

ECE 3100 - Functions, Formulas, and Definitions

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1 Pre - Prelim 1

1.1 Lecture 1 - What is Probability?

Probability is a way of mathematically modelling situations involving uncertainty with the goal of making predictions decisions and models. Probability can be understood in many ways, such as:

1. Frequency of Occurrence: Or percentage of successes in a moderately large number of similar situations.
2. Subjective belief: Or certainty based on other understood facts about a claim.

For our Probability Models, we define the set of all outcomes to be Ω , better known as the **sample space** of an experiment. All subsets of Ω are called **events**. These are both sets and can be understood using default set notation.

1.2 Lecture 2 - Probability Law

Given Ω chosen, a **probability law** on Ω is a mapping \mathbb{P} that assigns a number for every event such that:

$$\begin{array}{l} \mathbb{P}(A) \geq 0 \quad \text{for every event } A \\ \mathbb{P}(\Omega) = 1 \quad (\text{normalization}) \end{array} \quad (\text{Kolmogorov's Axioms})$$

1.2.1 Additivity rules:

- If $A \cap B = \emptyset$, (A, B) events, then:

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) \quad (1)$$

- If events A_1, A_2, \dots are all disjoint, then:

$$\mathbb{P}\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mathbb{P}(A_n) \quad (2)$$

By these rules, we can surmise that $\mathbb{P}(\emptyset) = 0$.

For any events A, B :

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B) \quad (\text{Event Union})$$

When we have a probability law on a finite Ω with all outcomes equally likely (i.e. $\mathbb{P}(\{s\}) = 1/\text{size}(\Omega)$), we call this probability law \mathbb{P} a **(discrete) uniform probability law**.

1.3 Lecture 3 - Conditional Prob & Product Rule

1.3.1 Conditional Probability

Conditional Probability is defined $\mathbb{P}(A | B) = \text{"Probability of } A \text{ given } B\text{"}$. It is understood as the likelihood that event A occurs, given that B also occurs.

$$\boxed{\mathbb{P}(A | B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}} \quad (\text{Conditional Probability Def})$$

If there is a finite number of different outcomes that are all equally likely, the conditional probability can be written as follows:

$$\boxed{\mathbb{P}(A | B) = \frac{\text{number of elements of } A \cap B}{\text{number of elements of } B}} \quad (3)$$

1.3.2 Product Rule

There are two main ways to write the product rule and they both have different setups.

- If we have events D_1 to D_n where $D_1 > D_2 > \dots > D_n$ (D_1 largest, D_n smallest), then we can apply the first form of the product rule:

$$\boxed{\mathbb{P}(D_n) = \mathbb{P}(D_1)\mathbb{P}(D_2 | D_1)\mathbb{P}(D_3 | D_2) \dots \mathbb{P}(D_n | D_{n-1})} \quad (\text{Product Rule 1})$$

- If we have events A_1 to A_n with non-empty intersection (i.e. $A_1 \cap A_2 \cap \dots \cap A_n$), let $D_k = A_1 \cap A_2 \cap \dots \cap A_k$, then $D_1 > D_2 > \dots > D_n$. If we then write the product rule on the events D_n in terms of A_n , we get:

$$\boxed{\mathbb{P}(A_1 \cap \dots \cap A_n) = \mathbb{P}(A_1)\mathbb{P}(A_2 | A_1)\mathbb{P}(A_3 | A_1 \cap A_2) \dots \mathbb{P}(A_n | A_1 \cap \dots)} \quad (\text{Product Rule 2})$$

1.4 Lecture 4 - Total Probability

Given an event B , say C_1, C_2, \dots, C_n (events) is a **partition** of B when:

- $B = C_1 \cup C_2 \cup \dots \cup C_n$
- C 's are all disjoint

If A_1, A_2, \dots, A_n is a partition of Ω , then C_1, C_2, \dots, C_n partitions B , where $C_k = B \cap A_k$ for $1 \leq k \leq n$.

From there we define the Total Probability Theorem:

$$\boxed{\mathbb{P}(B) = \mathbb{P}(B | C_1)\mathbb{P}(C_1) + \mathbb{P}(B | C_2)\mathbb{P}(C_2) + \dots + \mathbb{P}(B | C_n)\mathbb{P}(C_n)} \quad (\text{Total Probability Theorem})$$

1.5 Lecture 5 - Bayes Law & Independence

1.5.1 Bayes Law

Bayes' Rule is defined by mixing the definition of Condition Probability, and the Total Probability Theorem.

