Aufgaben 3

11. Aufgabe

$$\int_1^e \frac{\ln(x)}{x \cdot \sqrt{1 + \left(\ln(x)\right)^2}} \, dx$$

$$\int \frac{\ln(x)}{x \cdot \sqrt{1 + \left(\ln(x)\right)^2}} \, dx \qquad\qquad u = \ln(x),\; du = \frac{1}{x} \, dx$$

$$\int \frac{u}{\sqrt{1 + u^2}} \, du$$

$$= \sqrt{1 + u^2} + C$$

$$\sqrt{1 + \left(\ln(x)\right)^2}$$

$$\int_1^e \frac{\ln(x)}{x \cdot \sqrt{1 + \left(\ln(x)\right)^2}} \, dx = \left[\sqrt{1 + \left(\ln(x)\right)^2} \right]_1^e$$

$$= \sqrt{1 + \left(\ln(e)\right)^2} - \sqrt{1 + \left(\ln(1)\right)^2}$$

$$= \sqrt{1 + 1^2} - \sqrt{1 + 0^2}$$

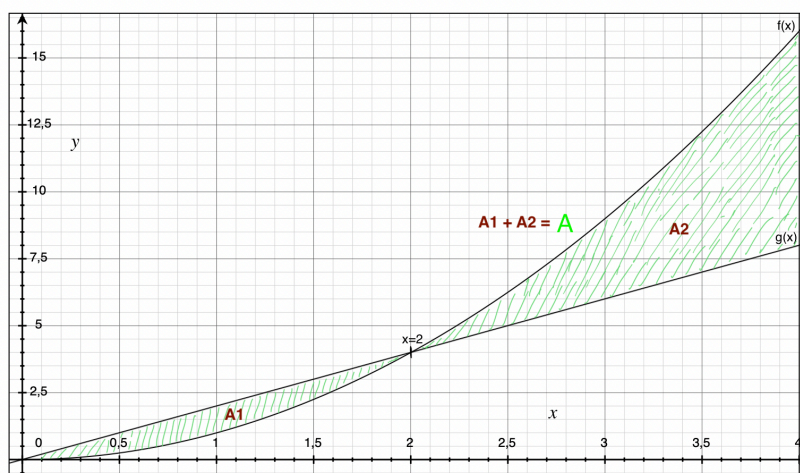$$= \sqrt{2} - \sqrt{1}$$

$$= \sqrt{2} - 1 \approx 0.4142$$

12. Aufgabe

$$f(x) = x^2, \quad g(x) = 2x \quad [0, 4]$$

$$A = A_1 + A_2$$

$$A_1 = \int_0^2 \left(g(x) - f(x) \right) \, dx$$

$$A_2 = \int_2^4 \left(f(x) - g(x) \right) \, dx$$

$$A_1 = \int_0^2 \left(g(x) - f(x) \right) \, dx$$

$$= \int_0^2 \left(2x - x^2 \right) \, dx$$

$$= \int_0^2 -x^2 + 2x \, dx$$

$$\int -x^2 + 2x \, dx$$

$$= -\frac{1}{3}x^3 + x^2 + C$$

$$= \left[-\frac{1}{3}x^3 + x^2 \right]_0^2$$

$$= \left(-\frac{1}{3}2^3 + 2^2 \right) - \left(-\frac{1}{3}0^3 + 0^2 \right)$$

$$= \left(-\frac{8}{3} + 4 \right) - (0 + 0)$$

$$= -\frac{8}{3} + \frac{12}{3} = \frac{12}{3} - \frac{8}{3} = \frac{4}{3}$$

$$A_2 = \int_2^4 \left(f(x) - g(x) \right) \, dx$$

$$= \int_2^4 x^2 - 2x \, dx$$

$$\int x^2 - 2x \, dx$$

$$= \frac{1}{3}x^3 - x^2 + C$$

$$= \left[\frac{1}{3}x^3 - x^2 \right]_2^4$$

$$= \left(\frac{1}{3}4^3 - 4^2 \right) - \left(\frac{1}{3}2^3 - 2^2 \right)$$

$$= \left(\frac{64}{3} - 16 \right) - \left(\frac{8}{3} - 4 \right)$$

$$= \left(\frac{64}{3} - \frac{48}{3} \right) - \left(\frac{8}{3} - \frac{12}{3} \right)$$

$$= \frac{16}{3} + \frac{4}{3} = \frac{20}{3}$$

$$A = A_1 + A_2 = \frac{4}{3} + \frac{20}{3} = \frac{24}{3} = 8$$

13. Aufgabe

$$\int_0^1 x \cdot \ln(x) \, dx$$

Unstetigkeitsstelle = 0, stetig im Intervall (0, 1]
Unbeschränkt für $x \to 0$

$$F(x) = \frac{1}{2} x^2 \ln(x) - \frac{1}{4} x^2 + C = x^2 \left(\frac{1}{2} \ln(x) - \frac{1}{4} + C \right)$$

