**Additivity rules:**

For any events $A, B$:

$$\boxed{\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)}$$

**Counting**

$$\boxed{\text{comb:} = \frac{n!}{(n-k)!} \text{ perm:} = \frac{n!}{k!(n-k)!} = \binom{n}{k}}$$

**Conditional Probability**

$$\boxed{\mathbb{P}(A \mid B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}}$$

$$\boxed{\begin{array}{l} \mathbb{P}(A \cap B \mid C) = \mathbb{P}(A \mid C)\mathbb{P}(B \mid C) \qquad \text{or} \\ \mathbb{P}(A \mid B \cap C) = \mathbb{P}(A \mid C), \mathbb{P}(B \cap C) > 0 \end{array}}$$

**Product Rule**

- With events $D_1$ to $D_n$ where $D_1 > D_2 > \cdots > D_n$ ($D_1$ largest, $D_n$ smallest):

$$\boxed{\mathbb{P}(D_n) = \mathbb{P}(D_1)\mathbb{P}(D_2 \mid D_1)\mathbb{P}(D_3 \mid D_2)\ldots\mathbb{P}(D_n \mid D_{n-1}} \tag{Product Rule 1}$$

- With events $A_1$ to $A_n$ with non-empty intersection, let $D_k = A_1 \cap A_2 \cap \cdots \cap A_k$, then $D_1 > D_2 > \cdots > D_n$:

$$\boxed{\mathbb{P}(A_1 \cap \cdots \cap A_n) = \mathbb{P}(A_1)\mathbb{P}(A_2 \mid A_1)\mathbb{P}(A_3 \mid A_1 \cap A_2)\ldots\mathbb{P}(A_n \mid A_1 \cap \ldots)} \tag{Product Rule 2}$$

**Total Probability Theorem in Bayes, & Independence**

$$\boxed{\mathbb{P}(A_k \mid B) = \frac{\mathbb{P}(B \mid A_k)P(A_k)}{\mathbb{P}(B \mid A_1)\mathbb{P}(A_1) + \cdots + \mathbb{P}(B \mid A_n) \text{or} \mathbb{P}(B)}}$$

$$\boxed{\begin{array}{l} \mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B) \qquad \text{or} \\ \mathbb{P}(A \mid B) = \mathbb{P}(A), \mathbb{P}(B) > 0 \end{array}}$$

**Covariance**

$$\boxed{\begin{aligned} &Cov(X,Y) = \mathbb{E}((X - \mathbb{E}(X))(Y - \mathbb{E}(Y))) \\ &\rho = \frac{Cov(X,Y)}{\sigma_X \sigma_Y} \\ &Cov = 0 \Rightarrow Var(X+Y) = Var(X) + Var(Y) \\ &\text{indep} \Rightarrow Cov = 0 \end{aligned}}$$

**Pmfs**

$$\boxed{\begin{aligned} &p_{X,Y}(x,y) = \mathbb{P}(\{X = x\} \cap \{Y = y\}) \\ &\mathbb{E}(Z) = \sum_{x \in X} \sum_{y \in Y} g(x,y) p_{X,Y}(x,y) \\ &p_{X|A}(x) = \frac{\mathbb{P}(\{X = x\} \cap A)}{\mathbb{P}(A)} \quad p_{X|Y}(x \mid y) = \frac{p_{X,Y}(x,y)}{p_Y(y)} \end{aligned}}$$

**Discrete Random Variables**

- Let $p \in [0,1]$; the **Bernoulli p pmf** be defined by:

$$p_X(k) = \begin{cases} p & \text{when } k = 1 \\ 1 - p & \text{when } k = 0 \end{cases}$$

E(X) = p; Var(X) = p(1 - p)

- Given positive integer $n$, some $p \in [0,1]$, the **Binomial(n,p) pmf** is defined as:

$$p_X(k) = \binom{n}{k} p^k (1-p)^{n-k} \quad 0 \le k \le n$$

- Poisson(X):

$$p_X(k) = e^{-\lambda} \frac{\lambda^k}{k!} \quad 0 \le k \le \infty (k \in \mathbb{N})$$

E(X) $= \lambda; Var(X) = \lambda$

**Expectation, Variance**

$$\boxed{\begin{aligned} \mathbb{E}(X) &= \sum_{x \in X} x p_X(x) \\ \mathbb{E}(Y) &= \sum_{x \in X} g(X) p_X(x) \\ \mathbb{E}(X \mid A) &= \int_{-\infty}^{\infty} x f_{X|A}(x) dx \\ \mathbb{E}(X \mid Y = y) &= \int_{-\infty}^{\infty} x f_{X|Y}(x \mid y) dx \forall y \\ \mathbb{E}(X) &= \mathbb{E}(\mathbb{E}(X \mid Y)) \end{aligned}}$$

$$\boxed{\begin{aligned} Var(X) &= \mathbb{E}((X - \mathbb{E}(X))^2) \\ Var(X) &= \mathbb{E}(X^2) - (\mathbb{E}(X))^2 \\ \sigma_X &= \sqrt{Var(X)} \\ Var(X) &= \mathbb{E}(Var(X \mid Y)) + Var(\mathbb{E}(X \mid Y)) \\ Var(X) &= \mathbb{E}[(X - \mathbb{E}[X])^2] \\ Var(X) &= \int_{-\infty}^{\infty} (X - \mathbb{E})^2 f_X(x) dx \\ Var(X \mid Y) &= \mathbb{E}((X - \mathbb{E}(X \mid Y))^2 \mid Y) \end{aligned}}$$

