A Feedback Theory of Two-Degree-of-Freedom Optimal Wiener–Hopf Design

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Abstract—**The design of linear two-degree-of-freedom stabilizing controllers is treated in a quadratic-cost setting. The class of all such controllers which give finite cost is established and the tradeoff possible between optimum performance, tracking-cost sensitivity, and stability margins is discussed.**

I. Introduction

In general, every control problem possesses two natural degrees of freedom corresponding to the availability of an exogenous set-point input $\boldsymbol{u}$ and a sensor output $\boldsymbol{w}$ (Fig. 1). However, in the conventional single-loop configuration of Fig. 2, the controller generates the plant input $\boldsymbol{r}$ either by processing only the difference $\boldsymbol{u}-\boldsymbol{w}(F=1)$, or by processing $\boldsymbol{u}$ and $\boldsymbol{w}$ in a special interdependent manner $(F\neq 1)$.[1] Unfortunately, this restriction has made it difficult, if not impossible, to incorporate essential sensitivity and stability-margin requirements into a quadratic-cost setting. It is therefore necessary to evolve an optimization procedure that allows the controller to process $\boldsymbol{u}$ and $\boldsymbol{w}$ independently. This paper contains several new results which provide a substantial theoretical framework for the execution of this basic idea.

1) The class of all two-degree-of-freedom stabilizing controllers for a prescribed plant and feedback sensor pair is identified in terms of two *free* matrix parameters (Theorem 1).

2) In (42) an explicit expression is obtained for the percent change in the closed-loop characteristic polynomial, due to a change in the plant transfer matrix. This formula reveals that the percent change involves only one of the free parameters in 1). Thus, this parameter can be used to adjust various stability margins.

3) Three new measures, one absolute and two relative, are introduced to describe the effect of plant sensitivity on closed-loop performance. The latter two are used to augment the standard cost-functional and the optimization problem is viewed as one of minimizing cost with a prescribed bound on tracking-cost sensitivity.

4) The augmented cost-functional E is shown to decompose into a sum $E = E_u + E_w$, where E_u is determined solely by R_u, the closed-loop transfer-matrix from $\boldsymbol{u}$ to $\boldsymbol{r}$, and E_w is determined solely by R_w, the closed-loop transfer-matrix from $-\boldsymbol{m}$ to $\boldsymbol{r}$ (Fig. 1). A complete characterization of all admissible R_u's which yield finite E_u and all admissible R_w's which yield finite E_w is obtained in Theorems 2 and 3, respectively. Once again, the representations are in terms of two free rational matrix parameters $Z_u(s)$ and $Z_w(s)$. Moreover, the unique R_u which minimizes E_u is paired with $Z_u = 0$ while the unique R_w which minimizes E_w is paired with $Z_w = 0$.

5) An illustrative example worked out in detail in Section III serves to clarify both the scope and potential of the method, and the discussion in Section IV offers a comparison with some of the existing literature.

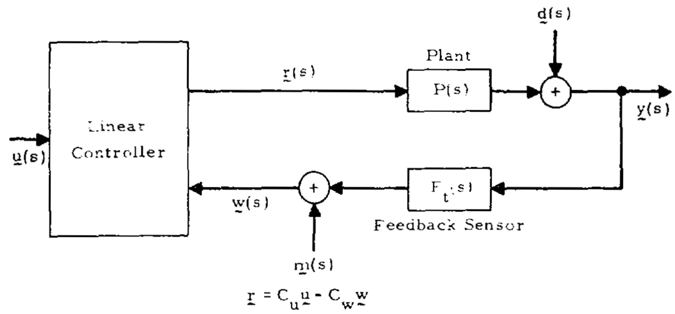

Fig. 1. The general servoregulator configuration.

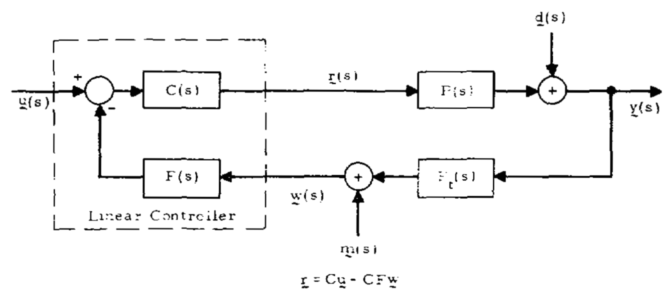

Fig. 2. The classical single-loop servoregulator configuration.

II. Problem Formulation and Results

Apart from all other considerations, the design of a linear controller has two clearly defined objectives. First,[2] transient acquisition of a prescribed set-point input $\boldsymbol{u}(s)$ by the plant output $\boldsymbol{y}(s)$ must be accomplished satisfactorily in the presence of measurement noise $\boldsymbol{m}(s)$ and a saturation constraint on the plant input $\boldsymbol{r}(s)$. Second, regulation in the face of load disturbance $\boldsymbol{d}(s)$ entering through the plant must be acceptable. In the most general situation both requirements are present simultaneously and the problem is one of servoregulation. Available to the designer are two degrees of freedom embodied in the exogeneous input $\boldsymbol{u}(s)$ and the feedback sensor output $\boldsymbol{w}(s)$ and the overall configuration may therefore be depicted schematically as shown in Fig. 1.

To exploit both degrees of freedom fully, the controller must be permitted to generate an output

$$\boldsymbol{r}(s) = C_u(s)\boldsymbol{u}(s) - C_w(s)\boldsymbol{w}(s) \tag{1}$$

in which the transfer matrices $C_u(s)$ and $C_w(s)$ are arbitrary

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[1] This interdependence is most subtle in the multivariable case.

[2] We work exclusively in the complex $s=\sigma+j\omega$ – domain and all matrices $A(s)$ are assumed to be real and rational. We also omit arguments wherever convenient.

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subject solely to the demand of closed-system asymptotic stability. But as has already been pointed out, the classical single-loop structure of Fig. 2 constrains the processing of $\boldsymbol{u}(s)$ and $\boldsymbol{w}(s)$ and the two inherent degrees of freedom are not completely utilized. Thus, it is not surprising to find that in a multivariable single-loop environment the two design objectives often conflict and even appear mutually contradictory. Moreover, to make matters worse, usually little or no design flexibility remains for the inclusion of any sensitivity or stability-margin considerations.[3] Our goal in this paper is to develop a rigorous theory of two-degree-of-freedom optimal controller design that subsumes these requirements analytically and also respects the notion of feedback from the outset.[4] Theorems 1, 2, and 3 constitute the core of this theory and their proofs are given in full in the Appendix.

The transfer matrices $P(s)$ of the plant and $F_t(s)$ of the sensor are assumed to be prescribed in advance. Clearly, according to (1)

$$T_c(s)=[C_u(s)|-C_w(s)] \tag{2}$$

is the controller transfer matrix from the pair of inputs $\boldsymbol{u}(s)$, $\boldsymbol{w}(s)$ to the output $\boldsymbol{r}(s)$. A straightforward analysis of Fig. 1 yields

$$\boldsymbol{r}=R_u\boldsymbol{u}-R_w(F_t\boldsymbol{d}+\boldsymbol{m}) \tag{3}$$

where

$$R_u=(1+C_wF_tP)^{-1}C_u \tag{4}$$

and

$$R_w=(1+C_wF_tP)^{-1}C_w. \tag{5}$$

It is not difficult to see from (3) that $R_u(s)$, $R_w(s)$ and

$$T(s)=P(s)R_u(s) \tag{6}$$

are closed-system transfer matrices from $\boldsymbol{u}$ to $\boldsymbol{r}$, $-\boldsymbol{m}$ to $\boldsymbol{r}$, and $\boldsymbol{u}$ to $\boldsymbol{y}$, respectively. Of course, in any stable design, all three must be analytic in Re $s \geq 0$.[5]

In principle, a meaningful cost-functional should incorporate those design factors that are judged to be of engineering significance. Although there exists considerable difference of opinion as to how they are to be assessed and rated in relative importance, it is nevertheless generally conceded that transient error, large rms plant input, and excessive closed-system sensitivity to plant uncertainty are all factors that deserve to be penalized.

Let $G_e(s)$ denote the spectral density of the error

$$\boldsymbol{e}(s)=\boldsymbol{u}(s)-\boldsymbol{y}(s) \tag{7}$$

$G_r(s)$ that of the input $\boldsymbol{r}(s)$ and let $Q(s)$ denote some suitably chosen full-rank para-Hermitian nonnegative-definite weight-matrix.[6] Then,[7] useful cost-functionals for tracking error and plant saturation are given by

$$E_t=\frac{1}{2\pi j}\int_{-j\infty}^{j\infty}\operatorname{tr}G_e(s)ds \tag{8}$$

and

$$E_s=\frac{1}{2\pi j}\int_{-j\infty}^{j\infty}\operatorname{tr}(QG_r)ds, \tag{9}$$

respectively [3]. As usual, to permit tradeoff between E_t and E_s, it proves more convenient to work with the single cost

$$E_0=E_t+kE_s \tag{10}$$

k an adjustable nonnegative constant. In terms of the individual spectral densities $G_u(s)$, $G_d(s)$, and $G_m(s)$ of $\boldsymbol{u}(s)$, $\boldsymbol{d}(s)$, and $\boldsymbol{m}(s)$, we obtain the expanded form

$$\begin{aligned}E_0=&\frac{1}{2\pi j}\int_{-j\infty}^{j\infty}\operatorname{tr}[(1-PR_u)G_u(1-PR_u)_*+kQR_uG_uR_{u*}]ds\\&+\frac{1}{2\pi j}\int_{-j\infty}^{j\infty}\operatorname{tr}[(1-PR_wF_t)G_d(1-PR_wF_t)_*\\&+kQR_wF_tG_dF_{t*}R_{w*}]ds\\&+\frac{1}{2\pi j}\int_{-j\infty}^{j\infty}\operatorname{tr}[R_wG_mR_{w*}(P_*P+kQ)]ds.\end{aligned} \tag{11}$$

It follows readily from (3)–(5) that any uncertainty $\delta P(s)$ in $P(s)$ produces respective first-order perturbations

$$\delta\boldsymbol{e}=-S(\delta P)\boldsymbol{r} \tag{12}$$

and

$$\delta\boldsymbol{r}=-R_wF_t(\delta P)\boldsymbol{r} \tag{13}$$

in $\boldsymbol{e}$ and $\boldsymbol{r}$, where

$$S(s)=1-P(s)R_w(s)F_t(s) \tag{14}$$

is the closed-system sensitivity matrix. Consequently, since (12) and (13) express $\delta\boldsymbol{e}$ and $\delta\boldsymbol{r}$ with respect to the *same* reference $\boldsymbol{r}$, the matrix multipliers

$$S(\delta P) \tag{15}$$

and

$$R_wF_t(\delta P) \tag{16}$$

serve as feedback indicators of the relative importance of plant-uncertainty on tracking-error and plant-saturation, respectively.

