Tilburg University

Inventory and Production Management

Group Assignment Report

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Preface

For the purpose of this assignment we've been provided with a dataset consisting of demand records for 104 weeks for Mr.Stark's coffee company. Our first task is to forecast the 6-week-ahead demand. To this end, we've decided to generate the desired estimates using two methods: Single Exponential Smoothing (referred to hereon as SES) and Holt's trend method. Before performing any computations, we visualise the demand Mr.Stark faced over the last 104 weeks below:

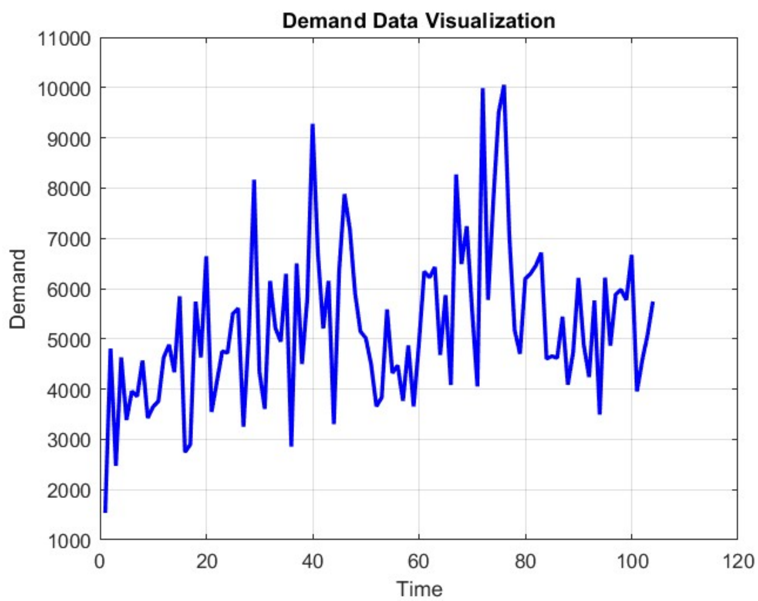

Figure 1: Demand past data

As portrayed by the figure, the demand curve appears to be rather inconsistent over time, with relatively steep peaks and troughs between consecutive time frames. Even before constructing the aforementioned forecasting methods, we suspect that such spread forebodes high forecast variance estimates.

Another essential ingredient to our preliminary analysis is to try and establish the approximate distribution of our demand. Such information is crucial for the forthcoming analysis. In this vein, we provide a histogram based on the demand data in figure 2 below:

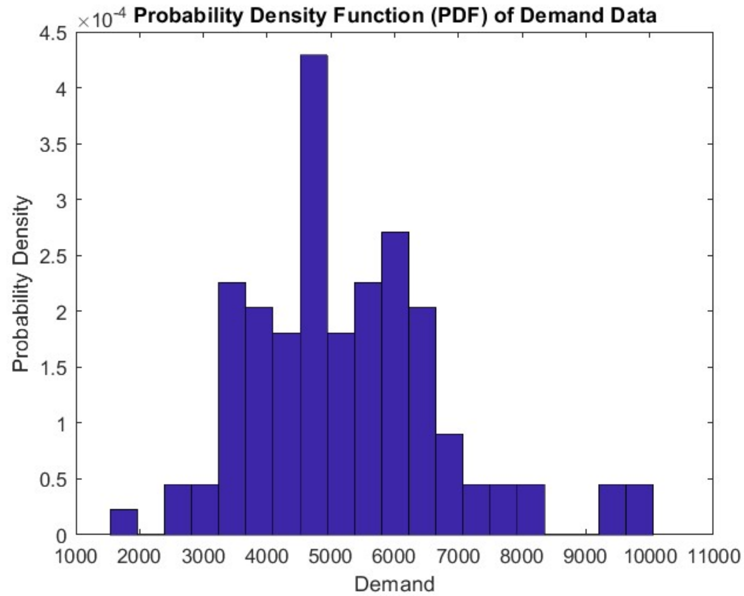

Figure 2: Demand Distribution

Drawing conclusions from the histogram, we posit that for the sake of this assignment it is viable to conclude that our data leans towards the gamma distributions family rather than the normal distributions, due to the discernible skew towards the left and elongated tail stretching rightwards. Thereby, throughout this report the derivations will adhere to the gamma pdf, CDF and loss functions hereon.

In order to avoid over fitting upon parameter optimisation in the sequel, we use 70 of the 104 observations (to come at roughly 70%) in our data as training set, leaving out the remaining 34 records ($\approx$ 30% of data) as our test set. We introduce this method as means to grant better robustness to our results and avoid anomalies in our forecasts due to deviant parameters.

1 Exercise 1

Adhering to the previous discussion, in this section we carry out SES and Holt estimations for the demand in weeks 105 through 110 inclusive.

1.1 SES Forecasting procedure

This forecasting method requires the quantification of two parameters: the smoothing parameter α and the initialisation level $\hat{y}_1$. Unlike Holt's setup, this method lacks a trend figure.

For the sake of obtaining coherent results, rather than prematurely fixing the level and α via rule of thumb or using certain average, we use values that prove optimal according to our models. The optimisation procedure we enact is as follows:

1. We start off by selecting conventional initialisation smoothing parameter $\alpha = 0.1$ and $\hat{y}_1$ is declared as the average of the first 10 observations in our dataset.

2. Constructing a function *SES*, we generate our forecasts for each time period $\hat{y}_i$ for $i = 1, 2, \ldots, 105$, where $\hat{y}_1$ is defined as mentioned above and the other data points are obtained via direct application of the SES algorithm:

$$\hat{y}_{t+1} = \alpha \cdot y_t + (1 - \alpha) \cdot \hat{y}_t$$

3. The SES forecast for h periods ahead here is given by $\hat{y}_{t+h|t} = \hat{y}_{t+1}$, which in our case implies $\hat{y}_{104+h|104} = \hat{y}_{105}$ for $h = 1, 2, \ldots, 6$.

