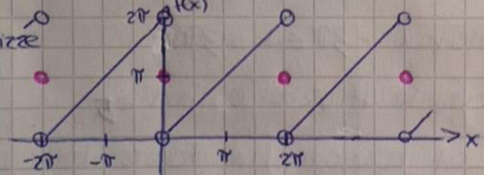
Lösung

1) $f(x) = x \quad , \quad 0 < x < 2\pi$

Skizze



a) Funktionswert an den Nahtstellen muss dem Wert entsprechen, gegen den die Fourier-Reihe konvergiert: dem arithmetischen Mittel aus links- und rechtsseitigem Grenzwert, da die Nahtstellen unstetige Stellen sind

Nahtstelle $f(0) \neq f(2\pi) \neq \lim\limits_{\substack{h \to 0 \\ h > 0}} \frac{f(0-h)+f(0+h)}{2} = \frac{2\pi + 0}{2} = \underline{\pi}$

$f(x) = \begin{cases} x, & 0 < x < 2\pi \\ \pi, & x = 0 \end{cases}$ damit MWE erfüllt

b) Symmetrie? Nein. $\left(\begin{array}{l} f(-x) \neq f(x) \\ f(-x) \neq -f(x) \end{array} \right)$; Integrationsbereich: $0 \ldots 2\pi$

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{2\pi}\int_0^{2\pi} x\,dx = \frac{1}{2\pi}\left[\frac{1}{2}x^2\right]_0^{2\pi} = \frac{1}{2\pi}\left(\frac{1}{2}\cdot 4\pi^2 - \frac{1}{2}\cdot 0\right)$$

$$= \frac{1}{2\pi}\cdot\frac{1}{2}\cdot 4\pi^2 = \pi$$

mit $\int x\cos(kx)\,dx$ partiell integriert $= \frac{\cos(kx)}{k^2} + \frac{x\sin(kx)}{k} + C$

$$a_k = \frac{1}{\pi}\int_0^{2\pi} x\cos(kx)\,dx = \frac{1}{\pi}\left[\frac{\cos(kx)}{k^2} + \frac{x\sin(kx)}{k}\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left(\frac{\overset{\to 1}{\cos(k\cdot 2\pi)}}{k^2} + \frac{\overset{\to 0}{2\pi\cdot\sin(k\cdot 2\pi)}}{k} - \left(\frac{\overset{\to 1}{\cos(0)}}{k^2} + 0\right)\right)$$

$$= \frac{1}{\pi}\left(\frac{1}{k^2} + 0 - \frac{1}{k^2}\right) = 0$$

$$b_k = \frac{1}{\pi}\int_0^{2\pi} x\sin(kx)\,dx = \frac{1}{\pi}\left[\frac{\sin(kx)}{k^2} - \frac{x\cos(kx)}{k}\right]_0^{2\pi}$$
$$= \frac{1}{\pi}\left(\frac{\overbrace{\sin(k\cdot 2\pi)}^{\to 0}}{k^2} - \frac{2\pi\cdot\overbrace{\cos(2\pi\cdot k)}^{\to 1}}{k} - \left(\frac{\overbrace{\sin(0)}^{\to 0}}{k^2} - 0\right)\right)$$
$$= \frac{1}{\pi}\left(0 - \frac{2\pi}{k} - 0\right)$$
$$= -\frac{2\pi}{\pi k} = -\frac{2}{k}$$

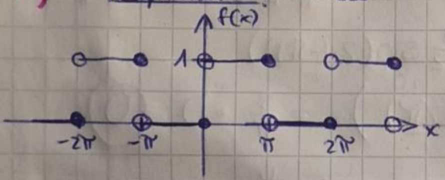
mit $\int x\sin(kx)dx$ partiell integriert
$= \frac{\sin(kx)}{k^2} - \frac{x\cos(kx)}{k} + C$

Fourier-Reihe von f: $\sim \pi + \sum_{k=1}^{\infty}\left(-\frac{2}{k}\cdot\sin(kx)\right)$