It is therefore consistent with our weighting of $\boldsymbol{e}$ and $\boldsymbol{r}$ in E_0 to introduce the plant-uncertainty spectral density[8]

$$G_s(s)=\langle(\delta P)(\delta P)_*\rangle, \tag{17}$$

another adjustable nonnegative constant μ, and

$$\begin{aligned}E=E_0&+\frac{\mu}{2\pi j}\int_{-j\infty}^{j\infty}\operatorname{tr}(SG_sS_*)ds\\&+\frac{\mu}{2\pi j}\int_{-j\infty}^{j\infty}\operatorname{tr}(kQR_wF_tG_sF_{t*}R_{w*})ds\end{aligned} \tag{18}$$

as our final candidate for the quadratic measure of total performance.[9] Explicitly,

$$\begin{aligned}E=&\frac{1}{2\pi j}\int_{-j\infty}^{j\infty}\operatorname{tr}[(1-PR_u)G_u(1-PR_u)_*+kQR_uG_uR_{u*}]ds\\&+\frac{1}{2\pi j}\int_{-j\infty}^{j\infty}\operatorname{tr}[(1-PR_wF_t)(G_d+\mu G_s)(1-PR_wF_t)_*\\&+kQR_wF_t(G_d+\mu G_s)F_{t*}R_{w*}]ds\\&+\frac{1}{2\pi j}\int_{-j\infty}^{j\infty}\operatorname{tr}[R_wG_mR_{w*}(P_*P+kQ)]ds.\end{aligned} \tag{19}$$

[3] The two early excellent texts by I. Horowitz [1] and S. Chang [2] are notable for their appreciation of this point.

[4] For example, if the plant and feedback-sensor are asymptotically stable and load disturbance is absent, the detrimental effects of sensor noise can be completely circumvented by using open-loop compensation. Thus, in this case there might be other reasons for insisting on feedback. In the design theory proposed in this paper, a feedback solution emerges naturally by incorporating two sensitivity measures in the cost functional.

[5] Re $a \equiv$ "real part" of a.

[6] $Q(s) = Q_*(s) \equiv Q'(-s)$, $Q(j\omega) \geq 0$ for all real ω and det $Q(s) \not\equiv 0$. (The prime denotes matrix transposition and det $A \equiv$ determinant A.)

[7] tr A = trace A and all random processes are assumed to be second-order stationary with possible shape-deterministic mean values.

[8] The ensemble average in (17) is over all parameters that contribute to $\delta P(s)$.

[9] The use of spectral densities to characterize plant-uncertainty is not new [4].

Observe that (19) is obtained from (11) by replacing G_d with $G_d + \mu G_s$. Hence, it is possible to account for *some* of the feedback effects of plant-uncertainty by increasing load disturbance and, in a sense, this conclusion justifies an old conjecture.

There are many pairs (R_u, R_w) that yield both finite cost E and an asymptotically stable closed-system design. Included in these are absolutely optimal ones that minimize E absolutely and conditionally optimal ones that minimize E, subject to prescribed bounds on stability margins and *tracking-cost* sensitivity.[10] To delineate either of these sets, it is first necessary to solve the intermediate problem of finding all pairs (R_u, R_w) *acceptable* for the given datum (P, F_t).

Definition 1: The two real rational matrices $R_u(s)$ and $R_w(s)$ are said to be acceptable for the given data $P(s)$ and $F_t(s)$, if there exists a controller which realizes them as the designated closed-system transfer matrices of an internally asymptotically stable configuration of generic type shown in Fig. 1.

This definition of acceptability does not exclude nondynamical controllers from consideration, so that $T_c(s)$ (2) is not necessarily proper. It has been our experience that properness is almost always guaranteed in correctly posed optimization problems. In any case, the matter can always be investigated after the larger class has been identified.

Assumption 1: The plant and feedback sensor are both free of hidden poles in Re $s \geq 0$ and any finite pole of $F_t(s)P(s)$ in Re $s \geq 0$ has a McMillan degree which is equal to the sum of its McMillan degrees as a pole of $F_t(s)$ and as a pole of $P(s)$. (Any such pair is said to admissible.)

It should be obvious that all parts of Assumption 1 are necessary for the existence of a stabilizing controller. Theorem 1 characterizes the pairs (R_u, R_w) acceptable for a prescribed admissible datum pair (P, F_t) in terms of two arbitrary real rational matrix parameters $H_1(s)$ and $K_1(s)$ analytic in Re $s \geq 0$. Let

$$A^{-1}B = F_tP = B_1A_1^{-1} \tag{20}$$

where (A, B) is any left and (B_1, A_1) is any right coprime pair of polynomial matrices. As is well known [3], [5], [6], there exist real polynomial matrices $X(s)$, $Y(s)$, $X_1(s)$, and $Y_1(s)$ such that $BA_1 = AB_1$, $A_1Y_1 = YA$,

$$AX + BY = 1 \tag{21}$$

$$X_1A_1 + Y_1B_1 = 1 \tag{22}$$

and

$$\det X \cdot \det X_1 \neq 0. \tag{23}$$

Theorem 1: Under Assumption 1, a pair (R_u, R_w) is acceptable for (P, F_t) iff

$$R_u = A_1H_1 \tag{24}$$

and

$$R_w = A_1(Y_1 + K_1A) \tag{25}$$

where $H_1(s)$ and $K_1(s)$ are *arbitrary* compatibly dimensioned real rational matrices analytic in Re $s \geq 0$ such that

$$\det (X_1 - K_1B) \neq 0. \tag{26}$$

The associated controller transfer matrix $T_c(s)$ that realizes the pair (R_u, R_w) is then given by

$$T_c = (X_1 - K_1B)^{-1}[H_1 | -(Y_1 + K_1A)] \tag{27}$$

and any one of its realizations (minimal or otherwise) that is free of unstable hidden modes defines an asymptotically stable closed-system design. More precisely, if

$$K_1 = L_1^{-1}M_1 \tag{28}$$

is any left-coprime decomposition of $K_1(s)$, the resultant closed-system characteristic polynomial $\Delta(s)$ is given by

$$\Delta = \frac{\Delta_c}{\det (L_1X_1 - M_1B)} \cdot \frac{\Delta_t\Delta_p}{\det A_1} \cdot \det L_1 \tag{29}$$

where $\Delta_c(s)$, $\Delta_t(s)$, and $\Delta_p(s)$ are the characteristic polynomials of the controller, feedback sensor, and plant, respectively.

The first factor in (29) is strict-Hurwitz because the controller is realized without unstable hidden modes, the second because (P, F_t) is an admissible pair, and the third because $K_1(s)$ is analytic in Re $s \geq 0$. (The parametrizations of $R_u(s)$, $R_w(s)$, and $T_c(s)$ given in (24)–(27) are useful generalizations of those presented in [3] for one-degree-of-freedom controllers.)[11]

Although many different generic controller topologies can be employed for the realization of $T_c(s)$, it is nonetheless easily seen that the single-loop configuration of Fig. 2 is inadequate, even in the scalar case.[12] For, observe that if $C_u = C$ and $C_w = CF$, then (27) leads to the equations

$$C = (X_1 - K_1B)^{-1}H_1 \tag{30}$$

and

$$CF = (X_1 - K_1B)^{-1}(Y_1 + K_1A) \tag{31}$$

for the determination of $C(s)$ and $F(s)$. Hence, any finite zero of $H_1(s)$ in Re $s \geq 0$ which is neither a zero of $X_1 - K_1B$ nor $Y_1 + K_1A$, is simultaneously a zero of $C(s)$ and a pole of $F(s)$, i.e., it is an unstable hidden mode in the realization of $T_c(s)$.

One possible remedy is to use a precompensator $C_f(s)$ outside the loop as shown in Fig. 3. Under these conditions (30) and (31) go into

$$CC_f = (X_1 + K_1B)^{-1}H_1 \tag{32}$$

and

$$CF = (X_1 - K_1B)^{-1}(Y_1 + K_1A), \tag{33}$$

respectively, and the troublesome zeros of $H_1(s)$ in Re $s \geq 0$ can be transferred into $C_f(s)$. Two obvious disadvantages of the method are that the precompensator must be stable and dynamical because it lies external to the loop. When necessary, such difficulties should be avoided by the use of a different structure and in our opinion, the topic of controller topology deserves more attention.

The problem of constructing tractable stability-margin measures for single-input, single-output plants is, of course, classical [1], [2]. Nevertheless, it is only recently [9]–[18], because of renewed interest in frequency-domain controller design, that the multivariable case has come under careful scrutiny. Our own approach attempts to exploit the double-parameter representation of $T_c(s)$.

Let $P(s)$ and $F_t(s)$ denote nominal transfer matrices and let

$$T_c(s) = [C_u(s) | -C_w(s)] \tag{34}$$

[10] To be defined.

[11] By using results contained in a recent comprehensive paper by Desoer and Gustafson [44], it should be possible to extend the parametrizations (24)–(27) beyond the rational domain. The two excellent papers [7], [8] by Pernebo also solve several important problems involving two-degree-of-freedom dynamical controller design. However, the restriction to dynamical controllers precludes the use of derivative compensation and appears to lose the convenient parametrization (27). Moreover, performance, at least in the sense that we have defined it, is not an explicit issue. Moreover, it is also possible to arrive at Theorem 1 directly from the class of stabilizing controllers derived in [3] using results available in [39].

[12] Our argument is scalar oriented but has a direct multivariable analog.

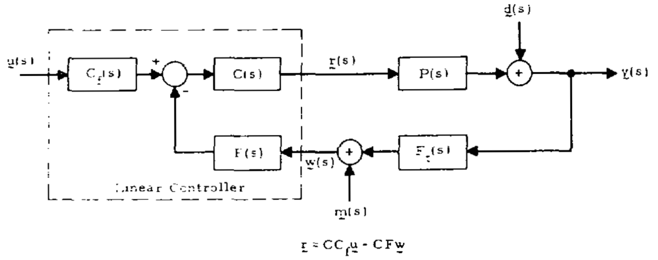

Fig. 3. The precompensated classical single-loop servoregulator configuration.

denote the transfer matrix of any companion stabilizing controller. Then, from Theorem 1,

$$C_w = (X_1 - K_1 B)^{-1}(Y_1 + K_1 A) \tag{35}$$

where $K_1(s)$ is any real rational matrix analytic in Re $s \geq 0$.

Consider a perturbation that modifies *only* the plant.[13] Under such a change, $F_t(s)$, $C_u(s)$, $C_w(s)$, $\Delta_t(s)$, and $\Delta_c(s)$ remain unaltered but the characteristic polynomials $\Delta_p(s)$ and $\Delta(s)$ can change. Let the transfer matrix and characteristic polynomial of the modified plant be denoted by $P(s) + \delta P(s)$ and $\Delta_{p'}(s)$, respectively. Concomitantly, let

$$F_t(P+\delta P) = (B_1 + \delta B_1)(A_1 + \delta A_1)^{-1} \tag{36}$$

where $B_1(s) + \delta B_1(s)$, $A_1(s) + \delta A_1(s)$ are right-coprime polynomial matrices. An explicit expression for the characteristic polynomial $\Delta_\delta(s)$ of the perturbed closed-system structure is easily derived.