**Pdf, Cdf Def**

$$\mathbb{P}(a \le X \le b) = \int_a^b f_X(x)dx$$

$$F_X(x) = \int_{-\infty}^x f_X(t)dt \quad \underline{\textbf{cont}}$$

$$f_X(x) = \frac{d}{dx}F_X(x)$$

$$F_X(x) = \mathbb{P}(\{X \le x\}) \quad \underline{\textbf{discr}}$$

$$\mathbb{P}(\{X \in V\}) = \int_V f_X(x)dx$$

$$F_X(x) = \sum_{\{x_k | x_k \le x\}} p_X(x_k)$$

$$p_X(x_k) = F_X(x_k) - F_X(x_{k-1})$$

**Mean and Variance: Continuous**

$$\mathbb{E}(X) = \int_{-\infty}^{+\infty} x f_X(x)dx$$

$$Var(X) = \mathbb{E}[(X - \mathbb{E}[X])^2]$$

**Conditional Prob Def**

$$\mathbb{P}(A \mid B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$$

**Types of continuous rvs**

- **X uniform on [a,b]:**

$$f_X(x) = \begin{cases} \frac{1}{b-a} & \text{when } x \in [a,b] \\ 0 & \text{else} \end{cases}$$

$$F_X(x) = \begin{cases} 0 & \text{when } x < a \\ \frac{x-a}{b-a} & \text{when } a \le x \le b \\ 1 & \text{when } x > b \end{cases}$$

$$\mathbb{E}[X] = \frac{b+a}{2} \qquad Var(X) = \frac{(b-a)^2}{12}$$

- **standard normal:**

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \qquad \therefore \mathbb{E}(X) = 0,\ Var = 1$$

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-t^2/2}dt$$

$$\mathbb{P}(\{X > x\}) = \mathbb{P}(\{\sigma + Y + M > x\}) = \mathbb{P}\Big(Y > \frac{x-M}{\sigma}\Big)$$

**Joint Pdf**

$$\mathbb{P}(\{(X,Y) \in V\}) = \iint_{-V} f_{X,Y}(x,y)dxdy$$

$$f_X(x) = \int_{-\infty}^{+\infty} f_{X,Y}(x,y)dy$$

$$\mathbb{E}[g(X,Y)] = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} g(X,Y) f_{X,Y}(x,y)dxdy$$

$$f_Y(y) = \int_{-\infty}^{+\infty} f_{X,Y}(x,y)dx$$

$$F_{X,Y}(x,y) = \mathbb{P}(X \le x, Y \le y)$$

$$f_{X,Y}(x,y) = \frac{\delta F_{X,Y}}{\delta x \delta y}(x,y)$$

**Indpendence: Continuous**

- $f_{X,Y}(x,y) = f_X(x) f_Y(y) \quad \forall x, y$
- $\mathbb{E}(XY) = \mathbb{E}(X)\mathbb{E}(Y)$
- $\mathbb{E}(g(X)h(Y)) = \mathbb{E}(g(X))\mathbb{E}(h(Y))$
- $Var(X+Y) = Var(X) + Var(Y)$

**Conditioning on Event**

$$\mathbb{P}(\{X \in V\} \mid A) = \int_V f_{X|A}(x)dx$$

$$f_{X|A}(x) = \begin{cases} \frac{f_X(x)}{\mathbb{P}(\{X \in W\})} & \text{when } X \in W \\ 0 & \text{otherwise} \end{cases}$$

$$\mathbb{E}(X \mid A) = \int_{-\infty}^{+\infty} x f_{X|A}(x)dx$$

$$\mathbb{E}(X \mid A) = \sum_{x \in X} x p_{X|A}(x)$$

**Conditioning on rv**

$$f_{X|Y}(x \mid y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

$$f_X(x) = \int_{-\infty}^{+\infty} f_Y(y) f_{X|Y}(x \mid y)dy$$

$$\mathbb{E}[X \mid Y = y] = \int_{-\infty}^{+\infty} x f_{X|Y}(x \mid y)dx$$

**Inequals, Moment Functions, Limit Theorems**

$$\mathbb{P}(\{|X - \mu| \ge c\}) \le \frac{Var(X)}{c^2}$$

$$\mathbb{P}(\{X \ge c\}) \le \frac{\mathbb{E}(X)}{c}$$

$$M_X(s) = \mathbb{E}(e^{sX})$$

$$M_{\alpha Y + \beta} = e^{\beta s} M_Y(\alpha s)$$

$$S_n = X_1 + \cdots + X_n \text{iid} \qquad \mathbb{E}(S_n) = n\mu \qquad Var(S_n) = n\sigma^2$$

$$M_n = \frac{1}{n} S_n \qquad \mathbb{E}(M_n) = \mu \qquad Var(M_n) = \frac{^2}{n}$$

$$Z_n = \frac{X_1 + \cdots + X_n - n\mu}{\sqrt{n}\sigma} \qquad \mathbb{E}(Z_n) = 0 \qquad Var(Z_n) = 1$$