$$\int_\alpha^1 x \cdot \ln(x) \, dx = \lim_{\alpha \to 0, \, \alpha > 0} \left[x^2 \left(\frac{1}{2} \ln(x) - \frac{1}{4} \right) \right]_\alpha^1$$

$$= 1^2 \left(\frac{1}{2} \ln(1) - \frac{1}{4} \right) - \lim_{\alpha \to 0, \, \alpha > 0} \frac{1}{2} \alpha^2 \ln(\alpha) - \frac{1}{4} \alpha^2$$

$$= -\frac{1}{4} - \lim_{\alpha \to 0, \, \alpha > 0} \alpha^2 \left(\frac{1}{2} \ln(\alpha) - \frac{1}{4} \right)$$

L'Hospital:

$$\lim_{\alpha \to 0, \, \alpha > 0} \alpha^2 \ln(\alpha) = \lim_{\alpha \to 0, \, \alpha > 0} \frac{\ln(\alpha)}{\frac{1}{\alpha^2}} = \lim_{\alpha \to 0, \, \alpha > 0} \frac{\frac{1}{\alpha}}{-\frac{2}{\alpha^3}} = \lim_{\alpha \to 0, \, \alpha > 0} -\frac{\alpha^3}{2\alpha} \to 0$$

$$= -\frac{1}{4} - 0 = -\frac{1}{4} \to \text{konvergent}$$

14. Aufgabe

$$\int_{\frac{1}{2}}^{\infty} \frac{1}{\sqrt{x\left(1+x^2\right)}} \, dx$$

$$= \int_{\frac{1}{2}}^{\infty} \frac{1}{\sqrt{x+x^3}} \, dx$$

Für $\int_{1}^{\infty}$:

$$\left| f(x) \right| = \frac{1}{\sqrt{x+x^3}} = \frac{1}{\left(x+x^3\right)^{\frac{1}{2}}} \leq \frac{1}{\left(x^3\right)^{\frac{1}{2}}} = \frac{1}{x^{3 \cdot \frac{1}{2}}} = \frac{1}{x^{\frac{3}{2}}}$$

$$\frac{3}{2} > 1 \rightarrow \text{Konvergenz (Majorantenkriterium)}$$

Da sowohl die Grenze $\left(\frac{1}{2} \text{ und } 1 \right)$ als auch der Bereich im Intervall $\left(\text{zwischen } \frac{1}{2} \text{ und } 1 \right)$ einsetzbar sind und keine Unstetigkeitsstellen existieren, ist dieser Abschnitt konvergent und somit das ganze umeigentliche Integral.

15. Aufgabe

$$f(x) = \begin{cases} 1, & -\pi < x \leq 0 \\ x, & 0 < x \leq \pi \end{cases}$$

$$a_0 = \frac{1}{2\pi} \cdot \int_{-\pi}^{\pi} f(x) \, dx = \frac{1}{2\pi} \cdot \left(\int_{-\pi}^{0} 1 \, dx + \int_{0}^{\pi} x \, dx \right)$$

$$= \frac{1}{2\pi} \left([x]_{-\pi}^{0} + \left[\frac{x^2}{2} \right]_{0}^{\pi} \right) = \frac{1}{2\pi} \left((0 - (-\pi)) + \left(\frac{\pi^2}{2} - \frac{0^2}{2} \right) \right) = \frac{1}{2\pi} \left(\pi + \frac{\pi^2}{2} \right)$$

$$= \frac{\pi}{2\pi} + \frac{\pi^2}{4\pi} = \frac{2\pi}{4\pi} + \frac{\pi^2}{4\pi} = \frac{2\pi + \pi^2}{4\pi} = \frac{\pi\left(2+\pi\right)}{4\pi} = \frac{2+\pi}{4} = \frac{\pi}{2}$$

$$a_k = \frac{1}{\pi} \cdot \int_{-\pi}^{\pi} f(x) \cdot \cos(kx)\, dx = \frac{1}{\pi}\left(\int_{-\pi}^{0} 1 \cdot \cos(k)\, dx + \int_{0}^{\pi} x \cdot \cos(kx)\, dx \right)$$