4. Having obtained the forecasts for each point in time considered, we generate the squared errors by first subtracting the SES estimates from each corresponding observation, and then squaring the respective differences.

5. Via taking the square root of the average of the squared errors **of our training set** (first 70 records), we quantify the RMSE of the training data.

6. Using the handle function opt_SES and the $fmincon$ functionality in Matlab, we minimise the training RMSE and extract the respective smoothing and initial level parameters. We use this specific function as it allows us to feed in lower and upper bounds for the variables of interest. For the level we only impose positiveness, whereas for the smoothing parameter we restrict the domain to $\alpha \in [0, 0.3]$ as we seek to avoid extreme smoothing that would potentially spark overfit, thus compromising the model estimates.As initial bracket $x0$ we supply an α of 0.1 and level equal to the average of the first 10 observations.

The optimal parameter values we obtain are now provided:

α	$\hat{y}_1$
0.156	3636.723

Table 1: Optimal SES parameters

Having fine-tuned these parameters, we roll out the SES equations for each time frame again (this time on the entire database), eventually arriving at the one-period-ahead forecast $\hat{\mu}_1(t)$, which due to the properties of SES is the predicted demand for all upcoming 6 weeks. The cumulative sum at each point in time gives us the desired $\hat{\mu}_\tau(t)$ for $\tau = 1, 2, \ldots, 6$. The results are displayed later in Table 2.

As for the forecast error standard deviation, we incorporate two methods for the sake of completeness-the simple naive estimator and the analytical method.

Prior to generating the sigmas, we need an initial value for the one-period-ahead sigma. To do this, we construct a function $get_smoothing_SES$ that uses the squared errors from the SES estimation and applies single exponential smoothing on them and the dataset (the smoothing parameter here is a different α_{SE}):

$$V_t = \alpha_{SE} e_t^2 + (1 - \alpha_{SE}) V_{t-1}$$

As initial level, we follow the convention of supplying the RMSE of the SES estimates, but for the sake of comprehensiveness we restrict the RMSE only to the test set (get the mean square of the errors in periods 1-70) to circumvent overfitting issues. The function returns the smoothed RMSE of the test set. As for the choice of the smoothing constant, we optimise the aforementioned function with respect to the returned variable (the training smoothed RMSE) to get the optimal value, using $\alpha_{SE} = 0.05$ as initial guess to supply to the minimisation function. In our case, the optimal smoothing constant emerged as $\alpha_{SE} = 0.0194$. Although low in magnitude, we believe that this constant is reasonable for too large figures would equivocally lead to overfitting (we would just have the smoothed errors following the original squared errors), whereas lower α_{SE} reflects rough gravitation towards the RMSE in general (the initial point), which is agreeable in this intermittent setup. Using this figure and the training set RMSE as initial level, we assemble the smoothed squared errors for each forecast. Using the square root of the very last smoothed squared error, $\sqrt{V_{100}}$, we arrive at the desired initial estimate for the one-step estimated error standard deviation $\hat{\sigma_1}$.

The naive method then dictates that the standard deviations for our forecasts are given by

$$\hat{\sigma}_\tau = \sqrt{\tau} \cdot \hat{\sigma_1} \text{ for } \tau = 1, 2, \ldots, 6.$$

However, despite it offering simplicity in computations and easier interpretability, the naive estimator rest on rather strong assumptions:

1. Forecasts need to be i.i.d. for all future periods.
2. The forecasts for multiple steps into the future are equally challenging to derive as the ones for only one frame ahead.

Considering how irregular the demand of Mr.Stark as depicted by the figures in the preface, we deem the second assumption particularly inapplicable for the investigated demand schedule here.

To tackle this, we embed an analytical sigma estimator in our model. To this end, it's apt to satisfy rather milder assumptions:

1. Correct specification of the underlying model.
2. Knowledge of the model parameters.
3. Access to data on the current state t.

Combining our work so far, we believe that all three boxes are ticked, given that we have obtained the optimal model parameters and are provided current data. To calculate the analytical estimator, we use the same $\hat{\sigma_1}$ optimised previously, but this time devise the rest using the formula

$$\hat{\sigma_\tau} = \hat{\sigma}_1 \cdot \sqrt{\sum_{j=0}^{\tau-1} C_j^2}, \text{ where } C_j = 1 + \sum_{i=1}^{j} \alpha$$

In this fashion, we obtain the desired forecast error standard deviation figures. Below we summarise both the μ and σ values that emerge as final results from our modelling:

τ	$\hat{\mu_\tau}$	$\hat{\sigma_\tau^N}$	$\hat{\sigma_\tau^A}$
1	5302.218	1403.813	1403.813
2	10604.436	1985.291	2146.057
3	15906.655	2431.475	2828.584
4	21208.873	2807.626	3500.424
5	26511.091	3139.021	4178.287
6	31813.309	3438.625	4869.550

Table 2: Forecasts and Errors SES Model

The superscript N above refers to the naive estimator and A-to the analytical one, respectively. Observing the latest trends in Mr.Stark's sales, we deem these results reasonable for the near future, considering how the demand figures fluctuate around the 5000 units mark. In the graph below, we visualise how close our SES forecasts are to the actual recorded data.

