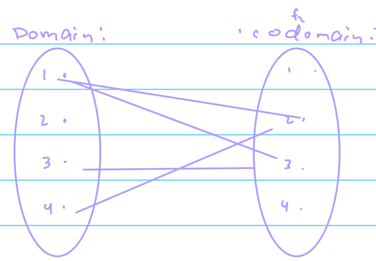
FUNCTIONS

A function $f: X \to Y$ is a relation $X \times Y$ s.t. for every $x \in X$ there is exactly one $y \in Y$ s.t. $f(x, y)$

$D = \{1, 2, 3, 4\}$



$f_1 = \{(1,1), (1,3), (2,3)\}$

× the input 1 has 2 outputs

× input "4", "3" has no output

$f_2 = \{(1,2), (2,3), (3,3), (4,2)\}$

$f: X \to Y$, $F(a,b)$ we write $F(a) = b$, $b = f(a)$ $\quad f(a) = f(c)$

$PL: \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$. $PL(a,b) = a+b$. $PL(1,2) = 3$

$t = 0 \quad r = 5$

$PL(t, r) = 15$

$D = \mathbb{Z}^+ \qquad \forall x.\ PL(x,x) > x$

- Constants impact bounded variable functions
- The defn. of a function only has "rules" about the input

Let $m3 = \mathbb{Z}^+ \to \{0, 1, 2\}$

$m3(x) = x \% 3$

$m3(3) = 0 \quad m3(5) = 2 \quad m3(6) = 0$

PROPERTIES OF FUNCTIONS

Let $f: X \to Y$ be a function

f is Onto (surjective)

if for every $y \in Y$, there is (at least one) $x \in X$ s.t. $f(x) = Y$.

f is Injective

for every $y \in Y$, y has at most one source, meaning, if $f(x_1) = y \wedge f(x_2) = y$ then $x_1 = x_2$

- $PL: \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ $\quad PL(x,y)$ is $x+y$

  (0, 1) , (-2, 3)

  → surjective

- f is surjective:

  $\forall y \in Y.\ \exists x \in X.\ (f(x) = y)$

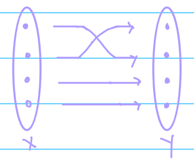
- $f: x \to y$ injective:

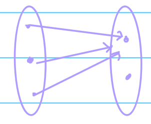
  $\forall y \in Y.\ \forall x_1 \in X.\ \forall x_2 \in X$

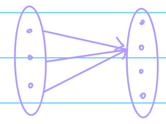
  $(f(x_1) = y \wedge f(x_2) = y) \to (x_1 = x_2)$

- **f is Bijective**

  if it is surjective and injective



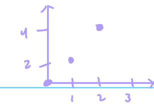




Let X, Y be sets

If there is a bijective $X \to Y$,

$|X| = |Y|$



Let $f: \mathbb{Z} \to \mathbb{Z}$ $f(x) = 2x$

Prove: ① f is not surjective:

Prove: $\neg(\forall y \in Y.\ \exists x \in X.\ (f(x)=y))$, $\exists y \in Y.\ \forall x \in X.\ (f(x) \neq y)$

Proof: Take $b=3$. Prove that for $\forall x \in X.\ f(x) \neq 3$

Let $a \in X$, $f(a) = 2a$

Assume $2a = 3$. Then $a = 1.5$, but $1.5 \notin \mathbb{Z}$, so

$\forall x \in X.\ 2x \neq 3$. Therefore

$\exists y \in Y.\ \forall x \in X.\ 2x \neq y$, so

$2x$ is NOT surjective

Prove: $f(x) = 2x$ is injective:

$\forall y \in Y.\ \forall x_1, x_2 \in X.\ (f(x_1) = y \wedge f(x_2) = y) \to x_1 = x_2$

Proof: Let $b \in Y$ $a_1, a_2 \in X$

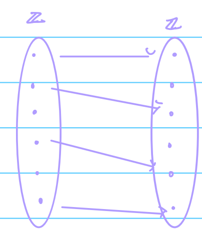
Assume: $f(a_1) = b$ $f(a_2) = b$

To show $a_1 = a_2$.

