# Properties of Binary Relations

Two Inputs Predicates

- Let $BT(x,y)$ be predicates over $\mathbb{Z} \times \mathbb{Z}$. $BT(x,y)$ is True iff $x \geq y$

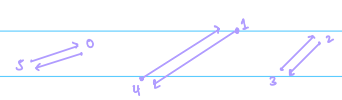
| | | |
|---|---|---|
| $BT(5,4)$ | $\neg BT(5,6)$ | $5 \geq 4$ |
| $(5,4) \in BT$ | $(5,6) \in BT$ | $5 \ngeq 6$ |

- Let $C_5(x,y)$ be a relation over $\{0,1,2,3,4,5\}$. $C_5(x,y)$ is True iff $x+y=5$

  $C_5 = \{(0,5),(5,0),(1,4),(4,1),(2,3),(3,2)\}$

- Let $R(x,y)$ be a binary predicate over D. Then we can Graph R.

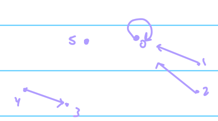
  Graph $C_5(x,y)$




- Let $R_2(x,y)$ be a relative over $\{0,1,2,3,4,5\}$

  $R_2 = \{(0,0),(1,0),(2,0),(4,3)\}$

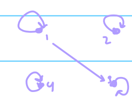
  $R_2$ Graph:



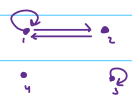
- Let $R(x,y)$ be a relation on $D$. $R$ is called Reflexive if $\forall x.\ R(x,x)$

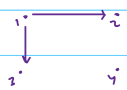
Ex: Draw a reflexive relation $R_3$ on $\{1,2,3,4\}$

$R_3 = \{(1,1),(2,2),(3,3),(4,4),(1,3)\}$



R is called symmetric if $\forall x.\ \forall y.\ (R(x,y) \rightarrow R(y,x))$



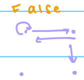


$R_1 \circ \{1,3\}$

R is Anti-symmetric if $\forall x.\ \forall y.\ ((x \neq y) \wedge R(x,y) \rightarrow R(y,x))$

no symmetries allowed in graph

False



symmetric

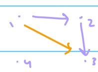
False

neither

R is Transitive if $\forall x.\ \forall y.\ \forall z.\ (R(x,y) \wedge R(y,z)) \rightarrow R(x,z)$

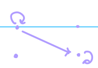
if there is a path in the graph, you also have an edge

False



no path from 2 to 3

$R_5$ :



$\cdot \{(1,1),(1,4),(4,4)\}$

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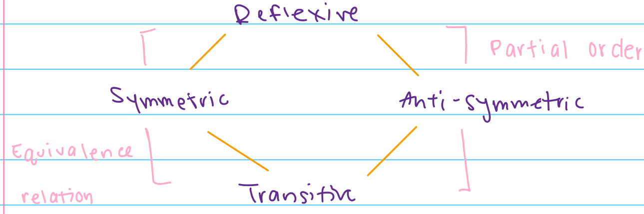



Prove that $R_1$ is NOT reflexive $\equiv$

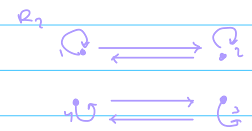
Disprove: $\forall x\, R_1(x,x)$, so prove $\neg \forall x.\ R_1(x,x)$, so prove $\exists x.\ \neg R(x,x)$

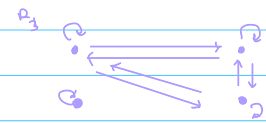
Proof: $a = 2$, $a \in \{1,2,3,4\}$ and $(2,2) \notin R_1 \iff \neg R(2,2)$

Therefore, $\exists x.\ \neg R_1(x,x)$, meaning $\neg(\forall . R_1(x,x))$

# Equivalence Relations

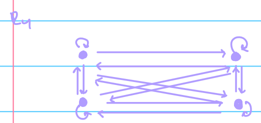
A binary relation $R(x,y)$ on $D$ is an equivalence relation if it is Reflexive, Symmetric, transitive



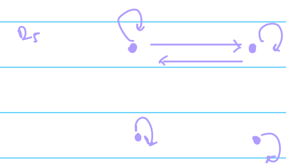




$[1]_{R_3} = \{1,2,3\} = [2]_{R_3} = [3]_{R_3}$

$[4]_{R_3} = \{4\}$




$[1]_{R_4} = \{1,2,3,4\}$




$[1]_{R_5} = \{1,2\}$

$[2]_{R_5} = \{1,2\}$

$[3]_{R_5} = \{3\}$

$[4]_{R_5} = \{4\}$

$R_5 = \{(1,1), (2,2), (3,3), (4,4), (1,2), 2,1)\}$

- Let $M3(x,y)$ on $\mathbb{N} \cup \{0\}$ $xM3y$ iff $x \% 3 = y \% 3$

  Prove M3 is a E.R.

Proof: ① M3 is Reflexive. Show $\forall x.\, M3(x,x)$. Let $a \in \mathbb{N} \cup \{0\}$. Show. $M3(a,a)$ [$\equiv aM3a$

$a \% 3 = a \% 3$ ✓

②

M3 is Symmetric: $\forall x. \forall y\, (x,y) \in M3 \rightarrow (y,x) \in M3$

Let $(a,b) \in (\mathbb{N} \cup \{0\}) \times (\mathbb{N} \cup \{0\})$

③ M3 is transitive: $\forall x \forall y \forall z\, (xM3y \wedge yM3z) \rightarrow (xM3z)$

Proof: Let $a, b, c \in \mathbb{N} \cup \{0\}$

| Data: | Conclusion: |
|---|---|
| $aM3b$ | $aM3c$ |
| $a \% 3 = b \%$ | $a \% 3 = c \% 3$ |
| $bM3c$ | by transitivity of "=" on $\mathbb{Z}$ |
| $b \% 3 = c \% 3$ | $a \% 3 = b \% 3 = c \% 3 \rightarrow a \% 3 = c \% 3$ |

We know $(a,b) \in M3$. Show $(b,a) \in M3$ $aM3b$ means $a \% 3 = b \% 3$ To show $bM3a$ means $b \% 3 = a \% 3$. By symmetry of "=" on $\mathbb{Z}$

So, by ①, ②, ③, conclude that M3 on $\mathbb{N} \cup \{0\}$ is a E.R.