2

$$f(x) = \begin{cases} 0, & -\pi < x \leq 0 \\ 1, & 0 < x \leq \pi \end{cases}$$

a) Symmetrie? Skizze:



keine Symmetrie, alle Koeffizienten berechnen

Integrationsbereich: $-\pi \ldots \pi$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi}\left(\int_{-\pi}^{0} 0\,dx + \int_0^{\pi} 1\,dx\right) = \frac{1}{2\pi}\int_0^{\pi} dx$$
$$= \frac{1}{2\pi}\left[x\right]_0^{\pi} = \frac{1}{2\pi}(\pi - 0) = \frac{1}{2}$$

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(kx)\,dx = \frac{1}{\pi}\left(\int_{-\pi}^{0} 0\cos(kx)dx + \int_0^{\pi} 1\cos(kx)dx\right)$$
$$= \frac{1}{\pi}\left[\frac{1}{k}\sin(kx)\right]_0^{\pi} = \frac{1}{k\pi}\left[\sin(kx)\right]_0^{\pi} = \frac{1}{k\pi}\left(\overbrace{\sin(k\pi)}^{\to 0} - \overbrace{\sin(0)}^{\to 0}\right)$$
$$= \frac{1}{k\pi}\cdot 0 = 0$$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(kx)\,dx = \frac{1}{\pi}\int_0^{\pi}\sin(kx)dx$$
$$= -\frac{1}{k\pi}\left[\cos(kx)\right]_0^{\pi} = -\frac{1}{k\pi}\left(\overbrace{\cos(\pi\cdot k)}^{\to(-1)^k} - \overbrace{\cos(0)}^{\to 1}\right)$$
$$= -\frac{1}{k\pi}\left((-1)^k - 1\right) = \frac{1}{k\pi}\cdot\left(1 - (-1)^k\right) = \begin{cases} 0, & k \text{ gerade} \\ \frac{2}{k\pi}, & k \text{ ungerade} \end{cases}$$

b) MWE an $x_0 = 0$ erfüllt $\Leftrightarrow f(x_0) = \lim\limits_{\substack{h\to 0 \\ h>0}} \frac{f(x_0-h)+f(x_0+h)}{2}$

$$f(x_0) = 0 \neq \lim_{\substack{h\to 0 \\ h>0}} \frac{f(-h)+f(h)}{2} = \frac{0+1}{2} = 1/2$$

$\to$ erfüllt nicht MWE an $x_0 = 0$

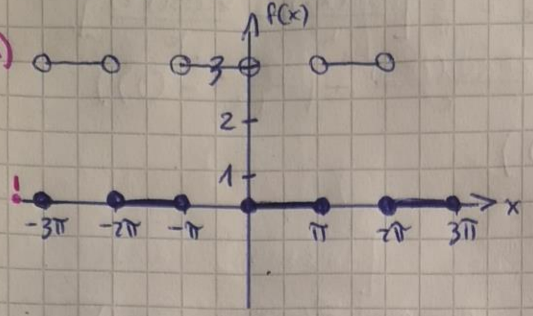
dasselbe für $x_1 = \pi$

$$f(\pi) = 1 \neq \lim_{\substack{h\to 0 \\ h>0}} \frac{f(\pi-h)+f(\pi+h)}{2} = \frac{1+0}{2} = 1/2$$

$\Rightarrow$ MWE nicht erfüllt

③ $f(x) = \begin{cases} 3 & , -\pi < x < 0 \\ 0 & , 0 \le x \le \pi \end{cases}$

a)



b) keine Symmetrie, Integrationsbereich: $-\pi \ldots \pi$

$$a_2 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(2x)\,dx = \frac{1}{\pi}\left(\int_{-\pi}^{0} 3\cos(2x)\,dx + \int_{0}^{\pi} 0 \cos(2x)\,dx\right)$$

$$= \frac{1}{\pi} \cdot 3 \cdot \int_{-\pi}^{0} \cos(2x)\,dx = \frac{3}{\pi}\left[\tfrac{1}{2}\sin(2x)\right]_{-\pi}^{0}$$

$$= \frac{3}{\pi}\left(\frac{1}{2}\cdot \underbrace{\sin(0)}_{\to 0} - \frac{1}{2}\sin(2\cdot(-\pi))\right)$$

$$= \frac{3}{\pi}\left(0 + \frac{1}{2}\underbrace{\sin(2\pi)}_{\to 0}\right) = \frac{3}{\pi} \cdot 0 = 0$$

$$b_5 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(5x)\,dx = \frac{1}{\pi}\int_{-\pi}^{0} 3 \cdot \sin(5x)\,dx$$

$$= \frac{3}{\pi}\left[-\frac{1}{5}\cos(5x)\right]_{-\pi}^{0} = \frac{3}{\pi}\left(-\frac{1}{5}\underbrace{\cos(0)}_{\to 1} + \frac{1}{5}\cos(-5\pi)\right)$$

