5. Aufgabe

Für welche reellen Werte q ist das uneigentliche Integral

$$\int_0^1 \frac{1}{x^q}\,dx$$

konvergent/divergent?

Lösung uneigentliches Integral 2.Art

1 Unstetigkeitsstelle von $f(x) = \frac{1}{x^q}$

f unbeschränkt für $x \to 0$

2 Stammfunktion von f bestimmen

$$\int \frac{1}{x^q}\,dx$$

2 Stammfunktion Fallunterscheidung bzgl. q erforderlich

$$\int \frac{1}{x^q}\,dx$$

$$\int \frac{1}{x^q}\,dx = \int x^{-q}dx = \frac{x^{-q+1}}{-q+1} + C, C \in \mathbb{R}, q \neq 1$$

$$\int \frac{1}{x}\,dx = \int x^{-1}dx = \ln(|x|) + C, C \in \mathbb{R}$$

3 Bestimmtes Integral

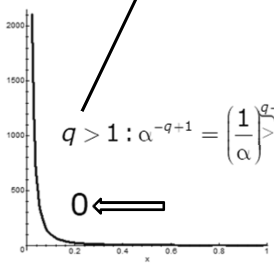
$$q \neq 1: \int_\alpha^1 \frac{1}{x^q}\,dx = \left[\frac{x^{-q+1}}{-q+1}\right]_\alpha^1 = \frac{1^{-q+1}}{-q+1} - \frac{\alpha^{-q+1}}{-q+1}$$

$$q = 1: \int_\alpha^1 \frac{1}{x^q}\,dx = [\ln(x)]_\alpha^1 = \ln(1) - \ln(\alpha)$$

④ Grenzwert

$$q \neq 1: \int_0^1 \frac{1}{x^q}\,dx = \lim_{\alpha\to 0,\alpha>0}\int_\alpha^1 \frac{1}{x^q}\,dx = \frac{1^{-q+1}}{-q+1} - \lim_{\alpha\to 0,\alpha>0}\frac{\alpha^{-q+1}}{-q+1}$$

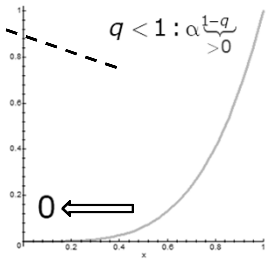
$$\boxed{\lim_{\alpha\to 0,\alpha>0}\frac{\alpha^{-q+1}}{-q+1}} = \begin{cases} \infty, q>1 \Rightarrow 0>1-q & \Rightarrow \displaystyle\int_0^1 \frac{1}{x^q}\,dx \ \text{divergent} \\ 0, q<1 \Rightarrow 0<1-q & \Rightarrow \displaystyle\int_0^1 \frac{1}{x^q}\,dx \ \text{konvergent} \end{cases}$$



$$q>1: \alpha^{-q+1} = \left(\frac{1}{\alpha}\right)^{\overbrace{q-1}^{>0}}$$

$$\underset{\alpha>0}{\alpha \to 0} \Rightarrow \frac{1}{\alpha} \to \infty$$

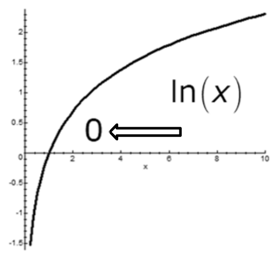
$$q<1: \alpha^{\overbrace{1-q}^{>0}}$$



$$q = 1: \int_0^1 \frac{1}{x^q}\,dx = \lim_{\alpha\to 0,\alpha>0}\int_\alpha^1 \frac{1}{x}\,dx$$

$$= \underbrace{\ln(1)}_{=0} - \boxed{\lim_{\alpha\to 0,\alpha>0}\ln(\alpha)} = -\infty$$

divergent



<u>Zusammenfassung</u>

$$\int_0^1 \frac{1}{x^q}\,dx = \begin{cases} \infty & \text{(divergent) für } q \geq 1 \\ \dfrac{1}{1-q} & \text{(konvergent) für } q < 1 \end{cases}$$