Lösung

(1) a. $\int_1^2 \frac{1}{x^2} dx = \int_1^2 x^{-2} dx = \left[-\frac{1}{x} \right]_1^2 = -\frac{1}{2} + \frac{1}{1} = \frac{1}{2}$

b. $\int_3^4 x^2 + e^x \, dx = \left[\frac{1}{3}x^3 + e^x \right]_3^4 = \frac{64}{3} + e^4 - \left(\frac{27}{3} + e^3 \right)$

$= \frac{64-27}{3} + e^4 - e^3 = \frac{37}{3} + e^3(e-1)$

c. $\int_1^4 \sqrt{x} \, dx = \int_1^4 x^{\frac{1}{2}} dx = \left[\frac{2}{3} x^{\frac{3}{2}} \right]_1^4 = \frac{2}{3} \cdot (\sqrt{4})^3 - \frac{2}{3} (\sqrt{1})^3$

$= \frac{16}{3} - \frac{2}{3} = \frac{14}{3}$

d. $\int_{-1}^3 \frac{1}{2x^4} dx = \frac{1}{2} \int_{-1}^3 x^{-4} dx = \frac{1}{2} \cdot \left[-\frac{1}{3} \frac{1}{x^3} \right]_{-1}^3 = \frac{1}{2}\left(-\frac{1}{81} - \frac{1}{3} \right)$

$= \frac{1}{2} \left(-\frac{1}{81} - \frac{27}{81} \right) = \frac{1}{\cancel{2}} \left(-\frac{\cancel{28}^{14}}{81} \right) = -\frac{14}{81}$

(2) a. $\int_0^1 (x^2+2)^3 \cdot 2x \, dx \quad , u = x^2 + 2$

Methode 1 : $\int (x^2+2)^3 \cdot 2x \, dx$ betrachten

Subst. $u = x^2 + 2 \quad , \quad g'(x) = \frac{du}{dx} = 2x \implies dx = \frac{du}{2x}$

$\int u^3 \cdot \cancel{2x} \cdot \frac{du}{\cancel{2x}} = \int u^3 \, du = \frac{1}{4} u^4 + C_1$

Resubst: $u = x^2 + 2$

$\int (x^2+2)^3 \cdot 2x \, dx = \frac{1}{4} (x^2+2)^4 + C_1$

$\int_0^1 (x^2+2)^3 \cdot 2x \, dx = \left[\frac{1}{4} (x^2+2)^4 \right]_0^1 = \frac{1}{4} (3)^4 - \frac{1}{4} \cdot 2^4$

$= \frac{81}{4} - \frac{16}{4} = \frac{65}{4}$

b. $\int_0^\pi 3x \sin x \, dx$

Methode 1 : $\int 3x \sin x \, dx \;\Big|\; \begin{matrix} u' = \sin x & u = -\cos x \\ v = 3x & v' = 3 \end{matrix}$

$\int 3x \sin x \, dx = -3x \cos x - \int -3 \cos x \, dx = -3x\cos x + 3 \int \cos x \, dx$

$= -3x \cos x + 3 \sin x + C_1$

$\int_0^\pi 3x \sin x \, dx = \left[-3x\cos x + 3 \sin x \right]_0^\pi = -3\pi \overbrace{\cos(\pi)}^{-1} + 3 \overbrace{\sin(\pi)}^{0} -$

$(-3 \cdot 0 \cdot \underbrace{\cos(0)}_{1} + 3 \cdot \underbrace{\sin(0)}_{0}) = 3\pi - 0 = 3\pi$

c. $\int_{\frac{\pi}{2}}^{\pi} \cos(3x) \, dx \quad , u = 3x$

Methode 2 :

Subst: $u = 3x = g(x) \quad , \quad g'(x) = \frac{du}{dx} = 3 \Rightarrow dx = \frac{du}{3}$