From first principles [3],

$$\Delta_\delta = \Delta_c \Delta_t \Delta_{p'} \cdot \det\,(1 + C_w F_t(P+\delta P)) \tag{37}$$

or, after some straightforward simplification with the aid of (22), (35), (36), $K_1 = L_1^{-1} M_1$ and the identity[14]

$$(L_1 X_1 - M_1 B) A_1 + (L_1 Y_1 + M_1 A) B_1 = L_1, \tag{38}$$

we obtain

$$\Delta_\delta = \frac{\Delta_c}{\det\,(L_1 X_1 - M_1 B)} \cdot \frac{\Delta_t \Delta_{p'}}{\det\,(A_1 + \delta A_1)} \cdot \det\,(L_1 + \delta L_1) \tag{39}$$

where

$$\delta L_1 = (L_1 X_1 - M_1 B) \cdot \delta A_1 + (L_1 Y_1 + M_1 A) \cdot \delta B_1. \tag{40}$$

Note, that with no plant change, (39) reduces to (29) and $\Delta_\delta(s)$ equals $\Delta(s)$. Each of the three multiplicative factors in (39) has a specific bearing on stability.

The first is strict-Hurwitz because the controller for the nominal design is realized without unstable hidden modes. As for the second, the *a priori* admissibility of the pair (P, F_t) guarantees only the strict-Hurwitz character of $\Delta_t \Delta_p/\det A_1$ and not that of $\Delta_t \Delta_{p'}/\det\,(A_1 + \delta A_1)$. To conclude that the latter is strict-Hurwitz it is necessary to assume admissibility of the perturbed pair $(P + \delta P, F_t)$.[15] If this is granted, there remains only the third factor $\det\,(L_1 + \delta L_1)$ to consider.

Clearly,

$$\det\,(L_1 + \delta L_1) = \det L_1 \cdot \det\,(1 + L_1^{-1}\delta L_1). \tag{41}$$

Since the nominal design is stable, $\det L_1(s)$ is a strict-Hurwitz polynomial which is invariant under the perturbation $\delta P(s)$. Hence, the dependence of the zeros of

$$\det\,(1 + L_1^{-1}\delta L_1) = \det\,(1 + (X_1 - K_1 B) \cdot \delta A_1 + (Y_1 + K_1 A) \cdot \delta B_1) \tag{42}$$

on the variations δA_1 and δB_1 reveals the tolerance of the nominal design to instability under the plant change. (We prove in the Appendix that these zeros are uniquely determined by $P(s)$ and $\delta P(s)$, so that the choice of coprime factors is immaterial.)

Ideally, to accommodate "large" excursions in $A_1(s)$ and $B_1(s)$, both $X_1(s) - K_1(s)B(s)$ and $Y_1(s) + K_1(s)A(s)$ must be made simultaneously "small" and unfortunately, in view of the identity

$$(X_1 - K_1 B) A_1 + (Y_1 + K_1 A) B_1 = 1 \tag{43}$$

this is impossible to achieve. Nevertheless, preliminary analysis suggests that regardless of their eventual definitions, all stability margins will involve constraints imposed either directly or indirectly on the two matrices $X_1 - K_1 B$ and $Y_1 + K_1 A$ through the choice of free parameters $K_1(s)$.

The two terms that have been added to E_0 in (18) to give the total cost E constitute only relative penalties of $\delta \boldsymbol{e}$ and $\delta \boldsymbol{r}$ with respect to $\boldsymbol{r}$ as reference. However, $\boldsymbol{r}$ also depends on the particular nominal design and what we now need is one additional performance measure that refers exclusively to itself. From (8) and (12) it follows that the *increment* $\delta \boldsymbol{e}$ *in* $\boldsymbol{e}$ produced by plant uncertainty $\delta P(s)$ gives rise to an increment

$$\delta E_t = \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} \mathrm{tr}[S\langle(\delta P)\boldsymbol{r}\boldsymbol{r}_*(\delta P)_*\rangle S_*]ds \tag{44}$$

in the tracking-cost E_t.[16] Or, since $\langle \boldsymbol{r}\boldsymbol{r}_* \rangle = G_r$,

$$\delta E_t = \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} \mathrm{tr}[S\langle(\delta P)G_r(\delta P)_*\rangle S_*]ds \tag{45}$$

where

$$G_r = R_u G_u R_{u*} + R_w(F_t G_d F_{t*} + G_m) R_{w*}. \tag{46}$$

By definition,

$$\eta_t = \left(\frac{\delta E_t}{E_t}\right)^{1/2} \tag{47}$$

is the tracking-cost sensitivity.

The calculation of η_t involves $R_u = A_1 H_1$, $R_w = A_1(Y_1 + K_1 A)$ and a rather difficult ensemble average $\langle(\delta P)G_r(\delta P)_*\rangle$.[17] In general, therefore, restrictions on η_t impose constraints on *both* of the free parameters $H_1(s)$ and $K_1(s)$. To summarize, our design procedure takes into explicit account performance, plant-saturation, and plant-uncertainty. The latter is penalized both relatively, by the addition of two extra terms to E_0, and absolutely by the imposition of bounds on η_t, the tracking-cost sensitivity. Moreover, a potential quantitative dependence of stability margin on the parameter $K_1(s)$ is also indicated.

The expression for E in (19) decomposes into the sum

$$E = E_u + E_w$$

in which

$$E_u = \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} \mathrm{tr}[(1 - PR_u)G_u(1 - PR_u)_* + kQR_u G_u R_{u*}]ds \tag{48}$$

[13] Thus far, the size of the perturbation is arbitrary.

[14] $BA_1 = AB_1$.

[15] For example, if $F_t(s) = 1$, admissibility is assured if the perturbed and unperturbed plants have identical hidden modes, etc.

[16] E_t is defined in (8) and $\langle \delta P(s) \rangle = 0$, all s.

[17] A convenient expression for this average in terms of G_r and $G_s = \langle (\delta P)(\delta P)_* \rangle$ is still not available.

includes all terms involving R_u and

$$E_w = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} \mathrm{tr}[(1-PR_wF_t)(G_d+\mu G_s)(1-PR_wF_t)_* + kQR_wF_t(G_d+\mu G_s)F_{t*}R_{w*}]ds + \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} \mathrm{tr}[R_wG_mR_{w*}(P_*P+kQ)]ds \tag{49}$$

all those involving R_w. Consequently, the problem of finding all acceptable pairs (R_u, R_w) that yield finite E, is equivalent to finding all acceptable R_u's that yield finite E_u and all acceptable R_w's that yield finite E_w. This separation is of great practical and theoretical significance. Theorems 2 and 3 completely solve the problem and rest on six important additional assumptions, many of which have been extensively discussed in [3].

Assumption 2: The spectral density $G_u(s)$ vanishes at least as fast as $1/s^2$ as $s \to \infty$, i.e.,[18]

$$G_u(s) \le 0(s^{-2}). \tag{50}$$

Assumption 3: $(F_t - 1)P$ and F_t are analytic on the finite part of the $s = j\omega$-axis.

Assumption 4: AG_uA_*, $A_{1*}(P_*P + kQ)A_1$ and Q are analytic and nonsingular on the finite part of the $s = j\omega$-axis.

Theorem 2: Suppose that Assumptions 1–4 are satisfied and let the matrices $\Lambda(s)$ and $\Omega_u(s)$ denote the Wiener–Hopf spectral solutions[19] of the two equations

$$A_{1*}(P_*P+kQ)A_1 = \Lambda_*\Lambda \tag{51}$$

and

$$G_u = \Omega_u\Omega_{u*}. \tag{52}$$

Let

$$\Gamma = \Lambda_*^{-1}A_{1*}P_*\Omega_u. \tag{53}$$

Then the following holds.

1) The set of all acceptable R_u's that yield finite E_u is generated by the formula[20]

$$R_u = A_1\Lambda^{-1}(Z_u + \{\Gamma\}_+)\Omega_u^{-1} \tag{54}$$

where $Z_u(s)$ is an arbitrary real rational matrix $\le 0(s^{-1})$ and analytic in Re $s \ge 0$.

2) The acceptable R_u that minimizes E_u is given by

$$\tilde{R}_u = A_1\Lambda^{-1}\{\Gamma\}_+\Omega_u^{-1} \tag{55}$$

and corresponds to the choice $Z_u(s) = 0$.

3) Let the minimum cost E_u be denoted by $\tilde{E}_u$. Then, for any $Z_u(s)$

$$E_u = \tilde{E}_u + \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} \mathrm{tr}(Z_uZ_{u*})ds \ge \tilde{E}_u. \tag{56}$$

4) Every R_u given by (54) preserves the steady-state tracking potential of the plant.

It is important to note that formulas (51)–(55) permit us to relate the poles of PR_u, which condition the transient response, to the choice of nonnegative constant k and weighting matrix Q. Thus, poor transient response can be predicted and avoided in advance. Of even greater practical significance is the fact that *no* coprime factorizations are needed if the plant and sensor transfer matrices are analytic in the *strict* right half-plane, Re $s > 0$!

Corollary: Let $P(s)$ and $F_t(s)$ be free of finite poles in Re $s > 0$ and let $\Lambda_r(s)$ denote the Wiener–Hopf solution of the "reduced" equation

$$P_*P + kQ = \Lambda_{r*}\Lambda_r. \tag{56a}$$

Then, under the conditions of Theorem 2, (54) and (55) simplify to

$$R_u = \Lambda_r^{-1}(Z_u + \{\Gamma\}_+)\Omega_u^{-1} \tag{56b}$$

and

$$\tilde{R}_u = \Lambda_r^{-1}\{\Gamma_r\}_+\Omega_u^{-1}, \tag{56c}$$

respectively, where

$$\Gamma_r = \Lambda_{r*}^{-1}P_*\Omega_u. \tag{56d}$$

Proof: Clearly, $F_t(s)P(s) = B_1(s)A_1^{-1}(s)$ is analytic in Re $s > 0$; hence, det $A_1(s) \neq 0$, Re $s > 0$. Consequently, from (51) and the special properties of the spectral factors $\Lambda_r(s)$ and $\Lambda(s)$, we obtain $\Lambda(s) = \Lambda_r(s)A_1(s)$. Lemma 1 now follows immediately by substitution into (54), (55), and (53). Q.E.D.

Assumption 5: The spectral densities $G_d(s)$ and $G_s(s)$ are both $\le 0(s^{-2})$ and $G_m(s)$ is analytic on the finite part of the $s = j\omega$-axis.

Assumption 6: Let

$$G = G_m + F_t(G_d + \mu G_s)F_{t*}. \tag{56e}$$

The matrix AGA_* is analytic and nonsingular on the finite part of the $s = j\omega$-axis.[21]

Assumption 7: $F_t(s)$ is nonsingular on the finite part of the $s = j\omega$-axis.

Theorem 3: Suppose that Assumptions 1–7 are satisfied and let $\Omega(s)$ denote the Wiener–Hopf spectral soltuion of the equation

$$AGA_* = \Omega\Omega_*. \tag{56f}$$

Let

$$I = AF_t(G_d + \mu G_s)PA_1. \tag{56g}$$

Then the following holds.

1) The set of all acceptable R_w's that yield finite E_w is generated by the formula

$$R_w = A_1\Lambda^{-1}(\{\Lambda_*^{-1}I_*\Omega_*^{-1}\}_+ + \{\Lambda A_1^{-1}Y\Omega\}_- + Z_w)\Omega^{-1}A \tag{56h}$$

where $Z_w(s)$ is an arbitrary real rational matrix $\le 0(s^{-1})$ and analytic in Re $s \ge 0$.