Given Ω, \mathbb{P} , if A_1, A_2, \dots, A_n are events that partition Ω , and have nonzero $\mathbb{P}(A)$, then for any event B ,

$$\boxed{\mathbb{P}(A_k | B) = \frac{\mathbb{P}(B | A_k)\mathbb{P}(A_k)}{\mathbb{P}(B | A_1)\mathbb{P}(A_1) + \dots + \mathbb{P}(B | A_n)\mathbb{P}(A_n)}} \quad (\text{Bayes' Law})$$

1.5.2 Independence

Given Ω, \mathbb{P} , any events A and B are **independent** when:

$\begin{aligned} \mathbb{P}(A \cap B) &= \mathbb{P}(A)\mathbb{P}(B) \\ \mathbb{P}(A \mid B) &= \mathbb{P}(A), \mathbb{P}(B) > 0 \end{aligned}$	or	
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(Independence Def)

1.6 Lecture 6 - Conditional Dependence & Counting

1.6.1 Conditional Dependence

Given Ω and \mathbb{P} : say that events A and B are **conditionally independent** given C when:

$\begin{aligned} \mathbb{P}(A \cap B \mid C) &= \mathbb{P}(A \mid C)\mathbb{P}(B \mid C) \\ \mathbb{P}(A \mid B \cap C) &= \mathbb{P}(A \mid C), \mathbb{P}(B \cap C) > 0 \end{aligned}$	or	
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(Conditional Independence Def)

1.6.2 Counting

Counting is the process of using the number of elements in the events to calculate probability. This technique mostly arises in situations where either:

- The sample space Ω has a finite number of equally likely outcomes. Then, for any event A ,

$$\mathbb{P}(A) = \frac{\# \text{ of elements of } A}{\# \text{ of elements of } \Omega}$$

- An event A has a finite number of equally likely outcomes with probability p . Then for that event A :

$$\mathbb{P}(A) = p \cdot (\# \text{ of elements of } A)$$

1.7 Lecture 7 - Counting

Counting Principle: in a process with a sequence of stages $1, 2, \dots, r$ with n_1 choices at stage 1 over to n_r at stage r ; # of outcomes is $n_1 n_2 \dots n_r$.

Can be used to rederive $(\# \text{ subsets of } \Omega) = 2^{\#(elem)}$.

1.7.1 k -permutations of n objects

We are given n distinct objects and a number $k \leq n$, and we want to find out the number of ways we could take k distinct objects from the group of n objects and arrange them in a sequence. By using the Counting Principle, we can find that the **number of k -permutations** of this set is:

$\begin{aligned} n(n-1) \dots (n-k+1) &= \frac{n(n-1) \dots (n-k+1)(n-k) \dots 2 \cdot 1}{(n-k) \dots 2 \cdot 1} \\ &= \frac{n!}{(n-k)!} \end{aligned}$		
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(K-permutations)

Special Case: If $k = n$, then the number of k -permutations of n objects is simply $n!$.

1.7.2 k -combinations of n objects

For finding the number of k -combinations, we can look back to our k -permutations and reason about them. Say we have the same setup as before but we are not arranging the items in a sequence. For each combination, we have $k!$ “duplicate” permutations. Thus, we can look at the number permutations and reason that the number of k -combinations should be that over $k!$, making the **number of k -combinations** of this set is:

$$\boxed{\frac{n!}{k!(n-k)!} = \binom{n}{k}} \quad (\text{K-combinations})$$

1.8 Lecture 8 - Discrete Random Variables

1.8.1 Random Variables

Given Ω and \mathbb{P} , a **discrete random variable (r.v.)** is a real valued function with domain Ω that takes on only finite or countably infinite number of different values (i.e. $X : \Omega \rightarrow \mathbb{R}$).

1.8.2 Probability Mass Functions

Given Ω, \mathbb{P} , associated with any discrete rv $X : \Omega \rightarrow \mathbb{R}$ is X 's **probability mass function (pmf)** - notation p_X .

$$\boxed{\forall x \text{ of } X, p_X(x) = \mathbb{P}(A_X) \text{ where } A_X = \{s \in \Omega : X(s) = x\}} \quad (\text{pmf Def})$$

Things to Note:

- $\mathbb{P}(A_X)$ can also be written as $\mathbb{P}(\{X = x\})$ or $\mathbb{P}(X = x)$.
- $p_X(x) \geq 0$ for all possible values of X .
- A pmf is essentially a probability law on the different values in the codomain of a random variable, so the same laws that apply to probability laws apply to pmfs:

$$p_X(x) \geq 0 \quad \text{for every } x \in X$$
$$\sum_{x \in X} p_X(x) = 1 \quad (\text{normalization})$$

- If V is any finite or countably inf. set of possible values of X , then if we set $B = \{\text{the event “} X \in V \text{”}\}$, (i.e. $B = \{s \in \Omega : X(s) \in V\}$), then $\mathbb{P}(B) = \sum_{x \in V} p_X(x)$.