$k = 1, 2, \ldots$

$$= \frac{1}{\pi}\left(\Big[\sin(k)\Big]_{-\pi}^{0} + \left[\frac{\cos(kx)}{k^2} + \frac{x \cdot \sin(kx)}{k} \right]_{0}^{\pi} \right)$$

$$= \frac{1}{\pi}\left(\big(\sin(k) - \sin(k)\big) + \left(\frac{\cos(k\pi)}{k^2} + \frac{\pi \cdot \sin(k\pi)}{k} - \left(\frac{\cos(k0)}{k^2} + \frac{0 \cdot \sin(k0)}{k} \right) \right) \right)$$

$$= \frac{1}{\pi}\left(\frac{\cos(k\pi)}{k^2} + \frac{\pi \cdot \sin(k\pi)}{k} - \frac{\cos(0)}{k^2} \right)$$

$$= \frac{\cos(k\pi) - 1}{k^2 \pi} + \frac{\pi \cdot \sin(k\pi)}{k \pi}$$

$$= \frac{(-1)^k - 1}{k^2 \pi} + \frac{\pi 0}{k}$$

$$= \frac{(-1)^k - 1}{k^2 \pi}$$

$$b_k = \frac{1}{\pi} \cdot \int_{-\pi}^{\pi} f(x) \cdot \sin(kx)\, dx = \frac{1}{\pi}\left(\int_{-\pi}^{0} 1 \cdot \sin(k)\, dx + \int_{0}^{\pi} x \cdot \sin(kx)\, dx \right)$$

$k = 1, 2, \ldots$

$$= \frac{1}{\pi}\left(\Big[-\cos(k)\Big]_{-\pi}^{0} + \left[\frac{\sin(kx)}{k^2} - \frac{x \cdot \cos(kx)}{k} \right]_{0}^{\pi} \right)$$

$$= \frac{1}{\pi}\left(\big(-\cos(k) - (-\cos(k))\big) + \left(\left(\frac{\sin(k\pi)}{k^2} - \frac{\pi \cdot \cos(k\pi)}{k} \right) \right.\right.$$

$$\left.\left. - \left(\frac{\sin(k0)}{k^2} - \frac{0 \cdot \cos(k0)}{k} \right) \right) \right)$$

$$= \frac{1}{\pi}\left(\big(-\cos(k) + \cos(k)\big) + \left(\left(\frac{\sin(k\pi)}{k^2} - \frac{\pi \cdot \cos(k\pi)}{k} \right) - \frac{\sin(0)}{k^2} \right) \right)$$

$$= \frac{1}{\pi}\left(\frac{\sin(k\pi)}{k^2} - \frac{\pi \cdot \cos(k\pi)}{k} \right)$$

$$= \frac{\sin(k\pi)}{k^2 \pi} - \frac{\pi \cdot \cos(k\pi)}{k \pi}$$

$$= \frac{\sin(k\pi)}{k^2\pi} - \frac{\cos(k\pi)}{k}$$

$$= \frac{0}{k^2\pi} - \frac{(-1)^k}{k}$$

$$= -\frac{(-1)^k}{k}$$

$$f \sim a_0 + \sum_{k=1}^{\infty} \left(a_k \cdot \cos(kx) + b_k \cdot \sin(kx) \right)$$

$$f \sim \frac{\pi}{2} + \sum_{k=1}^{\infty} \left(\frac{(-1)^k - 1}{k^2\pi} \cdot \cos(kx) - \frac{(-1)^k}{k} \cdot \sin(kx) \right)$$

Stelle $x = 0$:

$$f \sim \frac{\pi}{2} + \sum_{k=1}^{\infty} \left(\frac{(-1)^k - 1}{k^2\pi} \cdot \cos(k0) - \frac{(-1)^k}{k} \cdot \sin(k0) \right)$$

$$= \frac{\pi}{2} + \sum_{k=1}^{\infty} \left(\frac{(-1)^k - 1}{k^2\pi} \cdot \cos(0) - \frac{(-1)^k}{k} \cdot \sin(0) \right)$$

$$= \frac{\pi}{2} + \sum_{k=1}^{\infty} \left(\frac{(-1)^k - 1}{k^2\pi} - \frac{(-1)^k}{k} \cdot 0 \right)$$

$$= \frac{\pi}{2} + \sum_{k=1}^{\infty} \left(\frac{(-1)^k - 1}{k^2\pi} \right)$$

$$\sum_{k=1}^{\infty} \left(\frac{(-1)^k - 1}{k^2\pi} \right)$$

$$\lim_{k\to\infty} \frac{1}{\sqrt[k]{\left| \frac{(-1)^k - 1}{k^2\pi} \right|}}$$

$$= \lim_{k\to\infty} \frac{1}{\sqrt[k]{\left| (-1)^k \cdot \frac{-1}{k^2\pi} \right|}}$$

$$= \lim_{k \to \infty} \frac{1}{\left| -1 \right| \cdot \sqrt[k]{\left| -\frac{1}{k^2 \pi} \right|}}$$

$$= \lim_{k \to \infty} \frac{1}{\sqrt[k]{\frac{1}{k^2 \pi}}}$$

$$= \lim_{k \to \infty} \frac{1}{\left(\frac{1}{k^2 \pi} \right)^{\frac{1}{k}}} = \frac{1}{1} = 1$$

$$= \frac{\pi}{2} + 1 = \frac{3\pi}{2}$$

Die Fourierreihe von f konvergiert an der Stelle $x = 0$ gegen den Wert $\frac{3\pi}{2}$.