$x_u = \frac{\pi}{2} \implies u_u = 3 \frac{\pi}{2}$

$x_o = \pi \implies u_o = 3 \cdot \pi$

$\int_{\frac{3\pi}{2}}^{3\pi} \cos(u) \cdot \frac{du}{3} = \frac{1}{3} \int_{\frac{3\pi}{2}}^{3\pi} \cos(u) \, du$

$$\frac{1}{3}\int_{\frac{3\pi}{2}}^{3\pi}\cos(u)\,du = \frac{1}{3}\left[\sin(u)\right]_{\frac{3\pi}{2}}^{3\pi} = \frac{1}{3}\left(\overset{\nearrow 0}{\sin(3\pi)} - \overset{\nearrow -1}{\sin\left(\frac{3\pi}{2}\right)}\right)$$

$$= \frac{1}{3}(0+1) = \frac{1}{3}$$

d. $\int_0^1 x e^x\,dx$

Methode 2: $\left| \begin{matrix} u' = e^x & u = e^x \\ v = x & v' = 1 \end{matrix} \right.$

$$\int_0^1 x e^x\,dx = \left[x\cdot e^x\right]_0^1 - \int_0^1 1\cdot e^x\,dx = e^1 - 0 - \left[e^x\right]_0^1$$

$$= e - (e-1) = e - e + 1 = 1$$

(3) $f(x) = -x^3 + 5x$, $f(-x) = -(-x)^3 + 5\cdot(-x)$

$$= x^3 - 5x = -(-x^3+5x) = -f(x)$$

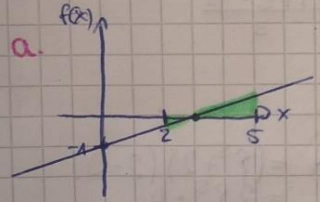
$\Rightarrow$ ungerade Funktion

$\int_{-a}^{a} f(x)\,dx = 0$, wenn $f(x)$ ungerade Funktion

$$\Rightarrow \int_{-3}^{3}(-x^3+5x)\,dx = 0$$

Skizze immer sinnvoll!

(4) a.



$f(x) = 0$ Nullstelle

$\frac{1}{3}x - 1 = 0$

$x = 3$

$$\int_2^5 |f(x)|\,dx = \int_2^3 |f(x)|\,dx + \int_3^5 |f(x)|\,dx$$

$$= \int_2^3 -f(x)\,dx + \int_3^5 f(x)\,dx$$

$$= -\int_2^3 \frac{1}{3}x - 1\,dx + \int_3^5 \frac{1}{3}x - 1\,dx$$

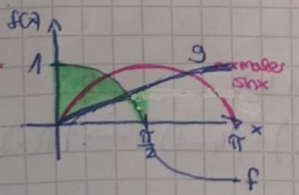
$$= -\left[\frac{1}{6}x^2 - x\right]_2^3 + \left[\frac{1}{6}x^2 - x\right]_3^5$$

$$= -\left[\left(\frac{9}{6} - 3\right) - \left(\frac{4}{6} - 2\right)\right] + \left(\frac{25}{6} - 5\right) - \left(\frac{9}{6} - 3\right)$$

$$= -\frac{9-18}{6} + \frac{4-12}{6} + \frac{25-30}{6} - \frac{9-18}{6}$$

$$= \frac{9}{6} - \frac{8}{6} - \frac{5}{6} + \frac{9}{6} = \frac{9-8-5+9}{6} = \frac{5}{6}$$

b.



$f(x) = g(x)$

$\cos(x) = \sin\left(\frac{1}{2}x\right)$

$\Leftrightarrow x = \frac{\pi}{3}$, $f(x) = g(x) = \frac{1}{2}$

Schnittpunkt

$$\int_0^{\pi/2} |f(x) - g(x)|\,dx = \int_0^{\pi/3} |f(x) - g(x)|\,dx + \int_{\pi/3}^{\pi/2} |f(x) - g(x)|\,dx$$

$$= \int_0^{\pi/3} \cos x - \sin\left(\frac{x}{2}\right)dx + \int_{\pi/3}^{\pi/2} \sin\left(\frac{x}{2}\right) - \cos x\,dx$$

$$= \left[\sin x + 2\cos\left(\frac{x}{2}\right)\right]_0^{\pi/3} + \left[-2\cos\left(\frac{x}{2}\right) - \sin x\right]_{\frac{\pi}{3}}^{\pi/2}$$