2) The acceptable R_w that minimizes E_w is given by

$$\tilde{R}_w = A_1\Lambda^{-1}(\{\Lambda_*^{-1}I_*\Omega_*^{-1}\}_+ + \{\Lambda A_1^{-1}Y\Omega)\}_-)\Omega^{-1}A \tag{56i}$$

and corresponds to the choice $Z_w = 0$.

3) Let the minimum cost E_w be denoted by $\tilde{E}_w$. Then, for any $Z_w(s)$,

$$E_w = \tilde{E}_w + \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} \mathrm{tr}(Z_wZ_{w*})ds \ge \tilde{E}_w. \tag{56j}$$

Corollary: Let $P(s)$ and $F_t(s)$ be free of finite poles in Re $s > 0$

[18] $A(s) \le 0(s^\nu)$ means that no entry in $A(s)$ grows faster than s^ν as $s \to \infty$.

[19] $\Lambda(s)$ and $\Omega_u(s)$ are square real rational matrices that are analytic together with their inverses in Re $s > 0$. They are unique up to constant orthogonal matrix multipliers [19].

[20] In the partial fraction expansion of $\Gamma(s)$, the contributions made by all its finite poles in Re $s \le 0$, Re $s > 0$ and at $s = \infty$ are denoted by $\{\Gamma\}_+$, $\{\Gamma\}_-$, and $\{\Gamma\}_\infty$, respectively. Clearly, $\{\Gamma\}_+$ is analytic in Re $s > 0$, $\{\Gamma\}_-$ in Re $s \le 0$, and both are $\le 0(s^{-1})$.

[21] Assumptions 1–7 are not completely independent. For example, as is shown in the Appendix (A-39) the $j\omega$-analyticity of $(F_t - 1)P$ implies that of PA_1. In turn, the $j\omega$-analyticity of PA_1 and Q implies that of $A_{1*}(P_*P + kQ)A_1$.

and let $\Omega_r(s)$ denote the Wiener-Hopf solution of the reduced equation

$$G = \Omega_r \Omega_{r*}. \tag{56k}$$

Then, under the conditions of Theorem 3, (56h) and (56i) simplify to

$$R_w = \Lambda_r^{-1}(\{\Lambda_{r*}^{-1} I_{r*} \Omega_{r*}^{-1}\}_+ + Z_w)\Omega_r^{-1} \tag{56l}$$

and

$$\tilde{R}_w = \Lambda_r^{-1}\{\Lambda_{r*}^{-1} I_{r*} \Omega_{r*}^{-1}\}_+ \Omega_r^{-1}, \tag{56m}$$

respectively, where

$$I_r = F_t(G_d + \mu G_s)P. \tag{56n}$$

Proof: By hypothesis, A_1 and A are nonsingular in Re $s > 0$. Thus, $\Lambda = \Lambda_r A_1$, $\Omega = A\Omega_r$,

$$\{\Lambda A_1^{-1} Y \Omega\}_- = 0 \tag{56o}$$

and

$$\Omega^{-1} I \Lambda^{-1} = \Omega_r^{-1} A^{-1} A F_t(G_d + \mu G_s) P A_1 A_1^{-1} \Lambda_r^{-1} \Omega_r^{-1} = I_r \Lambda_r^{-1}. \tag{56p}$$

The conclusions now follow immediately by direct substitution into (56h) and (56i). Q.E.D.

Comment: The importance of the corollaries to Theorems 2 and 3 cannot be overemphasized, since the majority of plants encountered in practice are modeled by transfer matrices that are analytic in Re $s > 0$ but usually possess one or more $j\omega$-axis poles. Our design formulas (56a)–(56d) and (56k)–(56n) incorporate all such poles naturally, without any need to move them first into Re $s < 0$.

III. Illustrative Example

For the pure servoproblem without measurement noise, both G_d and G_m are zero, $G_r = R_u G_u R_{u*}$,

$$E = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} \mathrm{tr}[(1-PR_u)G_u(1-PR_u)_* + kQR_uG_uR_{u*}]ds + \frac{\mu}{2\pi j}\int_{-j\infty}^{j\infty} \mathrm{tr}[(1-PR_wF_t)G_s(1-PR_wF_t)_* + kQR_wF_tG_sF_{t*}R_{w*}]ds \tag{57}$$

and

$$\eta_t^2 = \left(\frac{\delta E_t}{E_t}\right) = \frac{\frac{1}{2\pi j}\int_{-j\infty}^{j\infty} \mathrm{tr}[(1-PR_wF_t)\langle(\delta P)R_uG_uR_{u*}(\delta P)_*\rangle(1-PR_wF_t)_*]ds}{\frac{1}{2\pi j}\int_{-j\infty}^{j\infty} \mathrm{tr}[(1-PR_u)G_u(1-PR_u)_*]ds}. \tag{58}$$

Clearly, in this case E_w reflects only plant-uncertainty so that $E_u = E_t + kE_s$ combines both the tracking and saturation costs. Note also, that because of the neglect of m, the choice of an acceptable R_w that yields finite E_w is independent of μ.

Consider a single-input, single-output plant with nominal transfer function given by

$$P(s) = \frac{s-1}{s(s-2)}. \tag{59}$$

Suppose also that

$$\delta P(s) = \frac{a_0 s - a_1}{s(s-2)} \tag{60}$$

where $\langle a_0\rangle = \langle a_1\rangle = 0$ and

$$\langle a_0^2\rangle = \sigma^2 = \langle a_1^2\rangle. \tag{61}$$

Under the conditions $k = Q = F_t = 1$, design a two-degree-of-freedom servo to track a step-input $u(s)$ whose spectral density $G_u(s) = -1/s^2$. Compare the results against the standard one-degree-of-freedom optimal design.

Solution: It is easily verified that Assumptions 1–7 are satisfied and that

$$A = s(s-2) = A_1, \qquad B = s-1 = B_1 \tag{62}$$

$$\Lambda = s^2 + \sqrt{7}s + 1, \qquad \Omega_u = \frac{1}{s}. \tag{63}$$

Let us first select the acceptable pair $(\tilde{R}_u, \tilde{R}_w)$ that minimizes both E_u and E_w in (57). Then, with the aid of (55) and the results in Theorem 3, we readily obtain[22]

$$\tilde{R}_u = -\frac{s(s-2)}{s^2+\sqrt{7}s+1}, \quad \tilde{R}_w = \frac{s(s-2)((8+3\sqrt{7})s-1)}{(s+1)(s^2+\sqrt{7}s+1)} \tag{64}$$

$$E_u = 3.65, \ E_t = 2.70, \ E_s = 0.95, \ \eta_t = 4.18\sigma. \tag{65}$$

The simple single-loop two-degree-of-freedom realization of the optimal pair $(\tilde{R}_u, \tilde{R}_w)$ that is shown in Fig. 4 corresponds to the controller transfer matrix

$$T_c(s) = \left[\frac{-(s+1)}{s-(5+2\sqrt{7})} \;\middle|\; \frac{(8+3\sqrt{7})s-1}{s-(5+2\sqrt{7})}\right]. \tag{66}$$

Under the assumption that $C(s)$, $P(s)$, and $F(s)$ are realized minimally,[23] $\Delta_t(s) = 1$, $\Delta_p(s) = s(s-2)$, and

$$\Delta_c(s) = (s+1)(s-(5+2\sqrt{7})). \tag{67}$$

Furthermore, since $K_1 = L_1^{-1}M_1$ and

$$\tilde{R}_w = A_1(Y_1 + K_1 A) = A_1 L_1^{-1}(L_1 Y_1 + M_1 A), \tag{68}$$

$L_1(s)$ is identified as the denominator of $\tilde{R}_w(s)$. Hence, from (64)

$$L_1(s) = (s+1)(s^2+\sqrt{7}s+1) \tag{69}$$

and (29) yields

$$\Delta(s) = (s+1)^2(s^2+\sqrt{7}s+1) \tag{70}$$

for the associated closed-system characteristic polynomial. [Note that an extra $s + 1$ factor is due to the presence of a hidden mode in the realization of $T_c(s)$.]

It is not possible to realize $\tilde{R}_u(s)$ with a single-loop one-degree-of-freedom configuration.[24] Indeed, under this restriction, $F_t(s) = F(s) = 1$ in Fig. 2,

$$\tilde{R}_u = (1+CP)^{-1}C, \tag{71}$$

[22] $G_s(s) = \dfrac{\sigma^2(1-s^2)}{s^2(s^2-4)}$.

[23] The McMillan degree of $T_c(s)$ equals one, but a minimal realization of the pair $C(s)$, $F(s)$ in Fig. 4 requires two integrators. Thus, $T_c(s)$ has not been realized minimally.

[24] Hence, *LQG* does not work either [3].

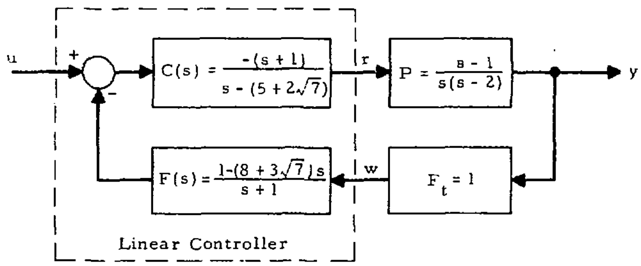

Fig. 4. Two-degree-of-freedom single-loop realization of the optimal pair $(\tilde{R}_u, \tilde{R}_w)$, (64).

$$C=\tilde{R}_u(1-P\tilde{R}_u)^{-1}=\frac{2-s}{s-(1+\sqrt{7})} \tag{72}$$

and closed-system stability is precluded by the Re $s > 0$ pole-zero cancellation between $P(s)$ and $C(s)$ at $s = 2$.

But if we insist on this topology, the smallest possible realizable cost is given by $E_u = 131$ and is achieved [3] by the design

$$R_u=\frac{s(s-2)((11+4\sqrt{7})s-2)}{(s+2)(s^2+\sqrt{7}s+1)}, \quad C=\frac{(11+4\sqrt{7})s-2}{s-(7+3\sqrt{7})} \tag{73}$$

$$E_t=42.35,\ E_s=88.65,\ \eta_t=3.65\sigma. \tag{74}$$

In this second design, the costs E_u, E_t, and E_s have been increased, respectively, by factors of 35.89, 15.69, and 93.32 for a mere 12.7 percent reduction in tracking-cost sensitivity! Actually, both $\eta_t = 4.18\sigma$ and $\eta_t = 3.65\sigma$ are totally unacceptable.[25] Yet, by exploiting the extra degree of freedom, it is very easy to reduce η_t to a value $\leq 0.25\sigma$ without any change in stability margins and at a total cost E_u approximately equal to E_t in the one-degree-of-freedom design (73), (74).