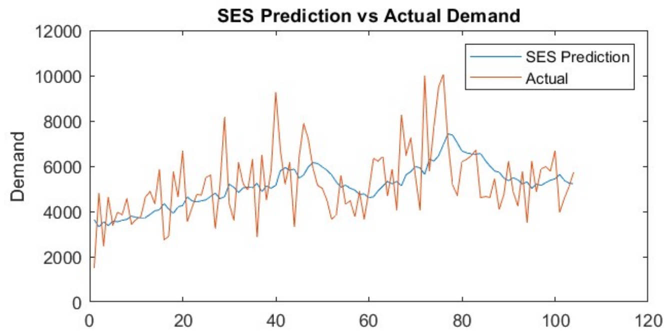

Figure 3: SES Forecast Accuracy

1.2 Holt's Trend Method

To provide thorough feedback on Mr.Stark's policy, we investigate the forecasting performance of a trend model as well. The environment here is governed by the following equations:

$$l_t = \alpha y_t + (1-\alpha)(l_{t-1} + b_{t-1}),$$
$$b_t = \beta^*(l_t - l_{t-1}) + (1-\beta^*)b_{t-1},$$
$$\hat{y}_{t+h|t} = l_t + hb_t,$$

where l_t is the level at time t, b_t is the corresponding period t trend and $\hat{y}_t$ is the demand forecast for t.

The algorithm we apply here completely coincides with the steps discussed in the previous section, hence the functions in our code are named in a similar fashion (only the 'SES' component is replaced with 'Holt'). First, we optimise the model parameters (smoothing constants and initialisation parameters) via training RMSE minimisation. As initial bracket we use intermediate smoothing constant values ($\alpha = 0.2$ and $\beta^* = 0.053$), the initial level is same as before (average of first 10 observations), and the trend figure is chosen to be the mean of the first 10 differences between two consecutive periods. We restrict $\alpha \in [0.02, 0.51] and \beta^* \in [0.005, 0.176]$ as sanity check (too high values would just wrap around the actual demand and lead to no smoothing). Optimisation yields the following results:

α	β^*	$\hat{l}_1$	$\hat{b}_1$
0.020	0.005	3924.241	30.159

Table 3: Optimal Holt parameters

The initial level is close to the first couple of observations, but the trend appears to be low compared to the disparities in demand between adjacent periods. Furthermore, both smoothing constants are ascribed the lowest feasible value, which indicates rather stark smoothing, portrayed in Figure 4 below. Incorporating these parameters, we quantify the Holt model predictions and generate the 6-period forecasts as tasked. As for the errors, we consider naive and analytical sigmas again, calculating both in a similar fashion to the one laid out for SES. There are two disparities that we point out:

1. The optimal smoothing α_{SE} now is $\alpha_{SE} = 0.0185$, which again is fairly low, so we corroborate similar justification as for SES.

2. In the analytical error calculation, although the same assumptions are in order, the way the constants C_j are formed departs from the previous declaration, now being

$$C_j = 1 + \sum_{i=1}^{j} i \cdot [\alpha + \tfrac{1}{2}\beta^*(i+1)]$$

Having mentioned these bookkeeping arrangements, we present the output for Holt below:

τ	$\hat{\mu_\tau}$	$\hat{\sigma_\tau^N}$	$\hat{\sigma_\tau^A}$
1	6601.806	1557.802	1557.802
2	13231.374	2203.064	2230.773
3	19888.705	2698.192	2770.808
4	26573.798	3115.603	3249.707
5	33286.654	3483.350	3695.850
6	40027.273	3815.819	4124.286

Table 4: Forecasts and Errors Holt Model

As visible, the predicted cumulative forecasts are generally larger than the SES counterparts, the naive sigma exceeds the SES naive estimator and the analytical standard error for Holt is lower for the longer horizons. However, the last note is mostly due to the different C_j derivation, which now depends on two smoothing constants, both of which low as previously discussed. We observe that unlike SES, the Holt model mostly brings around a best fit line, which we deem redundant. Thereby, we conclude that SES offers higher accuracy for Mr.Stark's current setup.

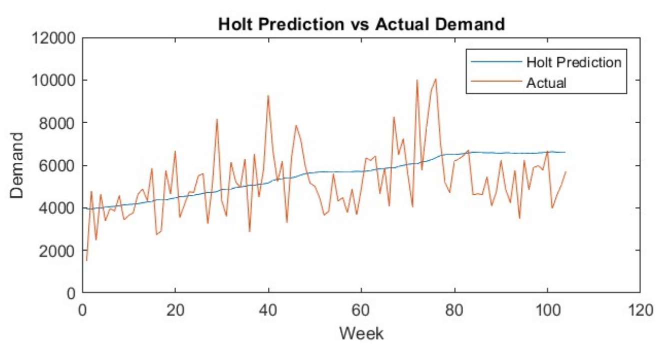

Figure 4: Holt Forecast Accuracy

2 Exercise 2

As stated in the end of the previous section, we believe that for the sake of our analysis the SES model provides better interpretability and appears to bode higher precision in capturing noisy data than the Holt model, the latter exhibiting extra smoothing throughout. Therefore, we posit that for the rest of the paper we rest our discussion on the SES model, which would also enable us to avoid the necessary alterations needed to incorporate Holt's trend factor below.

Before initiating the numerical research, we extract all relevant parameters from the description of Mr.Stark's policy. Our review period is $R = 4$ weeks, lead time is $L = 2$ weeks, initial order-up-to level is $S = 32,000$ units, the purchase price of each item is given by $v = \$120$, sale price is $P = \$132$, the annual holding cost comes at $r = 0.25$ per item per dollar, the fixed costs associated with order placement and delivery are $A = \$500$, the desired fill rate is $P_2 = 0.99$ and the customer discount in case of stockout captures the B_2 cost, given here by $B_2 = 0.05$.

