Lösungen

① a. $\vec{x} = \begin{pmatrix} -6 \\ 3 \\ -2 \end{pmatrix}$

$$\|\vec{x}\|_1 = \sum_{k=1}^{3} |x_k| = |-6| + |3| + |-2| = 11$$

$$\|\vec{x}\|_2 = \sqrt{\sum_{k=1}^{3} |x_k|^2} = \sqrt{|-6|^2 + |3|^2 + |-2|^2} = \sqrt{36+9+4} = \sqrt{49} = 7$$

$$\|\vec{x}\|_\infty = \max_{k=1,2,3} |x_k| = \max\{|-6|, |3|, |-2|\} = 6$$

b. $\vec{x} = \begin{pmatrix} 2-i \\ 4+3i \\ 7 \end{pmatrix}$

$$|x_1| = |2-i| = \sqrt{2^2 + (-1)^2} = \sqrt{5}$$
$$|x_2| = |4+3i| = \sqrt{4^2 + 3^2} = 5$$
$$|x_3| = |7| = 7$$

$$\|\vec{x}\|_1 = \sum_{k=1}^{3} |x_k| = \sqrt{5} + 5 + 7 = 12 + \sqrt{5}$$

$$\|\vec{x}\|_2 = \sqrt{\sum_{k=1}^{3} |x_k|^2} = \sqrt{\sqrt{5}^2 + 5^2 + 7^2} = \sqrt{5+25+49} = \sqrt{79}$$

$$\|\vec{x}\|_\infty = \max_{k=1,2,3} |x_k| = \max\{\sqrt{5}, 5, 7\} = 7$$

② a.

$$\|A\|_1 = \max_{k=1,\dots,4} \sum_{i=1}^{3} |a_{ik}| = \max\{4 + |-2| + 1,\ 3 + |-10| + |-9|,\ 0 + 2 + 7,\ 1 + 1 + 3\}$$
$$= \max\{7, 22, 9, 5\} = 22$$

$$\|A\|_\infty = \max_{i=1,\dots,3} \sum_{k=1}^{4} |a_{ik}| = \max\{4+3+0+1,\ |-2| + |-10| + 2 + 1,\ 1 + |-9| + 7 + 3\}$$
$$= \max\{8, 15, 20\} = 20$$

b.

$$|1-2i| = \sqrt{1^2 + (-2)^2} = \sqrt{5}$$
$$|3+6i| = \sqrt{3^2 + 6^2} = \sqrt{45} = 3\sqrt{5}$$
$$|-8| = 8$$

$$|2| = 2$$
$$|2+i| = \sqrt{2^2 + 1^2} = \sqrt{5}$$
$$|3-4i| = \sqrt{3^2 + 4^2} = 5$$

$$\|A\|_1 = \max\{\sqrt{5} + 2,\ 3\sqrt{5} + \sqrt{5},\ 8 + 5\} = \max\{\overset{\approx 4}{\sqrt{5} + 2},\ \overset{\approx 9}{4\sqrt{5}},\ 13\} = 13$$

$$\|A\|_\infty = \max\{\sqrt{5} + 3\sqrt{5} + 8,\ 2 + \sqrt{5} + 5\} = \max\{\underset{\approx 17}{4\sqrt{5} + 8},\ \underset{\approx 9}{7 + \sqrt{5}}\} = 4\sqrt{5} + 8$$

③ a. $\vec{x} = \begin{pmatrix} 3-4i \\ 3 \end{pmatrix}$ $\quad |3-4i| = \sqrt{3^2+4^2} = 5$
$|3| = 3$

$\|x\|_1 = 5+3 = 8 \longrightarrow \frac{\vec{x}}{\|\vec{x}\|_1} = \frac{\begin{pmatrix} 3-4i \\ 3 \end{pmatrix}}{8} = \begin{pmatrix} 3/8 - 1/2\, i \\ 3/8 \end{pmatrix} = \vec{v}$

