Aufgaben 3

11. Aufgabe

$$\int_1^e \frac{\ln(x)}{x \cdot \sqrt{1+(\ln(x))^2}}\ dx$$

$$\int \frac{\ln(x)}{x \cdot \sqrt{1+(\ln(x))^2}}\ dx \qquad u = \ln(x),\ du = \frac{1}{x}\ dx$$

$$\int \frac{u}{\sqrt{1+u^2}}\ du$$

$$= \sqrt{1+u^2} + C$$

$$\sqrt{1+(\ln(x))^2}$$

$$\int_1^e \frac{\ln(x)}{x \cdot \sqrt{1+(\ln(x))^2}}\ dx = \left[\sqrt{1+(\ln(x))^2}\right]_1^e$$

$$= \sqrt{1+(\ln(e))^2} - \sqrt{1+(\ln(1))^2}$$

$$= \sqrt{1+1^2} - \sqrt{1+0^2}$$

$$= \sqrt{2} - \sqrt{1}$$

$$= \sqrt{2} - 1 \approx 0.4142$$

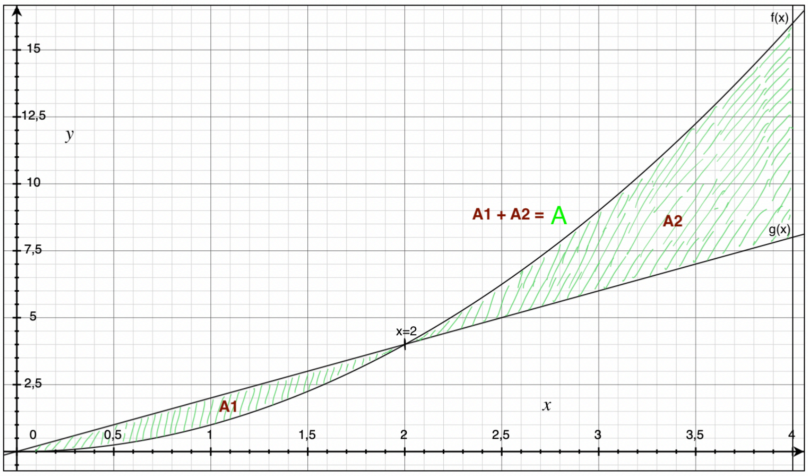
12. Aufgabe

$$f(x) = x^2, \quad g(x) = 2x \quad [0,\ 4]$$

$$A = A_1 + A_2$$

$$A_1 = \int_0^2 \big(g(x) - f(x)\big)\ dx$$

$$A_2 = \int_2^4 \big(f(x) - g(x)\big)\ dx$$



$$A_1 = \int_0^2 \left(g(x) - f(x)\right)\ dx$$

$$= \int_0^2 \left(2x - x^2\right)\ dx$$

$$= \int_0^2 -x^2 + 2x\ dx$$

$$\int -x^2 + 2x\ dx$$

$$= -\frac{1}{3}x^3 + x^2 + C$$

$$= \left[-\frac{1}{3}x^3 + x^2\right]_0^2$$

$$= \left(-\frac{1}{3}2^3 + 2^2\right) - \left(-\frac{1}{3}0^3 + 0^2\right)$$

$$= \left(-\frac{8}{3} + 4\right) - (0 + 0)$$

$$= -\frac{8}{3} + \frac{12}{3} = \frac{12}{3} - \frac{8}{3} = \frac{4}{3}$$

$$A_2 = \int_2^4 \left(f(x) - g(x)\right)\ dx$$

$$= \int_2^4 x^2 - 2x\ dx$$

$$\int x^2 - 2x\ dx$$

$$= \frac{1}{3}x^3 - x^2 + C$$

$$= \left[\frac{1}{3}x^3 - x^2\right]_2^4$$

$$= \left(\frac{1}{3}4^3 - 4^2\right) - \left(\frac{1}{3}2^3 - 2^2\right)$$

$$= \left(\frac{64}{3} - 16\right) - \left(\frac{8}{3} - 4\right)$$

$$= \left(\frac{64}{3} - \frac{48}{3}\right) - \left(\frac{8}{3} - \frac{12}{3}\right)$$