The last necessary element we require to conduct the research are the expected demand readings for all review, lead time and review plus lead time periods. Because nowhere it is explicitly outlined when the review period starts (it could start in the forecasted week 1,2,3,etc.), we cannot directly extrapolate weeks 1-4 to be our review period forecast horizon and 5-6 to be the lead time. To accommodate this adjustment, we decide to take the last forecasted cumulative demand, namely $\hat{\mu}_6$, as the demand for the whole review and lead time cycle, i.e. $E[D_{R+L}] = \hat{\mu}_6$. That being said, the standard deviation choice follows an identical line of thought, hence we deduce that $\sigma_{R+L} = \hat{\sigma}_6$ is a suitable choice for the cycle standard deviation. From these figures it is easy to extrapolate the review- and lead-time-only expectations, summarised in the table below: Given that the expected

$E[D_R]$	21208.873
σ_R	3975.971
$E[D_L]$	10604.436
σ_L	2811.436
$E[D_{R+L}]$	31813.309
σ_{R+L}	4869.550

Table 5: Demand Expectations and Standard Deviation

demand for the whole cycle is a few hundred units short of the order-up-to level S and the forecast error is reasonably high given the wide prediction horizon, we expect that the Key Performance Indicators (KPIs hereon) won't be perfect. As declared previously, we postulate that the demand Mr.Stark has faced over the two years of study appears to be gamma-distributed, so we initialise the relevant shape and scale parameters $\rho_R = (\frac{\mu_R}{\sigma_R})^2$, $\rho_L = \rho_R \frac{L}{R}$, $\rho_{R+L} = \rho_R(1 + \frac{L}{R})$ and $\lambda = \mu_R/\sigma_R^2$.

We then introduce various anonymous functions in our code that calculate each of the requested KPIs, including one for the safety factor k. Because the follow-up items require the assessment of optimal order-up-to levels, the argument of these anonymous functions is defined to be be S in particular.

The service level is found via direct application of $P_1 = P[D_{R+L} < S]$, which is easily quantified using the gamma cdf that captures the demand during review plus lead time, or simply $gamcdf(S, rho\_RL, 1/lambda)$. For the fill rate we should first introduce the Expected Stockouts Per Replenishment Cycle (ESPRC) figure. Here, we emphasize that we implement the ESPRC definition **with correction for shortages at the beginning of a replenishment cycle**. Given the irregularity of the demand, we cannot ascertain positive net stock just before replenishment ($NS_{-} > 0$), hence we adhere to the corrected ESPRC formula ($ESPRC = \mathcal{GL}(S, \rho_{R+L}, \lambda) - \mathcal{GL}(S, \rho_L, \lambda)$). We then plug this expression into the service equation to obtain the fill rate $P_2 = 1 - \frac{ESPRC}{E[D_R]}$. The remaining KPIs of interest allude to the ordering, holding and shortage costs, which we define as follows:

1. $ordering\_costs = A\frac{52}{R}$, where A is the fixed cost paid for the placement of each order and $\frac{52}{R}$ depicts the annual frequency of the order placements.

2. $holding\_costs = (k \cdot \sigma_{R+L} + \frac{1}{2}\mu_R) \cdot v \cdot r$, where the expression within the parentheses captures the expected on-hand stock throughout the year, which the company pays storage quotes on of scale $v \cdot r$ on.

3. $shortage\_costs = B_2 \cdot v \cdot ESPRC \cdot \frac{52}{R}$, where we incorporate the annual (hence the $\frac{52}{R}$ term) of incurred costs when shortage is expected to arise under the given fractional charge per unit short B_2.

Pooling the three types of costs above into a single expression gives us the total costs figure, which is the last of the KPIs considered. Implementing these expressions with the available parameters (and current order-up-to level), we achieve the following scores:

P_1	0.536
ESPRC	1850.757
P_2	0.913
Ordering costs	6500
Holding costs	323733.819
Shortage costs	144359.058
Total costs	474592.878

Table 6: KPIs Initial Setup

Considering our previous observation, the P_1 rate of 0.536 appears reasonable due to the likely fluctuations in the longer-run (towards the later weeks) caused by the high standard deviation figures for them. The service level depicts the probability of stockout, and given that our expected demand for review and lead time together is almost on a par with the current order-up-to level, it is normal to expect that Mr.Stark could go short on supply come actual demand exceeds the expected one a little. The ESPRC and fill rate are also feasible because of the correction we introduce (the correction generally reduces the ESPRC, which in tern results in a slightly raised P_2 figure). Although misleading at first, considering that the setup evaluates shortages and satisfied-from-shelf demand during the current replenishment cycle (hence review period), the 32,000 units are ample to provide a thin buffer for Mr.Stark (but the performance is still noticeably poorer than the desired fill rate). The ordering costs are a byproduct of the review period and for now do not depend on S. Holding costs are relatively high due to the magnitude of the expected on-hand stock (around 10800 units), which entails higher storage costs. However, the holding-shortage costs trade-off implies that Mr.Stark will incur lesser shortage costs as evident from the table. The total costs come at approximately 474,600, so Mr.Stark should pursue cost optimisation directives.

3 Exercise 3

In this section, we reevaluate the KPIs computed previously. Here, the anonymous functions constructed in the foregoing section play an essential role, because their format allows us to minimise the total costs straight away via the $fminsearch$ functionality. The initial guess we supply is μ_{R+L}. After initiating the optimisation function, we retrieve the order-up-to level which minimises the total costs figure as desired. Plugging this S-value into the equations defined for exercise 2 yields the following output:

S	33009
P_1	0.615
P_2	0.933
Ordering costs	6500
Holding costs	353996.947
Shortage costs	110987.246
Total costs	471484.193

Table 7: KPIs for Optimal Total Costs

The optimal level of S is approximately 33,009 units. As displayed above, both the service level and the fill rate increase a little. That is justified in our opinion, because higher S presupposes lower probability of stockout during the replenishment cycle (as depicted by the improvement in P_1), whereas for the fill rate larger order-up-to level presupposes more customers being served directly 'from shelf,' so the figure there is also in line with our interpretation. Ordering costs remain unchanged, which ensues from the fixed R value (they depend on R alone). As expected, holding costs go up due to the larger order quantities-expected on-hand stock rises, hence more units need to be stored in warehouses, giving rise to additional expenditures for Mr.Stark. However, this increase is counteracted in value by the decrease in shortage costs-when Mr.Starks places orders with higher volume, his business will naturally face less shortages (ESPRC drops), so the stockout occasions shrink in number. Therefore, we believe that all projections are intact and agree with the anticipated reallocation of expenses.

