COMBINATORICS

THE MULTIPLICATION PRINCIPLE

Shoes : boots , black shoes — 2 ×
pants : Blue jeans — 1 × = 6 outfits
shirts : red , blue , yellow — 3

How many bit strings of size 4?

$\underset{3}{\underline{2}} \cdot \underset{2}{\underline{2}} \cdot \underset{1}{\underline{2}} \cdot \underset{0}{\underline{2}}$ ← can contain 16 numbers

0 - 15 $[2^0 + 2^1 + 2^2 + 2^3 = 15]$

Write a 3-letter string from {A, B, C, D}

w/ repetition: $\underline{\ \ }_4\ \underline{\ \ }_4\ \underline{\ \ }_4 = 64$

ABC ≠ BCA

w/out repetion: $\underline{\ \ }_4\ \underline{\ \ }_3\ \underline{\ \ }_2 = 24$

If we have k steps to complete a task,
In step 1, choose 1 of n_1
In step 2, " " n_2
⋮
In Step k, " " n_k options:
overall, we have $n_1 \cdot n_2 \cdot n_3 \cdot \ldots \cdot n_k$ possibilities to complete

THE ADDITION PRINCIPLE

If A, B are sets

$A \cap B = \emptyset$ Then $|A \cup B| = |A| + |B|$

How many 3 letter strings over $\{A, B, C, D\}$ (with repetition)

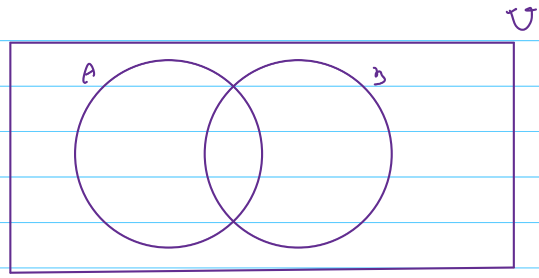
Start with A or starts with B:

Starts w/ A: $\underline{A}\ \underline{\ \ }_4\ \underline{\ \ }_4 = 16$ Answer $= 16 + 16 = 32$

Starts w/ B: 16

ex: AAA, BAA, ACB, ABC

INCLUSION-EXCLUSION PRINCIPLE

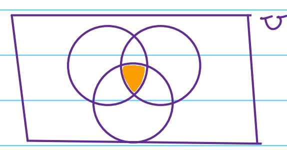


$|A \cup B| = |A| + |B| - |A \cap B|$

ex: $|A \cup (B \cup C)| = |A| + |B \cup C| - \underline{|A \cap (B \cup C)|}$

$|(A - B) \cup (A \cap C)|$

$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$



$\{A, B, C, D\}$; 3-letter strings; start or end w/ A (or both)

starts w/ A: $\underline{A}\ \underline{\ \ }\ \underline{\ \ }_4 = 16$

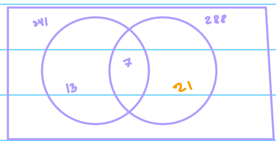
+

ends w/ A: $\underline{\ \ }\ \underline{\ \ }\ \underline{A} = 16$ $= \boxed{28}$

+

Intersection $|X \cap Y| = \underline{A}\ \underline{\ \ }\ \underline{A} = 4$

CS 241 has 20 students and 7 students also take CS 288. In both classes, there are 41 students. How many students in 288?



$|A \cup B| = 41$ $\qquad |A \cup B| = |A| + |B| - |A \cap B|$

$|A| = 20$

$|A \cap B| = 7$

$|B| = 41 - 20 + 7 = 28$

PERMUTATIONS

ordered selections

$\{1, 2, 3, 4\}$

$\underline{1}$ $\underline{2}$ $\underline{3}$ $\underline{4}$

4 3 2 1

1 3 4 2

a set of n distinct elements has n! permutations

$\underset{12}{\_} \ \underset{11}{\_} \ \underset{10}{\_} \ \ldots \ \underset{1}{\_} = 12!$

$\underset{12}{\_} \ \underset{11}{\_} \ \underset{10}{\_} \ \underset{9}{\_} \ \underset{8}{\_} = 12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 = \frac{12!}{7!} = P(12, 5)$ size of permutation

A set of n distinct elements have many k-permutations $(0 \le k \le n)$

we have: $\frac{n!}{(n-k!)} = p(n, k)$

How many 4-letter words can we make from A, B, C, D, E, F, G

$\underline{7}$ $\underline{6}$ $\underline{5}$ $\underline{4}$

How many 3 letter words can we make from A, B, C (no rep)