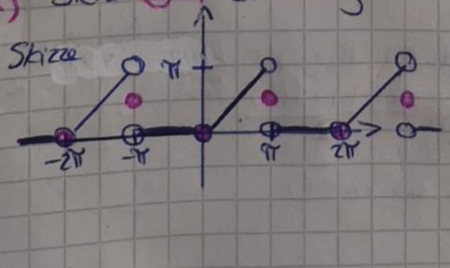
$$= \frac{3}{\pi}\left(-\frac{1}{5} + \frac{1}{5}\underbrace{\cos(5\pi)}_{\to -1}\right) = \frac{3}{\pi}\left(-\frac{1}{5} - \frac{1}{5}\right) = \frac{3}{\pi}\left(-\frac{2}{5}\right) = -\frac{6}{5\pi}$$

c) Die Fourier-Reihe konvergiert immer gegen das arithm. Mittel aus links- und rechtsseitigem Grenzwert.

$$\lim_{\substack{h \to 0 \\ h > 0}} \frac{f(-h) + f(h)}{2} = \frac{3+0}{2} = 3/2$$

④ $f(x) = \begin{cases} x & , 0 < x < \pi \\ 0 & , \pi < x < 2\pi \end{cases}$

a) siehe ①a) Erklärung



Nahtstellen $x_0 = 0$ (oder 2π)

$$f(x_0) = \lim_{\substack{h \to 0 \\ h > 0}} \frac{f(-h) + f(h)}{2} = \frac{0+0}{2} = 0$$

und $x_1 = \pi$

$$f(x_1) = \lim_{\substack{h \to 0 \\ h > 0}} \frac{f(\pi - h) + f(\pi + h)}{2} = \frac{\pi + 0}{2} = \frac{\pi}{2}$$

b) Symmetrie? Nein.

Integrationsbereich: $0 \ldots 2\pi$

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{2\pi}\left(\int_0^{\pi} x\,dx + \int_{\pi}^{2\pi} 0\,dx\right) = \frac{1}{2\pi}\left[\frac{1}{2}x^2\right]_0^{\pi}$$
$$= \frac{1}{2\pi}\left(\frac{1}{2}\pi^2 - 0\right) = \pi/4$$

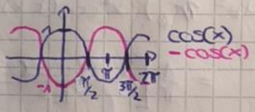
$$a_k = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos(kx)\,dx = \frac{1}{\pi}\int_0^{\pi} x\cos(kx)\,dx$$
$$= \frac{1}{\pi}\left[\frac{\cos(kx)}{k^2} + \frac{x\sin(kx)}{k}\right]_0^{\pi} = \frac{1}{\pi}\left(\frac{\overset{\nearrow(-1)^k}{\cos(\pi k)}}{k^2} + \frac{\pi\cdot\overset{\nearrow 0}{\sin(k\pi)}}{k} - \left(\frac{\overset{\nearrow 1}{\cos(0)}}{k^2} + 0\right)\right)$$
$$= \frac{1}{\pi}\left(\frac{(-1)^k}{k^2} + 0 - \frac{1}{k^2}\right) = \frac{1}{\pi}\cdot\frac{(-1)^k - 1}{k^2} = \begin{cases} 0, & k \text{ gerade} \\ \frac{-2}{\pi k^2}, & k \text{ ungerade}\end{cases}$$

$$b_k = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin(kx)\,dx = \frac{1}{\pi}\int_0^{\pi} x\sin(kx)\,dx$$
$$= \frac{1}{\pi}\left[\frac{\sin(kx)}{k^2} - \frac{x\cos(kx)}{k}\right]_0^{\pi} = \frac{1}{\pi}\left(\frac{\overset{\nearrow 0}{\sin(\pi k)}}{k^2} - \frac{\pi\overset{\nearrow(-1)^k}{\cos(k\pi)}}{k} - \left(\frac{\overset{\nearrow 0}{\sin(0)}}{k^2} - 0\right)\right)$$
$$= \frac{1}{\pi}\left(-\frac{\pi\cdot(-1)^k}{k}\right) = \frac{1}{\pi}\cdot\frac{\pi\cdot(-(-1)^k)}{k} = \frac{(-1)^{k+1}}{k} \quad \begin{cases} -1/k, & k \text{ gerade} \\ 1/k, & k \text{ ungerade}\end{cases}$$