4 Exercise 4

In the setup of this exercise, we preset the fill rate to be 0.99. In order to extrapolate the order-up-to level which accomplishes this P_2, we recall the equation which delineates the fill rate under every (R,S) policy:

$$P_2 = 1 - \frac{ESPRC}{\mu_R} \iff ESPRC - (1 - P_2) \cdot \mu_R = 0$$

The only enter in the equation above which depends on the choice of S is the ESPRC, so we introduce another anonymous function of S which depicts the left-hand side of the latter equation. Finding the $fzero$ of this anonymous function retrieves the desired order-up-to level that gives us the fill rate of 0.99. The relevant KPIs here become:

S	38842
P_1	0.9191
Ordering costs	6500
Holding costs	528975.203
Shortage costs	16542.921
Total costs	552018.124

Table 8: KPIs for $P_2 = 0.99$

The results portray an increase in the order-up-to level, which we expected given the fill rates over the previous items. In order to serve 99% of the customer base without reverting to back orders requires higher overall stock availability in-store, which in its turn imposes greater order quantities. The service level is the highest one observed this far, coming at 91.91%. The grounds for this increase rest in the increase in S-it is less likely to experience stockouts when the order size is of such volume. The ordering costs remain steady yet again due to identical considerations to the preceding discussion (independent of S). As implied by the new order-up-to level, the holding costs have undergone a steep increase-when Mr.Stark orders larger quantities, he handles more unsold items which require storage, compounding costs for the vendor. The shortage costs are substantially lower than the ones observed from the outset, given that the new order-up-to level presupposes considerably fewer stockout occasion (as captured by the improved P_1. Overall, the total costs for Mr.Stark peak among the considered schedules so far as the inflated holding costs significantly outweigh the plummet in shortage costs, thus inducing a net spike in total costs.

5 Exercise 5

In this section, we administer the goodwill costs Mr.Stark is bound to face whenever stockout occurs. Given the essence of the goodwill cost as introduced in the exercise, we believe that it alludes to the B_1 cost notion. To estimate these expenditures, we need to refine the shortage costs formula to now incorporate B_1 instead of B_2. We arrive at

$$shortage\_costs = \tfrac{52}{R} \cdot B_1 \cdot P[stockoutperRC]$$

Since the probability of stockout taking place during the replenishment cycle can be expressed in terms of the net stock just before the arrival of the ordered replenishment, we can perform the following manipulations:

$$\begin{aligned} P[\text{stockout per RC}] &= P[NS_- < 0] \\ &= P[S - D_{R+L} < 0] \\ &= 1 - P[D_{R+L} < S] \\ &= 1 - \Gamma(S, \rho_{R+L}, 1/\lambda_{R+L}) \end{aligned}$$

Above, Γ signifies the CDF of a gamma distribution with shape parameter ρ_{R+L} and scale λ_{R+L}, evaluated at point S.

We now have a new expression for the total costs function, namely

$$TC(S) = A \cdot \tfrac{52}{R} + (S - \mu_{R+L} + \tfrac{1}{2}\mu_R) \cdot v \cdot r + \tfrac{52}{R} \cdot B_1 \cdot [1 - \Gamma(S, \rho_{R+L}, 1/\lambda_{R+L})]$$

To obtain a closed-form solution for the B_1 cost, we need to take the derivative of the total costs function and set it equal to 0.

$$\tfrac{dTC}{dS} = v \cdot r - \tfrac{52}{R} \cdot B_1 \cdot \gamma(S, \rho_{R+L}, 1/\lambda_{R+L}) = 0,$$

which eventually yields

$$B_1 = \tfrac{v \cdot r \cdot R}{52 \cdot \gamma(S, \rho_{R+L}, 1/\lambda_{R+L})},$$

where γ refers to the pdf of the gamma distribution with shape parameter ρ_{R+L} and scale λ_{R+L}, evaluated at point S. However, we note that the foregoing prescription encapsulates the total goodwill costs, not the ones per item short. In order to retrieve the individual cost, we believe that the most intuitive approach is to divide the estimated total goodwill expenses by the ESPRC, for the ESPRC depicts precisely the expected number of shortage occurrences.

We now need to establish a linkage between B_1 and P_2 costs, which we do in the following fashion:

1. We first declare a vector with entries all integer fill rates between 1% and 99% inclusive. We've decided to exclude the extreme points (0% and 100%), because they are unrealistic in practice and produce asymptotic results, sending the cost functions astray (e.g. holding costs go exuberant when we consider a fill rate of 100%).

2. For each entry in the vector of fill rates, we compute the corresponding order-up-to level via solving the service equation reflecting the fill rate of interest.

3. We plug the resulting S into the goodwill cost formula as defined previously.
4. Having generated the total B_1 costs for the given P_2 level, we divide by the ESPRC to get the per-item penalty.