$\left.\begin{matrix} \underline{A} & \underline{B} & \underline{C} \\ \underline{B} & \underline{A} & \underline{C} \\ \underline{A} & \underline{C} & \underline{B} \end{matrix}\right\} 3!$

$\vdots$

How many sets can we make from A, B, C $\{A, B, C\}$

$\{B, A, C\}$

How many sets of size k are subsets of a set size n $(0 \le k \le n)$

$$P(n,k) = \frac{\frac{n!}{(n-k)!}}{k!} = \frac{n!}{(n-k)! \cdot k!} = \binom{n}{k} = C(n,k)$$

Choose operator

241 has 20 students. How many comittees of 5 can we choose?

$$\frac{20!}{5!(20-5)!} = \binom{20}{5}$$

$\{A, B, C, D, E, F, G\}$

3 people w/ B or w/ C

3 per committee w/ B (no C)

$\{B, \_, \_\}$ $\binom{5}{2} = 10$

C and no B = 10

C and B = 5

Answer = 25

ex:

4 Ants: A_1, A_2, A_3, A_4

3 bees: B_1, B_2, B_3

5 flies: f_1, f_2, f_3, f_4, f_5

lines: 12!

lines where all bugs are the same type are standing together: $4! \cdot 3! \cdot 5! \cdot 3!$

Committee of 3 flies & up to 1 bee:

$$\binom{5}{3}\binom{3}{0} + \binom{5}{3}\binom{3}{1}$$

permutations of $\{f_1, f_2, f_3, f_4, f_5\}$ where f_1, f_2 are next to each other

$$P = \frac{5!}{3!} \cdot 4$$

THE BINOMIAL THEOREM

$$(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^{n-i} y^i = \binom{n}{0} x^n y^0 + \binom{n}{1} x^{n-1} y^1 + \dots + \binom{n}{n} x^{n-n} y^n$$

Coeff. of a^3b^2 in the expansion of $(3a-2b)^5$?

$x = 3a$ $\qquad \binom{5}{2}(3a)^3(-b)^2 = \binom{5}{2} 27a^3 \cdot 4b^2 = 10 \cdot 27 \cdot 4 \; a^3b^2$

$y = -2b$ $\qquad \boxed{1080}\, a^3b^2$

Identity: $(k \le n)$

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$$

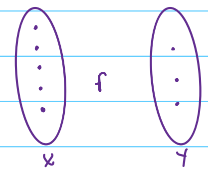
$\{A, B, C, D\} = \{A, C, D\} \cup \{B\}$ $\qquad \binom{4}{2} = \binom{3}{2} + \binom{3}{1}$

$\binom{3}{2}$ $\{A, C\}$ $\qquad \{B, A\}$ $\qquad \binom{3}{1}$

$\{A, D\}$ $\qquad \{B, C\}$

$\{C, D\}$ $\qquad \{B, D\}$

THE PIGEONHOLE PRINCIPLE



$|X| > |Y|$

f is not injective

There are $x_1 \neq x_2$ s.t. $f(x_1) = f(x_2)$

ex. $\{1, 2, 3, 4, 5, 6\}$

A subset of size 4 $\{\underline{6}\ \underline{5}\ \underline{4}\ \underline{\ }\ \underline{\ }\}$ s.t. the sum of every 2 elements is not 7

NO

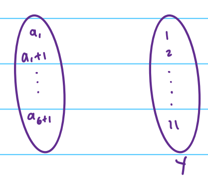
ex. $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

A subset of 6, no consecutive numbers

$\{1, 3, 5, 7, 9, x\}$

Proof: say $\{a_1, a_2, a_3, a_4, a_5, a_6\} \subseteq D$ w/ no consecutive #s

look at: $\{a_1+1, a_2+1, a_3+1, a_4+1, a_5+1, a_6+1\}_{1-11}$



2 variables from $\{a_1, a_1+1, a_2, a_2+1, \ldots a_6+1\}$ must have the same value

So one of $\{a_1, a_2, \ldots, a_6\}$ is the same as $\{a_1+1, a_2+1, \ldots, a_6+1\}$

It is not possible to have 6 diff #, one must be taken

$\{1, 3, 5, 7, 9, \underline{2}\}$ $\{2, 4, 6, 8, 10\}$

Prove that there is no subset of size 151 of $\{1, 2, 3 \cdots, 300\}$ w/ no two consecutive #s

Proof: say $\{a_1, a_2, \ldots a_{151}\}$ is such set, then

$\{a_1+1, \cdots a_{151}+1$

So we have 302 variables w/ 301 values so by the pigeonhole principle, $a_i+1 = a_j$ for some $1 \leq i, j \leq 151$

So the set has 2 consecutive numbers