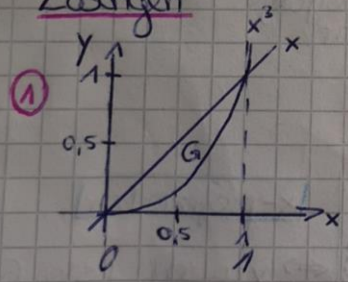
Lösungen

1



x zwischen festen Grenzen
y zwischen variablen Grenzen
→ y-projizierbar
→ erst nach y integrieren

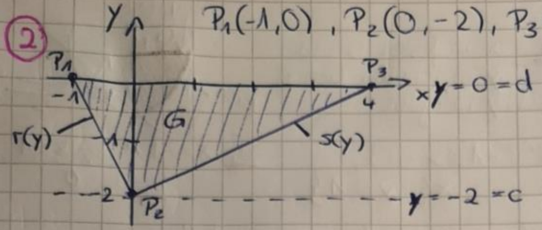
$$\int_0^1 \int_{x^3}^{x} (x^2+y)\,dy\,dx = \int_0^1 \left[x^2 y + \frac{1}{2}y^2\right]_{y=x^3}^{y=x} dx$$

$$= \int_0^1 \left(x^3 + \frac{1}{2}x^2 - \left(x^5 + \frac{1}{2}x^6\right)\right) dx$$

$$= \left[\frac{1}{4}x^4 + \frac{1}{6}x^3 - \frac{1}{6}x^6 - \frac{1}{14}x^7\right]_0^1 = \left[-\frac{1}{14}x^7 - \frac{1}{6}x^6 + \frac{1}{4}x^4 + \frac{1}{6}x^3\right]_0^1$$

$$= -\frac{1}{14} - \cancel{\frac{1}{6}} + \frac{1}{4} + \cancel{\frac{1}{6}} - (0+0+0+0) = \frac{14}{56} - \frac{4}{56} = \frac{10}{56}$$

② $P_1(-1,0)$, $P_2(0,-2)$, $P_3(4,0)$



→ x-projizierbar,
da x in variablen Grenzen
y in festen Grenzen
→ erst nach x integrieren

$$G=\{(x,y)\in\mathbb{R}^2: -2\le y\le 0,\ r(y)\le x\le s(y)\}$$

r(y): Gerade durch P_1 und P_2

$$\frac{y-y_1}{x-x_1}=\frac{y_2-y_1}{x_2-x_1}:\quad \frac{y-0}{x+1}=\frac{-2-0}{0+1}$$

$$x=\frac{0+1}{-2-0}(y-0)+(-1)$$

$$x=-\frac{1}{2}y-1$$

s(y): Gerade durch P_2 und P_3

$$x=\frac{4-0}{0+2}(y+2)+0=2(y+2)=2y+4$$

$$G=\{(x,y)\in\mathbb{R}^2: -2\le y\le 0,\ -\tfrac{1}{2}y-1\le x\le 2y+4\}$$

$$S=\iint_G 3y\,d(x,y)=\int_{-2}^{0}\int_{-\frac{1}{2}y-1}^{2y+4}3y\,dx\,dy$$

$$=\int_{-2}^{0}\left[3xy\right]_{x=-1/2y-1}^{x=2y+4}dy=\int_{-2}^{0}3y(2y+4)-3y\left(-\tfrac{1}{2}y-1\right)dy$$

$$=\int_{-2}^{0}6y^2+12y+\tfrac{3}{2}y^2+3y\,dy=\int_{-2}^{0}\tfrac{15}{2}y^2+15y\,dy$$

$$=\left[\tfrac{15}{8}y^3+\tfrac{15}{2}y^2\right]_{-2}^{0}=\left[\tfrac{5}{2}y^3+\tfrac{15}{2}y^2\right]_{-2}^{0}$$

$$=0-\left(\tfrac{5}{2}\cdot(-8)+\tfrac{15}{2}\cdot 4\right)=20-30=-10$$

③ $$\int_0^{\pi}\int_1^{2}\int_{-2}^{0}(\sin(x)+xy-2z^2)\,dz\,dy\,dx$$

$$=\int_0^{\pi}\int_1^{2}\left[\sin(x)z+xyz-\tfrac{2}{3}z^3\right]_{-2=z}^{0=z}dy\,dx$$

$$=\int_0^{\pi}\int_1^{2}0-\left(-2\sin(x)-2xy+\tfrac{16}{3}\right)dy\,dx$$

$$=\int_0^{\pi}\int_1^{2}\left(2\sin(x)+2xy-\tfrac{16}{3}\right)dy\,dx$$

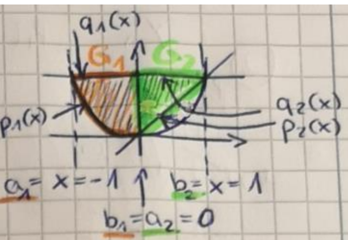
$$=\int_0^{\pi}\left[2\sin(x)y-\tfrac{16}{3}y+xy^2\right]_{1=y}^{2=y}dx$$

$$=\int_0^{\pi}\left(4\sin(x)-\tfrac{32}{3}+4x-\left(2\sin(x)-\tfrac{16}{3}+x\right)\right)dx$$

$$=\int_0^{\pi}\left(2\sin(x)+3x-\tfrac{16}{3}\right)dx=\left[-2\cos(x)+\tfrac{3}{2}x^2-\tfrac{16}{3}x\right]_0^{\pi}$$

$$=-2\cdot(-1)+\tfrac{3}{2}\pi^2-\tfrac{16}{3}\pi-(-2+0-0)=\tfrac{3}{2}\pi^2-\tfrac{16}{3}\pi+4$$

