matrices of the form

$$A = \begin{pmatrix} 1 & a \\ b & 2 \end{pmatrix}$$

are invertible over the field $\mathbf{Z}_p$?

Solution: A matrix is invertible if and only if it is row equivalent to the identity matrix. Applying the row operation $R_2 \to R_2 - bR_1$, we obtain that A is row equivalent to the matrix

$$B = \begin{pmatrix} 1 & a \\ 0 & 2-ab \end{pmatrix}$$

Clearly, the matrix B is row equivalent to the identity matrix if and only if $ab \neq 2$. To count how many possibilities there are for $ab \neq 2$, it is easier to count the number of solutions of $ab = 2$. Clearly $a, b \neq 0$. Given $b \neq 0$ it determines a uniquely, that is, we have $a = 2b^{-1}$. Hence, the equation $ab = 2$ has exactly $p - 1$ solutions. Therefore , the number of choices for $ab \neq 2$ is $p^2 - (p-1) = p^2 - p + 1$.