Pooling all these results together in a graph, we visualise the relationship between the fill rate of 99% and the goodwill costs below:

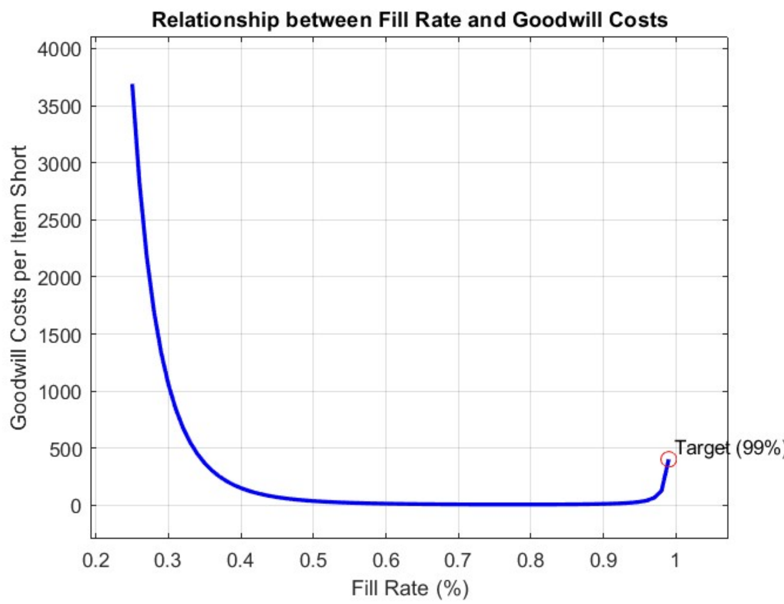

Figure 5: P_2 vs B_1 Costs Relationship

Note that we've deliberately left out the occurrences for $P_2 < 25\%$: the associated expenditures become rather exorbitant for Mr.Stark due to the summit in shortage costs, which exerts great pressure on the total costs in turn. What's more, it is counter intuitive for Mr.Stark to strive for such low fill rates in the first place for he aims at stable customer satisfaction, so omitting this information shouldn't affect his decision at all.

The figure suggests that the goodwill costs gradually shrink up until $P_2 = 0.90$ (approximately), and then rise again, undergoing a steep incline beyond $P_2 = 0.96$. The decline follows from a straightforward comparison of the two costs at hand-the more purchases are directly granted ('off-shelf sales), the higher the less customers lost and the higher the client satisfaction. As for the rise past the 90% mark, we justify it in the following way: beyond this fill rate threshold, it becomes very costly to fail to serve your customer base, because the reputation of immediately serving customer demand off the shelf poses the threat of losing loyal customers in case of shortage occurrences. If these people are used to being served on the spot, failure to attend to their demand could generate huge losses for Mr.Stark, especially if on average they purchase multiple items. If Mr.Starks wants to maintain the 99% P_2-level, the relative goodwill cost we obtain is \$403.523, which we deem expensive, but reason in the previously described manner.

As reference point, the goodwill cost associated with the initial order-up-to level and it's previously discussed KPIs is only \$15.35, but the shortage costs related to the customer discount of 5% pile up (as calculated in exercise 2) and generate additional expenditures for Mr.Stark.

6 Exercise 6

In this exercise, we aim to determine the optimal review period for Mr.Stark, given the initial order-up-to level of 32,000 units. We tackle the task in two consecutive steps: first, we disregard the previously discussed goodwill costs, and follow through to account for them as well in the sequel.

As a preparatory step to our study, we ought to introduce a number of rectifications to the methodology laid out hitherto. All anonymous functions originally defined as conditional on a particular S-level are now rearranged so that the variable of interest is also the review period R. Furthermore, due to the variability in the review window, an adjustment of the demand during review and during review plus lead time is in order. In this respect, we take the SES predictions as groundwork and reason as follows: due to the constant future period forecast assumption of the model (the demand during the $t+h$ time frame is the same as the one constructed for $t+1$), we can represent the review-period demand for an arbitrary R' as $\mu_{R'} = R' \cdot \hat{\mu}_1$, where $\hat{\mu}_1$ is the one-week-ahead forecast as generated using SES in exercise 1. Having established this reading, the demand during review plus lead time can be quantified as $\mu_{R'+L} = (R' + L) \cdot \hat{\mu}_1$. Note that as the lead time remains unchanged, the forecasted demand for it also undergoes no alterations.

To generate the relevant sigma estimates, we first reintroduce the analytical sigma definition from exercise 1, namely

$$\hat{\sigma}_{R'+L} = \hat{\sigma}_1 \cdot \sqrt{\sum_{j=0}^{R'+L-1} C_j^2}, \text{ where } C_j = 1 + \sum_{i=1}^{j} \alpha$$

Note that contrary to our foregoing formulation, the sum under the square root has upper bound $R' + L - 1$. This adaptation has been introduce to account for every possible choice of review period length. Longer review predisposes higher expected forecast error. Thereby, we introduce interconnected functions which first calculate the cumulative sigma vector for review and lead time demand, end then withdraw the last entry of this vector as our estimate for $\hat{\sigma}_{R'+L}$. The grounds for this methodology follow the same vein as exercise 1 (we have no information as to when the starting point of review or lead time is, so we need to address every possibility). Extrapolating the standard deviation estimate in this fashion, we incorporate the review- and lead-time-only analytical sigmas via simple manipulation techniques:

$$\hat{\sigma}_{R'} = \frac{\hat{\sigma}_{R'+L}}{\sqrt{1+\frac{L}{R'}}}$$
$$\hat{\sigma}_L = \hat{\sigma}_{R'} \cdot \sqrt{\frac{R'}{L}}$$

Having introduced all these refinements, it is rather straightforward to compute the ρ's, λ and all relevant KPI figures, this time declaring them as dependent on both R and S rather than S alone-for this sake, we construct the updated versions of the preceding anonymous functions (mentioned in the discussion of exercise 2) to now feature the R-dependent μ, σ, ρ and λ parameters.