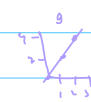
$f(a_1) = b$ means $2a_1 = b$

$f(a_2) = b$ means $2a_2 = b$

$\Big\}$ $2a_1 = 2a_2$, $a_1 = a_2$



$f(x) = 2x$ is injective



Let $g: \mathbb{R} \to \mathbb{R}$ $h: \mathbb{R} \to \mathbb{R}$ $g(x) = 2x$

- g is a bijection

Prove: g is surjective

Let $b \in \mathbb{R}$ show $\exists a \in \mathbb{R}.\ f(a) = b$

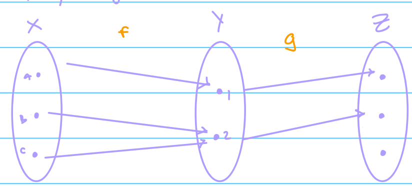
Take $a = \frac{b}{2}$ $b \in \mathbb{R}$ therefore, so $a \in \mathbb{R}$.

$f(a) = 2a = 2(\frac{1}{2}b) = b$

A function $f: X \to Y$ is a binary relation s.t. for every $x \in X$ there is exactly one $y \in Y$
s.t. $f(x,y)$ $f(x)=y$

Let $f: X \to Y$, $g: Y \to Z$
Then we define $g \circ f: X \to Z$
by $g \circ f(x) = g(f(x))$

Let $f: X \to Y$, $g: Y \to Z$



$g \circ f(a) = \alpha$

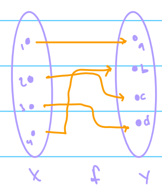
$g \circ f(b) = \beta$

$g \circ f(c) = \beta$

$g \circ f(x) = g(f(x))$

function composition

Let $f: X \to Y$ be a bijection f^{-1} is the inverse function of f, $f^{-1}: Y \to X$ is a bijection
for $x \in X$: $f(f(x)) = x$ $[f^{-1} \circ f(x) = x]$
for $y \in Y$: $[f \circ f^{-1}(y) = y]$



bijection:
Surj: every $y \in Y$ is reachable
+
injective: if " ", then it only has one source

$(f^{-1})^{-1} = f$

$R \to R$ $f(x) = x+1$ $g(x) = x^2$
$f \circ g(3) = f(3^2) = 3^2 + 1 = 10$
$g \circ f(3) = 16$
$f^{-1}(y) = y-1$

$f(f^{-1}(y)) = f(y-1) = (y-1)+1 = y$
$f^{-1}(f(x)) = f^{-1}(x+1) = (x+1)-1 = x$

let $b : N \cup \{0\} \to \mathbb{Z}$

write: b^{-1}

$b^{-1}(0) = 0$

$b^{-1}(2) = 3$

$b^{-1}(-2) = 4$

$\vdots$

$0 \to 0$

$1 \to 1$

$2 \to -1$

$3 \to 2$

$4 \to -2$

$5 \to 3$

$5 \to -3$

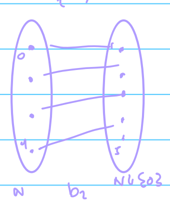
$$b(n) : \begin{cases} -\frac{n}{2} & \\ \frac{n+1}{2} & \text{else} \end{cases}$$

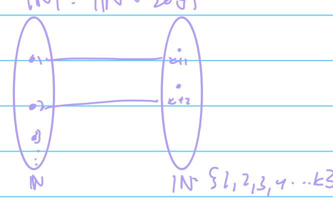
Let X, Y be sets $|X| = |Y|$ if there is a bijection $X \to Y$, so

$|N \cup \{0\}| = |\mathbb{Z}|$

find bijection $N \to N \cup \{0\}$ so

$b_2(n) = n-1$ $|N| = |N \cup \{0\}|$





$|N| = \aleph_0$ ← aleph

- bijectives for $N \to$ {positive even numbers}
  - $b_3(n) = 2n$
  - $b_3^{*}(n) = 3n$
  - $b_3^{**}(n) = 4n - 3$

$|N| = |N \cup \{0\}| = |\{3n \mid n \in N\}| = |\text{Evens}| \leftarrow \aleph_0$