HW7: Measuring VaR and Expected Shortfall using Extreme Value Theory

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Introduction

Extreme Value Theory (EVT): What's happening in the tails of distributions.

Why EVT

- Center of distribution have specific form which might be different from the tails.
- Measures of risk (VaR, ES) are influenced by the tail distribution.
- Usual methods (Gaussian), undershoot the true value of risk measures.

Peaks-Over-Threshold (POT)

The Peaks-Over-Threshold (POT) approach assumes that the losses Y , conditionally on being above a threshold u, are distributed as a generalised Pareto distribution (GPD), hence

$$Z \sim GPD(\xi, u, \sigma)$$

with cumulative density function:

$$F_{\xi,u,\sigma}(z) = 1 - \left(1 + \xi \left(\frac{z-u}{\sigma}\right)\right)^{-1/\xi}, \quad \text{for} \quad \xi \neq 0,$$

I the case of this HW the location parameter u, i.e. is the threshold value, is such that $u = 10 \cdot 10^6$, or $u = 20 \cdot 10^6$. It's the threshold at which we consider an observation "extreme".

We have two parameters to estimate: $\hat{\xi}$ and $\hat{\sigma}$.

Methods of Moments Estimator (MM)

Estimators

$$\widehat{\xi}_{MM} = \frac{1}{2}\left(1 - \frac{(\bar{m}-u)^2}{S^2}\right), \qquad \widehat{\sigma}_{MM} = \frac{\bar{m}-u}{2}\left(\frac{(\bar{m}-u)^2}{S^2} + 1\right), \qquad (1)$$

where $\bar{m}$ is the empirical mean and S^2 the empirical variance.

Implementation

1. Compute the Empirical mean with numpy functions mean and var.
2. Compute the closed form estimators as described in Equation (1).

Maximum Likelihood Estimator (MLE)

Derive from the CDF the following PDF for the GPD

Estimators

$$\frac{1}{\sigma}(1+\xi z)^{-\left(\frac{1+\xi}{\xi}\right)}, \tag{2}$$

and compute $\hat{\sigma}_{MLE}$ and $\hat{\xi}_{MLE}$.

Implementation

1. Create a function **GPDLogLikelihood**, witht the Log-Likelihood function derived from Equation (7), with parameter σ, ξ, and y, i.e. the observations $(z-u)$.
2. Optimize the function **GPDLogLikelihood** using **fmin**.

VaR and ES estimators

Having computed the estimators for $\hat{\xi}$ and $\hat{\sigma}$ either by *MLE* or *MM*, we can plugin those estimates in the following closed form solutions for the VaR and ES:

VaR

$$VaR = \tilde{\mu} + \frac{\tilde{\sigma}}{\xi}\left(\left(1-p\right)^{-\xi} - 1\right).$$

where $\alpha = 1\%$ and

$$\tilde{\sigma} = \sigma\left(1 - P[Y \leq u]\right)^{\xi}$$

$$\tilde{\mu} = u - \frac{\tilde{\sigma}\left(\left(1 - P[Y \leq u]\right)^{-\xi} - 1\right)}{\xi}$$

where $1 - P[Y \leq u]$ is the proportion of of $Y = z - u$ above the threshold u.

VaR and ES estimators

Finally, we can compute the estimator for the ES as:

ES

$$ES = VaR - \frac{\tilde{\sigma}}{(\xi - 1)}\alpha^{-\xi}$$