$\vec{v}$ ist Einheitsvektor, da $\|\vec{v}\|_1 = |3/8 - 1/2\, i| + |3/8|$
$= \sqrt{(3/8)^2 + (1/2)^2} + 3/8$
$= \sqrt{9/64 + 1/4} + 3/8 = \sqrt{9/64 + 16/64} + 3/8$
$= \sqrt{25/64} + 3/8 = 5/8 + 3/8 = 1$

b. $\vec{x} = \begin{pmatrix} -3 \\ 2 \\ 5 \end{pmatrix}$ $\quad \|\vec{x}\|_2 = \sqrt{3^2+2^2+5^2} = \sqrt{38}$

$\longrightarrow \frac{\vec{x}}{\|\vec{x}\|_1} = \frac{1}{\sqrt{38}} \cdot \begin{pmatrix} -3 \\ 2 \\ 5 \end{pmatrix} = \vec{v}$

$\vec{v}$ ist Einheitsvektor, da $\|\vec{v}\|_2 = \sqrt{\left(\frac{1}{\sqrt{38}}\right)^2 \cdot (3^2+2^2+5^2)} = \sqrt{\frac{1}{38} \cdot (9+4+25)}$
$= \sqrt{\frac{1}{38} \cdot 38} = 1$

④ $\vec{x} \perp \vec{y} \Leftrightarrow \langle \vec{x}, \vec{y} \rangle = 0$

da nichts angegeben mit Standard-Skalarprodukt $\langle \vec{x}, \vec{y} \rangle = \sum_{k=1}^{n} x_k \cdot \overline{y_k}$ wenn komplexer Vektorraum

a. $\langle \vec{x}, \vec{y} \rangle = 1 \cdot 0 + 3 \cdot 1 + (-2) \cdot 7 + 5 \cdot (-2)$
$= 0 + 3 - 14 - 10$
$= -21 \neq 0 \longrightarrow \vec{x} \not\perp \vec{y}$

b. mit $\overline{\vec{y}} = \begin{pmatrix} 5 \\ 2-3i \\ 1+2i \end{pmatrix}$

$\langle \vec{x}, \vec{y} \rangle = (3+i) \cdot 5 + 2 \cdot (2-3i) + (1-2i)(1+2i)$
$= 15 + 5i + 4 - 6i + 1^2 + 2^2$
$= 24 - i \neq 0 \longrightarrow \vec{x} \not\perp \vec{y}$

⑤ $\vec{a} \perp \vec{x}$ und $\vec{a} \perp \vec{y} \Leftrightarrow \langle \vec{a}, \vec{x} \rangle = \langle \vec{a}, \vec{y} \rangle = 0$

I $\quad 1 \cdot a_1 + 3 \cdot a_2 - 2a_3 + 5a_4 = 0$
II $\quad 0 \cdot a_1 + 1 \cdot a_2 + 7a_3 - 2a_4 = 0$

$\left(\begin{array}{cccc|c} 1 & 3 & -2 & 5 & 0 \\ 0 & 1 & 7 & -2 & 0 \end{array}\right)$ schon Stufenform, kein Umformen nötig

$a_4 = t \quad , t \in \mathbb{R}$
$a_3 = s \quad , s \in \mathbb{R}$

$a_2 + 7a_3 - 2a_4 = 0$
$a_2 = -7s + 2t$

$a_1 + 3a_2 - 2a_3 + 5a_4 = 0$
$a_1 = -3(-7s+2t) - 2s + 5t$
$a_1 = 21s - 6t - 2s + 5t$
$a_1 = 19s - t$

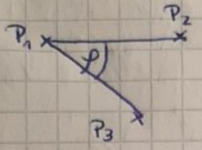
$\vec{a} = s \begin{pmatrix} 19 \\ -7 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -1 \\ 2 \\ 0 \\ 1 \end{pmatrix} \quad , s,t \in \mathbb{R}$