# Rational Numbers

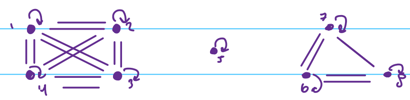
$Q = \{(n,m) \mid n \in \mathbb{Z} \wedge m \in \mathbb{Z} \wedge m \neq 0\}$

$\mathbb{Q}$

| | | |
|---|---|---|
| (1,2) | (-1,2) | (1,3) |
| (3,6) | (1,2) | (3,9) |
| (4,8) (-10,-20) | (-4,8) | (10,30) |
| ⋮ | ⋮ | ⋮ |

Let $REQ(q_1, q_2)$ on $\mathbb{Q}$

$(q_1, q_2) \in REQ$ iff $q_1 = (a,b)$ and $a \cdot d = b \cdot c$
$q_2 = (c,d)$



# ORDER RELATIONS

A Partial order is a binary relation on D that is Reflexive, Anti-Sym, transitive

- let $STE(x,y)$ on $\mathbb{Z}$ $x\,STE\,y$ if $x \leq y$.

  $\forall x. \forall y. ((x \neq y) \wedge R(x,y)) \rightarrow \neg R(y,x))$

  For $x, y \in \mathbb{Z}$ if $x \neq y \wedge x \leq y$ then $y \not\leq x$

Reflexive graph:

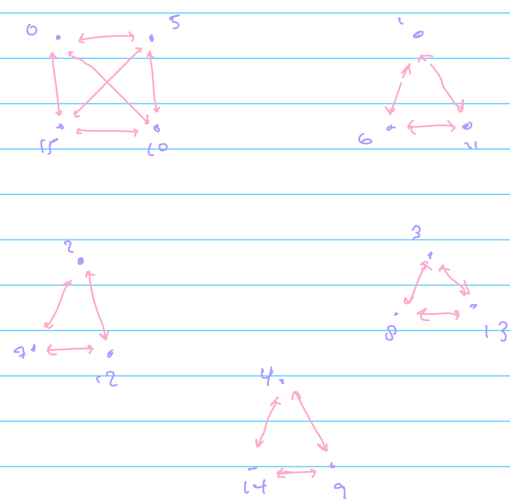
$D = \{0, 1, 2, \ldots 15\}$

D

| | | | | |
|---|---|---|---|---|
| 0 | 1 | 2 | 3 | 4 |
| 5 | 6 | 7 | 8 | 9 |
| 10 | 11 | 12 | 13 | 14 |
| 15 | | | | |

$M5(x,y)$ $x\,M5\,y$ iff $x \% 5 = y \% 5$

M5 on D



# PARTIAL ORDERS

$D = P(\{0,1,2\}) = \{\emptyset, \{0\}, \{1\}, \{2\}, \{0,1\}, \{0,2\}, \{1,2\}, \{0,1,2\}\}$

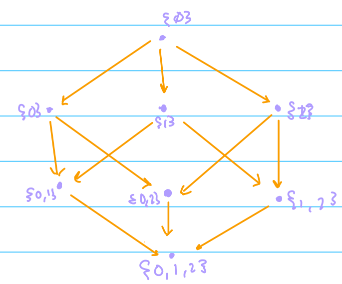
Look at "$\subseteq$"

For all $A \in D$, $A \subseteq A$ (prove)

prove $\forall A.\ A \subseteq A$

Let $A \in D$. Show $A \subseteq A$

Let $i \in A$ so $A \subseteq A$




reflexive, transitive graph

① $\subseteq$ ② $\supseteq$

$\{0\} \subseteq \{0,1\}$ $\{0\} \underset{\subseteq}{\supseteq} \{0\}$

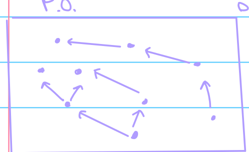
$\{0,2\} \supseteq \{2\}$ $\{0\} \underset{\not\supseteq}{\not\subseteq} \{1\}$

$\{1\}$ $\{0,1\}$

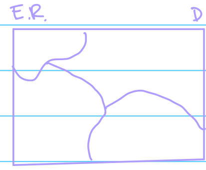
A total order is a partial order where for every $x, y \in D$ $R(x,y) \vee R(y,x)$

$D = \mathbb{Z}$, relation "$\leq$" for every $i, j \in \mathbb{Z}$ we have $x \leq y$ or $y \leq x$, if $x \leq y$ or $y \leq x$, if $x \leq y$ and $y \geq x$ then $x = y$

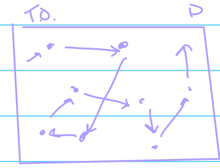
P.O. D



E.R. D



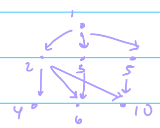
T.O. D



$D = \mathbb{Z}^+$ $3|3$, $6|6$, $14|14$, $3|6$, $6|3$, $12|48$, $(x|y \wedge y|x) \rightarrow (x=y)$

$x/y$ $x|y \wedge y|z \rightarrow x|z$

reflexive & transitive




$2 \nmid 3$ $3 \nmid 2$