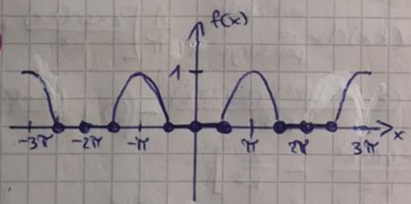
$$f \sim \frac{\pi}{4} + \sum_{k=1}^{\infty}\left[\frac{1}{\pi k^2}\left((-1)^k - 1\right)\cos(kx) + \frac{1}{k}(-1)^{k+1}\sin(kx)\right]$$

5

$$f(x) = \begin{cases} 0, & 0 \le x < \pi/2 \\ \cos(x), & \pi/2 \le x \le 3\pi/2 \\ 0, & 3\pi/2 < x < 2\pi \end{cases}$$



a)



b) symmetrisch zur y-Achse → $f(-x) = f(x)$ gerade Funktion $\Rightarrow b_k = 0$ → c) alle $b_k = 0$

Integrationsbereich: $0 \ldots \pi$ reicht

$$a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\,dx = \frac{1}{\pi}\left(\int_0^{\pi/2} 0\,dx + \int_{\pi/2}^{\pi} -\cos(x)\,dx\right)$$
$$= -\frac{1}{\pi}\int_{\pi/2}^{\pi}\cos(x)\,dx = -\frac{1}{\pi}\left[\sin(x)\right]_{\pi/2}^{\pi}$$
$$= -\frac{1}{\pi}\left(\overset{\nearrow 0}{\sin(\pi)} - \overset{\nearrow 1}{\sin(\pi/2)}\right) = -\frac{1}{\pi}(0 - 1) = \frac{1}{\pi}$$

NR partielle Integration

$\int \cos x \cdot \cos x\,dx$

$\begin{bmatrix} u' = \cos x & u = \sin x \\ v = \cos x & v' = -\sin x \end{bmatrix}$

$= \sin x\cos x + \int \sin^2 x\,dx$

$= \sin x\cos x + \int 1 - \cos^2 x\,dx$

$\Rightarrow 2\cdot\int \cos^2 x\,dx = \sin x\cos x + \int 1\,dx$

$\int \cos^2 x\,dx = \frac{\sin x\cos x}{2} + \frac{x}{2} + C$

$$a_1 = \frac{2}{\pi}\cdot\int_0^{\pi} f(x)\cos(x)\,dx = \frac{2}{\pi}\cdot\int_{\pi/2}^{\pi} -\cos(x)\cdot\cos(x)\,dx$$
$$= -\frac{2}{\pi}\int_{\pi/2}^{\pi}\cos^2(x) \overset{\boxed{NR}}{=} -\frac{2}{\pi}\left[\frac{\sin x\cos x}{2} + \frac{x}{2}\right]_{\pi/2}^{\pi}$$
$$= -\frac{2}{\pi}\left(\frac{\overset{\nearrow 0}{\sin(\pi)}\overset{\nearrow -1}{\cos(\pi)}}{2} + \frac{\pi}{2} - \left(\frac{\overset{\nearrow 1}{\sin(\pi/2)}\overset{\nearrow 0}{\cos(\pi/2)}}{2} + \frac{\pi}{4}\right)\right) = -\frac{2}{\pi}\left(\frac{\pi}{2} - \frac{\pi}{4}\right) = -\frac{2}{\pi}\cdot\frac{\pi}{4}$$
$$= -1/2$$

$a_5 = \frac{2}{\pi}\int_0^{\pi} f(x)\cos(5x)\,dx = \frac{2}{\pi}\int_{\pi/2}^{\pi} -\cos(x)\cdot\cos(5x)\,dx$

$= -\frac{1}{\pi}\int_{\pi/2}^{\pi} 2\cdot\cos x\cdot\cos(5x)\,dx = -\frac{1}{\pi}\left(\int_{\pi/2}^{\pi}\cos((5+1)x)\,dx + \int_{\pi/2}^{\pi}\cos((5-1)x)\,dx\right)$