To preserve stability margins, the free parameter $K_1(s)$ must be left intact; hence, $R_w(s) = \tilde{R}_w(s)$. It follows then, in view of (58), that to reduce η_t we must vary $R_u(s)$ and this we accomplish by a suitable choice of $Z_u(s)$ in (54). In particular, for

$$Z_u(s)=\frac{1}{s+\omega_0},\ \omega_0=0.0126, \tag{75}$$

$$E_u=\tilde{E}_u+\frac{1}{2\pi j}\int_{-j\infty}^{j\infty} Z_u Z_{u*}\,ds=\tilde{E}_u+\frac{1}{2\omega_0}=\tilde{E}_u+0.0063 \tag{76}$$

and we obtain the third design

$$R_w=\tilde{R}_w,\ R_u=\tilde{R}_u\cdot\frac{\omega_0}{s+\omega_0} \tag{77}$$

$$E_u=48.3333,\ E_t=43.3038,\ E_s=0.02874,\ \eta_t=0.243\sigma. \tag{78}$$

The simple realization shown in Fig. 5 is suggested immediately by (77).[26]

The extremely small 3 dB bandwidth $\omega_0 = 0.0126$ rad/s of the precompensator $C_f(s)$ leads to a significant decrease in the bandwidth of the overall transfer function and a corresponding erosion in transient performance. This sacrifice of bandwidth for tracking-cost sensitivity explains why almost all of $E_u = 43.3333$ is now contributed by $E_t = 43.3038$. Nevertheless, and this a key point, except for a truly negligible increase in E_t, this suboptimal two-degree-of-freedom design (77) + (78) is, in almost every respect, vastly superior to the optimal one-degree-of-freedom design (73), (74).[27] Furthermore, the two designs have almost identical traditional gain and phase margins. In fact, in both cases the loop transfer function is given by

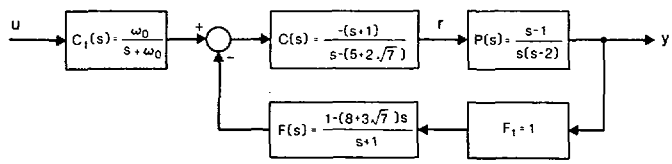

Fig. 5. Two-degree-of-freedom single-loop realization of the pair (R_u, R_w), (76).

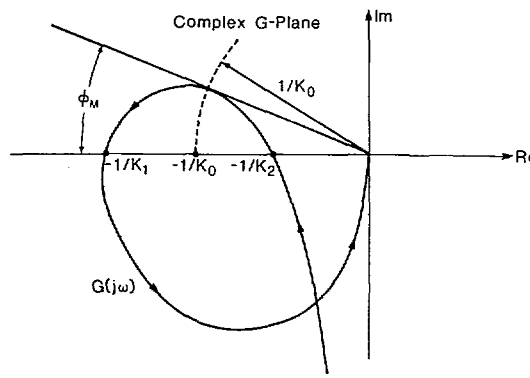

Fig. 6. Form of the Nyquist diagram for the examples.

$$KG(s)=\frac{K(s-1)(s-\sigma_0)}{s(s-2)(s-\sigma_p)} \tag{78a}$$

where σ_0 and σ_p are positive and K is a real number. It follows that the Nyquist plot of $G(j\omega)$ for $\omega > 0$ has the form shown in Fig. 6. Since the loop transfer function possesses poles in Re $s > 0$ at $s = 2$ and $s = \sigma_p$, the closed-loop system is stable iff the Nyquist diagram encircles the $-1/K$ point in the complex G-plane twice in a counterclockwise direction, i.e., iff

$$0<K_1<K<K_2. \tag{78b}$$

Thus, the one and two-degree-of-freedom optimal designs are stable iff

$$0.791<\frac{K}{K_0}<1.26 \tag{78.c}$$

and

$$0.780<\frac{K}{K_0}<1.21, \tag{78d}$$

respectively, where $K_0 = 11 + 4\sqrt{7}$ and $K_0 = 8 + 3\sqrt{7}$ are the corresponding nominal gains. Observe that the stability intervals in (78c) and (78d) are practically the same.

The angle ϕ_M shown in Fig. 6 is usually accepted as a conventional measure of stability phase margin and equals 10.4° for the first design and 9.5° for the second. Note that the agreement is again very close. In view of the difficulties associated with controlling this particular plant [47], [48], and [57], it is not at all clear that these phase margins can be significantly improved without accepting a serious deterioration in performance. In fact, we have explored a class of suboptimal two-degree-of-freedom designs with $Z_u(s) \equiv 0$ and

$$\frac{\sigma_o}{\sqrt{\mu}}\cdot Z_w(s)=\frac{\sigma\alpha}{s+\beta},\qquad \beta>0 \tag{78e}$$

and compared them using $\max_\omega |T(j\omega)|$ as a criterion, where $T = PR_wF_t$. (This stability margin measure is now receiving a great

[25] For a (reasonable) 10 percent variation in the numerator coefficients of $P(s)$, $\sigma \approx 0.1$ and $\eta_t = 3.65\sigma = 0.365$ represents a (most unreasonable) 36.5 percent change in tracking-cost about the nominal.

[26] The associated closed-loop characteristic polynomial $\Delta(s)$ is again given by (70).

[27] For example, a root mean-square deviation $\sigma = 0.1$ in the numerator coefficients of $P(s)$ now entails only a 2.43 percent change in tracking cost, etc.

deal of attention [9]–[13], [20]. For the optimal design ($\alpha = 0$), $\max_\omega |T(j\omega)| = 15.6$ dB which is only slightly larger than

$$\inf_{\substack{-\infty<\alpha<\infty \\ \beta>0}} (\sup_\omega |T(j\omega)|) = (\max_\omega |T(j\omega)|)\Big|_{\substack{\beta=4.28\alpha \\ \alpha\to\infty}} = 11.7 \text{ dB}. \quad (78f)$$

However, since

$$\frac{\sigma_Q^2}{\mu} \cdot (E_w - \tilde{E}_w) = \frac{(\sigma\alpha)^2}{2\beta} \quad (78g)$$

we see that this 3.9 dB improvement can only be obtained at the expense of an enormous increase in cost. Clearly then, the decision to retain the stability margins realized with the optimum two-degree-of-freedom design is justified.

The free parameter $Z_w(s)$ is expected to play a much more active design role after suitable margins have been precisely defined. In any case, we believe that our examples should dispel any remaining doubts about either the superiority or practical feasibility of a two-degree-of-freedom design theory of optimal controllers.

IV. Discussion and Suggestions for Future Research

The analytic design technique described in Sections II and III is quite comprehensive and takes performance, plant saturation, and plant sensitivity into account *explicitly*, with stability margins playing a passive background role. Naturally, its real significance can only be judged by its success in practical applications. In our opinion, its main features are the following.

1) Every component of the cost-functional has a clear physical significance.
2) The two available degrees of freedom are fully exploited.
3) Tradeoffs are easily recognized and included in the design.
4) There are no restrictions on the plant except that it be finite-dimensional and free of unstable hidden modes.

A comparison with some other published work in this area reveals many points of contact but also some differences in philosophy. Of the references we cite, it appears that only [9], [11], [12], and [18] consider the possibility of sacrificing cost to enhance feedback sensitivity and stability margins. In [18], state-variable feedback is used exclusively and stability margin is traded off against optimal regulator performance. In [12], essentially the same problem is considered and its LQG formulation is discussed briefly. However, as noted in [12], the unrealistic assumption is made that the observer corresponds to the perturbed plant and not the nominal. In both [9] and [11] a procedure for regulator design is outlined based on qualitative physical considerations similar to those employed by us. Namely, reasonable stability margin criteria are first translated into restrictions on the singular values of the two sensitivity matrices $S(j\omega)$ and $1 - S(j\omega)$ and a quadratic cost-functional is then used to penalize tracking error and saturation. If the optimal design obtained by minimizing the cost fails to satisfy the constraints imposed on the singular values, it is discarded, and the process is repeated with a different set of weighting matrices until a satisfactory solution is reached [11]. The similarities in the above approach and that taken in our paper should be evident, despite the fact that we bound tracking-cost instead of singular values, do not incorporate the effects of sensitivity in the same way and have not yet suggested precise quantitative stability margins. But in any case, since [9] and [11] also gauge performance in terms of a quadratic cost which must be finite to be meaningful, all acceptable R_w's are necessarily given by (56h). Hence, instead of changing weighting matrices in the cost, it may be easier and perhaps more efficient to go to a suboptimal design by choosing a nonzero $Z_w(s)$ and then to calculate the loss in performance by means of (56j).

A relatively new development is the use of nonquadratic norms to cope with the problem of sensitivity minimization [33]–[35], [46]. In [33], disturbance rejection and the problem of plant uncertainty are considered mainly in a one-degree-of-freedom setting. Unfortunately, the two-degree-of-freedom treatment of plant uncertainty neglects both measurement noise and disturbance inputs and places most of the emphasis on plants that are either stable or have been stabilized from the outset. In [34], [35], and [46] unstable single-input-output plants are admitted but attention is restricted to one-degree-of-freedom designs. Although it is probably true that minmax worst-case designs cannot be completely avoided if analytic models for plant uncertainty are not available, it also seems likely that a quadratic cost-functional must play some role in the evaluation of performance. The expressions (54) and (56h) for all acceptable R_u's and R_w's will then subsume all possible designs through the choice of free parameters $Z_u(s)$ and $Z_w(s)$. Clearly, only further study can determine whether a tractable comprehensive theory of optimal design based on nonquadratic norms is really feasible and advantageous. The growing number of papers on this subject [49]–[56] suggests that an answer may soon be forthcoming.

The treatment of stability margins for linear time-invariant finite-dimensional multivariable systems given in [9]–[12], [14], and [20] employs only feedback structures of the type shown in Fig. 3, and envisages either additive or multiplicative perturbations from a nominal loop transfer matrix $G_0(s)$, i.e.,

$$G(s) = G_0(s) + \delta G_0(s) \quad (79)$$

or

$$G(s) = [1 + \delta G_0(s)]G_0(s). \quad (80)$$

Correspondingly,

$$S_0 = [1 + G_0(s)]^{-1} \quad (81)$$

and

$$T_0 = G_0(s)[1 + G_0(s)]^{-1} \quad (82)$$

are the respective sensitivity matrices to be considered. But, as our work in Section III indicates, *both* $X_1 - K_1B$ and $Y_1 + K_1A$ must play an important role, so that it is probable that neither (79) nor (80), taken alone, suffices. Since $X_1 - K_1B$ and $Y_1 + K_1A$ cannot be made "too small," simultaneously, it is anticipated that some of the gain-bandwidth ideas described in [21] will prove helpful. In addition, there is reason to believe that the Re $s \geq 0$ analyticity of $L_1^{-1}\delta L_1$ in (42) can be exploited to develop sharp $j\omega$-axis stability-margin criteria.

In view of the computational complexities in multivariable system design, it is fortunate to have available computer algorithms for spectral factorization [22], [23] and the determination of matrix fraction descriptions [24]–[26]. Other important references [27], [40]–[42], [55] contain additional results which should ease the burden of numerical computation. As we have seen in the lemmas to Theorems 2 and 3, for plants with left-half plane and $j\omega$-axis poles only, it is possible to eliminate all coprime decompositions.

The polynomial approach advocated by Kučera [27], [28] dispenses with partial fraction expansions and deserves to be considered. In fact, Šebek [45] has applied this approach to single-input–output systems to obtain an optimal two-degree-of-freedom controller for the case $G_d = G_m = F_t = 1, E = E_0$. At this point, we are not certain that their method can be employed in our general formulation of the design problem but the matter is not closed.