The last remark we point out in the preliminary discussion is the way we specify our holding costs function in this item. For given review period R', we construct the function as

$$holding\_costs = \max(0, S - \mu_{R'+L} + \frac{1}{2}\mu_{R'}) \cdot v \cdot r$$

The reason we make this adjustment is that too large optimal review period estimates could potentially wind the holding costs negative, which is infeasible for Mr.Stark. To this end, whenever the $\mu_{R'+L}$ term grows to the extent where it counteracts the positive effect of $S + \frac{1}{2}\mu_{R'}$ within the $max()$ function, the holding costs ascribed to Mr.Stark's name are set to 0.

6.1 Methodology Walk-Through

As means to vindicate our choice of optimal review period span, we focus on total cost optimisation. Taking Mr.Stark's current order-up-to level as fixed, we monitor the motion of the total costs figure for any R' between 1 and 52 weeks (1 calendar year). We acknowledge that more advanced approaches exist, but for the purpose of this paper we deem this approach efficient enough.

Before studying the numerical output, we anticipate the ordering and holding costs to decline. The former goes down due to the less frequent order placements and hence fewer related costs, whereas the latter hypothesis is a byproduct of the previous introduction of the holding costs function-longer review periods predetermine a drastic spike in shortage occurrences (the 32,000 units will be hugely insufficient for review longer than 4 weeks given the current SES demand forecasts), so there will be a negligible number of items to be stored (if any). Consequently, the expected shortage costs increase in R exponentially, given the soaring probability of placing back orders. This issue will be prominent in the goodwill costs discussion in subsection 6.3. Finally, after obtaining the optimal review period for each scenario, we re estimate the optimal S so that Mr.Stark can sustain the desired fill rate of 99%.

6.2 No Goodwill Costs Analysis

Here, we disregard goodwill costs in the shortage cost function, meaning that the only expenses during shortage are associated with the $B1$-stemming losses. Estimating the total costs following the aforedescribed procedure, the optimal review period we fetch is 3 weeks. If Mr.Stark decides to foster his customer satisfaction levels and bring down his expenditures as much as possible, he should reschedule his ordering point to be a week earlier than his current policy. Then, if opting for $R' = 3$ weeks, whilst still seeking to accomplish a P_2 level of 0.99, Mr.Stark needs to readjust his order-up-to level accordingly. The estimated KPIs would be: The returned KPI's are all reasonable from our perspective. The retrieved order-up-to level is sensible for Mr.Stark given his aim to confine his sales to 99% 'off the shelf.' Considering the optimal S for this fill rate we obtained in the original setup (when review was 4 weeks), coming at just shy of 39,000 units, the 32859 observed in the 3-week-review setup here appear ordinary. The service level is now a whit below 93%, which implies a stockout probability of just over 7%. Given the forecast errors, this figure

S	32858.352
P_1	0.9284
Ordering costs	8666.667
Holding costs	429017.656
Shortage costs	16542.921
Goodwill costs	0
Total costs	454227.243

Table 9: KPIs for $R' = 3$ Weeks, No Goodwill Costs and $P_2 = 0.99$

remains in line with our interpretation. Due to this enhanced P_1 level, shortage costs have declined-logical given the reduced probability of running out of stock. Nevertheless, the holding costs have increased, but that also aligns with the novel policy's defining characteristics-restricting the fill rate to 99% disposes Mr.Stark to keeping larger on-hand stock volumes, which if not sold need be stored in a warehouse against an according payment. The ordering costs have gone up a little due to the more regular placement of replenishment orders. Altogether, the total costs have lessened compared to the previous optimums for the initial review period.

6.3 Goodwill Costs Included Analysis

In this final section of exercise 6, we accommodate the goodwill costs in our model. We follow an identical optimisation scheme as in the previous subsection (where no goodwill costs were incurred), but in the novel setup we include the goodwill expenditures as a separate KPI embedded into the total costs.

$$goodwill\_costs = B_1 \cdot ESPRC \cdot [1 - \Gamma(S, \rho_{R+L}, 1/\lambda)] \cdot \tfrac{52}{R},$$

where the Γ term reflects the gamma CDF with the respective parameters included in the parentheses. As goodwill cost figure we take the \$403.523 penalty per item shored defined earlier for the $99\% fillrate$, and multiply it by the ESPRC to get the total expected loss of goodwill. Looping through the feasible review period lengths (between 1 and 52 weeks), we obtain the latest optimum to be for $R' = 2$ weeks. The relevant KPIs if Mr.Stark reorganises his business to adhere to this specific policy and still achieve a fill rate of 99% are as follows: The order-up-to level changes to 26972 units now, which provides a signif-

S	26972.790
P_1	0.9419
Ordering costs	13000
Holding costs	331954.079
Shortage costs	16542.921
Goodwill costs	64622.839
Total costs	426119.838

Table 10: KPIs for $R' = 2$ Weeks, Including Goodwill Costs and $P_2 = 0.99$

icant buffer for Mr.Stark, encapsulated by the high service level value of over 94%. This means that stockouts are considerably less probable, implying high holding costs. As depicted in the table, the costs for storing unsold items are exorbitant considering the short 2-week long review period, especially if we refer to the previous reorder schemes studied above. The annual goodwill costs are $\$65,000$, which suggests that Mr.Star doesn't face excessive shortages on annual basis. This note is further consolidated by the remarkably low shortage costs per annul, adhering to very high 'off-the-shelf' demand satisfaction. The ordering costs have naturally doubled referring to the outset position, given that the review period has been narrowed down in half. Overall, this setup has improved Mr.Stark's total costs in comparison with the previous subsection where we overlooked lost goodwill. This illustrates that knowledge of all relevant costs and optimising the order schedule with respect to all of them is crucial for Mr.Stark's business, especially if he aims to foster the highest revenue and customer satisfaction levels possible. Considering all possible channels for expenses, it is easier for Mr.Stark to obtain the best balance between each family of costs and maximise his gains.