(6) a. $\|\vec{x}\| = \sqrt{\langle \vec{x}, \vec{x} \rangle} = \sqrt{1 \cdot 1 + (-4) \cdot (-4) + 7 \cdot 7} = \sqrt{1 + 16 + 49} = \sqrt{66}$

b. $\|\vec{x}\| = \sqrt{\langle \vec{x}, \vec{x} \rangle} = \sqrt{(3-2i)(3+2i) + (4+10i)(4-10i)}$

oder $= \sqrt{3^2 + (-2)^2 + 4^2 + 10^2} = \sqrt{9 + 4 + 16 + 100} = \sqrt{129}$

(7) Skizze:



$$\overline{P_1 P_2} = \begin{pmatrix} 0-0 \\ 4-1 \\ 8-2 \\ 3-1 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ 6 \\ 2 \end{pmatrix} = \vec{x}$$

$$\overline{P_1 P_3} = \begin{pmatrix} 2-0 \\ 1-1 \\ 4-2 \\ 0-1 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 2 \\ -1 \end{pmatrix} = \vec{y}$$

$$\cos(\varphi) = \frac{\langle \vec{x}, \vec{y} \rangle}{\|\vec{x}\| \cdot \|\vec{y}\|} \leftarrow \text{kanonische Norm}$$

$\langle \vec{x}, \vec{y} \rangle$ mit Standard-Skalarprodukt

$$= 0 \cdot 2 + 3 \cdot 0 + 6 \cdot 2 + 2 \cdot (-1)$$
$$= 0 + 0 + 12 - 2 = 10$$

$$\|\vec{x}\| = \sqrt{\langle \vec{x}, \vec{x} \rangle} = \sqrt{0^2 + 3^2 + 6^2 + 2^2} = \sqrt{9 + 36 + 4} = 7$$

$$\|\vec{y}\| = \sqrt{\langle \vec{y}, \vec{y} \rangle} = \sqrt{2^2 + 0^2 + 2^2 + (-1)^2} = \sqrt{4 + 4 + 1} = 3$$

$$\Rightarrow \cos(\varphi) = \frac{10}{7 \cdot 3} = \frac{10}{21}$$

$$\rightarrow \arccos\left(\frac{10}{21}\right) \approx 61{,}6^\circ$$

(8) a. $\vec{u} = \frac{\langle \vec{a}, \vec{b} \rangle}{\|\vec{b}\|^2} \cdot \vec{b} \longrightarrow \langle \vec{a}, \vec{b} \rangle = (-2) \cdot 9 + 4 \cdot 2 + \frac{1}{6} \cdot (-6)$

$= -18 + 8 - 1 = -11$

$\vec{v} = \vec{a} - \vec{u}$

$\|\vec{b}\| = \sqrt{\langle \vec{b}, \vec{b} \rangle} = \sqrt{9^2 + 2^2 + (-6)^2}$

$= \sqrt{81 + 4 + 36} = \sqrt{121} = 11$

$$\Rightarrow \vec{u} = \frac{-11}{11^2} \cdot \begin{pmatrix} 9 \\ 2 \\ -6 \end{pmatrix} = -\frac{1}{11} \cdot \begin{pmatrix} 9 \\ 2 \\ -6 \end{pmatrix}$$

$$\vec{v} = \vec{a} - \vec{u} = \frac{11}{11} \vec{a} - \left(-\frac{1}{11} \begin{pmatrix} 9 \\ 2 \\ -6 \end{pmatrix} \right)$$

$$= \frac{1}{11} \cdot 11\vec{a} + \frac{1}{11} \cdot \begin{pmatrix} 9 \\ 2 \\ -6 \end{pmatrix}$$

$$= \frac{1}{11} \left(11\vec{a} + \begin{pmatrix} 9 \\ 2 \\ -6 \end{pmatrix} \right) = \frac{1}{11} \begin{pmatrix} -22 + 9 \\ 44 + 2 \\ 11/6 - 6 \end{pmatrix} = \frac{1}{11} \begin{pmatrix} -13 \\ 46 \\ -25/6 \end{pmatrix}$$

b. $\vec{u} = \frac{\langle \vec{a}, \vec{b} \rangle}{\|\vec{b}\|^2} \cdot \vec{b} \longrightarrow \langle \vec{a}, \vec{b} \rangle = (4-2i)(4+2i) + (4+9i) \cdot 4$

