6. Aufgabe

Für welche reellen Werte p ist das uneigentliche Integral

$$\int_1^\infty \frac{1}{x^p}\,dx$$

konvergent/divergent?

Lösung uneigentliches Integral 1.Art

$$f(x) = \frac{1}{x^p}$$

Bereits in Aufgabe 5 berechnet

1 Stammfunktion von f bestimmen

2 Bestimmtes Integral

3 Grenzwert

1 Stammfunktion

Fallunterscheidung bzgl. p erforderlich

$$\int \frac{1}{x^p}\,dx = \frac{x^{-p+1}}{-p+1} + C, C \in \mathbb{R}, p \neq 1$$

$$\int x^{-1}dx = \ln(|x|) + C, C \in \mathbb{R}$$

2 Bestimmtes Integral $[1, \beta]$

$$p \neq 1: \int_1^\beta \frac{1}{x^p}\,dx = \left[\frac{x^{-p+1}}{-p+1}\right]_1^\beta = \frac{\beta^{-p+1}}{-p+1} - \frac{1^{-p+1}}{-p+1}$$

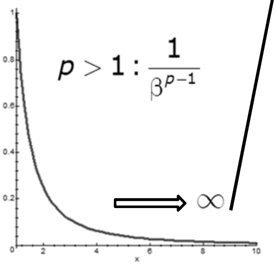
$$p = 1: \int_1^\beta \frac{1}{x}\,dx = [\ln(x)]_1^\beta = \ln(\beta) - \ln(1)$$

Prof. Dr. Hans-Jürgen Dobner, HTWK Leipzig, MNZ

③ Grenzwert

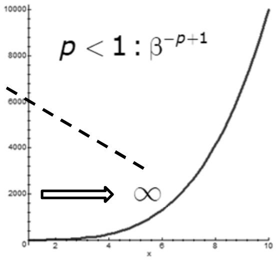
$$p \neq 1: \int_1^\infty \frac{1}{x^p}\,dx = \lim_{\beta\to\infty} \int_1^\beta \frac{1}{x^p}\,dx = \lim_{\beta\to\infty} \frac{\beta^{-p+1}}{-p+1} - \frac{1^{-p+1}}{-p+1}$$

$$\boxed{\lim_{\beta\to\infty} \frac{\beta^{-p+1}}{-p+1}} = \begin{cases} 0, p > 1 \Rightarrow 0 > 1-p & \Longrightarrow \int_1^\infty \frac{1}{x^p}\,dx \text{ konvergent} \\ \infty, p < 1 \Rightarrow 0 < 1-p & \Longrightarrow \int_1^\infty \frac{1}{x^p}\,dx \text{ divergent} \end{cases}$$



$p > 1: \frac{1}{\beta^{p-1}}$ $\Longrightarrow \infty$

$\beta \to \infty \Rightarrow \frac{1}{\beta} \to 0$

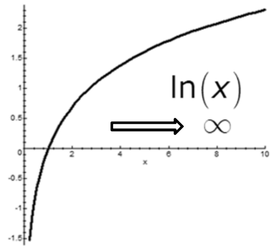


$p < 1: \beta^{-p+1}$ $\Longrightarrow \infty$

$$p = 1: \int_1^\infty \frac{1}{x^p}\,dx = \lim_{\beta\to\infty} \int_1^\beta \frac{1}{x}\,dx$$

$$= \boxed{\lim_{\beta\to\infty} \ln(\beta)} - \underbrace{\ln(1)}_{=0}$$

divergent $\equiv \infty$



$\ln(x)$ $\Longrightarrow \infty$

<u>Zusammenfassung</u>

$$\int_1^\infty \frac{1}{x^p}\,dx = \begin{cases} \frac{1}{p-1} \text{ (konvergent) für } p > 1 \\ \infty \text{ (divergent) für } p \leq 1 \end{cases}$$