$$= \frac{16}{3} + \frac{4}{3} = \frac{20}{3}$$

$$A = A_1 + A_2 = \frac{4}{3} + \frac{20}{3} = \frac{24}{3} = 8$$

13. Aufgabe

$$\int_0^1 x \cdot \ln(x)\ dx$$

Unstetigkeitsstelle = 0, stetig im Intervall (0, 1]
Unbeschränkt für $x \to 0$

$$F(x) = \frac{1}{2}x^2\ln(x) - \frac{1}{4}x^2 + C = x^2\left(\frac{1}{2}\ln(x) - \frac{1}{4} + C\right)$$

$$\int_\alpha^1 x \cdot \ln(x)\ dx = \lim_{\alpha\to 0,\ \alpha>0}\left[x^2\left(\frac{1}{2}\ln(x) - \frac{1}{4}\right)\right]_\alpha^1$$

$$= 1^2\left(\frac{1}{2}\ln(1) - \frac{1}{4}\right) - \lim_{\alpha\to 0,\ \alpha>0}\frac{1}{2}\alpha^2\ln(\alpha) - \frac{1}{4}\alpha^2$$

$$= -\frac{1}{4} - \lim_{\alpha\to 0,\ \alpha>0}\alpha^2\left(\frac{1}{2}\ln(\alpha) - \frac{1}{4}\right)$$

L'Hospital:

$$\lim_{\alpha\to 0,\ \alpha>0}\alpha^2\ln(\alpha) = \lim_{\alpha\to 0,\ \alpha>0}\frac{\ln(\alpha)}{\frac{1}{\alpha^2}} = \lim_{\alpha\to 0,\ \alpha>0}\frac{\frac{1}{\alpha}}{-\frac{2}{\alpha^3}} = \lim_{\alpha\to 0,\ \alpha>0} -\frac{\alpha^3}{2\alpha} \to 0$$

$= -\frac{1}{4} - 0 = -\frac{1}{4} \to$ konvergent

14. Aufgabe

$$\int_{\frac{1}{2}}^{\infty} \frac{1}{\sqrt{x\left(1+x^2\right)}}\ dx$$

$$= \int_{\frac{1}{2}}^{\infty} \frac{1}{\sqrt{x+x^3}}\ dx$$

Für $\int_{1}^{\infty}$:

$$\left|f(x)\right| = \frac{1}{\sqrt{x+x^3}} = \frac{1}{\left(x+x^3\right)^{\frac{1}{2}}} \leq \frac{1}{\left(x^3\right)^{\frac{1}{2}}} = \frac{1}{x^{3\cdot\frac{1}{2}}} = \frac{1}{x^{\frac{3}{2}}}$$

$\frac{3}{2} > 1 \rightarrow$ Konvergenz (Majorantenkriterium)

Da sowohl die Grenze $\left(\frac{1}{2} \text{ und } 1\right)$ als auch der Bereich im Intervall $\left(\text{zwischen } \frac{1}{2} \text{ und } 1\right)$ einsetzbar sind und keine Unstetigkeitsstellen existieren, ist dieser Abschnitt konvergent und somit das ganze umeigentliche Integral.