$= 4^2 + 2^2 + 16 + 36i$

$= 16 + 4 + 16 + 36i$

$= 36 + 36i$

$\|\vec{b}\| = \sqrt{\langle \vec{b}, \vec{b} \rangle} = \sqrt{|4-2i| + |4|^2}$

$= \sqrt{4^2 + (-2)^2 + 4^2}$

$= \sqrt{16 + 4 + 16} = \sqrt{36} = 6$

$$\Rightarrow \vec{u} = \frac{36 + 36i}{6^2} \vec{b} = (1+i) \begin{pmatrix} 4-2i \\ 4 \end{pmatrix} = \begin{pmatrix} 4 - 2i + 4i + 2 \\ 4 + 4i \end{pmatrix} = \begin{pmatrix} 6+2i \\ 4+4i \end{pmatrix}$$

$$\vec{v} = \vec{a} - \vec{u} = \begin{pmatrix} 4-2i - (6+2i) \\ 4+9i - (4+4i) \end{pmatrix} = \begin{pmatrix} -2-4i \\ 5i \end{pmatrix}$$

9

$\vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$, $\|\vec{a}\| = \sqrt{66}$, $\vec{a} = \|\vec{a}\| \cdot \lambda \cdot \overbrace{\begin{pmatrix} \varrho \\ 2 \\ -16 \end{pmatrix}}^{\text{Länge } 1} \to \vec{\gamma} = \sqrt{66} \cdot \lambda \cdot \begin{pmatrix} \varrho \\ 2 \\ -16 \end{pmatrix}$

$= \|\vec{a}\| \cdot \vec{e} = \sqrt{66} \cdot \vec{e}$

$\|\vec{e}\| = 1$

$\vec{b} = \begin{pmatrix} 2 \\ -3 \\ 6 \end{pmatrix}$, $\|\vec{b}\| = 7$

$\left\| \frac{\langle \vec{a}, \vec{b} \rangle}{\|\vec{b}\|^2} \cdot \vec{b} \right\| = 7$

$\frac{|\langle \vec{a}, \vec{b} \rangle|}{\|\vec{b}\|^2} \cdot \|\vec{b}\| = 7$

$\frac{|\langle \vec{a}, \vec{b} \rangle|}{\|\vec{b}\|} = 7$

$\vec{e} = \frac{\vec{\gamma}}{\|\vec{\gamma}\|}$ Einheitsvektor zu $\vec{\gamma}$

$= \frac{1}{\left\| \begin{pmatrix} \varrho \\ 2 \\ -16 \end{pmatrix} \right\|} \cdot \begin{pmatrix} \varrho \\ 2 \\ -16 \end{pmatrix} = \frac{1}{\sqrt{\varrho^2 + 2^2 + (-16)^2}} \begin{pmatrix} \varrho \\ 2 \\ -16 \end{pmatrix} = \frac{1}{\sqrt{\varrho^2 + 260}} \begin{pmatrix} \varrho \\ 2 \\ -16 \end{pmatrix}$

$\vec{a} = \frac{\sqrt{66}}{\sqrt{\varrho^2 + 260}} \begin{pmatrix} \varrho \\ 2 \\ -16 \end{pmatrix} \longrightarrow |\langle \vec{a}, \vec{b} \rangle| = \left| \left\langle \frac{\sqrt{66}}{\sqrt{\varrho^2 + 260}} \cdot \begin{pmatrix} \varrho \\ 2 \\ -16 \end{pmatrix}, \begin{pmatrix} 2 \\ -3 \\ 6 \end{pmatrix} \right\rangle \right|$

$= \left| \frac{\sqrt{66}}{\sqrt{\varrho^2 + 260}} \cdot (2 \cdot \varrho - 6 - 96) \right|$