④ **a** y-projizierbar: x in festen Grenzen
y in variablen Grenzen
→ erst nach y integrieren



$G_1 = \{(x,y) \in \mathbb{R}^2 : -1 \le x \le 0,\ x^2 \le y \le 1\}$

$G_2 = \{(x,y) \in \mathbb{R}^2 : 0 \le x \le 1,\ x \le y \le 1\}$

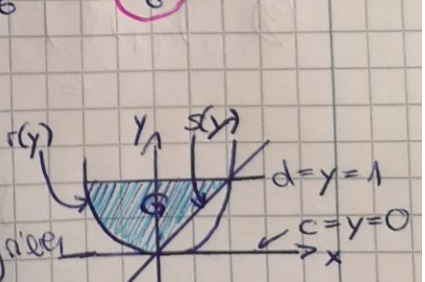
$\longrightarrow G = G_1 + G_2$

$$S_1 = \int_{-1}^{0}\int_{x^2}^{1} (2y + 3x + 2)\, dy\, dx$$
$$= \int_{-1}^{0} \left[y^2 + 3xy + 2y\right]_{y=x^2}^{y=1} dx$$
$$= \int_{-1}^{0} 1 + 3x + 2 - (x^4 + 3x^3 + 2x^2)\, dx$$
$$= \int_{-1}^{0} (-x^4 - 3x^3 - 2x^2 + 3x + 3)\, dx$$
$$= \left[-\frac{1}{5}x^5 - \frac{3}{4}x^4 - \frac{2}{3}x^3 + \frac{3}{2}x^2 + 3x\right]_{-1}^{0}$$
$$= 0 - \left(\frac{1}{5} - \frac{3}{4} + \frac{2}{3} + \frac{3}{2} - 3\right)$$
$$= -\left(\frac{12 - 45 + 40 + 90 - 180}{60}\right) = \frac{83}{60}$$

$$S_2 = \int_{0}^{1}\int_{x}^{1} (2y + 3x + 2)\, dy\, dx$$
$$= \int_{0}^{1} \left[y^2 + 3xy + 2y\right]_{y=x}^{y=1} dx$$
$$= \int_{0}^{1} 1 + 3x + 2 - (x^2 + 3x^2 + 2x)\, dx$$
$$= \int_{0}^{1} -4x^2 + x + 3\, dx$$
$$= \left[-\frac{4}{3}x^3 + \frac{1}{2}x^2 + 3x\right]_{0}^{1}$$
$$= -\frac{4}{3} + \frac{1}{2} + 3 - (0 + 0 + 0)$$
$$= -\frac{8}{6} + \frac{3}{6} + \frac{18}{6} = \frac{18 + 3 - 8}{6} = \frac{13}{6}$$

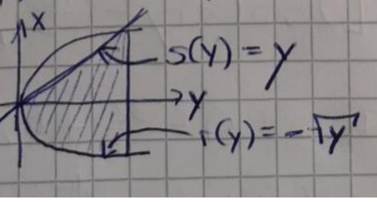
$$S = \frac{83}{60} + \frac{130}{60} = \frac{213}{60} = \frac{71}{20}$$

b x-projizierbar: y feste Grenzen
x variable Grenzen
→ erst nach x integrieren



$G = \{(x,y) \in \mathbb{R}^2 : 0 \le y \le 1,\ r(y) \le y \le s(y)\}$

Skizze (drehen)



$$S = \int_0^1 \int_{-\sqrt{y}}^{y} (2y + 3x + 2)\, dx\, dy$$

$$= \int_0^1 \left[2xy + \frac{3}{2}x^2 + 2x \right]_{x=-\sqrt{y}}^{x=y} dy$$

$$= \int_0^1 \left(2y^2 + \frac{3}{2}y^2 + 2y - \left(-2y \cdot \sqrt{y} + \frac{3}{2}y - 2\sqrt{y} \right) \right) dy$$

$$= \int_0^1 \left(\frac{7}{2}y^2 + \frac{1}{2}y + \underbrace{2y \cdot \sqrt{y}}_{2 \cdot y^{3/2}} + \underbrace{2\sqrt{y}}_{2 \cdot y^{1/2}} \right) dy$$

$$= \left[\frac{7}{6}y^3 + \frac{1}{4}y^2 + \frac{4}{5}y^{5/2} + \frac{4}{3}y^{3/2} \right]_0^1$$

$$= \left[\frac{7}{6}y^3 + \frac{1}{4}y^2 + \frac{4}{5}\sqrt{y^5} + \frac{4}{3}\sqrt{y^3} \right]_0^1$$

$$= \frac{7}{6} + \frac{1}{4} + \frac{4}{5} + \frac{4}{3} - (0)$$

$$= \frac{70 + 15 + 48 + 80}{60} = \frac{213}{60} = \frac{71}{20}$$ (Probe: $= S$ aus **a**)