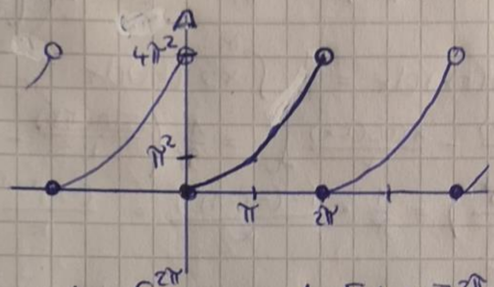
$= -\frac{1}{\pi}\left(\left[\frac{1}{6}\sin(6x)\,dx\right]_{\pi/2}^{\pi} + \left[\frac{1}{4}\sin(4x)\right]_{\pi/2}^{\pi}\right)$

$= -\frac{1}{\pi}\left(\frac{1}{6}\underbrace{\sin(6\pi)}_{\nearrow 0} - \frac{1}{6}\underbrace{\sin(3\pi)}_{\nearrow 0} + \frac{1}{4}\underbrace{\sin(4\pi)}_{\nearrow 0} - \frac{1}{4}\underbrace{\sin(2\pi)}_{\nearrow 0}\right)$

$= 0$

⑥ $f(x) = x^2,\ 0 \leq x < 2\pi$

a)



keine Symmetrie

Integrationsbereich: $0 .. 2\pi$

$a_0 = \frac{1}{2\pi}\int_0^{2\pi} x^2\,dx = \frac{1}{2\pi}\left[\frac{1}{3}x^3\right]_0^{2\pi} = \frac{1}{\cancel{2\pi}}\left(\frac{1}{3}\cdot\cancel{8}^{4}\cancel{\pi^3}^{2} - 0\right) = \frac{4}{3}\pi^2$

$a_k = \frac{1}{\pi}\int_0^{2\pi} x^2\cos(kx)\,dx \overset{\text{mit 2facher part. Integration}}{=} \frac{1}{\pi}\left[\frac{2x\cos(kx)}{k^2} + \frac{(2-k^2x^2)\sin(kx)}{k^3}\right]_0^{2\pi}$

$= \frac{1}{\cancel{\pi}}\left[\frac{2\cdot 2\cancel{\pi}\cdot \underbrace{\cos(2\pi k)}_{\nearrow 1}}{k^2} + 0 - (0+0)\right] = \frac{4}{k^2}$

$b_k = \frac{1}{\pi}\int_0^{2\pi} x^2\sin(kx)\,dx \overset{\text{mit 2facher part. Integration}}{=} \frac{1}{\pi}\left[\frac{2x\sin(kx)}{k^2} + \frac{(2-k^2x^2)\cos(kx)}{k^3}\right]_0^{2\pi}$

$= \frac{1}{\pi}\left[0 + \frac{(2-k^2\cdot 4\pi^2)\cdot \underbrace{\cos(2\pi k)}_{\nearrow 1}}{k^3} - \left(0 + \frac{(2-0)\underbrace{\cos(0)}_{\nearrow 1}}{k^3}\right)\right]$

$= \frac{1}{\pi}\left[\frac{2-4\pi^2k^2}{k^3} - \frac{2}{k^3}\right]$

$= \frac{1}{\cancel{\pi}}\left(\frac{\cancel{2}-4\pi^{\cancel{2}}k^2-\cancel{2}}{k^3}\right) = -\frac{4\pi k^{\cancel{2}}}{k^{\cancel{3}}} = -\frac{4\pi}{k}$

Fourier-Reihe von f: $\frac{4}{3}\pi^2 + \sum_{k=1}^{\infty}\left(\frac{4}{k^2}\cos(kx) - \frac{4\pi}{k}\sin(kx)\right)$

b) $x_0 = 0$: $f(x_0) = 0 \neq \lim_{\substack{h\to 0\\ h>0}} \frac{f(-h)+f(h)}{2} = \frac{4\pi^2+0}{2} = 2\pi^2$

→ MWE gilt hier nicht

$x_1 = 1$: Hier handelt es sich um eine stetige Stelle, die Mittelwert-eigenschaft gilt hier sowieso [die Fourier-Reihe konvergiert gegen den Funktionswert $f(x_1) = 1^2 = 1$]