One aspect of multivariable design that has not been addressed in this paper is that of robustness with respect to tracking and regulation [29], [30]. The reason for this omission is the following. In our development the only ramp-like signals that can be tracked and regulated are those whose Laplace transforms have poles that coincide with the finite $j\omega$-axis poles of $P(s)$. To attempt to track or regulate any other type of ramp requires the presence of such a ramp at the plant input and plant saturation is usually inevitable. Hence, the performance index (19) is infinite and design is meaningless.

Attempts to circumvent the difficulty by modifying the saturation part of the cost functional only avoid the issue. Clearly, the saturation constraint must be respected if design is based on a linearized model. At present, it appears to us that the inclusion of Re $s > 0$ poles in the controller to ensure robust asymptotic tracking and regulation can conflict with the requirement of no plant saturation. (Thus, the feedback problems treated in [32], [39], and [43] may need elaboration.)

Consequently, in our opinion, if plant saturation is a primary concern, robustness only makes sense if the plant already possesses the necessary $j\omega$-axis poles. But then, since such poles are usually structural in origin, they already exhibit great resistance to changes in plant parameters. For example, in the positioning of an inertial load J with viscous damping B, the transfer function relating angular displacement θ to applied torque λ is given by

$$\frac{\theta(s)}{\lambda(s)}=\frac{1}{s(Js+B)}. \tag{83}$$

Evidently, the pole at the origin is firmly placed and totally insensitive to variations in the parameters J and B. In this case, the plant is already able to track step inputs robustly and there is no need to include an additional pole in the controller. However, if plant saturation presents no problem the notion of robustness is meaningful and its inclusion is accomplished by simply setting $kQ = 0$ in the cost-functional (19).

Naturally, the selection of a suitable controller for the realization of an optimal pair (R_u, R_w) is an important problem. That one exists is assured, but many structures are possible. In [1, p. 241] eight different two-degree-of-freedom configurations are given which include, in particular, those shown in Figs. 2 and 3. Can they all be generalized to the multivariable case and, if so, is one of them preferable? It may be possible to single one out on the basis of system sensitivity to controller uncertainty, a consideration we have ignored. Qualitatively, it seems that the most desirable topology is the one shown in Fig. 2 with $F(s) \equiv 1$ because the sensitivity with respect to $C(s)$ is the same as that with respect to $P(s)$. Unfortunately, this is a one-degree-of-freedom unity-feedback structure and is only applicable if $R_u = R_w$. (If $R_u \neq R_w$, a two-degree-of-freedom configuration is compulsory.) References [7], [8], and [33]–[39] may prove helpful. The following is a list of some future research objectives.

1) Define physically meaningful stability margins and translate them into constraints on Z_w involving the plant uncertainties δA_1 and δB_1.

2) Determine, if possible, an acceptable pair (R_u, R_w) that minimizes the cost-functional E, maintains prescribed stability margins, and also keeps tracking-cost sensitivity within given bounds.

3) Investigate the potential role played by robustness in the context of our design philosophy.

4) Study the impact of sensitivity on the choice of controller structure used to realize an acceptable pair (R_u, R_w).

Appendix

Proof of Theorem 1

Necessity: Let $\Delta_c(s)$, $\Delta_t(s)$, and $\Delta_p(s)$ denote the characteristic polynomials of the controller, feedback sensor, and plant, respectively. Then [3]

$$\Delta=\Delta_c\Delta_t\Delta_p \cdot \det\,(1+C_wF_tP) \tag{A-1}$$

is the characteristic polynomial of the general two-degree-of-freedom structure shown in Fig. 1. Since $T_c = [C_u| -C_w]$,

$$1+C_wF_tP=1+T_c\begin{bmatrix}0\\ \hline -1\end{bmatrix}F_tP. \tag{A-2}$$

Thus, in terms of the coprime decompositions $F_tP = B_1A_1^{-1}$ and $T_c = A_c^{-1}B_c$,

$$\Delta=\frac{\Delta_c\Delta_t\Delta_p}{\det A_c \cdot \det A_1}\cdot \det\left(A_cA_1+B_c\begin{bmatrix}0\\ \hline -1\end{bmatrix}B_1\right). \tag{A-3}$$

Clearly, $\det A_c$ divides Δ_c and $\Delta_t\Delta_p/\det A_1$ is finite and $\neq 0$ in Re $s \geq 0$ because of the assumed admissibility of the pair (P, F_t). Consequently, the polynomial

$$g(s)= \det\left(A_cA_1+B_c\begin{bmatrix}0\\ \hline -1\end{bmatrix}B_1\right) \tag{A-4}$$

must be strict-Hurwitz if the structure is to be asymptotically stable.

It now follows that

$$R_u=(1+C_wF_tP)^{-1}C_u=\left(1+T_c\begin{bmatrix}0\\ \hline -1\end{bmatrix}F_tP\right)^{-1}T_c\begin{bmatrix}1\\ \hline 0\end{bmatrix} \tag{A-5}$$

$$=A_1\left(A_cA_1+B_c\begin{bmatrix}0\\ \hline -1\end{bmatrix}B_1\right)^{-1}B_c\begin{bmatrix}1\\ \hline 0\end{bmatrix}=A_1H_1 \tag{A-6}$$

where $H_1(s)$ is analytic in Re $s \geq 0$. Similarly,

$$R_w=(1+C_wF_tP)^{-1}C_w$$

$$=A_1\left(A_cA_1+B_c\begin{bmatrix}0\\ \hline -1\end{bmatrix}B_1\right)^{-1}B_c\begin{bmatrix}0\\ \hline -1\end{bmatrix} \tag{A-7}$$

$$=A_1(A_cA_1+\tilde{B}_cB_1)^{-1}\tilde{B}_c \tag{A-8}$$

in which

$$\tilde{B}_c=B_c\begin{bmatrix}0\\ \hline -1\end{bmatrix}. \tag{A-9}$$

Let us define $K_1(s)$ by means of the equation

$$Y_1+K_1A=(A_cA_1+\tilde{B}_cB_1)^{-1}\tilde{B}_c. \tag{A-10}$$

Then, $R_w = A_1(Y_1 + K_1A)$ and it remains to be shown that $K_1(s)$ is analytic in Re $s \geq 0$. (The Re $s \geq 0$ analyticity of $Y_1 + K_1A$ is obvious.)

From (A-10),

$$X_1-K_1B=X_1-((A_cA_1+\tilde{B}_cB_1)^{-1}\tilde{B}_c-Y_1)A^{-1}B. \tag{A-11}$$

Since $F_tP = A^{-1}B = B_1A_1^{-1}$ and $A_1^{-1} = X_1 + Y_1B_1A_1^{-1}$, (A-11) and (A-2) yield

$$X_1-K_1B=A_1^{-1}-(A_cA_1+\tilde{B}_cB_1)^{-1}\tilde{B}_cF_tP$$

$$=A_1^{-1}-A_1^{-1}(1+C_wF_tP)^{-1}C_wF_tP$$

$$=A_1^{-1}(1+C_wF_tP)^{-1}=(A_cA_1+\tilde{B}_cB_1)^{-1}A_c. \tag{A-12}$$

Therefore, $X_1 - K_1B$ is also analytic in Re $s \geq 0$ and the identity

$$(Y_1+K_1A)X-(X_1-K_1B)Y=Y_1X-X_1Y+K_1 \tag{A-13}$$

immediately establishes the Re $s \geq 0$ analyticity of $K_1(s)$. Note that

$$\det\,(X_1-K_1B)=\frac{\det A_c}{g}\not\equiv 0 \tag{A-14}$$

is necessary.

Sufficiency: Let $R_u = A_1H_1$ and let $R_w = A_1(Y_1 + K_1A)$ where H_1 and K_1 are real rational matrices analytic in Re $s \geq 0$. Suppose also that det $(X_1 - K_1B) \not\equiv 0$. Our task is to prove that R_u, R_w are acceptable for the given admissible pair (P, F_t) in the sense of Definition 1.

From (5), $C_w = (1 + C_wF_tP)R_w$, hence,

$$\begin{aligned} C_w &= R_w(1-F_tPR_w)^{-1} \\ &= A_1(Y_1+K_1A)(1-B_1(Y_1+K_1A))^{-1} \\ &= A_1(1-(Y_1+K_1A)B_1)^{-1}(Y_1+K_1A) \\ &= A_1(X_1A_1-K_1BA_1)^{-1}(Y_1+K_1A) \\ &= (X_1-K_1B)^{-1}(Y_1+K_1A) \end{aligned} \tag{A-15}$$

since $X_1A_1 + Y_1B_1 = 1$. In the same way, from (4),

$$\begin{aligned} C_u &= (1+C_wF_tP)R_u \\ &= (X_1-K_1B)^{-1}(X_1-K_1B+(Y_1+K_1A)A^{-1}B)A_1H_1 \\ &= (X_1-K_1B)^{-1}(X_1+Y_1B_1A_1^{-1})A_1H_1 \\ &= (X_1-K_1B)^{-1}H_1. \end{aligned} \tag{A-16}$$

Thus,

$$T_c = [C_u | -C_w] = (X_1-K_1B)^{-1}[H_1 | -(Y_1+K_1A)]. \tag{A-17}$$

Let $K_1 = L_1^{-1}M_1$ be any left-coprime decomposition of K_1. Then,

$$T_c = (L_1X_1-M_1B)^{-1}[L_1H_1 | -(L_1Y_1+M_1A)]. \tag{A-18}$$

The pair $(L_1X_1 - M_1B)$, $(L_1Y_1 + M_1A)$ is left-coprime. Indeed, assume that for some fixed s there exists a vector $\boldsymbol{a}$ such that

$$\boldsymbol{a}'[L_1X_1-M_1B | L_1Y_1+M_1A] = \boldsymbol{O}'. \tag{A-19}$$

Then, in view of the identity

$$(L_1X_1-M_1B)A_1+(L_1Y_1+M_1A)B_1 = L_1 \tag{A-20}$$

we have $\boldsymbol{a}'L_1 = \boldsymbol{O}'$ and (A-19) reduces to

$$\boldsymbol{a}'M_1[-B|A] = \boldsymbol{O}'. \tag{A-21}$$

But A, B is left-coprime, hence, $\boldsymbol{a}'M_1 = \boldsymbol{O}'$ also. In short,

$$\boldsymbol{a}'[L_1|M_1] = \boldsymbol{O}' \tag{A-22}$$

which implies $\boldsymbol{a} = \boldsymbol{O}$ because L_1, M_1 is left-coprime and our assertion is established.

It is now easy to conclude from (A-18) that *every* finite pole of $T_c(s)$ in Re $s \geq 0$ of McMillan degree m, is a zero of det $(L_1Y_1 - M_1B)$ of multiplicity precisely equal to m. Consequently, with the aid of (A-15), we find that

$$\begin{aligned} \Delta &= \Delta_c\Delta_t\Delta_p \cdot \det\ (1+C_wF_tP) \\ &= \frac{\Delta_c\Delta_t\Delta_p}{\det\ (L_1X_1-M_1B)} \cdot \det\ (L_1X_1-M_1B+L_1Y_1B_1A_1^{-1}+M_1B) \\ &= \frac{\Delta_c}{\det\ (L_1X_1-M_1B)} \cdot \frac{\Delta_t\Delta_p}{\det\ A_1} \cdot \det\ L_1. \end{aligned} \tag{A-23}$$

The first factor in (A-23) is strict-Hurwitz because det $(L_1X_1 - M_1B)$ divides Δ_c and $T_c(s)$ is realized without unstable hidden modes, the second because the pair (P, F_t) is admissible, and the third because $K_1(s)$ is analytic in Re $s \geq 0$. Hence, the controller $T_c(s)$ yields an asymptotically stable design and the proof of Theorem 1 is complete. Q.E.D.