Note: for a given pmf, there are multiple Ω 's, \mathbb{P} 's, X 's that lead to that PMF.

1.8.3 Common PMFs

- **Discrete uniform pmf of interval** $a \leq k \leq b$, $a, b \in \mathbb{N}$:

$$p_X(k) = \begin{cases} \frac{1}{b-a+1} & \text{when } a \leq k \leq b \\ 0 & \text{all over } k \end{cases}$$

- Let $p \in [0, 1]$; the **Bernoulli p pmf** be defined by:

$$p_X(k) = \begin{cases} p & \text{when } k = 1 \\ 1 - p & \text{when } k = 0 \end{cases}$$

- Given positive integer n , some $p \in [0, 1]$, the **Binomial(n, p) pmf** is defined as:

$$p_X(k) = \binom{n}{k} p^k (1 - p)^{n-k} \quad 0 \leq k \leq n$$

This pmf tends to show up in situations involving sequences of independent trials, such as coin flips. Useful if you are trying to find the probability of k heads in n coin flips.

- Given $p \in (0, 1)$ the **geometric pmf** defined by:

$$p_X(k) = p(1 - p)^{k-1} \quad \text{for all } 1 \leq k \leq \infty \text{ positive integers}$$

This pmf tends to show up in situations such as $\mathbb{P}(\text{it takes } k \text{ flips to flip a heads})$.

- Poisson(λ):

$$p_X(k) = e^{-\lambda} \frac{\lambda^k}{k!} \quad 0 \leq k \leq \infty (k \in \mathbb{N})$$

1.9 Lecture 9 - Expectation, Variance

1.9.1 Function of a Random Variable

Given a random variable X and any function $g : \mathbb{R} \rightarrow \mathbb{R}$, can define another r.v. $Y = g(X)$:

$$\forall s \in \Omega, Y(s) = g(X(s))$$

The function of a discrete r.v. is another discrete r.v.. Generally, it is non-trivial to get the pmf of $Y = g(X)$, but it is sometimes easy. (See examples)

1.9.2 Expected Value

Given a discrete r.v. X with $p_X(x)$ pmf, we define the **expected value (expectation)**:

$$\boxed{\mathbb{E}(X) = \sum_{x \in X} x p_X(x)} \quad (\text{Expected Value Definition})$$

Given $X, Y = g(X)$, what is $\mathbb{E}(Y)$? One way is to figure out $p_Y(y)$ for all possible values of $y \in Y$ and then find it through:

$$\mathbb{E}(Y) = \sum_{y \in Y} y p_Y(y)$$

and get P_y through p_x , though that is generally a non-trivial solution. Another possible solution is to use the Expected Value Rule.

1.9.3 Expected Value Rule

Given X, p_X , and $Y = g(X)$,

$$\boxed{\mathbb{E}(Y) = \sum_{x \in X} g(x)p_X(x)} \quad (\text{Expected Value Rule})$$

Special Case: $Y = \alpha X + \beta$

$$\begin{aligned} \mathbb{E}(Y) &= \sum_{x \in X} g(x)p_X(x) \\ &= \sum_{x \in X} (\alpha x + \beta)p_X(x) \\ &= \alpha \sum_{x \in X} xp_X(x) + \beta \sum_{x \in X} p_X(x) \\ &= \alpha \mathbb{E}(X) + \beta \end{aligned}$$

1.9.4 Variance

Given an rv X with pmf p_X , we define **Variance** to be:

$$\boxed{Var(X) = \mathbb{E}((X - \mathbb{E}(X))^2)} \quad (\text{Variance Def})$$

Off of this definition, we also define standard deviation to be $\sigma_X = \sqrt{Var(X)}$.

1.10 Lecture 10 - Expected Value and Variance Examples

2 Prelim 1 - Prelim 2

- 2.1 Lecture 11 - Multiple Discrete RVs, Joint pmf's, Conditionals**
- 2.2 Lecture 12 - Conditional Probability for RV's, Conditional Expectation**
- 2.3 Lecture 13 - Conditional Expectation, Independence of RVs**
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