2D Square Force Sensor Model

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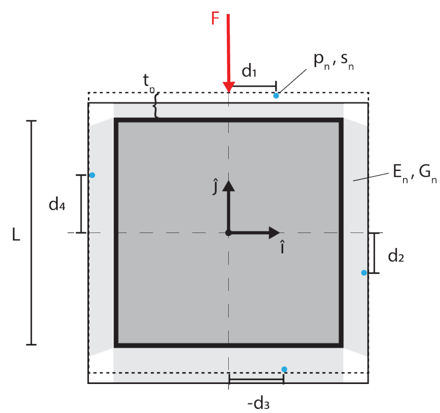
1 General 2D with proportional coupling



Assumptions:

- Fully rigid cube and shell.
- Cube is stationary relative to reference frame.
- Arbitrary 2D Force/Torque load
- Sensors output a single signal, proportional to the displacement of the sensors, with sensitivities p and s in normal and shear directions respectively. The system is customized to exclusive normal or shear shear sensors by setting one of these to zero.
- Assembly is sufficiently rigid that the small angle approximation is appropriate.

For the case of prebuilt strain elements, the compliant layer parameter and sensor are tied.

Model

The model is centered around the relation: $\vec{S} = [C]\vec{F}$, where S is the array of individual sensor measurements, F is the external load vector and $[C]$ is a transformation matrix that maps the two.

$$\vec{S} = [C]\vec{F}$$

$$\vec{F} = \begin{bmatrix} F_x \\ F_y \\ M_z \end{bmatrix}$$

$$\vec{S} = \begin{bmatrix} S_1 \\ S_2 \\ \dots \\ S_n \end{bmatrix}$$

$[C]$ itself is a product of two lower-level transformations matrices: $[M]$ & $[K]$

$$\vec{S} = [C]\vec{F} = [M][K]\vec{F}$$

$$[C] = [M][K]$$

Where $[K]$ is the inverse of a matrix of linear spring-like transformations, in that it maps forces to displacements.

$$\vec{x} = [K]\vec{F}$$

$$\vec{x} = \begin{bmatrix} x \\ y \\ \theta \end{bmatrix}$$

$[M]$ maps displacements to sensor readings via some known transduction principle, and additionally converts θ to x and y displacements first in the rightmost column.

$$\vec{S} = [M]\vec{x}$$

Parameters: For $i \in \{1, 2, 3, 4\}$:

- s_i Is the relative sensitivity to shear loading for sensor i
- p_i Is the relative sensitivity to axial loading for sensor i
- d_i Is the displacement from the central axis for sensor i
- G_i Is the Shear modulus of compliant i
- E_i Is the Young's modulus of compliant i
- t_i Is the height of compliant i

- w_i Is the width of compliant i

We find: $[M] = \begin{bmatrix} s_1 & p_1 & p_1 d_1 - s_1(\frac{L}{2} + h_1) \\ p_2 & -s_2 & -p_2 d_2 - s_2(\frac{L}{2} + h_2) \\ -s_3 & -p_3 & -p_3 d_3 - s_3(\frac{L}{2} + h_3) \\ -p_4 & s_4 & p_4 d_4 - s_4(\frac{L}{2} + h_4) \end{bmatrix}$

Next we determine the entries of the diagonal matrix $[K]$ using strain mechanics:

$$K = \begin{bmatrix} \frac{1}{K_x} & 0 & 0 \\ 0 & \frac{1}{K_y} & 0 \\ 0 & 0 & \frac{1}{K_z} \end{bmatrix}$$

From the linear strain relations:

$$\Delta l = \frac{F/A}{E/l}$$

$$\Delta x = \frac{Fh}{AG}$$

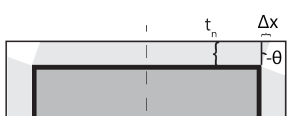
We can find stiffness in x and y as:

$$K_x = \frac{L^2 G_1}{t_1} + \frac{L^2 E_2}{t_2} + \frac{L^2 G_3}{t_3} + \frac{L^2 E_4}{t_4}$$

$$K_y = \frac{L^2 E_1}{t_1} + \frac{L^2 G_2}{t_2} + \frac{L^2 E_3}{t_3} + \frac{L^2 G_4}{t_4}$$

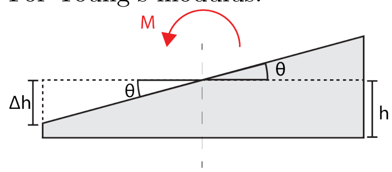
Where variables are chosen as the parameters strain model: $A = w_n^2$, $l = h = t_n$
position model: $A = L^2, l = h = h_n$

For strain relations in θ to calculate K_z, both the Shear and Young's moduli resist rotation. For the Shear modulus:



Substituting F with $\frac{M}{L/2+t_n}$ in the shear equation yields
$\frac{\Delta x AG}{t_n} = \frac{M}{L/2+t_n}$
and
$tan(\theta) = \frac{\Delta x}{t_n}$
$\theta \simeq \frac{\Delta x}{t_n}$
$\theta = \frac{M}{(L/2+t_n)w_1^2 G_3}$

For Young's modulus:



$$M = \frac{2}{h}\int_{-d}^{d} x\Delta h\frac{x}{d}Ew\ dx = \frac{4}{h}\frac{\Delta h}{d}Ew\frac{d^3}{3}$$

$$\rightarrow \Delta h = \frac{3h}{4Ewd^2}M$$

$$tan\theta = \frac{\Delta h}{d}$$

$$\theta \simeq \frac{\Delta h}{d}$$

$$\theta = \frac{3h}{4Ewd^3}M$$

Combining the two yields:

$$K_z = \frac{4E_1w_1d_1^3}{3h_1} + \frac{4E_2w_2d_2^3}{3h_2} + \frac{4E_3w_3d_3^3}{3h_3} + \frac{4E_4w_4d_4^3}{3h_4}$$
$$+(L/2+t_1)w_1^2G_1 + (L/2+t_2)w_2^2G_2 + (L/2+t_3)w_3^2G_3 + (L/2+t_4)w_4^2G_4$$

With these defined $[M]$ and $[K]$ we can now calculate a $[C]$ for different sensor and compliance configurations and assess its inaccuracy.

2 Assessment of Inaccuracy:

The objective function in the optimization of this model was based on Bicchi's generalized error bound:

$$\epsilon_p \leq \{[2+N_c]\frac{|\delta\mathbf{C}|}{|\mathbf{C}|} + N_c\frac{|\delta\mathbf{v}|}{|\mathbf{v}|}\}N_c^2$$

Where $\mathbf{v}$ is the vector of sensing element readings, $\mathbf{C}$ is the compliance matrix, N_C is its condition number, and ϵ_p is the relative error in the assembly's force readings. This model does not calculate sensor errors, and furthermore since the compliance matrix error term is separable from the sensing element error term, to assess the quality of a sensor configuration we use the corresponding section from Bicchi's objective:

$$\epsilon_{config} \leq [2+N_c]\frac{|\delta\mathbf{C}|}{|\mathbf{C}|}$$

By design this error bound is a function of system parameters, and hence it may be used to analyze the change in performance with respect to certain parameters. This permits both a systematic optimization procedure provided a sufficiently constrained system, in addition to a more general analysis of the variation of system performance with respect to certain parameters.

The latter is explored in more detail in the attached graphs.