Lösungen

(1) a. $\int x^2+2x+\frac{1}{x}\,dx = \int x^2dx + \int 2x\,dx + \int \frac{1}{x}dx = \underline{\frac{1}{3}x^3 + x^2 + \ln|x| + C}$

b. $\int \frac{1}{\sqrt{x}}dx = \int x^{-\frac{1}{2}}dx = 2x^{\frac{1}{2}} + C = \underline{2\sqrt{x} + C}$

c. $\int \frac{10x^8+3}{x^4}dx = \int \frac{10x^{\cancel{8}4}}{\cancel{x^4}}dx + \int \frac{3}{x^4}dx = \int 10x^4dx + \int 3x^{-4}dx = \underline{2x^5 - \frac{1}{x^3} + C}$

d. $\int e^{-x}dx = \frac{1}{-1}e^{-x} + C = \underline{-e^{-x} + C}$

e. $\int \frac{1}{(x+1)^2}dx = \int (x+1)^{-2}dx = \frac{1}{-1}\cdot(x+1)^{-1}\cdot\frac{1}{1}\boxed{+C} = \underline{-\frac{1}{x+1} + C}$

f. $\int 2\cdot 3^x dx = \underline{\frac{2}{\ln(3)}\cdot 3^x + C}$

(2) a. $\int x^2\sin x\,dx \quad \begin{bmatrix} u' = \sin x & u = -\cos x \\ v = x^2 & v' = 2x \end{bmatrix}$

$= -x^2\cos x - \int -\cos x \cdot 2x\,dx = -x^2\cos x + 2\int x\cos x\,dx + C_1$

$\int x\cos x\,dx \quad \begin{bmatrix} u' = \cos x & u = \sin x \\ v = x & v' = 1 \end{bmatrix}$

$= x\sin x - \int \sin x\,dx = x\sin x + \cos x + C_2$ einsetzen

$\Rightarrow \int x^2\sin x\,dx = -x^2\cos x + 2\int x\cos x\,dx + C_1$

$= -x^2\cos x + 2(x\sin x + \cos x + C_2) + C_1$

$= -x^2\cos x + 2x\sin x + 2\cos x + \underbrace{2C_2 + C_1}_{C}$

$= \underline{(2-x^2)\cos x + 2x\sin x + C}$

b. $\int 2^x \cdot 3x\,dx \quad \begin{bmatrix} u' = 2^x & u = \frac{1}{\ln(2)} \cdot 2^x \\ v = 3x & v' = 3 \end{bmatrix}$

$= \frac{3x}{\ln(2)} \cdot 2^x - \int \frac{3}{\ln(2)} \cdot 2^x\,dx = \frac{3x}{\ln(2)} \cdot 2^x - \frac{3}{(\ln(2))^2} \cdot 2^x + C$

$= \left(x - \frac{1}{\ln(2)}\right) \frac{3}{\ln(2)} \cdot 2^x + C$

c. $\int \ln(2x) \cdot (4x^3 - 1)\,dx \quad \begin{bmatrix} u' = 4x^3 - 1 & u = x^4 - x \\ v = \ln(2x) & v' = \frac{1}{2x} \cdot 2 = \frac{1}{x} \end{bmatrix}$

$= \ln(2x)(x^4 - x) - \int \frac{x^4 - x}{x}\,dx = \ln(2x)(x^4 - x) - \int x^3 - 1\,dx$

$= \ln(2x)(x^4 - x) - \frac{1}{4}x^4 + x + C$

③ a. $\int \sin^2(x)\cos(x)\,dx$

① Subst.: $u = \sin x = g(x)$

$\frac{du}{dx} = g'(x) = \cos x \implies dx = \frac{du}{\cos x}$

② $\int u^2 \cdot \cancel{\cos x} \cdot \frac{du}{\cancel{\cos x}} = \int u^2\,du = \frac{1}{3}u^3 + C$

③ Resubst.: $u = \sin x$

$\int \sin^2(x)\cos(x)\,dx = \frac{1}{3}(\sin x)^3 + C = \frac{1}{3}\sin^3 x + C$

b. $\int \sqrt{4x-1}\,dx$

① Subst. $u = 4x - 1 = g(x)$

$\frac{du}{dx} = g'(x) = 4 \implies dx = \frac{du}{4}$

② $\int \sqrt{u} \cdot \frac{du}{4} = \frac{1}{4}\int u^{\frac{1}{2}}\,du = \frac{1}{4} \cdot \frac{1}{\frac{3}{2}} \cdot u^{\frac{3}{2}} + C = \frac{1}{2\cancel{4}} \cdot \frac{\cancel{2}}{3} \cdot \sqrt{u^3} + C$

$= \frac{1}{6} \cdot \sqrt{u^3} + C$

③ Resubst. $u = 4x - 1$

$\int \sqrt{4x-1}\,dx = \frac{1}{6} \cdot \sqrt{(4x-1)^3} + C$

c. $\int \frac{x+1}{\sqrt{x^2+2x+3}}\,dx$

① Subst. $u = x^2 + 2x + 3 = g(x)$

$\frac{du}{dx} = g'(x) = 2x + 2 \implies dx = \frac{du}{2x+2}$

② $\int \frac{1}{\sqrt{u}} \cdot (x+1) \cdot \frac{du}{2x+2} = \int \frac{1}{\sqrt{u}} \cdot \frac{\cancel{x+1}}{2(\cancel{x+1})} \cdot du = \frac{1}{2} \cdot \int u^{-\frac{1}{2}}\,du = \sqrt{u} + C$