$= \frac{\sqrt{66}}{\sqrt{\varrho^2 + 260}} \cdot |2 \cdot \varrho - 102|$

$\|\vec{b}\| = \sqrt{2^2 + (-3)^2 + 6^2} = \sqrt{4 + 9 + 36} = 7$

$\|\vec{c}\| = \frac{1}{7} \cdot \frac{\sqrt{66}}{\sqrt{\varrho^2 + 260}} \cdot |2 \cdot \varrho - 102| = 7 \quad | \cdot 7 \quad | : \sqrt{66} \quad | \cdot \sqrt{\varrho^2 + 260}$

$|2 \cdot \varrho - 102| = \frac{49}{\sqrt{66}} \cdot \sqrt{\varrho^2 + 260}$

$= 49 \cdot \sqrt{\frac{\varrho^2 + 260}{66}}$ mit $\varrho^2 = 2^2$ oder $(-2)^2 = 4$

$= 49 \cdot \sqrt{\frac{264}{66}} = \sqrt{4} = 49 \cdot 2$

$|2\varrho - 102| = 98$

mit $\varrho_1 = 2$: $|2 \cdot 2 - 102| = |-98| = 98 \quad = 98$

$\varrho_2 = -2$: $|2 \cdot (-2) - 102| = |-106| = 106 \neq 98$

$\rightarrow \varrho_1 = 2$ ist korrekt

$\rightarrow \underline{\vec{a}} = \frac{\sqrt{66}}{\sqrt{2^2 + 264}} \begin{pmatrix} 2 \\ 2 \\ -16 \end{pmatrix} = \sqrt{\frac{66}{264}} \begin{pmatrix} 2 \\ 2 \\ -16 \end{pmatrix} = \sqrt{1/4} \begin{pmatrix} 2 \\ 2 \\ -16 \end{pmatrix} = 1/2 \cdot \begin{pmatrix} 2 \\ 2 \\ -16 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -8 \end{pmatrix}$

(10) $\vec{x} \perp \vec{y} \iff \|\vec{x}+\vec{y}\|^2 = \|\vec{x}\|^2 + \|\vec{y}\|^2$

a. $\|\vec{x}+\vec{y}\| = \left\|\begin{pmatrix} 3 - 10 \\ 5 + 6 \end{pmatrix}\right\| = \sqrt{(-7)^2 + 11^2}$

$= \sqrt{49+121} = \sqrt{170}$

$\|\vec{x}\| = \sqrt{3^2+5^2} = \sqrt{9+25} = \sqrt{34}$

$\|\vec{y}\| = \sqrt{(-10)^2+6^2} = \sqrt{136}$

$\vec{x} \perp \vec{y} \iff \sqrt{170}^2 = \sqrt{34}^2 + \sqrt{136}^2$

$170 = 34 + 136 = 170$

→ $\vec{x}$ und $\vec{y}$ sind orthogonal zueinander

b. $|2-i|^2 = 2^2 + (-1)^2 = 5 \rightarrow \|\vec{x}\| = \sqrt{5+3^2} = \sqrt{14}$

$|3+i|^2 = 3^2 + 1^2 = 10 \rightarrow \|\vec{y}\| = \sqrt{5^2+10} = \sqrt{35}$

$\|\vec{x}+\vec{y}\| = \left\|\begin{pmatrix} 2-i+5 \\ 3+3+i \end{pmatrix}\right\| = \left\|\begin{pmatrix} 7-i \\ 6+i \end{pmatrix}\right\| = \sqrt{7^2+(-1)^2+6^2+1^2}$

$= \sqrt{49+1+36+1} = \sqrt{87}$

$\vec{x} \perp \vec{y} \iff \sqrt{87}^2 = \sqrt{14}^2 + \sqrt{35}^2$

$87 \neq 14 + 35 = 49$

→ $\vec{x}$ und $\vec{y}$ sind nicht orthogonal zueinander