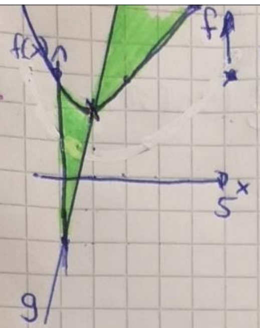
$$= \left(\overset{\frac{\sqrt{3}}{2}}{\sin\left(\frac{\pi}{3}\right)} + 2\cdot\overset{\frac{\sqrt{3}}{2}}{\cos\left(\frac{\pi}{6}\right)}\right) - \left(\overset{\nearrow 0}{\sin(0)} + 2\cdot\overset{\nearrow 1}{\cos(0)}\right)$$

$$+ \left(\left(-2\cdot\underset{\frac{\sqrt{2}}{2}}{\cos\left(\frac{\pi}{4}\right)}\right) - \underset{1}{\sin\left(\frac{\pi}{2}\right)}\right) - \left(-2\cdot\underset{\frac{\sqrt{3}}{2}}{\cos\left(\frac{\pi}{6}\right)} - \underset{\frac{\sqrt{3}}{2}}{\sin\left(\frac{\pi}{3}\right)}\right)$$

$$= \frac{\sqrt{3}}{2} + \sqrt{3} - 2 - \sqrt{2} - 1 + \sqrt{3} + \frac{\sqrt{3}}{2}$$

$$= 3\sqrt{3} - \sqrt{2} - 3$$

c.



ungenau gezeichnet, Hauptsache quadr. Fkt.

$f(x)=g(x)$

$x^2-2x+3=4x-2$

$x^2-6x+5=0$

$x_{1,2}=3\pm\sqrt{9-5}=3\pm2$

$x_1=1 \quad x_2=5$ Schnittpunkte

$$\int_0^5 |f(x)-g(x)|\,dx=\int_0^1 |f(x)-g(x)|\,dx+\int_1^5 |f(x)-g(x)|\,dx$$

$$=\int_0^1 (x^2-2x+3-(4x-2))\,dx+\int_1^5 (4x-2-(x^2-2x+3))\,dx$$

$$=\int_0^1 x^2-6x+5\,dx+\int_1^5 -x^2+6x-5\,dx$$

$$=\left[\tfrac{1}{3}x^3-3x^2+5x\right]_0^1+\left[-\tfrac{1}{3}x^3+3x^2-5x\right]_1^5$$

$$=\left(\tfrac{1}{3}-3+5\right)-(0)+\left(-\tfrac{125}{3}+\tfrac{75}{1}-25\right)-\left(-\tfrac{1}{3}+3-5\right)$$

$$=\frac{1-9+15}{3}+\frac{-125+225-75}{3}-\frac{-1+9-15}{3}$$

$$=\frac{7+25+7}{3}=\frac{39}{3}=13$$

5 Nach dem MWS gilt: $\int_0^2 x^5+2\,dx=(\xi^5+2)(2-0)$
$=2\xi^5+4$

$$\int_0^2 x^5+2\,dx=\left[\tfrac{1}{6}x^6+2x\right]_0^2=\frac{64}{6}+4-(0)$$

$$\Longrightarrow 2\xi^5\cancel{+4}=\frac{\cancel{64}^{32}}{\cancel{6}_3}\cancel{+4} \quad |:2$$

$$\xi^5=\frac{\cancel{32}^{16}}{\cancel{6}_3}=\frac{16}{3}$$

$$\xi=\sqrt[5]{\frac{16}{3}}$$

6 Nach dem MWS gilt: $\exists\xi\in[1,4]$:

$$\int_1^4 1/2\,x^2+3\,dx=\left(\tfrac{1}{2}\cdot\xi^2+3\right)\cdot(4-1)$$ $\longrightarrow$ $|f(x)|\le M$ herausfinden

da ξ^2 wächst mit größerem ξ:

$$|f(\xi)|=\left|\tfrac{1}{2}\cdot\xi^2+3\right|$$

$$\tfrac{1}{2}\cdot\xi^2+3\underset{\xi=4}{\le}\tfrac{16}{2}+3=11$$

$$\tfrac{1}{2}\cdot\xi^2+3\underset{\xi=1}{\ge}\tfrac{1}{2}+3=3{,}5$$

Abschätzung: $3{,}5\cdot(4-1)\le\int_1^4 \tfrac{1}{2}x^2+3\,dx\le 11\cdot(4-1)$

$10{,}5\le\int_1^4 \tfrac{1}{2}x^2+3\,dx\le 33$ [tatsächlich: $\frac{117}{6}$]