③ Resubst. $u = x^2 + 2x + 3$

$\int \frac{x+1}{\sqrt{x^2+2x+3}}\,dx = \sqrt{x^2+2x+3} + C$

④ a. $\int 2\sin x \cdot \cos x\, dx \left[\begin{matrix} u' = \cos x & u = \sin x \\ v = 2\sin x & v' = 2\cos x \end{matrix}\right]$

$= 2\sin^2 x - \int 2\sin x \cos x\, dx + C_1$

$\Rightarrow \int 2\sin x \cdot \cos x = 2\sin^2 x - \int 2\sin x\cos x\, dx + C_1 \;\Big|+\int 2\sin x \cos x\, dx$

$2 \cdot \int 2\sin x \cos x = 2\sin^2 x + C_1 \quad |:2$

$\int 2\sin x \cos x = \sin^2 x + \frac{1}{2} C_1 = \underline{\sin^2 x + C}$

ODER

$\int 2\sin x \cos x\, dx \left[\begin{matrix} u' = 2\sin x & u = -2\cos x \\ v = \cos x & v' = -\sin x \end{matrix}\right]$

$= -2\cos^2 x - \int 2\sin x \cos x\, dx + C_1$

$\Rightarrow \int 2\sin x \cos x\, dx = -2\cos^2 x - \int 2\sin x\cos x\, dx + C_1 \;\Big|+\int 2\sin x\cos x\, dx$

$2\int 2\sin x \cos x = -2\cos^2 x + C_1 \quad |:2$

$\int 2\sin x \cos x = -\cos^2 x + \frac{1}{2} C_1 = \underline{-\cos^2 x + C}$

$\left[\begin{matrix} \sin^2(x) + C' = -\cos^2(x) + C'' & \text{mit } \sin^2(x) + \cos^2(x) = 1 \\ & \rightarrow -\cos^2(x) = \sin^2(x) - 1 \\ \sin^2(x) + C' = \sin^2(x) \underbrace{-1 + C''}_{C'} & \end{matrix}\right]$

b. $\int 2\sin x \cos x\, dx$

① Subst: $u = \cos x = g(x)$

$\frac{du}{dx} = g'(x) = -\sin x \Longrightarrow dx = \frac{du}{-\sin x}$

② $\int 2 \cdot \cancel{\sin x} \cdot u \cdot \frac{du}{-\cancel{\sin x}} = \int -2u\, du = \frac{-2}{2} u^2 + C = -u^2 + C$

③ Resubst: $\int 2\sin x \cos x\, dx = -(\cos x)^2 + C = \underline{-\cos^2 x + C}$

(Probe: siehe a)

⑤ $F(x)$ Stammfkt. von $f(x)$, wenn $F'(x) = f(x)$

bzw. $F(x) = \int f(x)\, dx$

$\int \frac{1-e^x}{1+e^x}\, dx \rightarrow$ Logarithm. Integration: $\int \frac{g'(x)}{g(x)}\, dx$

$g(x) = 1 + e^x, \quad g'(x) = e^x \qquad \left[1 - e^x \xrightarrow{+2e^x - 2e^x} \underline{e^x} - 2e^x + 1 \right]$

$\int \frac{1-e^x}{1+e^x}\, dx = \int \frac{e^x - 2e^x + 1}{1+e^x}\, dx = \int \cancel{\frac{e^x + 1}{1+e^x}} - 2 \cdot \frac{e^x}{1+e^x}\, dx$

$= \int 1 - 2 \cdot \frac{e^x}{1+e^x}\, dx = \int 1\, dx - 2 \cdot \int \frac{e^x}{1+e^x}\, dx = \underline{x - 2 \cdot (\ln(1+e^x)) + C}$

⑥ $f(x) = ?$

$f''(x) = \frac{1}{\sqrt{x-1}} = (x-1)^{-1/2}$

$f(2)=2$

$f'(2)=2$

$f'(x) = \int f''(x)dx = \int (x-1)^{-1/2} dx$

$= 2\cdot(x-1)^{1/2} + C' = 2\cdot\sqrt{x-1} + C'$

$f'(2) = 2$

$2\cdot\sqrt{2-1} + C' = 2$

$\cancel{2} + C' = \cancel{2}$

$C' = 0$

$f'(x) = 2\cdot\sqrt{x-1} + 0$

$= 2\cdot\sqrt{x-1}$

$f(x) = \int f'(x)dx = \int 2\sqrt{x-1}\, dx = \int 2\cdot(x-1)^{1/2}dx$

$= \frac{2}{3/2}(x-1)^{3/2} + C = \frac{4}{3}\cdot\sqrt{(x-1)^3} + C$

$f(2) = 2$

$4/3\cdot\sqrt{(2-1)^3} + C = 2$

$4/3 + C = 2$

$C = 2 - 4/3 = 6/3 - 4/3 = 2/3$

$f(x) = 4/3\cdot\sqrt{(x-1)^3} + 2/3$

$\underline{f(x) = 4/3\sqrt{(x-1)^3} + 2/3}$