$\{a\}_{n=0}^{\infty}$ $\quad a_n = C_1 \cdot a_{n-1} + C_2 a_{n-2} \quad n \geq 2$

$a_0 = A_0 \quad a_1 = A_1 \quad C_1, C_2, A_0, A_1$: constants

[Find explicit formula]

ex: 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, ...

$f_n = f_{n-1} + f_{n-2}, \ n \geq 2 \quad f_0 = 0 \ f_1 = 1$

Step 1:

Solve for t: $\quad t^2 - C_1 t - C_2 = 0 \quad$ (solutions r_1, r_2)

Case 1: $\quad r_1 \neq r_2$

The explicit formula take form:

$a_n = B r_1^n + D r_2^n \quad$ (B, D constants)

Case 2: $\quad r_1 = r_2$

The explicit formula will be like:

$a_n = B r_1^n + D r_1^n \cdot n \quad$ (B, D constants)

Step 2

Case 1: $\quad r_1 \neq r_2$

for $n = 0$: $\quad a_0 = B r_1^0 + D r_2^0 = A_0$

for $n = 1$: $\quad a_1 = B r_1^1 + D r_2^1 = A_1$

Case 2: $\quad r_1 = r_2$ $\qquad$ Solve for B and D

$a_0 = B \cdot 1 + D \cdot 1 \cdot 0 = A_0$

$a_1 = B r_1^1 + D r_1^1 \cdot 1 = A_1$

ex:

Let $a_n = 5a_{n-1} \cdot 6a_{n-2}$ $\quad n \geq 2$, $a_0 = 7$, $a_1 = 16$

$7, 16, \underbrace{38}_{5\cdot16-6\cdot7}, \underbrace{94}_{5\cdot38-6\cdot16}, \ldots$

step 1:

solve: $t^2 - 5t + 6 = 0$

$(t-2)(t-3) = 0 \qquad r_1 = 2, r_2 = 3$

so:

$a_n = B2^n + D3^n$

Step 2:

$n=0$

$a_0 = B\cdot2^0 + D\cdot3^0 = B+D = A_0 \Rightarrow B+D=7$

$n=1$:

$a_1 = B\cdot2^1 + D\cdot3^1 = 2B+3D = 16$

$$\begin{cases} B+D=7 \rightarrow B=7-D \\ 2B+3D=16 \end{cases}$$

$2(7-D)+3D=16$

$14 - 2D + 3D = 16 \rightarrow \boxed{D=2} \quad \boxed{B=5}$

ex.

Let $f_n = 1 \cdot f_{n-1} + 1 \cdot f_{n-2}$ for $n \geq 2$, $a_0 = 0$ $a_1 = 1$

Step 1:

Solve: $t^2 - t - 1 = 0$ $\quad \frac{1 \pm \sqrt{1 - 4(1)(-1)}}{2} = \frac{1 \pm \sqrt{5}}{2}$

so:

$$a_n = B\left(\frac{1+\sqrt{5}}{2}\right)^n + D\left(\frac{1-\sqrt{5}}{2}\right)^n$$

a_0:

$a_0 = B + D = 0 \Rightarrow B = -D$

a_1:

$a_1 = B\left(\frac{1+\sqrt{5}}{2}\right) + D\left(\frac{1-\sqrt{5}}{2}\right) = 1 \Rightarrow \frac{B + B\sqrt{5}}{2} + \frac{D - D\sqrt{5}}{2} = 1$

$\Rightarrow B + B\sqrt{5} + D - D\sqrt{5} = 2$

Solve:

$\cancel{-D} - D\sqrt{5} + \cancel{D} - D\sqrt{5} = 2 \Rightarrow -2D\sqrt{5} = 2 \Rightarrow D = \frac{-1}{\sqrt{5}} =$

$B = \frac{1}{\sqrt{5}}$ $\quad D = -\frac{1}{\sqrt{5}}$

Into Fibonacci sequence

$$f_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

Solution:

$f_0 = 0$

$f_1 = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right) - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right) = \frac{1+\sqrt{5}}{2\sqrt{5}} - \frac{1-\sqrt{5}}{2\sqrt{5}} = \frac{2\sqrt{5}}{2\sqrt{5}} = 1$

$f_2 = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^2 - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^2$

ex:

$a_n = 4a_{n-1} - 4a_{n-2}$ $\quad a_0 = 0 \quad a_1 = 4$

Find the explicit formula:

step1: Solve

$t^2 - 4t + 4 = 0$

$(t-2)^2 = 0 \quad t = 2$

$r_1 = r_2 = 2$

2nd case !

$a_n = B \cdot 2^n + D \cdot n \cdot 2^n$ for some constants B, D

$a_0 = B = 0$

$a_1 = 2B + 2D = 4 \Rightarrow B + D = 2$

$B = 0 \quad D = 2$

formula:

$a_n = 2n2^n = n2^{n-1}$

a_0, a_1, a_2, a_3

$0, 4, \underbrace{16}_{4\cdot 4 - 4\cdot 0}, \underbrace{48}_{4\cdot 16 - 4\cdot 4}, \ldots$

A sequence : $a_n = 5 \cdot 2^n + 2 \cdot 3^n$ for all $n > 0$

$a_0 = 7, \quad a_1 = 16, \quad a_2 = 5 \cdot 2^2 + 2 \cdot 3^2 = 38$

$a_i = 5 \cdot 2^i + 2 \cdot 3^i$

$a_{i-1} = 5 \cdot 2^{(i-1)} + 2 \cdot 3^{i-1}$

$a_{i-2} = 5 \cdot 2^{(i-2)} + 2 \cdot 3^{(i-2)}$

Validate (True / False)

$a_n = 5a_{n-1} - 6a_{n-2}$ for $n \geq 2$, and $a_0 = 7 \quad a_1 = 1$