15. Aufgabe

$$f(x) = \begin{cases} 1, & -\pi < x \leq 0 \\ x, & 0 < x \leq \pi \end{cases}$$

$$a_0 = \frac{1}{2\pi} \cdot \int_{-\pi}^{\pi} f(x)\ dx = \frac{1}{2\pi} \cdot \left(\int_{-\pi}^{0} 1\ dx + \int_{0}^{\pi} x\ dx\right)$$

$$= \frac{1}{2\pi}\left([x]_{-\pi}^{0} + \left[\frac{x^2}{2}\right]_0^{\pi}\right) = \frac{1}{2\pi}\left(\left(0-(-\pi)\right) + \left(\frac{\pi^2}{2} - \frac{0^2}{2}\right)\right) = \frac{1}{2\pi}\left(\pi + \frac{\pi^2}{2}\right)$$

$$= \frac{\pi}{2\pi} + \frac{\pi^2}{4\pi} = \frac{2\pi}{4\pi} + \frac{\pi^2}{4\pi} = \frac{2\pi+\pi^2}{4\pi} = \frac{\pi\,(2+\pi)}{4\pi} = \frac{2+\pi}{4} = \frac{\pi}{2}$$

$$a_k = \frac{1}{\pi} \cdot \int_{-\pi}^{\pi} f(x) \cdot \cos(kx)\ dx = \frac{1}{\pi}\left(\int_{-\pi}^{0} 1 \cdot \cos(kx)\ dx + \int_{0}^{\pi} x \cdot \cos(kx)\ dx\right)$$

$k = 1, 2, \ldots$

$$= \frac{1}{\pi}\left(\left[\frac{\sin(kx)}{k}\right]_{-\pi}^{0} + \left[\frac{\cos(kx)}{k^2} + \frac{x \cdot \sin(kx)}{k}\right]_{0}^{\pi}\right)$$

$$= \frac{1}{\pi}\left(\left(\frac{\sin(k0)}{k} - \frac{\sin\left(k(-\pi)\right)}{k}\right) + \left(\frac{\cos(k\pi)}{k^2} + \frac{\pi \cdot \sin(k\pi)}{k} - \left(\frac{\cos(k0)}{k^2} + \frac{0 \cdot \sin(k0)}{k}\right)\right)\right)$$

$$= \frac{1}{\pi}\left(\frac{\sin(0)}{k} - \frac{\sin\left(k(-\pi)\right)}{k} + \frac{\cos(k\pi)}{k^2} + \frac{\pi \cdot \sin(k\pi)}{k} - \frac{\cos(0)}{k^2}\right)$$

$$= \frac{1}{\pi}\left(\frac{0}{k} - \frac{0}{k} + \frac{(-1)^k}{k^2} + \frac{\pi \cdot 0}{k} - \frac{1}{k^2}\right)$$

$$= \frac{1}{\pi}\left(\frac{(-1)^k}{k^2} - \frac{1}{k^2}\right)$$

$$= \frac{(-1)^k}{k^2\pi} - \frac{1}{k^2\pi}$$

$$= \frac{(-1)^k - 1}{k^2\pi}$$

$$b_k = \frac{1}{\pi} \cdot \int_{-\pi}^{\pi} f(x) \cdot \sin(kx)\ dx = \frac{1}{\pi}\left(\int_{-\pi}^{0} 1 \cdot \sin(kx)\ dx + \int_{0}^{\pi} x \cdot \sin(kx)\ dx\right)$$

$k = 1, 2, \ldots$

$$= \frac{1}{\pi}\left(\left[-\frac{\cos(kx)}{k}\right]_{-\pi}^{0} + \left[\frac{\sin(kx)}{k^2} - \frac{x \cdot \cos(kx)}{k}\right]_{0}^{\pi}\right)$$

$$= \frac{1}{\pi}\left(\left(-\frac{\cos(k0)}{k} - \left(-\frac{\cos\left(k(-\pi)\right)}{k}\right)\right) + \left(\left(\frac{\sin(k\pi)}{k^2} - \frac{\pi \cdot \cos(k\pi)}{k}\right) - \left(\frac{\sin(k0)}{k^2} - \frac{0 \cdot \cos(k0)}{k}\right)\right)\right)$$