7 Exercise 7

To answer this question, we first discuss some of the potential shortcomings of the (R,S) inventory policy in Mr.Stark's case. As previously posited, Figure 1 (on page 2 of the current report) clearly indicates high variance and irregularity of the product demand. To this end, implementing inventory policies with continuous inventory review appears reasonable to ensure higher customer satisfaction. This means that an (s,Q) or (s,S) inventory policy could be a viable solution given the improved flexibility of replenishment order placements. Taking into account the aforementioned exuberance of Mr.Stark's incoming demand, regularly monitoring the inventory position and recharging it whenever the available stock falls below a predetermined level should reduce the stockout occasions, enhancing the company's fill rates and service levels.

First, we investigate the (s,Q) inventory policy. We deem this policy infeasible because of the format of demand reporting. Local sales offices throughout Europe compile their client orders and send Mr. Stark their aggregate demand at the end of each week. Since the quantities are reported only once a week, the system falls vulnerable to conceding undershoots during weeks with spikes in customer demand. When not addressed properly and if higher demand levels persist, the undershoot could become a costly issue for Mr.Stark as more back orders would be in order, rotting customer satisfaction. Thereby, because the (s,Q) inventory policy ignores undershoots for the most part, we postulate that it is yet suboptimal for the company and doesn't portray a substantial improvement to the current state of affairs.

Focusing on the targets fostered by Mr.Stark, we believe that the (s,S) inventory policy could curb many of the setbacks of the current management scheme. Generally speaking, (s,S) policies are theoretically superior with high demand variance to (s,Q) and (R,S), for it grants higher customer satisfaction while maintaining lower inventory. In union, these advantages would increase the profitability of the company as undershoots are easier to screen, allowing the service levels and fill rates to be preserved at the desired levels. However, the implementation of this policy by Mr.Stark isn't strictly straightforward:

1. First, because demands are only provided on a weekly basis, therefore they cannot be continuously monitored. This premises that the company could only adopt an (R,s,S) inventory policy with R=1 week as the lowest possible review period.

2. Second, applying such policies is generally more complex. Configuring and implementing the system in practice is rather cumbersome from the perspective of the company. However, since some clients moved from Kava to another supplier (as provided in the exercise), we extrapolate that 'Easy Coffee' falls into the A item class category, for which items the complications around (s,S) policies are justified in the pursual of customer satisfaction.

8 Exercise 8

In order to provide an overview of the KPIs of all inventory control policies from the previous parts, we first need to settle a few assumptions. The KPIs we will discuss are the $ESPRC$, ordering costs, holding costs, shortage costs, total costs, P_1- and P_2-levels. The models of interest are (s,Q), (R,S) and (s,S). For the computations of the KPIs for these models, the B_2 shortage costs are used as previously specified.

In the (s,Q) model, the annual demand level D is required. Since in Kava's case it is non-stationary, in order to conduct all relevant calculations we summed up the weekly figures for all 104 weeks of record and then divided them by 2 to get an annual value. We chose this approach as we have no specification of when the calendar year starts in the database, so we aim to administer all possibilities. It is a stark assumption to make (demand stationarity), but is nevertheless needed in the (s,Q) setup. To calculate the order quantity Q, we've decided to estimate it using EOQ. To retrieve s and the safety factor k later on, total costs were minimised in a similar fashion to, say, exercise 6.

For the (R,S) model, very little adjustments had to be made as the dataset provided to us was already suited for this exact model. The only manipulation we needed was the annualisation of all relevant KPIs, which follows the same vein as the foregoing discussion in the report.

Last, in the (s,S) model we require a specific P_2 value to conduct our computations. We choose the same figure as in (s,Q) model due to the similarities between the two models (both rely on the threshold s to place a restock order). Another motivation to choose this value is that it provided a much better costs allocation than the original 99% fill rate (after all, Mr.Stark pursues the highest revenues possible). Using the service equation and this fill rate, we extracted the corresponding S-value and computed the relevant KPIs using it. The other assumptions coincide with the remarks made on the (s,Q) policy above-demand stationarity, the way annual demand is computed, etc. The major drawback for us when it comes to using this policy is that the data provided to us doesn't enable us to estimate the undershoots whatsoever. They pose a key element of the (s,S) computations, so without the relevant records of those the (s,S) estimates are imprecise.

	(s,Q)	(R,S)	(s,S)
Ordering costs	45311	6500	45311
Holding costs	180905	323733	271527
Shortage costs	50763	144359	7239
Total costs	276979	474592	324077
P1	0.934	0.535	0.9895
P2	0.969	0.912	0.969
ESPRC	93.36	1850.75	983.25

Table 11: KPIs for various policies

9 Exercise 9

First, in order to achieve better forecast accuracy and secure the fostered customer satisfaction levels, Mr.Stark could increase the frequency of demand data collection. Introducing the practice of recording the values with a higher frequency, say twice or thrice a week, would slim down the interrecord variance magnitudes and allow for more certainty in the predicted demand figures. From a computational perspective, this practice would enlarge the data set and, therefore, ensure the availability of a larger training set for the optimisation of the model parameters. This should not instigate large costs for Mr.Stark due to the nature of the adjustment-the only potential expenditure could be related to data storage (say on a server or cloud) in the long term, come he decides to handle all past data. Inspecting demand on a more often basis would also allow the company to be more flexible and absorb fluctuations in the market faster.

Another refinement Mr.Stark could consider is to try and negotiate a shorter lead time for the Pokald deliveries. Observing the irregularity in weekly demand, a two-week window is rather tricky for on-hand stock maintenance and could prove expensive in holding costs terms. Also, such long waiting times increase the probability of running short of stock, potentially jeopardising consumer satisfaction. If Pokald could offer a shortened lead time (even at a slightly higher ordering cost), or another company could provide it at similar rates, Mr.Stark's business could only benefit from it.