Proof of Theorem 2

Since $G_u = \Omega_u\Omega_{u*}$ and all matrices are rational, it is clear from (48) that E_u is finite iff $(1 - PR_u)\Omega_u \leq 0(s^{-1})$, $QR_uG_uR_{u*} \leq 0(s^{-2})$ and both are $j\omega$-analytic. But from Assumption 2, $\Omega_u \leq 0(s^{-1})$; hence, $(1 - PR_u)\Omega_u \leq 0(s^{-1})$ iff $PR_u\Omega_u \leq 0(s^{-1})$ which implies $PR_u\Omega_u\Omega_{u*} = PR_uG_u \leq 0(s^{-2})$. Let us rewrite E_u as

$$E_u = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} \text{tr}\ [G_u-2PR_uG_u+(P_*P+kQ)R_uG_uR_{u*}]\ ds. \tag{A-24}$$

Then, G_u and $PR_uG_u \leq 0(s^{-2}) \rightarrow$ tr $[(P_*P + kQ)R_uG_uR_{u*}] \leq 0(s^{-2})$. Equivalently, if we cycle the trace and use (51), (52), and (54),

$$\text{tr}\ (\Omega_{u*}R_{u*}A_{1*}^{-1}\Lambda_*\Lambda A_1^{-1}R_u\Omega_u) \leq 0(s^{-2}); \tag{A-25}$$

therefore,[28]

$$\Lambda A_1^{-1}R_u\Omega_u = Z + \{\Gamma\}_+ \leq 0(s^{-1}). \tag{A-26}$$

But by definition, $\{\Gamma\}_+ \leq 0(s^{-1})$ so that $Z \leq 0(s^{-1})$. In brief, if the integrand ρ of E_u is $\leq 0(s^{-2})$, then $Z \leq 0(s^{-1})$. The converse is also true.

In fact, it suffices to verify that $Z \leq 0(s^{-1})$ implies $PR_u\Omega_u \leq 0(s^{-1})$. Observe first that

$$PR_u\Omega_u = PA_1\Lambda^{-1}(Z+\{\Gamma\}_+); \tag{A-27}$$

hence, $PA_1\Lambda^{-1}$ proper $\rightarrow PR_u\Omega_u \leq 0(s^{-1})$. However, from (51),

$$1 = (PA_1\Lambda^{-1})_*(PA_1\Lambda^{-1}) + k(A_1\Lambda^{-1})_*Q(A_1\Lambda^{-1}), \tag{A-28}$$

the sum of two para-Hermitian nonnegative-definite matrices. Consequently, $PA_1\Lambda^{-1}$ must be bounded at $s = \infty$.

Let us say that a rational matrix is "good" if it is analytic on the finite part of the $s = j\omega$-axis. We will now prove that ρ is good iff Z is good. As we have already indicated, ρ is good iff both $(1 - PR_u)\Omega_u$ and $QR_uG_uR_{u*}$ are good. By Assumption 4, Q is good and nonsingular for all finite $s = j\omega$; thus, $QR_uG_uR_{u*}$ is good iff $R_u\Omega_u$ is good. In short, ρ is good iff $\Psi_1 \equiv (1 - PR_u)\Omega_u$ and $\Psi_2 \equiv R_u\Omega_u$ are good.

Let $R_u = (Y + A_1K)A$. Then, Ψ_1 and Ψ_2 are good iff K is good. Indeed, it follows easily from Assumption 4 that $A\Omega_u$ and Λ are good and nonsingular for all finite $s = j\omega$. Hence, Ψ_2 is good iff

$$A_1K = \Psi_2(A\Omega_u)^{-1} - Y \tag{A-29}$$

is good. Similarly, from the identity

$$\Psi_1 = (1-PR_u)\Omega_u = (1-F_tPR_u+(F_t-1)PR_u)\Omega_u \tag{A-30}$$

and the postulated goodness of $(F_t - 1)P$ in Assumption 3, it is evident, given that Ψ_1 is good, iff

$$(1-F_tPR_u)\Omega_u = (X-B_1K)A\Omega_u \tag{A-31}$$

is good. That is, iff B_1K is good. Consequently, ρ is good iff

$$\begin{bmatrix} A_1 \\ \hline B_1 \end{bmatrix} K \tag{A-32}$$

is good, i.e., iff K is good because the pair B_1, A_1 is right-coprime.

Let us substitute $R_u = (Y + A_1K)A$ into (54) and solve for Z to obtain

$$Z = \Lambda A_1^{-1}(Y+A_1K)A\Omega_u - \{\Gamma\}_+. \tag{A-33}$$

[28] The subscript u on Z_u has been dropped.

Since Λ^{-1} and $(A\Omega_u)^{-1}$ are good (Assumption 4), Z will be good iff

$$\Lambda^{-1}Z(A\Omega_u)^{-1}=A_1^{-1}Y+K-\Lambda^{-1}\{\Gamma\}_+(A\Omega_u)^{-1} \quad \text{(A-34)}$$

is good, i.e., iff

$$\begin{aligned} &A_1^{-1}Y-\Lambda^{-1}\{\Gamma\}_+(A\Omega_u)^{-1} \\ &=A_1^{-1}Y-\Lambda^{-1}(\Gamma-\{\Gamma\}_--\{\Gamma\}_\infty)(A\Omega_u)^{-1} \end{aligned} \quad \text{(A-35)}$$

is good.[29]

Clearly, by definition, $[\Gamma]_-$ and $[\Gamma]_\infty$ are good, so that Z is good iff

$$\Lambda^{-1}\Gamma(A\Omega_u)^{-1}-A_1^{-1}Y=(\Lambda_*\Lambda)^{-1}(PA_1)_*A^{-1}-A_1^{-1}Y \quad \text{(A-36)}$$

is good. Or, with the aid of (51), iff

$$(\Lambda_*\Lambda)^{-1}A_{1*}[P_*(A^{-1}-PY)-kQY] \quad \text{(A-37)}$$

is good. Since $A^{-1} = X + F_tPY$, the matrix in (A-37) is good iff

$$(\Lambda_*\Lambda)^{-1}[(PA_1)_*(X+(F_t-1)PY)-kA_{1*}QY] \quad \text{(A-38)}$$

is good and this is surely true if PA_1 is good. But by Assumption 3,

$$PA_1=F_tPA_1-(F_t-1)PA_1=B_1-(F_t-1)PA_1 \quad \text{(A-39)}$$

is obviously good and the implication ρ good $\rightarrow$ Z good has been established. Conversely, by retracing our steps we can also show that Z good $\rightarrow$ ρ good.

Thus, let Z be good. Then, from (A-33),

$$K=\Lambda^{-1}(Z+\{\Gamma\}_+)(A\Omega_u)^{-1}-A_1^{-1}Y \quad \text{(A-40)}$$

is evidently good because

$$\Lambda^{-1}\{\Gamma\}_+(A\Omega_u)^{-1}-A_1^{-1}Y \quad \text{(A-41)}$$

is good. Finally, K good implies Ψ_1 and Ψ_2 both good which implies ρ good.

Consider the collection of all matrices $R_u = A_1H_1$, where

$$H_1=\Lambda^{-1}(Z+\{\Gamma\}_+)\Omega_u^{-1} \quad \text{(A-42)}$$

and Z is analytic in Re $s \geq 0$. All such R_u's are acceptable. For clearly, due to the properties of Λ, Ω_u and $\{\Gamma\}_+$, H_1 is analytic in Re $s > 0$ iff Z is analytic in Re $s > 0$. Futhermore, since Z good implies K good and

$$H_1=KA+A_1^{-1}YA \quad \text{(A-43)}$$

Z good implies H_1 good if it can be shown that $A_1^{-1}YA$ is good. This we accomplish by noting that $X_1A_1+Y_1B_1 = 1$ yields

$$\begin{aligned} A_1^{-1}Y&=X_1Y+Y_1B_1A_1^{-1}Y=XY_1+Y_1A^{-1}BY \\ &=X_1Y+Y_1A^{-1}(1-AX)=X_1Y+Y_1A^{-1}-Y_1X; \end{aligned} \quad \text{(A-44)}$$

hence,

$$A_1^{-1}YA=X_1YA+Y_1-Y_1XA, \quad \text{(A-45)}$$

a polynomial matrix. To summarize, R_u is acceptable and yields finite cost E_u iff it has the form $R_u = A_1H_1$, where H_1 is given in (A-42) and Z is $\leq 0(s^{-1})$ and analytic in Re $s \geq 0$.

Let E_u and $\tilde{E}_u$ denote the costs associated with the choices Z and $Z = 0$, respectively. Then, after some tedious but straightforward algebra we obtain

$$2\pi j(E_u-\tilde{E}_u)=\int_{-j\infty}^{j\infty} \text{tr } (\Delta_*\Omega_{u*}^{-1}Z_*\Lambda_*^{-1})\ ds+\int_{-j\infty}^{j\infty} \text{tr}(ZZ_*)ds \quad \text{(A-46)}$$

where

$$\Delta_*=2(A_{1*}(P_*P+kQ)A_1\tilde{H}_1-A_{1*}P_*)G_u \quad \text{(A-47)}$$

and

$$\tilde{H}_1=\Lambda^{-1}\{\Gamma\}_+\Omega_u^{-1}. \quad \text{(A-48)}$$

It follows easily from (A-47) and (53) that

$$\begin{aligned} \Delta_*\Omega_{u*}^{-1}&=\Lambda_*\{\Lambda_*^{-1}A_{1*}P_*\Omega_u\}_+-A_{1*}P_*\Omega_u \\ &=-\Lambda_*(\{\Gamma\}_-+\{\Gamma\}_\infty). \end{aligned} \quad \text{(A-49)}$$

Hence, $\Delta_*\Omega_{u*}^{-1}$ and $Z_*\Lambda_*^{-1}$ are free of poles in Re $s \leq 0$.

Now, we have already established that E_u and $\tilde{E}_u$ are both finite because they correspond to Z's that are $\leq 0(s^{-1})$ and good. Since the second integral on the right side of (A-46) is clearly finite, the same must be true of the first one. Thus, its integrand is $\leq 0(s^{-2})$ and analytic in Re $s \leq 0$. By Cauchy's theorem, this integral equals zero; hence,

$$E_u-\tilde{E}_u=\frac{1}{2\pi j}\int_{-j\infty}^{j\infty} \text{tr } (ZZ_*)\ ds \geq 0. \quad \text{(A-50)}$$

Therefore, $\tilde{E}_u$ is the minimum value of the cost E_u and is realized by the unique acceptable choice $\tilde{R}_u = A_1\tilde{H}_1$.

Lastly, it remains to prove that an optimal design preserves the steady-state tracking capability of the plant. Stated differently (see [3, pp. 322 and 323]) every finite $j\omega$-axis pole of $P(s)$ of McMillan degree ν is a zero of the determinant of the closed-system error transfer matrix of multiplicity at least equal to ν.