$$= \frac{1}{\pi}\left[\left(-\frac{\cos(0)}{k} + \frac{\cos\left(k(-\pi)\right)}{k}\right) + \left(\left(\frac{\sin(k\pi)}{k^2} - \frac{\pi \cdot \cos(k\pi)}{k}\right) - \frac{\sin(0)}{k^2}\right)\right]$$

$$= \frac{1}{\pi}\left[\left(-\frac{1}{k} + \frac{0}{k}\right) + \left(\left(\frac{\sin(k\pi)}{k^2} - \frac{\pi \cdot \cos(k\pi)}{k}\right) - \frac{0}{k^2}\right)\right]$$

$$= \frac{1}{\pi}\left(-\frac{1}{k} + \frac{\sin(k\pi)}{k^2} - \frac{\pi \cdot \cos(k\pi)}{k}\right)$$

$$= \frac{1}{\pi}\left(\frac{\sin(k\pi)}{k^2} - \frac{\pi \cdot \cos(k\pi)}{k} - \frac{1}{k}\right)$$

$$= \frac{\sin(k\pi)}{k^2\pi} - \frac{\pi \cdot \cos(k\pi)}{k\pi} - \frac{1}{k\pi}$$

$$= \frac{0}{k^2\pi} - \frac{\pi \cdot \cos(k\pi) - 1}{k\pi}$$

$$= -\frac{(-1)^k\pi - 1}{k\pi}$$

$$f \sim a_0 + \sum_{k=1}^{\infty}\left(a_k \cdot \cos(kx) + b_k \cdot \sin(kx)\right)$$

$$f \sim \frac{\pi}{2} + \sum_{k=1}^{\infty}\left(\frac{(-1)^k - 1}{k^2\pi} \cdot \cos(kx) - \frac{(-1)^k\pi - 1}{k\pi} \cdot \sin(kx)\right)$$

Stelle $x = 0$:

$$f \sim \frac{\pi}{2} + \sum_{k=1}^{\infty}\left(\frac{(-1)^k - 1}{k^2\pi} \cdot \cos(kx) - \frac{(-1)^k\pi - 1}{k\pi} \cdot \sin(kx)\right)$$

$$= \frac{\pi}{2} + \sum_{k=1}^{\infty}\left(\frac{(-1)^k - 1}{k^2\pi} \cdot \cos(0) - \frac{(-1)^k\pi - 1}{k\pi} \cdot \sin(0)\right)$$

$$= \frac{\pi}{2} + \sum_{k=1}^{\infty}\left(\frac{(-1)^k - 1}{k^2\pi} - \frac{(-1)^k\pi - 1}{k\pi} \cdot 0\right)$$

$$= \frac{\pi}{2} + \sum_{k=1}^{\infty}\left(\frac{(-1)^k - 1}{k^2\pi}\right)$$

$$\sum_{k=1}^{\infty}\left(\frac{(-1)^k-1}{k^2\pi}\right)$$

$$\lim_{k\to\infty}\frac{1}{\sqrt[k]{\left|\frac{(-1)^k-1}{k^2\pi}\right|}}$$

$$=\lim_{k\to\infty}\frac{1}{\sqrt[k]{\left|(-1)^k\cdot\frac{-1}{k^2\pi}\right|}}$$

$$=\lim_{k\to\infty}\frac{1}{\left|-1\right|\cdot\sqrt[k]{\left|-\frac{1}{k^2\pi}\right|}}$$

$$=\lim_{k\to\infty}\frac{1}{\sqrt[k]{\frac{1}{k^2\pi}}}$$

$$=\lim_{k\to\infty}\frac{1}{\left(\frac{1}{k^2\pi}\right)^{\frac{1}{k}}}=\frac{1}{1}=1$$

$$=\frac{\pi}{2}+1=\frac{3\pi}{2}$$

Die Fourierreihe von f konvergiert an der Stelle $x=0$ gegen den Wert $\frac{3\pi}{2}$.