4. Aufgabe

Bestimmen Sie mit der Inversenformel die Inverse der Matrix

$$\mathbf{A}=\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

Lösung

$$\mathbf{A}^{-1}=\frac{1}{\det(\mathbf{A})}\cdot adj(\mathbf{A})^T$$

$$adj(\mathbf{A})=\left[\left((-1)^{i+j}\det\left(\mathbf{A}_{i,j}\right)\right)_{i,j=1,2,\ldots,n}\right]$$

$\mathbf{A}_{i,j}$ entsteht aus $\mathbf{A}$ durch Streichen der i-ten Zeile und j-ten Spalte

$$\mathbf{A}=\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$ **A** regulär?

$$|\mathbf{A}|=\begin{vmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{vmatrix}$$

Entwickeln nach der ersten Zeile

$(-1)^{1+1}$ $$\begin{vmatrix} \cancel{2} & \cancel{-1} & \cancel{0} \\ \cancel{-1} & 2 & -1 \\ \cancel{0} & -1 & 2 \end{vmatrix}$$

$(-1)^{1+2}$ $$\begin{vmatrix} \cancel{2} & \cancel{-1} & \cancel{0} \\ -1 & \cancel{2} & -1 \\ 0 & \cancel{-1} & 2 \end{vmatrix}$$

$$|\mathbf{A}|=(-1)^{1+1}\cdot 2\cdot\begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix}+(-1)^{1+2}\cdot(-1)\cdot\begin{vmatrix} -1 & -1 \\ 0 & 2 \end{vmatrix}$$

$$2\cdot\left(2\cdot 2-(-1)\cdot(-1)\right)+\left((-1)\cdot 2-0\cdot(1)\right)=4\neq 0$$ $\Rightarrow$**A** regulär

$$adj\left(\mathbf{A}\right)=\left[\left((-1)^{i+j}\det\left(\mathbf{A}_{i,j}\right)\right)_{i,j=1,2,\ldots,n}\right]$$

$$\mathbf{A}=\begin{pmatrix}2 & -1 & 0\\ -1 & 2 & -1\\ 0 & -1 & 2\end{pmatrix}$$

$$\mathbf{A}_{1,1}=\begin{pmatrix}2 & -1 & 0\\ -1 & 2 & -1\\ 0 & -1 & 2\end{pmatrix}\qquad \mathbf{A}_{1,2}=\begin{pmatrix}2 & -1 & 0\\ -1 & 2 & -1\\ 0 & -1 & 2\end{pmatrix}\qquad \mathbf{A}_{1,3}=\begin{pmatrix}2 & -1 & 0\\ -1 & 2 & -1\\ 0 & -1 & 2\end{pmatrix}$$

$$\left|\mathbf{A}_{1,1}\right|=2\cdot 2-(-1)\cdot(-1)=3\qquad \left|\mathbf{A}_{1,2}\right|=(-1)\cdot 2-(-1)\cdot 0=-2\qquad \left|\mathbf{A}_{1,3}\right|=(-1)\cdot(-1)-0\cdot 2=1$$

$$\mathbf{A}_{2,1}=\begin{pmatrix}2 & -1 & 0\\ -1 & 2 & -1\\ 0 & -1 & 2\end{pmatrix}\qquad \mathbf{A}_{2,2}=\begin{pmatrix}2 & -1 & 0\\ -1 & 2 & -1\\ 0 & -1 & 2\end{pmatrix}\qquad \mathbf{A}_{2,3}=\begin{pmatrix}2 & -1 & 0\\ -1 & 2 & -1\\ 0 & -1 & 2\end{pmatrix}$$

$$\left|\mathbf{A}_{2,1}\right|=(-1)\cdot 2-(-1)\cdot 0=-2\qquad \left|\mathbf{A}_{2,2}\right|=2\cdot 2-0\cdot 0=4\qquad \left|\mathbf{A}_{2,3}\right|=2\cdot(-1)-0\cdot(-1)=-2$$

$$\mathbf{A}_{3,1}=\begin{pmatrix}2 & -1 & 0\\ -1 & 2 & -1\\ 0 & -1 & 2\end{pmatrix}\qquad \mathbf{A}_{3,2}=\begin{pmatrix}2 & -1 & 0\\ -1 & 2 & -1\\ 0 & -1 & 2\end{pmatrix}\qquad \mathbf{A}_{3,3}=\begin{pmatrix}2 & -1 & 0\\ -1 & 2 & -1\\ 0 & -1 & 2\end{pmatrix}$$

$$\left|\mathbf{A}_{3,1}\right|=(-1)\cdot(-1)-2\cdot 0=1\qquad \left|\mathbf{A}_{3,2}\right|=2\cdot(-1)-(-1)\cdot 0=-2\qquad \left|\mathbf{A}_{3,3}\right|=2\cdot 2-(-1)\cdot(-1)=3$$

$$adj\left(\mathbf{A}\right)=\left[\left((-1)^{i+j}\det\left(\mathbf{A}_{i,j}\right)\right)_{i,j=1,2,\ldots,n}\right]$$

$$\mathbf{A}=\begin{pmatrix}2 & -1 & 0\\ -1 & 2 & -1\\ 0 & -1 & 2\end{pmatrix}$$

$$\begin{bmatrix}\left|\mathbf{A}_{1,1}\right|=3 & \left|\mathbf{A}_{1,2}\right|=-2 & \left|\mathbf{A}_{1,3}\right|=1\\ \left|\mathbf{A}_{2,1}\right|=-2 & \left|\mathbf{A}_{2,2}\right|=4 & \left|\mathbf{A}_{2,3}\right|=-2\\ \left|\mathbf{A}_{3,1}\right|=1 & \left|\mathbf{A}_{3,2}\right|=-2 & \left|\mathbf{A}_{3,3}\right|=3\end{bmatrix}\quad \Rightarrow adj\left(\mathbf{A}\right)=\begin{pmatrix}3 & 2 & 1\\ 2 & 4 & 2\\ 1 & 2 & 3\end{pmatrix}=adj\left(\mathbf{A}\right)^{T}$$

$$\mathbf{A}^{-1}=\frac{1}{\det\left(\mathbf{A}\right)}\cdot adj\left(\mathbf{A}\right)^{T}=\frac{1}{4}\begin{pmatrix}3 & 2 & 1\\ 2 & 4 & 2\\ 1 & 2 & 3\end{pmatrix}=\begin{pmatrix}\frac{3}{4} & \frac{1}{2} & \frac{1}{4}\\ \frac{1}{2} & 1 & \frac{1}{2}\\ \frac{1}{4} & \frac{1}{2} & \frac{3}{4}\end{pmatrix}$$

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