Since PR_u is the system transfer matrix from $\boldsymbol{u}$ to $\boldsymbol{r}$, $1 - PR_u$ is the error matrix. Moreover,

$$1-PR_u=\Psi_1\Omega_u^{-1}=\Psi_1(A\Omega_u)^{-1}A \quad \text{(A-51)}$$

and

$$\det\ (1-PR_u)=\frac{\det\ \Psi_1\cdot\det\ A}{\det\ (A\Omega_u)}. \quad \text{(A-52)}$$

But for any acceptable R_u that yields finite E_u, Ψ_1 is good. Moreover, $A\Omega_u$ is nonsingular on the finite $s = j\omega$-axis. Consequently, every purely imaginary zero of det A is a zero of the left side of (A-52) of at least the same multiplicity. Q.E.D.

Proof of Theorem 3

Let ρ denote the integrand of E_w, (49). Then, E_w finite $\rightarrow \rho \leq 0(s^{-2})$. In particular,

$$\text{tr } [(1-PR_wF_t)G_{ds}(1-PR_wF_t)_*] \leq 0(s^{-2}) \quad \text{(A-53)}$$

where

$$G_{ds}=G_d+\mu G_s. \quad \text{(A-54)}$$

But by Assumption 5, $G_{ds} \leq 0(s^{-2})$, hence, invoking (A-53),

$$PR_wF_tG_{ds} \leq 0(s^{-2}). \quad \text{(A-55)}$$

In turn, (A-55), (56h) and the alternative expression[30]

$$\begin{aligned} \rho=&\text{tr } G_{ds}-2\ \text{tr } (PR_wF_tG_{ds}) \\ &+\text{tr } [(F_tG_{ds}F_{t*}+G_m)R_{w*}(P_*P+kQ)R_w] \end{aligned} \quad \text{(A-56)}$$

[29] Actually, Assumption 4 implies $\{\Gamma\}_\infty = 0$ but we do not need this result.

[30] $\int_{-j\infty}^{j\infty} \text{tr } (PR_wF_tG_{ds})\ ds=\int_{-j\infty}^{j\infty} \text{tr } (PR_wF_tG_{ds})_*\ ds.$

imply that

$$\begin{aligned}\text{tr }&[(F_tG_{ds}F_{t*}+G_m)R_{w*}(P_*P+kQ)R_w]\\&=\text{tr }(A^{-1}\Omega\Omega_*A_*^{-1}R_{w*}A_{1*}^{-1}\Lambda_*\Lambda A_1^{-1}R_w)\\&=\text{tr }(\Sigma\Sigma_*)\leq 0(s^{-2})\end{aligned}\tag{A-58}$$

in which[31]

$$\Sigma=\{\Lambda_*^{-1}I_*\Omega_*^{-1}\}_+ +\{\Lambda A_1^{-1}Y\Omega\}_- +Z.\tag{A-59}$$

Consequently, $\Sigma \leq 0(s^{-1})$, i.e., $Z \leq 0(s^{-1})$. Conversely, $Z \leq 0(s^{-1}) \rightarrow \rho \leq 0(s^{-2})$. In fact, it suffices to show that $Z \leq 0(s^{-1}) \rightarrow \text{tr }(PR_wF_tG_{ds}) \leq 0(s^{-2})$.

Clearly, since $R_w = A_1\Lambda^{-1}\Sigma\Omega^{-1}A$,

$$PR_wF_tG_{ds}=PA_1\Lambda^{-1}\Sigma\Omega^{-1}AF_tG_{ds}.\tag{A-60}$$

However, $AGA_* = \Omega\Omega_*$ gives

$$1=(\Omega^{-1}A)G_m(\Omega^{-1}A)_*+(\Omega^{-1}AF_t)G_{ds}(\Omega^{-1}AF_t)_*\tag{A-61}$$

and a familiar argument permits us to infer that $\Omega^{-1}AF_tG_{ds} \leq 0(s^{-1})$.[32] Furthermore, according to (A-28), $PA_1\Lambda^{-1}$ is proper and since $Z \leq 0(s^{-1}) \rightarrow \Sigma \leq 0(s^{-1})$, the assertion follows immediately from (A-60).

R_w is acceptable, i.e.,

$$R_w=A_1(Y_1+K_1A)\tag{A-62}$$

where $K_1(s)$ is analytic in Re $s \geq 0$, iff $Z(s)$ is analytic in Re $s \geq 0$. In fact, by solving (A-62) for K_1 we obtain, with the help of the identity $A_1Y_1 = YA$ [3]

$$\begin{aligned}K_1&=k_1+\Lambda^{-1}\{\Lambda A_1^{-1}Y\Omega\}_\infty\Omega^{-1} &\text{(A-63)}\\&=\Lambda^{-1}(\{\Lambda_*^{-1}I_*\Omega_*^{-1}\}_+ -\{\Lambda A_1^{-1}Y\Omega\}_+ +Z)\Omega^{-1} &\text{(A-64)}\\&=\Lambda^{-1}\{\Lambda_*^{-1}I_*\Omega_*^{-1}-\Lambda A_1^{-1}Y\Omega\}_+\Omega^{-1}+\Lambda^{-1}Z\Omega^{-1}. &\text{(A-65)}\end{aligned}$$

Clearly, K_1 analytic in Re $s \geq 0 \rightarrow Z$ analytic in Re $s > 0$ and, in addition, $\{\ \}_+$ good implies Z good.

Now, since Λ_*^{-1} and Ω_*^{-1} are good and nonsingular (Assumptions 4 and 6),

$$\begin{aligned}\{\ \}_+&=\{\Lambda_*^{-1}(I_*-\Lambda_*\Lambda A_1^{-1}Y\Omega\Omega_*)\Omega_*^{-1}\}_+\\&=\{\Lambda_*^{-1}(A_{1*}P_*G_{ds}F_{t*}A_*-A_{1*}(P_*P+kQ)YAGA_*)\Omega_*^{-1}\}_+\end{aligned}\tag{A-67}$$

will be good iff the matrix function in round parentheses is good. That is, iff

$$(PA_1)_*F_t^{-1}(G-G_m)A_*-A_{1*}(P_*P+kQ)YAGA_*\tag{A-68}$$

$$\begin{aligned}&=(PA_1)_*F_t^{-1}(A^{-1}-F_tPY)(AGA_*)\\&\quad-(PA_1)_*F_t^{-1}G_mA_*-kA_{1*}QY(AGA_*)\\&=(PA_1)_*F_t^{-1}X(AGA_*)-(PA_1)_*F_t^{-1}G_mA_*-kA_{1*}Q(AGA_*)\end{aligned}\tag{A-69}$$

is good. And this is obviously true because PA_1 is good and because Assumptions 3 and 7 imply that F_t^{-1} is good.

R_w acceptable implies ρ good. To see this, we must examine (49) carefully. It is not difficult to verify directly that ρ is good iff

$$(I-PR_wF_t)J,\ \text{tr }[QR_wGR_{w*}],\ \text{tr }(PR_wG_mR_{w*}P_*)\tag{A-70}$$

are all good ($G_{ds} = JJ_*$). Clearly, $A_1(Y_1 + K_1A) = (Y + A_1K_1)A$ follows from $A_1Y_1 = YA$. Thus,

$$QR_wGR_{w*}=Q(Y+A_1K_1)(AGA_*)(Y+A_1K_1)_*\tag{A-71}$$

is good since Q, AGA_* and K_1 are good. Similarly, PA_1 good and G_m good (Assumption 5) imply $PR_wG_mR_{w*}P_*$ good. Lastly, AGA_* good $\rightarrow AF_tJ$ good;[33] hence,

$$(1-PR_wF_t)J=(1-P(Y+A_1K_1)AF_t)J\tag{A-72}$$

is good iff $(1 - PYAF_t)J = (1 - PYAF_t)(AF_t)^{-1}(AF_tJ)$ is good. But

$$\begin{aligned}&(1-PYAF_t)(AF_t)^{-1}\\&=(AF_t)^{-1}(1-AF_tPY)=(AF_t)^{-1}(1-BY)=F_t^{-1}X\end{aligned}\tag{A-73}$$

is good. To sum up, R_w is acceptable and yields finite cost E_w iff it has the form (56h), where $Z_w(s)$ is an arbitrary real rational matrix $\leq 0(s^{-1})$ and analytic in Re $s \geq 0$.

Let E_w and $\tilde{E}_w$ denote the costs associated with the choices Z and $Z = 0$. Straightforward analysis yields

$$\begin{aligned}2\pi j(E_w-\tilde{E}_w)=-2\int_{-j\infty}^{j\infty}&\text{tr}[Z_*(\{\Lambda_*^{-1}I_*\Omega_*^{-1}\}_-\\&+\{\Lambda_*^{-1}I_*\Omega_*^{-1}\}_\infty]ds+\int_{-j\infty}^{j\infty}\text{tr}(ZZ_*)ds.\end{aligned}\tag{A-74}$$

From the finiteness of the second integral on the right side of (A-74) and the guaranteed finiteness of E_w and $\tilde{E}_w$, we easily conclude by Cauchy's theorem that the first integral vanishes. Therefore,

$$E_w=\tilde{E}_w+\frac{1}{2\pi j}\int_{-j\infty}^{j\infty}\text{tr}(ZZ_*)ds\geq\tilde{E}_w.\tag{A-75}$$

Q.E.D.

Finally, to conclude, we will show that the zeros of det $(1 + L_1^{-1}\delta L_1)$, (42), depend solely on $P(s)$, $\delta P(s)$ and the nominal design and not on the choice of coprime factorizations. From (29) and (39) we obtain

$$\frac{\Delta_\delta}{\Delta}=\frac{\Delta_{p'}}{\Delta_p}\cdot\frac{\det A_1}{\det(A_1+\delta A_1)}\cdot\det(1+L_1^{-1}\delta L_1).\tag{A-76}$$

Clearly, by their very definitions, Δ, Δ_δ, Δ_p and $\Delta_{p'}$ are uniquely specified by $P(s)$, $\delta P(s)$ and the nominal design for $T_c(s)$. However, A_1 and $A_1 + \delta A_1$, being right-coprime matrix denominator polynomials for F_tP and $F_t(P + \delta P)$, respectively, are only determined up to right elementary polynomial matrix multipliers [6]. Thus, it follows immediately from (A-76) that det $(1 + L_1^{-1}\delta L_1)$ is uniquely determined by the data up to a nonzero multiplicative constant. Q.E.D.

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[31] We also drop the subscript w on Z_w.

[32] Let $G_{ds} = JJ_*$. Then, $J \leq 0(s^{-1})$ and $\Omega^{-1}AF_tJ$ is proper, so that

$$\Omega^{-1}AF_tJJ_*=\Omega^{-1}AF_tG_{ds}\leq 0(s^{-1}).$$

[33] Due to the impossibility of any $j\omega$-axis pole cancellation in the sum

$$AGA_*=AF_tG_{ds}F_{t*}A_*+AG_mA_*$$

each term must be good if AGA_* is good.

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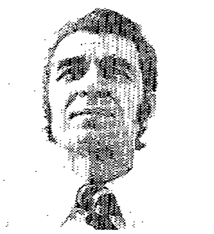

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